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Abstract

This paper treats a distributed optimal control problem for a tumor growth model of viscous Cahn–Hilliard type. The evolution of the tumor fraction is governed by a thermodynamic force induced by a double-well potential of logarithmic type. The cost functional contains a nondifferentiable term like the L^1–norm in order to enhance the occurrence of sparsity effects in the optimal controls, i.e., of subdomains of the space-time cylinder where the controls vanish. In the context of cancer therapies, sparsity is very important in order that the patient is not exposed to unnecessary intensive medical treatment. In this work, we focus on the derivation of second-order sufficient optimality conditions for the optimal control problem. While in previous works on the system under investigation such conditions have been established for the case without sparsity, the case with sparsity has not been treated before. The results obtained in this paper also improve the known results on this phase field model for the case without sparsity.

1 Introduction

Let $\alpha > 0,\ \beta > 0,\ \chi > 0$, and let $\Omega \subset \mathbb{R}^3$ denote some open and bounded domain having a smooth boundary $\Gamma = \partial\Omega$ and the unit outward normal $\mathbf{n}$ with associated outward normal derivative $\partial_{\mathbf{n}}$. Moreover, we fix some final time $T > 0$ and introduce for every $t \in (0, T)$ the sets $Q_t := \Omega \times (0, t)$ and $Q^t := \Omega \times (t, T)$. We also set, for convenience, $Q := Q_T$ and $\Sigma := \Gamma \times (0, T)$. We then consider the following optimal control problem:

(CP) Minimize the cost functional

$$
\begin{aligned}
J((\mu, \varphi, \sigma), \mathbf{u}) :=\ & \frac{b_1}{2} \iint_Q |\varphi - \widehat{\varphi}_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi(T) - \widehat{\varphi}_\Omega|^2 + \frac{b_3}{2} \iint_Q (|u_1|^2 + |u_2|^2) \\
& + \kappa \iint_Q (|u_1| + |u_2|) \\
=:\ & J_1((\mu, \varphi, \sigma), \mathbf{u}) + \kappa g(\mathbf{u})
\end{aligned} \tag{1.1}
$$

subject to the state system

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = P(\varphi)(\sigma + \chi(1-\varphi) - \mu) - \mathbb{h}(x,t)u_1 \quad \text{in } Q\,, \tag{1.2}$$

$$\beta\partial_t\varphi - \Delta\varphi + F_1'(\varphi) + F_2'(\varphi) = \mu + \chi\,\sigma \quad \text{in } Q\,, \tag{1.3}$$

$$\partial_t\sigma - \Delta\sigma = -\chi\Delta\varphi - P(\varphi)(\sigma + \chi(1-\varphi) - \mu) + u_2 \quad \text{in } Q\,, \tag{1.4}$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\sigma = 0 \quad \text{on } \Sigma\,, \tag{1.5}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega\,, \tag{1.6}$$

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and to the control constraint

$$\mathbf{u} = (u_1, u_2) \in \mathcal{U}_{\rm ad}\,. \tag{1.7}$$

Here, b_1 and b_2 are nonnegative constants, while b_3 and κ are positive; $\widehat{\varphi}_Q$ and $\widehat{\varphi}_\Omega$ are given target functions. The term $g(\mathbf{u})$ accounts for possible sparsity effects. Moreover, the set of admissible controls $\mathcal{U}_{\rm ad}$ is a nonempty, closed and convex subset of the control space

$$\mathcal{U} := L^\infty(Q)^2. \tag{1.8}$$

The state system (1.2)–(1.6) constitutes a simplified and relaxed version of the four-species thermodynamically consistent model for tumor growth originally proposed by Hawkins-Daruud et al. in [38] that additionally includes the chemotaxis-like terms $\chi\sigma$ in (1.3) and $-\chi\Delta\varphi$ in (1.4). Let us briefly review the role of the occurring symbols. The primary (state) variables φ, μ, σ denote the tumor fraction, the associated chemical potential, and the nutrient concentration, respectively. Furthermore, the additional term $\alpha\partial_t\mu$ corresponds to a parabolic regularization of equation (1.2), while $\beta\partial_t\varphi$ is the viscosity contribution to the Cahn–Hilliard equation. The nonlinearity P denotes a proliferation function, whereas the positive constant χ represents the chemotactic sensitivity and provides the system with a cross-diffusion coupling.

The evolution of the tumor fraction is mainly governed by the nonlinearities F_1 and F_2 whose derivatives occur in (1.3). Here, F_2 is smooth, typically a concave function. As far as F_1 is concerned, we admit in this paper functions of logarithmic type such as

$$F_{1,\log}(r) = \begin{cases} (1+r)\,\ln(1+r) + (1-r)\,\ln(1-r) & \text{for } r \in (-1,1) \\ 2\,\ln(2) & \text{for } r \in \{-1,1\}, \\ +\infty & \text{for } r \notin [-1,1] \end{cases} \tag{1.9}$$

We assume that $F = F_1 + F_2$ is a double-well potential. This is actually the case if $F_2(r) = k(1-r^2)$ with a sufficiently large $k > 0$. Note also that $F'_{1,\log}(r)$ becomes unbounded as $r \searrow -1$ and $r \nearrow 1$.

The control variable u_2 occurring in (1.4) can model either an external medication or some nutrient supply, while u_1, which occurs in the phase equation (1.2), models the application of a cytotoxic drug to the system. Usually, u_1 is multiplied by a truncation function $\mathbb{h}(\varphi)$ in order to have the action only in the spatial region where the tumor cells are located. Typically, one assumes that $\mathbb{h}(-1) = 0$, $\mathbb{h}(1) = 1$, and $\mathbb{h}(\varphi)$ is in between if $-1 < \varphi < 1$; see [29, 35, 41, 42] for some insights on possible choices of $\mathbb{h}$. Also in [15, 17, 54], this kind of nonlinear coupling between u_1 and φ has been admitted. For our following analysis, this nonlinear coupling is too strong, and, only for technical reasons, we have chosen to simplify the original system somewhat by assuming that $\mathbb{h} = \mathbb{h}(x,t)$ is a bounded nonnegative function that does not depend on φ. We stress the fact that this simplification does not have any impact on the validity of the results from [17] to be used below.

As far as well-posedness is concerned, the above model was already investigated in the case $\chi = 0$ in [6–9], and in [25] with $\alpha = \beta = \chi = 0$. There the authors also pointed out how the relaxation parameters α and β can be set to zero, by providing the proper framework in which a limit system can be identified and uniquely solved. We also note that in [13] a version has been studied in which the Laplacian in the equations (1.2)–(1.4) has been replaced by fractional powers of a more general class of selfadjoint operators having compact resolvents. A model which is similar to the one studied in this note was the subject of [15, 54].

For some nonlocal variations of the above model we refer to [27, 28, 47]. Moreover, in order to better emulate in-vivo tumor growth, it is possible to include in similar models the effects generated by the fluid flow development by postulating a Darcy law or a Stokes–Brinkman law. In this direction, we refer to [11,21,24,27,29–33,35,59], and we also mention [36], where elastic effects are included. For further models, discussing the case of multispecies, we refer the reader to [21,27]. The investigation of associated optimal control problems also presents a wide number of results of which we mention [10,13,15,22,23,28,34,37,42,48–52,54,56].

Sparsity in the optimal control theory of partial differential equations is a very active field of research. The use of sparsity-enhancing functionals goes back to inverse problems and image processing. Soon after the seminal paper [57], many results were published. We mention only very few of them with closer relation to our paper, in particular [1,39,40], on directional sparsity, and [5] on a general theorem for second-order conditions; moreover, we refer to some new trends in the investigation of sparsity, namely, infinite horizon sparse optimal control (see, e.g., [43, 44]), and fractional order optimal control (cf. [46], [45]). These papers concentrated on first-order optimality conditions for sparse optimal controls of single elliptic and parabolic equations. In [3, 4], first- and second-order optimality conditions have been discussed in the context of sparsity for the (semilinear) system of FitzHugh–Nagumo equations. Moreover, we refer to the measure control of the Navier–Stokes system studied in [2].

The optimal control problem **(CP)** has recently been investigated in [17] for the case of logarithmic potentials F_1 and without sparsity terms, where second-order sufficient optimality conditions have been derived using the τ–critical cone and the splitting technique as described in the textbook [58]. In [54] and [18], sparsity terms have been incorporated, where in the latter paper not only logarithmic nonlinearities but also nondifferentiable double obstacle potentials have been admitted. However, second-order sufficient optimality conditions have not been derived.

The derivation of meaningful second-order conditions for locally optimal controls of **(CP)** in the logarithmic case with sparsity term is the main object of this paper. In particular, we aim at constructing suitable critical cones which are as small as possible. In our approach, we follow the recent work [55] on the sparse optimal control of Allen–Cahn systems, which was based on ideas developed in [4].

The paper is organized as follows. In the next section, we list and discuss our assumptions, and we collect known results from [18] concerning the properties of the state system (1.2)–(1.6) and of the control-to-state operator. In Section 3, we study the optimal control problem. We derive first-order necessary optimality conditions and results concerning the full sparsity of local minimizers, and we establish second-order sufficient optimality conditions for the optimal control problem **(CP)**. In an appendix, we prove auxiliary results that are needed for the main theorem on second-order sufficient conditions.

Prior to this, let us fix some notation. For any Banach space X, we denote by $\|\cdot\|_X$ the norm in the space X, by X^* its dual space, and by $\langle\cdot\,,\cdot\rangle_X$ the duality pairing between X^* and X. For any $1 \le p \le \infty$ and $k \ge 0$, we denote the standard Lebesgue and Sobolev spaces on Ω by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, and the corresponding norms by $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. For $p = 2$, they become Hilbert spaces, and we employ the standard notation $H^k(\Omega) := W^{k,2}(\Omega)$. As usual, for Banach spaces X and Y that are both continuously embedded in some topological vector space Z, we introduce the linear space $X \cap Y$ which becomes a Banach space when equipped with its natural norm $\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y$, for $v \in X \cap Y$. Moreover, we recall the definition (1.8) of the control space $\mathcal{U}$ and introduce the spaces

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega)\,, \quad W_0 := \{v \in H^2(\Omega) :\ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}. \tag{1.10}$$

Furthermore, by $(\cdot\,,\cdot)$ and $\|\cdot\|$ we denote the standard inner product and related norm in H, and, for simplicity, we also set $\langle\cdot\,,\cdot\rangle := \langle\cdot\,,\cdot\rangle_V$.

Throughout the paper, we make repeated use of Hölder's inequality, of the elementary Young inequality

$$ab \le \delta|a|^2 + \frac{1}{4\delta}|b|^2 \quad \forall\, a, b \in \mathbb{R}, \quad \forall\, \delta > 0, \tag{1.11}$$

as well as the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$.

We close this section by introducing a convention concerning the constants used in estimates within this paper: we denote by C any positive constant that depends only on the given data occurring in the state system and in the cost functional, as well as on a constant that bounds the $(L^\infty(Q) \times L^\infty(Q))$–norms of the elements of $\mathcal{U}_{\rm ad}$. The actual value of such generic constants C may change from formula to formula or even within formulas. Finally, the notation C_δ indicates a positive constant that additionally depends on the quantity δ.

2 General setting and properties of the control-to-state operator

In this section, we introduce the general setting of our control problem and state some results on the state system (1.2)–(1.6) and the control-to-state operator that in its present form have been established in [17, 18].

We make the following assumptions on the data of the system.

(A1) $\alpha,\, \beta,\, \chi$ are positive constants.

(A2) $F = F_1 + F_2$, where $F_2 \in C^5(\mathbb{R})$ has a Lipschitz continuous derivative F_2', and where $F_1 : \mathbb{R} \to [0, +\infty]$ is convex and lower semicontinuous and satisfies $F_1(0) = 0$, $F_{1_{|(-1,1)}} \in C^5(-1, 1)$, as well as

$$\lim_{r \searrow -1} F_1'(r) = -\infty \quad \text{and} \quad \lim_{r \nearrow 1} F_1'(r) = +\infty\,. \tag{2.1}$$

(A3) $P \in C^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ and $\mathbb{h} \in L^\infty(Q)$ are nonnegative and bounded.

(A4) The initial data satisfy $\mu_0, \sigma_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi_0 \in W_0$, as well as

$$-1 < \min_{x \in \overline{\Omega}} \varphi_0(x) \le \max_{x \in \overline{\Omega}} \varphi_0(x) < 1\,. \tag{2.2}$$

(A5) With fixed given constants $\underline{u}_i, \overline{u}_i$ satisfying $\underline{u}_i < \overline{u}_i$, $i = 1, 2$, we have

$$\mathcal{U}_{\rm ad} = \{\mathbf{u} = (u_1, u_2) \in \mathcal{U} : \underline{u}_i \le u_i \le \overline{u}_i \text{ a.e. in } Q \text{ for } i = 1, 2\}\,. \tag{2.3}$$

(A6) $R > 0$ is a constant such that $\mathcal{U}_{\rm ad} \subset \mathcal{U}_R := \{\mathbf{u} \in \mathcal{U} :\, \|\mathbf{u}\|_{\mathcal{U}} < R\}$.

