Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

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submitted: March 21, 2023

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No. 3000
Berlin 2023

2020 *Mathematics Subject Classification.* 35B10, 35B40, 35Q30, 35R37, 76D05, 76D07.

Key words and phrases. Time-periodic solutions, moving boundary, exterior domain, maximal regularity, spatial decay.

Y. Shibata: Adjunct faculty member in the Department of Mechanical Engineering and Materials Science, University of Pittsburgh, USA; Partially supported by Top Global University Project, JSPS Grant-in-aid for Scientific Research (A) 17H0109, and Toyota Central Research Institute Joint Research Fund.

Edited by
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Abstract

We study an incompressible viscous flow around an obstacle with an oscillating boundary that moves by a translational periodic motion, and we show existence of strong time-periodic solutions for small data in different configurations: If the mean velocity of the body is zero, existence of time-periodic solutions is provided within a framework of Sobolev functions with isotropic pointwise decay. If the mean velocity is non-zero, this framework can be adapted, but the spatial behavior of flow requires a setting of anisotropically weighted spaces. In the latter case, we also establish existence of solutions within an alternative framework of homogeneous Sobolev spaces. These results are based on the time-periodic maximal regularity of the associated linearizations, which is derived from suitable $\mathscr{R}$-bounds for the Stokes and Oseen resolvent problems. The pointwise estimates are deduced from the associated time-periodic fundamental solutions.

1 Introduction

We consider a body with an oscillating boundary that moves through the three-dimensional space, which is filled with an incompressible viscous fluid. The fluid motion is described by the Navier–Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_t, \quad \mathbf{u}|_{\Gamma_t} = \mathbf{h}, \quad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0, \tag{1.1}$$

where Ω_t is the fluid domain with boundary Γ_t at time $t \in \mathbb{R}$. The velocity field $\mathbf{u} = (u_1, u_2, u_3)^\top$ and the pressure field $\mathfrak{p}$ are unknown, while we prescribe the external force $\mathbf{f} = (f_1, f_2, f_3)^\top$ and the fluid velocity at the boundary $\mathbf{h} = (h_1, h_2, h_3)^\top$. The constant $\mu > 0$ denotes the kinematic viscosity of the fluid. Let $\mathcal{T} > 0$ be the time period of the boundary oscillation. The body motion consists of a prescribed translation with velocity $\mathbf{v}_\mathcal{B}(t) \in \mathbb{R}^3$ such that the fluid domain after one period is given by

$$\Omega_{t+\mathcal{T}} = \Omega_t + \int_t^{t+\mathcal{T}} \mathbf{v}_\mathcal{B}(\tau)\,\mathrm{d}\tau \tag{1.2}$$

for all times $t \in \mathbb{R}$. The translational velocity $\mathbf{v}_\mathcal{B}(t)$ is assumed to be time-periodic, that is, $\mathbf{v}_\mathcal{B}(t + \mathcal{T}) = \mathbf{v}_\mathcal{B}(t)$, so that the displacement vector in (1.2) is independent of the time t. Moreover, by changing the frame of coordinates, we may assume that it is directed along the x_1-axis such that the mean velocity over one period is given by

$$\frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} \mathbf{v}_\mathcal{B}(\tau)\,\mathrm{d}\tau = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \mathbf{v}_\mathcal{B}(\tau)\,\mathrm{d}\tau = \kappa e_1$$

for some $\kappa \in \mathbb{R}$, where e_1 denotes unit vector in x_1-direction. Then κ corresponds to the mean translational speed of the body. If the data $\mathbf{f}$ and $\mathbf{h}$ are time-periodic with period $\mathcal{T}$, then the whole system (1.1) is time periodic in a frame moving with velocity κe_1. In this article, we show existence of

DOI 10.20347/WIAS.PREPRINT.3000

time-periodic solutions to this problem in suitable functional frameworks. Observe that a natural choice of boundary conditions in system (1.1) would be classical no-slip conditions, where $\mathbf{h}$ is determined by the motion of the obstacle and its boundary. In order to handle this choice, it is necessary to keep track of the dependence on the parameter κ in the final existence result, see also Remark 4.4 below.

