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Quenched homogenization of infinite range random conductance model on stationary point processes

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Abstract

We prove homogenization for elliptic long-range operators in the random conductance model on random stationary point processes in d dimensions with Dirichlet boundary conditions and with a jointly stationary coefficient field. Doing so, we identify 4 conditions on the point process and the coefficient field that have to be fulfilled at different stages of the proof in order to pass to the homogenization limit. The conditions can be clearly attributed to concentration of support, Rellich-Poincaré inequality, non-degeneracy of the homogenized matrix and ergodicity of the elliptic operator.

1 Introduction

We consider a stationary ergodic point process in $\mathbb{R}^d$ or $\mathbb{Z}^d$ with realization $\mathbb{x} = (\mathbf{x}_i)_{i\in\mathbb{N}}$. A definition of this concept is recalled in Section 2.3 below. We furthermore assume to be given a random coefficient field jointly stationary with $\mathbb{x}$:

$$\alpha : \mathbb{x} \times \mathbb{x} \to [0,1]\,, \qquad (x,y) \mapsto \alpha_{x,y}\,. \tag{1.1}$$

We denote $\mathbb{E}$ the classical expectation and $\mathbb{E}_0$ the conditional expectation $0 \in \mathbb{x}$. Since $\mathbb{x}$ is stationary we find for every jointly stationary random variable f that $\mathbb{E}f = \mathbb{E}_0 f$ and we demand

$$\alpha_{x,y} = \alpha_{y,x}\,, \quad \alpha_{x,x} = 0 \quad \text{and} \qquad 0 < \mathbb{E}_0\left(\sum_{z\in\mathbb{x}\setminus\{0\}} \alpha_{0,z}\right) < \infty\,. \tag{1.2}$$

Given $\varepsilon > 0$ we consider the sets $\mathbb{x}^\varepsilon$, and the functions $\alpha^\varepsilon : \mathbb{x}^\varepsilon \times \mathbb{x}^\varepsilon \to \mathbb{R}$

$$\mathbb{x}^\varepsilon := \varepsilon\mathbb{x} = (\varepsilon\mathbf{x}_i)_{i\in\mathbb{N}} = (\mathbf{x}_i^\varepsilon)_{i\in\mathbb{N}}\,, \quad \alpha^\varepsilon_{x,y} = \alpha_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\,.$$

Introducing the function spaces

$$\mathcal{S}^\varepsilon_{\mathbb{x}} := \{\mathbb{x}^\varepsilon \to \mathbb{R}\} \quad \text{and} \quad \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q}) := \{u \in \mathcal{S}^\varepsilon_{\mathbb{x}} \,:\, \forall \mathbf{x}_i^\varepsilon \in \mathbb{x}^\varepsilon \backslash \boldsymbol{Q} \;\; u(\mathbf{x}_i^\varepsilon) = 0\}$$

we write $u_i := u(\mathbf{x}_i^\varepsilon)$ for every $u \in \mathcal{S}^\varepsilon_{\mathbb{x}}$ and introduce the linear operator on $\mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$:

$$\forall \mathbf{x}_i^\varepsilon \in \boldsymbol{Q} \cap \mathbb{x}^\varepsilon : \qquad \left(\mathcal{L}^\varepsilon_{\mathbb{x},\alpha} u\right)_i := \varepsilon^{-2} \sum_{j\neq i} \alpha^\varepsilon_{\mathbf{x}_j^\varepsilon,\mathbf{x}_i^\varepsilon} \frac{(u_j - u_i)}{|\mathbf{x}_j - \mathbf{x}_i|^2}\,.$$

We are particularly interested in the limit behavior of the discrete differential equation

$$-\mathcal{L}^\varepsilon_{\mathbb{x},\alpha} u^\varepsilon = f^\varepsilon \quad \text{in } \boldsymbol{Q} \cap \mathbb{x}^\varepsilon\,, \qquad u^\varepsilon(\mathbf{x}_i^\varepsilon) = 0 \quad \text{if } \mathbf{x}_i^\varepsilon \in \mathbb{x}^\varepsilon \setminus \boldsymbol{Q}\,. \tag{1.3}$$

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where $f^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$ is a sequence that converges weakly in a sense to be specify below.

The homogenization of problem (1.3) has been studied successfully first in [9] for $\mathbb{x} = \mathbb{Z}^d$. Writing $e_1, \dots e_d$ for the canonical basis of $\mathbb{R}^d$, in [9] some additional condition of the type

$$\mathbb{E}\left(\sum_i \alpha_{0,e_i}^{-1}\right)^{\frac{d}{2}} < \infty \tag{1.4}$$

is needed. The condition imposed in [9] is more general, but reads similar. Recently, a more general result has been optained in [3] under the condition that for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ it holds

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d} \alpha_{0,z}\,|z|^2\right)^p < \infty \qquad \text{and} \qquad \mathbb{E}\sum_{\substack{z\in\mathbb{Z}^d\\|z|=1}} \alpha_{0,z}^{-q} < \infty\,. \tag{1.5}$$

Since all recent results work on $\mathbb{x} = \mathbb{Z}^d$ and with $\alpha_{x,y} > 0$ for $|x-y| = 1$, our result is indeed new. Like in our previous work [9] we use stochastic two-scale methods developed in [9] to show that stationary ergodic point processes $\mathbb{x}$ in $\mathbb{Z}^d$ with weights $\alpha_{x,y}$ satisfying Assumptions 1.1 below lead to a homogenization result for (1.3). In this context we further use recent results from finite volume analysis [2, 5] to prove our compactness and uniform Poincaré inequalities in $\varepsilon > 0$. Furthermore, the proof that the support of $u \in \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$ – regarded as a function in $L^2(\boldsymbol{Q})$ – lies within a small ball around $\boldsymbol{Q}$ is inspired by recently developed ideas by the author for continuous homogenization [13, 15].

We note at this point that up to now only little is known on long-range interaction besides the two recent work [3, 9]. Another approach in terms of a random resistor network [8] was recently established by Faggionato. It however separates the edges inside $\boldsymbol{Q}$ and accounts only for interaction between points inside $\boldsymbol{Q}$ with those outside $\boldsymbol{Q}$ but not for “inside –inside”, e.g. nearest neighbor interaction. We further note that in a previous work of the second author (e.g. the preprint [14]) the same problem was considered in less generality for point processes in $\mathbb{Z}^d$.

1.1 Notation

We write $\mathbb{B}_R(x) := \left\{y \in \mathbb{R}^d :\ |x-y| < R\right\}$ for the open ball of radius R around $x \in \mathbb{R}^d$ and more general

$$\mathbb{B}_R(\boldsymbol{Q}) := \left\{y \in \mathbb{R}^d :\ \exists x \in \boldsymbol{Q} \text{ s.t. } |x-y| < R\right\}\,.$$

Given $\mathbb{x}^\varepsilon = \left(\mathbf{x}_j^\varepsilon\right)_{j\in\mathbb{N}}$ we construct a Voronoi tessellation of cells g_j^ε with center $\mathbf{x}_j^\varepsilon$ and with mass $m_j^\varepsilon = \left|\mathrm{g}_j^\varepsilon\right|$ (the Lebesgue measure) respectively. We say that the Voronoi cells g_i^ε and g_j^ε are neighbored if the $d-1$ dimensional Hausdorff measure $m_{i,j}^\varepsilon := \left|\partial\mathrm{g}_i^\varepsilon \cap \partial\mathrm{g}_j^\varepsilon\right|$ is positive. We write

$$i \sim j \text{ or } \mathbf{x}_i^\varepsilon \sim \mathbf{x}_j^\varepsilon \text{ if the cells } \mathrm{g}_i^\varepsilon \text{ and } \mathrm{g}_j^\varepsilon \text{ are neighbored and}$$

$$\mathcal{N}(\mathbf{x}_i^\varepsilon, \mathbb{x}^\varepsilon) := \left\{\mathbf{x}_j^\varepsilon \in \mathbb{x}^\varepsilon :\ i \sim j\right\} \quad \text{with} \quad \mathcal{N}(\mathbf{x}_i^\varepsilon, \mathbb{x}^\varepsilon) := \varepsilon\mathcal{N}(\frac{\mathbf{x}_i^\varepsilon}{\varepsilon}, \mathbb{x})\,. \tag{1.6}$$

Given the random point process $\mathbb{x}$ we define

$$\Gamma = \{\gamma_{ij} := \frac{1}{2}(\mathbf{x}_i + \mathbf{x}_j) \ :\ i \neq j\}\,, \qquad \Gamma^\varepsilon = \{\gamma_{ij}^\varepsilon := \varepsilon\gamma_{ij}\}\,.$$

We furthermore use the following short notations for sums

$$\alpha_{ij} := \alpha_{\mathbf{x}_i,\mathbf{x}_j}\,, \qquad \alpha_{ij}^\varepsilon := \alpha^\varepsilon_{\mathbf{x}_i^\varepsilon,\mathbf{x}_j^\varepsilon}\,, \qquad \sum_i := \sum_{i\in\mathbb{N}}, \qquad \sum_{i,j} := \sum_{\substack{i,j\in\mathbb{N}\\ i\neq j}}$$

and introduce the spaces

$$\mathcal{G}_{\mathbb{x}}^\varepsilon := \{\Gamma^\varepsilon \to \mathbb{R}\} \quad \text{and} \quad \mathcal{G}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q}) := \left\{u \in \mathcal{G}_{\mathbb{x}}^\varepsilon \,:\, \forall \gamma_{ij}^\varepsilon \in \Gamma^\varepsilon\backslash\boldsymbol{Q}\;\; u(\gamma_{ij}^\varepsilon) = 0\right\}$$

By construction, it holds $\alpha_{ij}^\varepsilon = \alpha_{ij}$ and we define the following semi-norm on $\mathcal{S}_{\mathbb{x}}^\varepsilon$:

$$\forall u \in \mathcal{S}_{\mathbb{x}}^\varepsilon : \qquad \lfloor u \rfloor_{\alpha\Gamma^\varepsilon} := \left(\varepsilon^{d-2}\sum_{i,j}\alpha_{ij}^\varepsilon\frac{(u_j-u_i)^2}{|\mathbf{x}_j-\mathbf{x}_i|^2}\right)^{\frac12}\,.$$

We will see below that under certain conditions, $\lfloor u \rfloor_{\alpha\Gamma^\varepsilon}$ indeed is a norm on $\mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$.

For every $\mathbb{x}$ and every $\varepsilon > 0$ as well as for positive numbers $(a_i^\varepsilon)_{i\in\mathbb{N}}$ we find the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{a\mathbb{x}^\varepsilon}$ and the corresponding norm $\|\cdot\|_{a\mathbb{x}^\varepsilon}$ on $\mathcal{S}_{\mathbb{x}}^\varepsilon$ given by

$$\langle u, v\rangle_{a\mathbb{x}^\varepsilon} := \varepsilon^d \sum_{\mathbf{x}_i\in\mathbb{x}^\varepsilon} a_i^\varepsilon u_i v_i\,.$$

Typical examples are the choices

$$a_i^\varepsilon \equiv 1 \qquad \text{or} \qquad a_i^\varepsilon = m_i^\varepsilon\,. \tag{1.7}$$

We define $\mathcal{R}_{\varepsilon,\mathbb{x}} :\, L^q_{\mathrm{loc}}(\mathbb{R}^d) \to \mathcal{S}_{\mathbb{x}}^\varepsilon$, $1\le q<\infty$ and its adjoint $\mathcal{R}^*_{\varepsilon,\mathbb{x}} : \mathcal{S}_{\mathbb{x}}^\varepsilon \to L^q_{\mathrm{loc}}(\mathbb{R}^d)$ through

$$\left(\mathcal{R}_{\varepsilon,\mathbb{x}}\phi\right)_i = |G_i^\varepsilon|^{-1}\int_{G_i^\varepsilon}\phi\,, \qquad \text{and} \qquad \left(\mathcal{R}^*_{\varepsilon,\mathbb{x}}u\right)[x] = u(\mathbf{x}_i^\varepsilon) \qquad \text{if} \quad x\in G_i^\varepsilon\,.$$

Again, we drop the index $\mathbb{x}$ if no confusion is possible.

The above (semi-) norms as well as $\mathcal{R}^*_{\varepsilon,\mathbb{x}}$ can be restricted to $\mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$ using $u_{\mathbf{x}_i} = 0$ for $\mathbf{x}_i \in \mathbb{x}^\varepsilon \setminus \boldsymbol{Q}$.

1.2 Our setting

We make the following crucial assumptions, where, in line with the general definition of $m_{i,j}^\varepsilon$ above, m_0 is the mass of the cell corresponding to $x_i = 0$ and $m_{0,j}$ is the Hausdorff mass of the Voronoi interface between the neighbors $x_i = 0$ and x_j. :

Assumption 1.1.

1 Support-Condition*: There exists $\beta_{\mathfrak{d}} > d+1$ such that*

$$f_{\mathfrak{d}}(R) < R^{-\beta_{\mathfrak{d}}}\,, \qquad \textit{where } f_{\mathfrak{d}}(R) := \frac{1}{2d}\mathbb{P}(\mathbb{B}_R(0)\cap\mathbb{x} = \emptyset) \tag{1.8}$$

2 Poincaré-Condition*: Defining*

$$\beta_{ij} := \begin{cases} |\partial m_{ij}|\,|\mathbf{x}_j-\mathbf{x}_i|\;, i\sim j\\ 0 \qquad\;\;\, , \textit{ else.}\end{cases} \tag{1.9}$$

there exists some $p \in \left(\frac{2d}{d+2},\,2\right)$ such that it holds

$$\mathbb{E}_0\sum_{0\sim j}\alpha_{0j}\left(\frac{\beta_{0j}}{\alpha_{0j}}\right)^{\frac{2}{2-p}} < \infty \tag{1.10}$$

3 Nondegeneracy-Condition*: Let* $(e_k)_{k=1,\dots,d}$ *be a orthonormal basis of* $\mathbb{R}^d$*. Then there exists* $C>0$ *such that for* $k=1,\dots,d$

$$\mathbb{E}_0 \sum_{j\sim 0} \beta_{0,j} \frac{|\mathbf{x}_j \cdot e_k|}{|\mathbf{x}_j|} > C \tag{1.11}$$

4 Ergodicity-Condition(s)*: There exist* $q_\alpha, q_m > 1$ *such that*

$$\mathbb{E}_0 m_0^{-q_m} + \mathbb{E}_0 \sum_j \alpha_{0j}^{q_\alpha} < \infty\,.$$

Remark 1.2*.* Condition (1.8) is new compared to [9, 3] and is solely due to the fact that $\mathbb{x} \neq \mathbb{Z}^d$.

Example 1.3*.* If $\mathbb{P}(x \in \mathbb{x}) = p_0 \in (0,1)$ is distributed i.i.d. among all $x \in \mathbb{Z}^d$ it is easy to see that

$$f_{\mathfrak{d}}(R) < C\,\exp(-R^d)$$

for some $C>0$ depending on p_0.

Remark 1.4*.* In case $\mathbb{x} = \mathbb{Z}^d$, we fall back to $\beta_{ij} = 1$ for $i \sim j$ and $m_i = 1$ and every cell has a fixed diameter. Consequently, Assumption 1.1 collapses to the following condition: There exists $p \in \left(\frac{2d}{d+2},\, 2\right)$ and $q > 1$ such that

$$\mathbb{E}\sum_{0\sim j} \alpha_{0j}^{\frac{-p}{2-p}} + \mathbb{E}_0 \sum_j \alpha_{0j}^q < \infty \tag{1.12}$$

This can be understood as a generalization of (1.4) by using $p = \frac{2d}{d+2}$ and $q>1$, even though we are slightly worse than (1.2).

Theorem 1.5. *Let* $\mathbb{x}$ *a stationary ergodic point process* $\mathbb{x} = (\mathbf{x}_i)_{i\in\mathbb{N}}$ *and* $\alpha \in \mathcal{G}_{\mathbb{x}}^\varepsilon$ *a jointly stationary random function such that Assumption 1.1 is satisfied. Then almost surely the following properties are satisfied by* $\mathbb{x}$*,* α *and* $\mathcal{L}_{\mathbb{x},\alpha}^\varepsilon$ *and* A_{hom} *given by* (5.1) *below:*

1 *For some* $c>0$ *it holds*

$$\forall x \in \boldsymbol{Q}: \qquad c\,|\xi|^2 \leq \xi\cdot A_{\mathrm{hom}}(x)\xi \leq c^{-1}\,|\xi|^2$$

and $\mathcal{L}_{\mathbb{x},\alpha}^\varepsilon$ *weakly G-converges to* $u \mapsto \nabla\cdot A_{\mathrm{hom}}\nabla u$ *in the following sense: If* $f^\varepsilon \in \mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$ *is a sequence and* $f \in L^2(\boldsymbol{Q})$ *such that* $\mathcal{R}_{\varepsilon,\mathbb{x}}^* f^\varepsilon \rightharpoonup f$ *weakly in* $L^2(\mathbb{R}^d)$ *and if* $u^\varepsilon \in \mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$ *is the solution to*

$$\forall \mathbf{x}_i^\varepsilon \in \boldsymbol{Q}\cap\mathbb{x}^\varepsilon: \qquad -\left(\mathcal{L}_{\mathbb{x},\alpha}^\varepsilon u\right)_i = f_i^\varepsilon\,,$$

then there exists a unique $u \in H_0^1(\boldsymbol{Q})$ *such that* $\mathcal{R}_{\varepsilon,\mathbb{x}}^* u^\varepsilon \to u$ *strongly in* $L^q(\boldsymbol{Q})$ *as* $\varepsilon \to 0$*,* q *given in Assumption 1.1.4, and* u *is the solution to*

$$-\nabla\cdot(A_{\mathrm{hom}}\nabla u) = f \quad \textit{in } \boldsymbol{Q} \qquad \textit{with } u|_{\partial\boldsymbol{Q}} \equiv 0\,. \tag{1.13}$$

2 *There exists* $\beta \in (0,1)$ *such that for every* $u \in \mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$ *it holds* $\mathrm{supp}\mathcal{R}_{\varepsilon,\mathbb{x}}^* u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$*. Furthermore it holds* $\mathcal{R}_{\varepsilon,\mathbb{x}}^*\mathcal{R}_{\varepsilon,\mathbb{x}}\phi \to \phi$ *strongly in* $L^q(\mathbb{R}^d)$ *for every* $\phi \in L^q(\boldsymbol{Q})$*.*

3 *There exists a constant* $C>0$ *such that for every* $\varepsilon>0$ *with* $\alpha_i^\varepsilon = m_{\mathbf{x}_i^\varepsilon}^\varepsilon$ *it holds*

$$\forall U \in \mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q}): \qquad \|U\|_{m\mathbb{x}^\varepsilon} \leq C\,\lfloor U \rfloor_{\alpha\Gamma^\varepsilon}$$

and boundedness of $\lfloor u^\varepsilon \rfloor_{\alpha\Gamma^\varepsilon}$ *implies precompactness of* $\mathcal{R}_{\varepsilon,\mathbb{x}}^* u^\varepsilon$ *in* $L^q(\mathbb{R}^d)$*.*

The proof of Theorem 1.5 will be given in Section 7.