Remark 2.1. Observe that **(A3)** implies that the functions P, P', P'' are Lipschitz continuous on $\mathbb{R}$. Let us also note that $F_1 = F_{1,\log}$ satisfies **(A2)**. Moreover, (2.2) implies that initially there are no pure phases. Finally, **(A6)** just fixes an open and bounded subset of $\mathcal{U}$ that contains $\mathcal{U}_{\rm ad}$.

The following result is a consequence of [18, Thm. 2.3].

Theorem 2.2. *Suppose that the conditions* **(A1)–(A6)** *are fulfilled. Then the state system* (1.2)–(1.6) *has for every* $\mathbf{u} = (u_1, u_2) \in \mathcal{U}_R$ *a unique strong solution* (μ, φ, σ) *with the regularity*

$$\mu \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W_0) \cap L^\infty(Q), \tag{2.4}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W_0) \cap C^0(\overline{Q}), \tag{2.5}$$

$$\sigma \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W_0) \cap L^\infty(Q). \tag{2.6}$$

Moreover, there is a constant $K_1 > 0$*, which depends on* $\Omega, T, R, \alpha, \beta$ *and the data of the system, but not on the choice of* $\mathbf{u} \in \mathcal{U}_R$*, such that*

$$\begin{aligned} &\|\mu\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)\cap L^\infty(Q)} \\ &+ \|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W_0)\cap C^0(\overline{Q})} \\ &+ \|\sigma\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)\cap L^\infty(Q)} \le K_1 . \end{aligned} \tag{2.7}$$

Furthermore, there are constants r_*, r^**, which depend on* $\Omega, T, R, \alpha, \beta$ *and the data of the system, but not on the choice of* $\mathbf{u} \in \mathcal{U}_R$*, such that*

$$-1 < r_* \le \varphi(x,t) \le r^* < 1 \quad \textit{for all } (x,t) \in \overline{Q}. \tag{2.8}$$

Also, there is some constant $K_2 > 0$ *having the same dependencies as* K_1 *such that*

$$\max_{i=0,1,2,3} \left\|P^{(i)}(\varphi)\right\|_{L^\infty(Q)} + \max_{i=0,1,2,3,4,5} \max_{j=1,2} \left\|F_j^{(i)}(\varphi)\right\|_{L^\infty(Q)} \le K_2 . \tag{2.9}$$

Finally, if $\mathbf{u}_i \in \mathcal{U}_R$ *are given controls and* $(\mu_i, \varphi_i, \sigma_i)$ *the corresponding solutions to* (1.2)–(1.6)*, for* $i = 1, 2$*, then, with a constant* $K_3 > 0$ *having the same dependencies as* K_1*,*

$$\begin{aligned} &\|\mu_1 - \mu_2\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)} + \|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)} \\ &+ \|\sigma_1 - \sigma_2\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)} \le K_3 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(Q)^2} . \end{aligned} \tag{2.10}$$

Remark 2.3. Condition (2.8), known as the *separation property*, is especially important for the case of singular potentials such as $F_1 = F_{1,\log}$, since it guarantees that the phase variable φ always stays away from the critical values $-1, +1$. The singularity of F_1' is therefore no longer an obstacle for the analysis, as the values of φ range in some interval in which F_1' is smooth.

Owing to Theorem 2.2, the control-to-state operator

$$\mathcal{S} : \mathbf{u} = (u_1, u_2) \mapsto (\mu, \varphi, \sigma)$$

is well defined as a mapping between $\mathcal{U} = L^\infty(Q)^2$ and the Banach space specified by the regularity results (2.4)–(2.6). We now discuss its differentiability properties. For this purpose, some functional analytic preparations are in order. We first define the linear spaces

$$\begin{aligned} \mathcal{X} &:= X \times \widetilde{X} \times X, \quad \text{where} \\ X &:= H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W_0) \cap L^\infty(Q), \\ \widetilde{X} &:= W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W_0) \cap C^0(\overline{Q}), \end{aligned} \tag{2.11}$$

which are Banach spaces when endowed with their natural norms. Next, we introduce the linear space

$$\mathcal{Y} := \big\{(\mu,\varphi,\sigma)\in\mathcal{X}:\ \alpha\partial_t\mu+\partial_t\varphi-\Delta\mu\in L^\infty(Q),\ \ \beta\partial_t\varphi-\Delta\varphi-\mu\in L^\infty(Q),\\ \partial_t\sigma-\Delta\sigma+\chi\Delta\varphi\in L^\infty(Q)\big\}, \tag{2.12}$$

which becomes a Banach space when endowed with the norm

$$\|(\mu,\varphi,\sigma)\|_{\mathcal{Y}} := \|(\mu,\varphi,\sigma)\|_{\mathcal{X}} + \|\alpha\partial_t\mu+\partial_t\varphi-\Delta\mu\|_{L^\infty(Q)} + \|\beta\partial_t\varphi-\Delta\varphi-\mu\|_{L^\infty(Q)}\\ + \|\partial_t\sigma-\Delta\sigma+\chi\Delta\varphi\|_{L^\infty(Q)}\,. \tag{2.13}$$

Finally, we put

$$Z := H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0), \tag{2.14}$$

$$\mathcal{Z} := Z\times Z\times Z. \tag{2.15}$$

For fixed $(\varphi^*,\mu^*,\sigma^*)$, we first discuss an auxiliary result for the linear initial-boundary value problem

$$\alpha\partial_t\mu+\partial_t\varphi-\Delta\mu = \lambda_1\left[P(\varphi^*)(\sigma-\chi\varphi-\mu)+P'(\varphi^*)(\sigma^*+\chi(1-\varphi^*)-\mu^*)\varphi\right]\\ -\lambda_2\,\mathbb{h}\,h_1+\lambda_3 f_1 \quad\text{in } Q, \tag{2.16}$$

$$\beta\partial_t\varphi-\Delta\varphi-\mu = \lambda_1\left[\chi\,\sigma-F''(\varphi^*)\varphi\right]+\lambda_3 f_2 \quad\text{in } Q, \tag{2.17}$$

$$\partial_t\sigma-\Delta\sigma+\chi\Delta\varphi = \lambda_1\left[-P(\varphi^*)(\sigma-\chi\varphi-\mu)-P'(\varphi^*)(\sigma^*+\chi(1-\varphi^*)-\mu^*)\varphi\right]\\ +\lambda_2 h_2+\lambda_3 f_3 \quad\text{in } Q, \tag{2.18}$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\sigma = 0 \quad\text{on } \Sigma, \tag{2.19}$$

$$\mu(0) = \lambda_4\mu_0,\quad \varphi(0) = \lambda_4\varphi_0,\quad \sigma(0) = \lambda_4\sigma_0,\quad\text{in } \Omega, \tag{2.20}$$

which for $\lambda_1=\lambda_2=1$ and $\lambda_3=\lambda_4=0$ coincides with the linearization of the state equation at $((\mu^*,\varphi^*,\sigma^*),(u_1^*,u_2^*))$. We have the following result.

Lemma 2.4. *Suppose that $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\{0,1\}$ are given and that the assumptions* **(A1)**–**(A6)** *are fulfilled. Moreover, let $\mathbf{u}^*=(u_1^*,u_2^*)\in\mathcal{U}_R$ be given and $(\mu^*,\varphi^*,\sigma^*)=\mathcal{S}(\mathbf{u}^*)$. Then* (2.16)–(2.20) *has for every $\mathbf{h}=(h_1,h_2)\in L^2(Q)^2$ and $(f_1,f_2,f_3)\in L^2(Q)^3$ a unique solution $(\mu,\varphi,\sigma)\in Z\times\widetilde{X}\times Z$. Moreover, the linear mapping*

$$((h_1,h_2),(f_1,f_2,f_3))\mapsto(\mu,\varphi,\sigma) \tag{2.21}$$

is continuous from $L^2(Q)^2\times L^2(Q)^3$ into $Z\times\widetilde{X}\times Z$. Moreover, if $\mathbf{h}\in L^\infty(Q)^2$ and $(f_1,f_2,f_3)\in L^\infty(Q)^3$, in addition, then it holds $(\mu,\varphi,\sigma)\in\mathcal{Y}$, and the mapping (2.21) *is continuous from $L^\infty(Q)^2\times L^\infty(Q)^3$ into $\mathcal{Y}$.*

Proof. The existence result and the continuity of the mapping (2.21) between the spaces $L^\infty(Q)^2\times L^\infty(Q)^3$ and $\mathcal{Y}$ directly follow from the statement of [17, Lem. 4.1 and Rem. 4.2]. Moreover, from the estimates (4.36)–(4.38) and (4.43) in [17] we can conclude that the mapping (2.21) is also continuous between the spaces $L^2(Q)^2\times L^2(Q)^3$ and $Z\times\widetilde{X}\times Z$. □

Now let $\mathbf{u}^*=(u_1^*,u_2^*)\in\mathcal{U}_R$ be arbitrary and $(\mu^*,\varphi^*,\sigma^*)=\mathcal{S}(\mathbf{u}^*)$. Then, according to [17, Thm. 4.4], the control-to-state operator $\mathcal{S}$ is twice continuously Fréchet differentiable at $\mathbf{u}^*$ as a mapping from $\mathcal{U}$ into $\mathcal{Y}$. Moreover, for every $\mathbf{h}=(h_1,h_2)\in\mathcal{U}$, the first Fréchet derivative $S'(\mathbf{u}^*)\in\mathcal{L}(\mathcal{U},\mathcal{Y})$ of $\mathcal{S}$ at $\mathbf{u}^*$ is given by the identity $\mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]=(\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}})$, where $(\eta^{\mathbf{h}},\rho^{\mathbf{h}},\theta^{\mathbf{h}})\in\mathcal{Y}$ is the unique solution to the linearization of the state system given by the initial-boundary value problem (2.16)–(2.20) with $\lambda_1=\lambda_2=1$ and $\lambda_3=\lambda_4=0$.

Remark 2.5. Observe that, in view of the continuity of the embedding $\mathcal{Y} \subset Z \times \widetilde{X} \times Z$, the operator $\mathcal{S}'(\mathbf{u}^*) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ also belongs to the space $\mathcal{L}(\mathcal{U}, Z \times \widetilde{X} \times Z)$ and, owing to the density of $\mathcal{U}$ in $L^2(Q)^2$, can be extended continuously to an element of $\mathcal{L}(L^2(Q)^2, Z \times \widetilde{X} \times Z)$ without changing its operator norm. Denoting the extended operator still by $\mathcal{S}'(\mathbf{u}^*)$, we see that the identity $\mathcal{S}'(\mathbf{u}^*)[\mathbf{h}] = (\eta^{\mathbf{h}}, \xi^{\mathbf{h}}, \theta^{\mathbf{h}})$ is also valid for every $\mathbf{h} \in L^2(Q)^2$, only that $(\eta^{\mathbf{h}}, \xi^{\mathbf{h}}, \theta^{\mathbf{h}}) \in Z \times \widetilde{X} \times Z$, in general. In addition, it also follows from the proof of [17, Lem. 4.1] that there is a constant $K_4 > 0$, which depends only on R and the data, such that

$$\|\mathcal{S}'(\mathbf{u})[\mathbf{h}]\|_{Z\times\widetilde{X}\times Z} \leq K_4 \|\mathbf{h}\|_{L^2(Q)^2} \quad \text{for all } \mathbf{u} \in \mathcal{U}_R \text{ and every } \mathbf{h} \in L^2(Q)^2 . \tag{2.22}$$

Next, we show a Lipschitz property for the extended operator $\mathcal{S}'$.

Lemma 2.6. *The mapping $\mathcal{S}' : \mathcal{U} \to \mathcal{L}(L^2(Q)^2, Z \times \widetilde{X} \times Z)$, $\mathbf{u} \mapsto \mathcal{S}'(\mathbf{u})$, is Lipschitz continuous in the following sense: there is a constant $K_5 > 0$, which depends only on R and the data, such that, for all controls $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}_R$ and all increments $\mathbf{h} \in L^2(Q)^2$,*

$$\| (\mathcal{S}'(\mathbf{u}_1) - \mathcal{S}'(\mathbf{u}_2)) [\mathbf{h}]\|_{\mathcal{Z}} \leq K_5 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(Q)^2} \|\mathbf{h}\|_{L^2(Q)^2} . \tag{2.23}$$

Proof. We put $(\mu_i, \varphi_i, \sigma_i) := \mathcal{S}(\mathbf{u}_i)$, $(\eta_i, \xi_i, \theta_i) := \mathcal{S}'(\mathbf{u}_i)[\mathbf{h}]$, $i = 1, 2$, as well as

$$\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mu := \mu_1 - \mu_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \sigma := \sigma_1 - \sigma_2,$$
$$\eta := \eta_1 - \eta_2, \quad \xi := \xi_1 - \xi_2, \quad \theta := \theta_1 - \theta_2 .$$

Then it follows from (2.10) in Theorem 2.2 that

$$\|(\mu, \varphi, \sigma)\|_{\mathcal{Z}} \leq K_3 \|\mathbf{u}\|_{L^2(Q)^2} . \tag{2.24}$$