A special case of the present situation is when the body does not oscillate and its shape is time independent. Then the problem reduces to the steady flow around a body that translates with constant speed κ. It is well known that the physical and mathematical properties of this problem strongly depend on whether $\kappa = 0$ or $\kappa \neq 0$. In particular, when $\kappa \neq 0$, one observes a wake region behind the moving body, which is reflected by an anisotropic decay of the velocity field. As was shown recently, the same behavior can be observed for the time-periodic flow past a rigid body [5]. This observation suggests the necessity to also distinguish the cases $\kappa = 0$ and $\kappa \neq 0$ in the presence of an oscillating boundary.

The mathematically rigorous study of time-periodic Navier–Stokes flows was initiated in the works of Serrin [29], Prodi [25], Yudovich [36] and Prouse [26]. While these articles focus on bounded domains, the first analytical result on time-periodic solutions to the Navier–Stokes equations in an unbounded domain was achieved decades later by Maremonti [21], who derived existence of time-periodic solutions in the three-dimensional whole space within an L^2 framework. In the case of the presence of an exterior domain, the first results on existence of time-periodic solutions are due to Salvi [28] and Maremonti and Padula [22] for $\kappa = 0$. The case $\kappa \neq 0$ is due to Galdi and Silvestre [16], who considered the situation of a general time-periodic rigid motion. Existence of time-periodic mild solutions in so-called weak Lebesgue spaces is due to the fundamental work by Yamazaki [35] for $\kappa = 0$, which was further developed to a general approach to time-periodic problems by Geissert, Hieber and Nguyen [18], who also treated the case $\kappa \neq 0$. However, the classes of solutions studied in these articles do not give suitable information on the decay of the flow far from the body. This issue was addressed by Galdi and Sohr [17] for $\kappa = 0$, and very recently by Galdi [14] for $\kappa \neq 0$, who established existence of regular solutions with pointwise spatial decay. The asymptotic behavior is also reflected in the framework of homogeneous Sobolev spaces introduced by Galdi and Kyed [15], who showed existence of time-periodic strong solutions in the case $\kappa \neq 0$ based on a framework of time-periodic maximal L^p regularity for the corresponding Oseen linearization. As was shown recently by Eiter, Kyed and Shibata [9] in a more general framework, a combination of this approach with suitable pointwise estimates leads to the existence of time-periodic solutions for $\kappa = 0$.

Nearly all of the previous articles are concerned with Navier–Stokes flows in a domain with a fixed boundary. Only in [28], the flow in an exterior domain with periodically moving boundary was considered, and existence of time-periodic weak solutions was shown. While the corresponding problem in a bounded domain has been addressed by several researchers in a framework of weak solutions [24, 23, 27], time-periodic mild solutions in a bounded domain were recently established by Farwig, Kozono, Tsuda and Wegmann [10] via a semigroup approach. These solutions were later shown to be strong [11]. Independently, Eiter, Kyed and Shibata [9] derived existence of strong solutions in a bounded domain from the aforementioned framework of time-periodic maximal regularity without relying on semigroup theory. This approach was also used to establish time-periodic solutions in the case of one-phase and two-phase flows [8]. In the present article, we follow this strategy to establish first results on the existence of time-periodic strong solutions to the Navier–Stokes equations in an exterior domain with an oscillating boundary.

We begin by transforming (1.1) to a problem in a time-independent reference domain Ω with boundary Γ. A suitable linearization leads to the system

$$\partial_t \mathbf{v} - \mu\Delta\mathbf{v} - \kappa\partial_1\mathbf{v} + \nabla\mathfrak{p} = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega\times\mathbb{T}, \qquad \mathbf{v}|_{\Gamma\times\mathbb{T}} = \mathbf{h}. \tag{1.3}$$

Here $\mathbb{T} = \mathbb{R}/\mathcal{T}\mathbb{Z}$ denotes the torus group associated with the given time period $\mathcal{T}$, and it indicates that all functions occurring in (1.3) are time periodic. To treat the full nonlinear problem by a fixed-point argument, we first derive a result on maximal regularity for this time-periodic linear problem, that is, the existence of unique solutions to (1.3) that satisfy an *a priori* estimate of the form

$$\begin{aligned}\|\partial_t \mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\nabla^2 \mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + |\kappa|\,\|\partial_1 \mathbf{v}_S\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\\ \leq C\big(\|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big)\end{aligned}$$

for suitable $p, q \in (1,\infty)$, where $\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})$ is a suitable trace space introduced below.