Remark 1.6. The most surprising part of Theorem 1.5 is probably part 2., i.e. $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbb{x}}u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$ instead of a result $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbb{x}}u \subset \mathbb{B}_{C\varepsilon}(\boldsymbol{Q})$ for some $C > 0$. The reason for that is that Voronoi cells G_i^ε in general might become arbitrary large, even for small ε. However, it is "very unlikely" that their diameter becomes larger than $C\varepsilon^\beta$. We highlight at this point that $\beta \in (0,1)$ implies $\varepsilon^\beta \gg \varepsilon$ as $\varepsilon \to 0$ but still $\varepsilon^\beta \to 0$.

Remark 1.7. The case of $\mathbb{x} \subset \mathbb{Z}^d$ is but a special case of the above setting, see Section 2.4.

2 Probability space and ergodic theorems

2.1 Random measures

In what follows, let $\mathcal{M}(\mathbb{R}^d)$ be the space of Radon measures (in $\mathbb{R}^d$ this equals to both Baire and Borel measures) equipped with the Vague topology. The Vague topology is metrizable and defined as the smallest topology such that for every $f \in C_c(\mathbb{R}^d)$ the map $\mu \mapsto \int_{\mathbb{R}^d} f\,\mathrm{d}\mu$ is continuous ([4]). We furthermore rely on the following definition (see [12]) of stationary random measures:

Definition 2.1. Let $\Omega \subset \mathcal{M}\big(\mathbb{R}^d\big)$ be an arbitrary (in particular also non-measurable) set and let $\sigma_\Omega := \{A \cap \Omega :\ A \in \sigma_{\mathcal{M}}\}$ be the Vague sigma-algebra restricted to Ω.

1 A random measure is a measurable surjective mapping $\mu :\ \tilde{\Omega} \to \Omega, \tilde{\omega} \mapsto \mu_{\tilde{\omega}}$, where $\left(\tilde{\Omega}, \tilde{\sigma}, \tilde{\mathbb{P}}\right)$ is a probability space. Since $\mathbb{P} := \tilde{\mathbb{P}} \circ \mu^{-1}$ is a probability measure on Ω, we also find the following:

2 A probability measure $\mathbb{P}$ on (Ω, σ_Ω) is called a random measure. In order to highlight the measure aspect, we write the identity on Ω as

$$\Omega \to \mathcal{M}\big(\mathbb{R}^d\big)\,, \qquad \omega \mapsto \mu_\omega := \omega\,.$$

We also say that $(\Omega, \sigma_\Omega, \mathbb{P})$ induces the random measure $\omega \mapsto \mu_\omega$.

3 For every bounded Borel sets $A_1, A_2, \dots, A_k \subset \mathbb{R}^d$, $k \in \mathbb{N}$ we denote by

$$F_k(A_1, \dots, A_k;\, x_1, \dots, x_k) = \mathbb{P}(\mu_\omega(A_i) \le x_i;\, i = 1, \dots, k)$$

the finite dimensional distributions (fidi distributions) of μ_ω.

4 A random measure $\mu_\bullet$ is called stationary if for every $x \in \mathbb{R}^d$ it holds $\tau_x\Omega \subset \Omega$ and the fidi distributions of $\mu_\bullet$ and $\tau_x\mu_\bullet$ are the same, i.e.

$$F_k(A_1, \dots, A_k;\, x_1, \dots, x_k) = F_k(A_1 + x, \dots, A_k + x;\, x_1, \dots, x_k)\,.$$

This is equivalent with $\mathbb{P} = \mathbb{P} \circ \tau_x$ for all $x \in \mathbb{R}^d$.

5 If $\Omega \subset \mathcal{M}\big(\mathbb{R}^d\big)^k$ and 1.–4. holds componentwise, we speak of a k-dimensional random measure.

In what follows, we summarize the theory outlined in the recent work [12].

Theorem 2.2. *Let μ_ω be a stationary k-dimensional random measure. Then there exists a precompact metric space Ω with a probability measue $\mathbb{P}$ and a family $(\tau_x)_{x\in\mathbb{Z}^d}$ of continuous bijective mappings $\tau_x:\Omega\mapsto\Omega$, having the properties of a* dynamical system *on $(\Omega,\mathscr{F},\mathcal{P})$, i.e. they satisfy (i)-(iii):*

(i) $\tau_x\circ\tau_y=\tau_{x+y}$, $\tau_0=id$ *(Group property)*

(ii) $\mathcal{P}(\tau_{-x}B)=\mathcal{P}(B)\quad\forall x\in\mathbb{R}^d$, $B\subset\Omega$ *measurable (Measure preserving)*

(iii) $A:\ \mathbb{R}^d\times\Omega\to\Omega\qquad(x,\omega)\mapsto\tau_x\omega$ *is continuous (Continuity of evaluation)*

Furthermore, for every Borel set $A\subset\mathbb{R}^d$ it holds $\mu_\omega(A-x)=\mu_{\tau_x\omega}(A)$ and there exists a measure $\mu_{\mathcal{P}}$ and Ω called Palm measure such that Campbells formula holds:

$$\forall g\in C_c(\mathbb{R}^d),\ \|g\|_{L^1(\mathbb{R}^d)}=1,\ f\in L^1(\Omega;\mu_{\mathcal{P}}):\qquad\int_\Omega f\,\mathrm{d}\mu_{\mathcal{P}}=\int_\Omega\int_{\mathbb{R}^d}g(x)f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)\,.$$

Finally if $a_\omega\in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mu_\omega)$ is a family of stationary random functions with $\mathbb{E}\int_{[-1,1]^d}a_\omega\,\mathrm{d}\mu_\omega<\infty$ then there exists a function $a:\ \Omega\to\mathbb{R}$ with $a\in L^1(\Omega;\mu_{\mathcal{P}})$ such that $a_\omega(x)\mathrm{d}\mu_\omega(x):=a(\tau_x\omega)\mathrm{d}\mu_\omega(x)$.

The Campbell formula gives us for any Ball B around 0 and $f\in L^p(\Omega;\mu_{\mathcal{P}})$, $1\le p<\infty$:

$$0=\lim_{k\to\infty}|B|\int_\Omega|f_k(\omega)-f(\omega)|^p d\mu_{\mathcal{P}}(\omega)=\lim_{k\to\infty}\int_\Omega\int_B|f_k(\tau_x\omega)-f(\tau_x\omega)|^p\mathrm{d}\mu_\omega(x)\mathrm{d}\mathbb{P}\,.$$

and hence we have the following:

Corollary 1. *If $f\in L^p(\Omega)$, $1\le p<\infty$ than a.s. $f(\tau_x\omega)\in L^p_{loc}(\mathbb{R}^d)$. If $f_k\to f$ in $L^p(\Omega)$, then a.s., for a subsequence,*

$$f_{k_n}(\tau.\omega)\to f(\tau.\omega)\text{ in }L^p_{loc}(\mathbb{R}^d).$$

Remark 2.3. The precompactness of Ω has many important implications: (i) First, the bounded continuous functions $C_b(\Omega)$ will be dense in any $L^p(\Omega;\mu)$, $1\le p<\infty$ and any Borel measure μ on Ω, including the probability measure $\mathbb{P}$ and any Palm measure $\mu_{\mathcal{P}}$. (ii) Second, the dynamical system allows us to take a Dirac sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n\in C_c^\infty(\mathbb{B}_{n^{-1}}(0))$ and any $f\in C_b(\Omega)$ and define

$$f_n(\omega):=\int_{\mathbb{R}}^n\varphi_n(x)f(\tau_x\omega)\,\mathrm{d}x\,.$$

Then for every ω it holds $x\mapsto f_n(\tau_x\omega)$ is Lipschitz and $|f_n(\tau_x\omega)-f_n(\tau_y\omega)|\le\|\nabla\varphi_n\|_\infty\|f_n\|_\infty|x-y|$. Furthermore $f_n\to f$ pointwise and, as shown in [12], also in every $L^p(\Omega;\mu)$, $1\le p<\infty$. Thus the following space is dense in every $L^p(\Omega;\mu)$, $1\le p<\infty$:

$$C_b^{0,1}(\Omega):=\left\{f\in C_b(\Omega):\ \exists C>0:\forall\omega\in\Omega,\ x,y\in\mathbb{R}^d:|f(x)-f(y)|<C|x-y|\right\}\,.$$

While the latter property is not used in this work, it could help to simplify some calculations and may still be helpful in future studies.

2.2 Ergodic theorems

Definition 2.4. A set $A \subset \Omega$ is *almost invariant* if $\mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0$ for every $x \in \mathbb{R}^d$. The family

$$\mathscr{I} = \left\{ A \in \mathscr{F} \ : \ \forall x \in \mathbb{R}^d \, \mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0 \right\} \tag{2.1}$$

of almost invariant sets is a σ-algebra and the probability measure is called *ergodic* if it holds: $A \in \mathscr{I} \Leftrightarrow \mathbb{P}(A) \in \{0, 1\}$. Often, also below, one says that τ is ergodic instead of $\mathbb{P}$.

There exists an equivalent definition of ergodicity (see [4] Proposition 10.3.III), namly a dynamical system τ is called *ergodic* w.r.t. $\mathbb{P}$ if

$$\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \mathbb{P}(A \cap \tau_x B) \mathrm{d}x = \mathbb{P}(A)\, \mathbb{P}(B)\,. \tag{2.2}$$

The importance of the concept of ergodicity stems from the following ergodic theorems:

Definition 2.5 (Intensity of a Measure)**.** Let μ be a stationary random measure on $\mathbb{R}^n$. The intensity I of μ is defined as

$$I := \mathbb{E}[\mu([0,1]^n)].$$

Lemma 2.6 (see Chapter 13 of [4])**.** *Let $\omega \mapsto \mu_\omega$ be a stationary random measure with finite intensity. Then their unique Palm measure $\mu_{\mathcal{P}}$ is finite e.g.*

$$\mu_{\mathcal{P}}(\Omega) < \infty.$$

Theorem 2.7 (Ergodic Theorem I, Chapter 13 of [4])**.** *Let the dynamical system τ_x be ergodic and assume that the Palm measure $\mu_{\mathcal{P}}$ of the stationary random measure μ_ω has finite intensity. Then, with $\mu_\omega^\varepsilon(B) := \varepsilon^d \mu_\omega(\varepsilon^{-1} B)$, for all $g \in L^1(\Omega, \mu_{\mathcal{P}})$, it holds*

$$\lim_{\varepsilon\to 0} \int_A g(\tau_{\frac{x}{\varepsilon}}\omega) d\mu_\omega^\varepsilon(x) = |A| \int_\Omega g(\omega) d\mu_{\mathcal{P}}(\omega)$$

for $\mathbb{P}$ almost every ω and for all bounded Borel sets A that contain an open ball around 0.

Theorem 2.8 (Ergodic Theorem II, [10])**.** *Let the dynamical system τ_x be ergodic and assume that the stationary random measure μ_ω has finite intensity. Then, defining $\mu_\omega^\varepsilon(B) := \varepsilon^d \mu_\omega(\varepsilon^{-1} B)$, it holds: for all $g \in L^1(\Omega, \mu_{\mathcal{P}})$ we find for $\mathbb{P}$-almost every ω, and all $\varphi \in C_c(\mathbb{R}^d)$ that*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} g(\tau_{\frac{x}{\varepsilon}}\omega)\varphi(x) d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^d} \int_\Omega g(\omega)\varphi(x) d\mu_{\mathcal{P}}(\omega) dx.$$

2.3 The coefficients α_{ij} and the notation μ_Γ and $\mu_{\Gamma,\mathcal{P}}$

A particular case of the above is a random point process, which is a random measure such that each realization is the sum of countably many delta distributions.

Definition 2.9 (Random point process)**.** A random measure μ_ω is a *random point process* if for every ω there exist countably many points $\mathbb{x}(\omega) = (\mathbf{x}_i(\omega))_{i\in\mathbb{N}}$ such that

$$\mu_{\mathbf{x}} := \mu_\omega(A) = \sum_i \delta_{\mathbf{x}_i(\omega)}(A) \qquad \text{with Palm measure} \quad \mu_{\mathbb{x},\mathcal{P}}\,.$$

Lemma 2.10. *We set $\boldsymbol{C}_R$ the cube with length R and center $0 \in \mathbb{R}^d$. If $\mathbb{E}_0 \sum_j \alpha_{0,j} < \infty$ then also*

$$\forall R > 0 : \qquad \mathbb{E} \sum_{\gamma_{ij} \in \boldsymbol{C}_R} \alpha_{ij} < \infty \,. \tag{2.3}$$

If α is stationary, then for almost every realization there exists a constant $C > 0$ such that

$$\lim_{R\to\infty} R^{-d} \sum_{\gamma_{ij} \in \boldsymbol{C}_R} \alpha_{ij} < C \,.$$

Proof. Without loss of generality, we prove (2.3) restricting ourselves to $R = 1$, since the other cases can be obtained from a scaling $\mathbb{x} \to R\mathbb{x}$. Since $\boldsymbol{C}_1$ has edgelength 1, we can split $\mathbb{R}^d$ into cubes $\boldsymbol{C}_1(z) = \boldsymbol{C}_1 + z$. Furthermore, we define

$$\mathbb{Z}^d_+ := \left\{ z = (z_1 \dots z_d) \in \mathbb{Z}^d : \, z_k < 0 \Rightarrow k > 2 \text{ and } \exists l < k : \, z_l > 0 \right\} \,.$$

In other words: $z \in \mathbb{Z}^d_+$ iff the first non-zero coordinate is positive. Thus it can be verified that for each $z \neq 0$ either $z \in \mathbb{Z}^d_+$ or $-z \in \mathbb{Z}^d_+$ but not both. By construction of Γ, for any two points $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{x}$ with $\gamma_{ij} \in \boldsymbol{C}_1$ and $\mathbf{x}_i \in \boldsymbol{C}_1(z)$ it has to hold that $\mathbf{x}_j \in \mathbb{B}_{\sqrt{d}}(\boldsymbol{C}_1(-z))$. Therefore we can find the following estimate:

$$\begin{aligned}
\mathbb{E} \sum_{\gamma_{ij} \in \boldsymbol{C}_1} \alpha_{ij} &\leq \mathbb{E} \sum_{\mathbf{x}_i \in \boldsymbol{C}_1} \sum_{\mathbf{x}_j \in \mathbb{B}_{\sqrt{d}}(C1)} \alpha_{ij} + \mathbb{E} \sum_{z \in \mathbb{Z}^d_+} \sum_{\mathbf{x}_i \in \boldsymbol{C}_1(z)} \sum_{\mathbf{x}_j \in \mathbb{B}_{\sqrt{d}}(\boldsymbol{C}_1(-z))} \alpha_{ij} \\
&\leq \mathbb{E} \sum_{\mathbf{x}_i \in \boldsymbol{C}_1} \sum_j \alpha_{ij} + \mathbb{E} \sum_{\mathbf{x}_i \in \boldsymbol{C}_1} \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \sum_{\mathbf{x}_j \in \mathbb{B}_{\sqrt{d}}(\boldsymbol{C}_1(z))} \alpha_{ij} \\
&\leq C \mathbb{E}_0 \sum_j \alpha_{0,j} \,.
\end{aligned}$$

The limit behaviour now follows from the ergodic theorem. □

Let

$$\eta_{ij} = \exp(-|\mathbf{x}_i - \mathbf{x}_j|) \tag{2.4}$$

which is jointly stationary with $\mathbb{x}$ and Γ and study

$$\mu_{\eta\Gamma}(A) := \sum_{i,j} \eta_{ij}(\omega) \delta_{\gamma_{ij}(\omega)}(A) \,.$$

Corollary 2. *$\mu_{\eta\Gamma}$ is a stationary random measure and thus has a Palm measure $\mu_{\eta\Gamma,\mathcal{P}}$.*

This is a consequence of Lemma 2.10. In fact, it holds more generally.

Theorem 2.11. *Let $\mathbb{x}$ be a stationary random point process and let $\alpha : \mathcal{G} \to [0, \infty)$. Then the following are equivalent.*
(i) $\alpha_{ij} : \Gamma \to \mathbb{R}$ is jointly stationary with $\mathbb{x}$ and satisfies $\mathbb{E}_0 \sum_j \alpha_{0,j} < \infty$.
(ii) $\mu_{\alpha\Gamma}(A) := \sum_{i,j} \alpha_{ij}(\omega) \delta_{\gamma_{ij}(\omega)}(A)$ is a stationary random measure and there exists $\beta \in L^1(\Omega; \mu_{\eta\Gamma,\mathcal{P}})$ such that almost surely $\alpha_{ij}(\omega) = \beta(\tau_{\gamma_{ij}}\omega)\eta_{ij}(\omega)$ and for the Palm measure $\mu_{\alpha\Gamma,\mathcal{P}}$ of $\mu_{\alpha\Gamma}$ holds $\mu_{\alpha\Gamma,\mathcal{P}} = \beta \mu_{\eta\Gamma,\mathcal{P}}$.