Moreover, (η, ξ, θ) solves the problem

$$\alpha\partial_t\eta + \partial_t\xi - \Delta\eta = P(\varphi_1)(\theta - \chi\xi - \eta) + P'(\varphi_1)(\sigma_1 + \chi(1 - \varphi_1) - \mu_1)\xi + f_1 , \tag{2.25}$$
$$\beta\partial_t\xi - \Delta\xi = \chi\theta - F''(\varphi_1)\xi + f_2 , \tag{2.26}$$
$$\partial_t\theta - \Delta\theta + \chi\Delta\xi = -P(\varphi_1)(\theta - \chi\xi - \eta) - P'(\varphi_1)(\sigma_1 + \chi(1 - \varphi_1) - \mu_1)\xi + f_3 , \tag{2.27}$$
$$\partial_{\mathbf{n}}\eta = \partial_{\mathbf{n}}\xi = \partial_{\mathbf{n}}\theta = 0 , \tag{2.28}$$
$$\eta(0) = \xi(0) = \theta(0) = 0 , \tag{2.29}$$

which is of the form (2.16)–(2.20) with $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \lambda_4 = 0$, and where

$$f_1 := -f_3 := ((P(\varphi_1) - P(\varphi_2))(\theta_2 - \chi\xi_2 - \eta_2) + P'(\varphi_1)(\sigma - \chi\varphi - \mu)\xi_2 + (P'(\varphi_1) - P'(\varphi_2))(\sigma_2 + \chi(1 - \varphi_2) - \mu_2)\xi_2 , \tag{2.30}$$
$$f_2 := -(F''(\varphi_1) - F''(\varphi_2))\xi_2 . \tag{2.31}$$

We therefore conclude from Lemma 2.4 that

$$\|(\eta, \xi, \theta)\|_{\mathcal{Z}} \leq C \left(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)}\right) .$$

Hence, the proof will be finished once we can show that

$$\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} \leq C \|\mathbf{u}\|_{L^2(Q)^2} \|\mathbf{h}\|_{L^2(Q)^2} . \tag{2.32}$$

To this end, we first use the mean value theorem, (2.9), Hölder's inequality, the continuity of the embedding $V \subset L^4(\Omega)$, as well as (2.10) and (2.24), to find that

$$\|f_2\|^2_{L^2(Q)} \le C \iint_Q |\varphi|^2 |\xi_2|^2 \le C \int_0^T \|\varphi\|_4^2 \|\xi_2\|_4^2 \, ds \le C \|\varphi\|^2_{C^0([0,T];V)} \|\xi_2\|^2_{C^0([0,T];V)}$$
$$\le C \|\varphi\|_Z^2 \|\mathcal{S}'(\mathbf{u}_2)[\mathbf{h}]\|^2_{\mathcal{Z}} \le C \|\mathbf{u}\|^2_{L^2(Q)^2} \|\mathbf{h}\|^2_{L^2(Q)^2} . \tag{2.33}$$

Here, we have for convenience omitted the argument s in the third integral. We will do this repeatedly in the following. For the three summands on the right-hand side of (2.30), which we denote by A_1, A_2, A_3, in this order, we obtain by similar reasoning the estimates

$$\iint_Q |A_1|^2 \le C \iint_Q |\varphi|^2 |\theta_2 - \chi\xi_2 - \eta_2|^2 \le C \int_0^T \|\varphi\|_4^2 \left(\|\theta_2\|_4^2 + \|\xi_2\|_4^2 + \|\eta_2\|_4^2\right) ds$$
$$\le C \|\mathbf{u}\|^2_{L^2(Q)^2} \|\mathbf{h}\|^2_{L^2(Q)^2} , \tag{2.34}$$

$$\iint_Q |A_2|^2 \le C \int_0^T |\xi_2|^2 \left(|\mu|^2 + |\varphi|^2 + |\sigma|^2\right) ds \le C \int_0^T \|\xi_2\|_4^2 \left(\|\mu\|_4^2 + \|\varphi\|_4^2 + \|\sigma\|_4^2\right) ds$$
$$\le C \|\mathbf{u}\|^2_{L^2(Q)^2} \|\mathbf{h}\|^2_{L^2(Q)^2} , \tag{2.35}$$

$$\iint_Q |A_3|^2 \le C \left(\|\sigma_2\|^2_{L^\infty(Q)} + \|\varphi_2\|^2_{L^\infty(Q)} + \|\mu_2\|^2_{L^\infty(Q)} + 1\right) \iint_Q |\varphi|^2 |\xi_2|^2$$
$$\le C \|\mathbf{u}\|^2_{L^2(Q)^2} \|\mathbf{h}\|^2_{L^2(Q)^2} , \tag{2.36}$$

where in the last estimate we also used (2.7) and (2.33). With this, the assertion is proved. □

Next, we turn our interest to the second Fréchet derivative $\mathcal{S}''(\mathbf{u}^*)$ of $\mathcal{S}$ at $\mathbf{u}^*$. Let $\mathbf{h} = (h_1, h_2) \in \mathcal{U}$ and $\mathbf{k} = (k_1, k_2) \in \mathcal{U}$. Then, $(\eta^{\mathbf{h}}, \xi^{\mathbf{h}}, \theta^{\mathbf{h}}) := \mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]$ and $(\eta^{\mathbf{k}}, \xi^{\mathbf{k}}, \theta^{\mathbf{k}}) := \mathcal{S}'(\mathbf{u}^*)[\mathbf{k}]$ both belong to $\mathcal{Y}$ and, by virtue of [17, Thm. 4.6], $(\nu, \psi, \rho) = \mathcal{S}''(\mathbf{u}^*)[\mathbf{h}, \mathbf{k}] \in \mathcal{Y}$ is the unique solution to the bilinearization of the state system at $((\mu^*, \varphi^*, \sigma^*), (u_1^*, u_2^*))$, which is given by the linear initial-boundary value problem

$$\alpha\partial_t\nu + \partial_t\psi - \Delta\nu = P(\varphi^*)(\rho - \chi\psi - \nu) + P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)\psi + f_1 , \tag{2.37}$$
$$\beta\partial_t\psi - \Delta\psi - \nu = \chi\rho - F''(\varphi^*)\psi + f_2 , \tag{2.38}$$
$$\partial_t\rho - \Delta\rho + \chi\Delta\psi = -P(\varphi^*)(\rho - \chi\psi - \nu) - P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)\psi + f_3 , \tag{2.39}$$
$$\partial_{\mathbf{n}}\nu = \partial_{\mathbf{n}}\psi = \partial_{\mathbf{n}}\rho = 0 , \tag{2.40}$$
$$\nu(0) = \psi(0) = \rho(0) = 0 , \tag{2.41}$$

and which is again of the form (2.16)–(2.20) with $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \lambda_4 = 0$, where

$$f_1 := -f_3 := P'(\varphi^*) \left(\xi^{\mathbf{k}} (\theta^{\mathbf{h}} - \chi\xi^{\mathbf{h}} - \eta^{\mathbf{h}}) + \xi^{\mathbf{h}} (\theta^{\mathbf{k}} - \chi\xi^{\mathbf{k}} - \eta^{\mathbf{k}})\right)$$
$$+ P''(\varphi^*)\, \xi^{\mathbf{k}} \xi^{\mathbf{h}} (\sigma^* + \chi(1-\varphi^*) - \mu^*) , \tag{2.42}$$

$$f_2 := -F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}} . \tag{2.43}$$

Now assume that $\mathbf{h}, \mathbf{k} \in L^2(Q)^2$ are given. Then the expressions $(\eta^{\mathbf{h}}, \xi^{\mathbf{h}}, \theta^{\mathbf{h}}) := \mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]$ and $(\eta^{\mathbf{k}}, \xi^{\mathbf{k}}, \theta^{\mathbf{k}}) := \mathcal{S}'(\mathbf{u}^*)[\mathbf{k}]$ are well-defined elements of the space $Z \times \tilde{X} \times Z$, where $\mathcal{S}'(\mathbf{u}^*)$ now denotes the extension of the Fréchet derivative introduced in Remark 2.5. We now claim that there is a constant $\widehat{C} > 0$ that depends only on R and the data, such that

$$\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} \,\le\, \widehat{C}\,\|\mathbf{h}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2}\,. \tag{2.44}$$

Indeed, arguing as in the derivation of the estimates (2.33)–(2.36), we obtain

$$\begin{aligned}
&\|f_1\|^2_{L^2(Q)} \,\le\, C\int_0^T \|\xi^{\mathbf{k}}\|_4^2\,\left(\|\theta^{\mathbf{h}}\|_4^2 + \|\xi^{\mathbf{h}}\|_4^2 + \|\eta^{\mathbf{h}}\|_4^2\right)\,ds\\
&\quad + C\int_0^T \|\xi^{\mathbf{h}}\|_4^2\,\left(\|\theta^{\mathbf{k}}\|_4^2 + \|\xi^{\mathbf{k}}\|_4^2 + \|\eta^{\mathbf{k}}\|_4^2\right)\,ds\\
&\quad + C\,\left(\|\sigma^*\|^2_{L^\infty(Q)} + \|\varphi^*\|^2_{L^\infty(Q)} + \|\mu^*\|^2_{L^\infty(Q)} + 1\right)\int_0^T \|\xi^{\mathbf{k}}\|_4^2\,\|\xi^{\mathbf{h}}\|_4^2\,ds\\
&\le\, C\,\|\mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]\|^2_{C^0([0,T];V)}\,\|\mathcal{S}'(\mathbf{u}^*)[\mathbf{k}]\|^2_{C^0([0,T];V)} \,\le\, C\,\|\mathbf{h}\|^2_{L^2(Q)^2}\,\|\mathbf{k}\|^2_{L^2(Q)^2}\,,
\end{aligned}$$

$$\|f_2\|^2_{L^2(Q)} \,\le\, C\int_0^T \|\xi^{\mathbf{h}}\|_4^2\,\|\xi^{\mathbf{k}}\|_4^2\,ds \,\le\, C\,\|\mathbf{h}\|^2_{L^2(Q)^2}\,\|\mathbf{k}\|^2_{L^2(Q)^2}\,,$$

which proves the claim. At this point, we can conclude from Lemma 2.4 that the system (2.37)–(2.41) has for every $\mathbf{h},\mathbf{k} \in L^2(Q)^2$ a unique solution $(\nu,\psi,\rho) \in Z \times \widetilde{X} \times Z$. Moreover, we have, with a constant $K_6 > 0$ that depends only on R and the data,

$$\|(\nu,\psi,\rho)\|_{Z\times\widetilde{X}\times Z} \,\le\, K_6\,\|\mathbf{h}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2} \quad \forall\,\mathbf{h},\mathbf{k} \in L^2(Q)^2. \tag{2.45}$$

Remark 2.7. Similarly as in Remark 2.5, the operator $\mathcal{S}''(\mathbf{u}^*) \in \mathcal{L}(\mathcal{U},\mathcal{L}(\mathcal{U},\mathcal{Y}))$ can be extended continuously to an element of $\mathcal{L}(L^2(Q)^2,\mathcal{L}(L^2(Q)^2, Z \times \widetilde{X} \times Z))$ without changing its operator norm. Denoting the extended operator still by $\mathcal{S}''(\mathbf{u}^*)$, we see that the identity $\mathcal{S}''(\mathbf{u}^*)[\mathbf{h},\mathbf{k}] = (\nu,\psi,\rho)$ is also valid for every $\mathbf{h},\mathbf{k} \in L^2(Q)^2$, only that $(\nu,\psi,\rho) \in Z \times \widetilde{X} \times Z$, in general. In addition, we have

$$\|\mathcal{S}''(\mathbf{u})[\mathbf{h},\mathbf{k}]\|_{Z\times\widetilde{X}\times Z} \,\le\, K_6\,\|\mathbf{h}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2} \quad \text{for all } \mathbf{u} \in \mathcal{U}_R \text{ and } \mathbf{h},\mathbf{k} \in L^2(Q)^2\,. \tag{2.46}$$

We conclude our preparatory work by showing a Lipschitz property for the extended operator $\mathcal{S}''$ that resembles (2.23).

