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Approximating dynamic phase-field fracture in viscoelastic materials with a first-order formulation for velocity and stress

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Abstract

We investigate a model for dynamic fracture in viscoelastic materials at small strains. While the sharp crack interface is approximated with a phase-field method, we consider a viscous evolution with a quadratic dissipation potential for the phase-field variable. A non-smooth constraint enforces a unidirectional evolution of the phase-field, i.e. material cannot heal. The viscoelastic equation of motion is transformed into a first order formulation and coupled in a nonlinear way to the non-smooth evolution law of the phase field. The system is fully discretized in space and time with a discontinuous Galerkin approach for the first-order formulation. Based on this, existence of discrete solutions is shown and, as the step size in space and time tends to zero, their convergence to a suitable notion of weak solution of the system is discussed.

1 Introduction

The propagation of dynamic cracks is a complex phenomenon and crucial for the understanding of materials under extreme loading conditions. Inertia and acceleration terms have to be taken into account in the model description in situations when fast external loadings are applied to the system or when interacting effects of elastic wave propagation and crack evolution are too substantial to be neclected. Our work is set in such a context. The material domain is $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$ and the deformation and damage of the body will be monitored over a time interval $[0,T] \subset \mathbb{R}$. The fracture, which appears as a spatial discontinuity of complex geometry in the domain, is approximated with a phase-field method: Instead of a lower-dimensional crack, the damage variable $z\colon [0,T] \times \Omega \to [0,1]$ is used to characterize the material failure. Here $z(t,x) = 1$ stand for the undamaged material state, whereas $z(t,x) = 0$ refers to the the state of maximal damage in in the material point $x \in \Omega$ at time $t \in [0,T]$. Additional complexity enters the mathematical description when covering time-dependent behaviour in viscoelastic materials where the material response does not only depend on actual loading conditions but also on the history of the deformation.

Existence of solutions in this setting has been shown in e.g. [9, 14, 10, 15, 17] with the momentum balance formulated as as second-order system. Time discretizations for these systems using finite differences or Newmark methods show a high numerical dissipation [18] and if the focus lies on the hyperbolic nature of the problem, numerical methods are required that are capable of accurately capturing the elastic wave propagation.

For this purpose, to prevent that energy is lost by the numerical implementation, the viscoelastic momentum balance is reformulated as a first-order system and combined with a space discretization using a discontinuous Galerkin approach. This method has been developed and tested in [18]. Our aim in this work is to provide a convergence analysis for the algorithm introduced in [18].

In the following we give more details about the setting considered in this paper.

We want to determine the displacement vector $\mathbf{u}\colon [0,T] \times \Omega \to \mathbb{R}^d$, the velocity $\mathbf{v} = \partial_t \mathbf{u}$, the linearized strain $\boldsymbol{\varepsilon} = \mathrm{sym}(\mathrm{D}\mathbf{u}) = \frac{1}{2}(\mathrm{D}u + \mathrm{D}u^T)$ and the strain rate $\dot{\boldsymbol{\varepsilon}} = \partial_t \boldsymbol{\varepsilon} = \mathrm{sym}(\mathrm{D}\mathbf{v})$, where $\dot{\boldsymbol{\varepsilon}}$ and $\partial_t \boldsymbol{\varepsilon}$ denote the time derivative of $\boldsymbol{\varepsilon}$, such that formally the elasticity system satisfies

$$\mathbf{0} = \varrho_0 \partial_t \mathbf{v} - \operatorname{div} \boldsymbol{\sigma} - \mathbf{f} \qquad \text{in } (0,T) \times \Omega$$

with mass density $\rho_0 > 0$, volume force $\mathbf{f}$ and degraded viscoelastic stress response $\boldsymbol{\sigma}$. With constant material tensors, $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ the Hookean elasticity tensor, damping tensor $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ and the degradation function $g \in \mathrm{C}^2(\mathbb{R})$, the stress field $\boldsymbol{\sigma}\colon (0,T) \times \Omega \to \mathbb{R}^{d\times d}$

is here defined by

$$\boldsymbol{\sigma} = \big(\mathbb{C}(z)\boldsymbol{\varepsilon} + \mathbb{D}(z)\partial_t \boldsymbol{\varepsilon}\big)\,.$$

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Given initial values $\mathbf{u}_0$ and ε_0 for displacement and strain where $\varepsilon_0 = \mathrm{sym}(\mathrm{D}\mathbf{u}_0)$, the displacement variable $\mathbf{u}$ can be reconstructed from the velocity $\mathbf{v}$ such that $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)\,\mathrm{d}s$. Then, the second-order momentum balance above can be rewritten as a first-order system using the state variables $(\mathbf{v}, \varepsilon, \boldsymbol{\sigma})$ as

$$\varrho_0 \partial_t \mathbf{v} - \mathrm{div}\,\boldsymbol{\sigma} = \mathbf{f} \qquad \text{in } Q\,, \tag{1.1a}$$

$$\partial_t \boldsymbol{\varepsilon} - \mathrm{sym}(\mathrm{D}\mathbf{v}) = \mathbf{0} \qquad \text{in } Q\,, \tag{1.1b}$$

$$\boldsymbol{\sigma} - \big(\mathbb{C}(z)\boldsymbol{\varepsilon} + \mathbb{D}(z)\partial_t\boldsymbol{\varepsilon}\big) = \mathbf{0} \qquad \text{in } Q\,, \tag{1.1c}$$

where $Q := (0,T)\times\Omega$ denotes the space-time cylinder for a given $T > 0$ and a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. System (1.1) is complemented by the following boundary and initial conditions on $\partial\Omega = \partial_{\mathsf{N}}\Omega \cup \partial_{\mathsf{D}}\Omega$ with $\partial_{\mathsf{N}}\Omega$ denoting the Neumann and $\partial_{\mathsf{D}}\Omega$ the Dirichlet boundary:

$$\mathbf{v} = \mathbf{v}_{\mathsf{D}} \text{ on } (0,T)\times\partial_{\mathsf{D}}\Omega\,, \qquad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g}_{\mathsf{N}} \text{ on } (0,T)\times\partial_{\mathsf{N}}\Omega\,, \tag{1.2a}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega\,, \qquad \varepsilon(0) = \varepsilon_0 \text{ in } \Omega\,. \tag{1.2b}$$

This configuration depends on initial data $\mathbf{v}_0$ and ε_0, volume forces $\mathbf{f}$, and boundary data $\mathbf{g}_{\mathsf{N}}$ and $\mathbf{v}_{\mathsf{D}}$.

We also investigate the case $\mathbb{D} = \mathbf{0}$ without viscosity, but then the regularity of the solution is reduced. As a measure for the material degradation in every point of the domain, the phase-field variable $z\colon [0,T]\times\Omega \to [0,1]$ is introduced where $z(t,x) = 0$ represents the damaged material and $z(t,x) = 1$ the totally undamaged state. Self-healing of the material is not allowed. Thus, a monotonically decreasing evolution of the phase-field variable is enforced using the characteristic function

$$\chi_{(-\infty,0]}(\dot z) := \begin{cases} \infty & \dot z > 0\,, \\ 0 & \dot z \le 0 \end{cases} \tag{1.3}$$

and the non-smoothness of the characteristic function leads to a subdifferential inclusion describing the propagation of the phase-field variable:

$$0 \in \beta_{\mathrm{r}}\partial_t z + \partial\chi_{(-\infty,0]}(\partial_t z) + \frac{1}{2}\mathbb{C}'(z)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - G_{\mathrm{c}}\big(1 - z + l_{\mathsf{c}}^2\Delta z\big) \qquad \text{in } (0,T)\times\Omega\,. \tag{1.4}$$

The phase-field evolution depends on a retardation time $\beta_{\mathrm{r}} > 0$, a length scale $l_{\mathsf{c}} > 0$ and a scaling factor $G_{\mathrm{c}} > 0$ that is a material parameter and encodes the energy release rate by crack opening. G_{c} is related to the Griffiths constant for brittle fracture and the length scale $1/l_{\mathsf{c}}$. Moreover, the term $-(1 - z + l_{\mathsf{c}}\Delta z)$ approximates the sharp crack surface in the sense of [1]. The subdifferential inclusion (1.4) is complemented by the following boundary and initial conditions

$$G_{\mathrm{c}} l_{\mathsf{c}}^2 \nabla z\cdot\mathbf{n} \;=\; 0 \quad \text{on } (0,T)\times\partial\Omega\,, \tag{1.5a}$$

$$z(0) \;=\; z_0 \quad \text{in } \Omega\,. \tag{1.5b}$$

Outline of the paper. We specify the problem setup in Sect. 2 and introduce the mathematical assumptions on the domain and given data. In Section 2.2 we introduce a suitable weak formulation in Definition 2.1 and summarize our main result. Subsequently, in Section 2.3 we give a comparison to previous results, in particular with regard to the weak formulation of the momentum balance as a first-order and as a second-order system. The discretized system is presented in Sect. 3. Here, a staggered implicit scheme is used for the time discretization and for the spatial approximation a discontinuous Galerkin approach is employed. The motivation of this approach originates in the solution theory of first-order hyperbolic systems.
Starting from this point, a comprehensive convergence analysis is carried out. In Proposition 3.5 we show the existence of discrete solutions and establish a discrete energy-dissipation estimate: In the elastic part, a linear system of equations has to be solved. For the phase-field evolution the treatment of a nonlinear system requires more advanced techniques. Here, a generalized Newton method with a descent approach is employed in Section 3.2 to ensure the energy stability estimate already on the fully discrete level, cf. [3, 17]. Beginning with Proopsition 4.1, the limit passage in the discrete evolution equations is discussed and the energy dissipation estimate for the solutions in the limit is established in Lemma 4.5 based on the results of Lemma 4.3, Lemma 4.4. Eventually, the existence of weak solutions in the sense of Def. 2.1 is concluded in Theorem 4.2.

2 Mathematical assumptions and weak formulation

In this section we derive and formulate a weak version of the momentum balance (1.1) and differential inclusion (1.4).

2.1 Basic assumptions on the domain and the given data

For this purpose, we agree on the following setup.

Assumptions on the domain: We assume that

$$\begin{aligned}&\Omega \subset \mathbb{R}^d,\ d \in \{2,3\} \text{ is a bounded Lipschitz domain with boundary } \partial\Omega = \bar{\Omega}\setminus\Omega \text{ and}\\ &\text{relatively open Dirichlet and Neumann boundaries } \Gamma_{\mathsf{D}}, \Gamma_{\mathsf{N}} \subset \partial\Omega \text{ such that } \overline{\Gamma_{\mathsf{D}} \cup \Gamma_{\mathsf{D}}} = \partial\Omega\,.\end{aligned} \tag{2.1a}$$

We denote the space-time cylinder by

$$Q = (0,T)\times\Omega\,. \tag{2.1b}$$

Assumptions on the tensors $\mathbb{C},\mathbb{D}$ and on the degradation functions: We assume that the symmetric material tensors $\mathbb{C},\mathbb{D} : z \mapsto \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ can be expressed as the products of sufficiently smooth degradation functions $g_{\mathbb{C}}, g_{\mathbb{D}} : z \mapsto [g_*, g^*]$ and constant tensors $\tilde{\mathbb{C}}, \tilde{\mathbb{D}} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$, i.e.,

$$\text{for all } z\in\mathbb{R}: \quad \mathbb{C}(z) := g_{\mathbb{C}}(z)\tilde{\mathbb{C}} \quad\text{and}\quad \tilde{\mathbb{C}}(z) := g_{\mathbb{D}}(z)\tilde{\mathbb{D}}\,. \tag{2.2a}$$

The constant fourth-order tensors $\tilde{\mathbb{C}}, \tilde{\mathbb{D}}$ are further assumed to be symmetric

$$\tilde{\mathbb{C}}, \tilde{\mathbb{D}} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}} \tag{2.2b}$$

and positive (semi-)definite. More precisely, we assume that $\tilde{\mathbb{C}}$ is uniformly positive definite, i.e.,

$$\text{there is a constant } c_{\tilde{\mathbb{C}}} > 0 \text{ such that for all } \mathbf{A}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}: \quad \tilde{\mathbb{C}}\mathbf{A} : \mathbf{A} \geq c_{\tilde{\mathbb{C}}}|A|^2\,, \tag{2.2c}$$

and that $\tilde{\mathbb{D}}$ is positive semi-definite, only, i.e.,

$$\text{there is a constant } c_{\tilde{\mathbb{D}}} \geq 0 \text{ such that for all } \mathbf{A}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}: \quad \tilde{\mathbb{D}}\mathbf{A} : \mathbf{A} \geq c_{\tilde{\mathbb{D}}}|A|^2\,. \tag{2.2d}$$

Observe that we explicitly allow for the case $c_{\tilde{\mathbb{D}}} = 0$, so that viscous dissipation cannot act as a regularization. We will therefore distinguish in our results the two cases $c_{\tilde{\mathbb{D}}} = 0$ and $c_{\tilde{\mathbb{D}}} > 0$.

According to (2.2a), in dependence of z, the material is degraded by the elastic and the viscous degradation functions $g_{\mathbb{C}}$ and $g_{\mathbb{D}}$. For the two degradation functions we impose the following assumptions:

Regularity: $g_{\mathbb{C}}, g_{\mathbb{D}} \in \mathrm{C}^2(\mathbb{R})\,,$ (2.2e)

Monotonicity: $g'_{\mathbb{C}}(z), g'_{\mathbb{D}}(z) \geq 0$ for all $z\in\mathbb{R},$ more precisely,

$$g'_{\mathbb{C}}(z), g'_{\mathbb{D}}(z) \begin{cases} > 0 & \text{for } z\in[0,1]\,,\\ \geq 0 & \text{for } z\in(z_*,0)\cup(1,z^*) \text{ for given } z_* < 0 < 1 < z^*\,,\\ = 0 & \text{for } z\in(-\infty,z_*]\cup[z^*,\infty)\,.\end{cases} \tag{2.2f}$$

Boundedness from below: there is a constant $g_* > 0$ such that $g(z) \geq g_*$ for all $z\in\mathbb{R}\,.$ (2.2g)

The monotonicity assumption (2.2f) ensures that increasing material damage decreases the stored elastic energy, i.e., if $0 < z_1 < z_2 \leq 1$ then $\mathcal{E}^{\mathrm{el}}(z_1,\boldsymbol{\varepsilon}) > \mathcal{E}^{\mathrm{el}}(z_2,\boldsymbol{\varepsilon})$ for any $\boldsymbol{\varepsilon}\in\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$, because $g(z_1) > g(z_2)$ for $g\in\{g_{\mathbb{C}}, g_{\mathbb{D}}\}$. This strict monotonicity is imposed on the physically relevant range $z\in[0,1]$, where z represents the volume fraction of undamaged material. Together with the monotonicity assumptions on the remaining intervals and the boundedness from below (2.2g) we may conclude that

$$\text{there are constants } 0 < g_* < g^* \text{ and } g^{**} \text{ such that } 0 < g_* \leq g(z) \leq g^*\,, \tag{2.2h}$$

so that $g(z), g(z)^{-1}\in\mathrm{L}^\infty(\Omega)$ for any $\mathcal{L}^d$-measurable function $z:\Omega\to\mathbb{R}$.

Assumptions on the given data: For the given data in the Cauchy problem (1.1)–(1.2) we make the following assumptions

$$\begin{aligned}
&\text{volume load:} && \mathbf{f} \in \mathrm{L}^2(Q;\mathbb{R}^d)\,, && (2.3a)\\
&\text{surface load:} && \mathbf{g}_\mathrm{N} \in \mathrm{L}^2((0,T)\times\Gamma_\mathrm{N};\mathbb{R}^d)\,, && (2.3b)\\
&\text{surface velocity:} && \mathbf{v}_\mathrm{D} \in \mathrm{L}^2((0,T)\times\Gamma_\mathrm{D};\mathbb{R}^d)\,, && (2.3c)\\
&\text{initial data:} && z_0 \in H^1(\Omega),\ \mathbf{v}_0 \in \mathrm{L}^2(\Omega;\mathbb{R}^d) \text{ and } \boldsymbol{\varepsilon}_0 \in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_\mathrm{sym})\,. && (2.3d)
\end{aligned}$$

2.2 Weak formulation of Cauchy problem (1.1)–(1.2) & (1.4)–(1.5) and main result

Throughout this paper we denote by

$$(\cdot,\cdot)_A \quad \text{with } A \in \{\Omega, Q\} \tag{2.4}$$

the inner product on $\mathrm{L}^2(A)$, $\mathrm{L}^2(A;\mathbb{R}^d)$ or $\mathrm{L}^2(A;\mathbb{R}^{d\times d}_\mathrm{sym})$ depending on the respective arguments.

Derivation of a weak formulation for Cauchy problem (1.1)–(1.2): Let z be a sufficiently smooth solution of (1.4) and let $(\mathbf{v},\varepsilon,\boldsymbol{\sigma})$ be a sufficiently smooth solution of system (1.1). To deduce a weak formulation for (1.1) in the spirit of a first-order system we test system (1.1) with sufficiently smooth test functions $(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi})$ and integrate the result over Q

$$\begin{aligned}
0 &= \big(\varrho_0\partial_t\mathbf{v} - \operatorname{div}\boldsymbol{\sigma} - \mathbf{f}, \mathbf{w}\big)_Q + \big(\partial_t\boldsymbol{\varepsilon} - \operatorname{sym}(\mathrm{D}\mathbf{v}), \boldsymbol{\Phi}\big)_Q + \big(\boldsymbol{\sigma} - \mathbb{C}(z)\boldsymbol{\varepsilon} - \mathbb{D}(z)\partial_t\boldsymbol{\varepsilon}, \boldsymbol{\Psi}\big)_Q\\
&= \big(\varrho_0\partial_t\mathbf{v}, \mathbf{w}\big)_Q + \big(\partial_t\boldsymbol{\varepsilon}, \boldsymbol{\Phi} - \mathbb{D}(z)\boldsymbol{\Psi}\big)_Q - \big(\operatorname{div}\boldsymbol{\sigma}, \mathbf{w}\big)_Q - \big(\operatorname{sym}(\mathrm{D}\mathbf{v}), \boldsymbol{\Phi}\big)_Q\\
&\quad + \big(\boldsymbol{\sigma} - \mathbb{C}(z)\boldsymbol{\varepsilon}, \boldsymbol{\Psi}\big)_Q - \big(\mathbf{f}, \mathbf{w}\big)_Q\,.
\end{aligned} \tag{2.5}$$

For this, we have used the symmetry (2.2b) of the tensor $\mathbb{D}(z)$. Next, we integrate in (2.5) by parts as often as to shift all appearing derivatives from the solution to the test functions. This gives

$$\begin{aligned}
0 = &-\big(\varrho_0\mathbf{v}, \partial_t\mathbf{w}\big)_Q + \big(\varrho_0\mathbf{v}(T), \mathbf{w}(T)\big)_\Omega - \big(\varrho_0\mathbf{v}(0), \mathbf{w}(0)\big)_\Omega - \big(\boldsymbol{\varepsilon}, \partial_t\boldsymbol{\Phi} - \partial_t\big(\mathbb{D}(z)\boldsymbol{\Psi}\big)\big)_Q\\
&+ \big(\boldsymbol{\varepsilon}(T), \boldsymbol{\Phi}(T) - \mathbb{D}(z(T))\boldsymbol{\Psi}(T)\big)_\Omega - \big(\boldsymbol{\varepsilon}(0), \boldsymbol{\Phi}(0) - \mathbb{D}(z(0))\boldsymbol{\Psi}(0)\big)_\Omega\\
&+ \big(\boldsymbol{\sigma}, \operatorname{sym}(\mathrm{D}\mathbf{w})\big)_Q - \big(\boldsymbol{\sigma}\mathbf{n}, \mathbf{w}\big)_{(0,T)\times\partial\Omega} + \big(\mathbf{v}, \operatorname{div}\boldsymbol{\Phi}\big)_Q - \big(\mathbf{v}, \boldsymbol{\Phi}\mathbf{n}\big)_{(0,T)\times\partial\Omega}\\
&+ \big(\boldsymbol{\sigma} - \mathbb{C}(z)\boldsymbol{\varepsilon}, \boldsymbol{\Psi}\big)_Q - \big(\mathbf{f}, \mathbf{w}\big)_Q\,.
\end{aligned}$$

Inserting the Cauchy data (1.2) yields

$$\begin{aligned}
0 = &-\big(\varrho_0\mathbf{v}, \partial_t\mathbf{w}\big)_Q + \big(\varrho_0\mathbf{v}(T), \mathbf{w}(T)\big)_\Omega - \big(\varrho_0\mathbf{v}(0), \mathbf{w}(0)\big)_\Omega - \big(\boldsymbol{\varepsilon}, \partial_t\boldsymbol{\Phi} - \partial_t\big(\mathbb{D}(z)\boldsymbol{\Psi}\big)\big)_Q\\
&+ \big(\boldsymbol{\varepsilon}(T), \boldsymbol{\Phi}(T) - \mathbb{D}(z(T))\boldsymbol{\Psi}(T)\big)_\Omega - \big(\boldsymbol{\varepsilon}_0, \boldsymbol{\Phi}(0) - \mathbb{D}(z(0))\boldsymbol{\Psi}(0)\big)_\Omega\\
&+ \big(\boldsymbol{\sigma}, \operatorname{sym}(\mathrm{D}\mathbf{w})\big)_Q - \big(\mathbf{g}_\mathsf{N}, \mathbf{w}\big)_{(0,T)\times\Gamma_\mathsf{N}} - \big(\boldsymbol{\sigma}\mathbf{n}, \mathbf{w}\big)_{(0,T)\times\Gamma_\mathsf{D}}\\
&+ \big(\mathbf{v}, \operatorname{div}\boldsymbol{\Phi}\big)_Q - \big(\mathbf{v}_\mathsf{D}, \boldsymbol{\Phi}\mathbf{n}\big)_{(0,T)\times\Gamma_\mathsf{D}} - \big(\mathbf{v}, \boldsymbol{\Phi}\mathbf{n}\big)_{(0,T)\times\Gamma_\mathsf{N}}\\
&+ \big(\boldsymbol{\sigma} - \mathbb{C}(z)\boldsymbol{\varepsilon}, \boldsymbol{\Psi}\big)_Q - \big(\mathbf{f}, \mathbf{w}\big)_Q\,.
\end{aligned} \tag{2.6}$$

With the additional conditions for the smooth test functions $(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi})$

$$\begin{aligned}
\mathbf{w}(T) = 0\,, \qquad \boldsymbol{\Phi}(T) = \boldsymbol{\Psi}(T) &= \mathbf{0} && \text{in } \Omega\,, && (2.7a)\\
\mathbf{w} &= \mathbf{0} && \text{on } (0,T)\times\Gamma_\mathsf{D}\,, && (2.7b)\\
\boldsymbol{\Phi}\mathbf{n} &= \mathbf{0} && \text{on } (0,T)\times\Gamma_\mathsf{N}\,, && (2.7c)
\end{aligned}$$

equation (2.6) results in the following variational characterization of Cauchy problem (1.1)–(1.2)

$$
\begin{aligned}
0 = &-\big(\varrho_0 \mathbf{v}, \partial_t \mathbf{w}\big)_Q - \big(\varrho_0 \mathbf{v}_0, \mathbf{w}(0)\big)_\Omega - \big(\boldsymbol{\varepsilon}, \partial_t \boldsymbol{\Phi} - \partial_t\big(\mathbb{D}(z)\boldsymbol{\Psi}\big)\big)_Q \\
&- \big(\boldsymbol{\varepsilon}_0, \boldsymbol{\Phi}(0) - \mathbb{D}(z(0))\boldsymbol{\Psi}(0)\big)_\Omega \\
&+ \big(\boldsymbol{\sigma}, \mathrm{sym}(\mathrm{D}\mathbf{w})\big)_Q - \big(\mathbf{g}_{\mathsf{N}}, \mathbf{w}\big)_{(0,T)\times\Gamma_{\mathsf{N}}} \\
&+ \big(\mathbf{v}, \operatorname{div} \boldsymbol{\Phi}\big)_Q - \big(\mathbf{v}_{\mathsf{D}}, \boldsymbol{\Phi}\mathbf{n}\big)_{(0,T)\times\Gamma_{\mathsf{D}}} \\
&+ \big(\boldsymbol{\sigma} - \mathbb{C}(z)\boldsymbol{\varepsilon}, \boldsymbol{\Psi}\big)_Q - \big(\mathbf{f}, \mathbf{w}\big)_Q
\end{aligned}
\tag{2.8}
$$

for all sufficiently smooth test functions $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ satisfying (2.7).

Function spaces for (1.4) **and** (2.8)**:** In view of conditions (2.7) and (1.4) we define on the time-space cylinder $Q = (0,T)\times\Omega$ the following function spaces for the smooth test functions:

$$
\begin{aligned}
\mathcal{V} &:= \mathrm{C}^1(\overline{Q};\mathbb{R}^d)\,, && \text{(2.9a)}\\
\mathcal{V}_{T,\mathsf{D}} &:= \big\{\mathbf{w}\in\mathcal{V}\colon\ \mathbf{w}(T)=\mathbf{0} \text{ in } \Omega\,,\ \mathbf{w}=\mathbf{0} \text{ on } (0,T)\times\Gamma_{\mathsf{D}}\big\}\,, && \text{(2.9b)}\\
\mathcal{W} &:= \mathrm{C}^1(\overline{Q};\mathbb{R}^{d\times d}_{\mathrm{sym}})\,, && \text{(2.9c)}\\
\mathcal{W}_T &:= \big\{\boldsymbol{\Psi}\in\mathcal{W}\colon\ \boldsymbol{\Psi}(T)=\mathbf{0} \text{ in } \Omega\big\}\,, && \text{(2.9d)}\\
\mathcal{W}_{T,\mathsf{N}} &:= \big\{\boldsymbol{\Phi}\in\mathcal{W}_T\colon\ \boldsymbol{\Phi}\mathbf{n}=\mathbf{0} \text{ on } (0,T)\times\Gamma_{\mathsf{N}}\big\}\,, && \text{(2.9e)}\\
\mathcal{Z} &:= \big\{\varphi\in\mathrm{C}^1(\overline{Q})\colon\ \varphi\le 0 \text{ a.e. in } Q\big\}\,. && \text{(2.9f)}
\end{aligned}
$$

Short-hand notation for the terms in (1.4) **and** (2.8)**:** Moreover, we introduce the following short-hand notation for the Ambrosio-Tortorelli phase-field term appearing in (1.4)

$$
b_Q(z,\varphi) := -G_{\mathrm{c}}\,\big(1-z,\varphi\big)_Q + G_{\mathrm{c}} l_{\mathsf{c}}^2\,\big(\nabla z,\nabla\varphi\big)_Q\,, \tag{2.10a}
$$

and for the bilinear forms appearing in (2.8)

$$
\begin{aligned}
m_Q\big((\mathbf{v},\boldsymbol{\varepsilon}),(\mathbf{w},\boldsymbol{\eta})\big) &:= \big(\varrho_0\mathbf{v},\mathbf{w}\big)_Q + \big(\boldsymbol{\varepsilon},\boldsymbol{\eta}\big)_Q\,, && \text{(2.10b)}\\
a_Q\big((\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\boldsymbol{\Phi})\big) &:= \big(\boldsymbol{\sigma},\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_Q + \big(\mathbf{v},\operatorname{div}\boldsymbol{\Phi}\big)_Q\,, && \text{(2.10c)}\\
r_Q\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\boldsymbol{\Psi}\big) &:= \big(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon},\boldsymbol{\Psi}\big)_Q\,, && \text{(2.10d)}\\
\ell_Q(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi}) &:= \big(\mathbf{f},\mathbf{w}\big)_Q + \big(\varrho_0\mathbf{v}_0,\mathbf{w}(0)\big)_\Omega + \big(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0)-\mathbb{D}(z)\boldsymbol{\Psi}(0)\big)_\Omega \\
&\qquad + \big(\mathbf{v}_{\mathsf{D}},\boldsymbol{\Phi}\mathbf{n}\big)_{(0,T)\times\Gamma_{\mathsf{D}}} + \big(\mathbf{g}_{\mathsf{N}},\mathbf{w}\big)_{(0,T)\times\Gamma_{\mathsf{N}}}\,. && \text{(2.10e)}
\end{aligned}
$$

Here, the linear form ℓ_Q depends on the given data $\mathbf{f}$, $\mathbf{v}_0$, $\boldsymbol{\varepsilon}_0$, $\mathbf{v}_{\mathsf{D}}$, and $\mathbf{g}_{\mathsf{N}}$.