Proof. Let (i) hold. Since the position of γ_{ij} and α_{ij} are linked to $\mathbf{x}_i$ and $\mathbf{x}_j$, $\mu_{\alpha\Gamma}$ is a stationary random measure provided it is almost surely a measre on $\mathbb{R}^d$.

Since $\mu_{\alpha\Gamma}$ is sigma additive by definition, this follows from Lemma 2.10.

Given $R > 0$, we set $\alpha_{R,ij} = \alpha_{ij}$ if $|\mathbf{x}_i - \mathbf{x}_j| < R$ and $\alpha_{R,ij} = \eta_{ij}$ else. It is again clear from the above that $\mu_{\alpha_R,\omega}$ is a stationary random measure and if $\mu_{\eta\Gamma,\mathcal{P}}$ is the Palm measure of $\mu_{\eta\Gamma}$ then Theorem 2.2 implies that there exists $\beta_R \in L^1(\Omega; \mu_{\eta\Gamma,\mathcal{P}})$ with $\alpha_{R,ij} = \beta_{R,ij}\eta_{ij}$. Also for every $R > 0$ we have

$$\int_\Omega \beta_R \mathrm{d}\mu_{\eta\Gamma,\mathcal{P}} = \mathbb{E}\sum_{\gamma_{ij}\in \boldsymbol{C}_1} \beta_{R,ij}\eta_{ij} = \mathbb{E}\sum_{\gamma_{ij}\in \boldsymbol{C}_1} \alpha_{R,ij} \leq \mathbb{E}\sum_{\gamma_{ij}\in \boldsymbol{C}_1} \alpha_{ij}$$
$$\leq C\mathbb{E}_0 \sum_j \alpha_{0,j}$$

Since for almost every ω and γ_{ij} it holds $\beta_{R,ij}(\omega)\eta_{ij} = \beta_R(\tau_{\gamma_{ij}}\omega)\eta_{ij} = \alpha_{R,ij} \leq \alpha_{ij}$, since $\mathbb{E}\sum_{\gamma_{ij}\in \boldsymbol{C}_1} \alpha_{ij} < \infty$ and since $\alpha_{R,ij} \nearrow \alpha_{ij}$, this implies also $\beta_R \nearrow \beta \in L^1(\Omega; \mu_{\eta\Gamma,\mathcal{P}})$ with $\beta\, \mathrm{d}\mu_{\eta\Gamma,\mathcal{P}} = \mathrm{d}\mu_{\alpha\Gamma,\mathcal{P}}$.

The opposite direction of the statement is straight forward to prove. □

We emphasize that

$$\mu_{\Gamma(\omega)}(A) := \sum_{i,j} \delta_{\gamma_{ij}(\omega)}(A)$$

never can be a measure as the mass of any unit cube is infinite. Thus, at first glance it makes no sense to consider a Palm measure of $\mu_{\Gamma(\omega)}$. However, Theorem 2.11 provides us with some additional degrees of freedom to interprete $\mu_{\Gamma(\omega)}$:

Definition 2.12. We say that a stationary random function f satisfies $f_\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^d; \mu_{\Gamma(\omega)})$ if $\mathbb{E}_0 \sum_j |f_{0,j}|^p < \infty$. Correspondingly, we say $f \in L^p(\Omega; \mu_{\Gamma,\mathcal{P}})$ if there exists a stationary f such that $f_\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^d; \mu_{\Gamma(\omega)})$ and $f_\omega(\gamma_{ij}) = f(\tau_{\gamma_{ij}}\omega)$. In this case, we interpret

$$\int_\Omega f \mathrm{d}\mu_{\Gamma,\mathcal{P}} := \lim_{R\to\infty} \frac{1}{|\mathbb{B}_R(0)|} \mu_{f,\Gamma(\omega)}\left(\mathbb{B}_R(0)\right) ,$$

for every ω such that the ergodic theorem holds.

Theorem 2.13 (Double ergodic theorem)**.** *Let α_{ij} be a non-negative stationary random field on $\Gamma(\omega)$ which is invariant under permutation $\alpha_{ij} = \alpha_{ji}$ and with $\mathbb{E}_0 \sum_j \alpha_{0,j} < \infty$. If $\varphi \in C_c(\mathbb{R}^d)$, $\psi \in C(\mathbb{R}^d)$, ψ^ε_{ij} is a sequence of functions $\mathcal{G}^\varepsilon \to \mathbb{R}$ and there exists a continuous strictly monotone increasing function $o : [0, \infty) \to [0, \infty)$ with $o(0) = 0$ such that $|\psi(\gamma^\varepsilon_{ij}) - \psi^\varepsilon_{ij}| \leq o(|\mathbf{x}^\varepsilon_i - \mathbf{x}^\varepsilon_j|)$ whenever $\varphi(\mathbf{x}^\varepsilon_i) \neq 0$ or $\varphi(\mathbf{x}^\varepsilon_j) \neq 0$ then*

$$\lim_{\varepsilon\to 0} \sum_i \varphi(\mathbf{x}^\varepsilon_i) \sum_{j\neq i} \alpha_{ij}\psi^\varepsilon_{ij} = \lim_{\varepsilon\to 0} \sum_{i\sim j} (\varphi(\mathbf{x}^\varepsilon_i) + \varphi(\mathbf{x}^\varepsilon_j))\alpha_{ij}\psi^\varepsilon_{ij} = 2\int_{\mathbb{R}^d} \psi\varphi\mu_{\alpha\Gamma,\mathcal{P}}(\Omega)\,.$$

Proof. Let Q be a bounded open domain around 0 with $\mathrm{supp}(\varphi) \subseteq Q$, and $\delta > 0$. Since $\mathbb{E}_0 \sum_j \alpha_{0j} < \infty$ we find a $D > 0$ such that

$$\mathbb{E}_0 \sum_{j:|\mathbf{x}_j|>D} \alpha_{0j} = \lim_{\varepsilon\to 0} \varepsilon^d \frac{1}{|Q|} \sum_{i\cap Q} \sum_{j:\|\mathbf{x}^\varepsilon_j - \mathbf{x}^\varepsilon_i\|>\varepsilon D} \alpha_{ij} < \delta.$$

If $i \in Q$ and $\|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon\| \leq \varepsilon D$ we find $|\psi(\gamma_{ij}^\varepsilon) - \psi_{ij}^\varepsilon| \leq o(\varepsilon D)$ and hence

$$\left|\varepsilon^d \sum_{i\cap Q} \sum_{j:\|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon\| \leq \varepsilon D} \alpha_{ij} \varphi(\mathbf{x}_i^\varepsilon)(\psi(\gamma_{ij}^\varepsilon) - \psi_{ij}^\varepsilon)\right| \\ < \varepsilon^d \|\varphi\|_\infty \sum_{i\cap Q} \sum_{j:\|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon\| \leq \varepsilon D} \alpha_{ij}\, o(\varepsilon D) \to 0.$$

while for the rest we obtain

$$\left|\varepsilon^d \frac{1}{|Q|} \sum_{i\cap Q} \sum_{j:\|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon\| > \varepsilon D} \alpha_{ij} \varphi(\mathbf{x}_i^\varepsilon)(\psi(\gamma_{ij}^\varepsilon) - \psi_{ij}^\varepsilon)\right| \leq \delta 4 \|\varphi\|_\infty \|\psi\|_\infty \to 0.$$

Repeating the argument with $2\varphi(\gamma_{ij}^\varepsilon) - (\varphi(\mathbf{x}_i^\varepsilon) + \varphi(\mathbf{x}_j^\varepsilon))$ we find with the ergodic theorem

$$\lim_{\varepsilon\to 0} \sum_{i\sim j} (\varphi(\mathbf{x}_i^\varepsilon) + \varphi(\mathbf{x}_j^\varepsilon)) \alpha_{ij} \psi_{ij}^\varepsilon = \lim_{\varepsilon\to 0} \sum_{i\sim j} 2\varphi(\gamma_{ij}^\varepsilon) \alpha_{ij} \psi(\gamma_{ij}^\varepsilon) = 2 \int_{\mathbb{R}^d} \psi\varphi \mu_{\alpha\Gamma,\mathcal{P}}(\Omega)\,.$$

□

2.4 Point processes on $\mathbb{Z}^d$

As mentioned in the introduction, there are also studies considering discrete elliptic operators on point processes on subsets of $\mathbb{Z}^d$.

A stationary random point process in $\mathbb{Z}^d$ is a random point process with $\mathbb{x} \subset \mathbb{Z}^d$ almost surely and the property that for every $z \in \mathbb{Z}^d$ the random measures μ_ω and $\mu_\omega(\,\cdot\, - z)$ have the same distribution.

Given a stationary ergodic point process in $\mathbb{Z}^d$, the above Definition 2.4 of ergodicity which is based on $\mathbb{R}^d$ makes no more sense. Therefore we introduce the following modified version:

Definition 2.14. Given Ω the probablity space of a stationary random point process in $\mathbb{Z}^d$, a set $A \subset \Omega$ is *almost invariant* if for every $x \in \mathbb{Z}^d$ it holds $\mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0$. The family

$$\mathscr{I} = \left\{A \in \mathscr{F} \,:\, \forall x \in \mathbb{Z}^d\, \mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0\right\} \tag{2.5}$$

of almost invariant sets is a σ-algebra and the probability measure is called *ergodic* if $\mathscr{I} = \{\Omega, \emptyset\}$.

Let $\mathbb{Y} := [0,1)^d$ be the half open unit cube with the topology of the torus and let

$$\tau: \quad \mathbb{R}^d \times \mathbb{Y} \to \mathbb{Y}\,, \qquad (x,y) \mapsto \tau_x(y) := (x+y) \mod \mathbb{Z}^d\,.$$

For every $x \in \mathbb{R}^d$ we write $\lfloor x \rfloor \in \mathbb{Z}^d$ the element that satisfies $x - \lfloor x \rfloor \in \mathbb{Y}$. Then we define $\tilde\Omega := \Omega \times \mathbb{Y}$ as well as $\tilde{\mathbb{P}} := \mathbb{P} \times \mathcal{L}^d$ where $\mathcal{L}^d$ is the Lebesgue measure and the following

$$\tau: \quad \mathbb{R}^d \times \tilde\Omega \to \tilde\Omega\,, \qquad (x,(\omega,y)) \mapsto (\tau_{\lfloor x \rfloor}\omega, \tau_x y)\,.$$

Furthermore, for every μ_ω and $y \in \mathbb{Y}$ we define $\mu_{\tau_y\omega} := \mu(\,\cdot\, - y)$ and extend this to $\mu_{\tau_x\omega} := \mu(\,\cdot\, - x)$ for every $x \in \mathbb{R}^d$.

Lemma 2.15. *The map $\Omega \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d)$, $(\omega, x) \to \mu_{\tau_x \omega}$ is continuous. Furthermore, the push forward of $\tilde{\mathbb{P}}$ from $\tilde{\Omega}$ to $\mathcal{M}(\mathbb{R}^d)$ using $(\omega, y) \to \mu_{\tau_y \omega}$ is ergodic.*

Proof. This follows from (2.2) using that the integer- and noninteger-parts of τ act solely on Ω resp. $\mathbb{Y}$. □

The implication of the last lemma is that we can consider any ergodic stationary random point process in $\mathbb{Z}^d$ as an ergodic stationary random point process in $\mathbb{R}^d$ by shifting it by $y \in \mathbb{Y}$ with a uniform distribution.

3 Discrete differential operators

3.1 Discrete gradients in $\mathbb{R}^d$

We introduce the following notation:

$$\partial_{ij}^\varepsilon u := \frac{u_j^\varepsilon - u_i^\varepsilon}{\varepsilon |\mathbf{x}_j - \mathbf{x}_i|}\,, \qquad \nu_{ij}^\varepsilon := \frac{\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon}{|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon|}.$$

Since $\nu_{ij} = \nu_{ij}^\varepsilon$ is independent from ε, we will always omit this index. Introducing

$$\nabla^\varepsilon u : \Gamma^\varepsilon \to \mathbb{R}^d : (\nabla^\varepsilon u)_{ij} := \partial_{ij}^\varepsilon u \nu_{ij}\,,$$

we observe that $(\nabla^\varepsilon u)_{ij}$ is invariant under the permutation of i and j. In order to turn $(\nabla^\varepsilon u)_{ij}$ into a permutation invariant scalar, we introduce the following:

Definition 3.1 (Normal Field)**.** Let $e_0 = 0$, $(e_i)_{i=1,\dots,n}$ be the canonical basis of $\mathbb{R}^d$, and let $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be unit n-sphere. Define

$$\begin{aligned}\mathcal{F} :=& \{\nu \in S^{n-1} \mid \exists m \in \{1, ...n\} : \nu \cdot e_i = 0 \\ & \forall i \in \{0, 1, ..., m-1\} \text{ and } \nu \cdot e_m > 0\}.\end{aligned}$$

Thus, for every $\nu \in S^{n-1}$, it holds $\nu \in \mathcal{F}$ if and only if $-\nu \notin \mathcal{F}$ and we can define

$$\forall (i,j) : \ \tilde{\nu}_{ij} := \nu_{ij} \text{ if } \nu_{ij} \in \mathcal{F} \quad \text{or} \quad \tilde{\nu}_{ij} = \nu_{ji} = -\nu_{ij} \text{ if } \nu_{ji} \in \mathcal{F}\,.$$

Hence $\tilde{\nu}_{ij} = \tilde{\nu}_{ji}$ is invariant under permutation of i and j and so is

$$(\tilde{\nabla}^\varepsilon u)(\gamma_{ij}^\varepsilon) := (\tilde{\nabla}^\varepsilon u)_{ij} := (\partial_{ij}^\varepsilon u \nu_{ij}) \cdot \tilde{\nu}_{ij}.$$

Lemma 3.2. *The operator $\tilde{\nabla}^\varepsilon$ is a linear operator $\mathcal{S}^\varepsilon \to \mathcal{G}^\varepsilon$ with adjoint $-(div^\varepsilon \,\cdot)$ defined by*

$$(div^\varepsilon\ \phi)(\mathbf{x}_i^\varepsilon) := (div^\varepsilon\ \phi)_i := \sum_{\mathbf{x}_j^\varepsilon} \frac{\nu_{ij}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|} \cdot \tilde{\nu}_{ij}\phi_{ij}^\varepsilon.$$

For $\varepsilon = 1$, we simply write $\tilde{\nabla}$ which is defined on $\mathcal{S}$ and div on $\mathcal{G}$.

Proof. Let $u \in \mathcal{S}^\varepsilon$ and $\phi \in \mathcal{G}^\varepsilon$. Then

$$\begin{aligned}\langle(\tilde{\nabla}^\varepsilon u), \phi\rangle_{\Gamma^\varepsilon(\omega)} &= \varepsilon^d \sum_{ij} (\tilde{\nabla}^\varepsilon u)_{ij}\phi^\varepsilon_{ij} = \varepsilon^d \sum_{ij} \frac{(u^\varepsilon_j \nu_{ij} + u^\varepsilon_i \nu_{ji})}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|} \cdot \tilde{\nu}_{ij}\phi^\varepsilon_{ij} \\ &= \varepsilon^d \sum_i u^\varepsilon_i (\sum_j \frac{\nu_{ji}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|} \cdot \tilde{\nu}_{ij}\phi^\varepsilon_{ij}) \\ &= -\varepsilon^d \sum_i u^\varepsilon_i (\sum_j \frac{\nu_{ij}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|} \cdot \tilde{\nu}_{ij}\phi^\varepsilon_{ij}) = \langle u, -div^\varepsilon\ \phi\rangle_{\mathbf{x}^\varepsilon(\omega)}.\end{aligned}$$

□

Lemma 3.3. *Let $A:\ \mathcal{G}^\varepsilon \to \mathcal{G}^\varepsilon$ given through $A:\ g_{ij} \mapsto \alpha_{ij} g_{ij}$. Then*

$$(div^\varepsilon\ A\tilde{\nabla}^\varepsilon u)_i = (\mathcal{L}^\varepsilon_\omega u)_i.$$

Proof. This follows from

$$\begin{aligned}(div^\varepsilon\ A\tilde{\nabla}^\varepsilon u)_i &= \sum_j \frac{\nu_{ij}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|} \cdot \tilde{\nu}_{ij} \frac{(u^\varepsilon_j - u^\varepsilon_i)}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|}\nu_{ij} \cdot \tilde{\nu}_{ij}\alpha_{ij} \\ &= \varepsilon^{-2} \sum_j \alpha_{ij} \frac{(u^\varepsilon_j - u^\varepsilon_i)}{|\mathbf{x}_j - \mathbf{x}_i|^2} = (\mathcal{L}^\varepsilon_\omega u)_i.\end{aligned}$$

□

In a similar way to the above proof, A computation for $u, \upsilon \in \mathcal{S}^\varepsilon$ shows

$$\begin{aligned}\langle -\mathcal{L}^\varepsilon_\omega u, \upsilon\rangle_{\mathbf{x}^\varepsilon(\omega)} &= \langle -div^\varepsilon(A\tilde{\nabla}^\varepsilon u), \upsilon\rangle_{\mathbf{x}^\varepsilon(\omega)} = \varepsilon^d \sum_i \upsilon_i(-div^\varepsilon\ A\tilde{\nabla}^\varepsilon u)_i \\ &= \varepsilon^n \sum_{ij} (\tilde{\nabla}^\varepsilon \upsilon)_{ij} A(\tilde{\nabla}^\varepsilon u)_{ij} \\ &= \varepsilon^{n-2} \sum_{ij} \alpha_{ij} \frac{(u^\varepsilon_j - u^\varepsilon_i)}{|\mathbf{x}_j - \mathbf{x}_i|} \frac{(\upsilon^\varepsilon_j - \upsilon^\varepsilon_i)}{|\mathbf{x}_j - \mathbf{x}_i|} = \langle \tilde{\nabla}^\varepsilon u, \tilde{\nabla}^\varepsilon \upsilon\rangle_{\alpha\Gamma^\varepsilon(\omega)}.\end{aligned}$$

This directly implies that $-\mathcal{L}^\varepsilon_\omega$ is strictly positive definite on non-constant functions.