Lemma 2.8. *The mapping $\mathcal{S}'' : \mathcal{U} \to \mathcal{L}(L^2(Q)^2,\mathcal{L}(L^2(Q)^2, Z\times\widetilde{X}\times Z))$, $\mathbf{u} \mapsto \mathcal{S}''(\mathbf{u})$, is Lipschitz continuous in the following sense: there is a constant $K_7 > 0$, which depends only on R and the data, such that, for all controls $\mathbf{u}_1,\mathbf{u}_2 \in \mathcal{U}_R$ and all increments $\mathbf{h},\mathbf{k} \in L^2(Q)^2$,*

$$\|\left(\mathcal{S}''(\mathbf{u}_1) - \mathcal{S}''(\mathbf{u}_2)\right)[\mathbf{h},\mathbf{k}]\|_{\mathcal{Z}} \,\le\, K_7\,\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(Q)^2}\,\|\mathbf{h}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2}\,. \tag{2.47}$$

Proof. We put $(\mu_i,\varphi_i,\sigma_i) := \mathcal{S}(\mathbf{u}_i)$, $(\eta_i^{\mathbf{h}},\xi_i^{\mathbf{h}},\theta_i^{\mathbf{h}}) := \mathcal{S}'(\mathbf{u}_i)[\mathbf{h}]$, $(\eta_i^{\mathbf{k}},\xi^{\mathbf{k}},\theta_i^{\mathbf{k}}) := \mathcal{S}'(\mathbf{u}_i)[\mathbf{k}]$, $(\nu_i,\psi_i,\rho_i) := \mathcal{S}''(\mathbf{u}_i)[\mathbf{h},\mathbf{k}]$, for $i = 1,2$, as well as

$$\begin{aligned}
&\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mu := \mu_1 - \mu_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \sigma := \sigma_1 - \sigma_2,\\
&\eta^{\mathbf{h}} := \eta_1^{\mathbf{h}} - \eta_2^{\mathbf{h}}, \quad \xi^{\mathbf{h}} := \xi_1^{\mathbf{h}} - \xi_2^{\mathbf{h}}, \quad \theta^{\mathbf{h}} := \theta_1^{\mathbf{h}} - \theta_2^{\mathbf{h}},\\
&\eta^{\mathbf{k}} := \eta_1^{\mathbf{k}} - \eta_2^{\mathbf{k}}, \quad \xi^{\mathbf{k}} := \xi_1^{\mathbf{k}} - \xi_2^{\mathbf{k}}, \quad \theta^{\mathbf{k}} := \theta_1^{\mathbf{k}} - \theta_2^{\mathbf{k}},\\
&\nu := \nu_1 - \nu_2, \quad \psi := \psi_1 - \psi_2, \quad \rho := \rho_1 - \rho_2.
\end{aligned}$$

Then it follows from (2.10) and (2.23) that

$$\|(\mu,\varphi,\sigma)\|_{\mathcal{Z}} \,\le\, C\,\|\mathbf{u}\|_{L^2(Q)^2}\,, \quad \|(\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}})\|_{\mathcal{Z}} \,\le\, C\,\|\mathbf{u}\|_{L^2(Q)^2}\,\|\mathbf{h}\|_{L^2(Q)^2}\,,$$
$$\|(\eta^{\mathbf{k}},\xi^{\mathbf{k}},\theta^{\mathbf{k}})\|_{\mathcal{Z}} \,\le\, C\,\|\mathbf{u}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2}\,. \tag{2.48}$$

We also recall the estimates (2.22) and (2.46). Moreover, (ν,ψ,ρ) solves the problem

$$\alpha\partial_t\nu+\partial_t\psi-\Delta\nu \,=\, P(\varphi_1)(\rho-\chi\psi-\nu)+P'(\varphi_1)(\sigma_1+\chi(1-\varphi_1)-\mu_1)\psi\,+\,g_1\,, \tag{2.49}$$
$$\beta\partial_t\psi-\Delta\psi=\chi\rho-F''(\varphi_1)\psi\,+\,g_2\,, \tag{2.50}$$
$$\partial_t\rho-\Delta\rho+\chi\Delta\psi=-P(\varphi_1)(\rho-\chi\psi-\nu)-P'(\varphi_1)(\sigma_1+\chi(1-\varphi_1)-\mu_1)\psi\,+\,g_3\,, \tag{2.51}$$
$$\partial_{\mathbf{n}}\nu=\partial_{\mathbf{n}}\psi=\partial_{\mathbf{n}}\rho=0\,, \tag{2.52}$$
$$\nu(0)=\psi(0)=\rho(0)=0\,, \tag{2.53}$$

which is again of the form (2.16)–(2.20) with $\lambda_1=\lambda_3=1$ and $\lambda_2=\lambda_4=0$, where

$$\begin{aligned}
g_1 &:= -g_3 := ((P(\varphi_1)-P(\varphi_2))(\rho_2-\chi\psi_2-\nu_2)\,+\,P'(\varphi_1)(\sigma-\chi\varphi-\mu)\psi_2\\
&+\,(P'(\varphi_1)-P'(\varphi_2))(\sigma_2+\chi(1-\varphi_2)-\mu_2)\psi_2\,+\,(P'(\varphi_1)-P'(\varphi_2))\,\xi_1^{\mathbf{k}}\,(\theta_1^{\mathbf{h}}-\chi\xi_1^{\mathbf{h}}-\eta_1^{\mathbf{h}})\\
&+\,P'(\varphi_2)\,\xi^{\mathbf{k}}\,(\theta_1^{\mathbf{h}}-\chi\xi_1^{\mathbf{h}}-\eta_1^{\mathbf{h}})\,+\,P'(\varphi_2)\,\xi_2^{\mathbf{k}}\,(\theta^{\mathbf{h}}-\chi\xi^{\mathbf{h}}-\eta^{\mathbf{h}})\\
&+\,(P'(\varphi_1)-P'(\varphi_2))\,\xi_1^{\mathbf{h}}\,(\theta_1^{\mathbf{k}}-\chi\xi_1^{\mathbf{k}}-\eta_1^{\mathbf{k}})\,+\,P'(\varphi_2)\,\xi^{\mathbf{h}}\,(\theta_1^{\mathbf{k}}-\chi\xi_1^{\mathbf{k}}-\eta_1^{\mathbf{k}})\\
&+\,P'(\varphi_2)\,\xi_2^{\mathbf{h}}\,(\theta^{\mathbf{k}}-\chi\xi^{\mathbf{k}}-\eta^{\mathbf{k}})\,+\,(P''(\varphi_1)-P''(\varphi_2))\,\xi_1^{\mathbf{h}}\,\xi_1^{\mathbf{k}}\,(\sigma_1+\chi(1-\varphi_1)-\mu_1)\\
&+\,P''(\varphi_2)\,\xi^{\mathbf{k}}\,\xi_1^{\mathbf{h}}\,(\sigma_1+\chi(1-\varphi_1)-\mu_1)\,+\,P''(\varphi_2)\,\xi_2^{\mathbf{k}}\,\xi^{\mathbf{h}}\,(\sigma_1+\chi(1-\varphi_1)-\mu_1)\\
&+\,P''(\varphi_2)\,\xi_2^{\mathbf{k}}\,\xi_2^{\mathbf{h}}\,(\sigma-\chi\varphi-\mu)\;\;=:\;\sum_{i=1}^{13}B_i\,,
\end{aligned} \tag{2.54}$$

$$\begin{aligned}
g_2 &:= -(F''(\varphi_1)-F''(\varphi_2))\psi_2-\left(F^{(3)}(\varphi_1)-F^{(3)}(\varphi_2)\right)\,\xi_1^{\mathbf{h}}\,\xi_1^{\mathbf{k}}\\
&-F^{(3)}(\varphi_2)\left(\xi^{\mathbf{h}}\,\xi_1^{\mathbf{k}}\,+\,\xi_2^{\mathbf{h}}\,\xi^{\mathbf{k}}\right)\,,
\end{aligned} \tag{2.55}$$

where B_i denotes the ith summand on the right-hand side of (2.54).

At this point, we infer from the proof of [17, Lem. 4.1] that the assertion follows once we can show that

$$\sum_{i=1}^{13}\,\|B_i\|_{L^2(Q)}\,+\,\|g_2\|_{L^2(Q)}\,\le\,C\,\|\mathbf{u}\|_{L^2(Q)^2}\,\|\mathbf{h}\|_{L^2(Q)^2}\,\|\mathbf{k}\|_{L^2(Q)^2}\,.$$

We only show the corresponding estimate for the terms B_1,B_4,B_{11} and leave the others to the interested reader. In the following, we make use of the mean value theorem, Hölder's inequality, the continuity of the embeddings $V\subset L^6(\Omega)\subset L^4(\Omega)$, and the global estimates (2.7), (2.8), (2.22), and (2.46). We have

$$\begin{aligned}
\|B_1\|^2_{L^2(Q)} &\le\, C\int_0^T\|\varphi\|_4^2\left(\|\rho_2\|_4^2+\|\psi_2\|_4^2+\|\nu_2\|_4^2\right)ds\\
&\le\, C\,\|\varphi\|^2_{C^0([0,T];V)}\,\|\mathcal{S}''(\mathbf{u}_2)[\mathbf{h},\mathbf{k}]\|^2_{C^0([0,T];V)^3}\,\le\,C\,\|\mathbf{u}\|^2_{L^2(Q)^2}\,\|\mathbf{h}\|^2_{L^2(Q)^2}\,\|\mathbf{k}\|^2_{L^2(Q)^2}\,,\\
\|B_4\|^2_{L^2(Q)} &\le\, C\int_0^T\|\varphi\|_6^2\,\|\xi_1^{\mathbf{k}}\|_6^2\left(\|\theta_1^{\mathbf{h}}\|_6^2+\|\xi_1^{\mathbf{h}}\|_6^2+\|\eta_1^{\mathbf{h}}\|_6^2\right)ds\\
&\le\, C\,\|\varphi\|^2_{C^0([0,T];V)}\,\|\xi_1^{\mathbf{k}}\|_Z^2\,\|\mathcal{S}'(\mathbf{u}_1)[\mathbf{h}]\|^2_{\mathcal{Z}}\,\le\,C\,\|\mathbf{u}\|^2_{L^2(Q)^2}\,\|\mathbf{h}\|^2_{L^2(Q)^2}\,\|\mathbf{k}\|^2_{L^2(Q)^2}\,,
\end{aligned}$$

as well as

$$\|B_{11}\|^2_{L^2(Q)} \le C\left(\|\sigma_1\|^2_{L^\infty(Q)} + \|\varphi_1\|^2_{L^\infty(Q)} + \|\mu_1\|^2_{L^\infty(Q)} + 1\right)\int_0^T \|\varphi\|_6^2\,\|\xi_1^{\mathbf{h}}\|_6^2\,\|\xi_1^{\mathbf{k}}\|_6^2\,ds$$
$$\le C\,\|\varphi\|^2_{C^0([0,T];V)}\,\|\xi_1^{\mathbf{h}}\|^2_{C^0([0,T];V)}\,\|\xi_1^{\mathbf{k}}\|^2_{C^0([0,T];V)} \le C\,\|\mathbf{u}\|^2_{L^2(Q)^2}\,\|\mathbf{h}\|^2_{L^2(Q)^2}\,\|\mathbf{k}\|^2_{L^2(Q)^2}\,.$$

The assertion of the lemma is thus proved. □

3 The optimal control problem

We now begin to investigate the control problem **(CP)**. In addition to **(A1)**–**(A6)**, we make the following assumptions:

(C1) The constants b_1, b_2 are nonnegative, while b_3, κ are positive.

(C2) It holds $\widehat{\varphi}_\Omega \in H^1(\Omega)$ and $\widehat{\varphi}_Q \in L^2(Q)$.

(C3) $g : L^2(Q)^2 \to \mathbb{R}$ is nonnegative, continuous and convex on $L^2(Q)^2$.

Observe that **(C3)** implies that g is weakly sequentially lower semicontinuous on $L^2(Q)^2$. Moreover, denoting in the following by ∂ the subdifferential mapping in $L^2(Q)^2$, it follows from standard convex analysis that ∂g is defined on the entire space $L^2(Q)^2$ and is a maximal monotone operator. In addition, the mapping $((\mu,\varphi,\sigma),\mathbf{u}) \mapsto J((\mu,\varphi,\sigma),\mathbf{u})$ defined by the cost functional (1.1) is obviously continuous and convex (and thus weakly sequentially lower semicontinuous) on the space $\big(L^2(Q) \times C^0([0,T];L^2(\Omega)) \times L^2(Q)\big) \times L^2(Q)^2$. From a standard argument (which needs no repetition here) it then follows that the problem **(CP)** has a solution.

In the following, we often denote by $\mathbf{u}^* = (u_1^*, u_2^*) \in \mathcal{U}_{\rm ad}$ a local minimizer in the sense of $\mathcal{U}$ and by $(\mu^*,\varphi^*,\sigma^*) = \mathcal{S}(\mathbf{u}^*)$ the associated state. The corresponding adjoint state variables solve the adjoint system, which is given by the backward-in-time parabolic system

$$-\partial_t p - \beta\partial_t q - \Delta q + \chi\Delta r + F''(\varphi^*)q - P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p-r)$$
$$+ \chi P(\varphi^*)(p-r) = b_1(\varphi^* - \widehat{\varphi}_Q) \quad \text{in } Q\,, \tag{3.1}$$
$$-\alpha\partial_t p - \Delta p - q + P(\varphi^*)(p-r) = 0 \quad \text{in } Q\,, \tag{3.2}$$
$$-\partial_t r - \Delta r - \chi q - P(\varphi^*)(p-r) = 0 \quad \text{in } Q\,, \tag{3.3}$$
$$\partial_{\mathbf{n}} p = \partial_{\mathbf{n}} q = \partial_{\mathbf{n}} r = 0 \quad \text{on } \Sigma\,, \tag{3.4}$$
$$(p+\beta q)(T) = b_2(\varphi^*(T) - \widehat{\varphi}_\Omega), \quad \alpha p(T) = 0, \quad r(T) = 0, \quad \text{in } \Omega\,. \tag{3.5}$$