With this, a weak formulation of the Cauchy problem (1.1)–(1.2) and (1.4)–(1.5) is introduced as follows.

Definition 2.1 (Weak solutions of system (1.1)–(1.2) & (1.4)–(1.5))**.** *A quadruple* $(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma},z)$ *with*

$$
(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma})\in \mathrm{L}^2(Q;\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}) \quad \text{and} \quad z\in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))\cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \tag{2.11}
$$

is a weak solution of the Cauchy problem (1.1)–(1.2) *&* (1.4)–(1.5) *if the following conditions are satisfied:*

- *weak formulation of the momentum balance*

$$
\begin{aligned}
-m_Q\big((\mathbf{v},\boldsymbol{\varepsilon}),(\partial_t\mathbf{w},\partial_t\boldsymbol{\Phi}-\partial_t\big(\mathbb{D}(z)\boldsymbol{\Psi}\big))\big) + a_Q\big((\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\boldsymbol{\Phi})\big)& \\
+ r_Q\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\boldsymbol{\Psi}\big) - \ell_Q(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi}) = 0&
\end{aligned}
\tag{2.12a}
$$

for all test functions $(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi})\in\mathcal{V}_{T,D}\times\mathcal{W}_{T,N}\times\mathcal{W}_T$,

- *one-sided variational inequality for* z

$$\beta_{\mathrm{r}}\left(\partial_t z,\varphi\right)_Q+\frac{1}{2}\left(\mathbb{C}'(z)\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},\varphi\right)_Q+b_Q(z,\varphi)\ \geq\ 0\quad\text{for all }\varphi\in\mathcal{Z}\,, \tag{2.12b}$$

$$z(0)\ =\ z_0\quad\text{in }H^1(\Omega)\,, \tag{2.12c}$$

- *unidirectionality:*

$$\text{for all } t_1<t_2\in[0,T]\text{ it is } z(t_2)\leq z(t_1)\text{ a.e. in }\Omega\,. \tag{2.12d}$$

Subsequently, in Sections 3–4 we will prove the existence of a weak solution in the sense of Def. 2.1. More precisely, our main result can be summarized as follows:

Theorem 2.2 (Existence of a weak solution)**.** *Let the assumptions* (2.1)–(2.3) *on the domain, the material tensors and the given data be satisfied. Then the following statements hold true:*

1. *There exists a quadruple* $(\mathbf{v},\varepsilon,\boldsymbol{\sigma},z)$ *of regularity* (2.11) *which is a weak solution of the Cauchy problem* (1.1)–(1.2) *&* (1.4)–(1.5) *in the sense of Definition 2.1.*
2. *Apart from* (2.12b)–(2.12d) *a solution* z *also satisfies* $z(t)\in[0,z_0]$ *for all* $t\in[0,T]$.

For the proof of Thm. 2.2 we will introduce for system (2.12) a fully discrete approximation in space and time in Section 3. We will show that the discrete system has a solution, see Propsition 3.5, and that, as the fineness of the discretization increases, the discrete solutions converge in a weak sense to a weak solution of (2.12), see. Yet, before we immerse into this task, we provide a comparison of the results in [17], which make use of a second-order weak formulation of the momentum balance with the weak formulation in the sense of Def. 2.1.

2.3 Comparison of the first-order and a second-order weak formulation of the momentum balance and the energy-dissipation (in)equality

Comparison of weak formulation of the momentum balance as a first-order and as second-order system: Already in [17] the existence of a weak solution to the dynamic phase-field fracture model was shown by means of a fully discrete scheme. Herein, a weak solution was introduced as pair $(\mathbf{u},z)$ of regularity

$$\mathbf{u}\in\mathrm{H}^1\big(0,T;\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\big),\ z\in\mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^2\big(0,\mathrm{T};\mathrm{H}^1(\Omega)\big) \tag{2.13}$$

and characterized by the conditions (2.12b)–(2.12d) for the phase field variable z in addition to a weak formulation of the momentum balance as a second-order system, i.e., for all $t\in[0,T]$:

$$\begin{aligned}&\rho_0(\dot{\mathbf{u}}(t),\boldsymbol{\nu}(t))_\Omega-\rho_0\int_0^t(\dot{\mathbf{u}}(s),\dot{\boldsymbol{\nu}}(s))_\Omega\,\mathrm{d}s+\int_0^t\big(\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u})+\mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}),\mathrm{sym}(\nabla\boldsymbol{\nu})\big)_\Omega\,\mathrm{d}s\\&\quad=\rho_0(\dot{\mathbf{u}}(0),\boldsymbol{\nu}(0))_\Omega+\int_0^t(\mathbf{f}(s),\boldsymbol{\nu}(s))_\Omega\,\mathrm{d}s\end{aligned} \tag{2.14}$$

for all test functions $\boldsymbol{\nu}\in L^2(0,T;H^1_{\mathsf{D}}(\Omega,\mathbb{R}^d))\cap H^1(0,T:L^2(\Omega,\mathbb{R}^d))$,

where $H^1_{\mathsf{D}}(\Omega,\mathbb{R}^d):=\{\tilde{\boldsymbol{\nu}}\in H^1(\Omega,\mathbb{R}^d),\ \tilde{\boldsymbol{\nu}}=\mathbf{0}\text{ on }\Gamma_{\mathsf{D}}\}$.

Hereby, for the results in [17], the positive definiteness of the viscous tensor $\mathbb{D}$ was assumed, i.e., that $c_{\tilde{\mathbb{D}}}>0$ in (2.2d).

In the following we establish the equivalence of the weak formulation (2.12a) of the first order system and of the weak formulation (2.14) of the second-order system under the restriction that $\mathbf{v}(T)=0$ for the test functions in (2.14).

Proposition 2.3 (Equivalence of the weak formulations (2.12a) and (2.14))**.** *Let the assumptions* (2.1)–(2.3) *on the domain, the material tensors and the given data be satisfied. Further assume that* $c_{\tilde{\mathbb{D}}}>0$ *in* (2.2d).

1. *Let the pair* (u,z) *be a weak solution in the sense of* (2.12b)–(2.14)*. Then the triple*

$$\big(\dot{\mathbf{u}},\mathrm{sym}(\nabla\mathbf{u}),(\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u})+\mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}))\big)$$

is also weak solution of the weak formulation (2.12a) *of the first order system.*

2. Let the quadruple $(\mathbf{v}, \varepsilon, \boldsymbol{\sigma}, z)$ be a weak solution in the sense of Def. 2.1. Then there additionally holds

$$\mathbf{v} \in L^2(0,T;H^1(\Omega,\mathbb{R}^d))\,. \tag{2.15}$$

Assume that the initial data $\mathbf{u}_0 \in H^1_D(\Omega,\mathbb{R}^d)$ and $\varepsilon_0 \in L^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfy

$$\varepsilon_0 = \mathrm{sym}(\nabla \mathbf{u}_0) \quad \text{in } L^2(\Omega;\mathbb{R}^{d\times d})\,. \tag{2.16}$$

Then it can be identified

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_0 + \int_0^{(\cdot)} \mathbf{v}(s)\,\mathrm{d}s \;\; \text{in } H^1(0,T;H^1(\Omega,\mathbb{R}^d))\,, && \text{(2.17a)}\\
\varepsilon &= \mathrm{sym}(\nabla\mathbf{u}) \;\; \text{in } L^2(Q,\mathbb{R}^{d\times d})\,, && \text{(2.17b)}\\
\dot\varepsilon &= \mathrm{sym}(\nabla\dot{\mathbf{u}}) = \mathrm{sym}(\nabla\mathbf{v}) \;\; \text{in } L^2(Q,\mathbb{R}^{d\times d})\,, && \text{(2.17c)}\\
& && \text{(2.17d)}
\end{aligned}$$

and the triple $(\mathbf{v}, \varepsilon, \boldsymbol{\sigma})$ can be identified with

$$(\mathbf{v},\varepsilon,\boldsymbol{\sigma}) = \big(\dot{\mathbf{u}}, \mathrm{sym}(\nabla\mathbf{u}), (\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u}) + \mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}))\big) \tag{2.17e}$$

and satisfies the weak formulation of the second-order system (2.14) *for all test functions*

$$\begin{gathered}\boldsymbol{\nu} \in L^2(0,T;H^1_D(\Omega,\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega,\mathbb{R}^d))\\ \text{with the restriction that } \boldsymbol{\nu}(s)=0 \text{ for all } s\in[t,T], \text{ for any } t\in(0,T).\end{gathered} \tag{2.18}$$

Proof. **To 1.:** For the weak solution $\mathbf{u}$ of (2.14) we observe that the corresponding triple satisfies

$$\big(\dot{\mathbf{u}}, \mathrm{sym}(\nabla\mathbf{u}), (\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u}) + \mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}))\big) \in L^2(Q:\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})\,,$$

which is the regularity (2.11), and we introduce the following notation

$$\begin{aligned}
\mathbf{v} &= \dot{\mathbf{u}}\,, && \text{(2.19a)}\\
\varepsilon &= \mathrm{sym}(\nabla\mathbf{u})\,, && \text{(2.19b)}\\
\dot\varepsilon &= \mathrm{sym}(\nabla\mathbf{v})\,, && \text{(2.19c)}\\
\sigma &= \mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u}) + \mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}) = \mathbb{C}(z)\varepsilon + \mathbb{D}(z)\dot\varepsilon\,, && \text{(2.19d)}\\
\mathbf{v}_{\mathsf{D}} &= \dot{\mathbf{u}}|_{\Gamma_{\mathsf{D}}} = \dot{\mathbf{u}}_{\mathsf{D}} = 0\,. && \text{(2.19e)}
\end{aligned}$$

Accordingly, testing (2.19c) by $\boldsymbol{\Phi} \in \mathcal{W}_{T,\mathsf{N}}$ and (2.19d) by $\Psi \in \mathcal{W}_T$ gives

$$(\dot\varepsilon - \mathrm{sym}(\nabla\mathbf{v}), \boldsymbol{\Phi})_Q = 0 \quad \text{for all } \boldsymbol{\Phi}\in\mathcal{W}_{T,\mathsf{N}} \quad \text{and} \quad (\sigma - (\mathbb{C}(z)\varepsilon + \mathbb{D}(z)\dot\varepsilon))_Q = 0 \quad \text{for all } \boldsymbol{\Psi}\in\mathcal{W}_T \tag{2.20}$$

Furthermore, we set $t = T$ in (2.14) and restrict (2.14) to test functions

$$\boldsymbol{\nu} \in C^2(\overline{Q},\mathbb{R}^d) \cap L^2(0,T;H^1_{\mathsf{D}}(\Omega,\mathbb{R}^d)) \cap H^1(0,T:L^2(\Omega,\mathbb{R}^d))$$

with the additional property that also

$$\dot{\boldsymbol{\nu}} = 0 \quad \text{on } (0,T)\times\Gamma_{\mathsf{D}}\,.$$

We use once more (2.19d) in order to introduce σ in (2.14) and add equations (2.20) to the result. Repeating the integration by parts and the arguments from (2.5)–(2.8) gives the weak formulation (1.1a) of the first-order system.

To 2.: Let the quadruple $(\mathbf{v}, \varepsilon, \boldsymbol{\sigma}, z)$ be a weak solution in the sense of Def. 2.1. Thanks to the uniform positive definiteness of the tensor $\mathbb{D}(z)$, energy estimates indeed result in the additional regularity (2.15), see Lemma 4.1 below. Accordingly, we can define the displacement field as in (2.17a) and regularity (2.15) further ensures that $\mathrm{sym}(\nabla\mathbf{u}) \in$

$H^1(0,T;L^2(\Omega,\mathbb{R}^d))$, cf. (2.17b), as well as the relations $\mathbf{v}=\dot{\mathbf{u}}\in L^2(0,T;H^1(\Omega,\mathbb{R}^d))$ and $\mathrm{sym}(\nabla\dot{\mathbf{u}})=\mathrm{sym}(\nabla\mathbf{v})\in L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, cf. (2.17c). Testing (2.12a) with the triple $(\mathbf{0},\Phi,\mathbf{0})\in\mathcal{V}_{T,\mathsf{N}}\times\mathcal{W}_{T,\mathsf{N}}\times\mathcal{W}_T$ and reversing the integration by parts in space provides

$$\begin{aligned}0&=-(\boldsymbol{\varepsilon},\partial_t\boldsymbol{\Phi})_Q-(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0))_\Omega+(\mathbf{v},\mathrm{div}\boldsymbol{\Phi})_Q-(\mathbf{v}_\mathsf{D},\boldsymbol{\Phi}\mathbf{n})_{(0,T)\times\Gamma_\mathsf{D}}\\&=-(\boldsymbol{\varepsilon},\partial_t\boldsymbol{\Phi})_Q-(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0))_\Omega-(\mathrm{sym}(\nabla\mathbf{v}),\boldsymbol{\Phi})_Q\end{aligned}\tag{2.21}$$

Using (2.17a) and performing another integration by parts in time on the third term gives

$$\begin{aligned}0&=-(\boldsymbol{\varepsilon},\partial_t\boldsymbol{\Phi})_Q-(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0))_\Omega+(\mathrm{sym}(\nabla\mathbf{u}),\partial_t\boldsymbol{\Phi})_Q+(\mathrm{sym}(\nabla\mathbf{u}_0),\boldsymbol{\Phi}(0))_\Omega\\&=-(\boldsymbol{\varepsilon}-\mathrm{sym}(\nabla\mathbf{u}),\partial_t\boldsymbol{\Phi})_Q\,,\end{aligned}\tag{2.22}$$

where we also have used (2.16). Thanks to the fundamental lemma of the calculus of variations this shows that $\boldsymbol{\varepsilon}-\mathrm{sym}(\nabla\mathbf{u})=c$ a.e. in Q for a constant $c\in\mathbb{R}$. By the compatibility of the initial data (2.16) we conclude that $c=0$.

Similarly, we test (2.12a) with the triple $(\mathbf{0},\mathbf{0},\Psi)\in\mathcal{V}_{T,\mathsf{N}}\times\mathcal{W}_{T,\mathsf{N}}\times\mathcal{W}_T$ to find

$$0=(\boldsymbol{\varepsilon},\partial_t(\mathbb{D}(z)\boldsymbol{\Psi}))_Q+(\boldsymbol{\varepsilon}_0,\mathbb{D}(z(0))\boldsymbol{\Psi}(0))_\Omega+(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon},\boldsymbol{\Psi})_Q\,.\tag{2.23}$$

By the density of $\mathcal{W}_T$, resp. $\mathcal{W}_{T,\mathsf{N}}$ in

$$\begin{aligned}\tilde{\mathcal{W}}_T&:=\{\tilde{\boldsymbol{\Psi}}\in H^1(0,T;H^1(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})),\ \tilde{\boldsymbol{\Psi}}(T)=\mathbf{0}\text{ in }\Omega\}\,,\text{ resp.}\\\tilde{\mathcal{W}}_{T,\mathsf{N}}&:=\{\tilde{\boldsymbol{\Phi}}\in\tilde{\mathcal{W}}_T,\ \tilde{\boldsymbol{\Phi}}\mathbf{n}=\mathbf{0}\text{ on }(0,T)\times\Gamma_\mathsf{N}\}\,,\end{aligned}$$

we conclude that relations (2.21), (2.22) also hold true in $\tilde{\mathcal{W}}_{T,\mathsf{N}}$ and that relation (2.23) also holds true in $\tilde{\mathcal{W}}_T$. Assuming further that $\boldsymbol{\Psi}$ in (2.23) is such that even $\mathbb{D}(z)\boldsymbol{\Psi}\in\tilde{\mathcal{W}}_{T,\mathsf{N}}$, we are allowed to make use of (2.21) and (2.22) and conclude

$$\begin{aligned}0&=(\boldsymbol{\varepsilon},\partial_t(\mathbb{D}(z)\boldsymbol{\Psi}))_Q+(\boldsymbol{\varepsilon}_0,\mathbb{D}(z(0))\boldsymbol{\Psi}(0))_\Omega+(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon},\boldsymbol{\Psi})_Q\\&=-(\mathrm{sym}(\nabla\mathbf{v}),\mathbb{D}(z)\boldsymbol{\Psi})_Q+(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon},\boldsymbol{\Psi})_Q\\&=\big(\boldsymbol{\sigma}-(\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u})+\mathbb{D}(z)\mathrm{sym}(\nabla\mathbf{v})),\boldsymbol{\Psi}\big)_Q\,,\end{aligned}\tag{2.24}$$

Note that $\mathbb{D}(z)\boldsymbol{\Psi}\in\tilde{\mathcal{W}}_{T,\mathsf{N}}$ is possible, since $z\in H^1(0,T;H^1(\Omega))$.

Now we choose two test functions

$$\hat{\boldsymbol{\nu}},\tilde{\boldsymbol{\nu}}\in\tilde{\mathcal{V}}_\mathsf{D}:=C^2(\overline{Q};\mathbb{R}^d)\cap L^2(0,T;H^1_\mathsf{D}(\Omega,\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^d))$$

with the additional properties

$$\hat{\boldsymbol{\nu}}(s)=\tilde{\boldsymbol{\nu}}(s)\quad\text{in }\Omega\ \text{ for all }s\in[t,T]\text{ and any }t\in(0,T]\,,\tag{2.25a}$$

$$\mathbb{D}(z)\mathrm{sym}(\nabla\hat{\boldsymbol{\nu}})\mathbf{n}=\mathbb{D}(z)\mathrm{sym}(\nabla\tilde{\boldsymbol{\nu}})\mathbf{n}\quad\text{on }(0,T)\times\Gamma_\mathsf{N}\,.\tag{2.25b}$$

From this we infer that a triple $(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi})$ given by

$$\mathbf{w}:=(\hat{\boldsymbol{\nu}}-\tilde{\boldsymbol{\nu}})\in\mathcal{V}_{T,\mathsf{D}}\quad\text{and}\quad\boldsymbol{\Phi}=\boldsymbol{\Psi}:=\mathrm{sym}(\nabla(\hat{\boldsymbol{\nu}}-\tilde{\boldsymbol{\nu}}))\in\tilde{\mathcal{W}}_{T,\mathsf{N}}\tag{2.26}$$

provides admissible test functions for (2.12a), such that also (2.21)–(2.24) are valid. Due to this, (2.12a) rewrites as

$$\begin{aligned}0=&-m_Q\big((\mathbf{v},\boldsymbol{\varepsilon}),(\partial_t\mathbf{w},\partial_t\boldsymbol{\Phi}-\partial_t\big(\mathbb{D}(z)\boldsymbol{\Psi}\big))\big)+a_Q\big((\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\boldsymbol{\Phi})\big)\\&+r_Q\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\boldsymbol{\Psi}\big)-\ell_Q(\mathbf{w},\boldsymbol{\Phi},\boldsymbol{\Psi})\\=&-(\rho_0\mathbf{v}(0),\mathbf{w}(0))_\Omega-(\rho_0\mathbf{v},\dot{\mathbf{w}})_Q+(\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u})+\mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}),\mathrm{sym}(\nabla\mathbf{w}))_Q\\&-(\mathbf{f},\mathbf{w})_Q-(\mathbf{g}_\mathsf{N},\mathbf{w})_{(0,T)\times\Gamma_\mathsf{N}}\qquad\text{for all }\mathbf{w}\in\mathcal{V}_{T,\mathsf{D}}\cap C^2(\overline{Q};\mathbb{R}^d)\,.\end{aligned}$$

Since $\mathcal{V}_{T,\mathsf{D}}$ is dense in the set of functions $\mathbf{w}=(\hat{\boldsymbol{\nu}}-\tilde{\boldsymbol{\nu}})\in L^2(0,T;H^1_\mathsf{D}(\Omega,\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^d))$ with the additional property (2.25a) this gives

$$\begin{aligned}0=&-(\rho_0\mathbf{v}(0),\mathbf{w}(0))_\Omega-(\rho_0\mathbf{v},\dot{\mathbf{w}})_{Q_t}+(\mathbb{C}(z)\mathrm{sym}(\nabla\mathbf{u})+\mathbb{D}(z)\mathrm{sym}(\nabla\dot{\mathbf{u}}),\mathrm{sym}(\nabla\mathbf{w}))_{Q_t}\\&-(\mathbf{f},\mathbf{w})_{Q_t}-(\mathbf{g}_\mathsf{N},\mathbf{w})_{(0,t)\times\Gamma_\mathsf{N}}\\&\qquad\text{for all }\mathbf{w}=(\hat{\boldsymbol{\nu}}-\tilde{\boldsymbol{\nu}})\in L^2(0,T;H^1_\mathsf{D}(\Omega,\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^d))\text{ with (2.25a)}\,,\end{aligned}$$

where $Q_t:=(0,t)\times\Omega$. This finishes the proof of statement 2. □

Discussion of the energy-dissipation (in)equality: Moreover it has to be mentioned that under the assumption of uniform positive definiteness of $\mathbb{D}(z)$, in addition to (2.12c)–(2.14) it was possible in [17] to show that a weak solution (u,z) also satisfies the following energy-dissipation balance for all $t \in [0,T]$

$$\begin{aligned}&\tilde{\mathcal{E}}^{\rm kin}\big(\dot{\mathbf{u}}(t)\big) + \tilde{\mathcal{E}}^{\rm el}\big(z(t),\mathbf{u}(t)\big) + \mathcal{E}^{\rm ext}\big(t,\mathbf{u}(t)\big) + \int_0^t 2\tilde{\mathcal{R}}^{\rm vis}\big(z(r);\dot{\mathbf{u}}(r)\big)\,\mathrm{d}r + \mathcal{R}^{\rm pf}\big(\dot z(r)\big)\,\mathrm{d}r\\ &= \tilde{\mathcal{E}}^{\rm kin}\big(\dot{\mathbf{u}}(0)\big) + \tilde{\mathcal{E}}^{\rm el}\big(z(0),\mathbf{u}(0)\big) + \tilde{\mathcal{E}}^{\rm ext}\big(0,\mathbf{u}(0)\big) + \int_0^t \partial_t\tilde{\mathcal{E}}^{\rm ext}\big(r,u(r)\big)\,\mathrm{d}r\,.\end{aligned} \tag{2.27}$$

Above, in (2.27) and in the discussion below we use the following notation for the energy functionals and dissipation potentials of the system

$$\tilde{\mathcal{E}}^{\rm el}(z,\mathbf{u}) = \mathcal{E}^{\rm el}(z,\boldsymbol{\varepsilon}) := \int_\Omega \frac{1}{2}\mathbb{C}(z)\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\,\mathrm{d}\mathbf{x}\,, \quad \text{where } \boldsymbol{\varepsilon} = \mathrm{sym}(\nabla\mathbf{u})\,, \tag{2.28a}$$

$$\tilde{\mathcal{E}}^{\rm kin}(\dot{\mathbf{u}}) = \mathcal{E}^{\rm kin}(\mathbf{v}) := \frac{\varrho_0}{2}\int_\Omega |\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,, \text{ where } \mathbf{v} = \dot{\mathbf{u}}\,, \tag{2.28b}$$

$$\mathcal{E}^{\rm pf}(z) := \frac{G_{\rm c}}{2}\int_\Omega \big((1-z)^2 + l_{\rm c}^2|\nabla z|^2\big)\,\mathrm{d}\mathbf{x}\,, \tag{2.28c}$$

$$\tilde{\mathcal{E}}^{\rm ext}(t,\mathbf{u}) := -\int_\Omega \mathbf{f}(t)\cdot\mathbf{u}(t)\,\mathrm{d}\mathbf{x} - \int_{\Gamma_{\rm N}} \mathbf{g}_{\rm N}(t)\cdot\mathbf{u}(t)\,\mathrm{d}\mathbf{x}\,, \tag{2.28d}$$

$$\partial_t\tilde{\mathcal{E}}^{\rm ext}(t,\mathbf{u}) := \int_\Omega \dot{\mathbf{f}}(t)\cdot\mathbf{u}(t)\,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\rm N}} \dot{\mathbf{g}}_{\rm N}(t)\cdot\mathbf{u}(t)\,\mathrm{d}\mathbf{x}\,, \tag{2.28e}$$

$$\mathcal{E}^{\rm ext}(t,\mathbf{v}) := \int_\Omega \mathbf{f}(t)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\rm N}} \mathbf{g}_{\rm N}(t)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,, \tag{2.28f}$$

$$\text{so that} \quad \int_0^t \mathcal{E}^{\rm ext}(r,\mathbf{v})\,\mathrm{d}r = \tilde{\mathcal{E}}^{\rm ext}(t,\mathbf{u}) - \tilde{\mathcal{E}}^{\rm ext}(0,\mathbf{u}) - \int_0^t \partial_t\tilde{\mathcal{E}}^{\rm ext}(r,\mathbf{u})\,\mathrm{d}r$$

$$\tilde{\mathcal{R}}^{\rm vis}(z;\dot{\mathbf{u}}) = \mathcal{R}^{\rm vis}(z;\dot{\boldsymbol{\varepsilon}}) := \int_\Omega \frac{1}{2}\mathbb{D}(z)\dot{\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\mathbf{x}\,, \quad \text{where } \dot{\boldsymbol{\varepsilon}} = \mathrm{sym}(\nabla\dot{\mathbf{u}})\,, \tag{2.28g}$$

$$\mathcal{R}^{\rm pf}(\dot z) := \int_\Omega \frac{\beta_{\rm r}}{2}|\dot z|^2 + \chi_{(-\infty,0]}(\dot z)\,\mathrm{d}\mathbf{x} \quad \text{with } \chi_{(-\infty,0]} \text{ from (1.3)}\,. \tag{2.28h}$$

Remark 2.4 (Energy-dissipation balance for weak solutions in Def. 2.1)**.** *For the viscous system, i.e. $c_{\mathbb{D}} > 0$ in* (2.2d)*, the proof of the* equality (2.27) *in [17] relies on testing the weak second-order momentum balance* (2.14) *by the the time-derivative $\dot{\mathbf{u}}$ of the solution $\mathbf{u}$. This is an admissible test function in the setting of* (2.14)*. However, in Proposition 2.3 equivalence of the weak first-order momentum balance* (2.12a) *and the weak second-order momentum balance* (2.14) *is only valid when restricting the test functions $\boldsymbol{\nu} \in L^2(0,T;H^1_D(\Omega,\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega,\mathbb{R}^d))$ in* (2.14) *by the additional property that $\nu(s) = 0$ for all $s \in [t,T]$ for some $t \in (0,T]$, cf.* (2.18)*. Hence, in the setting of Proposition 2.3, 2. $\dot{\mathbf{u}}$ is no longer a suitable test function for the momentum balance, since $\dot{\mathbf{u}}(t) \neq \mathbf{0}$, in general. Therefore, we cannot expect to establish an energy* balance *for a weak solution $(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma},z)$ in the sense of Def. 2.1, even if $c_{\mathbb{D}} > 0$ in* (2.2d)*.*

As explained in Remark 2.4, instead of the balance (2.27) we can only deduce the following upper energy-dissipation estimate for weak solutions in the sense of Def. 2.1:

Proposition 2.5 (Energy-dissipation inequality for weak solutions in Def. 2.1)**.** *Let the assumptions* (2.1)–(2.3) *on the domain, the material tensors and the given data be satisfied and assume that the quadruple $(\mathbf{v},\varepsilon,\boldsymbol{\sigma},z)$ of regularity* (2.11) *is a weak solution of the Cauchy problem* (1.1)–(1.2) *&* (1.4)–(1.5) *in the sense of Definition 2.1. Assume that $\mathbb{D}$ is uniformly positive definite, i.e., that $c_{\tilde{\mathbb{D}}} > 0$ in* (2.2d)*. Then the energy-dissipation estimate*

$$\begin{aligned}&\mathcal{E}^{\rm kin}\big(\mathbf{v}(t)\big) + \mathcal{E}^{\rm el}\big(z(t),\boldsymbol{\varepsilon}(t)\big) + \mathcal{E}^{\rm pf}\big(z(t)\big) + \int_0^t 2\mathcal{R}^{\rm vis}(\dot{\boldsymbol{\varepsilon}}) + \mathcal{R}^{\rm pf}(\partial_s z)\,\mathrm{d}s\\ &\qquad\le \mathcal{E}^{\rm kin}\big(\mathbf{v}(0)\big) + \mathcal{E}^{\rm el}\big(z(0),\boldsymbol{\varepsilon}(0)\big) + \mathcal{E}^{\rm pf}\big(z(0)\big) + \int_0^t \mathcal{E}^{\rm ext}\big(s,\mathbf{v}(s)\big)\,\mathrm{d}s\end{aligned} \tag{2.29}$$

holds true for all $t \in (0,T]$.