3.2 Gradients on Ω

We recall the notation $\mu_\omega = \mu_{\mathbf{x}}$ and $\mu_{\mathbb{x},\mathcal{P}}$ from Definition 2.9 and also that due to Theorem 2.11 and Definition 2.12 we can make sense of

$$\mu_{\alpha\Gamma^\varepsilon} := \alpha\mu_{\Gamma^\varepsilon} \text{ with Palm measure } \mu_{\alpha\Gamma,\mathcal{P}} := \alpha\mu_{\Gamma,\mathcal{P}}\ .$$

Since ν_{ij} and $\tilde{\nu}_{ij}$ are jointly stationary with γ_{ij} Theorem 2.2 yields that

there exists a measurable ν on Ω such that a.s. for every γ_{ij} it holds $\nu_{ij}(\omega) = \nu(\tau_{\gamma_{ij}}\omega)$,

and similar for $\tilde{\nu}$. Also, for every $f \in C_b(\Omega)$ and fixed $\omega \in \Omega$ the following functions are continuous:

$$f_{\omega,\varepsilon}(x) := f(\tau_{\frac{x}{\varepsilon}}\omega), \quad \text{and} \quad f_\omega(x) := f(\tau_x\omega)$$

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Provided now that $0 \in \Gamma(\omega)$, which we can assume due to stationarity of Γ, for every $f \in C_b(\Omega)$ and almost every $\omega \in \Omega$ the expression

$$(\tilde{\nabla}_{Om} f)(\omega) := \varepsilon(\tilde{\nabla}^{\varepsilon} f_{\omega,\varepsilon})(0) = (\tilde{\nabla} f_{\omega})(0)$$

is well-defined and yields us a notion of a discrete gradient on Ω with the property

$$(\tilde{\nabla}_{Om} f)(\tau_{\gamma_{ij}}\omega) = (\tilde{\nabla} f_{\omega})(\gamma_{ij}) \,.$$

We observe that $\tilde{\nabla}_{Om}$ is a linear operator on $C_b(\Omega)$ and define

$$(div_{Om} f)_{\omega,\varepsilon}(\mathbf{x}_i^{\varepsilon}) := \sum_j \rho(\tau_{\gamma_{ij}}\omega)^{-1}\nu(\tau_{\gamma_{ij}}\omega)\cdot\tilde{\nu}(\tau_{\gamma_{ij}}\omega) f(\tau_{\gamma_{ij}}\omega) = \varepsilon(div^{\varepsilon} f_{\omega,\varepsilon})(\mathbf{x}_i^{\varepsilon}) \tag{3.1}$$

where

$$\rho_{\omega}(\gamma_{ij}^{\varepsilon}) := \varepsilon|\mathbf{x}_j(\omega) - \mathbf{x}_i(\omega)|.$$

Then (3.1) is consistent with

$$\begin{aligned}
\varepsilon(div^{\varepsilon} f_{\omega,\varepsilon})(\mathbf{x}_i^{\varepsilon}) &= \sum_j \varepsilon(\rho_{\omega}(\gamma_{ij}^{\varepsilon}))^{-1}\nu_{\omega}(\gamma_{ij}^{\varepsilon})\cdot\tilde{\nu}_{\omega}(\gamma_{ij}^{\varepsilon}) f_{\omega,\varepsilon}(\gamma_{ij}^{\varepsilon}) \\
&= \sum_j \varepsilon\frac{1}{\varepsilon}\rho(\tau_{\gamma_{ij}}\omega)^{-1}\nu(\tau_{\gamma_{ij}}\omega)\cdot\tilde{\nu}(\tau_{\gamma_{ij}}\omega) f(\tau_{\varepsilon^{-1}\gamma_{ij}^{\varepsilon}}\omega) \\
&= \sum_j \rho(\tau_{\gamma_{ij}}\omega)^{-1}\nu(\tau_{\gamma_{ij}}\omega)\cdot\tilde{\nu}(\tau_{\gamma_{ij}}\omega) f(\tau_{\gamma_{ij}}\omega) \\
&= (div_{Om} f)_{\omega,\varepsilon}(\mathbf{x}_i^{\varepsilon}).
\end{aligned}$$

We will now show that $-div_{Om} = (\tilde{\nabla}_{Om})^*$ holds true on Ω, at least in a modified sense. For this reason, we define the scalar products

$$\langle u, \upsilon\rangle_{\mathbb{x},\mathcal{P}} := \int_{\Omega} u\upsilon d\mu_{\mathbb{x},\mathcal{P}}, \qquad \langle f, g\rangle_{\alpha\Gamma,\mathcal{P}} := \int_{\Omega} fg d\mu_{\alpha\Gamma,\mathcal{P}} \,,$$

as well as the bilinear form

$$\langle f, g\rangle_{\Gamma,\mathcal{P}} := \int_{\Omega} fg d\mu_{\Gamma,\mathcal{P}} \,.$$

Theorem 3.4. *For every $u \in C_b(\Omega),\ f \in L^1(\Omega, \mu_{\Gamma,\mathcal{P}}; \mathbb{R}^d)$ with $div_{Om} f \in L^2(\Omega, \mu_{\mathbb{x},\mathcal{P}})$ and every $\varphi \in C_c(\mathbb{R}^d)$, it holds for almost every $\omega \in \Omega$*

$$\lim_{\varepsilon\to 0} \varepsilon\langle div^{\varepsilon}(f_{\omega,\varepsilon}\varphi), u_{\omega,\varepsilon}\rangle_{\mathbb{x}^{\varepsilon}} = \int_{\mathbb{R}^d} \varphi(x)\langle div_{Om} f, u\rangle_{\mathbb{x},\mathcal{P}} dx.$$

Proof. Let us first define some abbreviations for the functions we are using, $f_{ij}(\omega) := f(\tau_{\gamma_{ij}}\omega) = f(\tau_{\varepsilon^{-1}\gamma_{ij}^{\varepsilon}}\omega)$, $u_i(\omega) := u(\tau_{\mathbf{x}_i}\omega)$, $\varphi_i^{\varepsilon} := \varphi(\mathbf{x}_i^{\varepsilon})$, and $\varphi_{ij}^{\varepsilon} := \varphi(\gamma_{ij}^{\varepsilon})$. For readability, we omit ω where it is possible. This leads to

$$\begin{aligned}
\varepsilon\langle div^{\varepsilon}(f_{\omega,\varepsilon}\varphi), u_{\omega,\varepsilon}\rangle_{\mathbb{x}^{\varepsilon}} &= \varepsilon\varepsilon^d \sum_{\mathbf{x}_i^{\varepsilon}} u_i \sum_{\mathbf{x}_j^{\varepsilon}} f_{ij}\frac{\nu_{ij}\cdot\tilde{\nu}_{ij}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|}\varphi_{ij}^{\varepsilon} \\
&= \varepsilon\varepsilon^d \sum_{\mathbf{x}_i^{\varepsilon}} u_i \sum_{\mathbf{x}_j^{\varepsilon}} f_{ij}\frac{\nu_{ij}\cdot\tilde{\nu}_{ij}}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|}\varphi_i^{\varepsilon} + \varepsilon^d \sum_{\mathbf{x}_i^{\varepsilon}} u_i \sum_{\mathbf{x}_j^{\varepsilon}} f_{ij}\frac{\nu_{ij}\cdot\tilde{\nu}_{ij}}{|\mathbf{x}_j - \mathbf{x}_i|}[\varphi_{ij}^{\varepsilon} - \varphi_i^{\varepsilon}].
\end{aligned}$$

For the first term on the RHS, we get

$$\varepsilon\varepsilon^d \sum_{\mathbf{x}_i^\varepsilon} u_i \sum_{\mathbf{x}_j^\varepsilon} f_{ij} \frac{\nu_{ij}\cdot\tilde{\nu}_{ij}}{\varepsilon|\mathbf{x}_j-\mathbf{x}_i|}\varphi_i^\varepsilon = \varepsilon\langle div^\varepsilon(f_{\omega,\varepsilon}), u_{\omega,\varepsilon}\varphi\rangle_{\mathbb{X}^\varepsilon(\omega)}$$
$$= \langle (div_{Om} f)_{\omega,\varepsilon} u_{\omega,\varepsilon}, \varphi\rangle_{\mathbb{X}^\varepsilon(\omega)}. \tag{3.2}$$

After Remark 2.3 (ii) $C_b(\Omega)$ lies dense in $L^2(\Omega,\mu_{\mathbb{x},\mathcal{P}})$. Therefore $u\, div_{Om} f \in L^1(\Omega,\mu_{\mathbb{x},\mathcal{P}})$, and via Theorem 2.8, for $\varepsilon\to 0$, converges (3.2) to

$$\int_{\mathbb{R}^d} \varphi(x)\langle div_{Om} f, u\rangle_{\mathbb{x},\mathcal{P}} dx.$$

It is left to show that the second term on the RHS vanishes for $\varepsilon\to 0$. Since $\varphi\in C_c(\mathbb{R}^d)$, $u\in C_b(\Omega)$, and $f\in L^1(\Omega,\mu_{\Gamma,\mathcal{P}};\mathbb{R}^d)$ we get

$$\varepsilon\varepsilon^d \sum_{\mathbf{x}_i^\varepsilon} u_i \sum_{\mathbf{x}_j^\varepsilon} f_{ij} \frac{\nu_{ij}\cdot\tilde{\nu}_{ij}}{\varepsilon|\mathbf{x}_j-\mathbf{x}_i|}[\varphi_{ij}^\varepsilon-\varphi_i^\varepsilon] \le \varepsilon^d \sum_{\mathbf{x}_i^\varepsilon}|u_i|\sum_{\mathbf{x}_j^\varepsilon}|f_{ij}|\varepsilon\|\nabla\varphi\|_\infty$$
$$\le \varepsilon^d \sum_{\mathbf{x}_i^\varepsilon}\sum_{\mathbf{x}_j^\varepsilon}\varepsilon|f_{ij}|\|u\|_\infty\|\nabla\varphi\|_\infty \le \varepsilon C\|f\|_{L^1(\Omega,\mu_{\Gamma,\mathcal{P}};\mathbb{R}^d)}\|u\|_\infty\|\nabla\varphi\|_\infty \to 0.$$

□

Corollary 3. *The linear operator* $\tilde{\nabla}_{Om}: L^2(\Omega,\mu_{\mathbb{x},\mathcal{P}};\mathbb{R}^d)\to L^2(\Omega,\mu_{\alpha\Gamma,\mathcal{P}})$ *is the adjoint of* $-div_{Om}(\alpha\cdot)$.

Proof. Let the functions be defined as in Theorem 3.4. A swift computation under the usage of the Ergodic theorem 2.8 and Theorem 3.4 gets us

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(x)\langle\alpha f,\tilde{\nabla}_{Om}u\rangle_{\Gamma,\mathcal{P}}dx &= \lim_{\varepsilon\to0}\langle\alpha_{\omega,\varepsilon}f_{\omega,\varepsilon}\varphi,(\tilde{\nabla}_{Om}u)_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)}\\ &= \lim_{\varepsilon\to0}\langle\alpha_{\omega,\varepsilon}f_{\omega,\varepsilon}\varphi,\varepsilon\tilde{\nabla}^\varepsilon u_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)}\\ &= \lim_{\varepsilon\to0}\varepsilon\langle -div^\varepsilon(\alpha_{\omega,\varepsilon}f_{\omega,\varepsilon}\varphi),u_{\omega,\varepsilon}\rangle_{\mathbb{X}^\varepsilon(\omega)}\\ &= \int_{\mathbb{R}^d}\varphi(x)\langle -div_{Om}\alpha f,u\rangle_{\mathbb{x},\mathcal{P}}dx.\end{aligned}$$

Since $\varphi\in C_c(\mathbb{R}^d)$, the claim follows. □

Finally, we introduce two subspaces of $L^2(\Omega,\mu_{a\Gamma,\mathcal{P}})$, the potential and solenoidal subspace.

Definition 3.5 (L^2_{pot} and L^2_{sol}). Given a stationary random point process $\mathbb{x}$ and a stationary random field $\alpha:\Gamma\to[0,\infty)$ with $\mathbb{E}_0\sum_j\alpha_{0j}<\infty$ we define the potentials and solenoidals on Ω as follows:

$$L^2_{\mathrm{pot}}(\alpha\Gamma) = \text{closure}_{L^2(\Omega,\mu_{a\Gamma,\mathcal{P}})}\{\tilde{\nabla}_{Om}f : f\in C_b(\Omega)\} \quad \text{and } L^2_{\mathrm{sol}}(\alpha\Gamma) = L^2_{\mathrm{pot}}(\alpha\Gamma)^\perp.$$

Remark 3.6. We could equally define

$$L^2_{\mathrm{pot}}(\alpha\Gamma) = \text{closure}_{L^2(\Omega,\mu_{a\Gamma,\mathcal{P}})}\{\tilde{\nabla}_{Om}f : f\in C_b^{0,1}(\Omega)\} \quad \text{and } L^2_{\mathrm{sol}}(\alpha\Gamma) = L^2_{\mathrm{pot}}(\alpha\Gamma)^\perp$$

as the closure of differences of $C_b^{0,1}(\Omega)$-functions. However, this is not needed in the calculations below.

Corollary 4. *$L^2_{\mathrm{pot}}(\alpha\Gamma)$ is well defined and not empty. Furthermore, if $\tilde\alpha \le \alpha$ pointwise almost surely then $L^2_{\mathrm{pot}}(\alpha\Gamma) \subset L^2_{\mathrm{pot}}(\tilde\alpha\Gamma)$.*

Proof. This is a straight forward calculation. □

Lemma 3.7. *For every $f \in L^2_{\mathrm{sol}}(\alpha\Gamma)$, it holds $div_{Om}(fa) = 0$ $\mu_{\Gamma,\mathcal{P}}-$almost surely. Hence for almost every realization $f_{\omega,\varepsilon}$ holds $div^\varepsilon(a_{\omega,\varepsilon}f_{\omega,\varepsilon}) = 0$ locally on $\mathbb{X}^\varepsilon(\omega)$.*

Proof. Let $f \in L^2_{\mathrm{sol}}(\alpha\Gamma)$ and let $\varphi \in C_c(\mathbb{R}^d)$. Then for every $u \in C_b(\Omega)$ we get with the help of Theorem 3.4 for some $\omega \in \Omega$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \varphi(x) \langle \tilde\nabla_{Om} u, af\rangle_{\Gamma,\mathcal{P}} dx = -\int_{\mathbb{R}^d} \varphi(x) \langle u, div_{Om} af\rangle_{\mathbb{X},\mathcal{P}} dx \\ &= -\lim_{\varepsilon\to 0} \langle u_{\omega,\varepsilon}\varphi, div^\varepsilon(a_{\omega,\varepsilon}f_{\omega,\varepsilon})\rangle_{\mathbb{X}^\varepsilon(\omega)}. \end{aligned}$$

This is true for every $\varphi \in C_c(\mathbb{R}^d)$ and every $u \in C_b(\Omega)$, hence the claim follows. □

4 Properties of $\mathcal{S}^\varepsilon_{\mathbb{X}}$

In this section we provide some fundamental properties of functions in $\mathcal{S}^\varepsilon_{\mathbb{X}}$, particularly a Poincaré inequality and a compact embedding result. For this we will use results from numerical analysis. Furthermore, we will show that the support of functions in $\mathcal{R}^*_{\varepsilon,\mathbb{X}}\mathcal{S}^\varepsilon_{\mathbb{X}}(\boldsymbol{Q})$ lies almost surely within a bounded region around $\boldsymbol{Q}$ while the support decreases towards $\boldsymbol{Q}$ as $\varepsilon \to 0$. This will imply for every $\phi \in L^2(\boldsymbol{Q})$ that $\mathcal{R}^*_{\varepsilon,\mathbb{X}}\mathcal{R}_{\varepsilon,\mathbb{X}}\phi \to \phi$ strongly in $L^2(\boldsymbol{Q})$ as $\varepsilon \to 0$.