To derive time-periodic maximal regularity, following the approach from [8, 9], we first investigate the associated resolvent problems. Note that the Fourier coefficients $(\hat{\mathbf{v}}_k, \hat{\mathfrak{p}}_k) = (\mathscr{F}_\mathbb{T}[\mathbf{v}](k), \mathscr{F}_\mathbb{T}[\mathfrak{p}](k))$, $k \in \mathbb{Z}$, of a time-periodic solution $(\mathbf{v}, \mathfrak{p})$ to (1.3) satisfy

$$i\frac{2\pi}{\mathcal{T}}\hat{\mathbf{v}}_k - \mu\Delta\hat{\mathbf{v}}_k - \kappa\partial_1\hat{\mathbf{v}}_k + \nabla\hat{\mathfrak{p}}_k = \hat{\mathbf{f}}_k, \qquad \operatorname{div}\hat{\mathbf{v}}_k = 0 \quad \text{in } \Omega, \qquad \hat{\mathbf{v}}_k|_\Gamma = \hat{\mathbf{h}}_k, \tag{1.4}$$

and if $\mathcal{A}(k)$ is a solution operator for (1.4) such that $(\hat{\mathbf{v}}_k, \hat{\mathfrak{p}}_k) = \mathcal{A}(k)(\hat{\mathbf{f}}_k, \hat{\mathbf{h}}_k)$, then a solution $(\mathbf{v}, \mathfrak{p})$ to (1.3) is formally given by

$$(\mathbf{v}, \mathfrak{p}) = \mathscr{F}_\mathbb{T}^{-1}\big[k \mapsto \mathcal{A}(k)\mathscr{F}_\mathbb{T}[(\mathbf{f}, \mathbf{h})](k)\big]. \tag{1.5}$$

This formula reduces the question of maximal regularity to the investigation of an operator-valued Fourier multiplier, for which we apply an abstract multiplier theorem based one the notion of $\mathscr{R}$-boundedness. This application is complicated by the fact that the required $\mathscr{R}$-bounds for the family $\{\mathcal{A}(k)\}$ are only available for large k. Therefore, we proceed as in [9] and decompose the time-periodic solution into a high-frequency part and a low-frequency part. While the first can then be treated by means of Fourier multipliers on $\mathbb{T}$, the latter consists of finitely many Fourier modes that can be handled separately.

As demonstrated in [9], the resulting framework of maximal regularity is suitable to treat the nonlinear problem (1.1) in the case of a bounded domain. Compared to this, the present setting of an exterior domain comes along with two difficulties: Firstly, for $k = 0$ the resolvent problem (1.5) is not uniquely solvable in a framework of classical Sobolev spaces. Therefore, we decompose the time-periodic linearized problem (1.3) into the associated steady-state problem and a purely oscillatory problem, which are studied in separate functional frameworks. Secondly, the treatment of the nonlinear problem requires suitable estimates of the nonlinear terms. Those cannot be derived within the resulting maximal-regularity function class, at least for $\kappa = 0$. Therefore, we complement the setting with pointwise estimates that are derived from the time-periodic fundamental solutions to (1.3), which were introduced by Eiter and Kyed [7]. In the resulting framework, the contraction mapping principle can be used to derive existence of time-periodic solutions to the full nonlinear problem for $\kappa = 0$, see Theorem 4.1. In the case $\kappa \neq 0$, the same method can be employed, which results in Theorem 4.2, but we have to take into account the anisotropic decay of the flow. However, since the maximal-regularity framework for $\kappa \neq 0$ leads to better integrability properties of solutions, we can also implement a fixed-point argument without enriching the functional setting with pointwise decay properties, see Theorem 4.3.