3 Discrete system

3.1 Discretization in space and time

In order to find a solution in the sense of Definition 2.1, the system (2.12a)–(2.12d) is fully approximated in space and time.

Approximation in space: The visco-elastic wave equation (2.12a) is approximated with a discontinuous Galerkin (DG) method, while the phase field evolution law (2.12b) is approximated with lowest order conforming finite elements.

On a mesh $\Omega_h = \bigcup_{K\in\mathcal{K}_h} K$ with elements K, let

$$V_h^{\mathrm{dg}} := \prod_{K\in\mathcal{K}_h} \mathbb{P}_k(K;\mathbb{R}^d) \quad \text{and} \quad W_h^{\mathrm{dg}} := \prod_{K\in\mathcal{K}_h} \mathbb{P}_k(K;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \tag{3.1a}$$

be the discontinuous finite element space of polynomial degree $k \geq 1$, and let

$$V_h^{\mathrm{cf}} \subset \mathbb{P}(\Omega_h)\cap \mathrm{C}^0(\overline{\Omega}) \quad \text{be the lowest order conforming finite elements,} \tag{3.1b}$$

so that $\varphi_h \in V_h^{\mathrm{cf}}$ is uniquely defined by the values $\big(\varphi_h(\mathbf{x})\big)_{\mathbf{x}\in\mathcal{N}_h}$ at the element vertices $\mathcal{N}_h = \bigcup_{K\in\mathcal{K}_h}\mathcal{N}_K \subset \overline{\Omega}$. Then, we have

$$\min_{\mathbf{x}\in\mathcal{N}_K}\varphi_h(\mathbf{x}) = \min_{\mathbf{x}\in\overline{K}}\varphi_h(\mathbf{x}) \quad \text{and} \quad \max_{\mathbf{x}\in\mathcal{N}_K}\varphi_h(\mathbf{x}) = \max_{\mathbf{x}\in\overline{K}}\varphi_h(\mathbf{x})\,, \qquad K\in\mathcal{K}_h\,.$$

We assume that the mesh is shape regular and that $\mathrm{diam}(K)\leq h$ for $K\in\mathcal{K}_h$.

For the discontinuous functions, we define jump terms on the faces $\mathcal{F}_h = \bigcup_K \mathcal{F}_K$, where $\mathcal{F}_K$ are the faces on every element K. For inner faces $f\in\mathcal{F}_h\cap\Omega$, let K_f be the neighboring cell such that $\bar{f} = \partial K\cap\partial K_f$. On boundary faces $f\in\mathcal{F}_h\cap\partial\Omega$ we set $K_f = K$. Let $\mathbf{n}_K$ be the outer unit normal vector on ∂K. We define the jump $[\mathbf{v}_h]_{K,f} = \mathbf{v}_{h,K_f} - \mathbf{v}_{h,K}$ on inner faces, where $\mathbf{v}_{h,K}$ denotes the continuous extension of $\mathbf{v}_h|_K$ to $\overline{K}$. In the same way, the jump for the stress tensor is defined. On Dirichlet boundary faces, we set $[\mathbf{v}_h]_{K,f} = -2\mathbf{v}_h$ and $[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n} = \mathbf{0}$. On Neumann boundaries, set $[\mathbf{v}_h]_{K,f} = \mathbf{0}$ and $[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n} = -2\,\boldsymbol{\sigma}_h\mathbf{n}$.

Following [11, 7, 18], this defines the DG approximation of the forms a_Q and ℓ_Q appearing in the weak momentum balance (2.12a), cf. also (2.10), which now involve the discontinuous functions $(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\in V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}$ depending on the phase field $z_h\in V_h^{\mathrm{cf}}$ with

$$\begin{aligned} a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) &= \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w}_h)\big)_{\Omega_h} + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Phi}_h\big)_{\Omega_h} \\ &\quad -\frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f \\ &\quad -\frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f \end{aligned} \tag{3.2a}$$

and the right-hand side

$$\begin{aligned} \ell_h^{\mathrm{dg}}\big(t,z_h;(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) &= \big(\mathbf{f}(t),\mathbf{w}_h\big)_\Omega + \big(\mathbf{v}_{\mathsf{D}}(t),\boldsymbol{\Phi}_h\mathbf{n}\big)_{\Gamma_{\mathsf{D}}} + \big(\mathbf{g}_{\mathsf{N}}(t),\mathbf{w}_h\big)_{\Gamma_{\mathsf{N}}} \\ &\quad - \big(\mathbf{v}_{\mathsf{D}}(t),Z_{\mathsf{P}}(z_h)(\mathbf{n}\cdot\mathbf{w}_h)\mathbf{n} + Z_{\mathsf{S}}(z_h)(\mathbf{n}\times\mathbf{w}_h)\big)_{\Gamma_{\mathsf{D}}} \\ &\quad - \big(\mathbf{g}_{\mathsf{N}}(t),Z_{\mathsf{P}}(z_h)(\mathbf{n}\cdot\boldsymbol{\Phi}_h\mathbf{n})\mathbf{n} + Z_{\mathsf{S}}(z_h)\mathbf{n}\times(\boldsymbol{\Phi}_h\mathbf{n})\big)_{\Gamma_{\mathsf{N}}}\,. \end{aligned} \tag{3.2b}$$

Following the methods of [7], we have introduced here the z_h-dependent impedances

$$Z_{\mathsf{P}}(z_h) = \sqrt{g(z_h)\varrho_0(2\mu+\lambda)} \quad \text{and} \quad Z_{\mathsf{S}}(z_h) = \sqrt{g(z_h)\varrho_0\mu} \tag{3.2c}$$

of compressional waves and shear waves, respectively, see Remark 3.1 for a discussion. Above, in (3.2c) $\lambda > 0$ and $\mu > 0$ denote the Lamé constants of the material tensor $\tilde{\mathbb{C}}$ from (2.2a).

Remark 3.1. *The form of the DG-operators and in particular the impedances are highly motivated by the solution theory of general Riemann problems for linear first-order hyperbolic conservation laws, see [7, Section 3, p. 22ff] and [11, Section 3, p. 4ff] for the construction of solutions. The Riemann problem is considered locally across the face $f \in \mathcal{F}_h$ between finite element cells $K \in \mathcal{K}_h$ and solutions of the (local) Riemann problem are used to approximate the flux across the interface f.*

The numerical method can be simplified by using fixed impedances $Z_P = \sqrt{\varrho_0(2\mu+\lambda)}$ and $Z_S = \sqrt{\varrho_0\mu}$ independently of the degradation. The arguments in the subsequent sections only rely on the monotonicity (3.3) *and the consistency* (3.4) *of the DG approximation.*

Proposition 3.2 (Monotonicity and consistency of the DG approximation)**.** *Let the assumptions* (2.1)–(2.3) *on the domain, the material tensors, and the given data hold true. Then, for all $h > 0$ the DG approximation introduced in* (3.2) *has the following properties:*

1. *For all $z_h \in V_h^{\mathrm{cf}}$ the bilinear form $a_h^{\mathrm{dg}}(z_h;\cdot,\cdot) : (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \times (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \to \mathbb{R}$ is monotone, i.e.,*

$$\begin{aligned} &a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{v}_h,\boldsymbol{\sigma}_h)\big) \\ &= \frac{1}{4}\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Big(\big\|Z_P(z_h)^{-1/2}\mathbf{n}_K\cdot[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\big\|_f^2 + \big\|Z_P(z_h)^{1/2}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,f}\big\|_f^2 \\ &\qquad + \big\|Z_S(z_h)^{-1/2}\mathbf{n}_K\times[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\big\|_f^2 + \big\|Z_S(z_h)^{1/2}\mathbf{n}_K\times[\mathbf{v}_h]_{K,f}\big\|_f^2\Big) \\ &\geq 0 \quad \text{for all } (\mathbf{v}_h,\boldsymbol{\sigma}_h) \in (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})\,, \end{aligned} \tag{3.3}$$

2. *For all $t \in (0,T)$ and for all $z_h \in V_h^{\mathrm{cf}}$ the bilinear form $a_h^{\mathrm{dg}}(z_h;\cdot,\cdot) : (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \times (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \to \mathbb{R}$ and the linear form $\ell_h^{\mathrm{dg}}(t,z_h;\cdot) : (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \to \mathbb{R}$ are consistent, i.e., they satisfy for all smooth test functions $(\mathbf{w},\boldsymbol{\Phi}) \in \mathcal{V}_{T,D} \times \mathcal{W}_N$:*

$$a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w},\boldsymbol{\Phi})(t)\big) = \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w})(t)\big)_\Omega + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Phi}(t)\big)_\Omega\,, \tag{3.4a}$$

$$\ell_h^{\mathrm{dg}}\big(t,z_h;(\mathbf{w},\boldsymbol{\Phi})(t)\big) = \big(\mathbf{f}(t),\mathbf{w}\big)_\Omega + \big(\mathbf{v}_D(t),\boldsymbol{\Phi}(t)\mathbf{n}\big)_{\Gamma_D} + \big(\mathbf{g}_N(t),\mathbf{w}(t)\big)_{\Gamma_N}\,. \tag{3.4b}$$

Proof. The proof of Prop. 3.2 is carried out in Appendix A. □

Approximation in time: In the discrete formulation, the condition $\partial_t z \leq 0$ is approximated using a Yosida regularization

$$Y_h(\dot z) := \frac{\delta_h}{2}M_+^2(\dot z) \quad \text{with } M_+(\dot z) := \max\{\dot z, 0\} \text{ and penalty parameter } \delta_h := \frac{\delta_0}{h} > 0\,, \tag{3.5}$$

for $\delta_0 > 0$ fixed. Note that the Yosida regularization is continuously differentiable with

$$Y_h{}'(u) = \frac{\mathrm{d}Y_h(u)}{\mathrm{d}u} = \begin{cases} u & \text{if } u > 0\,, \\ 0 & \text{if } u \leq 0\,. \end{cases} \tag{3.6}$$

In this way we regularize the viscous dissipation potential for the phase field variable by

$$\mathcal{R}_h^{\mathrm{pf}}(\dot z) = \int_\Omega \left(\frac{\beta_{\mathrm{r}}}{2}|\dot z|^2 + Y_h(\dot z)\right)\mathrm{d}\mathbf{x}\,. \tag{3.7}$$

Moreover, depending on $z_h^{n-1} \in V_h^{\mathrm{cf}}$ and $\boldsymbol{\varepsilon}_h^{n-1}$ and with the notation (2.28), we also introduce the functional

$$\begin{aligned} \mathcal{G}_h^n(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1};z_h) &:= \frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(z_h - z_h^{n-1}) + \mathcal{E}^{\mathrm{el}}(z_h,\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\mathrm{pf}}(z_h) \\ &= \int_\Omega \Big(\frac{\beta_{\mathrm{r}}}{2\triangle t_h^n}\big(z_h - z_h^{n-1}\big)^2 + \frac{1}{\triangle t_h^n}Y_h\big(z_h - z_h^{n-1}\big) + \frac{1}{2}\mathbb{C}(z)\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1} \\ &\qquad + \frac{G_{\mathrm{c}}}{2}\big((1-z_h)^2 + l_{\mathrm{c}}^2|\nabla z_h|^2\big)\Big)\,\mathrm{d}\mathbf{x} \end{aligned} \tag{3.8}$$

In analogy to (2.10) we define in in Ω the following linear and bilinear forms

$$m_\Omega\big((\mathbf{v},\boldsymbol{\varepsilon}),(\mathbf{w},\boldsymbol{\eta})\big) := \big(\varrho_0\mathbf{v},\mathbf{w}\big)_\Omega + \big(\boldsymbol{\varepsilon},\boldsymbol{\eta}\big)_\Omega \qquad \text{for all } \mathbf{v},\mathbf{w}\in \mathrm{L}^2(\Omega;\mathbb{R}^d) \text{ and all } \boldsymbol{\varepsilon},\boldsymbol{\eta}\in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})\,, \tag{3.9a}$$

$$r_\Omega\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\boldsymbol{\Psi}\big) := \big(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon},\boldsymbol{\Psi}\big)_\Omega \qquad \text{for all } z\in \mathrm{L}^\infty(\Omega) \text{ and } \boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\Psi}\in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})\,, \tag{3.9b}$$

$$b_\Omega(z,\varphi) := -G_{\mathrm{c}}\big((1-z),\varphi\big)_\Omega + G_{\mathrm{c}} l_{\mathrm{c}}^2\big(\nabla z,\nabla\varphi\big)_\Omega \qquad \text{for all } z,\varphi\in \mathrm{H}^1(\Omega)\,, \tag{3.9c}$$

Time discretization: We assume that the loading is much slower than the wave speed. Thus we start with large time steps $\triangle t_{\text{qs}} > 0$ for quasi-static increments. If waves are initiated by crack opening, the time step is decreased to

$$\triangle t_{\text{pf}} \in (0,\triangle t_{\text{qs}}) \quad \text{such that} \quad c_{\text{P}}\triangle t_{\text{pf}} \approx h \quad \text{with wave speed } c_{\text{P}} = \sqrt{(2\mu+\lambda)/\rho}\,. \tag{3.10}$$

Staggered time-discrete scheme: Let $t_h^0 = 0$, $t_h^1 = \triangle t_{\text{qs}}$, and $\triangle t_h^1 = t_h^1 - t_h^0$. Given the quadruple of initial values $(\mathbf{v}_h^0,\boldsymbol{\varepsilon}_h^0,\boldsymbol{\sigma}_h^0,z_h^0)$ with $(\mathbf{v}_h^0,\boldsymbol{\varepsilon}_h^0,\boldsymbol{\sigma}_h^0)\in V_h^{\text{dg}}\times W_h^{\text{dg}}\times W_h^{\text{dg}}$ and $z_h^0\in V_h^{\text{cf}}$ with $z_h^0\in(0,1]$ we proceed as follows in every time step $n=1,2,3,\ldots$:

(S1) Depending on $(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1})$ from the previous time step, we approximate the phase field $z_h^n\in V_h^{\text{cf}}$ by the implicit Euler method, i.e., by computing a critical point of $\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};\cdot)$ by solving the nonlinear equation

$$\begin{aligned}\frac{\beta_{\mathrm{r}}}{\triangle t_h^n}\big(z_h^n - z_h^{n-1},\varphi_h\big)_\Omega &+ \frac{1}{\triangle t_h^n}\big({Y_h}'(z_h^n - z_h^{n-1}),\varphi_h\big)_\Omega \\ &+\frac12\big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^{n-1}:\boldsymbol{\varepsilon}_h^{n-1},\varphi_h\big)_\Omega + b_\Omega(z_h^n,\varphi_h) = 0 \\ &\qquad\qquad \text{for all } \varphi_h\in V_h^{\text{cf}}\end{aligned} \tag{3.11a}$$

such that

$$\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^n) \le \mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^{n-1})\,. \tag{3.11b}$$

We introduce in Section 3.2 an iterative solution method for (3.11a) and show that (3.11b) is satisfied when starting the iteration procedure with z_h^{n-1}.

(S2) Depending on $(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1},\boldsymbol{\sigma}_h^{n-1})\in V_h^{\text{dg}}\times W_h^{\text{dg}}\times W_h^{\text{dg}}$ and $z_h^n\in V_h^{\text{cf}}$ we compute the solution $(\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n)\in V_h^{\text{dg}}\times W_h^{\text{dg}}\times W_h^{\text{dg}}$ for time step n by the implicit Euler method, i.e., by solving the linear equation

$$\begin{aligned}m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{w}_h,\boldsymbol{\Phi}_h-\mathbb{D}(z_h^n)\boldsymbol{\Psi}_h)\big) &+ \triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\boldsymbol{\Psi}_h\big) \\ &+ \triangle t_h^n a_h^{\text{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) \\ = m_\Omega\big((\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{w}_h,\boldsymbol{\Phi}_h-\mathbb{D}(z_h^n)\boldsymbol{\Psi}_h)\big) &+ \triangle t_h^n \ell_h^{\text{dg}}\big(t_h^n,z_h^n;(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big)\end{aligned} \tag{3.12}$$

for all $(\mathbf{w}_h,\boldsymbol{\Phi}_h,\boldsymbol{\Psi}_h)\in V_h^{\text{dg}}\times W_h^{\text{dg}}\times W_h^{\text{dg}}$.

(S3) If the relaxed energy is small and $z_h^n\approx z_h^{n-1}$, we expect that the next time step will also be quasi-static, and we set $\triangle t_h^{n+1} = \triangle t_{\text{qs}}$; otherwise, we set $\triangle t_h^{n+1} = \triangle t_{\text{pf}}$.
Then, we set $t_h^{n+1} = t_h^n + \triangle t_h^{n+1}$, and we continue with the next time step $n := n+1$ proceeding with (S1).

For simplicity of the presentation, we consider in the following only the case of homogeneous boundary data $\mathbf{v}_{\text{D}} = \mathbf{0}$ and $\mathbf{g}_{\text{N}} = \mathbf{0}$, and the volume forces are approximated by the L^2 projection $\mathbf{f}_h^n\in V_h^{\text{dg}}$ in $(t_h^{n-1},t_h^n)\times\Omega$, i.e.,

$$\big(\mathbf{f}_h^n,\mathbf{w}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega} = \big(\mathbf{f},\mathbf{w}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega}\,,\qquad \mathbf{w}_h\in V_h^{\text{dg}}\,, \tag{3.13}$$

and we use for the subsequent analysis the discrete right-hand side

$$\ell_h^{\text{dg}}\big(t_h^n,z_h^n;(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) = \big(\mathbf{f}_h^n,\mathbf{w}_h^n\big)_\Omega\,. \tag{3.14}$$

We also assume that $(\mathbf{v}_h^0,\boldsymbol{\varepsilon}_h^0)$ are the L^2 projections of the initial values $(\mathbf{v}_0,\boldsymbol{\varepsilon}_0)$.

For later reference we collect the main properties of the functional $\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};\cdot)$ in the following lemma

Lemma 3.3 (Properties of $\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; \cdot)$). *Let the assumptions* (2.1)–(2.3) *hold true. Then, for all $h > 0$, all $n \in \mathbb{N}$ and all $(\varepsilon_h^{n-1}, z_h^{n-1}) \in W_h^{\mathrm{cf}} \times V_h^{\mathrm{cf}}$ the functional $\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; \cdot) : V_h^{\mathrm{cf}} \to \mathbb{R}$ is continuous and coercive, i.e., there are constants $c_{\mathcal{G}_1},\ c_{\mathcal{G}_2} > 0$ such that*

$$\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h) \geq c_{\mathcal{G}_1} \|z_h\|^2_{H^1(\Omega)} - c_{\mathcal{G}_2}\,. \tag{3.15}$$

Proof. **Coercivity:** Omitting quadratic terms of lower order wrt. z_h in $\mathcal{G}_h^n$ and using the positivity of the degradation function $g_{\mathbb{C}}$ in (2.2a) we conclude

$$\begin{aligned}
\mathcal{G}_h^n(z_h^{n-1}, \varepsilon_h^{n-1}; z_h) &= \int_\Omega \Big(\frac{\beta_{\mathrm{r}}}{2\triangle t_h^n} \big(z_h - z_h^{n-1}\big)^2 + \frac{1}{\triangle t_h^n} Y_h\big(z_h - z_h^{n-1}\big) + \frac{1}{2}\mathbb{C}(z_h)\varepsilon_h^{n-1} : \varepsilon_h^{n-1} \\
&\qquad + \frac{G_{\mathrm{c}}}{2}\big((1-z_h)^2 + l_{\mathsf{c}}^2 |\nabla z_h|^2\big)\Big)\,\mathrm{d}\mathbf{x} \\
&\geq \int_\Omega \Big(\frac{G_{\mathrm{c}}}{2}\big((1-z_h)^2 + l_{\mathsf{c}}^2 |\nabla z_h|^2\big)\Big)\,\mathrm{d}\mathbf{x} \\
&\geq \int_\Omega \bigg(\frac{G_{\mathrm{c}}}{2}\Big(\frac{z_h^2}{2} - 1 + l_{\mathsf{c}}^2 |\nabla z_h|^2\Big)\bigg)\,\mathrm{d}\mathbf{x} \\
&\geq \int_\Omega \Big(\frac{G_{\mathrm{c}}}{4} z_h^2 - \frac{G_{\mathrm{c}}}{2} + \frac{G_{\mathrm{c}}}{2} l_{\mathsf{c}}^2 |\nabla z_h|^2\Big)\,\mathrm{d}\mathbf{x} \\
&\geq c_{\mathcal{G}_1}\|z_h\|^2_{H^1(\Omega)} - c_{\mathcal{G}_2}
\end{aligned}$$

with $c_{\mathcal{G}_1} = c_{\mathcal{G}_1}(G_{\mathrm{c}}, l_{\mathsf{c}})$ and $c_{\mathcal{G}_2} = c_{\mathcal{G}_1}(G_{\mathrm{c}}, |\Omega|)$.

Continuity: The functional $\mathcal{G}_h^n(z_h^{n-1}, \varepsilon_h^{n-1}; \bullet)$ consists of linear and quadratic terms in z_h and in addition $z_h \mapsto g_{\mathbb{C}}(z_h)$ is by definition in (2.2a) smooth. This ensures the continuity of the mapping $z_h \mapsto \mathcal{G}_h^n(z_h^{n-1}, \varepsilon_h^{n-1}; z_h)$. □

3.2 Computation of a critical point in (S1)

We show that the discrete nonlinear problem in (S1) can be solved iteratively such that, additionally, for all $n \in \{1, \ldots, N\}$, the discrete solutions z_h^n and z_h^{n-1} at steps n and $n-1$ also satisfy the condition

$$\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^n) \leq \mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n-1})\,, \tag{3.16}$$

where $(\varepsilon_h^{n-1}, z_h^{n-1})$ are the solutions computed in the previous step $n-1$. This result is accomplished by *constructing* a minimizing sequence for the coercive and continuous functional $\mathcal{G}_h^n$ in the finite-dimensional space V_h^{cf}.

Given the solutions $(\varepsilon_h^{n-1}, z_h^{n-1}) \in V_h^{\mathrm{cf}} \times W_h^{\mathrm{dg}}$ from the previous step $n-1$, recall from (3.15) that

$$\begin{aligned}
\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h) := \int_\Omega \Big(&\frac{\beta_{\mathrm{r}}}{2\triangle t_h^n}\big(z_h - z_h^{n-1}\big)^2 + \frac{1}{\triangle t_h^n} Y_h(z_h - z_h^{n-1}) \\
&+ \frac{1}{2}\mathbb{C}(z_h)\varepsilon_h^{n-1} : \varepsilon_h^{n-1} + \frac{G_{\mathrm{c}}}{2}\big((1-z_h)^2 + l_{\mathsf{c}}^2|\nabla z_h|^2\big)\Big)\,\mathrm{d}\mathbf{x}\,.
\end{aligned} \tag{3.17}$$

for any $z_h \in V_h^{\mathrm{cf}}$. With this, one calculates

$$\begin{aligned}
\mathrm{D}\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h)[\varphi_h] = &\frac{\beta_{\mathrm{r}}}{\triangle t_h^n}\big(z_h - z_h^{n-1}, \varphi_h\big)_\Omega + \frac{1}{\triangle t_h^n}\big(Y_h'(z_h - z_h^{n-1}), \varphi_h\big)_\Omega \\
&+ \frac{1}{2}\big(\mathbb{C}'(z_h)\varepsilon_h^{n-1} : \varepsilon_h^{n-1}, \varphi_h\big)_\Omega + b_\Omega(z_h, \varphi_h)
\end{aligned}$$

for all $z_h, \varphi_h \in V_h^{\mathrm{cf}}$. To find a zero of $\mathrm{D}\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; \cdot)$ a generalized Newton method is used. This is applicable since $\dim V_h^{\mathrm{cf}} < \infty$ and $\mathrm{D}\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; \cdot)$ is semi-smooth.