4.1 Support of $\mathcal{R}^*_{\varepsilon,\mathbb{X}}\mathcal{S}^\varepsilon_{\mathbb{X}}(\boldsymbol{Q})$

Lemma 4.1. *Let $\mathbb{X}$ be a stationary point process in $\mathbb{R}^d$ with $f_{\mathfrak{d}}$ given in* (1.8). *Then, if $\boldsymbol{G} := (\mathrm{g}_i)_{i\in\mathbb{N}}$ is the Voronoi tessellation for $\mathbb{X} = (\mathbf{x}_i)_{i\in\mathbb{N}}$ with maximal diameter*

$$\mathfrak{d}(\mathbf{x}_i) := \max_{x,y\in \mathrm{g}_i} |x-y| \,, \tag{4.1}$$

then

$$\mathbb{P}(\mathfrak{d} > D) < f_{\mathfrak{d}}\left(\frac{1}{6}D\right) \tag{4.2}$$

Proof. We define for a unit vector ν of unit length, $0 < \alpha < \frac{\pi}{2}$ and $R > 0$ the cone

$$\mathbb{C}_{\nu,\alpha,R}(x) := \{z \in \mathbb{B}_R(x) \,:\, z\cdot\nu > |z|\cos\alpha\} \,.$$

Because of the stationarity and because of $\mathbb{P}(A\cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ it holds for $R \in \mathbb{Z}$ and $\mathbf{E} := \{e_1, \dots e_d\} \cup \{-e_1, \dots - e_d\}$ ($\{e_1, \dots e_d\}$ being the canonical basis of $\mathbb{R}^d$)

$$\mathbb{P}(\exists e \in \mathbf{E} \,:\, \mathbb{B}_R(2Re) \cap \mathbb{X} = \emptyset) \le \sum_{i=1}^d \sum_{\pm} \mathbb{P}((\mathbb{B}_R(\pm 2Re_i) \cap \mathbb{X} = \emptyset) \le f_{\mathfrak{d}}(R)\,.$$

In particular, for $\alpha = \arctan\sqrt{1/3} = \frac{\pi}{6}$ we have the smallest opening angle such that $\mathbb{B}_R(2Re)$ lies completely inside $\mathbb{C}_{e,\alpha,3R}(0)$ and we discover

$$\mathbb{P}\big(\forall e \in \mathbf{E}:\ \mathbb{x} \cap \mathbb{C}_{e,\frac{\pi}{6},3R}(0) \neq \emptyset\big) \geq 1 - f_{\mathfrak{d}}(R)\,. \tag{4.3}$$

Now we take arbitrary points $\mathbf{x}_{\pm j} \in C_{\pm e_j,\alpha,3R}(0) \cap \mathbb{x}$. Then the planes given by the respective equations $\left(x - \frac{1}{2}\mathbf{x}_{\pm j}\right)\cdot \mathbf{x}_{\pm j} = 0$ define a bounded cell around 0, with a maximal diameter $D(\alpha, R) = CR$ which is proportional to R. The constant $C > 1$ depends solely on the opening angle $\alpha = \frac{\pi}{6}$ of the cones and can be shown from some trigonometric calculations to be smaller than 6. Estimate (4.2) now follows from

$$\mathbb{P}(\mathfrak{d} > D) = \mathbb{P}(\mathfrak{d} > CR) \leq \mathbb{P}\left(\exists e \in \mathbf{E}\ :\ \mathbb{x} \cap \mathbb{C}_{e,\alpha,3R}(0) = \emptyset\right) \leq f_{\mathfrak{d}}(R) = f_{\mathfrak{d}}(\frac{1}{6}D)\,.$$

□

Lemma 4.2. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f_{\mathfrak{d}}$ satisfy* (1.8)*. Then for every $\beta \in (0, 1 - \frac{d}{\beta_{\mathfrak{d}}})$ there exists almost surely $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and every $u \in \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$ it holds $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbb{x}} u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$. Furthermore, for a given bounded Lipschitz domain $\boldsymbol{Q}$ we define*

$$\mathcal{N}(\boldsymbol{Q}, \mathbb{x}^\varepsilon) := \{x \in \mathbb{x}^\varepsilon \backslash \boldsymbol{Q} : \mathcal{N}(x, \mathbb{x}^\varepsilon) \cap \boldsymbol{Q} \neq \emptyset\}\ . \tag{4.4}$$

Then there exists almost surely $\varepsilon_0 > 0$ and $\beta \in (0,1)$ such that for every $\varepsilon_0 > \varepsilon$ it holds $\mathcal{N}(\boldsymbol{Q}, \mathbb{x}^\varepsilon) \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$. Furthermore, for every domain $\tilde{\boldsymbol{Q}} \subset \boldsymbol{Q}$ with $\overline{\tilde{\boldsymbol{Q}}} \subset \boldsymbol{Q}$ there exists $\tilde{\varepsilon} > 0$ such that for every $\varepsilon < \tilde{\varepsilon}$ and every $\phi \in C_c(\boldsymbol{Q})$ it holds $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbb{x}}\phi \subset \boldsymbol{Q}$.

Proof. Let $u^\varepsilon_1(x) = 1$ if $x \in \boldsymbol{Q} \cap \mathbb{x}^\varepsilon$ and $u^\varepsilon_1(x) = 0$ else. Given $N := \varepsilon^{-1}$, $\beta_0 = 1 - \beta$ the event

$$B_N := \left(\bigcup_{\mathbf{x}_i \in \mathbb{x} \cap N\boldsymbol{Q}} G_i \subset \mathbb{B}_{N^{\beta_0}}(N\boldsymbol{Q})\right)$$

is equivalent with the event

$$\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbb{x}} u^\varepsilon_1 \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})\,.$$

For the complementary event $\neg B_N$ of B_N it holds

$$\begin{aligned}\mathbb{P}(\neg B_N) &\leq \mathbb{P}(\exists \mathbf{x}_i \in \mathbb{x} \cap N\boldsymbol{Q} : \mathbb{B}_{\mathfrak{d}_i}(\mathbf{x}_i) \not\subset \mathbb{B}_{N^{\beta_0}}(N\boldsymbol{Q})) \\ &\leq \sum_{\mathbb{x} \cap N\boldsymbol{Q}} \mathbb{P}\big(\mathfrak{d} \geq N^{\beta_0}\big) \leq C\,|\boldsymbol{Q}|\, N^d f_{\mathfrak{d}}\big(N^{\beta_0}\big) \\ &\leq CN^{d - \beta_0\beta_{\mathfrak{d}}}\,.\end{aligned}$$

If $\beta_0 \in (\frac{d}{\beta_{\mathfrak{d}}}, 1)$ the support-condition (1.8) implies $N^{d-\beta_0\beta_{\mathfrak{d}}} \to 0$ as $N \to \infty$ and hence for almost every ω there exists N_0 such that $\omega \in B_N$ for every $N > N_0$ and the first statement of the lemma holds.

The second statement can be proved similarly taking into account that every $x \in \mathcal{N}(N\boldsymbol{Q}, \mathbb{x})$ satisfies $x \in \mathbb{B}_{2\mathfrak{d}_i}(\mathbf{x}_i)$ for some $\mathbf{x}_i \in \mathbb{x} \cap N\boldsymbol{Q}$. The last statement follows from the positive distance of $\partial\tilde{\boldsymbol{Q}}$ and $\partial\boldsymbol{Q}$ as well as the first part applied to $\tilde{\boldsymbol{Q}}$. □

Lemma 4.3. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f_{\mathfrak{d}}$ satisfy* (1.8)*. Then for every $1 \leq q < \infty$ almost surely for every $\phi \in L^q(\boldsymbol{Q})$ it holds $\mathcal{R}^*_{\varepsilon,\mathbb{x}}\mathcal{R}_{\varepsilon,\mathbb{x}}\phi \to \phi$ in $L^q(\boldsymbol{Q})$ as $\varepsilon \to 0$.*

Proof. Let $\tilde{\boldsymbol{Q}} \supset \mathbb{B}_1(\boldsymbol{Q})$ be a large ball that contains 0. Given $\phi \in C_c^1(\tilde{\boldsymbol{Q}})$ and using the notation (4.1) we find

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\mathcal{R}_{\varepsilon,\mathbb{x}}^* \mathcal{R}_{\varepsilon,\mathbb{x}}\phi - \phi\right)^q &\le \sum_{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \left(\varepsilon \mathfrak{d}(\mathbf{x}_i) \left\|\nabla\phi\right\|_\infty\right)^q \varepsilon^d \left|G_i\right| \\ &\le \varepsilon^{q+d} \left\|\nabla\phi\right\|_\infty^q \sum_{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(\mathbf{x}_i)^q \left|G_i\right| \,.\end{aligned}$$

Because of Lemma 4.2 we know that almost surely for ε_0 independent from ϕ and every $\varepsilon < \varepsilon_0$ it holds $\varepsilon\mathfrak{d}(\mathbf{x}_i) < \mathrm{diam}\tilde{\boldsymbol{Q}} + 1$ for every $\varepsilon\mathbf{x}_i \in \tilde{\boldsymbol{Q}}$. Hence for every $D > 1$ we find from the ergodic theorem

$$\begin{aligned}\varepsilon^{q+d} \sum_{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(\mathbf{x}_i)^q \left|G_i\right| &= \varepsilon^{q+d} \sum_{\substack{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}} \\ \mathfrak{d}(\mathbf{x}_i) \le D}} \mathfrak{d}(\mathbf{x}_i)^q \left|G_i\right| + \varepsilon^d \sum_{\substack{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}} \\ \mathfrak{d}(\mathbf{x}_i) > D}} \left|\varepsilon\mathfrak{d}(\mathbf{x}_i)\right|^q \left|G_i\right| \\ &\le \varepsilon^q \left|\tilde{\boldsymbol{Q}}\right| D^q + \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q \sum_{k=0}^\infty \varepsilon^d \sum_{\substack{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}} \\ D+k < \mathfrak{d}(\mathbf{x}_i) < D+k+1}} \mathfrak{d}(\mathbf{x}_i)^d \\ &\le \varepsilon^q \left|\tilde{\boldsymbol{Q}}\right| D^q + \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q \sum_{k=0}^\infty \varepsilon^d \sum_{\substack{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}} \\ D+k < \mathfrak{d}(\mathbf{x}_i) < D+k+1}} (D+k+1)^d \\ &\to \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q \sum_{k=0}^\infty (D+k+1)^d \mathbb{P}(D+k < \mathfrak{d}(\,\cdot\,) < D+k+1) \\ &\le 2^d \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q \sum_{k=0}^\infty (D+k)^d f_{\mathfrak{d}}(\frac{1}{6}(D+k)) \\ &\le 2^d \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q \left(\frac{1}{6}\right)^{\beta_{\mathfrak{d}}} \sum_{k=0}^\infty (D+k)^{d-\beta_{\mathfrak{d}}} \,.\end{aligned}$$

Since $\beta_{\mathfrak{d}}(D) > d+1$ it follows

$$\lim_{\varepsilon\to 0} \varepsilon^{q+d} \sum_{\mathbf{x}_i \in \mathbb{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(\mathbf{x}_i)^q \left|G_i\right| \le 2^d \left(\frac{1}{6}\right)^{\beta_{\mathfrak{d}}} \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^q D^{d+1-\beta_{\mathfrak{d}}} \to 0$$

as $D \to \infty$ and we obtain that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \left(\mathcal{R}_{\varepsilon,\mathbb{x}}^* \mathcal{R}_{\varepsilon,\mathbb{x}}\phi - \phi\right)^q \le 0 \,. \tag{4.5}$$

Furthermore for every $\phi \in L^q(\boldsymbol{Q})$ it holds

$$\int_{\mathbb{R}^d} \left(\mathcal{R}_{\varepsilon,\mathbb{x}}^* \mathcal{R}_{\varepsilon,\mathbb{x}}\phi\right)^q = \sum_i \int_{G_i^\varepsilon} \left(\frac{1}{m_i^\varepsilon} \int_{G_i^\varepsilon} \phi\right)^q \le \sum_i \int_{G_i^\varepsilon} \frac{1}{m_i^\varepsilon} \int_{G_i^\varepsilon} \phi^q = \int_{\boldsymbol{Q}} \phi^q \,. \tag{4.6}$$

Now let $\phi \in L^q(\boldsymbol{Q})$ and let $(\phi_k)_{k\in\mathbb{N}} \subset C_c^1(\boldsymbol{Q})$ be a sequence with $\|\phi - \phi_k\|_{L^q(\boldsymbol{Q})} < \frac{1}{k}$. Given $\delta > 0$ we find

$$\left(\int_{\mathbb{R}^d} \left(\mathcal{R}_{\varepsilon,\mathbb{x}}^* \mathcal{R}_{\varepsilon,\mathbb{x}}\phi - \phi\right)^q\right)^{\frac{1}{q}} \le \|\phi - \phi_k\|_{L^q(\boldsymbol{Q})} + \left\|\mathcal{R}_{\varepsilon,\mathbb{x}}^* \mathcal{R}_{\varepsilon,\mathbb{x}}\left(\phi - \phi_k\right)\right\|_{L^q(\boldsymbol{Q})}$$

$$+\left(\int_{\mathbb{R}^d}\left(\mathcal{R}^*_{\varepsilon,\mathbb{x}}\mathcal{R}_{\varepsilon,\mathbb{x}}\phi_k-\phi_k\right)^q\right)^{\frac{1}{q}}.$$

We chose $k\in\mathbb{N}$ such that $\|\phi-\phi_k\|_{L^q(\boldsymbol{Q})}\le\frac{1}{3}\delta$ and with help of (4.5) we choose ε_0 such that for every $\varepsilon<\varepsilon_0$ it holds $\left\|\mathcal{R}^*_{\varepsilon,\mathbb{x}}\mathcal{R}_{\varepsilon,\mathbb{x}}\phi_k-\phi_k\right\|_{L^q(\boldsymbol{Q})}<\frac{1}{3}\delta$.
Due to (4.6) it also holds $\left\|\mathcal{R}^*_{\varepsilon,\mathbb{x}}\mathcal{R}_{\varepsilon,\mathbb{x}}\left(\phi-\phi_k\right)\right\|_{L^q(\boldsymbol{Q})}<\frac{1}{3}\delta$. Then in total for every $\varepsilon<\varepsilon_0$ it holds

$$\left(\int_{\mathbb{R}^d}\left(\mathcal{R}^*_{\varepsilon,\mathbb{x}}\mathcal{R}_{\varepsilon,\mathbb{x}}\phi-\phi\right)^q\right)^{\frac{1}{q}}<\delta.$$

□

4.2 Rellich-Sobolev-Poincare inequalities

In the following, we provide results from [2] and [5], which were formulated for a general family of (non-random) grids that include Voronoi grids, but we will adapt them to our setting for the readers convenience. To this aim we recall the definition of β in (1.9) and define the following (semi-) norms on $\mathcal{S}^\varepsilon_{\mathbb{x}}$:

$$\begin{aligned}
\|u\|_{m^\varepsilon\mathbb{x}^\varepsilon,p} &:= \left(\varepsilon^d\sum_i m_i^\varepsilon\left|u_i\right|^p\right)^{\frac{1}{p}}\\
\lfloor u\rfloor_{\beta\Gamma^\varepsilon,p} &:= \left(\varepsilon^{d-p}\sum_{i\sim j}\beta_{i,j}\frac{\left|u_j-u_i\right|^p}{\left|\mathbf{x}_j-\mathbf{x}_i\right|^p}\right)^{\frac{1}{p}},\\
\|u\|_{m^\varepsilon\mathbb{x}^\varepsilon,\beta\Gamma^\varepsilon,p} &:= \|u\|_{m^\varepsilon\mathbb{x}^\varepsilon,p}+\lfloor u\rfloor_{m^\varepsilon\mathbb{x}^\varepsilon,\beta\Gamma^\varepsilon,p}.
\end{aligned}$$

Theorem 4.4 (Discrete Sobolev-Poincare Inequality). *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded domain and let $\mathbf{x}$ be a stationary and ergodic point process in $\mathbb{R}^d$. Then for every $1\le p,q<\infty$ with $1-\frac{d}{p}>-\frac{d}{q}$ there exists a constant $C>0$ which only depends on p,q,d, and $\boldsymbol{Q}$ such that*

$$\|u\|_{m^\varepsilon\mathbb{x}^\varepsilon,q}\le C\lfloor u\rfloor_{\beta\Gamma^\varepsilon,p}$$

Proof. This is a direct consequence of [2], Theorem 6 Section 4.2. □

Theorem 4.5 (Discrete Gagliardo-Nirenberg-Sobolev Inequality). *Let $\mathbf{x}$ be a stationary and ergodic point process in $\mathbb{R}^d$. Then for any $1\le p<n$ and $1\le q\le m\le\frac{pn}{n-p}$, there exists a constant $C>0$ which only depends on p,q,n, and $\boldsymbol{Q}$ such that*

$$\|u\|_{m^\varepsilon\mathbb{x}^\varepsilon,m}\le C\lfloor u\rfloor^\theta_{\beta\Gamma^\varepsilon,p}\|u\|^{1-\theta}_{m^\varepsilon\mathbb{x}^\varepsilon,m}$$

where $\theta=(\frac{1}{q}-\frac{1}{m})(\frac{1}{q}+\frac{1}{n}-\frac{1}{p})^{-1}$.

Proof. A proof can be found in [2], Theorem 7 Section 4.3. □

Theorem 4.6 (Discrete Rellich Theorem). *Let $p\in[1,\infty)$, $\mathbf{x}$ be a stationary and ergodic point process in $\mathbb{R}^d$. Then for any $u^\varepsilon\in\mathcal{S}^\varepsilon_0(\boldsymbol{Q})(\varepsilon^{-1}\in\mathbb{N})$ such that $\sup_{\varepsilon>0}\lfloor u\rfloor_{\beta\Gamma^\varepsilon,p}<\infty$, the sequence $(\mathcal{R}^*_\varepsilon u^\varepsilon)_{\varepsilon>0}$ is precompact in $L^p(\mathbb{R}^n)$.*

Proof. A proof can be found in [5], Lemma B.19. □

Theorem 4.7 (Poincare Rellich Theorem). *Let $\mathbf{x}$ a stationary ergodic point process $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ with points solely in $\mathbb{Z}^d$ with α satisfying* (1.10)*. Then for every $q \in (2, \frac{dp}{d-p})$ there almost surely exists a constant $C_{\alpha,\mathbb{x}} > 0$ such that for every $\varepsilon > 0$ and every $u^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$ it holds*

$$\|u^\varepsilon\|_{m^\varepsilon \mathbb{x}^\varepsilon, q} \leq C_{\alpha,\mathbb{x}} \lfloor u^\varepsilon \rfloor_{\alpha\Gamma^\varepsilon} . \tag{4.7}$$

Furthermore, any sequence $u_k^{\varepsilon_k} \in \mathcal{S}^{\varepsilon_k}_{\mathbb{x}}(\boldsymbol{Q})$, $k \in \mathbb{N}$, with $\sup_k \lfloor u_k^{\varepsilon_k} \rfloor_{\alpha\Gamma^{\varepsilon_k}} < \infty$ is precompact in the sense that $\mathcal{R}^*_{\varepsilon_k,\mathbf{x}} u_k^{\varepsilon_k}$ is precompact in $L^q(\mathbb{R}^d)$.