According to [17, Thm. 5.2], the adjoint system has a unique weak solution (p,q,r) satisfying

$$p + \beta q \in H^1(0,T;V^*), \tag{3.6}$$
$$p \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W_0) \cap L^\infty(Q), \tag{3.7}$$
$$q \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.8}$$
$$r \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W_0) \cap L^\infty(Q), \tag{3.9}$$

as well as

$$
\begin{aligned}
&-\langle\partial_t(p+\beta q), v\rangle + \int_\Omega \nabla q\cdot\nabla v - \chi\int_\Omega \nabla r\cdot\nabla v + \int_\Omega F''(\varphi^*)\,q\,v \\
&- \int_\Omega P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p-r)v \,+\, \chi\int_\Omega P(\varphi^*)(p-r)v = b_1\int_\Omega(\varphi^* - \widehat{\varphi}_Q)v,
\end{aligned}
\tag{3.10}
$$

$$
-\alpha\int_\Omega \partial_t p\, v + \int_\Omega \nabla p\cdot\nabla v - \int_\Omega q\,v + \int_\Omega P(\varphi^*)\,(p-r)\,v = 0, \tag{3.11}
$$

$$
-\int_\Omega \partial_t r\, v + \int_\Omega \nabla r\cdot\nabla v - \chi\int_\Omega q\,v - \int_\Omega P(\varphi^*)\,(p-r)\,v = 0, \tag{3.12}
$$

for every $v\in V$ and almost every $t\in(0,T)$, and

$$
(p+\beta q)(T) = b_2(\varphi^*(T) - \widehat{\varphi}_\Omega), \quad p(T)=0, \quad r(T)=0 \quad \text{a.e. in } \Omega\,. \tag{3.13}
$$

Moreover, it follows from the proof of [17, Thm. 5.2] that there exists a constant $K_8 > 0$, which depends only on R and the data (but not on the special choice of $\mathbf{u}^*\in\mathcal{U}_{\rm ad}$), such that

$$
\begin{aligned}
&\|p\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)\cap L^\infty(Q)} \,+\, \|q\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\quad +\, \|r\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W_0)\cap L^\infty(Q)} \\
&\le\, K_8\left(\|\varphi^* - \widehat{\varphi}_Q\|_{L^2(Q)} \,+\, \|\varphi^*(T) - \widehat{\varphi}_\Omega\|_V\right)\,.
\end{aligned}
\tag{3.14}
$$

3.1 First-order necessary optimality conditions

In this section, we aim at deriving associated first-order necessary optimality conditions for local minima of the optimal control problem **(CP)**. We assume that **(A1)–(A6)** and **(C1)–(C3)** are fulfilled and define the reduced cost functionals associated with the functionals J and J_1 introduced in (1.1) by

$$
\widehat{J}(\mathbf{u}) = J(\mathcal{S}(\mathbf{u}),\mathbf{u}), \quad \widehat{J}_1(\mathbf{u}) = J_1(\mathcal{S}(\mathbf{u}),\mathbf{u})\,. \tag{3.15}
$$

Since $\mathcal{S}$ is twice continuously Fréchet differentiable from $\mathcal{U}$ into $\mathcal{Y}$ and $\mathcal{Y}$ is continuously embedded in $C^0([0,T];L^2(Q)^3)$, $\mathcal{S}$ is also twice continuously Fréchet differentiable from $\mathcal{U}$ into $C^0([0,T];L^2(Q)^3)$. It thus follows from the chain rule that the smooth part $\widehat{J}_1$ of $\widehat{J}$ is a twice continuously Fréchet differentiable mapping from $\mathcal{U}$ into $\mathbb{R}$, where, for every $\mathbf{u}^* = (u_1^*, u_2^*)\in\mathcal{U}$ and every $\mathbf{h} = (h_1,h_2)\in\mathcal{U}$, it holds with $(\mu^*,\varphi^*,\sigma^*) = \mathcal{S}(\mathbf{u}^*)$ that

$$
\begin{aligned}
\widehat{J}_1'(\mathbf{u}^*)[\mathbf{h}] \,=\, & b_1\iint_Q \xi^{\mathbf{h}}(\varphi^* - \widehat{\varphi}_Q) \,+\, b_2\int_\Omega \xi^{\mathbf{h}}(T)(\varphi^*(T) - \widehat{\varphi}_\Omega) \\
&+ b_3\iint_Q (u_1^* h_1 \,+\, u_2^* h_2),
\end{aligned}
\tag{3.16}
$$

where $(\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}}) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]$ is the unique solution to the linearized system (2.16)–(2.20), with $\lambda_1=\lambda_2=1$ and $\lambda_3=\lambda_4=0$, associated with $\mathbf{h}$.

Remark 3.1. Observe that the right-hand side of (3.16) is meaningful also for arguments $\mathbf{h}=(h_1,h_2)\in L^2(Q)^2$, where in this case $(\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}}) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]$ with the extension of the operator $\mathcal{S}'(\mathbf{u}^*)$ to $L^2(Q)^2$ introduced in Remark 2.5. Hence, by means of the identity (3.16) we can extend the operator $\widehat{J}_1'(\mathbf{u}^*)\in\mathcal{U}^*$ to $L^2(Q)^2$. The extended operator, which we again denote by $\widehat{J}_1'(\mathbf{u}^*)$, then becomes an element of $(L^2(Q)^2)^*$. In this way, expressions of the form $\widehat{J}_1'(\mathbf{u}^*)[\mathbf{h}]$ have a proper meaning also for $\mathbf{h}\in L^2(Q)^2$.

In the following, we assume that $\mathbf{u}^* = (u_1^*, u_2^*) \in \mathcal{U}_{\rm ad}$ is a given locally optimal control for $(\mathbf{CP})$ in the sense of $\mathcal{U}$, that is, there is some $\varepsilon > 0$ such that

$$\widehat{J}(\mathbf{u}) \geq \widehat{J}(\mathbf{u}^*) \quad \text{for all } \mathbf{u} \in \mathcal{U}_{\rm ad} \text{ satisfying } \|\mathbf{u} - \mathbf{u}^*\|_{\mathcal{U}} \leq \varepsilon. \tag{3.17}$$

Notice that any locally optimal control in the sense of $L^p(Q)^2$ with $1 \leq p < \infty$ is also locally optimal in the sense of $\mathcal{U}$. Therefore, a result proved for locally optimal controls in the sense of $\mathcal{U}$ is also valid for locally optimal controls in the sense of $L^p(Q)^2$. It is of course also valid for (globally) optimal controls.

Now, in the same way as in [55], we infer that then the variational inequality

$$\widehat{J}_1'(\mathbf{u}^*)[\mathbf{u} - \mathbf{u}^*] + \kappa\,(g(\mathbf{u}) - g(\mathbf{u}^*)) \geq 0 \quad \forall\, \mathbf{u} \in \mathcal{U}_{\rm ad} \tag{3.18}$$

is satisfied. Moreover, denoting by the symbol ∂ the subdifferential mapping in $L^2(Q)^2$ (recall that g is a convex continuous functional on $L^2(Q)^2$), we conclude from [55, Thm. 4.5] that there is some $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*) \in \partial g(\mathbf{u}^*) \subset L^2(Q)^2$ such that

$$\widehat{J}_1'(\mathbf{u}^*)[\mathbf{u} - \mathbf{u}^*] + \iint_Q \kappa\,(\lambda_1^*(u_1 - u_1^*) + \lambda_2^*(u_2 - u_2^*)) \geq 0 \quad \forall\, \mathbf{u} = (u_1, u_2) \in \mathcal{U}_{\rm ad}. \tag{3.19}$$

As usual, we simplify the expression $\widehat{J}_1'(\mathbf{u}^*)[\mathbf{u} - \mathbf{u}^*]$ in (3.19) by means of the adjoint state variables defined in (3.1)–(3.5). A standard calculation (see the proof of [17, Thm. 5.4]) then leads to the following result.

Theorem 3.2. (Necessary optimality condition) *Suppose that* **(A1)–(A6)** *and* **(C1)–(C3)** *are fulfilled. Moreover, let* $\mathbf{u}^* = (u_1^*, u_2^*) \in \mathcal{U}_{\rm ad}$ *be a locally optimal control of* **(CP)** *in the sense of* $\mathcal{U}$ *with associated state* $(\mu^*, \varphi^*, \sigma^*) = \mathcal{S}(\mathbf{u}^*)$ *and adjoint state* (p^*, q^*, r^*)*. Then there exists some* $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*) \in \partial g(\mathbf{u}^*)$ *such that, for all* $\mathbf{u} = (u_1, u_2) \in \mathcal{U}_{\rm ad}$,

$$\iint_Q (-\mathbb{h}\, p^* + \kappa\lambda_1^* + b_3 u_1^*)(u_1 - u_1^*) + \iint_Q (r^* + \kappa\lambda_2^* + b_3 u_2^*)(u_2 - u_2^*) \geq 0\,. \tag{3.20}$$

Remark 3.3. We underline again that (3.20) is also necessary for all globally optimal controls and all controls which are even locally optimal in the sense of $L^p(Q) \times L^p(\Sigma)$ with $p \geq 1$. Observe also that the variational inequality (3.20) is equivalent to two independent variational inequalities for u_1^* and u_2^* that have to hold simultaneously, namely,

$$\iint_Q (-\mathbb{h}\, p^* + \kappa\lambda_1^* + b_3 u_1^*)\,(u_1 - u_1^*) \geq 0 \quad \forall\, u_1 \in U_{\rm ad}^1, \tag{3.21}$$

$$\iint_Q (r^* + \kappa\lambda_2^* + b_3 u_2^*)\,(u_2 - u_2^*) \geq 0 \quad \forall\, u_2 \in U_{\rm ad}^2, \tag{3.22}$$

where

$$U_{\rm ad}^i := \{u_i \in L^\infty(Q) : \underline{u}_i \leq u_i \leq \overline{u}_i \text{ a.e. in } Q\}, \quad i = 1, 2. \tag{3.23}$$

3.2 Sparsity of controls

The convex function g in the objective functional accounts for the sparsity of optimal controls, i.e., any locally optimal control can vanish in some region of the space-time cylinder Q. The form of this region depends on the particular choice of the functional g which can differ in different situations. The sparsity properties can be deduced from the variational inequalities (3.21) and (3.22) and the form of the subdifferential ∂g. In this paper, we restrict our analysis to the case of *full sparsity* which is characterized by the functional (recall (1.1))

$$g(\mathbf{u}) = g(u_1, u_2) := \iint_Q (|u_1| + |u_2|) \ . \tag{3.24}$$

Other important choices leading to the *directional sparsity with respect to time* and the *directional sparsity with respect to space* are not considered here. It is well known (see, e.g., [54]) that the subdifferential of g is given by

$$\partial g(\mathbf{u}) = g(u_1, u_2)$$

$$:= \left\{ (\lambda_1, \lambda_2) \in L^2(Q)^2 : \ \lambda_i \in \left\{ \begin{array}{ll} \{1\} & \text{if } u_i > 0 \\ [-1,1] & \text{if } u_i = 0 \\ \{-1\} & \text{if } u_i < 0 \end{array} \right\} \text{ a.e. in } Q, \quad i = 1,2 \right\} . \tag{3.25}$$

The following sparsity result can be proved in exactly the same way as [55, Thm. 4.9].

Theorem 3.4. (Full sparsity) *Suppose that the assumptions* **(A1)**–**(A6)** *and* **(C1)**–**(C3)** *are fulfilled, and assume that* $\underline{u}_i < 0 < \overline{u}_i$*,* $i = 1, 2$*. Let* $\mathbf{u}^* = (u_1^*, u_2^*) \in \mathcal{U}_{\rm ad}$ *be a locally optimal control in the sense of* $\mathcal{U}$ *for the problem* **(CP)** *with the sparsity functional* g *defined in* (3.24)*, and with associated state* $(\mu^*, \varphi^*, \sigma^*) = \mathcal{S}(\mathbf{u}^*)$ *solving* (1.2)–(1.6) *and adjoint state* (p^*, q^*, r^*) *solving* (3.1)–(3.5)*. Then there exists some* $(\lambda_1^*, \lambda_2^*) \in \partial g(\mathbf{u}^*)$ *such that* (3.21)–(3.22) *are satisfied. In addition, we have that*

$$u_1^*(x,t) = 0 \quad \Longleftrightarrow \quad |-\mathbb{h}(x,t) p^*(x,t)| \le \kappa, \quad \textit{for a.e. } (x,t) \in Q, \tag{3.26}$$

$$u_2^*(x,t) = 0 \quad \Longleftrightarrow \quad |r^*(x,t)| \le \kappa, \quad \textit{for a.e. } (x,t) \in Q. \tag{3.27}$$

Moreover, if (p^*, q^*, r^*) *and* $(\lambda_1^*, \lambda_2^*)$ *are given, then* (u_1^*, u_2^*) *is obtained from the projection formulas*

$$u_1^*(x,t) = \max \left\{ \underline{u}_1, \min \left\{ \overline{u}_1, -{b_3}^{-1} \left(-\mathbb{h}\, p^* + \kappa\, \lambda_1^* \right)(x,t) \right\} \right\} \quad \textit{for a.e. } (x,t) \in Q, \tag{3.28}$$

$$u_2^*(x,t) = \max \left\{ \underline{u}_2, \min \left\{ \overline{u}_2, -{b_3}^{-1} \left(r^* + \kappa \lambda_2^* \right)(x,t) \right\} \right\} \quad \textit{for a.e. } (x,t) \in Q. \tag{3.29}$$

The projection formulas above are standard conclusions from the variational inequalities (3.21)–(3.22).

3.3 Second-order sufficient optimality conditions

In this section, we establish the main results of this paper, using auxiliary results collected in the Appendix. We provide conditions that ensure local optimality of pairs $\mathbf{u}^* = (u_1^*, u_2^*)$ obeying the first-order necessary optimality conditions of Theorem 3.2. Second-order sufficient optimality conditions are based on a condition of coercivity that is required to hold for the smooth part $\widehat{J}_1$ of $\widehat{\mathcal{J}}$ in a certain critical cone. The nonsmooth part g contributes to sufficiency by its convexity. In the following, we generally assume that **(A1)**–**(A6)**, **(C1)**–**(C3)**, and the conditions $\underline{u}_1 < 0 < \overline{u}_1$ and $\underline{u}_2 < 0 < \overline{u}_2$ are fulfilled. Our analysis will follow closely the lines of [55], which in turn follows [4], where a second-order analysis was performed for sparse control of the FitzHugh–Nagumo system. In particular, we adapt the proof of [4, Thm. 3.4] to our setting of less regularity.