For all $z_h, \psi_h, \varphi_h \in V_h^{\mathrm{cf}}$ we define the approximative linearization c_Ω by

$$c_\Omega(z_h; \psi_h, \varphi_h) := \frac{\beta_{\mathrm{r}}}{\triangle t_h^n}\big(\phi_h, \varphi_h\big)_\Omega + \frac{\delta_h}{\triangle t_h^n}\big(\psi_h, \varphi_h\big)_\Omega + G_{\mathrm{c}}\left(\psi_h, \varphi_h\right)_\Omega + G_{\mathrm{c}} l_{\mathrm{c}}^2\left(\nabla\psi_h, \nabla\varphi_h\right)_\Omega ,$$

where in the linearization the derivative $g_{\mathbb{C}}''(z_h)$ is omitted and the subdifferential of ${Y_h}'(z_h)$ is replaced by $\frac{\delta_h}{\triangle t_h^n}\big(\psi_h, \varphi_h\big)_\Omega$. Thus, for all $z_h \in V_h^{\mathrm{cf}}$ the bilinear form $c_\Omega(z_h; \cdot, \cdot) : V_h^{\mathrm{cf}} \times V_h^{\mathrm{cf}} \to \mathbb{R}$ is positive definite and continuous.

For all $n \in \{1, \dots, N\}$ the algorithm to compute z_h^n is the following:

1. Set $z_h^{n,0} := z_h^{n-1}$.
2. For $k = 1, 2, 3, \dots$ compute a direction of descent $d_h^{n,k} \in V_h$ by solving

$$c_\Omega(z_h^{n,k-1}; d_h^{n,k}, \varphi_h) = -\mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1})[\varphi_h] , \qquad \text{for all } \varphi_h \in V_h^{\mathrm{cf}} . \tag{3.18}$$

3. Stop if $\mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) = 0$.
4. Otherwise, $d_h^{n,k} \neq 0$, and we show that there exists a step size $\alpha_{n,k} > 0$ such that

$$\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k}) \leq \mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) . \tag{3.19}$$

5. Set $z_h^{n,k} := z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k}$. Proceed with step $k+1$ using 2..

In order to verify (3.19), we discuss each of the five summands appearing in $\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k})$ separately, cf. (3.17). For the first, the fourth, and the fifth summand we observe

$$\frac{1}{2}\big\|z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k} - z_h^{n-1}\big\|_\Omega^2 = \frac{1}{2}\big\|z_h^{n,k-1} - z_h^{n-1}\big\|_\Omega^2 + \alpha_{n,k}\big(z_h^{n,k-1} - z_h^{n-1}, d_h^{n,k}\big)_\Omega + \frac{1}{2}\alpha_{n,k}^2\big\|d_h^{n,k}\big\|_\Omega^2 , \tag{3.20}$$

$$\frac{1}{2}\big\|1 - \big(z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k}\big)\big\|_\Omega^2 = \frac{1}{2}\big\|1 - z_h^{n,k-1}\big\|_\Omega^2 - \alpha_{n,k}\big(1 - z_h^{n,k-1}, d_h^{n,k}\big)_\Omega + \frac{1}{2}\alpha_{n,k}^2\big\|d_h^{n,k}\big\|_\Omega^2 , \tag{3.21}$$

$$\frac{1}{2}\big\|\nabla z_h^{n,k-1} + \alpha_{n,k}\nabla d_h^{n,k}\big\|_\Omega^2 = \frac{1}{2}\big\|\nabla z_h^{n,k-1}\big\|_\Omega^2 + \alpha_{n,k}\big(\nabla z_h^{n,k-1}, \nabla d_h^{n,k}\big)_\Omega + \frac{1}{2}\alpha_{n,k}^2\big\|\nabla d_h^{n,k}\big\|_\Omega^2 . \tag{3.22}$$

For the third term in (3.17), remember that $\mathbb{C}(z) = g_{\mathbb{C}}(z)$. We make a Taylor expansion of $g_{\mathbb{C}} \in C^2(\mathbb{R})$ around $z_h^{n,k-1}(x)$ for a.a. $x \in \Omega$. This yields

$$\begin{aligned} &g_{\mathbb{C}}\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x})\big) \\ &= g_{\mathbb{C}}\big(z_h^{n,k-1}(\mathbf{x})\big) + \alpha_{n,k} g_{\mathbb{C}}'\big(z_h^{n,k-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2 g_{\mathbb{C}}''\big(\tilde{z}_h^{n,k-1}(\mathbf{x})\big)\big(d_h^{n,k}(\mathbf{x})\big)^2 \end{aligned} \tag{3.23}$$

with an intermediate value such that $\big|\tilde{z}_h^{n,k-1}(\mathbf{x}) - z_h^{n,k-1}(\mathbf{x})\big| \leq \alpha_{n,k}\big|d_h^{n,k}(\mathbf{x})\big|$.

Finally, we verify for the second term in (3.17) that

$$\begin{aligned} &Y_h\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) \\ &\leq Y_h\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) + \alpha_{n,k}{Y_h}'\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 . \end{aligned} \tag{3.24}$$

For this, we distinguish the following four cases:

1) $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$,
2) $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$,
3) $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$,
4) $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$.

In case 1), i.e., if $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$, we have

$$\begin{aligned} Y_h\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) &= \frac{1}{2}\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big)^2 \\ &= \frac{1}{2}\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big)^2 + \alpha_{n,k}\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 \\ &= Y_h\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) + \alpha_{n,k} Y_h{}'\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 , \end{aligned}$$

hence (3.24) for case 1).

In case 2), i.e., if $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$, we have

$$\max\big\{z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}), 0\big\} \leq \alpha_{n,k} d_h^{n,k}(\mathbf{x})$$

and thus

$$\begin{aligned} 0 < Y_h\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) &\leq \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 \\ &= Y_h\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) + \alpha_{n,k} Y_h{}'\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 , \end{aligned}$$

which is (3.24) for case 2).

In case 3), i.e., if $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) > 0$, we have

$$Y_h\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) = 0$$

and

$$\begin{aligned} &Y_h\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) + \alpha_{n,k} Y_h{}'\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 \\ &\quad= \frac{1}{2}\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big)^2 + \alpha_{n,k}\big(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big) d_h^{n,k}(\mathbf{x}) + \frac{1}{2}\alpha_{n,k}^2\big(d_h^{n,k}(\mathbf{x})\big)^2 \\ &\quad= \frac{1}{2}\big(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})\big)^2 > 0 , \end{aligned}$$

which yields (3.24) for case 3).

In case 4), i.e., if $z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$ and $z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}) \leq 0$ we observe that

$$Y_h(z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})) = 0 , \; Y_h(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})) = 0 , \text{ and } Y_h{}'(z_h^{n,k-1}(\mathbf{x}) - z_h^{n-1}(\mathbf{x})) = 0 ,$$

and thus we have (3.24) also in case 4).

Collecting all estimates (3.20)–(3.24) for the five contributions of $\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}(\mathbf{x}) + \alpha_{n,k} d_h^{n,k}(\mathbf{x}) - z_h^{n-1}(\mathbf{x}))$, we find

$$
\begin{aligned}
\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k}) &= \frac{\beta_\mathrm{r}}{2\triangle t_h^n}\big\|z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k} - z_h^{n-1}\big\|_\Omega^2 \\
&\quad + \int_\Omega \Big(\frac{1}{\triangle t_h^n} Y_h\big(z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k} - z_h^{n-1}\big) \\
&\qquad + \frac{1}{2} g_\mathbb{C}(z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1}\Big)\,\mathrm{d}\mathbf{x} \qquad (3.26) \\
&\quad + \frac{G_\mathrm{c}}{2}\Big(\big\|1 - \big(z_h^{n,k-1} + \alpha_{n,k} d_h^{n,k}\big)\big\|_\Omega^2 + l_\mathsf{c}^2\big\|\nabla z_h^{n,k-1} + \alpha_{n,k}\nabla d_h^{n,k}\big\|_\Omega^2\Big) \\
&\le \frac{\beta_\mathrm{r}}{2\triangle t_h^n}\big\|z_h^{n,k-1} - z_h^{n-1}\big\|_\Omega^2 + \alpha_{n,k}\frac{\beta_\mathrm{r}}{\triangle t_h^n}\big(z_h^{n,k-1} - z_h^{n-1}, d_h^{n,k}\big)_\Omega + \alpha_{n,k}^2\frac{\beta_\mathrm{r}}{2\triangle t_h^n}\big\|d_h^{n,k}\big\|_\Omega^2 \\
&\quad + \frac{1}{\triangle t_h^n}\int_\Omega Y_h\big(z_h^{n,k-1} - z_h^{n-1}\big)\,\mathrm{d}\mathbf{x} + \alpha_{n,k}\big({Y_h}'(z_h^{n,k-1} - z_h^{n-1}), d_h^{n,k}\big)_\Omega + \frac{\alpha_{n,k}^2}{2}\big\|d_h^{n,k}\big\|_\Omega^2 \\
&\quad + \frac{1}{2}\int_\Omega g_\mathbb{C}(z_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1}\,\mathrm{d}\mathbf{x} + \frac{\alpha_{n,k}}{2}\big(g_\mathbb{C}'(z_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1}, d_h^{n,k}\big)_\Omega \\
&\qquad + \frac{\alpha_{n,k}^2}{4}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \\
&\quad + \frac{G_\mathrm{c}}{2}\Big(\big\|1 - z_h^{n,k-1}\big\|_\Omega^2 + l_\mathsf{c}^2\big\|\nabla z_h^{n,k-1}\big\|_\Omega^2 - 2\alpha_{n,k}\Big(\big(1 - z_h^{n,k-1}, d_h^{n,k}\big)_\Omega - l_\mathsf{c}^2\big(\nabla z_h^{n,k-1}, \nabla d_h^{n,k}\big)_\Omega\Big) \\
&\qquad + \alpha_{n,k}^2\Big(\big\|d_h^{n,k}\big\|_\Omega^2 + l_\mathsf{c}^2\big\|\nabla d_h^{n,k}\big\|_\Omega^2\Big)\Big) \\
&= \mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) + \alpha_{n,k}\mathrm{D}\mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1})[d_h^{n,k}] + \frac{\alpha_{n,k}^2}{2} c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) \\
&\quad + \frac{\alpha_{n,k}^2}{4}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \qquad (3.27) \\
&= \mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) + \Big(\frac{\alpha_{n,k}^2}{2} - \alpha_{n,k}\Big) c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) \\
&\quad + \frac{\alpha_{n,k}^2}{4}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \\
&=: \mathcal{G}_h^n(\varepsilon_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) + R_h^n(d_h^{n,k}; \alpha_{n,k})\,. \qquad (3.28)
\end{aligned}
$$

In order to deduce (3.19) from this estimate it now has to be shown that a step size $\alpha_{n,k}$ can be chosen such that

$$
R_h^n(d_h^{n,k}; \alpha_{n,k}) := \Big(\frac{\alpha_{n,k}^2}{2} - \alpha_{n,k}\Big) c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) + \frac{\alpha_{n,k}^2}{4}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \overset{!}{\le} 0\,. \qquad (3.29)
$$

For this, we recall that $g_\mathbb{C}''$ is continuous and that $g_\mathbb{C}' = 0$ in $\mathbb{R}\setminus[0,2]$, so that $g_\mathbb{C}''$ is bounded. Thus, there exists a constant $0 < C_{h,n} = C_{h,n}(\varepsilon_h^{n-1}, \tilde{\mathbb{C}}, g_\mathbb{C}'')$ depending on h but **independent of** k such that

$$
\begin{aligned}
&\frac{1}{2}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \le C_{h,n}\,\big\|d_h^{n,k}\big\|_\Omega^2 \le \tilde{C}_{h,n}\, c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) \qquad (3.30) \\
&\text{with } \tilde{C}_{h,n} = \frac{2\triangle t_h^n C_{h,n}}{\beta_\mathrm{r}}\,,
\end{aligned}
$$

where we also used the positive definiteness of $c_\Omega(z_h^{n,k-1}; \cdot, \cdot)$ for the second estimate in (3.30). With this, we estimate

$$
\begin{aligned}
R_h^n(d_h^{n,k}; \alpha_{n,k}) &:= \Big(\frac{\alpha_{n,k}^2}{2} - \alpha_{n,k}\Big) c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) + \frac{\alpha_{n,k}^2}{4}\big(g_\mathbb{C}''(\tilde{z}_h^{n,k-1})\tilde{\mathbb{C}}\varepsilon_h^{n-1} : \varepsilon_h^{n-1} d_h^{n,k}, d_h^{n,k}\big)_\Omega \\
&\le \Big((1 + \tilde{C}_{h,n})\frac{\alpha_{n,k}^2}{2} - \alpha_{n,k}\Big) c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k})\,. \qquad (3.31)
\end{aligned}
$$

We are now able to select

$$\alpha_n := \frac{1}{1+\tilde{C}_{h,n}} \quad\text{and}\quad \alpha_{n,k} := \alpha_n\,, \qquad \text{so that}\quad \Big((1+\tilde{C}_{h,n})\frac{\alpha_{n,k}^2}{2} - \alpha_{n,k}\Big) = -\frac{1}{2(1+\tilde{C}_{h,n})} < 0\,. \tag{3.32}$$

Inserting this in (3.31), thanks to the positive definiteness of $c_\Omega(z_h^{n,k-1};\cdot,\cdot)$, we conclude (3.29).

Moreover, inserting our choice of α_n from (3.32) into (3.28) we deduce with $z_h^{n,k} = z_h^{n,k-1} + \alpha_n d_h^{n,k}$ that

$$\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k}) \le \mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) - \frac{1}{2(1+\tilde{C}_{h,n})} c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k})\,. \tag{3.33}$$

In consequence, the sequence $(\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k}))_{k\in\mathbb{N}}$ is (strictly) monotonically decreasing. Since $\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1};\cdot)$ is also bounded from below (by 0), $(\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k}))_{k\in\mathbb{N}}$ converges to its infimum. Rearranging terms in (3.33) and making once more use of the positive definiteness of $c_\Omega(z_h^{n,k-1};\cdot,\cdot)$ we infer that

$$\begin{aligned} c_*\|d_h^{n,k}\|_\Omega^2 &\le c_\Omega(z_h^{n,k-1}; d_h^{n,k}, d_h^{n,k}) \\ &\le (2(1+\tilde{C}_{h,n}))\Big(\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k-1}) - \mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k})\Big) \\ &\qquad\qquad \longrightarrow 0 \quad \text{as } k\to\infty\,. \end{aligned} \tag{3.34}$$

This implies that $d_h^{n,k} \to 0$ as $k\to\infty$.

By coercivity of $\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1};\cdot)$, the sequence $(z_h^{n,k})_{k\in\mathbb{N}}$ is bounded in the finite dimensional space V_h^{cf}. Thus, there exists a converging subsequence $(z_h^{n,k_j})_{j\in\mathbb{N}}$ and a limit z_h^n in V_h^{cf}.

Now we show by contradiction, that z_h^n is a critical point in (S1):
Suppose that z_h^n is not a critical point. Then $\mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^n)[\varphi_h] \neq 0$ for some $\varphi_h \in V_h^{\mathrm{cf}}$ and as a consequence of (3.18), also $c_\Omega(z_h^n; d_h^n, \varphi_h) \neq 0$ for some $\varphi_h \in V_h^{\mathrm{cf}}$. By continuity of $\mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1};\cdot)[\varphi_h]$ and $c_\Omega(\cdot;\cdot,\varphi_h)$, since $z_h^{n,k_j} \to z_h^n$ in the finite dimensional space V_h^{cf}, we also have that

$$0 \neq \mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^{n,k_j})[\varphi_h] \to \mathrm{D}\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1}; z_h^n)[\varphi_h] \neq 0$$

from some index $j_0\in\mathbb{N}$ on. Consequently also $c_\Omega(z_h^{n,k_j}; d_h^{n,k_j}, d_h^{n,k_j}) \not\to 0$ as $j\to\infty$, in contradiction to (3.34).

This concludes the proof. □

Remark 3.4. *This result is only required to guarantee that the numerical algorithm in (S1) is well-defined. For the existence proof of a weak solution only the existence of a critical point of the functional $\mathcal{G}_h^n(\cdot)$ is required, and this holds by coercivity and is independent from the numerical realization of (S1).*

3.3 Existence of discrete solutions

In this section we show that the *discrete* problems (3.11) and (3.12) defined in the staggered scheme (S1)–(S3) admit a solution. For shorter notation we set

$$\triangle\boldsymbol{\varepsilon}_h^n = \boldsymbol{\varepsilon}_h^n - \boldsymbol{\varepsilon}_h^{n-1} \quad\text{and}\quad \triangle z_h^n = z_h^n - z_h^{n-1} \text{ for } n = 1,\dots,N\,, \tag{3.35}$$

and define the projection

$$\Pi_h^{\mathrm{dg}}\colon \mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \to W_h^{\mathrm{dg}}\,,\quad \big(\Pi_h^{\mathrm{dg}}\boldsymbol{\Phi}, \boldsymbol{\Psi}_h\big)_\Omega = \big(\boldsymbol{\Phi}, \boldsymbol{\Psi}_h\big)_\Omega \quad \text{for all } \boldsymbol{\Phi}\in \mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\ \boldsymbol{\Psi}_h \in W_h^{\mathrm{dg}}\,. \tag{3.36}$$

Proposition 3.5 (Existence of discrete solutions). *Let the assumptions* (2.1)–(2.3) *as well as the setup for the discretization introduced in Sec. 3.1 hold true. Then the following statements hold true:*

1. *For every* $h > 0, n = 1, \dots, N$ *and* $N \in \mathbb{N}$ *fixed there exists a solution* $z_h^n \in V_h^{\mathrm{cf}}$ *of the nonlinear problem* (3.11) *in* (S1) *and a unique solution* $(\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ *of the linear problem* (3.12) *in* (S2).
2. *For every* $h > 0, n = 1, \dots, N$ *and* $N \in \mathbb{N}$ *the discrete stress tensor* $\boldsymbol{\sigma}_h^n \in W_h^{\mathrm{dg}}$ *is characterized by the identity*

$$\Big(\boldsymbol{\sigma}_h^n, \boldsymbol{\Psi}_h\Big)_\Omega = \Big(\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n) + \frac{1}{\triangle t_h^n}\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\triangle\boldsymbol{\varepsilon}_h^n), \boldsymbol{\Psi}_h\Big)_\Omega \quad \text{for all } \boldsymbol{\Psi}_h \in W_h^{\mathrm{dg}}\,. \tag{3.37}$$

3. *In case of homogeneous boundary data* $\mathbf{v}_\mathrm{D} = \mathbf{0}$, $\mathbf{g}_\mathrm{N} = \mathbf{0}$, *the discrete solution is bounded by the discrete energy-dissipation inequality*

$$\begin{aligned} &\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{pf}}(z_h^n) + \sum_{k=1}^n \left(\frac{2}{\triangle t_h^k}\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^k) + \frac{1}{\triangle t_h^k}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^k)\right) \\ &\le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \sum_{k=1}^n \triangle t_h^k \big(\mathbf{f}_h^k, \mathbf{v}_h^k\big)_\Omega\,. \end{aligned} \tag{3.38}$$

Proof. **To 1.: Existence of a solution** $z_h^n \in V_h^{\mathrm{cf}}$ **of** (3.11a)**:** By Lemma 3.3 the functional $\mathcal{G}_h^n(\cdot)$ is coercive and continuous, thus lower semicontinous, and therefore has a minimizer z_h^n in V_h^{cf}. The minimizer is a critical point of $\mathcal{G}_h^n$ and solves the nonlinear equation (3.11a) in (S1). On the other hand, a critical point and local minimizer for (S1) can be calculated iteratively as has been shown in Section 3.2.

Existence and uniqueness of a solution $(\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ **for** (3.12)**:** We show that the discrete linear system (3.12) has a unique solution. Since (3.12) is finite dimensional and linear, it suffices to prove that the kernel of the mapping

$$(\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0) \mapsto m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), \boldsymbol{\cdot}\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0), \boldsymbol{\cdot}\big) + \triangle t_h^n r_\Omega\big(z_h^n; (\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0), \boldsymbol{\cdot}\big)$$

related to the homogeneous problem is given by $\{(\mathbf{0}, \mathbf{0}, \mathbf{0}) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}\}$. It follows immediately that the dimension of the image is equal to the dimension of the domain, i.e., the existence of a solution is clear.

Thus, assume that $(\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ solves the homogeneous problem

$$m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h, \boldsymbol{\Phi}_h - \mathbb{D}(z_h^n)\boldsymbol{\Psi}_h)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0), (\mathbf{w}_h, \boldsymbol{\Phi}_h)\big) + \triangle t_h^n r_\Omega\big(z_h^n; (\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0), \boldsymbol{\Psi}_h\big) = 0 \tag{3.39}$$

for all $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$. We show in the following that necessarily $(\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

Choosing $(\mathbf{0}, \mathbf{0}, \boldsymbol{\Psi}_h)$ as a test function in (3.39) yields

$$\begin{aligned} 0 &= -\big(\boldsymbol{\Phi}_h^0, \mathbb{D}(z_h^n)\boldsymbol{\Psi}_h\big)_\Omega + \triangle t_h^n \big(\boldsymbol{\Psi}_h^0 - \mathbb{C}(z_h^n)\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h\big)_\Omega \\ &= -\big(\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\boldsymbol{\Phi}_h^0), \boldsymbol{\Psi}_h\big)_\Omega + \triangle t_h^n \big(\boldsymbol{\Psi}_h^0 - \Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\Phi}_h^0), \boldsymbol{\Psi}_h\big)_\Omega \qquad \text{for all } \boldsymbol{\Psi}_h \in W_h^{\mathrm{dg}}\,. \end{aligned} \tag{3.40}$$

This shows that

$$\boldsymbol{\Psi}_h^0 = \Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\Phi}_h^0) + (\triangle t_h^n)^{-1}\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\boldsymbol{\Phi}_h^0)\,. \tag{3.41}$$

Next, testing (3.39) with $(\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0, \mathbf{0})$ and exploiting the monotonicity (3.3) of a_h^{dg} as well as (3.41) and assumptions (2.2) on the tensor $\mathbb{C}$, results in

$$\begin{aligned} 0 &= m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0), (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0)\big) \\ &\ge m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0)\big) = \varrho_0 \left(\mathbf{w}_h^0, \mathbf{w}_h^0\right)_\Omega + \big(\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0\big)_\Omega \\ &= \varrho_0 \left(\mathbf{w}_h^0, \mathbf{w}_h^0\right)_\Omega + \big(\boldsymbol{\Phi}_h^0, \mathbb{C}(z_h^n)\boldsymbol{\Phi}_h^0\big)_\Omega + (\triangle t_h^n)^{-1}\big(\boldsymbol{\Phi}_h^0, \mathbb{D}(z_h^n)\boldsymbol{\Phi}_h^0\big)_\Omega \\ &\ge \varrho_0 \left\|\mathbf{w}_h^0\right\|_\Omega^2 + \big\|\mathbb{C}(z_h^n)^{1/2}\boldsymbol{\Phi}_h^0\big\|_\Omega^2 \ge \varrho_0 \left\|\mathbf{w}_h^0\right\|_\Omega^2 + g_* c_{\tilde{\mathbb{C}}}\big\|\boldsymbol{\Phi}_h^0\big\|_\Omega^2\,. \end{aligned}$$

Thanks to $\rho_0, g_* c_{\tilde{\mathbb{C}}} > 0$ this implies

$$\mathbf{w}_h^0 = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Phi}_h^0 = \mathbf{0}\,. \tag{3.42}$$

Inserting this into (3.40) yields

$$\mathbf{\Psi}_h^0 = \mathbf{0}\,, \tag{3.43}$$

so that we can indeed conclude that the solution of the homogeneous problem is $(\mathbf{0},\mathbf{0},\mathbf{0})$.

To 2. Characterization (3.37) **of the discrete stress tensor** $\boldsymbol{\sigma}_h^n$**:** To determine the form of the discrete stress $\boldsymbol{\sigma}_h^n$, we test (3.12) for the unique solution $\big(\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n\big)$ with a test function $(\mathbf{0},\mathbf{0},\mathbf{\Psi}_h)$. This yields for all $\mathbf{\Psi}_h\in W_h^{\mathrm{dg}}$

$$m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{0},-\mathbb{D}(z_h^n)\mathbf{\Psi}_h)\big)+\triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\mathbf{\Psi}_h\big)=m_\Omega\big((\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{0},-\mathbb{D}(z_h^n)\mathbf{\Psi}_h)\big)\,.$$

Rearranging terms and exploiting definitions (2.10) gives for all $\mathbf{\Psi}_h\in W_h^{\mathrm{dg}}$

$$\begin{aligned}0&=\triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\mathbf{\Psi}_h\big)-\big(\boldsymbol{\varepsilon}_h^n-\boldsymbol{\varepsilon}_h^{n-1},\mathbb{D}(z_h^n)\mathbf{\Psi}_h\big)_\Omega\\&=\triangle t_h^n\big(\boldsymbol{\sigma}_h^n-\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n,\mathbf{\Psi}_h\big)_\Omega-\big(\triangle\boldsymbol{\varepsilon}_h^n,\mathbb{D}(z_h^n)\mathbf{\Psi}_h\big)_\Omega\\&=\triangle t_h^n\big(\boldsymbol{\sigma}_h^n-\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n),\mathbf{\Psi}_h\big)_\Omega-\big(\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\triangle\boldsymbol{\varepsilon}_h^n),\mathbf{\Psi}_h\big)_\Omega\,.\end{aligned}$$

From this we infer that

$$\boldsymbol{\sigma}_h^n=\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n)+\frac{1}{\triangle t_h^n}\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\triangle\boldsymbol{\varepsilon}_h^n)\,. \tag{3.44}$$

To 3. Energy-dissipation inequality (3.38)**:** The discrete energy-dissipation estimate (3.38) is deduced in the following by testing (3.12) in (S2) with the triple $\big(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n,(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n\big)$. This results in the following expression

$$\begin{aligned}&m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n-\mathbb{D}(z_h^n)(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n)\big)+\triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n\big)\\&\qquad+\triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\\&=m_\Omega\big((\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n-\mathbb{D}(z_h^n)(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n)\big)+\triangle t_h^n\ell_h^{\mathrm{dg}}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\end{aligned} \tag{3.45}$$

and the expressions herein have to be further rearranged using the definitions (2.10) of the quadratic forms and of the energies and dissipation potentials (2.28).