Proof. Due to Lemma 4.2 we can assume w.l.o.g that for $\varepsilon > 0$ small enough it holds $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}} u \subset \mathbb{B}_1(\boldsymbol{Q})$ for every $u \in \mathcal{S}^\varepsilon_{\mathbb{x}}$. Since $p > \frac{2d}{d+2}$ it holds $\frac{dp}{d-p} > 2$ and we infer from the discrete Sobolev-Poincaré inequality in Theorem 4.4 that for some constant $C > 0$ depending only on p, q, d, δ and $\boldsymbol{Q}$

$$\|u\|_{p,\mathbb{x}^\varepsilon,m^\varepsilon} \leq \|u\|_{m^\varepsilon \mathbb{x}^\varepsilon, q} \leq C \lfloor u \rfloor_{\beta\Gamma,q} . \tag{4.8}$$

Using the discrete Gagliardo-Nirenberg-Sobolev inequality of Theorem 4.5 we furthermore infer the existence of $C > 0$ depending only on p, q, d, δ and $\boldsymbol{Q}$ such that

$$\|u\|_{m^\varepsilon \mathbb{x}^\varepsilon, q,} \leq C \lfloor u \rfloor^{1-\theta}_{\beta\Gamma^\varepsilon,p} \|u\|^\theta_{m^\varepsilon \mathbb{x}^\varepsilon, p} \tag{4.9}$$

where $\theta = d(1/p - 1/q) < 1$. Finally, we obtain from Hölders inequality for $\frac{2-p}{2}$

$$\lfloor u \rfloor_{\beta\Gamma^\varepsilon,p} \leq \left(\sum_{x_i^\varepsilon \in \boldsymbol{Q}\cap\mathbf{x}^\varepsilon} \sum_{i\sim j} \alpha_{ij} \left(\frac{\beta_{ij}}{\alpha_{ij}}\right)^{\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \lfloor u \rfloor_{\alpha\Gamma^\varepsilon} . \tag{4.10}$$

The ergodic theorem together with the assumed Poincaré condition (1.10) on α and β imply the bound $\lfloor u \rfloor_{\beta\Gamma^\varepsilon,p} \leq C_{\alpha,\mathbb{x}} \lfloor u \rfloor_{\alpha\Gamma^\varepsilon}$ for $C_{\alpha,\mathbb{x}}$ independent from ε.

Now let $u_k^{\varepsilon_k} \in \mathcal{S}^{\varepsilon_k}_{\mathbb{x}}$, $k \in \mathbb{N}$, be a sequence with $\sup_k \lfloor u_k^{\varepsilon_k} \rfloor_{\alpha\Gamma^{\varepsilon_k}} < \infty$ then inequalities (4.8) and (4.10) imply

$$\sup_k \|u_k^{\varepsilon_k}\|_{m^{\varepsilon_k}\Gamma^{\varepsilon_k},q} < \infty . \tag{4.11}$$

From Theorem 4.6 we infer that $\mathcal{R}^*_{\varepsilon_k,\mathbf{x}} u_k^{\varepsilon_k}$ is precompact in $L^p(\boldsymbol{Q})$. Hence (4.11)–(4.9) imply also precompactness of $\mathcal{R}^*_{\varepsilon_k,\mathbf{x}} u_k^{\varepsilon_k}$ in $L^q(\boldsymbol{Q})$. □

5 The homogenized matrix

Let $(e_k)_{k=1,\dots,d}$ be a orthonormal basis of $\mathbb{R}^d$, and let $\chi_k \in L^2_{\mathrm{pot}}(\alpha\Gamma)$ be the unique minimizers of the functional

$$E_k : L^2_{\mathrm{pot}}(\alpha\Gamma) \to \mathbb{R}, \qquad \chi \to \int_\Omega \alpha |\tilde{\nu} \cdot e_k + \chi|^2 d\mu_{\Gamma,\mathcal{P}} .$$

With this, we can finally define the matrix A_{hom} through

$$A_{hom} = (A_{k,m})_{k,m=1,\dots,d}, \tag{5.1}$$

$$\text{with } A_{k,m} = \int_\Omega \alpha(\tilde{\nu} \cdot e_k + \chi_k)(\tilde{\nu} \cdot e_m + \chi_m) d\mu_{\Gamma,\mathcal{P}}.$$

In the next theorem, we want to prove that the homogenized matrix is positive definite.

Theorem 5.1. *Let* $\mathbb{X}$ *satisfy* (1.8) *and let* α_{ij} *satisfy the Poincaré-condition* (1.10) *and the nondegeneracy condition* (1.11)*. Then the matrix* A_{hom} *is positive definite.*

Proof. We recall β defined in (1.9) to show in the following that $\tilde{\nu}\cdot e_k \in L^2_{\mathrm{sol}}(\beta\Gamma)$. For $u \in C_b(\Omega)$, an open ball $A = \mathbb{B}_1(0)$ and $\phi \in C_c^\infty(A)$ we obtain from the double ergodic Theorem 2.13 (where $\psi = 1$, $\psi_{ij} = 1$)

$$\left|\int_A \phi\langle\tilde{\nabla}_{Om} u, \beta\tilde{\nu}\cdot e_k\rangle_{\Gamma,\mathcal{P}}\mathrm{d}x\right| = \lim_{\varepsilon\to 0}\left|\varepsilon^d \sum_{(ij)\cap A}\beta_{ij}\frac{1}{2}(\phi_i+\phi_j)\nu_{ij}\cdot e_k\frac{(u_j(\omega)-u_i(\omega))}{|\mathbf{x}_j-\mathbf{x}_i|}\right| \tag{5.2}$$

We add $0 = \sum \pm S_\varepsilon$, where $S_\varepsilon = \varepsilon^d\sum_{(ij)\cap A}\beta_{ij}\frac{1}{2}(u_i+u_j)\nu_{ij}\cdot e_k\frac{\varepsilon(\phi_j(\omega)-\phi_i(\omega))}{\varepsilon|\mathbf{x}_j-\mathbf{x}_i|}$ has the property $|S_\varepsilon| \le \varepsilon C\|u\|_\infty\|\nabla\phi\|_\infty$. Furthermore, with help of the Support Lemma 4.2 it holds $\phi_i = 0$ as soon as $\mathbf{x}_i^\varepsilon \notin A$ and ε is small enough and we observe with help of $(a_1+a_2)(b_1-b_2)+(b_1+b_2)(a_1-a_2) = a_1b_1 - a_2b_2$ that

$$\left|\int_A \phi\langle\tilde{\nabla}_{Om} u, \beta\tilde{\nu}\cdot e_k\rangle_{\Gamma,\mathcal{P}}\mathrm{d}x\right| \le \lim_{\varepsilon\to 0}\left|\varepsilon^d \sum_{(ij)\cap A}\beta_{ij}\nu_{ij}\cdot e_k\frac{(\phi_j u_j(\omega)-\phi_i u_i(\omega))}{|\mathbf{x}_j-\mathbf{x}_i|}\right| + C\varepsilon \tag{5.3}$$

$$\le \lim_{\varepsilon\to 0}\left|\varepsilon^d\sum_{i\cap A}u_i(\omega)\phi_i\sum_j\frac{\beta_{ij}\nu_{ij}\cdot e_k}{|\mathbf{x}_j-\mathbf{x}_i|}\right| \le \lim_{\varepsilon\to 0}\varepsilon^d\sum_{i\cap A}\left|\sum_{(j\sim i)}|\partial\mathbb{G}_{ij}|\nu_{ij}\cdot e_k\right|\|u\phi\|_\infty$$

$$= \lim_{\varepsilon\to 0}\varepsilon^d\sum_{i\cap A}\left|\int_{\partial\mathbb{G}_i}\nu\cdot e_k\right|\|u\phi\|_\infty = \lim_{\varepsilon\to 0}\varepsilon^d\sum_{i\cap A}\left|\int_{\mathbb{G}_i} div\, e_k\right|\|u\phi\|_\infty = 0.$$

In the next step, we will use the nondegeneracy condition (1.11) which reads for some $C > 0$

$$C_\xi := \int_\Omega \beta|\tilde{\nu}\cdot\xi|^2 d\mu_{\Gamma,\mathcal{P}} \ge C\,\|\xi\|^2.$$

In particular

$$\|\xi\|^2 \le C^{-1}\int_\Omega \beta|\tilde{\nu}\cdot\xi|^2\mathrm{d}\mu_{\Gamma,\mathcal{P}} = C^{-1}\int_\Omega\beta\left(\sum_k^d\tilde{\nu}\cdot e_k\xi_k\right)^2\mathrm{d}\mu_{\Gamma,\mathcal{P}}$$

$$= C^{-1}\int_\Omega\beta\sum_m^d\sum_k^d(\tilde{\nu}\cdot e_k)(\tilde{\nu}\cdot e_m)\xi_k\xi_m\mathrm{d}\mu_{\Gamma,\mathcal{P}}$$

$$= C^{-1}\int_\Omega\beta\left(\sum_{k=1}^d\xi_k\underbrace{\tilde{\nu}\cdot e_k}_{|\ldots|\le 1}\right)\left(\sum_{m=1}^d\xi_m\left(\tilde{\nu}\cdot e_m+\chi_m\right)\right)\mathrm{d}\mu_{\Gamma,\mathcal{P}} \tag{5.4}$$

$$\le C^{-1}\left(\int_\Omega\frac{\beta^2}{\alpha}\left(\sum_{k=1}^d\xi_k\right)^2\mathrm{d}\mu_{\Gamma,\mathcal{P}}\right)^{\frac{1}{2}} \tag{5.5}$$

$$\cdot\left(\int_\Omega\sum_{m,k=1}^d\alpha\xi_m(\tilde{\nu}\cdot e_m+\chi_m)\xi_k(\tilde{\nu}\cdot e_k+\chi_k)\mathrm{d}\mu_{\Gamma,\mathcal{P}}\right)^{\frac{1}{2}}$$

$$\le M\|\xi\|\left(\sum_{m,k}^d\int_\Omega\alpha\xi_k\xi_m(\tilde{\nu}\cdot e_m+\chi_m)(\tilde{\nu}\cdot e_k+\chi_k)\mathrm{d}\mu_{\Gamma,\mathcal{P}}\right)^{\frac{1}{2}} \tag{5.6}$$

$$= M\|\xi\|\bigg(\sum_{m,k}^{d}\xi_k\xi_m A_{k,m}\bigg)^{\frac{1}{2}},$$

where we first used that $\tilde{\nu}\cdot e_k \in L^2_{\text{sol}}(\beta\Gamma)$ and $\chi_m \in L^2_{\text{pot}}(\beta\Gamma)$, therefore $\langle \chi_m, \beta\tilde{\nu}\cdot e_k\rangle_{\Gamma,\mathcal{P}} = 0$ for every $k, m = 1, ..., n$ in (5.4), Cauchy-Schwartz to arrive at (5.5), and lastly used the finiteness of the first term in (5.5) (Due to the Poincaré-condition), combined it with C^{-1} and abbreviated it to $M\|\xi\|$ in (5.6).
In total, we arrive at

$$\|\xi\| \le M\sqrt{\xi\cdot A_{hom}\xi}.$$

Since $C > 0$ it follows that A_{hom} is positive definite. □

Lemma 5.2. *Under the assumption of Theorem 5.1 It holds* $\mathbb{R}^d = span\{\int_\Omega b\tilde{\nu}d\mu_{\Gamma,\mathcal{P}} : \ b \in L^2_{\text{sol}}(\alpha\Gamma)\}$.

Proof. We proceed similarly as the proof of Lemma 4.5 in [7].
Since $\chi_k \in L^2_{\text{pot}}(\alpha\Gamma)$ is the unique minimizer to the functional E_k, $(\tilde{\nu}\cdot e_k + \chi_k) \in L^2_{\text{sol}}(\alpha\Gamma)$, i.e

$$\int_\Omega (\tilde{\nu}\cdot e_k + \chi_k)\chi_m \alpha d\mu_{\Gamma,\mathcal{P}} = 0, \ \ \forall m, k. \tag{5.7}$$

Next we define $V := span\{\int_\Omega b\tilde{\nu}d\mu_{\Gamma,\mathcal{P}} : \ b \in L^2_{\text{sol}}(\alpha\Gamma)\} \subseteq \mathbb{R}^d$ and choose $\xi \in V^\perp \setminus \{0\}$. Then

$$\int_\Omega \xi\cdot\tilde{\nu}(\tilde{\nu}\cdot e_k + \chi_k)\alpha d\mu_{\Gamma,\mathcal{P}} = \int_\Omega \sum_{m=1}^{d}\xi_m(e_m\cdot\tilde{\nu})(\tilde{\nu}\cdot e_k + \chi_k)\alpha d\mu_{\Gamma,\mathcal{P}} = 0, \tag{5.8}$$

for all $k = 1, ..., n$.
Multiplying (5.7) with ξ_m and adding it to (5.8) leads to

$$\int_\Omega \sum_{m=1}^{d}\xi_m(\tilde{\nu}\cdot e_m + \chi_m)(\tilde{\nu}\cdot e_k + \chi_k)\alpha d\mu_{\Gamma,\mathcal{P}} = 0.$$

Multiplying this equation now with ξ_k and then summing over k gets us to

$$\int_\Omega \sum_{k=1}^{d}\sum_{m=1}^{d}\xi_k\xi_m(\tilde{\nu}\cdot e_m + \chi_m)(\tilde{\nu}\cdot e_k + \chi_k)\alpha d\mu_{\Gamma,\mathcal{P}} = \xi A_{hom}\xi = 0.$$

This is a contradiction since A_{hom} is positive definite. Therefore $V^\perp = 0$ and thus the claim is proven. □

6 Stochastic two-scale convergence

We introduce stochastic two-scale convergence relying on previous works [9, 11, 16], but mostly on [12]. Since $C_b(\Omega)$ lies densely in the separable space $L^2(\Omega, \mu_{\alpha\Gamma,\mathcal{P}})$, see remark 2.3, which is a separable space, we can choose a countable dense family

$$\Phi = (\phi_i)_{i\in\mathbb{N}} \subset L^2(\Omega, \mu_{\alpha\Gamma,\mathcal{P}}) \text{ with } \phi_i \in C_b(\Omega).$$

More precisely, since $L^2_{\mathrm{pot}}(\alpha\Gamma)$ is the closure of discrete gradients of $C_b(\Omega)$-functions, the selection of a countable dense family of $C_b(\Omega)$-functions in $L^2_{\mathrm{pot}}(\alpha\Gamma)$ is straight forward. and also for subspaces of $L^2(\Omega,\mu_{\alpha\Gamma,\mathcal{P}})$, $\Phi_{\mathrm{sol}}\subset L^2_{\mathrm{sol}}(\alpha\Gamma)$ and $\Phi_{\mathrm{pot}}\subset L^2_{\mathrm{pot}}(\alpha\Gamma)$ where $\phi\in\Phi_{\mathrm{pot}}$ if $\phi=\tilde{\nabla}_{Om}u$ for some $u\in C_b(\Omega)$, which is possible because of the definition of L^2_{pot}. We assume that

$$\Phi=\Phi_{\mathrm{pot}}\otimes\Phi_{\mathrm{sol}}.$$

Further, we find a countable dense family

$$\Psi=(\psi_i)_{i\in\mathbb{N}}\subset C_0^\infty(\boldsymbol{Q})\text{ with }\psi_i\in C_c^\infty(\boldsymbol{Q}).$$

Let $\Omega_\Phi\subset\Omega$ be the set of all ω such that the ergodic theorems, 2.7, 2.8, and 2.13, hold for all $\phi\in\Phi$ and $\psi\in\Psi$. We call Ω_Φ the set of *typical realizations*.

Definition 6.1 (Two-Scale Convergence)**.** Let $\omega\in\Omega_\Phi$ and let $g^\varepsilon\in L^2(\mathbb{R}^d,\mu^\varepsilon_{\alpha\Gamma(\omega)})$ be a sequence such that

$$\sup_{\varepsilon>0}\|g^\varepsilon\|_{\alpha\Gamma^\varepsilon(\omega)}<\infty$$

and let $g\in L^2(\mathbb{R}^d;L^2(\Omega;\mu_{\alpha\Gamma,\mathcal{P}}))$. We say that g^ε converges in two-scales to g, written $g^\varepsilon\xrightarrow{2s}_\omega g$ if for every $\phi\in\Phi$ and every $\psi\in\Psi$ it holds

$$\lim_{\varepsilon\to0}\langle g^\varepsilon,\phi_{\omega,\varepsilon}\psi\alpha_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)}=\int\langle g(x,\cdot),\phi\alpha\rangle_{\Gamma,\mathcal{P}}\psi(x)dx.$$

Lemma 6.2 (Existence of Two-Scale Limits [12])**.** *For every $\omega\in\Omega_\Phi$ it holds: Let $g^\varepsilon\in L^2(\mathbb{R}^n)$ be a sequence of functions such that*

$$\sup_{\varepsilon>0}\|g^\varepsilon\|^2_{\alpha\Gamma^\varepsilon(\omega)}=\sup_{\varepsilon>0}\sum_{ij}\alpha_{ij}(g^\varepsilon(\gamma^\varepsilon_{ij}))^2\le C. \tag{6.1}$$

for some $C>0$ independent from ε. Then there exists a subsequence g^{ε_k} and $g\in L^2(\mathbb{R}^d;L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$ such that $g^{\varepsilon_k}\xrightarrow{2s}_\omega g$.

In the following, we study the limit of the discrete gradient within two-scale convergence. For that, we have to do some preparatory work.