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To this end, we fix a pair of controls $\mathbf{u}^* = (u_1^*, u_2^*)$ that satisfies the first-order necessary optimality conditions, and we set $(\mu^*, \varphi^*, \sigma^*) = \mathcal{S}(\mathbf{u}^*)$. Then the cone

$$C(\mathbf{u}^*) = \{(v_1, v_2) \in L^2(Q)^2 \text{ satisfying the sign conditions (3.30) a.e. in } Q\},$$

where

$$v_i(x,t) \begin{cases} \geq 0 & \text{if } u_i^*(x,t) = \underline{u}_i \\ \leq 0 & \text{if } u_i^*(x,t) = \overline{u}_i \end{cases}, \quad i = 1, 2\,, \tag{3.30}$$

is called the *cone of feasible directions*, which is a convex and closed subset of $L^2(Q)^2$. We also need the directional derivative of g at $\mathbf{u} \in L^2(Q)^2$ in the direction $\mathbf{v} = (v_1, v_2) \in L^2(Q)^2$, which is given by

$$g'(\mathbf{u}, \mathbf{v}) = \lim_{\tau \searrow 0} \frac{1}{\tau}(g(\mathbf{u} + \tau\mathbf{v}) - g(\mathbf{u}))\,. \tag{3.31}$$

Following the definition of the critical cone in [4, Sect. 3.1], we define

$$C_{\mathbf{u}^*} = \{\mathbf{v} \in C(\mathbf{u}^*) : D\widehat{J}(\mathbf{u}^*)[\mathbf{v}] + \kappa g'(\mathbf{u}^*, \mathbf{v}) = 0\}\,, \tag{3.32}$$

which is also a closed and convex subset of $L^2(Q)^2$. According to [4, Sect. 3.1], it consists of all $\mathbf{v} = (v_1, v_2) \in C(\mathbf{u}^*)$ satisfying

$$v_1(x,t) \begin{cases} = 0 & \text{if } |-\mathbb{h}(x,t)p^*(x,t) + b_3 u_1^*(x,t)| \neq \kappa \\ \geq 0 & \text{if } u_1^*(x,t) = \underline{u}_1 \text{ or } (-\mathbb{h}(x,t)p^*(x,t) = -\kappa \text{ and } u_1^*(x,t) = 0) \\ \leq 0 & \text{if } u_1^*(x,t) = \overline{u}_1 \text{ or } (-\mathbb{h}(x,t)p^*(x,t) = \kappa \text{ and } u_1^*(x,t) = 0) \end{cases}, \tag{3.33}$$

$$v_2(x,t) \begin{cases} = 0 & \text{if } |r^*(x,t) + b_3 u_2^*(x,t)| \neq \kappa \\ \geq 0 & \text{if } u_2^*(x,t) = \underline{u}_2 \text{ or } (r^*(x,t) = -\kappa \text{ and } u_2^*(x,t) = 0) \\ \leq 0 & \text{if } u_2^*(x,t) = \overline{u}_2 \text{ or } (r^*(x,t) = \kappa \text{ and } u_2^*(x,t) = 0) \end{cases}. \tag{3.34}$$

Remark 3.5. Let us compare the first condition in (3.33) with the situation in the differentiable control problem without sparsity terms obtained for $\kappa = 0$. Then this condition leads to the requirement that $v_1(x,t) = 0$ if $|-\mathbb{h}(x,t)p^*(x,t) + b_3 u_1^*(x,t)| > 0$, or, since $\kappa = 0$,

$$v_1(x,t) = 0 \text{ if } |-\mathbb{h}(x,t)p^*(x,t) + \kappa\lambda_1^*(x,t) + b_3 u_1^*(x,t)| > 0. \tag{3.35}$$

An analogous condition results for v_2.

One might be tempted to define the critical cone using (3.35) and its counterpart for v_2 also in the case $\kappa > 0$. This, however, is not a good idea, because it leads to a critical cone that is larger than needed, in general. As an example, we mention the particular case when the control $\mathbf{u}^* = \mathbf{0}$ satisfies the first-order necessary optimality conditions and when $|-\mathbb{h}\, p^*| < \kappa$ and $|r^*| < \kappa$ hold a.e. in Q. Then the upper relation of (3.33), and its counterpart for v_2, lead to $C_{\mathbf{u}^*} = \{\mathbf{0}\}$, the smallest possible critical cone.

However, thanks to $u_1^* = 0$, the variational inequality (3.21) implies that $-\mathbb{h}\, p^* + \kappa\lambda_1^* + b_3 u_1^* = 0$ a.e. in Q, i.e., the condition $|-\mathbb{h}(x,t)p^*(x,t) + \kappa\lambda_1^*(x,t) + b_3 u_1^*(x,t)| > 0$ can only be satisfied on a set of measure zero. Moreover, also the sign conditions (3.30) do not restrict the critical cone. Hence, the largest possible critical cone $C_{\mathbf{u}^*} = L^2(Q)^2$ would be obtained, provided that analogous conditions hold for u_2^* and r^* in Q.

In this example, the quadratic growth condition (3.43) below is valid for the choice (3.32) as critical cone even without assuming the coercivity condition (3.42) below (here the so-called first-order sufficient conditions apply), while the use of a cone based on (3.35) leads to postulating (3.42) on the whole space $L^2(Q)^2$ for the quadratic growth condition to be valid. This shows that the choice of (3.32) as critical cone is essentially better than of one based on (3.35).

At this point, we derive an explicit expression for $\widehat{J}_1''(\mathbf{u})[\mathbf{v},\mathbf{w}]$ for arbitrary $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in \mathcal{U}$. In the following, we argue similarly as in [58, Sect. 5.7] (see also [17, Sect. 6]). At first, we readily infer that, for every $((\mu,\varphi,\sigma),\mathbf{u}) \in (C^0([0,T];H))^3 \times \mathcal{U}$ and $\mathbf{v} = (v_1,v_2,v_3), \mathbf{w} = (w_1,w_2,w_3)$ such that $(\mathbf{v},\mathbf{h}), (\mathbf{w},\mathbf{k}) \in (C^0([0,T];H))^3 \times \mathcal{U}$, we have

$$J_1''((\mu,\varphi,\sigma),\mathbf{u})[(\mathbf{v},\mathbf{h}),(\mathbf{w},\mathbf{k})] = b_1 \iint_Q v_2 w_2 \, + \, b_2 \int_\Omega v_2(T) w_2(T) \, + \, b_3 \iint_Q \mathbf{h}\cdot\mathbf{k}, \quad (3.36)$$

where the dot denotes the euclidean scalar product in $\mathbb{R}^2$. For the second-order derivative of the reduced cost functional $\widehat{J}_1$ at a fixed control $\mathbf{u}^*$ we then find with $(\mu^*,\varphi^*,\sigma^*) = \mathcal{S}(\mathbf{u}^*)$ that

$$\begin{aligned}\widehat{J}_1''(\mathbf{u}^*)[\mathbf{h},\mathbf{k}] &= D_{(\mu,\varphi,\sigma)} J_1((\mu^*,\varphi^*,\sigma^*),\mathbf{u}^*)[(\nu,\psi,\rho)] \\ &\quad + \, J_1''((\mu^*,\varphi^*,\sigma^*),\mathbf{u}^*)[((\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}}),\mathbf{h}),((\eta^{\mathbf{k}},\xi^{\mathbf{k}},\theta^{\mathbf{k}}),\mathbf{k})], \qquad (3.37)\end{aligned}$$

where $(\eta^{\mathbf{h}},\xi^{\mathbf{h}},\theta^{\mathbf{h}})$, $(\eta^{\mathbf{k}},\xi^{\mathbf{k}},\theta^{\mathbf{k}})$, and (ν,ψ,ρ) stand for the unique corresponding solutions to the linearized system associated with $\mathbf{h}$ and $\mathbf{k}$, and to the bilinearized system, respectively. From the definition of the cost functional (1.1) we readily infer that

$$D_{(\mu,\varphi,\sigma)} J_1((\mu^*,\varphi^*,\sigma^*),\mathbf{u}^*)[(\nu,\psi,\rho)] = b_1 \iint_Q (\varphi^* - \widehat{\varphi}_Q)\psi \, + \, b_2 \int_\Omega (\varphi^*(T) - \widehat{\varphi}_\Omega)\psi(T). \qquad (3.38)$$

We now claim that, with the associated adjoint state (p^*,q^*,r^*),

$$\begin{aligned}& b_1 \iint_Q (\varphi^* - \widehat{\varphi}_Q)\psi \, + b_2 \int_\Omega (\varphi^*(T) - \widehat{\varphi}_\Omega)\psi(T) \\ &= \iint_Q \Big[P'(\varphi^*)\big(\xi^{\mathbf{k}}(\theta^{\mathbf{h}} - \chi\xi^{\mathbf{h}} - \eta^{\mathbf{h}}) + \xi^{\mathbf{h}}(\theta^{\mathbf{k}} - \chi\xi^{\mathbf{k}} - \eta^{\mathbf{k}})\big)(p^* - r^*) \\ &\qquad + P''(\varphi^*)\xi^{\mathbf{k}}\xi^{\mathbf{h}}(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*) \, - \, F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}} q^* \Big]. \qquad (3.39)\end{aligned}$$

To prove this claim, we multiply (2.37) by p^*, (2.38) by q^*, (2.39) by r^*, add the resulting equalities, and integrate over Q, to obtain that

$$\begin{aligned}0 = & \iint_Q p^* \Big[\alpha\partial_t\nu + \partial_t\psi - \Delta\nu - P(\varphi^*)(\rho - \chi\psi - \nu) \\ &\qquad - P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)\psi \, - \, f_1 \Big] \\ &+ \iint_Q q^* \Big[\beta\partial_t\psi - \Delta\psi - \nu - \chi\rho + F''(\varphi^*)\psi + F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}} \Big] \\ &+ \iint_Q r^* \Big[\partial_t\rho - \Delta\rho + \chi\Delta\psi + P(\varphi^*)(\rho - \chi\psi - \nu) \\ &\qquad + P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)\psi \, + \, f_1 \Big]\end{aligned}$$

with the function f_1 defined in (2.42). Then, we integrate by parts and make use of the initial and terminal conditions (2.41) and (3.13) to find that

$$
\begin{aligned}
0 = & \iint_Q \nu \Big[- \alpha\partial_t p^* - \Delta p^* - q^* + P(\varphi^*)(p^* - r^*)\Big] \\
& + \int_0^T \langle -\partial_t (p^* + \beta q^*)(t), \psi(t)\rangle \, dt \, + \, b_2 \int_\Omega (\varphi^*(T) - \widehat{\varphi}_\Omega)\psi(T) \\
& + \iint_Q \psi \Big[- \Delta q^* + \chi\Delta r^* + F''(\varphi^*)q^* + \chi P(\varphi^*)(p^* - r^*) \\
& \qquad\qquad - P'(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*)\Big] \\
& + \iint_Q \rho \Big[- \partial_t r^* - \Delta r^* - \chi q^* - P(\varphi^*)(p^* - r^*)\Big] \\
& + \iint_Q \Big[- P'(\varphi^*) \left(\xi^{\mathbf{k}}(\theta^{\mathbf{h}} - \chi\xi^{\mathbf{h}} - \eta^{\mathbf{h}}) + \xi^{\mathbf{h}}(\theta^{\mathbf{k}} - \chi\xi^{\mathbf{k}} - \eta^{\mathbf{k}})\right)(p^* - r^*) \\
& \qquad - P''(\varphi^*)\xi^{\mathbf{k}}\xi^{\mathbf{h}}(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*) + F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}}q^*\Big],
\end{aligned}
$$

whence the claim follows, since (p^*, q^*, r^*) solves the adjoint system (3.10)–(3.13). From this characterization, along with (3.37) and (3.38), we conclude that

$$
\begin{aligned}
\widehat{J}_1''(\mathbf{u}^*)[\mathbf{h},\mathbf{k}] \, = \, & b_1 \iint_Q \xi^{\mathbf{h}}\, \xi^{\mathbf{k}} \, + \, b_2 \int_\Omega \xi^{\mathbf{h}}(T)\xi^{\mathbf{k}}(T) \, + \, b_3 \iint_Q \mathbf{h}\cdot\mathbf{k} \\
+ \iint_Q & \Big[P'(\varphi^*) \left(\xi^{\mathbf{k}}(\theta^{\mathbf{h}} - \chi\xi^{\mathbf{h}} - \eta^{\mathbf{h}}) + \xi^{\mathbf{h}}(\theta^{\mathbf{k}} - \chi\xi^{\mathbf{k}} - \eta^{\mathbf{k}})\right)(p^* - r^*) \\
& + P''(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}} \, - \, F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}}q^*\Big] . \qquad (3.40)
\end{aligned}
$$