For this, we insert the characterization (3.44) of $\boldsymbol{\sigma}_h^n$ into (3.45) and find

$$\begin{aligned}&m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n))\big)+r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big)\\&\qquad+\triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\\&=m_\Omega\big((\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{v}_h^n,\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n))\big)+\triangle t_h^n\ell_h^{\mathrm{dg}}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\,.\end{aligned} \tag{3.46}$$

Herein, we further observe with the aid of (2.10) and (2.28) that

$$\begin{aligned}m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n))\big)&=\varrho_0\big(\mathbf{v}_h^n,\mathbf{v}_h^n\big)_\Omega+\big(\boldsymbol{\varepsilon}_h^n,\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n)\big)_\Omega\\&=\varrho_0\big(\mathbf{v}_h^n,\mathbf{v}_h^n\big)_\Omega+\big(\boldsymbol{\varepsilon}_h^n,\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n\big)_\Omega\\&=2\,\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+2\,\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)\,,\end{aligned} \tag{3.47}$$

and similarly, that

$$\begin{aligned}m_\Omega\big(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{v}_h^n,\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n))\big)&=\varrho_0\big(\mathbf{v}_h^{n-1},\mathbf{v}_h^n\big)_\Omega+\big(\boldsymbol{\varepsilon}_h^{n-1},\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n\big)_\Omega\\&\le\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^{n-1})+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1})\,,\end{aligned} \tag{3.48}$$

where we have used Young's inequality. Furthermore, again in view of (2.10) and (2.28), we find for r_Ω in (3.46) that

$$\begin{aligned}r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big)&=\big(\boldsymbol{\sigma}_h^n-\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n,\triangle\boldsymbol{\varepsilon}_h^n\big)_\Omega=\big(\boldsymbol{\sigma}_h^n-\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big)_\Omega\\&=\Big(\frac{1}{\triangle t_h^n}\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\triangle\boldsymbol{\varepsilon}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\Big)_\Omega=\Big(\frac{1}{\triangle t_h^n}\mathbb{D}(z_h^n)\triangle\boldsymbol{\varepsilon}_h^n,\triangle\boldsymbol{\varepsilon}_h^n\Big)_\Omega\\&=\frac{2}{\triangle t_h^n}\,\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)\,.\end{aligned} \tag{3.49}$$

Here we have also exploited the definition (3.36) of the projection operator Π_h^{dg}.

Inserting (3.47)–(3.49) into (3.46) and rearranging terms results in the estimate

$$\begin{aligned}\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\frac{2}{\triangle t_h^n}\,\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)+\triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\\ \leq \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^{n-1})+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1})+\triangle t_h^n\ell_h^{\mathrm{dg}}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\,.\end{aligned} \tag{3.50}$$

In view of (3.2) and the properties of the DG-approximation stated in Prop. 3.2 we further have that

$$\ell_h^{\mathrm{dg}}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)=(\mathbf{f}_h^n,\mathbf{v}_h^n)_\Omega \tag{3.51a}$$

and

$$a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big)\geq 0 \tag{3.51b}$$

by the monotonicity (3.3) of the DG-operator.

With the aid of the findings (3.51) relation (3.50) can be further estimated by

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\frac{2}{\triangle t_h^n}\,\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)\leq \text{(3.50)}=\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^{n-1})+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1})+\triangle t_h^n(\mathbf{f}_h^n,\mathbf{v}_h^n)_\Omega\,. \tag{3.52}$$

In order to complete the energy-dissipation estimate (3.38) it remains to deduce from the discrete phase-field equation (3.11) in (S1) an estimate corresponding to (3.52). For this, we exploit that a discrete solution $z_h^n\in V_h^{\mathrm{cf}}$ in particular satisfies (3.11b), i.e., we assume

$$\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^n)\leq\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^{n-1})\,,$$

as we have verified in Section 3.2. Thanks to this property we obtain

$$\begin{aligned}\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1})+\mathcal{E}^{\mathrm{pf}}(z_h^n)&=\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^n)\\ &\leq\mathcal{G}_h^n(\boldsymbol{\varepsilon}_h^{n-1},z_h^{n-1};z_h^{n-1})=\mathcal{E}^{\mathrm{el}}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1})+\mathcal{E}^{\mathrm{pf}}(z_h^{n-1})\,,\end{aligned}$$

and rearranging terms gives

$$\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^n)+\mathcal{E}^{\mathrm{pf}}(z_h^n)\leq\mathcal{E}^{\mathrm{el}}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1})-\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1})+\mathcal{E}^{\mathrm{pf}}(z_h^{n-1})\,. \tag{3.53}$$

Putting (3.53) together with (3.52) ultimately yields

$$\begin{aligned}\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\mathcal{E}^{\mathrm{pf}}(z_h^n)+\frac{2}{\triangle t_h^n}\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)+\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^n)\\ \leq\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^{n-1})+\mathcal{E}^{\mathrm{el}}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1})+\mathcal{E}^{\mathrm{pf}}(z_h^{n-1})+\triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega\,.\end{aligned} \tag{3.54}$$

Summing (3.54) over n results in telescopic terms and thus proves the assertion. □

3.4 Discrete version of the weak formulation (2.12)

Let $\{t_h^0=0<t_h^1<\ldots<t_h^{n-1}<t_h^n<\ldots<t_h^{N_h}=T\}$ with $N_h\in\mathbb{N}$ be a discretization of the time interval $[0,T]$. For all $n\in\{0,1,\ldots,N_h\}$ we now define piecewise constant interpolants in time

$$\begin{aligned}(z_h,\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\boldsymbol{\sigma}_h)\in\mathrm{L}^2(0,T;V_h^{\mathrm{cf}}\times V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}})\quad\text{by}\\ (z_h,\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\boldsymbol{\sigma}_h)(t):=(z_h^n,\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n)\quad\text{for all } t\in(t_h^{n-1},t_h^n]\,,\end{aligned} \tag{3.55a}$$

piecewise constant interpolants evaluating in a previous time step

$$
\begin{aligned}
(\underline{z}_h, \underline{\mathbf{v}}_h, \underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{\sigma}}_h) &\in \mathrm{L}^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \quad \text{by} \\
(\underline{z}_h, \underline{\mathbf{v}}_h, \underline{\boldsymbol{\varepsilon}}_h, \underline{\boldsymbol{\sigma}}_h)(t) &:= (z_h^{n-1}, \mathbf{v}_h^{n-1}, \boldsymbol{\varepsilon}_h^{n-1}, \boldsymbol{\sigma}_h^{n-1}) \quad \text{for all } t \in (t_h^{n-1}, t_h^n]\,,
\end{aligned}
\tag{3.55b}
$$

and the piecewise linear interpolants by

$$
\begin{aligned}
(\dot{z}_h, \dot{\boldsymbol{\varepsilon}}_h) &\in \mathrm{L}^2(0,T; V_h^{\mathrm{cf}} \times W_h^{\mathrm{dg}}) \quad \text{by} \\
(\dot{z}_h, \dot{\boldsymbol{\varepsilon}}_h)(t) &:= \frac{1}{\triangle t_h^n} (\triangle z_h^n, \triangle \boldsymbol{\varepsilon}_h^n) \quad \text{for all } t \in (t_h^{n-1}, t_h^n).
\end{aligned}
\tag{3.55c}
$$

These interpolants in time can be used both for the solutions and for the test functions of the discrete scheme (3.11)–(3.12) and also for the given data. Based on this we conclude the following result:

Proposition 3.6 (Properties of the interpolants of the discrete solutions)**.** *Let the assumptions of Prop. 3.5 be satisfied. Keep $h > 0$ and $N_h \in \mathbb{N}$ fixed. Let $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h) \in L^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ be the interpolants of the discrete solutions obtained by* (3.11)–(3.12) *of the staggered time-discrete scheme. Then the following statements hold true:*

1. *The quadruple $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h) \in L^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ satisfies*
   - *the discrete phase field evolution*

   $$
   \big(\beta_{\mathrm{r}} \dot{z}_h, \varphi_h\big)_Q + \big(Y_h{}'(\dot{z}_h), \varphi_h\big)_Q + \frac{1}{2}\big(\mathbb{C}'(z_h)\underline{\boldsymbol{\varepsilon}}_h : \underline{\boldsymbol{\varepsilon}}_h, \varphi_h\big)_Q + b_Q(z_h, \varphi_h) = 0 \tag{3.56a}
   $$

   for all $(\varphi_h^n)_{n=1}^{N_h} \subset V_h^{\mathrm{cf}},\ \varphi_h(s) = \varphi_h^n$ for $s \in (t_h^{n-1}, t_h^n]$.
   - *the discrete weak momentum balance*

   $$
   \begin{aligned}
   &m_Q\big((\dot{\mathbf{v}}_h, \dot{\boldsymbol{\varepsilon}}_h), (\mathbf{w}_h, \boldsymbol{\Phi}_h - \mathbb{D}(z_h)\boldsymbol{\Psi}_h)\big) + r_Q\big(z_h; (\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h), \boldsymbol{\Psi}_h\big) + \int_0^T a_h^{\mathrm{dg}}\big(z_h; (\mathbf{v}_h, \boldsymbol{\sigma}_h), (\mathbf{w}_h, \boldsymbol{\Phi}_h)\big)\,\mathrm{d}t \\
   &= \sum_{n=1}^{N_h} \ell_h^{\mathrm{dg}}\big(t_h^n, z_h^n; (\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n)\big)
   \end{aligned}
   \tag{3.56b}
   $$

   for all $(\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n)_{n=1}^{N_h} \subset V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}},\ (\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h)(s) = (\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n)$ if $s \in (t_h^{n-1}, t_h^n]$.
2. *The discrete stress tensor $\boldsymbol{\sigma}_h \in L^2(0,T; W_h^{\mathrm{dg}})$ is characterized by the identity*

   $$
   \Big(\boldsymbol{\sigma}_h, \boldsymbol{\Psi}_h\Big)_Q = \Big(\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h)\boldsymbol{\varepsilon}_h) + \Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h)\dot{\boldsymbol{\varepsilon}}_h), \boldsymbol{\Psi}_h\Big)_Q \quad \text{for all } (\boldsymbol{\Psi}_h^n)_{n=1}^{N_h} \subset W_h^{\mathrm{dg}},\ \boldsymbol{\Psi}_h(s) = \boldsymbol{\Psi}_h^n \ \text{if } s \in (t_h^{n-1}, t_h^n]\,. \tag{3.57}
   $$

3. *In case of homogeneous boundary data $\mathbf{v}_{\mathrm{D}} = \mathbf{0}$, $\mathbf{g}_{\mathrm{N}} = \mathbf{0}$, the discrete solution is bounded by the discrete energy-dissipation inequality for all $t \in (0,T]$*

   $$
   \begin{aligned}
   &\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h(t)) + \mathcal{E}^{\mathrm{el}}(z_h(t), \boldsymbol{\varepsilon}_h(t)) + \mathcal{E}^{\mathrm{pf}}(z_h(t)) + \int_0^t \Big(2\mathcal{R}^{\mathrm{vis}}(\dot{\boldsymbol{\varepsilon}}_h) + \mathcal{R}_h^{\mathrm{pf}}(\dot{z}_h)\Big)\,\mathrm{d}s \\
   &\le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \int_0^t \big(\mathbf{f}_h, \mathbf{v}_h\big)_\Omega\,.
   \end{aligned}
   \tag{3.58}
   $$

4. *The quadruple $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h) \in L^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ with $\boldsymbol{\sigma}_h = \Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h)\boldsymbol{\varepsilon}_h) + \Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h)\dot{\boldsymbol{\varepsilon}}_h)$ satisfies the following uniform a priori bounds*

   $$
   \begin{aligned}
   &\frac{\varrho_0}{4}\big\|\mathbf{v}_h\big\|_Q^2 + \frac{1}{2} g_* c_{\tilde{\mathbb{C}}} \big\|\boldsymbol{\varepsilon}_h\big\|_Q^2 + \frac{G_{\mathrm{c}}}{2}\Big(\big\|1 - z_h\big\|_Q^2 + l_c^2 \big\|\nabla z_h\big\|_Q^2\Big) \\
   &\quad + g_* c_{\tilde{\mathbb{D}}} \big\|\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \frac{\beta_{\mathrm{r}}}{2}\big\|\dot{z}_h\big\|_Q^2 + \frac{\delta_h}{2}\big\|\max\{\dot{z}_h, 0\}\big\|_Q^2 \\
   &\le \max\{T,1\}\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0)\Big) + \frac{\max\{T,1\}^2}{\varrho_0}\big\|\mathbf{f}\big\|_Q^2\,,
   \end{aligned}
   \tag{3.59}
   $$

   hence the sequence $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \dot{z}_h, c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}_h)_h$ is uniformly bounded in $L^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times V_h^{\mathrm{cf}} \times W_h^{\mathrm{dg}})$.

Proof. Statements 1.–3. are direct translations of Prop. 3.5, 1.–3., when using the definition of the interpolants (3.55) in time of the solutions obtained at every time step by the staggered scheme (3.11)–(3.12).

For statement 4., we observe for the total energy

$$\begin{aligned}&\frac{\varrho_0}{2}\big\|\mathbf{v}_h\big\|_Q^2+\frac{1}{2}\big\|\mathbb{C}(z_h)^{1/2}\boldsymbol{\varepsilon}_h\big\|_Q^2+\frac{G_\mathrm{c}}{2}\Big(\big\|1-z_h\big\|_Q^2+l_\mathsf{c}^2\,\big\|\nabla z_h\big\|_Q^2\Big)\\&=\sum_{n=1}^N\triangle t_h^n\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\mathcal{E}^{\mathrm{pf}}(z_h^n)\Big)\end{aligned}$$

and for the dissipation

$$\begin{aligned}&\big\|\mathbb{D}(z_h)^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2+\frac{\beta_\mathrm{r}}{2}\big\|\dot z_h\big\|_Q^2+\frac{\delta_h}{2}\big\|\max\{\dot z_h,0\}\big\|_Q^2\\&=\sum_{n=1}^N\left(\big\|\mathbb{D}(z_h)^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2+\frac{\beta_\mathrm{r}}{2}\big\|\dot z_h\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2+\frac{\delta_h}{2}\big\|\max\{\dot z_h,0\}\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2\right)\\&=\sum_{n=1}^N\frac{1}{\triangle t_h^n}\left(\big\|\mathbb{D}(z_h^n)^{1/2}\triangle\boldsymbol{\varepsilon}_h^n\big\|_\Omega^2+\frac{\beta_\mathrm{r}}{2}\big\|\triangle z_h^n\big\|_\Omega^2+\frac{\delta_h}{2}\big\|\max\{\triangle z_h^n,0\}\big\|_\Omega^2\right)\\&=\sum_{n=1}^N\left(\frac{2}{\triangle t_h^n}\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)+\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^n)\right)\end{aligned}$$

Using (3.13), we get $\sum_{n=1}^N\triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega=\sum_{n=1}^N\big(\mathbf{f},\mathbf{v}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega}=\big(\mathbf{f},\mathbf{v}_h\big)_Q$.

Together, the estimate (3.38) for the energy ($n=1,\dots,N$) and for the dissipation ($n=N$) yields the assertion by

$$\begin{aligned}&\frac{\varrho_0}{2}\big\|\mathbf{v}_h\big\|_Q^2+\frac{1}{2}\big\|\mathbb{C}(z_h)^{1/2}\boldsymbol{\varepsilon}_h\big\|_Q^2+\frac{G_\mathrm{c}}{2}\Big(\big\|1-z_h\big\|_Q^2+l_\mathsf{c}^2\,\big\|\nabla z_h\big\|_Q^2\Big)\\&\qquad+\big\|\mathbb{D}(z_h)^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2+\frac{\beta_\mathrm{r}}{2}\big\|\dot z_h\big\|_Q^2+\frac{\delta_h}{2}\big\|\max\{\dot z_h,0\}\big\|_Q^2\\&=\sum_{n=1}^N\triangle t_h^n\left(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n)+\mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n)+\mathcal{E}^{\mathrm{pf}}(z_h^n)\right)+\sum_{n=1}^N\left(\frac{2}{\triangle t_n^h}\mathcal{R}^{\mathrm{vis}}(\triangle\boldsymbol{\varepsilon}_h^n)+\frac{1}{\triangle t_k}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^n)\right)\\&\le\max\Big\{\sum_{n=1}^N\triangle t_h^n,1\Big\}\left(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0)+\mathcal{E}^{\mathrm{el}}(z_h^0,\boldsymbol{\varepsilon}_h^0)+\mathcal{E}^{\mathrm{pf}}(z_h^0)+\big(\mathbf{f},\mathbf{v}_h\big)_Q\right)\\&\le\max\{T,1\}\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0)+\mathcal{E}^{\mathrm{el}}(z_h^0,\boldsymbol{\varepsilon}_h^0)+\mathcal{E}^{\mathrm{pf}}(z_h^0)\Big)+\frac{\max\{T,1\}^2}{\varrho_0}\big\|\mathbf{f}\big\|_Q^2+\frac{\varrho_0}{4}\big\|\mathbf{v}_h\big\|_Q^2\,.\end{aligned}$$

□

4 Limit passage in the discrete systems (3.56) and existence of weak solutions for system (2.12)

We consider a shape-regular family $\big(\Omega_h\big)_{h\in\mathcal{H}}$ of meshes with $0 \in \overline{\mathcal{H}}$, e.g., obtained by uniform refinement of a coarse mesh. For simplicity, we consider uniform time steps

$$\triangle t_h^n = \triangle t_h = T/N_h \quad \text{with } N_h \in \mathbb{N} \quad \text{such that} \quad c_{\mathsf{ws}}\triangle t_h \approx h \tag{4.1a}$$

with respect to a reference wave speed $c_{\mathsf{ws}} > 0$. We set

$$t_h^n = n\triangle t_h \quad \text{and} \quad t_h^{n-1/2} = \frac{1}{2}(t_h^{n-1} + t_h^n). \tag{4.1b}$$

By Prop. 3.6 the discrete solutions $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \dot z_h, \mathbb{D}(z_h)^{\frac12}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}}$ with $\mathbb{D}(z_h) = g_{\mathbb{D}}(z_h)\tilde{\mathbb{D}}$ are uniformly bounded so that we obtain the compactness result stated below in Prop. 4.1. We point out that subsequently we write $c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}_h$ in order to indicate that we also include the case that $\mathbb{D}(z_h)$ is not positive definite, so that $c_{\tilde{\mathbb{D}}} = 0$ and then

$$\mathbb{D}(z_h)\dot{\boldsymbol{\varepsilon}}_h = c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}_h = \mathbf{0} = \mathbb{D}(z)\dot{\boldsymbol{\varepsilon}} = c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}\,, \tag{4.2}$$

see (4.5e) below. While convergence statements (4.5a)–(4.5e) follow from standard compactness arguments, convergence result (4.5f) is a consequence of (a discrete version of) the Aubin-Lions Lemma, cf. e.g. [16, Lem. 7.7], which guarantees that the embedding

$$\mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big) \cap \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \longrightarrow \mathrm{L}^2(Q)$$

is compact. This yields strong convergence (4.5f) of the discrete phase field approximations, which, in turn facilitates the convergence and identifaction (4.5g) of the stress tensor. Let us also point out that (4.6) states the unidirectionality of the phase-field evolution in a weaker way as the (2.12d). The improved result (2.12d) is obtained in Theorem 4.2.

Proposition 4.1. *Let the assumptions of Prop. 3.5 be satisfied. For all $h \in$ and $N_h \in \mathbb{N}$ let $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h) \in L^2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ be the interpolants of the discrete solutions obtained by* (3.11)–(3.12) *of the staggered time-discrete scheme with $\boldsymbol{\sigma}_h = (\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h)\boldsymbol{\varepsilon}_h + \Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h)\dot{\boldsymbol{\varepsilon}}_h))$. Assume that the discrete initial data converge strongly, i.e., that*

$$(z_h^0, \mathbf{v}_h^0) \to (z_0, \mathbf{v}_0) \quad \textit{in } H^1(Q) \times L^2(Q;\mathbb{R}^d) \tag{4.3}$$

Then there exists a (not relabelled) subsequence $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ with $\mathcal{H}_0 \subset \mathcal{H}$ and $0 \in \overline{\mathcal{H}_0}$ and a weak limit

$$(z, \mathbf{v}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \times \mathrm{L}^2(Q;\mathbb{R}^d) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \tag{4.4}$$

such that the following convergence statements hold true:

$$z_h \rightharpoonup z \qquad \textit{in } \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)\,, \tag{4.5a}$$

$$\mathbf{v}_h \rightharpoonup \mathbf{v} \qquad \textit{in } \mathrm{L}^2(Q;\mathbb{R}^d)\,, \tag{4.5b}$$

$$\boldsymbol{\varepsilon}_h \rightharpoonup \boldsymbol{\varepsilon} \qquad \textit{in } \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,, \tag{4.5c}$$

$$\dot z_h \rightharpoonup \dot z \qquad \textit{in } \mathrm{L}^2(Q;\mathbb{R}^d)\,, \tag{4.5d}$$

$$c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}_h \rightharpoonup c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}} \qquad \textit{in } \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,, \tag{4.5e}$$

$$z_h \to z \qquad \textit{strongly in } \mathrm{L}^2(Q;\mathbb{R}^d)\,, \tag{4.5f}$$

$$\boldsymbol{\sigma}_h \rightharpoonup \boldsymbol{\sigma} = \mathbb{C}(z)\boldsymbol{\varepsilon} + \mathbb{D}(z)\dot{\boldsymbol{\varepsilon}} \qquad \textit{in } \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,. \tag{4.5g}$$

Moreover, the following statements hold true:

1. *For the limit z the weak derivative $\partial_t z$ exists, so that $\partial_t z = \dot z$ from* (4.5d)*, and*

$$z \in \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big) \quad \textit{with} \quad z(0) = z_0 \quad \textit{and} \quad \partial_t z = \dot z \le 0 \textit{ a.e. in } Q\,, \tag{4.6}$$

i.e., the initial datum is attained and a weaker version of the unidirectionality (2.12d) *of the phase-field evolution is ensured.*

2. *Assume in addition that* $\tilde{\mathbb{D}}$ *is positive definite, i.e.* $c_{\tilde{\mathbb{D}}} > 0$ *in* (2.2d) *and that the initial datum is well prepared, i.e.*

$$\boldsymbol{\varepsilon}_h^0 \to \boldsymbol{\varepsilon}_0 \quad \textit{in } \; L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \tag{4.7}$$

also the weak derivatives $\partial_t \boldsymbol{\varepsilon}$ *and* $\mathrm{sym}(\mathrm{D}\mathbf{v})$ *exist, and*

$$\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0 \quad \textit{and} \quad \partial_t \boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}} = \mathrm{sym}(\mathrm{D}\mathbf{v})\,. \tag{4.8}$$

Proof. **To convergence statements** (4.5a)–(4.5e)**:** By (3.59) in Prop. 3.6, the discrete solutions $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \dot{z}_h, c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}}$ are uniformly bounded by

$$\big\|\mathbf{v}_h\big\|_Q^2 + \big\|\boldsymbol{\varepsilon}_h\big\|_Q^2 + \Big(\big\|1-z_h\big\|_Q^2 + \big\|\nabla z_h\big\|_Q^2\Big) + \big\|\sqrt{c_{\tilde{\mathbb{D}}}}\,\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \big\|\dot{z}_h\big\|_Q^2 + \delta_h\big\|\max\{\dot{z}_h,0\}\big\|_Q^2 \le C$$

with a constant $C > 0$ independent of $h \in \mathcal{H}$ but depending on the initial data $\mathbf{v}_0,\ z_0\ \boldsymbol{\varepsilon}_0$, the load $\mathbf{f}$, the lower bound $g(z_h) \ge g_* > 0$, and the material parameters. Thus, by compactness, there exist subsequences $(z_h)_{h\in\mathrm{H}_0} \subset \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)$ and $(\mathbf{v}_h, \boldsymbol{\varepsilon}_h, \dot{z}_h, \sqrt{c_{\tilde{\mathbb{D}}}}\,\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ that converge weakly in L^2 to a limit $(z, \mathbf{v}, \boldsymbol{\varepsilon}, \dot{z}, \sqrt{c_{\tilde{\mathbb{D}}}}\,\dot{\boldsymbol{\varepsilon}})$, which proves convergence statements (4.5a)–(4.5e).

To statement 1.: Since $\delta_h \to \infty$ for $h \to 0$, we obtain for the limit

$$\big\|\max\{\dot{z},0\}\big\|_Q \le \lim_{h\in\mathcal{H}_0} \big\|\max\{\dot{z}_h,0\}\big\|_Q \le \lim_{h\in\mathcal{H}_0} \frac{C}{\delta_h} = 0 \quad \text{and thus} \quad \dot{z} \le 0 \text{ a.e. in } Q\,.$$

Now we show that the weak time-derivative of the limit z exists and that it can be identified with $\dot{z}$. Thus, checking the definition of the weak derivative for smooth test functions $\phi \in \mathrm{C}^1(Q)$ with $\phi(T) = 0$ gives

$$\begin{aligned}
\big(z_h, \partial_t\phi\big)_Q &= \sum_{n=1}^{N_h} \big(z_h^n, \partial_t\phi\big)_{(t_h^{n-1},t_h^n)\times\Omega} \\
&= \sum_{n=1}^{N_h} \big(z_h^n, \phi(t_h^n) - \phi(t_h^{n-1})\big)_\Omega \\
&= -\big(z_h^0, \phi(0)\big)_\Omega + \sum_{n=1}^{N_h} \big(z_h^{n-1} - z_h^n, \phi(t_h^{n-1})\big)_\Omega \\
&= -\big(z_h^0, \phi(0)\big)_\Omega - \sum_{n=1}^{N_h} \big(\triangle z_h^n, \phi(t_h^{n-1})\big)_\Omega \\
&= -\big(z_h^0, \phi(0)\big)_\Omega - \sum_{n=1}^{N_h} \big(\dot{z}_h, \phi(t_h^{n-1})\big)_{(t_h^{n-1},t_h^n)\times\Omega}\,.
\end{aligned}$$

By the well-preparedness of the initial datum (4.3) and the fact that $\|\phi(t_h^{n-1}) - \phi(t)\|_\Omega \to 0$ as $t_h^{n-1} \to t$ by the smoothness of ϕ, we conclude

$$\begin{aligned}
\big(z_0, \phi(0)\big)_\Omega + \big(z, \partial_t\phi\big)_Q &= \lim_{h\in\mathcal{H}_0} \Big(\big(z_h^0, \phi(0)\big)_\Omega + \big(z_h, \partial_t\phi\big)_Q\Big) \\
&= -\lim_{h\in\mathcal{H}_0} \sum_{n=1}^{N_h} \big(\dot{z}_h, \phi(t_h^{n-1})\big)_{(t_h^{n-1},t_h^n)\times\Omega} \\
&= -\lim_{h\in\mathcal{H}_0} \sum_{n=1}^{N_h} \big(\dot{z}_h, \phi\big)_{(t_h^{n-1},t_h^n)\times\Omega} \\
&= -\big(\dot{z}, \phi\big)_Q\,.
\end{aligned} \tag{4.9}$$

Testing (4.9) with $\phi \in \mathrm{C}^1_{\mathsf{c}}(Q)$ shows that the weak derivative in time of z exists and identifies $\partial_t z = \dot{z}$. This implies that $z \in \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)$. By the regularity of Bochner spaces we thus also have $z \in \mathrm{C}^0\big([0,T];\mathrm{L}^2(\Omega)\big)$. Hence, testing (4.9) with $\phi(0) \neq 0$ and $\phi(T) = 0$ shows that the initial datum $z(0) = z_0$ is attained. This finishes the proof of statement 1..