Lemma 6.3. *For all typical realisations $\omega\in\Omega_\Phi$ and all Lipschitz functions $\upsilon:\mathbb{R}^d\to\mathbb{R}$ there exists a $C\in(0,\infty)$ such that*

$$\sup_{\varepsilon>0}\|\tilde{\nabla}^\varepsilon\upsilon\|^2_{\alpha\Gamma^\varepsilon(\omega)}\le C\|\nabla\upsilon\|^2_\infty.$$

Proof. We have due to the ergodic theorem applied to α

$$\|\tilde{\nabla}^\varepsilon\upsilon\|^2_{\alpha\Gamma^\varepsilon(\omega)}=\varepsilon^d\sum_{\gamma^\varepsilon_{ij}}\alpha_{ij}(\tilde{\nabla}^\varepsilon\upsilon)^2_{ij}=\varepsilon^d\sum_{\gamma^\varepsilon_{ij}}\alpha_{ij}((\partial^\varepsilon_{ij}\upsilon\nu_{ij})\cdot\tilde{\nu}_{ij})^2\le C\|\nabla\upsilon\|^2_\infty,\ .$$

□

Lemma 6.4. *Let β be a stationary random field with $\mu_{\beta\Gamma,\mathcal{P}}(\Omega)<\infty$. Then for every typical realisation $\omega\in\Omega_\Phi$ and every $\upsilon\in C_c^\infty(\mathbb{R}^d)$ it holds:*

$$\limsup_{\varepsilon\to0}\|\tilde{\nabla}^\varepsilon\upsilon-(\nabla\upsilon)\cdot\tilde{\nu}\|_{\beta\Gamma^\varepsilon(\omega)}=0.$$

Proof. Let Q be a bounded open domain around 0 with $supp\ \upsilon \subseteq Q$. If $\mathbf{x}_i \in Q$, $\|\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon\| \leq \varepsilon d$ and $\upsilon \in C_c^\infty(Q)$, we use a Taylor expansion around γ_{ij}^ε such that

$$\begin{aligned}
&\left|\frac{\upsilon_j^\varepsilon - \upsilon_i^\varepsilon}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|}\nu_{ij}\cdot\tilde{\nu}_{ij} - \nabla\upsilon(\gamma_{ij}^\varepsilon)\cdot\tilde{\nu}_{ij}\right| \\
=&\left|\frac{1}{\varepsilon|\mathbf{x}_j - \mathbf{x}_i|}\Bigg((\mathbf{x}_j^\varepsilon - \mathbf{x}_i^\varepsilon)^T\nabla\upsilon_{ij}^\varepsilon + \frac{1}{2}(\frac{1}{2}\mathbf{x}_j^\varepsilon - \frac{1}{2}\mathbf{x}_i^\varepsilon)^T\nabla^2\upsilon_{ij}^\varepsilon(\frac{1}{2}\mathbf{x}_j^\varepsilon - \frac{1}{2}\mathbf{x}_i^\varepsilon)\right. \\
&\left. - \frac{1}{2}(\frac{1}{2}\mathbf{x}_i^\varepsilon - \frac{1}{2}\mathbf{x}_j^\varepsilon)^T\nabla^2\upsilon_{ij}^\varepsilon(\frac{1}{2}\mathbf{x}_i^\varepsilon - \frac{1}{2}\mathbf{x}_j^\varepsilon) + \mathcal{O}(\varepsilon^3)\Bigg)\nu_{ij}\cdot\tilde{\nu}_{ij} - \nabla\upsilon(\gamma_{ij}^\varepsilon)\cdot\tilde{\nu}_{ij}\right| \\
\leq& \|\nabla^2\upsilon\|_\infty\varepsilon D\,.
\end{aligned} \tag{6.2}$$

The statement now follows from Theorem 2.13 applied to some non-negative $\varphi \in C_c(\mathbb{R}^d)$ with $\varphi = 1$ on the support of υ, as well as the choice $\psi = 0$ and $\psi^\varepsilon = \tilde{\nabla}^\varepsilon\upsilon - (\nabla\upsilon)\cdot\tilde{\nu}$. □

Lemma 6.5. *For all typical realisations $\omega \in \Omega_\Phi$ it holds, if $g^\varepsilon \xrightarrow{2s}_\omega g$, then*

$$\forall\upsilon \in C_c^\infty(\mathbb{R}^d): \quad \lim_{\varepsilon\to0}\langle g^\varepsilon, \alpha_{\omega,\varepsilon}\tilde{\nabla}^\varepsilon\upsilon\rangle_{\Gamma^\varepsilon(\omega)} = \int\langle g(x,\cdot), \alpha\nabla\upsilon(x)\cdot\tilde{\nu}\rangle_{\Gamma,\mathcal{P}}dx\,.$$

Proof. Let $\tilde{\nu}\cdot e_k := \tilde{\nu}_k \in \Phi_\Omega$ and $\nabla\upsilon \in \Psi$ for $k = 1, ..., n$. For $g^\varepsilon \xrightarrow{2s}_\omega g$ it follows

$$\lim_{\varepsilon\to0}\langle g^\varepsilon, \tilde{\nu}_k\partial_{e_k}\upsilon\alpha_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = \int\langle g(x,\cdot), \tilde{\nu}_k\alpha\rangle_{\Gamma,\mathcal{P}}\partial_{e_k}\upsilon(x)dx.$$

Summing over k we get

$$\lim_{\varepsilon\to0}\langle g^\varepsilon, \tilde{\nu}\cdot\nabla\upsilon\alpha_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = \int\langle g(x,\cdot), \tilde{\nu}\cdot\nabla\upsilon(x)\alpha\rangle_{\Gamma,\mathcal{P}}dx$$

first for our special choice of υ but by approximation for all $\upsilon \in C_c^\infty(Q)$. It remains to show that

$$\lim_{\varepsilon\to0}\langle g^\varepsilon, (\tilde{\nabla}^\varepsilon\upsilon - \tilde{\nu}\cdot\nabla\upsilon)\alpha_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = 0.$$

Using the Cauchy-Schwartz inequality and the ergodic theorem for g^ε and Lemma 6.4 it follows that

$$\begin{aligned}
&\varepsilon^d\sum_{ij}\alpha_{ij}((\tilde{\nabla}^\varepsilon\upsilon)_{ij} - \nabla\upsilon\cdot\tilde{\nu}_{ij})g_{ij}^\varepsilon \\
&\qquad\leq\left(\varepsilon^d\sum_{ij}\alpha_{ij}(g_{ij}^\varepsilon)^2\right)^{\frac{1}{2}}\left(\varepsilon^d\sum_{ij}\alpha_{ij}((\tilde{\nabla}^\varepsilon\upsilon)_{ij} - \nabla\upsilon\cdot\tilde{\nu}_{ij})^2\right)^{\frac{1}{2}} \\
&\qquad=\|g^\varepsilon\|_{\alpha\Gamma^\varepsilon(\omega)}\|\tilde{\nabla}^\varepsilon\upsilon - (\nabla\upsilon)\cdot\tilde{\nu}\|_{\alpha\Gamma^\varepsilon(\omega)} \to 0.
\end{aligned}$$

□

Due to Lemma 3.7 it follows that

$$\langle\tilde{\nabla}^\varepsilon u, \alpha_{\omega,\varepsilon}b_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = \langle u, -div^\varepsilon(\alpha_{\omega,\varepsilon}b_{\omega,\varepsilon})\rangle_{\mathbf{x}^\varepsilon(\omega)} = 0.$$

and hence we find the following result.

Corollary 5. *For all $\omega \in \Omega_\Phi$ and all $b \in \Phi_{\mathrm{sol}}$ the following is true:*

$$\langle \tilde{\nabla}^\varepsilon u, \alpha_{\omega,\varepsilon} b_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = 0 \text{ for all } u \in \mathcal{S}^\varepsilon(\omega) \text{ with bounded support.}$$

With all that done we can now prove the most important result of this chapter: the two-scale limit of a discrete gradient.

Theorem 6.6 (Two-Scale Convergence for Gradients)**.** *Let Assumption 1.1 1.–3. hold and $2 < q \leq \frac{dp}{d-p}$. For all typical $\omega \in \Omega_\Phi$ such that $u^\varepsilon \in \mathcal{S}_0^\varepsilon(\omega, \boldsymbol{Q})$ is a family of functions with $supp(u^\varepsilon) \subseteq \boldsymbol{Q} \cap \mathbb{X}^\varepsilon(\omega)$ for all ε and*

$$\|\tilde{\nabla}^\varepsilon u^\varepsilon\|_{\alpha\Gamma^\varepsilon(\omega)} + \|u^\varepsilon\|_{\mathbf{x}^\varepsilon} \leq C\,. \tag{6.3}$$

Finally let either Assumption 1.1.4. hold with $\sup_\varepsilon \|u^\varepsilon\|_\infty < \infty$ or let there be $q_\alpha^*, q_m^* > 2$ such that $\frac{1}{2} + \frac{1}{q_\alpha^*} + \frac{1}{q_m^*} + \frac{1}{q} < 1$ and $\mathbb{E}_0 m_0^{\frac{-q_m^*}{q}} < \infty$.
Then there exists a subsequence u^ε, not relabeled, $u \in H_0^1(\boldsymbol{Q}) \cap L^q(\boldsymbol{Q})$ and $\phi \in L^2(\mathbb{R}^n; L^2_{\mathrm{pot}}(\alpha\Gamma))$ such that

$$\mathcal{R}_\varepsilon^* u^\varepsilon \to u \text{ strongly in } L^q(\boldsymbol{Q}), \qquad \tilde{\nabla}^\varepsilon u^\varepsilon \xrightarrow{2s}_\omega \nabla u \cdot \tilde{\nu} + \phi \text{ as } \varepsilon \to 0. \tag{6.4}$$

Proof. Step 1: Due to (6.3) and Theorem 4.7 we get that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^q(\boldsymbol{Q})$ for $2 < q \leq \frac{dp}{d-p}$. Furthermore, if $\sup_\varepsilon \|u^\varepsilon\|_\infty < \infty$ we obtain from Lebesgues dominated convergence theorem that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^q(\boldsymbol{Q})$ for $2 < q < \infty$. So it remains to prove $u \in H_0^1(\boldsymbol{Q})$ and (6.4).
By Lemma 6.2 there exists $g \in L^2(\mathbb{R}^d; L^2(\Omega; \mu_{\Gamma,\mathcal{P}}))$ and a subsequence of $\tilde{\nabla}^\varepsilon u^\varepsilon$ s.t.

$$\tilde{\nabla}^\varepsilon u^\varepsilon \xrightarrow{2}_\omega g,$$

Now choose $b_{\omega,\varepsilon} \in \Phi_{\mathrm{sol}}$, where $b_{ij} := b_{\omega,\varepsilon}(\gamma_{ij}^\varepsilon)$ and $\upsilon \in C_c^\infty(\boldsymbol{Q})$ according to Corollary 5 and use the discrete product rule

$$(\tilde{\nabla}^\varepsilon \upsilon u^\varepsilon)_{ij} = \frac{1}{2}((u_j^\varepsilon + u_i^\varepsilon)(\tilde{\nabla}^\varepsilon \upsilon)_{ij} + (\upsilon_j^\varepsilon + \upsilon_i^\varepsilon)(\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij}),$$

and get

$$0 = \varepsilon^d \sum_{ij} \alpha_{ij} b_{ij} \frac{1}{2}((u_j^\varepsilon + u_i^\varepsilon)(\tilde{\nabla}^\varepsilon \upsilon)_{ij} + (\upsilon_j^\varepsilon + \upsilon_i^\varepsilon)(\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij}). \tag{6.5}$$

Step 2: We write $\overline{u}_{ij}^\varepsilon := \frac{1}{2}(u_j^\varepsilon + u_i^\varepsilon)$ and $\overline{\upsilon}_{ij}^\varepsilon := \frac{1}{2}(\upsilon_j^\varepsilon + \upsilon_i^\varepsilon)$. Since $\upsilon \in C_c^\infty(\boldsymbol{Q})$ we get $|\overline{\upsilon}_{ij}^\varepsilon - \upsilon_{ij}^\varepsilon| < \varepsilon|\mathbf{x}_i - \mathbf{x}_j|$ which implies by Theorem 2.13 that $\limsup_{\varepsilon\to 0} \|\overline{\upsilon}_{ij}^\varepsilon - \upsilon_{ij}^\varepsilon\|_{\alpha\Gamma^\varepsilon} = 0$ and therefore

$$\varepsilon^d \sum_{ij} \alpha_{ij} \overline{\upsilon}_{ij}^\varepsilon (\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij} b_{ij} \to \int_{\boldsymbol{Q}} \langle g(x,\cdot), b\alpha\rangle_{\Gamma,\mathcal{P}} \upsilon(x) dx. \tag{6.6}$$

Step 3: Next, we study the first part of (6.5). First we write $I_\varepsilon := \{i \in \mathbb{N} : \ x_i^\varepsilon \in \boldsymbol{Q}\}$ and observe that for $\psi^\varepsilon \in \mathcal{G}^\varepsilon$ (that is $\psi_{ij}^\varepsilon = \psi_{ji}^\varepsilon$) and $q_\alpha^*, q_m^* > 2$ such that $\frac{1}{2} + \frac{1}{q_\alpha^*} + \frac{1}{q_m^*} + \frac{1}{q} \leq 1$ the general Hölder inequality yields

$$\left|\varepsilon^d \sum_{ij} \alpha_{ij} \overline{u}_{ij}^\varepsilon \psi_{ij}^\varepsilon\right| \leq \varepsilon^d \sum_i |u_i^\varepsilon| \sum_j \alpha_{ij} \psi_{ij}^\varepsilon \leq \varepsilon^d \sum_i |u_i^\varepsilon| (\sum_j \alpha_{ij} (\psi_{ij}^\varepsilon)^2)^{\frac{1}{2}} (\sum_j \alpha_{ij})^{\frac{1}{2}} \tag{6.7}$$

$$\leq \left(\sum_{i\in I_\varepsilon} m_i^\varepsilon |u_i^\varepsilon|^q\right)^{\frac{1}{q}} \left(\varepsilon^d \sum_{i\in I_\varepsilon} m_i^{\frac{-q_m^*}{q}}\right)^{\frac{1}{q_m^*}} \left(\varepsilon^d \sum_{i\in I_\varepsilon}\sum_j \alpha_{ij}(\psi_{ij}^\varepsilon)^2\right)^{\frac{1}{2}} \left(\varepsilon^d \sum_{i\in I_\varepsilon}(\sum_j \alpha_{ij})^{\frac{q_\alpha^*}{2}}\right)^{\frac{1}{q_\alpha^*}}$$

We make use of the last inequality by setting $\psi_{ij}^\varepsilon = b_{ij}((\tilde\nabla^\varepsilon \upsilon)_{ij} - \nabla\upsilon_{ij}^\varepsilon \cdot \tilde\nu_{ij})$ and observe that the right hand side of (6.7) goes to 0: Lemma 6.4 applied to $\beta_{ij} = \alpha_{ij} b_{ij}^2$ together with boundedness of $\mathcal{R}_\varepsilon^* u^\varepsilon$ in $L^q(\boldsymbol{Q})$ and our assumptions on the distribution of α and m then ultimately yields that

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{ij} \alpha_{ij}\overline{u}_{ij}^\varepsilon (\tilde\nabla^\varepsilon \upsilon)_{ij} b_{ij} = \lim_{\varepsilon\to 0} \varepsilon^d \sum_{ij} \alpha_{ij}\overline{u}_{ij}^\varepsilon \nabla\upsilon_{ij}^\varepsilon \cdot \tilde\nu_{ij} b_{ij}. \tag{6.8}$$

We now set $\psi_{ij}^\varepsilon = b_{ij}\nabla\upsilon_{ij}^\varepsilon \cdot \tilde\nu_{ij}$. Estimate (6.7) yields that

$$L^\varepsilon : \quad \mathcal{S}_{\mathbf{x}}^\varepsilon(\boldsymbol{Q}) \to \mathbb{R}\,, \qquad u \mapsto \varepsilon^d \sum_{ij} \alpha_{ij}\overline{u}_{ij} b_{ij} \nabla\upsilon_{ij}^\varepsilon \cdot \tilde\nu_{ij}$$

satisfies for some $C > 0$ independent from ε that

$$L^\varepsilon(u) \leq C\|\mathcal{R}_\varepsilon^* u\|_{L^q(\boldsymbol{Q})}\,.$$

If we write for a continuous function $\phi \in C_c(\boldsymbol{Q})$ both $\tilde\phi_i^\varepsilon := \phi(\mathbf{x}_i^\varepsilon)$ and $\bar\phi_i^\varepsilon := \fint_{\mathfrak{g}_i^\varepsilon} \phi$ we obtain from the double ergodic theorem 2.13 and Lemma 4.3

$$\lim_{\varepsilon\to 0} L^\varepsilon(\tilde\phi^\varepsilon) = \lim_{\varepsilon\to 0} L^\varepsilon(\bar\phi^\varepsilon) = \int_{\boldsymbol{Q}} \phi\nabla\upsilon \int_\Omega \alpha b\tilde\nu \mathrm{d}\mu_{\Gamma,\mathcal{P}} \mathrm{d}x\,. \tag{6.9}$$

From the boundedness of L^ε in the dual of $L^q(\boldsymbol{Q})$ and the convergence (6.9) on a dense subset we infer

$$\lim_{\varepsilon\to 0} L^\varepsilon(u^\varepsilon) = \int_{\boldsymbol{Q}} u\nabla\upsilon \int_\Omega \alpha b\tilde\nu \mathrm{d}\mu_{\Gamma,\mathcal{P}} \mathrm{d}x\,.$$

In case of Assumption 1.1.4. we note that we can choose q and q_m^* arbitrarily large, and hence $q_m = \frac{q_m^*}{q}$ and $q_\alpha = \frac{q_\alpha^*}{2}$ can be arbitrary numbers larger than 1.