Observe that the expression on the right-hand side of (3.40) is meaningful also for increments $\mathbf{h},\mathbf{k} \in L^2(Q)^2$. Indeed, in this case the expressions $(\eta^{\mathbf{h}}, \xi^{\mathbf{h}}, \theta^{\mathbf{h}}) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{h}]$, $(\eta^{\mathbf{k}}, \xi^{\mathbf{k}}, \theta^{\mathbf{k}}) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{k}]$, and $(\nu, \psi, \rho) = \mathcal{S}''(\mathbf{u}^*)[\mathbf{h},\mathbf{k}]$ have an interpretation in the sense of the extended operators $\mathcal{S}'(\mathbf{u}^*)$ and $\mathcal{S}''(\mathbf{u}^*)$ introduced in Remark 2.5 and Remark 2.7. Therefore, the operator $\widehat{J}_1''(\mathbf{u}^*)$ can be extended by the identity (3.40) to the space $L^2(Q)^2 \times L^2(Q)^2$. This extension, which will still be denoted by $\widehat{J}_1''(\mathbf{u}^*)$, will be frequently used in the following. We now show that it is continuous. Indeed, we claim that for all $\mathbf{h},\mathbf{k} \in L^2(Q)^2$ it holds

$$
\left|\widehat{J}_1''(\mathbf{u}^*)[\mathbf{h},\mathbf{k}]\right| \, \le \, \widehat{C}\, \|\mathbf{h}\|_{L^2(Q)^2}\, \|\mathbf{k}\|_{L^2(Q)^2} \, , \qquad (3.41)
$$

where the constant $\widehat{C} > 0$ is independent of the choice of $\mathbf{u}^* \in \mathcal{U}_R$. Obviously, only the last integral on the right-hand side of (3.40) needs some treatment, and we estimate just its third summand, leaving the others as an exercise to the reader. We have, by virtue of Hölder's inequality, the continuity of the embedding $V \subset L^4(\Omega)$, and the global bounds (2.9), (2.22), and (3.14),

$$
\begin{aligned}
& \Big| \iint_Q F^{(3)}(\varphi^*)\xi^{\mathbf{h}}\xi^{\mathbf{k}}q^*\Big| \, \le \, C \int_0^T \|\xi^{\mathbf{h}}\|_4 \, \|\xi^{\mathbf{k}}\|_4 \, \|q^*\|_2 \, dt \\
& \le \, C\, \|\xi^{\mathbf{h}}\|_{C^0([0,T];V)}\, \|\xi^{\mathbf{k}}\|_{C^0([0,T];V)}\, \|q^*\|_{L^\infty(0,T;H)} \, \le \, C\, \|\mathbf{h}\|_{L^2(Q)^2}\, \|\mathbf{k}\|_{L^2(Q)^2} \, ,
\end{aligned}
$$

as asserted.

In the following, we will employ the following coercivity condition:

$$\widehat{J}_1''(\mathbf{u}^*)[\mathbf{v},\mathbf{v}] > 0 \quad \forall\, \mathbf{v} \in C_{\mathbf{u}^*} \setminus \{\mathbf{0}\}\,. \tag{3.42}$$

Condition (3.42) is a direct extension of associated conditions that are standard in finite-dimensional nonlinear optimization. In the optimal control of partial differential equation, it was first used in [5]. As in [4, Thm 3.3] or [5], it can be shown that (3.42) is equivalent to the existence of a constant $\delta > 0$ such that $\widehat{J}_1''(\mathbf{u}^*)[\mathbf{v},\mathbf{v}] \ge \delta\,\|\mathbf{v}\|^2_{L^2(Q)^2}$ for all $\mathbf{v} \in C_{\mathbf{u}^*}$.

We have the following result.

Theorem 3.6. (Second-order sufficient condition) *Suppose that* **(A1)**–**(A6)** *and* **(C1)**–**(C3)** *are fulfilled and that* $\underline{u}_i < 0 < \overline{u}_i,\ i = 1, 2$. *Moreover, let* $\mathbf{u}^* = (u_1^*, u_2^*) \in \mathcal{U}_{\rm ad}$, *together with the associated state* $(\mu^*, \varphi^*, \sigma^*) = \mathcal{S}(\mathbf{u}^*)$ *and the adjoint state* (p^*, q^*, r^*), *fulfill the first-order necessary optimality conditions of Theorem 3.2. If, in addition,* $\mathbf{u}^*$ *satisfies the coercivity condition* (3.42), *then there exist constants* $\varepsilon > 0$ *and* $\tau > 0$ *such that the quadratic growth condition*

$$\widehat{J}(\mathbf{u}) \ge \widehat{J}(\mathbf{u}^*) + \tau\,\|\mathbf{u} - \mathbf{u}^*\|^2_{L^2(Q)^2} \tag{3.43}$$

holds for all $\mathbf{u} \in \mathcal{U}_{\rm ad}$ *with* $\|\mathbf{u} - \mathbf{u}^*\|_{L^2(Q)^2} < \varepsilon$. *Consequently,* $\mathbf{u}^*$ *is a locally optimal control in the sense of* $L^2(Q)^2$.

Proof. The proof follows that of [4, Thm. 3.4]. We argue by contradiction, assuming that the claim of the theorem is not true. Then there exists a sequence of controls $\{\mathbf{u}_k\} \subset \mathcal{U}_{\rm ad}$ such that, for all $k \in \mathbb{N}$,

$$\|\mathbf{u}_k - \mathbf{u}^*\|_{L^2(Q)^2} < \frac{1}{k} \quad \text{while} \quad \widehat{J}(\mathbf{u}_k) < \widehat{J}(\mathbf{u}^*) + \frac{1}{2k}\|\mathbf{u}_k - \mathbf{u}^*\|^2_{L^2(Q)^2}\,. \tag{3.44}$$

Noting that $\mathbf{u}_k \ne \mathbf{u}^*$ for all $k \in \mathbb{N}$, we define

$$r_k = \|\mathbf{u}_k - \mathbf{u}^*\|_{L^2(Q)^2} \quad \text{and} \quad \mathbf{v}_k = \frac{1}{r_k}(\mathbf{u}_k - \mathbf{u}^*)\,.$$

Then $\|\mathbf{v}_k\|_{L^2(Q)^2} = 1$ and, possibly after selecting a subsequence, we can assume that

$$\mathbf{v}_k \to \mathbf{v} \ \text{weakly in } L^2(Q)^2$$

for some $\mathbf{v} \in L^2(Q)^2$. As in [4], the proof is split into three parts.

(i) $\mathbf{v} \in C_{\mathbf{u}^*}$: Obviously, each $\mathbf{v}_k$ obeys the sign conditions (3.30) and thus belongs to $C(\mathbf{u}^*)$. Since $C(\mathbf{u}^*)$ is convex and closed in $L^2(Q)^2$, it follows that $\mathbf{v} \in C(\mathbf{u}^*)$. We now claim that

$$\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}] + \kappa g'(\mathbf{u}^*, \mathbf{v}) = 0. \tag{3.45}$$

Notice that by Remark 3.1 the expression $\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}]$ is well defined. For every $r \in (0,1)$ and all $\mathbf{v} = (v_1, v_2),\ \mathbf{u} = (u_1, u_2) \in L^2(Q)^2$, we infer from the convexity of g that

$$\begin{aligned} g(\mathbf{v}) - g(\mathbf{u}) \ &\ge\ \frac{g(\mathbf{u} + r(\mathbf{v} - \mathbf{u})) - g(\mathbf{u})}{r} \ \ge\ g'(\mathbf{u}, \mathbf{v} - \mathbf{u}) \\ &=\ \max_{(\lambda_1,\lambda_2)\in\partial g(\mathbf{u})} \iint_Q \Big(\lambda_1(v_1 - u_1) + \lambda_2(v_2 - u_2)\Big). \end{aligned} \tag{3.46}$$

In particular, with $\mathbf{u}_k = (u_{k_1}, u_{k_2})$,

$$
\begin{aligned}
&\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}] + \kappa g'(\mathbf{u}^*, \mathbf{v}) \ \geq\ \widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}] + \iint_Q \kappa\big(\lambda_1^* v_1 \ + \ \lambda_2^* v_2\big) \\
&= \iint_Q \big((-\mathbb{h} p^* + b_3 u_1^* + \kappa\lambda_1^*) v_1 \ + \ (r^* + b_3 u_2^* + \kappa\lambda_2^*) v_2\big) \\
&= \lim_{k\to\infty} \frac{1}{r_k} \iint_Q \big((-\mathbb{h} p^* + b_3 u_1^* + \kappa\lambda_1^*)(u_{k_1} - u_1^*) \ + \ (r^* + b_3 u_2^* + \kappa\lambda_2^*)(u_{k_2} - u_2^*)\big) \\
&\geq 0\,, \qquad (3.47)
\end{aligned}
$$

by the variational inequality (3.20). Next, we prove the converse inequality. By (3.44), we have

$$
\widehat{J}_1(\mathbf{u}_k) - \widehat{J}_1(\mathbf{u}^*) + \kappa\left(g(\mathbf{u}_k) - g(\mathbf{u}^*)\right) < \frac{1}{2k} r_k^2\,,
$$

whence, owing to the mean value theorem, and since $\mathbf{u}_k = \mathbf{u}^* + r_k \mathbf{v}_k$, we get

$$
\widehat{J}_1(\mathbf{u}^*) + r_k \widehat{J}_1'(\mathbf{u}^* + \theta_k r_k \mathbf{v}_k)[\mathbf{v}_k] + \kappa g(\mathbf{u}^* + r_k \mathbf{v}_k) < \widehat{J}_1(\mathbf{u}^*) + \kappa g(\mathbf{u}^*) + \frac{1}{2k} r_k^2
$$

with some $0 < \theta_k < 1$. From (3.46), we obtain $\kappa(g(\mathbf{u}^* + r_k \mathbf{v}_k) - g(\mathbf{u}^*)) \geq \kappa g'(\mathbf{u}^*, r_k \mathbf{v}_k)$, and thus

$$
r_k \widehat{J}_1'(\mathbf{u}^* + \theta_k r_k \mathbf{v}_k)[\mathbf{v}_k] + r_k \kappa g'(\mathbf{u}^*, \mathbf{v}_k) < \frac{r_k^2}{2k}\,.
$$

We divide this inequality by r_k and pass to the limit $k \to \infty$. Here, we invoke Corollary 4.2 of the Appendix, and we use that $g'(\mathbf{u}^*, \mathbf{v}_k) \to g'(\mathbf{u}^*, \mathbf{v})$. We then obtain the desired converse inequality

$$
\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}] + \kappa g'(\mathbf{u}^*, \mathbf{v}) \leq 0\,,
$$

which completes the proof of (i).

(ii) $\mathbf{v} = \mathbf{0}$: We again invoke (3.44), now performing a second-order Taylor expansion on the left-hand side,

$$
\begin{aligned}
&\widehat{J}_1(\mathbf{u}^*) + r_k \widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}_k] + \frac{r_k^2}{2} \widehat{J}_1''(\mathbf{u}^* + \theta_k r_k \mathbf{v}_k)[\mathbf{v}_k, \mathbf{v}_k] + \kappa g(\mathbf{u}^* + r_k v_k) \\
&< \widehat{J}_1(\mathbf{u}^*) + \kappa g(\mathbf{u}^*) + \frac{r_k^2}{2k}\,.
\end{aligned}
$$

We subtract $\widehat{J}_1(\mathbf{u}^*) + \kappa g(\mathbf{u}^*)$ from both sides and use (3.46) once more to find that

$$
r_k \left(\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}_k] + \kappa g'(\mathbf{u}^*, \mathbf{v}_k)\right) + \frac{r_k^2}{2} \widehat{J}_1''(\mathbf{u}^* + \theta_k r_k \mathbf{v}_k)[\mathbf{v}_k, \mathbf{v}_k] < \frac{r_k^2}{2k}\,. \qquad (3.48)
$$

From the right-hand side of (3.46), and the variational inequality (3.20), it follows that

$$
\widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}_k] + \kappa g'(\mathbf{u}^*, \mathbf{v}_k) \geq 0\,,
$$

and thus, by (3.48),

$$
\widehat{J}_1''(\mathbf{u}^* + \theta_k r_k \mathbf{v}_k)[\mathbf{v}_k, \mathbf{v}_k] < \frac{1}{k}\,. \qquad (3.49)
$$

Passing to the limit $k \to \infty$, we apply Lemma 4.3 and deduce that $\widehat{J}_1''(\mathbf{u}^*)[\mathbf{v},\mathbf{v}] \le 0$. Since we know that $\mathbf{v} \in C_{\mathbf{u}^*}$, the second-order condition (3.42) implies that $\mathbf{v} = \mathbf{0}$.