To statement 2.: If, in addition, $\tilde{\mathbb{D}}$ is positive definite, i.e., $c_{\tilde{\mathbb{D}}} > 0$ in (2.2d), also the subsequence $(\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ itself weakly converges in L^2 to a limit $\dot{\boldsymbol{\varepsilon}}$, and, thanks to the well-preparedness (4.7) of the initial datum, one can show with the same arguments as above that the weak derivative in time of $\boldsymbol{\varepsilon}$ exists, that the inital datum is attained, and that one can identify

$$\partial_t \boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}}\,.$$

It remains to verify for the limit velocity that $\mathrm{sym}(\nabla \mathbf{v})$ exists and that one can identify $\mathrm{sym}(\nabla \mathbf{v}) = \dot{\boldsymbol{\varepsilon}}$.

To do so, we select a smooth test functions $\boldsymbol{\Phi} \in \mathrm{C}^1_{\mathsf{c}}(Q;\mathbb{R}^{d\times d}_{\mathsf{sym}})$, and let $\boldsymbol{\Phi}^n_h \in W^{\mathrm{dg}}_h \cap \mathrm{H}^1_0(\Omega;\mathbb{R}^{d\times d}_{\mathsf{sym}})$ be an approximation of $\boldsymbol{\Phi}$ that is piecewise constant in the time intervals (t^{n-1}_h, t^n_h) with $\lim\limits_{h\longrightarrow 0}\Big(\big\|\boldsymbol{\Phi}^n_h - \boldsymbol{\Phi}\big\|_Q + \big\|\operatorname{div}(\boldsymbol{\Phi}^n_h - \boldsymbol{\Phi})\big\|_Q\Big) = 0$.

Then, testing (3.12) in (S2) with $(\mathbf{0}, \boldsymbol{\Phi}^n_h, \mathbf{0})$ yields

$$m_\Omega\big((\mathbf{v}^n_h, \boldsymbol{\varepsilon}^n_h), (\mathbf{0}, \boldsymbol{\Phi}^n_h)\big) + \triangle t^n_h a^{\mathrm{dg}}_h\big(z^n_h; (\mathbf{v}^n_h, \boldsymbol{\sigma}^n_h), (\mathbf{0}, \boldsymbol{\Phi}^n_h)\big) = m_\Omega\big(\mathbf{v}^{n-1}_h, \boldsymbol{\varepsilon}^{n-1}_h), (\mathbf{0}, \boldsymbol{\Phi}^n_h)\big)\,,$$

and further using the definitions (3.9) and (3.2) of the discrete bilinear forms results in

$$\begin{aligned}
&\frac{1}{\triangle t_h}\big(\triangle \boldsymbol{\varepsilon}^n_h, \boldsymbol{\Phi}^n_h\big)_\Omega + \big(\mathbf{v}_h, \operatorname{div} \boldsymbol{\Phi}^n_h\big)_{\Omega_h} \\
&= \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big), \mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K\Big)_f \\
&\qquad\qquad + \Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big), \mathbf{n}_K\times\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K\Big)_f\Bigg)\,.
\end{aligned}$$

Since the approximations $\boldsymbol{\Phi}^n_h \in W^{\mathrm{dg}}_h \cap \mathrm{H}^1_0(\Omega;\mathbb{R}^{d\times d}_{\mathsf{sym}})$ satisfy $[\boldsymbol{\Phi}^n_h]_{K,f} = \mathbf{0}$, we obtain

$$\begin{aligned}
\big(\dot{\boldsymbol{\varepsilon}}, \boldsymbol{\Phi}\big)_Q + \big(\mathbf{v}, \operatorname{div} \boldsymbol{\Phi}\big)_Q &= \lim_{h\in\mathcal{H}_0}\Big(\big(\dot{\boldsymbol{\varepsilon}}_h, \boldsymbol{\Phi}_h\big)_Q + \big(\mathbf{v}_h, \operatorname{div} \boldsymbol{\Phi}_h\big)_Q\Big) \\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\Big(\big(\triangle \boldsymbol{\varepsilon}^n_h, \boldsymbol{\Phi}^n_h\big)_\Omega + \triangle t_h\big(\mathbf{v}_h, \operatorname{div} \boldsymbol{\Phi}^n_h\big)_{\Omega_h}\Big) \\
&= 0\,,
\end{aligned} \tag{4.10}$$

so that, in case of positive viscosity, the weak symmetric gradient in space exists for $\mathbf{v}$ and can be identified with $\dot{\boldsymbol{\varepsilon}} = \mathrm{sym}(\mathrm{D}\mathbf{v})$. This finishes the proof of statement 2..

To convergence statement (4.5f)**:** The proof of this convergence result is based on the Aubin-Lions Lemma, cf. e.g. [16, Lem. 7.7]. Yet, since z_h is discontinuous in time, the Aubin-Lions Lemma cannot be applied directly. Thus we define

$$\hat{z}_h(t) = z^0 + \int_0^t \dot{z}_h(s)\,\mathrm{d}s \in V^{\mathrm{cf}}_h \qquad \text{for } t \in [0,T]\,,$$

so that $\hat{z}_h \in \mathrm{H}^1(0,T;V^{\mathrm{cf}}_h)$ and $\partial_t \hat{z}_h = \dot{z}_h$; from $\dot{z}^n_h = \frac{1}{\triangle t^n_h}(z^n_h - z^{n-1}_h)$ we get $\hat{z}_h(t^n_h) = z^n_h$ for $n = 0,\ldots,N_h$, and using uniform time step sizes $\triangle t^n_h = \triangle t_h$ we obtain

$$\begin{aligned}
\|z_h - \hat{z}_h\|^2_Q &= \sum_{n=1}^{N_h}\int_{t^{n-1}_h}^{t^n_h}\Big\|z^n_h - z^{n-1}_h - \frac{t - t^{n-1}_h}{\triangle t_h}(z^n_h - z^{n-1}_h)\Big\|^2_\Omega\,\mathrm{d}t = \sum_{n=1}^{N_h}\int_{t^{n-1}_h}^{t^n_h}\frac{(t^n_h - t)^2}{(\triangle t_h)^2}\big\|z^n_h - z^{n-1}_h\big\|^2_\Omega\,\mathrm{d}t \\
&= \sum_{n=1}^{N_h}\frac{\triangle t_h}{3}\|z^{n-1}_h - z^n_h\|^2_\Omega = \sum_{n=1}^{N_h}\frac{(\triangle t_h)^3}{3}\|\dot{z}^n_h\|^2_\Omega = \frac{(\triangle t_h)^2}{3}\|\dot{z}_h\|^2_Q\,.
\end{aligned} \tag{4.11}$$

Since $(z_h)_{h\in\mathcal{H}_0}$ converges weakly to z in $L^2(Q)$ by (4.5a) and since $(\dot z_h)_{h\in\mathcal{H}_0}$ is uniformly bounded in $\mathrm{L}^2(Q)$ by (3.59), also $(\hat z_h)_{h\in\mathcal{H}_0}$ converges weakly to z in $\mathrm{L}^2(Q)$. Then, we obtain

$$\begin{aligned}
0 &= \lim_{h\in\mathcal{H}_0}\big(\nabla z-\nabla z_h,\varphi\big)_Q\\
&= -\lim_{h\in\mathcal{H}_0}\big(z-z_h,\operatorname{div}\varphi\big)_Q\\
&= -\lim_{h\in\mathcal{H}_0}\big(z-\hat z_h,\operatorname{div}\varphi\big)_Q\\
&= \lim_{h\in\mathcal{H}_0}\big(\nabla z-\nabla \hat z_h,\varphi\big)_Q \qquad \text{for any } \varphi\in\mathrm{C}^1_{\mathrm{c}}(Q)\,.
\end{aligned}$$

This implies that $(\nabla\hat z_h)_{h\in\mathcal{H}_0}\rightharpoonup\nabla z$ in $L^2(Q)$. Alltogether, this gives

$$\hat z_h\rightharpoonup z \quad \text{in } \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)\,.$$

Since also $\dot z_h\rightharpoonup\partial_t z=\dot z$ in $\mathrm{L}^2(Q)$ by (4.5d) and statement 1., we conclude that

$$\hat z_h\rightharpoonup z \quad \text{in } \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big) \text{ and in } \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)\,.$$

Since, by the Aubin-Lions Lemma the embedding of $\mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)$ to $\mathrm{L}^2(Q)$ is compact, we obtain strong convergence of $(\hat z_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^2(Q)$. By (4.11) this also holds true for the sequence of piecewise constant interpolants $(z_h)_h$, which finishes the proof of statement (4.5f).

To convergence statement (4.5g)**:** The strong L^2-convergence of the interpolants $(z_h)_h$ obtained in (4.5f) also implies strong convergence of $\big(g_J(z_h)\big)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^2(Q)$ for $J\in\{\mathbb{C},\mathbb{D}\}$. In addition, we have $g_J(z_h)\in\mathrm{L}^\infty(Q)$ for all $h\in\mathcal{H}_0$. Together with the weak convergence of $(\boldsymbol{\varepsilon}_h,\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\text{sym}},\mathbb{R}^{d\times d}_{\text{sym}})$ this yields for all $\boldsymbol{\Psi}\in\mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\text{sym}})$

$$\begin{aligned}
\lim_{h\in\mathcal{H}_0}\big(\boldsymbol{\sigma}_h,\boldsymbol{\Psi}\big)_Q &= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\boldsymbol{\sigma}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\Pi_h^{\mathrm{dg}}(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n)+\Pi_h^{\mathrm{dg}}(\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n),\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n+\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega+\triangle t_h^n\big(\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(g_{\mathbb{C}}(z_h^n)\tilde{\mathbb{C}}\boldsymbol{\varepsilon}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega+\triangle t_h^n\big(g_{\mathbb{D}}(z_h^n)\tilde{\mathbb{D}}\dot{\boldsymbol{\varepsilon}}_h^n,\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\tilde{\mathbb{C}}\boldsymbol{\varepsilon}_h^n,g_{\mathbb{C}}(z_h^n)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega+\triangle t_h^n\big(\tilde{\mathbb{D}}\dot{\boldsymbol{\varepsilon}}_h^n,g_{\mathbb{D}}(z_h^n)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}^n\big)_\Omega\\
&= \lim_{h\in\mathcal{H}_0}\big(\tilde{\mathbb{C}}\boldsymbol{\varepsilon}_h,g_{\mathbb{C}}(z_h)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}\big)_Q+\big(\tilde{\mathbb{D}}\dot{\boldsymbol{\varepsilon}}_h,g_{\mathbb{D}}(z_h)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}\big)_Q\\
&\overset{(*)}{=} \big(\tilde{\mathbb{C}}\boldsymbol{\varepsilon},g_{\mathbb{C}}(z)\boldsymbol{\Psi}\big)_Q+\big(\tilde{\mathbb{D}}\partial_t\boldsymbol{\varepsilon},g_{\mathbb{D}}(z)\boldsymbol{\Psi}\big)_Q\\
&= \big(g_{\mathbb{C}}(z)\tilde{\mathbb{C}}\boldsymbol{\varepsilon},\boldsymbol{\Psi}\big)_Q+\big(g_{\mathbb{D}}(z)\tilde{\mathbb{D}}\partial_t\boldsymbol{\varepsilon},\boldsymbol{\Psi}\big)_Q
\end{aligned}$$

with $\boldsymbol{\Psi}^n=\frac{1}{\triangle t_h^n}\int_{t_{n-1}}^{t_n}\boldsymbol{\Psi}(t)\,\mathrm{d}t$, so that $(\boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ converges weakly in $L^2(Q;\mathbb{R}^{d\times d}_{\text{sym}})$. In $(*)$, weak/strong convergence arguments are used: We have that both $g_J(z_h)\to g_J(z)$ and $\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}\to\boldsymbol{\Psi}$ strongly in $L^2(Q)$ as $h\to 0$. This implies convergence in measure. Additionally we also have that $|g_J(z_h)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}|\le g^*|\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}|$ pointwise a.e. in Q by the growth properties of g_J, cf. (2.2h). Thus, $g_J(z_h)\Pi_h^{\mathrm{dg}}\boldsymbol{\Psi}\to g_J(z)\boldsymbol{\Psi}$ strongly in $\mathrm{L}^2(Q:\mathbb{R}^{d\times d}_{\text{sym}}$ by the dominated convergence theorem. □

Convergence results (4.5) enable us to pass to the limit $h \to 0$ in the discrete version (3.56) of the weak formulation when using suitable approximations of admissible test functions. This this way we establish now that the weak limit of the discrete solutions obtained in Lem. 4.1 is a weak solution of the elastodynamic phase field model in the sense of Def. 2.1.

Theorem 4.2 (Existence of weak solutions in the sense of Def. 2.1)**.** *Let the assumptions of Proposition 4.1 be satisfied. The weak limit*

$$(z, \mathbf{v}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \times \mathrm{L}^2(Q;\mathbb{R}^d) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\text{sym}}) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\text{sym}})$$

approximated by the sequence $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ *of discrete solutions constructed by the staggered time-discrete scheme* (3.11a)–(3.12) *is a solution of the weak formulation* (2.12).

Proof. **Proof of the attainment of the initial value** z_0 (2.12c)**:** This property has already been verified in (4.6).

Proof of the unidirectionality property (2.12d)**:** So far, we have only verified the weaker unidirectionality property stated in (4.6). For the pointwise-in-time unidirectionality property (2.12d) we refer to [17, 5.2.1 Unidirectionality (10b), p. 33]. The proof of this pointwise condition relies on the uniform bound in (4.24g) which does not depend on the retardation β_{r}, and the additional convergence property (4.26f), which we establish in Lemma 4.4 below.

Proof of the one-sided variational inequality (2.12b)**:** We show that the weak limit $(z, \boldsymbol{\varepsilon}) \in \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big) \cap \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\text{sym}})$ of the sequence of interpolants $(z_h, \boldsymbol{\varepsilon}_h)_{h\in\mathcal{H}_0}$ solves (2.12b) by passing to the limit $h \to 0$ in (3.56a).

For this, we introduce suitable approximations $\varphi_h \in \mathrm{H}^1(0,T;V_h^{\mathrm{cf}})$ of the admissible test functions $\varphi \in \mathcal{Z}$, cf. (2.9). This is done by nodal interpolation in space such that

$$\varphi_h(t_n, \mathbf{x}) = \varphi(t_n, \mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{N}_h, \text{and } n = 0, \ldots, N_h\,, \tag{4.12a}$$

and by linear interpolation in time

$$\varphi_h(t) = \frac{1}{\Delta t_h^n}\Big((t_n - t)\varphi_h(t_{n-1}) + (t - t_{n-1})\varphi_h(t_n)\Big)\,, \qquad t \in (t_{n-1}, t_n)\,,\ n = 1, \ldots, N_h\,. \tag{4.12b}$$

Since $\varphi \le 0$ a.e. in Q by definition of $\mathcal{Z}$ and since we use lowest order finite elements, we also ensure that

$$\varphi_h \le 0 \quad \text{in } Q\,. \tag{4.13}$$

By construction, since φ is smooth, we also have the following strong convergence results for the interpolants

$$\varphi_h \to \varphi \quad \text{in } \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \quad \text{and} \quad \varphi_h \to \varphi \quad \text{in } \mathrm{L}^\infty(Q)\,. \tag{4.14}$$

Using these interpolants we rewrite (3.56a) as follows

$$\big(\beta_{\mathrm{r}}\dot{z}_h, \varphi_h\big)_Q + \big({Y_h}'(\dot{z}_h), \varphi_h\big)_Q + b_Q(z_h, \varphi_h) = \frac{1}{2}\big(\mathbb{C}'(z_h)\underline{\boldsymbol{\varepsilon}}_h : \underline{\boldsymbol{\varepsilon}}_h, -\varphi_h\big)_Q \tag{4.15}$$

and next we discuss the limit passage for each of the terms separately.

For the first term in (4.15) we observe by the weak L^2-convergence (4.5d) of $(\dot{z}_h)_h$ and the strong convergence (4.14) of the interpolants

$$\big(\beta_{\mathrm{r}}\dot{z}_h, \varphi_h\big)_Q \to \big(\beta_{\mathrm{r}}\dot{z}, \varphi\big)_Q\,. \tag{4.16}$$

For the second term in (4.15) we use that $\varphi_h \le 0$ in Q, whereas ${Y_h}'(\dot{z}_h) \ge 0$ a.e. in Q. This allows for the estimate

$$\big({Y_h}'(\dot{z}_h), \varphi_h\big)_Q \le 0\,. \tag{4.17}$$

For the third term in (4.15) we argue by the weak $L^2(0,T;H^1(\Omega))$-convergence (4.5a) and the strong convergence of the interpolants (4.14) to conclude that

$$b_Q(z_h, \varphi_h) \to b_Q(z, \varphi)\,. \tag{4.18}$$

For the fourth term in (4.15) we observe that the integrand has nonnegative sign, i.e. $-\varphi_h\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h \geq 0$ a.e. in Q, thanks to the growth properties of the material tensor (2.2f) and the fact that $\varphi_h \leq 0$ a.e. in Q. Due to this, the integrand is also convex in $\underline{\boldsymbol{\varepsilon}}_h$. Therefore we would like to pass to the limit in the fourth term of (4.15) with the aid of a lower semicontinuity result [6, Thm. 3.23]. For this we define the map $f : \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$, $f(y,\boldsymbol{\xi}) = y\tilde{\mathbb{C}}\boldsymbol{\xi} : \boldsymbol{\xi}$. We observe that $f(\cdot,\cdot)$ is a Carathéodory function and convex in the second variable $\boldsymbol{\xi}$. With this we define the functional

$$\begin{aligned}
&\mathcal{J} \colon \mathrm{L}^2(Q) \cap \mathrm{L}^\infty(Q) \times \mathrm{L}^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \longrightarrow \mathbb{R}\,,\\
&\mathcal{J}(g'_{\mathbb{C}}(z)\varphi,\boldsymbol{\varepsilon}) := \int_Q f(-g'_{\mathbb{C}}(z)\varphi,\boldsymbol{\varepsilon})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \big(g'_{\mathbb{C}}(z)\tilde{\mathbb{C}}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, -\varphi\big)_Q = \big(\mathbb{C}'(z)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, -\varphi\big)_Q\,.
\end{aligned}$$

The strong convergence (4.5f) of $(z_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^2(Q)$, hence also of $(g'_{\mathbb{C}}(z_h))_{h\in\mathcal{H}_0}$, and the strong convergence (4.14) of $(\varphi_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^\infty(Q)$ by construction implies the strong convergence of $\big(g'_{\mathbb{C}}(z_h)\varphi_h\big)_{h\in\mathcal{H}_0}$ in $\mathrm{L}^2(Q)$. Then, weak $L^2(Q)$-convergence in the variable ε enables us to invoke [6, Thm. 3.23] and to conclude lower semicontinuity of the functional $\mathcal{J}$.

However, observe that the fourth term in (4.15) involves the left-continuous piecewise constant interpolants $(\underline{\boldsymbol{\varepsilon}}_h)_h$, for which we have not yet established the required $L^2(Q)$-convergence. This convergence result has only been established for the right-continuous piecewise constant interpolants $(\boldsymbol{\varepsilon}_h)_h$ in (4.5c). Therefore we show below in (4.23) that we may replace $\underline{\boldsymbol{\varepsilon}}_h$ by $\boldsymbol{\varepsilon}_h$ upon creating an error term $ERR(h) = |\mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\underline{\boldsymbol{\varepsilon}}_h) - \mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\boldsymbol{\varepsilon}_h)|$ that tends to zero as $h \to 0$. Accordingly, we can pass to the limit in the fourth term in (4.15)

$$\begin{aligned}
\frac{1}{2}\big(\mathbb{C}'(z_h)\underline{\boldsymbol{\varepsilon}}_h : \underline{\boldsymbol{\varepsilon}}_h, -\varphi_h\big)_Q &= \mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\underline{\boldsymbol{\varepsilon}}_h) = \mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\boldsymbol{\varepsilon}_h) + ERR(h)\,, \quad \text{and hence}\\
\liminf_{h\to 0}\frac{1}{2}\big(\mathbb{C}'(z_h)\underline{\boldsymbol{\varepsilon}}_h : \underline{\boldsymbol{\varepsilon}}_h, -\varphi_h\big)_Q &= \liminf_{h\to 0}\big(\mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\boldsymbol{\varepsilon}_h) + ERR(h)\big) \geq \big(\mathbb{C}'(z)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, -\varphi\big)_Q
\end{aligned} \tag{4.19}$$

by the above established lower semicontinuity and the convergence of the error term (4.23).

Now, (4.15) together with (4.16)–(4.19) ultimately results in

$$\frac{1}{2}(\mathbb{C}'(z)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, -\varphi)_Q \leq \big(\beta_{\mathrm{r}}\dot{z},\varphi\big)_Q + b_Q(z,\varphi) \quad \text{for all } \varphi \in \mathcal{Z}\,, \tag{4.20}$$

which is (2.12b).

To conclude the proof of (2.12b) it remains to discuss the error term $ERR(h)$ introduced in (4.19).

For this, we insert $\varphi_h^n = \varphi_h(t_h^{n-1/2})$ and observe that

$$\triangle t_h^n\big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^n\big)_\Omega = \big(\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_{(t_h^{n-1},t_h^n)\times\Omega}$$

since φ_h is linear and z_h and $\boldsymbol{\varepsilon}_h$ are constant in time in every interval (t_h^{n-1},t_h^n), so that we have

$$\big(\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q = \sum_{n=1}^{N_h}\triangle t_h^n\big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^n\big)_\Omega\,. \tag{4.21}$$

Moreover, we have

$$\big(\mathbb{C}'(z_h)\underline{\boldsymbol{\varepsilon}}_h : \underline{\boldsymbol{\varepsilon}_h}, -\varphi_h\big)_Q = \sum_{n=1}^{N_h}\triangle t_h^n\big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega\,. \tag{4.22}$$

The error term $ERR(h) = |\mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\underline{\boldsymbol{\varepsilon}}_h) - \mathcal{J}(g'_{\mathbb{C}}(z_h)\varphi_h,\boldsymbol{\varepsilon}_h)|$ is thus given by the difference of (4.21) and (4.22). To further express it we observe that

$$\begin{aligned}
&\big(\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q - \sum_{n=1}^{N_h}\triangle t_h^n\big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega\\
&\quad - \triangle t_h^{N_h}\big(\mathbb{C}'(z_h^{N_h})\boldsymbol{\varepsilon}_h^{N_h} : \boldsymbol{\varepsilon}_h^{N_h}, -\varphi_h^{N_h}\big)_\Omega\\
&= \triangle t_h^1\big(\mathbb{C}'(z_h^1)\boldsymbol{\varepsilon}_h^0 : \boldsymbol{\varepsilon}_h^0, -\varphi_h^1\big)_\Omega + \sum_{n=1}^{N_h-1}\triangle t_h^n\Big(\big(\mathbb{C}'(z_h^{n+1})\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^{n+1}\big)_\Omega - \big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^n\big)_\Omega\Big)\,.
\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
ERR(h) =& \Big|\big(\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q - \sum_{n=1}^{N_h} \triangle t_h^n \big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega\Big| \\
=& \Big|\triangle t_h^{N_h}\big(\mathbb{C}'(z_h^{N_h})\boldsymbol{\varepsilon}_h^{N_h} : \boldsymbol{\varepsilon}_h^{N_h}, -\varphi_h^{N_h}\big)_\Omega + \triangle t_h^1\big(\mathbb{C}'(z_h^1)\boldsymbol{\varepsilon}_h^0 : \boldsymbol{\varepsilon}_h^0, -\varphi_h^1\big)_\Omega \\
&+ \sum_{n=1}^{N_h-1} \triangle t_h^n \Big(\big(\mathbb{C}'(z_h^{n+1})\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^{n+1}\big)_\Omega - \big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^n : \boldsymbol{\varepsilon}_h^n, -\varphi_h^n\big)_\Omega\Big)\Big| .
\end{aligned}$$

Since φ is smooth, we obtain for the interpolation $\lim_{h\in\mathcal{H}_0} \big(\varphi_h^{n+1} - \varphi_h^n\big) = 0$ in $\mathrm{L}^\infty(\Omega)$, and since $z \in \mathrm{H}^1(0,T;\Omega)$ and thus continuous in time, and g' is continuous and bounded, we also observe $\lim_{h\in\mathcal{H}_0} \big(g'(z_h^{n+1}) - g'(z_h^n), \psi\big)_\Omega = 0$ for all $\psi \in \mathrm{L}^2(\Omega)$. Moreover, $\big(\tilde{\mathbb{C}}\boldsymbol{\varepsilon}_h^n, \boldsymbol{\varepsilon}_h^n\big)_\Omega$ is uniformly bounded, so that the terms on the right-hand side tend to zero. This proves that

$$ERR(h) := \Big|\big(\mathbb{C}'(z_h)\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q - \sum_{n=1}^{N_h} \triangle t_h^n \big(\mathbb{C}'(z_h^n)\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega\Big| \to 0 \quad \text{as } h \to \infty . \tag{4.23}$$

This concludes the proof of (2.12b).

Proof of the weak momentum balance (2.12a)**:** For $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi}) \in \mathcal{V}_{T,\mathsf{D}} \times \mathcal{W}_T \times \mathcal{W}_\mathsf{N}$ let $(\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n) \in (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \cap \mathrm{C}^0(\Omega; \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be the nodal interpolant in space of $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})(t_h^n)$ defined by

$$(\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n)(t_n, \mathbf{x}) = (\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})(t_h^n, \mathbf{x}) \quad \text{for } \mathbf{x} \in \mathcal{N}_h \text{ and } n = 0, \dots, N_h,$$

and let $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h) \in \mathrm{H}^1(0,T; V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ be the linear interpolation in time, cf. (4.12b), so that we have strong convergence of $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h)_{h\in\mathcal{H}_0}$ to $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$.