Step 4: Going back to (6.5) we get in total

$$\int_{\boldsymbol{Q}} \langle g(x,\cdot), b\alpha\rangle_{\Gamma,\mathcal{P}} \upsilon(x)dx = -\int_{\boldsymbol{Q}} \langle b\alpha, \nabla\upsilon(x)\cdot\tilde\nu\rangle_{\Gamma,\mathcal{P}} u(x)dx. \tag{6.10}$$

Then with Lemma 5.2 it follows now that for any $k = 1, ..., n$ we can choose b^k such that $\langle b^k\alpha, \tilde\nu\rangle_{\Gamma,\mathcal{P}} = e_k$. Therefore

$$\begin{aligned}\left|\int_{\boldsymbol{Q}} u(x)\nabla\upsilon(x)\cdot e_k dx\right| &= \left|\int_{\boldsymbol{Q}} \upsilon(x)\langle g(x,\cdot), \alpha b^k\rangle dx\right| \\ &\leq \|\alpha b\|_{L^2(\boldsymbol{Q},\mu_{\Gamma,\mathcal{P}})} \int_{\boldsymbol{Q}} |\upsilon(x)| \|g(x,\cdot)\|_{L^2(\boldsymbol{Q},\mu_{\Gamma,\mathcal{P}})} dx \\ &\leq \|\upsilon\|_{L^2(\boldsymbol{Q})} \|\alpha b\|_{L^2(\boldsymbol{Q},\mu_{\Gamma,\mathcal{P}})} \int_{\boldsymbol{Q}} \|g(x,\cdot)\|_{L^2(\boldsymbol{Q},\mu_{\Gamma,\mathcal{P}})} dx\end{aligned}$$

And since $g \in L^2(\boldsymbol{Q}, L^2(\boldsymbol{Q}, \mu_{\Gamma,\mathcal{P}}))$ there exists a $C < \infty$ s.t.

$$\left|\int_{\boldsymbol{Q}} u(x)\nabla\upsilon(x)\cdot e_k dx\right| \leq C\|\upsilon\|_{L^2(\boldsymbol{Q})} \;\forall\; k = 1, ..., n.$$

Hence $u \in H^1(\boldsymbol{Q})$ and since $u_{|\mathbb{R}^d\setminus\boldsymbol{Q}} = 0$, $u \in H^1_0(\boldsymbol{Q})$.
Integration by parts on the RHS of (6.10) now gives us

$$\int_{\boldsymbol{Q}} \upsilon(x)\langle(g(x,\cdot) - \nabla u\cdot\tilde{\nu}), b\alpha\rangle_{\Gamma,\mathcal{P}}dx = 0.$$

Since this holds for all $\upsilon \in \Psi$ and $b \in \Phi_{\mathrm{sol}}$ we find

$$g = \nabla u\cdot\tilde{\nu} + \phi, \text{ with } \phi \in L^2(\mathbb{R}^n; L^2_{\mathrm{pot}}(\alpha\Gamma))$$

and therefore prove the claim. □

7 Proof of Theorem 1.5

Proof of Theorem 1.5. In what follows, we make use of the concept of Dirichlet energy which is defined as

$$\mathcal{E}^\varepsilon_\omega(u) := \langle -\mathcal{L}^\varepsilon_\omega u, u\rangle_{\mathbb{x}^\varepsilon}\,.$$

Furthermore, in Steps 2.–3. we assume

$$f_\infty := \sup_\varepsilon \sup_{\mathbf{x}^\varepsilon_i\in\boldsymbol{Q}} \frac{f^\varepsilon_i}{m^\varepsilon_i} < \infty. \tag{7.1}$$

In order for the last condition to make sense to the reader, simply consider the right hand side to be given in a weak form as $\langle f^\varepsilon, g^\varepsilon\rangle_{\mathbb{X}^\varepsilon} := \int_{\mathbb{R}^d} f\mathcal{R}^*_\varepsilon g^\varepsilon$, i.e. $f^\varepsilon_i := m^\varepsilon_i \fint_{\mathrm{g}^\varepsilon_i} f$.

Step 1: L^2-estimates We observe for every test function $g^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbb{x}}(\boldsymbol{Q})$

$$\langle -\mathcal{L}^\varepsilon_\omega u^\varepsilon, g^\varepsilon\rangle_{\mathbb{X}^\varepsilon} = \langle A\tilde{\nabla}^\varepsilon u^\varepsilon, \tilde{\nabla}^\varepsilon g^\varepsilon\rangle_{\Gamma^\varepsilon} = \langle f^\varepsilon, g^\varepsilon\rangle_{\mathbb{X}^\varepsilon}. \tag{7.2}$$

We now choose $g^\varepsilon = u^\varepsilon$ and apply the Poincare inequality 4.7 and Cauchy-Schwartz

$$\|u^\varepsilon\|^2_{\mathbf{x}^\varepsilon} \le C\mathcal{E}^\varepsilon_\omega(u^\varepsilon) = C\langle f^\varepsilon, u^\varepsilon\rangle_{\mathbf{x}^\varepsilon} \le C\|u^\varepsilon\|_{\mathbf{x}^\varepsilon}\|f^\varepsilon\|_{\mathbf{x}^\varepsilon}.$$

Hence

$$\|\tilde{\nabla}^\varepsilon u^\varepsilon\|^2_{\alpha\Gamma^\varepsilon} + \|u^\varepsilon\|^2_{\mathbf{x}^\varepsilon} \le C\|f^\varepsilon\|^2_{\mathbf{x}^\varepsilon}.$$

Step 2: Moser iteration and L^∞ estimate Now, let (7.1) hold. We insert $u = |u^\varepsilon|^\kappa\mathrm{sign}(u^\varepsilon)$ into the Dirichlet energy functional $\mathcal{E}^\varepsilon_\omega$. Using the Poincaré inequality in Theorem 4.7, inequality (A2) from [1], $f_\infty < \infty$ and Jensen's inequality together with the support Lemma 4.2 we find C_0 dependent on f_∞ and the constant $C_{\alpha,\mathbb{x}}$ in Theorem 4.7, but independent from κ, u^ε, ε such that

$$\begin{aligned}
\|\mathcal{R}^*_\varepsilon|u^\varepsilon|^{\kappa q}\|_{L^{\kappa q}(\mathbb{R}^d)} &\le C\mathcal{E}^\varepsilon_\omega(|u^\varepsilon|^\kappa\mathrm{sign}(u^\varepsilon)) = \varepsilon^d\sum_{i,j}\sum_j \alpha_{ij}\frac{(u^\varepsilon_i|^\kappa\mathrm{sign}(u^\varepsilon_i) - |u^\varepsilon_i|^\kappa\mathrm{sign}(u^\varepsilon_i))^2}{|\mathbf{x}^\varepsilon_i| - \mathbf{x}^\varepsilon_j|^2}\\
&\le \frac{\kappa^2}{2\kappa-1}\varepsilon^d\sum_i (u^\varepsilon_i)^{2\kappa-1}\left(-\mathcal{L}^\varepsilon_\omega u^\varepsilon\right)(\mathbf{x}^\varepsilon_i)\\
&\le \frac{\kappa^2}{2\kappa-1}C_0\,\|\mathcal{R}^*_\varepsilon u^\varepsilon\|^{2\kappa-1}_{L^{2\kappa-1}(\mathbb{R}^d)} \le \frac{\kappa^2}{2\kappa-1}C_0\,\|\mathcal{R}^*_\varepsilon u^\varepsilon\|^{2\kappa-1}_{L^{2\kappa}(\mathbb{R}^d)}\ .
\end{aligned}$$

We make use of $\rho := \frac{q}{2} > 1$ and set $\kappa = \kappa_j = \rho^j$. Then the above inequality reads

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^{2\rho^{j+1}}(\mathbb{R}^d)} \leq \exp\left(\frac{j \ln \rho}{\rho^j}\right) C_0^{\frac{1}{2\rho^j}} \, \|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^{2\rho^{j-1}}(\mathbb{R}^d)}^{1-\frac{1}{2\rho^j}} \tag{7.3}$$

We can iterate (7.3) over $j = 1, \ldots, J$ and obtain

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^{2\rho^J}(\mathbb{R}^d)} \leq \exp\left(\ln \rho \, \sum_{j=1}^{J-1} \frac{j}{\rho^j} + \frac{\ln C_0}{2} \sum_{j=1}^{J-1} \rho^{-j}\right) \|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^2(\mathbb{R}^d)}^{\tilde{\kappa}} \tag{7.4}$$

where $\tilde{\kappa} = \prod_{j=0}^{J-1}\left(1 - \frac{1}{2\rho^j}\right) < 1$. Put differently, there exists $0 < C < \infty$ independent from κ, u^ε, ε, such that for every $J > 1$ it holds

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^{2\rho^J}(\mathbb{R}^d)} \leq C \, \|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^2(\mathbb{R}^d)} \,. \tag{7.5}$$

However, for every $v \in \mathcal{S}_{\mathbb{x}}^\varepsilon(\boldsymbol{Q})$, we find $v_\infty := \sup_i |v(\mathbf{x}_i^\varepsilon)| < \infty$. Let $m_\infty(v)$ be the positive mass $0 < m_\infty(v) < |\mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})|$ of the support of v_∞. Then there exists $r > 0$ such that $\|\mathcal{R}_\varepsilon^* v\|_{L^\infty} < 2|m_\infty(v)|^{\frac{1}{r}} \, v_\infty < 2 \, \|\mathcal{R}_\varepsilon^* v\|_{L^r}$. Since there exists J_r with $2\rho^{J_r} > r$ it follows from (7.5) that

$$\sup_\varepsilon \|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^\infty(\mathbb{R}^d)} < \infty \,. \tag{7.6}$$

Step 3: Homogenization Let still $f_\infty < \infty$ such that (7.6) holds. Then by Theorem 6.6 and compact embedding Theorem 4.7 there exists a $u \in H_0^1(\boldsymbol{Q})$ and $w \in L^2(\boldsymbol{Q}; L^2_{\mathrm{pot}}(\alpha\Gamma))$ and a subsequence, not relabeled, s.t.

$$\mathcal{R}_\varepsilon^* u^\varepsilon \to u \text{ strongly in } L^2(\boldsymbol{Q}) \text{ and } \tilde{\nabla}^\varepsilon u^\varepsilon \stackrel{2s}{\rightharpoonup}_\omega \nabla u \cdot \tilde{\nu} + w \text{ as } \varepsilon \to 0$$

for all typical $\omega \in \Omega_\Phi$.
We now choose $v \in C_c^\infty(\mathbb{R}^n)$ with $supp\ v \subseteq \boldsymbol{Q}$ and $\varphi \in C_b(\Omega)$ with $\tilde{\nabla}_{Om}\varphi \in \Phi_{\mathrm{pot}}$ and substitute $g^\varepsilon = \varepsilon v \varphi_{\omega,\varepsilon}$ in (7.2), again we use the standard abbreviations, then we observe for all $\varepsilon > 0$

$$\begin{aligned}
\langle -\mathcal{L}_\omega^\varepsilon u^\varepsilon, \varepsilon v \varphi_{\omega,\varepsilon}\rangle_{\mathbf{x}^\varepsilon} &= \langle f^\varepsilon, \varepsilon v \varphi_{\omega,\varepsilon}\rangle_{\mathbf{x}^\varepsilon} \\
&= \varepsilon^d \sum_{ij} (\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij} \alpha_{ij} \frac{1}{\varepsilon} \frac{\nu_{ij} \cdot \tilde{\nu}_{ij}}{|\mathbf{x}_j - \mathbf{x}_i|} \varepsilon (v_j^\varepsilon \varphi_j - v_i^\varepsilon \varphi_i) \\
&= \varepsilon^d \sum_{ij} (\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij} \alpha_{ij} \frac{1}{\varepsilon} \frac{\nu_{ij} \cdot \tilde{\nu}_{ij}}{|\mathbf{x}_j - \mathbf{x}_i|} \varepsilon \\
&\quad \cdot \frac{1}{2}((v_j^\varepsilon - v_i^\varepsilon)(\varphi_i + \varphi_j) + (\varphi_j - \varphi_i)(v_i^\varepsilon + v_j^\varepsilon)) \\
&= \varepsilon^d \sum_{ij} \frac{1}{2} (\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij} \alpha_{ij} (\varepsilon (\tilde{\nabla}^\varepsilon v^\varepsilon)_{ij} (\varphi_i + \varphi_j) + (\tilde{\nabla}_{Om}\varphi)_{ij} (v_j^\varepsilon + v_i^\varepsilon)).
\end{aligned} \tag{7.7}$$

Then the first term on RHS vanishes for $\varepsilon \to 0$ since Lemma 6.3 yields $|\alpha \tilde{\nabla}^\varepsilon v| \leq C \|\nabla v\|_\infty$ and therefore bounded and φ is bounded by definition. For the second term follows that

$$\varepsilon^d \sum_{ij} (\tilde{\nabla}^\varepsilon u^\varepsilon)_{ij} \alpha_{ij} (\tilde{\nabla}_{Om}\varphi)_{ij} \frac{1}{2}(v_j^\varepsilon + v_i^\varepsilon) \to \int_{\boldsymbol{Q}} \langle \nabla u \cdot \tilde{\nu} + w, \alpha \tilde{\nabla}_{Om}\varphi\rangle_{\Gamma,\mathcal{P}} v(x) dx,$$

where the $\frac{1}{2}(v_j^\varepsilon + v_i^\varepsilon) \to v_{ij}^\varepsilon$, pointwise for fixed ij, as $\varepsilon \to 0$, same reasoning as in Theorem 6.6 Equation (6.6), and the expression is rewritten as its two-scale limit. With Equation (7.7) it follows

$$\int_{\boldsymbol{Q}} \langle \nabla u \cdot \tilde{\nu} + w, \alpha \tilde{\nabla}_{Om}\varphi \rangle_{\Gamma,\mathcal{P}} v(x) dx = 0. \tag{7.8}$$

Since Φ_{pot} is dense in $L^2_{\text{pot}}(\alpha\Gamma)$ and Ψ is dense in $L^2(\boldsymbol{Q})$(compactly supported continuous functions are dense in L^p), equation (7.8) holds for all $\tilde{\nabla}_{Om}\varphi \in L^2_{\text{pot}}(\alpha\Gamma)$ and $v \in L^2(\boldsymbol{Q})$.
Let now χ_k be defined as in Chapter 5 and let $\chi := (\chi_1, ..., \chi_n)^T \in (L^2_{\text{pot}}(\alpha\Gamma))^n$. Since $u \in H^1_0(\boldsymbol{Q})$, (7.8) admits the solution $w = \nabla u \cdot \chi$. The uniqueness of this solution follows directly from the Lax-Milgram theorem.
In the next step we test again equation (7.2) with a function $g \in C_c^\infty(\mathbb{R}^n)$ with $supp\ g = \boldsymbol{Q}$. Then we get

$$\langle A\tilde{\nabla}^\varepsilon u^\varepsilon, \tilde{\nabla}^\varepsilon g \rangle_{\Gamma^\varepsilon} = \langle f^\varepsilon, g \rangle_{\mathbb{X}^\varepsilon}.$$

Using Lemma 6.5, Theorem 6.6 and the unique solution for $w = \nabla u \cdot \chi$ on the LHS, as $\varepsilon \to 0$. On the RHS we multiply it with $\mathbb{1}_\Omega(\tau_x\omega) = 1$ and use the Ergodic theorem 2.8. Together we arrive at

$$\int_{\boldsymbol{Q}} \langle (\nabla u(x)) \cdot (\tilde{\nu} + \chi), \alpha \nabla g(x) \cdot \tilde{\nu} \rangle_{\Gamma,\mathcal{P}} dx = \mu_{\mathbb{X},\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} f(x) g(x) dx. \tag{7.9}$$

Next we set $v = \partial_k g$ and $\tilde{\nabla}_{Om}\varphi = \chi_k$ for $k = 1, .., n$ in Equation (7.8), sum up over k and add this result to (7.9) and we obtain

$$\int_{\boldsymbol{Q}} \langle (\nabla u(x)) \cdot (\tilde{\nu} + \chi), \alpha \nabla g(x) \cdot (\tilde{\nu} + \chi) \rangle_{\Gamma,\mathcal{P}} dx = \mu_{\mathbb{X},\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} f(x) g(x) dx.$$

By nature of the definition of A_{hom} we get

$$\int_{\boldsymbol{Q}} \nabla u \cdot (A_{hom} \nabla g) = \mu_{\mathbb{X},\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} fg \text{ for all } g \in C_c^\infty(\boldsymbol{Q}).$$

The fact that A_{hom} is nonsingular yields that the above equation is the weak formulation of (1.13). Hence from elliptic regularity theory, we get that $u \in H^2(\boldsymbol{Q}) \cap H^1_0(\boldsymbol{Q})$. Since the solution u is unique, the used subsequence is the entire sequence.

Step 4: General f^ε We now drop the assumption $f_\infty < \infty$. The operator $-\mathcal{L}_\alpha^\varepsilon$ is strictly positive definite on $\mathcal{S}_{\mathbb{X}}^\varepsilon(\boldsymbol{Q})$. It follows that on $\boldsymbol{Q}$ its inverse $\mathcal{B}^\varepsilon\colon\ \mathcal{S}_{\mathbb{X}}^\varepsilon(\boldsymbol{Q}) \to \mathcal{S}_{\mathbb{X}}^\varepsilon(\boldsymbol{Q})$ is well-defined. Similarly, the inverse $\mathcal{B}_0\colon\ L^2(\boldsymbol{Q}) \to L^2(\boldsymbol{Q})$ of $-\nabla \cdot (A_{\text{hom}} \nabla u)$, is well-defined.

Since A_{hom} is positive definite and symmetric, $\mathcal{B}_0$ is positive, compact and self-adjoint by the theory of elliptic partial differential equation, see e.g. [6, Chapter 6].

The operators $\mathcal{B}^\varepsilon$ are uniformly bounded in ε by virtue of Step 1. Moreover, $\mathcal{B}^\varepsilon$ are real and symmetric by construction and therefore self-adjoint. Finally, their range are finite-dimensional and thus $\mathcal{B}^\varepsilon$ are compact.

Let $v \in C(\overline{\boldsymbol{Q}})$. Then

$$\int_{\mathbb{R}^d} \left(\mathcal{R}_\varepsilon^* u^\varepsilon\right) v = \int_{\mathbb{R}^d} \left(\mathcal{R}_\varepsilon^* \mathcal{B}^\varepsilon f^\varepsilon\right) v = \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon^* f^\varepsilon \mathcal{R}_\varepsilon^* \left(\mathcal{B}^\varepsilon \mathcal{R}_\varepsilon v\right).$$

Since $\mathcal{R}_\varepsilon^*\mathcal{R}_\varepsilon v \rightharpoonup v$ in L^2 and $\sup_{\varepsilon>0}\|\mathcal{R}_\varepsilon v\|_\infty < \infty$, Step 3. implies that $\mathcal{B}^\varepsilon\mathcal{R}_\varepsilon v$ converges strongly in L^2 to $\mathcal{B}_0 v$. It follows that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathcal{R}_\varepsilon^* f^\varepsilon \mathcal{R}_\varepsilon^*\big(\mathcal{B}^\varepsilon\mathcal{R}_\varepsilon v\big) = \int_{\mathbb{R}^d} f\left(\mathcal{B}_0 v\right) = \int_{\mathbb{R}^d}\big(\mathcal{B}_0 f\big)\, v = \int_{\mathbb{R}^d} uv\,,$$

where we have used that the operator $\mathcal{B}_0$ is self-adjoint and where $u \in H^2(\boldsymbol{Q}) \cap H_0^1(\boldsymbol{Q})$ is the solution to (1.13). Since $C(\overline{\boldsymbol{Q}})$ is dense in $L^2(\boldsymbol{Q})$, it thus follows that $\mathcal{R}_\varepsilon^* u^\varepsilon \rightharpoonup u$. By virtue of Theorem 4.7 we conclude that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in L^q. □

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