(iii) *Contradiction:* From the previous step we know that $\mathbf{v}_k \to \mathbf{0}$ weakly in $L^2(Q)^2$. Moreover, (3.40) yields that

$$\begin{aligned}\widehat{J}_1''(\mathbf{u}^*)[v_k, v_k] &= b_3 \iint_Q |\mathbf{v}_k|^2 + b_1 \iint_Q |\xi_k|^2 + b_2 \int_\Omega |\xi_k(T)|^2 \\ &+ \iint_Q \Big[2P'(\varphi^*)\xi_k(\theta_k - \chi\xi_k - \eta_k)(p^* - r^*) - F^{(3)}(\varphi^*)q^* |\xi_k|^2 \Big] \\ &+ \iint_Q P''(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*)|\xi_k|^2 , \end{aligned} \tag{3.50}$$

where we have set $(\eta_k, \xi_k, \theta_k) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{v}_k]$, for $k \in \mathbb{N}$. By virtue of Lemma 4.3, the sum of the last four integrals on the right-hand side converges to zero. On the other hand, $\|\mathbf{v}_k\|_{L^2(Q)^2} = 1$ for all $k \in \mathbb{N}$, by construction. The weak sequential semicontinuity of norms then implies that

$$\liminf_{k\to\infty} \widehat{J}_1''(\mathbf{u}^*)[\mathbf{v}_k, \mathbf{v}_k] \ge \liminf_{k\to\infty} b_3 \iint_Q |\mathbf{v}_k|^2 = b_3 > 0 .$$

On the other hand, it is easily deduced from (3.49) and (2.47) that

$$\liminf_{k\to\infty} \widehat{J}_1''(\mathbf{u}^*)[\mathbf{v}_k, \mathbf{v}_k] \le 0 ,$$

a contradiction. The assertion of the theorem is thus proved. □

Remark 3.7. For the particular case $\kappa = 0$ without sparsity functional, Theorem 3.6 improves the second-order sufficient condition [17, Thm. 6.1]: indeed, our coercivity condition (3.42) is required on a smaller critical cone (compare (3.35) with the condition [17, (6.10)]), and we have local optimality in an L^2-neighborhood, hence in a larger set than in an L^∞-neighborhood as in [17]. We note at this place that the formula (6.5) in [17], which resembles (3.50), contains three sign errors: indeed, the term in the second line of [17, (6.5)] involving P'' should carry a "plus" sign, while the two terms in the third line should carry "minus" signs. These sign errors, however, do not have an impact on the validity of the results established in [17].

4 Appendix

In the following, we assume that **(A1)–(A6)** and **(C1)–(C3)** are fulfilled and that $\mathbf{u}^* \in \mathcal{U}_{\rm ad}$ is fixed with associated state $(\mu^*, \varphi^*, \sigma^*) = \mathcal{S}(\mathbf{u}^*)$ and adjoint state (p^*, q^*, r^*). We also recall the definitions of the spaces used below given in (2.11), (2.14), and (2.15).

Lemma 4.1. *Let $\{\mathbf{u}_k\} \subset \mathcal{U}_{\rm ad}$ converge strongly in $L^2(Q)^2$ to $\mathbf{u}^*$, and let $(\mu_k, \varphi_k, \sigma_k) = \mathcal{S}(\mathbf{u}_k)$ and (p_k, q_k, r_k), $k \in \mathbb{N}$, denote the associated states and adjoint states. Then*

$$\mu_k \to \mu^* \quad \text{strongly in } Z, \tag{4.1}$$

$$\varphi_k \to \varphi^* \quad \text{strongly in } Z \cap C^0(\overline{Q}), \tag{4.2}$$

$$\sigma_k \to \sigma^* \quad \text{strongly in } Z, \tag{4.3}$$

$$p_k \to p^* \quad \text{weakly-star in } Z \text{ and strongly in } C^0([0,T]; L^p(\Omega)) \text{ for } 1 \le p < 6, \tag{4.4}$$

$$q_k \to q^* \quad \text{weakly-star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.5}$$

$$r_k \to r^* \quad \text{weakly-star in } Z \text{ and strongly in } C^0([0,T]; L^p(\Omega)) \text{ for } 1 \le p < 6. \tag{4.6}$$

Proof. The strong convergence $\|\mathbb{S}(\mathbf{u}_k) - \mathbb{S}(\mathbf{u}^*)\|_{\mathcal{Z}} \to 0$ follows directly from (2.10). In addition, the global bound (2.7) implies that $\{\varphi_k\}$ is bounded in the space $\widetilde{X}$ defined in (2.11), which, thanks to the compactness of the embedding $W_0 \subset C^0(\overline{\Omega})$ and [53, Sec. 8, Cor. 4], is compactly embedded in $C^0(\overline{Q})$. Therefore it holds $\|\varphi_k - \varphi^*\|_{C^0(\overline{Q})} \to 0$ (at first only for a suitable subsequence, but then, owing to the uniqueness of the limit point, eventually for the entire sequence). The convergence properties (4.1)–(4.3) of the state variables are thus shown. In addition, it immediately follows from the mean value theorem and (2.9) that, as $k \to \infty$,

$$\max_{i=1,2,3} \|F^{(i)}(\varphi_k) - F^{(i)}(\varphi^*)\|_{C^0(\overline{Q})} \to 0,$$
$$\max_{i=0,1,2} \|P^{(i)}(\varphi_k) - P^{(i)}(\varphi^*)\|_{C^0(\overline{Q})} \to 0. \tag{4.7}$$

Next, we conclude from the bounds (3.14) and (2.7) that there are a subsequence, which is again labeled by $k \in \mathbb{N}$, and some triple (p, q, r) such that, as $k \to \infty$,

$$p_k \to p \quad \text{weakly-star in } Z \cap L^\infty(Q), \tag{4.8}$$
$$q_k \to q \quad \text{weakly-star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.9}$$
$$r_k \to r \quad \text{weakly-star in } Z \cap L^\infty(Q). \tag{4.10}$$

Moreover, by [53, Sect. 8, Cor. 4] and the compactness of the embedding $V \subset L^p(\Omega)$ for $1 \le p < 6$, we also have

$$p_k \to p, \quad r_k \to r, \quad \text{both strongly in } C^0([0,T];L^p(\Omega)) \text{ for } 1 \le p < 6. \tag{4.11}$$

From these estimates and (4.7) we can easily conclude that, as $k \to \infty$,

$$F''(\varphi_k)q_k \to F''(\varphi^*)q, \quad P(\varphi_k)(p_k - r_k) \to P(\varphi^*)(p - r),$$
$$P'(\varphi_k)(\sigma_k - \chi(1-\varphi_k) - \mu_k)(p_k - r_k) \to P'(\varphi^*)(\sigma^* - \chi(1-\varphi^*) - \mu^*)(p - r), \tag{4.12}$$

all weakly in $L^2(Q)$.

At this point, we consider the time-integrated version of the adjoint system (3.10)–(3.13) with test functions in $L^2(0,T;V)$, written for φ_k, p_k, q_k, r_k, $k \in \mathbb{N}$. Passage to the limit as $k \to \infty$, using the above convergence results, immediately leads to the conclusion that (p, q, r) solves the time-integrated version of (3.10)–(3.13) with test functions in $L^2(0,T;V)$, which is equivalent to saying that (p, q, r) is a solution to (3.10)–(3.13). By the uniqueness of this solution, we must have $(p, q, r) = (p^*, q^*, r^*)$. The convergence properties (4.4)–(4.6) are therefore valid for a suitable subsequence, and since the limit is uniquely determined, also for the entire sequence. ☐

Corollary 4.2. *Let $\{\mathbf{u}_k\} \subset \mathcal{U}_{\rm ad}$ converge strongly in $L^2(Q)^2$ to $\mathbf{u}^*$, and let $\{\mathbf{v}_k\}$ converge weakly to $\mathbf{v}$ in $L^2(Q)^2$. Then*

$$\lim_{k\to\infty} \widehat{J}_1'(\mathbf{u}_k)[\mathbf{v}_k] \;=\; \widehat{J}_1'(\mathbf{u}^*)[\mathbf{v}]\,. \tag{4.13}$$

Proof. We have, with $\mathbf{u}_k = (u_{k_1}, u_{k_2})$ and $\mathbf{v}_k = (v_{k_1}, v_{k_2})$,

$$\widehat{J}_1'(\mathbf{u}_k)[\mathbf{v}_k] = \iint_Q (-\mathbb{h}\, p_k + b_3 u_{k_1}) v_{k_1} + \iint_Q (r_k + b_3 u_{k_2}) v_{k_2}.$$

Owing to Lemma 4.1, we have, in particular, that $\{-\mathbb{h}\, p_k + b_3 u_{k_1}\}$ and $\{r_k + b_3 u_{k_2}\}$ converge strongly in $L^2(Q)$ to $-\mathbb{h}\, p^* + b_3 u_1^*$ and $r^* + b_3 u_2^*$, respectively, whence the assertion immediately follows. ☐

Lemma 4.3. *Let* $\{\mathbf{u}_k\}$ *and* $\{\mathbf{v}_k\}$ *satisfy the conditions of Corollary 4.2, and assume that* $b_3 = 0.$ *Then*

$$\lim_{k\to\infty} \widehat{J}_1''(\mathbf{u}_k)[\mathbf{v}_k,\mathbf{v}_k] = \widehat{J}_1''(\mathbf{u}^*)[\mathbf{v},\mathbf{v}]. \tag{4.14}$$

Proof. Let $\mathbf{v}_k = (v_{k_1}, v_{k_2})$, $\mathbf{v} = (v_1, v_2)$, $(\eta_k, \xi_k, \theta_k) = \mathcal{S}'(\mathbf{u}_k)[\mathbf{v}_k]$, and $(\eta, \xi, \theta) = \mathcal{S}'(\mathbf{u}^*)[\mathbf{v}]$. Since $b_3 = 0$, we infer from (3.50) that we have to show that, as $k \to \infty$,

$$\begin{aligned} & b_1 \iint_Q |\xi_k|^2 \,+\, b_2 \int_\Omega |\xi_k(T)|^2 \,+ \iint_Q 2P'(\varphi_k)\xi_k(\theta_k - \chi\xi_k - \eta_k)(p_k - r_k) \\ & + \iint_Q \Big[P''(\varphi_k)(\sigma_k + \chi(1-\varphi_k) - \mu_k)|\xi_k|^2 \,-\, F^{(3)}(\varphi_k) q_k\, |\xi_k|^2 \Big] \\ & \to\, b_1 \iint_Q |\xi^*|^2 \,+\, b_2 \int_\Omega |\xi^*(T)|^2 \,+ \iint_Q 2P'(\varphi^*)\xi^*(\theta^* - \chi\xi^* - \eta^*)(p^* - r^*) \\ & \quad + \iint_Q \Big[P''(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*)|\xi^*|^2 \,-\, F^{(3)}(\varphi^*) q^*\, |\xi^*|^2 \Big], \end{aligned} \tag{4.15}$$

where (p_k, q_k, r_k) and (p^*, q^*, r^*) are the associated adjoint states. By Lemma 4.1 and its proof, the convergence properties (4.1)–(4.6) and (4.7) are valid. Moreover, we have

$$(\eta_k, \xi_k, \theta_k) - (\eta^*, \varphi^*, \theta^*) = \left(\mathcal{S}'(\mathbf{u}_k) - \mathcal{S}'(\mathbf{u}^*)\right)[\mathbf{v}_k] \,+\, \mathcal{S}'(\mathbf{u}^*)[\mathbf{v}_k - \mathbf{v}]\,.$$

By virtue of (2.23) and the boundedness of $\{\mathbf{v}_k\}$ in $L^2(Q)^2$, the first summand on the right-hand side of this identity converges strongly to zero in $\mathcal{Z}$. The second converges to zero weakly in $Z \times \widetilde{X} \times Z$. Hence, thanks to the compactness of the embedding $Z \subset C^0([0,T]; L^p(\Omega))$ for $1 \le p < 6$ (see, e.g., [53, Sect. 8, Cor. 4]),

$$(\eta_k, \xi_k, \theta_k) \to (\eta^*, \xi^*, \theta^*) \quad \text{strongly in } C^0([0,T]; L^p(\Omega))^3 \quad \text{for } 1 \le p < 6. \tag{4.16}$$

In particular, as $k \to \infty$,

$$b_1 \iint_Q |\xi_k|^2 \,+\, b_2 \int_\Omega |\xi_k(T)|^2 \;\to\; b_1 \iint_Q |\xi^*|^2 \,+\, b_2 \int_\Omega |\xi^*(T)|^2\,. \tag{4.17}$$

Moreover, owing to the strong convergences in $C^0([0,T]; L^p(\Omega))$ for $1 \le p < 6$, it is easily checked, using Hölder's inequality, that

$$\begin{aligned} & P'(\varphi_k)\xi_k(\theta_k - \chi\xi_k - \eta_k)(p_k - r_k) \;\to\; P'(\varphi^*)\xi^*(\theta^* - \chi\xi^* - \eta^*)(p^* - r^*)\,, \\ & P''(\varphi_k)(\sigma_k + \chi(1-\varphi_k) - \mu_k)(p_k - r_k)|\xi_k|^2 \to P''(\varphi^*)(\sigma^* + \chi(1-\varphi^*) - \mu^*)(p^* - r^*)|\xi^*|^2, \end{aligned} \tag{4.18}$$

both strongly in $L^1(Q)$. It remains to show that, as $k \to \infty$,

$$\iint_Q F^{(3)}(\varphi_k) q_k\, |\xi_k|^2 \;\to\; \iint_Q F^{(3)}(\varphi^*) q^*\, |\xi^*|^2\,.$$

Since $q_k \to q^*$ weakly in $L^2(Q)$ by (4.9), it thus suffices to show that $F^{(3)}(\varphi_k)\,|\xi_k|^2 \;\to\; F^{(3)}(\varphi^*)\,|\xi^*|^2$ strongly in $L^2(Q)$. However, this is a simple consequence of (4.7) and (4.16). The assertion is thus proved. □

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