We set $(\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1}, \boldsymbol{\Psi}_h^{n-1}) = (\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h)(t_h^{n-1/2})$ and observe $\partial_t \mathbf{w}_h(t) = \frac{1}{\triangle t_h^n} \triangle \mathbf{w}_h^n$ for $\triangle \mathbf{w}_h^n = \mathbf{w}_h^n - \mathbf{w}_h^{n-1}$ and $t \in (t_{n-1}, t_n)$, $n = 1, \dots, N_h$. Using $\mathbf{w}_h^{N_h} = \mathbf{0}$, we obtain

$$\begin{aligned}
-\big(\varrho_0 \mathbf{v}_h, \partial_t \mathbf{w}_h\big)_Q = -\sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \triangle \mathbf{w}_h^n\big)_\Omega &= -\sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \mathbf{w}_h^n\big)_\Omega + \sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega \\
&= \big(\varrho_0 \mathbf{v}_h^0, \mathbf{w}_h^0\big)_\Omega + \sum_{n=1}^{N_h} \big(\varrho_0 \triangle \mathbf{v}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega
\end{aligned}$$

and for $\triangle \boldsymbol{\Phi}_h^n = \boldsymbol{\Phi}_h^n - \boldsymbol{\Phi}_h^{n-1}$ analogously, i.e., $-\big(\boldsymbol{\varepsilon}_h, \partial_t \boldsymbol{\Phi}_h\big)_Q = \big(\boldsymbol{\varepsilon}_h^0, \boldsymbol{\Phi}^0\big)_\Omega + \sum_{n=1}^{N_h} \big(\triangle \boldsymbol{\varepsilon}_h^n, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega$.

Since for $(\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})$ all jump terms and boundary terms vanish, we obtain consistency (3.4) for the DG bilinear form

$$a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{v}_h^n, \boldsymbol{\sigma}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) = \big(\boldsymbol{\sigma}_h^n, \mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega + \big(\mathbf{v}_h^n, \mathrm{div}\, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega .$$

Thus we obtain (3.12) in (S2), since we assume homogenous boundary conditions $\mathbf{v}_\mathsf{D} = \mathbf{0}$ and $\mathbf{g}_\mathsf{N} = \mathbf{0}$,

$$\begin{aligned}
& m_\Omega\big((\triangle \mathbf{v}_h^n, \triangle \boldsymbol{\varepsilon}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1} - \mathbb{D}(z_h^n)\boldsymbol{\Psi}_h^{n-1})\big) \\
&= \triangle t_h^n \Big(\ell_h^{\mathrm{dg}}\big(t_h^n, z_h^n; (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) - a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{v}_h^n, \boldsymbol{\sigma}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) \\
&\qquad - r_\Omega\big(z_h^n; (\boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n), \boldsymbol{\Psi}_h^{n-1}\big)\Big) \\
&= \triangle t_h^n \Big(\big(\mathbf{f}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega - \big(\boldsymbol{\sigma}_h^n, \mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega - \big(\mathbf{v}_h^n, \mathrm{div}\, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega \\
&\qquad - \big(\boldsymbol{\sigma}_h^n - \mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n, \boldsymbol{\Psi}_h^{n-1}\big)_\Omega\Big) .
\end{aligned}$$

Together with

$$m_\Omega\big((\triangle\mathbf{v}_h^n,\triangle\boldsymbol{\varepsilon}_h^n),(\mathbf{0},\mathbb{D}(z_h^n)\boldsymbol{\Psi}_h^{n-1})\big)=\big(\triangle\boldsymbol{\varepsilon}_h^n,\mathbb{D}(z_h^n)\boldsymbol{\Psi}_h^{n-1}\big)_\Omega=\triangle t_h^n\big(\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n,\boldsymbol{\Psi}_h^{n-1}\big)_\Omega$$

this yields

$$\begin{aligned}
&\big(\varrho_0\mathbf{v}_h,\partial_t\mathbf{w}_h\big)_Q+\big(\boldsymbol{\varepsilon}_h,\partial_t\boldsymbol{\Phi}_h\big)_Q+\big(\varrho_0\mathbf{v}_h^0,\mathbf{w}_h^0\big)_\Omega+\big(\boldsymbol{\varepsilon}_h^0,\boldsymbol{\Phi}^0\big)_\Omega\\
&=-\sum_{n=1}^{N_h}\Big(\big(\varrho_0\triangle\mathbf{v}_h^n,\mathbf{w}_h^{n-1}\big)_\Omega+\big(\triangle\boldsymbol{\varepsilon}_h^n,\boldsymbol{\Phi}_h^{n-1}\big)_\Omega\Big)\\
&=-\sum_{n=1}^{N_h}m_\Omega\big((\triangle\mathbf{v}_h^n,\triangle\boldsymbol{\varepsilon}_h^n),(\mathbf{w}_h^{n-1},\boldsymbol{\Phi}_h^{n-1})\big)\\
&=\sum_{n=1}^{N_h}\triangle t_h^n\Big(\big(\boldsymbol{\sigma}_h^n,\mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega+\big(\mathbf{v}_h^n,\mathrm{div}\,\boldsymbol{\Phi}_h^{n-1}\big)_\Omega\\
&\qquad+\big(\boldsymbol{\sigma}_h^n-\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n-\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n,\boldsymbol{\Psi}_h^{n-1}\big)_\Omega-\big(\mathbf{f}_h^n,\mathbf{w}_h^{n-1}\big)_\Omega\Big)\,.
\end{aligned}$$

Using strong convergence of the test functions and of $\big(g(z_h)\big)_{h\in\mathcal{H}_0}$ we find in the limit

$$\begin{aligned}
&\varrho_0\left(\mathbf{v},\partial_t\mathbf{w}\right)_Q+\left(\boldsymbol{\varepsilon},\partial_t\boldsymbol{\Phi}\right)_Q+\varrho_0\left(\mathbf{v}_0,\mathbf{w}(0)\right)_\Omega+\left(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0)\right)_\Omega\\
&=\lim_{h\in\mathcal{H}_0}\Big(\big(\varrho_0\mathbf{v}_h,\partial_t\mathbf{w}_h\big)_Q+\big(\boldsymbol{\varepsilon}_h,\partial_t\boldsymbol{\Phi}_h\big)_Q+\big(\varrho_0\mathbf{v}_h^0,\mathbf{w}_h^0\big)_\Omega+\big(\boldsymbol{\varepsilon}_h^0,\boldsymbol{\Phi}^0\big)_\Omega\Big)\\
&=\lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\Big(\big(\boldsymbol{\sigma}_h^n,\mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega+\big(\mathbf{v}_h^n,\mathrm{div}\,\boldsymbol{\Phi}_h^{n-1}\big)_\Omega\\
&\qquad+\big(\boldsymbol{\sigma}_h^n-\mathbb{C}(z_h^n)\boldsymbol{\varepsilon}_h^n-\mathbb{D}(z_h^n)\dot{\boldsymbol{\varepsilon}}_h^n,\boldsymbol{\Psi}_h^{n-1}\big)_\Omega-\big(\mathbf{f}_h^n,\mathbf{w}_h^{n-1}\big)_\Omega\Big)\\
&=\lim_{h\in\mathcal{H}_0}\Big(\big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w}_h)\big)_Q+\big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Phi}_h\big)_Q\\
&\qquad+\big(\boldsymbol{\sigma}_h-\mathbb{C}(z_h)\boldsymbol{\varepsilon}_h-\mathbb{D}(z_h)\dot{\boldsymbol{\varepsilon}}_h,\boldsymbol{\Psi}_h\big)_Q-\big(\mathbf{f}_h,\mathbf{w}_h\big)_Q\Big)\\
&=\big(\boldsymbol{\sigma},\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_Q+\big(\mathbf{v},\mathrm{div}\,\boldsymbol{\Phi}\big)_Q+\big(\boldsymbol{\sigma}-\mathbb{C}(z)\boldsymbol{\varepsilon}-\mathbb{D}(z)\dot{\boldsymbol{\varepsilon}},\boldsymbol{\Psi}\big)_Q-\big(\mathbf{f},\mathbf{w}\big)_Q\,.
\end{aligned}$$

This shows that the weak limit solves (2.12a). □

Improved convergence results and energy-dissipation estimate for the weak solutions

In the remaining part of the section we will discuss the energy-dissipation estimate (2.29) and show how to obtain this estimate by investigating the limit passage in the discrete energy estimate in (3.38).
We remark that for sufficiently regular solutions the energy-dissipation balance [17, Def. 1.3]

$$\begin{aligned}
&\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t))+\mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t))+\mathcal{E}^{\mathrm{pf}}(z(t))-\mathcal{E}^{\mathrm{ext}}(t,\mathbf{u}(t))+\int_0^t\big(\mathcal{R}^{\mathrm{vis}}(\dot{\boldsymbol{\varepsilon}}(s))+\mathcal{R}^{\mathrm{pf}}(\dot z(s))\big)\,\mathrm{d}s\\
&=\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0)+\mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0)-\mathcal{E}^{\mathrm{ext}}(0,\mathbf{u}(0))-\int_0^t\dot{\mathcal{E}}^{\mathrm{ext}}(s,\mathbf{u}(s))\,\mathrm{d}s
\end{aligned}$$

with

$$\dot{\mathcal{E}}^{\mathrm{ext}}(s,\mathbf{u}(s))=\big(\partial_t\mathbf{f}(s),\mathbf{u}(s)\big)_\Omega\,\mathrm{d}\mathbf{x}+\big(\partial_t\mathbf{g}_{\mathsf{N}}(s),\mathbf{u}(s)\big)_{\partial_{\mathsf{N}}\Omega}$$

can be established [17, Thm. 5.1]. An integration by parts yields

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t)) + \mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t)) + \mathcal{E}^{\mathrm{pf}}(z(t)) + \int_0^t \big(\mathcal{R}^{\mathrm{vis}}(\dot{\boldsymbol{\varepsilon}}(s)) + \mathcal{R}^{\mathrm{pf}}(\dot z(s))\big)\,\mathrm{d}s$$
$$= \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0) + \mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0) + \int_0^t \mathcal{E}^{\mathrm{ext}}(s,\mathbf{v}(s))\,\mathrm{d}s$$

and this is the relaxed form we are considering here with less regularity.

To pass to the limit in the discrete energy inequality (3.38), in particular pointwise in time convergences for the state variables are needed to estimate the first three energies $\mathcal{E}^{\mathrm{kin}}$, $\mathcal{E}^{\mathrm{el}}$ and $\mathcal{E}^{\mathrm{pf}}$ on the left-hand side.
This demands having uniform bounds also pointwise in time at hand.

Lemma 4.3. *Let the assumptions of Proposition 4.1 be satisfied. For the discrete solutions $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}, c_{\tilde{\mathbb{D}}}\dot{\boldsymbol{\varepsilon}})_h$ obtained by the staggered time-discrete scheme* (3.11a)–(3.12) *the additional estimates hold true with a constant $C > 0$ independent of $h > 0$:*

$$\|\mathbf{v}_h\|_{\mathrm{B}(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^d))} \le C, \tag{4.24a}$$
$$\|\boldsymbol{\varepsilon}_h\|_{\mathrm{B}(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C, \tag{4.24b}$$
$$\|\dot{\boldsymbol{\varepsilon}}_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C, \tag{4.24c}$$
$$\|\boldsymbol{\varepsilon}_h\|_{\mathrm{BV}(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C, \tag{4.24d}$$
$$\|z_h\|_{B(0,T;\mathrm{H}^1(\Omega))} \le C, \tag{4.24e}$$
$$\|\dot z_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \le \frac{C}{\sqrt{\beta_{\mathrm{r}}}}, \tag{4.24f}$$
$$\|z_h\|_{\mathrm{BV}(0,T;\mathrm{L}^1(\Omega))} \le C, \tag{4.24g}$$

Proof. The discrete energy dissipation estimate (3.38) gives

$$\begin{aligned}\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) = \frac{1}{2}\int_\Omega \rho_0|\mathbf{v}_h^n|^2\,\mathrm{d}\mathbf{x} &\le \tilde C + \sum_k \Delta t_h^k\big(\mathbf{f}_h^k,\mathbf{v}_h^k\big)_\Omega\\ &= \tilde C + \big(\mathbf{f},\mathbf{v}_h\big)_Q\\ &\le \tilde C + \|\mathbf{f}\|_Q\|\mathbf{v}_h\|_Q\\ &\le C\end{aligned}$$

with $\tilde C = \tilde C(\mathbf{v}_0,\boldsymbol{\varepsilon}_0,z_0)$ and since by Lemma 3.6 $\mathbf{v}_h$ is uniformly bounded in $\mathrm{L}^2(Q)$. This shows (4.24a) and in a similar way the bounds in (4.24b)-(4.24e). Notice, that for (4.24d), the bound on $\dot{\boldsymbol{\varepsilon}}_h$ in Lemma 3.6 is used to find

$$\begin{aligned}\sum_{n=1}^{N_h}\|\boldsymbol{\varepsilon}_h^k - \boldsymbol{\varepsilon}^{k-1}\|_{\mathrm{L}^2} = \sum_{k=1}^{N_h}\Delta t_h^n\Big\|\frac{\boldsymbol{\varepsilon}_h^n - \boldsymbol{\varepsilon}_h^{n-1}}{\Delta t_h^n}\Big\|_{\mathrm{L}^2} = \int_0^T \|\dot{\boldsymbol{\varepsilon}}_h(r)\|_{\mathrm{L}^2}\,\mathrm{d}r\\ = \|\dot{\boldsymbol{\varepsilon}}_h\|_{\mathrm{L}^1(0,T;\mathrm{L}^2)} \le \sqrt{T}\|\dot{\boldsymbol{\varepsilon}}_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le C\,.\end{aligned} \tag{4.25}$$

The uniform bounds on the z-dissipation gives (4.24f) where the bound still depends on β_{r}. To find bounds not depending on the parameter β_{r}, we concentrate on the Yosida part of the dissipation and proceed like in [17, proof of (48h), p. 26/27]. In this way, from Lemma 3.5 one can derive an uniform bound on the total (pointwise) variation $\sum_{k=1}^{N_h}\|z_h^{k-1} - z_h^k\|_{\mathrm{L}^1(\Omega)}$ of z_h such that together with (4.24e) it follows (4.24g).

□

We discuss now the limit passage in the discrete energy dissipation estimate of Lemma 3.5 which are based on the results hereafter.

Lemma 4.4. *Let the assumptions of Lemma 4.3 be satisfied. The following convergence statements are valid.*

$$\mathbf{v}_h \rightharpoonup \mathbf{v} \quad \textit{weakly in } \mathrm{L}^2\big(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^d)\big)\,, \tag{4.26a}$$

$$\mathbf{v}_h(t) \rightharpoonup \mathbf{v}(t) \quad \textit{weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^d) \textit{ for all } t\in[0,T]\,, \tag{4.26b}$$

$$\boldsymbol{\varepsilon}_h \rightharpoonup \boldsymbol{\varepsilon} \quad \textit{weakly in } \mathrm{L}^2\big(0,T;\mathrm{L}^2(\mathbb{R}^{d\times d}_{\mathrm{sym}})\big)\,, \tag{4.26c}$$

$$\boldsymbol{\varepsilon}_h(t) \rightharpoonup \boldsymbol{\varepsilon}(t) \quad \textit{weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})) \textit{ for all } t\in[0,T], \tag{4.26d}$$

$$z_h(t) \rightharpoonup z(t) \quad \textit{weakly in } \mathrm{H}^1(\Omega) \textit{ for all } t\in[0,T], \tag{4.26e}$$

$$z_h(t) \to z(t) \quad \textit{strongly in } \mathrm{L}^2(\Omega) \textit{ for all } t\in[0,T], \tag{4.26f}$$

$$z_h \to z \quad \textit{strongly in } \mathrm{L}_p\big(0,T;\mathrm{L}^2(\Omega)\big) \textit{ for all } p\in[1,\infty), \tag{4.26g}$$

$$z_h \overset{*}{\rightharpoonup} z \quad \textit{weakly-}{*} \textit{ in } \mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega)\big), \tag{4.26h}$$

Proof. As a consequence of (4.24g), a variant of Helly's Theorem in eg. [12, Theorem 2.1.24, p. 74]

can be used to see that (4.26e) and by compactness (4.26f) are true. Thus, in case of a vanishing mobility parameter $\beta_{\mathrm{r}} \to 0$, the damage variable z has a time derivative in the sense of measures with values in $\mathrm{L}^1(\Omega)$, i.e. $z\in \mathrm{BV}(0,T;\mathrm{L}^1(\Omega))$, although the pointwise convergences are with respect to $\mathrm{H}^1(\Omega)$, respectively $\mathrm{L}^2(\Omega)$ in a strong sense. The bound in (4.24e) is here crucial.
(4.26a) follows from the uniform bound in (4.24a). Applying dominated convergence in conjunction with (4.26f) and using the bound (4.24e) it follows (4.26g) and in addition that the limit is equal to (4.26h). The weak-$*$ convergence is here clear by (4.24e).
Eventually, (4.26c) is a consequence of (4.24b), while (4.26d) follows from (4.24d) by a version of Helly's Theorem [13, Theorem 6.1]. At first, we find

$$\boldsymbol{\varepsilon}_h(t) \rightharpoonup \tilde{\boldsymbol{\varepsilon}}(t) \quad \text{weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})) \text{ for all } t\in[0,T] \tag{4.27}$$

and it has to be shown that $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}$. For this, a discrete compactness result in [8, Theorem 1, p. 2] is used:
Since $\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}) \subset\subset \mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ compactly, it is with (4.24b) and (4.24c)

$$\|\dot{\boldsymbol{\varepsilon}}_h\|_{\mathrm{L}^2\left(0,T;\mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})\right)} + \|\boldsymbol{\varepsilon}_h\|_{\mathrm{L}^\infty\left(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})\right)} \le 2C\,.$$

Thus,

$$\boldsymbol{\varepsilon}_h \to \hat{\boldsymbol{\varepsilon}} \;\text{ strongly in } \mathrm{L}_q\big(0,T;\mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})\big) \;\text{ for all } q\in[1,\infty) \tag{4.28}$$

and $\hat{\boldsymbol{\varepsilon}} \in \mathrm{C}^0([0,T];\mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. This implies with (4.26c) and (4.27) by uniquenes of strong/weak limits that $\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})\big)$ since (4.28) implies

$$\boldsymbol{\varepsilon}_h(t) \to \boldsymbol{\varepsilon}(t) \;\text{ strongly in } \mathrm{H}^{-1}(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}}) \;\text{ for almost all } t\in[0,T]\,.$$

If we now choose $\tilde{\boldsymbol{\varepsilon}}$ as representative for this class, (4.26d) can be concluded.

It remains to show the pointwise convergence $\mathbf{v}_h(t) \rightharpoonup \mathbf{v}(t)$ weakly in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ to pass to the limit in the discrete energy estimate (3.38). To find uniform bounds, see Lemma 4.6 below.
Now, having a bound on $\dot{\mathbf{v}}_h$ in $\mathrm{L}^2(0,T;\mathrm{H}^1_D(\Omega,\mathbb{R}^d)^*)$ uniformly for all $h>0$, a first consequence is the uniform boundedness of the total variation of $\mathbf{v}_h$, implying with (4.24a) and Helly's selection principle [12, Theorem B.5.10] that we find a subsequence such that

$$\mathbf{v}_h(t) \to \hat{\mathbf{v}}(t) \text{ weakly-}{*} \text{ in } \mathrm{H}^1_D(\Omega,\mathbb{R}^d)^* \text{ for every } t\in[0,T] \tag{4.29}$$

and

$$\hat{\mathbf{v}} \in \mathrm{BV}\big(0,T;\mathrm{H}^1_D(0,T;\mathbb{R}^d)^*\big)\,. \tag{4.30}$$

Furthermore, with (4.24a) it is

$$\|\dot{\mathbf{v}}_h\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_D(\Omega,\mathbb{R}^d)^*)} + \|\mathbf{v}_h\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega,\mathbb{R}^d))} \le C$$

and by [8, Theorem 1, p. 2], passing to a subsequence, it can be concluded that

$$\mathbf{v}_h \to \tilde{\mathbf{v}} \tag{4.31}$$

strongly in every $\mathrm{L}_q(0,T;\mathrm{H}^1(\Omega,\mathbb{R}^d)^*)$ where $\tilde{\mathbf{v}} \in \mathrm{C}^0([0,T];\mathrm{H}^1_D(\Omega,\mathbb{R}^d)^*)$.

Since (4.26a) implies $\mathbf{v}_h \rightharpoonup \mathbf{v}$ weakly in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega,\mathbb{R}^d)$ we find $\mathbf{v} = \tilde{\mathbf{v}}$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega,\mathbb{R}^d))$ and by (4.31) $\mathbf{v}_h(t) \to \tilde{\mathbf{v}}(t)$ in $\mathrm{H}^{-1}(\Omega,\mathbb{R}^d)$ for a.a. $t \in [0,T]$. With (4.29), this implies

$$\mathbf{v} = \tilde{\mathbf{v}} = \hat{\mathbf{v}} \text{ in } \mathrm{L}^1\big(0,T;\mathrm{H}^1_D(\Omega,\mathbb{R}^d)^*\big)\,. \tag{4.32}$$

Now, for arbitrary $t \in [0,T]$, every subsequence of $(\mathbf{v}_h(t))_h$ is uniformly bounded in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ by (4.24a) and thus admits a subsequence weakly converging in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ to some $v_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$. By (4.29) it is $\mathbf{v}_t = \hat{\mathbf{v}}(t)$ in $\mathrm{H}^1_D(\Omega,\mathbb{R}^D)^*$, i.e. $\|\mathbf{v}_t - \hat{\mathbf{v}}(t)\|_{\mathrm{H}^{-1}} = 0$ and the convergence is true for the whole sequence (and everyt $t \in [0,T]$).

This can be seen as follows: Assume that the statement for $t \in [0,T]$ is not true for the *whole* sequence. Then there is a subsequence such that $\mathbf{v}_{h2}(t) \rightharpoonup \mathbf{v}_{t2}$ weakly in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Again we conclude that $\mathbf{v}_{t2} = \hat{\mathbf{v}}(t)$ identified in $\mathrm{H}^{-1}(\Omega,\mathbb{R}^d)$ and thus $\|\mathbf{v}_{t2} - \mathbf{v}_t\|_{\mathrm{H}^{-1}} = 0$. We want to show that this is also true with respect to the L^2-Norm. For that, let $\mathbf{w} \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ and $(\mathbf{w}_h)_h \subset \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ s.t. $\|\mathbf{w}_h - \mathbf{w}\|_{\mathrm{L}^2} \to 0$ as $h \to 0$. Then, $\langle \mathbf{v}_t - \mathbf{v}_{t2}, \mathbf{w}\rangle_{\mathrm{L}^2} = \lim_{h\to 0}\langle \mathbf{v}_t - \mathbf{v}_{t2}, \mathbf{w}_h\rangle_{\mathrm{L}^2} = 0$ which shows that the identity $\mathbf{v}_t = \mathbf{v}_{t2}$ is also true in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ and we conclude that for the *whole* sequence and every $t \in [0,T]$ it is

$$\mathbf{v}_h(t) \rightharpoonup \mathbf{v}_t \text{ weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^d)\,. \tag{4.33}$$

Since by (4.26a) we already know that for a.a. $t \in [0,T]$ we can identify $\mathbf{v}(t)$ with a function in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$, it follows like above that $\mathbf{v}_t = \mathbf{v}(t)$ in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Thus, we find for almost all $t \in [0,T]$ the convergence $\mathbf{v}_h(t) \rightharpoonup \mathbf{v}(t)$ weakly in L^2. Since the limit function $\mathbf{v}$ in (4.26a) is defined almost everywhere only, we make a choice for the remaining $t \in [0,T]$:

$$\mathbf{v}(t) = \mathbf{v}_t$$

and can state the L^2-convergence in (4.26b) since by (4.33) the *whole* sequence converges for every $t \in [0,T]$.

It follows that for the first three energies on the left-hand side of the discrete energy estimate (3.38) an estimate from below can be performed using (4.26b), (4.26e) for first and third, and (4.26f), (4.26d) in conjunction with [5, Thm. 3.4 Dacorogna, p. 74] for the middle term. The dissipation potentials on the left-hand side can be estimated by convergences $\dot{\boldsymbol{\varepsilon}}_h \rightharpoonup \dot{\boldsymbol{\varepsilon}}$, $\dot{z}_h \rightharpoonup \dot{z}$, lower semi-continuity and non-negativity of the Yosida-part. Eventually, for the remaining parts on the right-hand side the strong convergences of the L^2-projected initial $\mathbf{v}^0_h,\ z^0_h,\ \boldsymbol{\varepsilon}^0_h$ are exploited for the first three terms while for the last a weak/strong argument is used. □

Finally, we present the energy-dissipation estimate (2.29) in the following Lemma.

Lemma 4.5. *Let the assumptions of Proposition 4.1 be satisfied. Then the limit quadruple* (4.4) *satisfies the energy-dissipation estimate*

$$\begin{aligned}&\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t)) + \mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t)) + \mathcal{E}^{\mathrm{pf}}(z(t)) + \int_0^t \big(\mathcal{R}^{\mathrm{vis}}(\dot{\boldsymbol{\varepsilon}}(s)) + \mathcal{R}^{\mathrm{pf}}(\dot{z}(s))\big)\,\mathrm{d}s\\ &\le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0) + \mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0) + \int_0^t \mathcal{E}^{\mathrm{ext}}(s,\mathbf{v}(s))\,\mathrm{d}s\end{aligned} \tag{4.34}$$

for all $t \in [0,T]$.

A crucial step in the preceding proofs, especially in the proof of (4.26b) in Lemma 4.4 needed the following bound:

Lemma 4.6. *Let* $\mathbf{v}_h^n \in V_h^{\mathrm{dg}}$ *be discrete solution of (S2) obtained in Lemma 3.5.Define* $\dot{\mathbf{v}}_h(t) = \frac{1}{\Delta t_h^n}(\mathbf{v}_h^n - \mathbf{v}_h^{n-1})$ *for* $t \in (t_h^{n-1}, t_h^n)$*. The sequence* $(\dot{\mathbf{v}}_h)_{h\in\mathcal{H}}$ *is uniformly bounded in* $\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)^*)$.

Proof. Following the argumentation in [17, proof of (76e) on p. 32], we want to show that $\dot{\mathbf{v}}_h$ defines an element of the space $\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)^*)$ for every $h \in \mathcal{H}$ and that $(\dot{\mathbf{v}}_h)_{h\in\mathcal{H}} \subset \mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)^*)$ is uniformly bounded. For this purpose, we assume that for $\mathbf{w} \in \mathcal{V} = \mathrm{C}^1(\bar{Q},\mathbb{R}^d)$ an approximation $\tilde{\mathbf{w}}_h \in \mathrm{L}^2(0,T;V_h^{\mathrm{dg}}) \subset \mathrm{L}^2(Q,\mathbb{R}^d)$ by H^1-stable L^2-projection in *space* exists such that

$$(\hat{\mathbf{v}}_h, \tilde{\mathbf{w}}_h)_Q = (\hat{\mathbf{v}}_h, \mathbf{w})_Q \quad \text{for all } \hat{\mathbf{v}}_h \in \mathrm{L}^2(0,T;V_h^{\mathrm{dg}}) \subset \mathrm{L}^2(Q,\mathbb{R}^d)$$

and the gradient of the projection can be estimated from above by the original function, i.e.

$$\|\nabla\tilde{\mathbf{w}}_h(t)\|_\Omega \le c_k \|\nabla\mathbf{w}(t)\|_\Omega \,, \tag{4.35}$$

see e.g. in [2, 4] where conforming and shape regular triangulations and continuous piecewise polynomial conforming elements guarantee these properties. In particular, continuous elements are contained in the spaces V_h^{dg} by definition in Section 3.

A test in (S2) with $(\tilde{\mathbf{w}}_h(t), \mathbf{0}, \mathbf{0})$, $t \in (t_h^{n-1}, t_h^n)$, yields:

$$\big(\dot{\mathbf{v}}_h(t), \tilde{\mathbf{w}}_h(t)\big)_\Omega = -\big(\boldsymbol{\sigma}_h^n, \boldsymbol{\varepsilon}(\tilde{\mathbf{w}}_h(t)\big)_\Omega + \big(\mathbf{f}_h(t), \tilde{\mathbf{w}}_h(t)\big)_\Omega$$

since the DG-terms vanish by estimate (4.35) and if $\|\tilde{\mathbf{w}}_h\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}})} \le 1$, we find

$$\begin{aligned}\big(\dot{\mathbf{v}}_h, \tilde{\mathbf{w}}_h\big)_Q &= -\big(\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\tilde{\mathbf{w}}_h)\big)_Q + \big(\mathbf{f}, \tilde{\mathbf{w}}_h\big)_Q \\ &\le \|\boldsymbol{\sigma}_h\|_Q \|\boldsymbol{\varepsilon}(\tilde{\mathbf{w}}_h)\|_Q + \|\mathbf{f}\|_Q \|\tilde{\mathbf{w}}_h\|_Q \le C\end{aligned}$$

by Lemma 3.6. We conclude for $\mathbf{w} \in \mathcal{V} = \mathrm{C}^1(\bar{Q},\mathbb{R}^d)$ with $\|\mathbf{w}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}})} \le 1$ that

$$\begin{aligned}\big(\dot{\mathbf{v}}_h, \mathbf{w}\big)_Q &= \big(\dot{\mathbf{v}}_h, \tilde{\mathbf{w}}_h\big)_Q + \big(\dot{\mathbf{v}}_h, \mathbf{w} - \tilde{\mathbf{w}}_h\big)_Q \\ &\le C + 0\end{aligned} \tag{4.36}$$

since $\mathbf{w}$ and L^2-projection $\tilde{\mathbf{w}}$ have the same action on elements of V_h^{dg}. Taking the supremum over all $\mathbf{w} \in \mathrm{C}^1(\bar{Q},\mathbb{R}^d)$ with $\|\mathbf{w}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_D)} \le 1$ shows that

$$\|\dot{\mathbf{v}}_h\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)^*)} = \sup_{\substack{w \in \mathrm{C}^1(0,T;\mathrm{H}^1_{\mathrm{D}}) \\ \|\mathbf{w}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_{\mathrm{D}})} \le 1}} \big(\dot{\mathbf{v}}_h, \mathbf{w}\big)_Q \le C\,.$$

□

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The results in Appendix A are taken from [7], and Appendix B provides a constructive proof of the Aubin-Lions Lemma which illustrates the connection to space-time finite elements.

Appendix A Consistency and stability of the DG approximation

For $(\mathbf{v}_h, \boldsymbol{\sigma}_h), (\mathbf{w}_h, \boldsymbol{\Psi}_h) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ depending on the phase field z_h we have

$$
\begin{aligned}
&a_h^{\mathrm{dg}}\big(z_h; (\mathbf{v}_h, \boldsymbol{\sigma}_h), (\mathbf{w}_h, \boldsymbol{\Psi}_h)\big)\\
&\quad= \big(\boldsymbol{\sigma}_h, \operatorname{sym}(\mathrm{D}\mathbf{w}_h)\big)_{\Omega_h} + \big(\mathbf{v}_h, \operatorname{div} \boldsymbol{\Psi}_h\big)_{\Omega_h}\\
&\quad\quad - \frac{1}{2} \sum_{K\in\mathcal{K}_h} \sum_{f\in\mathcal{F}_K} \Bigg(\Big(\mathbf{n}_K \cdot \big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big), \mathbf{n}_K \cdot \big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f\\
&\quad\quad\quad + \Big(\mathbf{n}_K \times \big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big), \mathbf{n}_K \times \big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f \Bigg)\\
&\quad= -\big(\operatorname{div} \boldsymbol{\sigma}_h, \mathbf{w}_h\big)_{\Omega_h} - \big(\operatorname{sym}(\mathrm{D}\mathbf{v}_h), \boldsymbol{\Psi}_h\big)_{\Omega_h} + \sum_{K\in\mathcal{K}_h} \sum_{f\in\mathcal{F}_K} \Big(\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{w}_{h,K}\big)_f + \big(\mathbf{v}_{h,K}, \boldsymbol{\Psi}_{h,K}\mathbf{n}_K\big)_f \Big)\\
&\quad\quad + \frac{1}{2} \sum_{K\in\mathcal{K}_h} \sum_{f\in\mathcal{F}_K} \Bigg(\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, [\mathbf{w}_h]_{K,f}\big)_f + \big(\mathbf{v}_{h,K}, [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f\\
&\quad\quad\quad - \big(\mathbf{n}_K \cdot \boldsymbol{\sigma}_{h,K}\mathbf{n}_K, Z_{\mathsf{P}}(z_h)^{-1}\mathbf{n}_K \cdot [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f - \big(Z_{\mathsf{P}}(z_h)\mathbf{n}_K \cdot \mathbf{v}_{h,K}, \mathbf{n}_K \cdot [\mathbf{w}_h]_{K,f}\big)_f\\
&\quad\quad\quad - \big(\mathbf{n}_K \times \boldsymbol{\sigma}_{h,K}\mathbf{n}_K, Z_{\mathsf{S}}(z_h)^{-1}\mathbf{n}_K \times [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f\\
&\quad\quad\quad - \big(Z_{\mathsf{S}}(z_h)\mathbf{n}_K \times \mathbf{v}_{h,K}, \mathbf{n}_K \times [\mathbf{w}_h]_{K,f}\big)_f \Bigg)
\end{aligned}
$$

and, using on inner faces

$$
\begin{aligned}
&\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{w}_{h,K}\big)_f + \big(\boldsymbol{\sigma}_{h,K_f}\mathbf{n}_{K_f}, \mathbf{w}_{h,K_f}\big)_f + \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, [\mathbf{w}_h]_{K,f}\big)_f + \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K_f}\mathbf{n}_{K_f}, [\mathbf{w}_h]_{K_f,f}\big)_f\\
&\quad= \big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{w}_{h,K}\big)_f - \big(\boldsymbol{\sigma}_{h,K_f}\mathbf{n}_K, \mathbf{w}_{h,K_f}\big)_f\\
&\quad\quad\quad + \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{w}_{K_f,h}\big)_f - \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{w}_{K,h}\big)_f - \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K_f}\mathbf{n}_K, \mathbf{w}_{K,h}\big)_f + \frac{1}{2}\big(\boldsymbol{\sigma}_{h,K_f}\mathbf{n}_K, \mathbf{w}_{K_f,h}\big)_f\\
&\quad= -\frac{1}{2}\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K, \mathbf{w}_{K,h}\big)_f - \frac{1}{2}\big([\boldsymbol{\sigma}_h]_{K_f,f}\mathbf{n}_{K_f}, \mathbf{w}_{K_f,h}\big)_f\\
&- \big(\mathbf{n}_K \cdot \boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{n}_K \cdot [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f - \big(\mathbf{n}_{K_f} \cdot \boldsymbol{\sigma}_{h,K_f}\mathbf{n}_{K_f}, \mathbf{n}_{K_f} \cdot [\boldsymbol{\Psi}_h]_{K_f,f}\mathbf{n}_{K_f}\big)_f\\
&\quad= -\big(\mathbf{n}_K \cdot \boldsymbol{\sigma}_{h,K}\mathbf{n}_K, \mathbf{n}_K \cdot [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f + \big(\mathbf{n}_K \cdot \boldsymbol{\sigma}_{h,K_f}\mathbf{n}_K, \mathbf{n}_K \cdot [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f\\
&\quad= \big(\mathbf{n}_K \cdot [\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K, \mathbf{n}_K \cdot [\boldsymbol{\Psi}_h]_{K,f}\mathbf{n}_K\big)_f\\
&\quad= -\big(\mathbf{n}_K \cdot [\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K, \mathbf{n}_K \cdot \boldsymbol{\Psi}_{K,h}\mathbf{n}_K\big)_f - \big(\mathbf{n}_{K_f} \cdot [\boldsymbol{\sigma}_h]_{K_f,f}\mathbf{n}_{K_f}, \mathbf{n}_{K_f} \cdot \boldsymbol{\Psi}_{K_f,h}\mathbf{n}_{K_f}\big)_f
\end{aligned}
$$

and correspondingly for the other terms and the boundary faces, so that

$$
\begin{aligned}
&a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\Psi}_h)\big)\\
&\quad= -\big(\operatorname{div}\boldsymbol{\sigma}_h,\mathbf{w}_h\big)_{\Omega_h} - \big(\operatorname{sym}(\mathrm{D}\mathbf{v}_h),\boldsymbol{\Psi}_h\big)_{\Omega_h}\\
&\qquad -\frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K,\mathbf{w}_{h,K}\big)_f + \big([\mathbf{v}_h]_{K,f},\boldsymbol{\Psi}_{h,K}\mathbf{n}_K\big)_f\\
&\qquad\qquad + \big(Z_{\mathsf{P}}(z_h)^{-1}\mathbf{n}_K\cdot[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K,\mathbf{n}_K\cdot\boldsymbol{\Psi}_{h,K}\mathbf{n}_K\big)_f + \big(Z_{\mathsf{P}}(z_h)\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,f},\mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_f\\
&\qquad\qquad + \big(Z_{\mathsf{P}}(z_h)^{-1}\mathbf{n}_K\times[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K,\mathbf{n}_K\times\boldsymbol{\Psi}_{h,K}\mathbf{n}_K\big)_f\\
&\qquad\qquad + \big(Z_{\mathsf{P}}(z_h)\mathbf{n}_K\times[\mathbf{v}_h]_{K,f},\mathbf{n}_K\times\mathbf{w}_{h,K}\big)_f\Bigg)\\
&\quad= -\big(\operatorname{div}\boldsymbol{\sigma}_h,\mathbf{w}_h\big)_{\Omega_h} - \big(\operatorname{sym}(\mathrm{D}\mathbf{v}_h),\boldsymbol{\Psi}_h\big)_{\Omega_h}\\
&\qquad -\frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\Big(\mathbf{n}_K\cdot\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K + Z_{\mathsf{P}}(z_h)[\mathbf{v}_h]_{K,f}\big),\mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}\boldsymbol{\Psi}_{h,K}\mathbf{n}_K+\mathbf{w}_{h,K}\big)\Big)_f\\
&\qquad\qquad + \Big(\mathbf{n}_K\times\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K + Z_{\mathsf{S}}(z_h)[\mathbf{v}_h]_{K,f}\big),\mathbf{n}_K\times\big(Z_{\mathsf{S}}(z_h)^{-1}\boldsymbol{\Psi}_{h,K}\mathbf{n}_K+\mathbf{w}_{h,K}\big)\Big)_f\Bigg).
\end{aligned}
$$

Together, we get

$$
\begin{aligned}
&a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{v}_h,\boldsymbol{\sigma}_h)\big)\\
&\quad= \frac12 a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{v}_h,\boldsymbol{\sigma}_h)\big) + \frac12 a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{v}_h,\boldsymbol{\sigma}_h)\big)\\
&\quad= -\frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K - [\mathbf{v}_h]_{K,f}\big)\Big)_f\\
&\qquad\qquad + \Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K - [\mathbf{v}_h]_{K,f}\big)\Big)_f\\
&\qquad\qquad + \Big(\mathbf{n}_K\cdot\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K + Z_{\mathsf{P}}(z_h)[\mathbf{v}_h]_{K,f}\big),\mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}\boldsymbol{\sigma}_{h,K}\mathbf{n}_K + \mathbf{v}_{h,K}\big)\Big)_f\\
&\qquad\qquad + \Big(\mathbf{n}_K\times\big([\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K + Z_{\mathsf{S}}(z_h)[\mathbf{v}_h]_{K,f}\big),\mathbf{n}_K\times\big(Z_{\mathsf{S}}(z_h)^{-1}\boldsymbol{\sigma}_{h,K}\mathbf{n}_K + \mathbf{v}_{h,K}\big)\Big)_f\Bigg)\\
&\quad= \frac14\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\big(Z_{\mathsf{P}}(z_h)^{-1}\mathbf{n}_K\cdot[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K,\mathbf{n}_K\cdot[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\big)_f + \big(Z_{\mathsf{P}}(z_h)\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,f},\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,f}\big)_f\\
&\qquad + \big(Z_{\mathsf{S}}(z_h)^{-1}\mathbf{n}_K\times[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K,\mathbf{n}_K\times[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\big)_f\\
&\qquad + \big(Z_{\mathsf{S}}(z_h)\mathbf{n}_K\times[\mathbf{v}_h]_{K,f},\mathbf{n}_K\times[\mathbf{v}_h]_{K,f}\big)_f\Bigg).
\end{aligned}
$$

For smooth test functions $(\mathbf{w},\boldsymbol{\Psi}) \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ we obtain

$$
\begin{aligned}
a_h^{\mathrm{dg}}&\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w},\boldsymbol{\Psi})\big)\\
&= \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_{\Omega_h} + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Psi}\big)_{\Omega_h}\\
&\quad -\frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Psi}]_{K,f}\mathbf{n}_K - [\mathbf{w}]_{K,f}\big)\Big)_f\\
&\qquad\qquad + \Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big(Z_{\mathsf{P}}(z_h)^{-1}[\boldsymbol{\Psi}]_{K,f}\mathbf{n}_K - [\mathbf{w}]_{K,f}\big)\Big)_f\Bigg)\\
&= \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_{\Omega_h} + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Psi}\big)_{\Omega_h}\\
&\qquad + \big(\mathbf{n}_K\cdot\boldsymbol{\sigma}_h\mathbf{n}_K - Z_{\mathsf{P}}(z_h)\mathbf{n}_K\cdot\mathbf{v}_h,\mathbf{n}_K\cdot\mathbf{w}\big)_{\Gamma_{\mathsf{D}}} + \big(\mathbf{n}_K\times\boldsymbol{\sigma}_h\mathbf{n}_K - Z_{\mathsf{S}}(z_h)\mathbf{n}_K\times\mathbf{v}_h,\mathbf{n}_K\times\mathbf{w}\big)_{\Gamma_{\mathsf{D}}}\\
&\qquad + \big(\mathbf{n}_K\cdot\mathbf{v}_h - Z_{\mathsf{P}}(z_h)^{-1}\mathbf{n}_K\cdot\boldsymbol{\sigma}_h\mathbf{n}_K,\mathbf{n}_K\cdot\boldsymbol{\Psi}\mathbf{n}_K\big)_{\Gamma_{\mathsf{N}}}\\
&\qquad + \big(\mathbf{n}_K\times\mathbf{v}_h - Z_{\mathsf{S}}(z_h)^{-1}\mathbf{n}_K\times\boldsymbol{\sigma}_h\mathbf{n}_K,\mathbf{n}_K\times\boldsymbol{\Psi}\mathbf{n}_K\big)_{\Gamma_{\mathsf{N}}}\\
&= \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_{\Omega_h} + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Psi}\big)_{\Omega_h}\\
&\qquad + \big(\boldsymbol{\sigma}_h\mathbf{n}_K - Z_{\mathsf{P}}(z_h)(\mathbf{n}_K\cdot\mathbf{v}_h)\mathbf{n}_K - Z_{\mathsf{S}}(z_h)\mathbf{n}_K\times\mathbf{v}_h,\mathbf{w}\big)_{\Gamma_{\mathsf{D}}}\\
&\qquad + \big(\mathbf{v}_h - Z_{\mathsf{P}}(z_h)^{-1}(\mathbf{n}_K\cdot\boldsymbol{\sigma}_h\mathbf{n}_K)\mathbf{n}_K - Z_{\mathsf{S}}(z_h)^{-1}\mathbf{n}_K\times\boldsymbol{\sigma}_h\mathbf{n}_K,\boldsymbol{\Psi}\mathbf{n}_K\big)_{\Gamma_{\mathsf{N}}}
\end{aligned}
$$

and thus $a_h^{\mathrm{dg}}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w},\boldsymbol{\Psi})\big) = \big(\boldsymbol{\sigma}_h,\mathrm{sym}(\mathrm{D}\mathbf{w})\big)_{\Omega_h} + \big(\mathbf{v}_h,\mathrm{div}\,\boldsymbol{\Psi}\big)_{\Omega_h}$ for test functions with $\mathbf{w}=\mathbf{0}$ on Γ_{D} and $\boldsymbol{\Psi}\mathbf{n}_K = \mathbf{0}$ on Γ_{N}.

Appendix B Approximation properties of space-time finite volumes

We define the projections

$$
\begin{aligned}
\Pi_n\colon \mathrm{L}^2(Q) &\longrightarrow \mathrm{L}^2(\Omega)\,, & (\Pi_n\varphi)(\mathbf{x}) &= \frac{1}{\triangle t_h^n}\int_{t_h^{n-1}}^{t_h^n}\varphi(t,\mathbf{x})\,\mathrm{d}t\,, & t &\in (t_h^{n-1},t_h^n)\,,\\
\Pi_{n,K}\colon \mathrm{L}^2(\Omega) &\longrightarrow \mathbb{R}\,, & \Pi_{n,K}\varphi &= \frac{1}{|K|}\int_K(\Pi_n\varphi)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, & \mathbf{x} &\in K\,,\\
\Pi_h\colon \mathrm{L}^2(Q) &\longrightarrow \mathrm{L}^2(Q)\,, & (\Pi_h\varphi)(t,\mathbf{x}) &= \Pi_{n,K}\varphi\,, & K &\in \mathcal{K}_h\,,
\end{aligned}
$$

so that Π_h is the L^2 projection to space-time finite volume aproximations in $\mathbb{P}_0(Q_h)$.

Lemma B.1. *For $\varphi \in \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)$ we have*

$$
\big\|\varphi - \Pi_h\varphi\big\|_Q \le \sqrt{\frac{T\max_n\triangle t_h^n}{6}}\big\|\partial_t\varphi\big\|_Q + \frac{h}{\sqrt{3}}\big\|\nabla\varphi\big\|_Q\,.
$$

Proof. For $\varphi \in \mathrm{C}^1(Q)$ define $\varphi_n = \Pi_n\varphi$ and $\varphi_h = \Pi_h\varphi$. Then, we obtain for $t \in (t_h^{n-1}, t_h^n)$ and $\mathbf{x} \in K$

$$\begin{aligned}
\varphi(t,\mathbf{x}) - \varphi_n(\mathbf{x}) &= \frac{1}{\triangle t_h^n}\int_{t_h^{n-1}}^{t_h^n} \big(\varphi(t,\mathbf{x}) - \varphi(s,\mathbf{x})\big)\mathrm{d}s \\
&= \frac{1}{\triangle t_h^n}\left(\int_{t_h^{n-1}}^{t} \big(\varphi(t,\mathbf{x}) - \varphi(s,\mathbf{x})\big)\mathrm{d}s - \int_{t}^{t_h^n} \big(\varphi(s,\mathbf{x}) - \varphi(t,\mathbf{x})\big)\mathrm{d}s\right) \\
&= \frac{1}{\triangle t_h^n}\left(\int_{t_h^{n-1}}^{t}\int_s^t \partial_\tau\varphi(\tau,\mathbf{x})\mathrm{d}\tau\mathrm{d}s - \int_{t}^{t_h^n}\int_t^s \partial_\tau\varphi(\tau,\mathbf{x})\mathrm{d}\tau\mathrm{d}s\right) \\
&= \frac{1}{\triangle t_h^n}\left(\int_{t_h^{n-1}}^{t} (s - t_h^{n-1})\partial_s\varphi(s,\mathbf{x})\mathrm{d}s - \int_{t}^{t_h^n} (t_h^n - s)\partial_s\varphi(s,\mathbf{x})\mathrm{d}s\right) \\
\varphi_n(\mathbf{x}) - \varphi_{n,K} &= \frac{1}{|K|}\int_K \big(\varphi_n(\mathbf{x}) - \varphi_n(\mathbf{y})\big)\,\mathrm{d}\mathbf{y} \\
&= \frac{1}{\triangle t_h^n}\int_{t_h^{n-1}}^{t_h^n}\frac{1}{|K|}\int_K\int_0^1 (\mathbf{x}-\mathbf{y})\cdot\nabla\varphi\big(t,\mathbf{y}+s(\mathbf{x}-\mathbf{y})\big)\,\mathrm{d}s\mathrm{d}\mathbf{y}\mathrm{d}t\,,
\end{aligned}$$

so that

$$\begin{aligned}
\big\|\varphi - \varphi_n\big\|_Q^2 &= \sum_{n=1}^{N_h}\int_\Omega\int_{t_h^{n-1}}^{t_h^n}\left(\frac{1}{\triangle t_h^n}\int_{t_h^{n-1}}^{t_h^n}\big(\varphi(s,\mathbf{x}) - \varphi_n(\mathbf{x})\big)^2\mathrm{d}s\right)^2\mathrm{d}t\mathrm{d}\mathbf{x} \\
&= \sum_{n=1}^{N_h}\int_\Omega\int_{t_h^{n-1}}^{t_h^n}\left(\frac{1}{\triangle t_h^n}\left(\int_{t_h^{n-1}}^{t}(s - t_h^{n-1})\partial_s\varphi(s,\mathbf{x})\mathrm{d}s - \int_t^{t_h^n}(t_h^n - s)\partial_s\varphi(s,\mathbf{x})\mathrm{d}s\right)\right)^2\mathrm{d}t\mathrm{d}\mathbf{x} \\
&\le \sum_{n=1}^{N_h}\int_\Omega\frac{1}{(\triangle t_h^n)^2}\int_{t_h^{n-1}}^{t_h^n}\left(\left(\int_{t_h^{n-1}}^{t}(s - t_h^{n-1})^2\mathrm{d}s + \int_t^{t_h^n}(t_h^n - s)^2\mathrm{d}s\right)\int_{t_h^{n-1}}^{t_h^n}\big(\partial_s\varphi(s,\mathbf{x})\big)^2\mathrm{d}s\right)\mathrm{d}t\mathrm{d}\mathbf{x} \\
&= \big\|\partial_t\varphi\big\|_Q^2\sum_{n=1}^{N_h}\frac{1}{3(\triangle t_h^n)^2}\int_{t_h^{n-1}}^{t_h^n}\left(\int_{t_h^{n-1}}^{t}(t - t_h^{n-1})^3 + (t_h^n - t)^3\right)\mathrm{d}t \le \frac{T}{6}\Big(\max_{n=1,\dots,N_h}\triangle t_h^n\Big)\big\|\partial_t\varphi\big\|_Q^2\,, \\
\big\|\varphi_n - \varphi_h\big\|_Q^2 &= \sum_{n=1}^{N_h}\sum_{K\in\mathcal{K}_h}\int_K\left(\frac{1}{\triangle t_h^n}\int_{t_h^{n-1}}^{t_h^n}\frac{1}{|K|}\int_K(\mathbf{x}-\mathbf{y})\int_0^1\nabla\varphi\big(t,\mathbf{y}+s(\mathbf{x}-\mathbf{y})\big)\,\mathrm{d}s\mathrm{d}\mathbf{y}\mathrm{d}t\right)^2\mathrm{d}\mathbf{x} \\
&\le \sum_{n=1}^{N_h}\sum_{K\in\mathcal{K}_h}\int_K\left(\frac{1}{\triangle t_h^n|K|}\int_{t_h^{n-1}}^{t_h^n}\int_K|\mathbf{x}-\mathbf{y}|^2\mathrm{d}\mathbf{y}\mathrm{d}t\int_{t_h^{n-1}}^{t_h^n}\int_K\int_0^1\big|\nabla\varphi\big(t,\mathbf{y}+s(\mathbf{x}-\mathbf{y})\big)\big|^2\,\mathrm{d}s\mathrm{d}\mathbf{y}\mathrm{d}t\right)\mathrm{d}\mathbf{x} \\
&\le \sum_{n=1}^{N_h}\sum_{K\in\mathcal{K}_h}\frac{h^{d+2}}{3\triangle t_h^n|K|}\int_{t_h^{n-1}}^{t_h^n}\int_K\int_K\int_0^1\big|\nabla\varphi\big(t,\mathbf{y}+s(\mathbf{x}-\mathbf{y})\big)\big|^2\,\mathrm{d}s\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}t \le \frac{h^2}{3}\big\|\nabla\varphi\big\|_Q^2\,.
\end{aligned}$$

This yields the assertion by $\big\|\varphi - \varphi_h\big\|_Q \le \big\|\varphi - \varphi_n\big\|_Q + \big\|\varphi_n - \varphi_h\big\|_Q$. □

Corollary B.2. *The embedding* $E\colon \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \longrightarrow \mathrm{L}^2(Q)$, $E\phi = \phi$ *is compact.*

Proof. By Lem. (B.1) we obtain

$$\begin{aligned}
\big\|E\varphi - \Pi_h\varphi\big\|_Q &= \big\|\varphi - \varphi_h\big\|_Q \le \big\|\varphi - \varphi_n\big\|_Q + \big\|\varphi_n - \varphi_h\big\|_Q \\
&\le \frac{\max_n \triangle t_h^n}{\sqrt{6}}\big\|\partial_t\varphi\big\|_Q + \frac{h}{\sqrt{3}}\big\|\nabla\varphi\big\|_Q \longrightarrow 0\,,\quad h\in\mathcal{H}\,,
\end{aligned}$$

so that E can be approximated by a sequence of mappings $\Pi_h\colon \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big)\cap\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \longrightarrow \mathrm{L}^2(Q)$ with finite dimensional range, which shows compactness of the embedding E. □

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