We want to emphasize that our analysis also leads to new existence results in the case of a moving body with a fixed boundary. In [15], existence of strong time-periodic solutions was established for $\kappa \neq 0$ under the assumption that the translational velocity has the same direction at all times, that is, $\mathbf{v}_\mathcal{B}(t) = v_\mathcal{B}(t)e_1$ for some scalar time-periodic function $v_\mathcal{B}$ with non-zero mean. This assumption was necessary to work with an Oseen linearization in a frame attached to the body, that is, moving with the actual translational velocity $\mathbf{v}_{\mathcal{B}}$. In contrast, we shall work in a frame moving with the *mean* translational velocity κe_1. Regarding the remainder $\mathbf{v}_{\mathcal{B}} - \kappa e_1$ as an oscillation of the boundary, we can omit the restrictions on $\mathbf{v}_{\mathcal{B}}$ from [15]. For more details, see Section 3.

The structure of the article is as follows: After introducing the general notation in Section 2, we reformulate the nonlinear system (1.1) as a problem on a fixed domain in Section 3. In Section 4, we then state our main results on the nonlinear and the linear problems. The proofs for the linear theory for $\kappa \neq 0$ are provided in Section 5, while they were provided in [9] for $\kappa = 0$. In Section 6 we conclude by the proofs of the existence results for the nonlinear problem (1.1).

2 Notation

For topological vector spaces X and Y, we denote the space of continuous linear operators from X to Y by $\mathscr{L}(X, Y)$. We write X' for the dual space of X, and when X is a normed space, then $\|\cdot\|_X$ denotes is norm.

By Ω we denote a three-dimensional exterior C^2-domain, that is, a domain that is the complement of a compact set with connected C^2-boundary $\Gamma = \partial\Omega$. Let $b > 0$ be a sufficiently large radius such that $\Gamma \subset B_b := \{x \in \mathbb{R}^3 \mid |x| < b\}$.

We write $\partial_j := \partial_{x_j}$ for partial derivatives in space, and ∇, div and Δ denote gradient, divergence and Laplace operator, which only act in spatial variables. For a sufficiently regular function u and $k \in \mathbb{N}$, we denote the collection of all k-th order derivatives by $\nabla^k u$.

For classical Lebesgue and Sobolev spaces we write $\mathrm{L}_q(\Omega)$ and $\mathrm{H}^k_q(\Omega)$, where $q \in [1, \infty]$ and $k \in \mathbb{N}$, and $\mathrm{L}_{q,\mathrm{loc}}(\Omega)$ and $\mathrm{H}^k_{q,\mathrm{loc}}(\Omega)$ denote their local variants. Homogeneous Sobolev spaces are defined via

$$\hat{\mathrm{H}}^k_q(\Omega) := \big\{u \in \mathrm{L}_{1,\mathrm{loc}}(\Omega) \mid \nabla^k u \in \mathrm{L}_q(\Omega)\big\}$$

When it is clear from the context, we sometimes use the same notation for spaces of vector-valued or matrix-valued functions. For example, we write $\mathrm{L}_q(\Omega)$ instead of $\mathrm{L}_q(\Omega)^3$ or $\mathrm{L}_q(\Omega)^{3\times 3}$.

For a given time period $\mathcal{T} > 0$ we let $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ denote the corresponding torus group. Functions on $\mathbb{T}$ can then be identified with $\mathcal{T}$-periodic functions on $\mathbb{R}$, which we do tacitly in what follows. We equip the topological group $\mathbb{T}$ with the normalized Haar measure defined via

$$\forall u \in C^0(\mathbb{T}): \quad \int_{\mathbb{T}} u(t)\,\mathrm{d}t = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} u(\tau)\,\mathrm{d}\tau.$$

Bochner–Lebesgue spaces are denoted by $\mathrm{L}_p(\mathbb{T}; X)$ for $p \in [1, \infty]$, and we set

$$\mathrm{H}^1_p(\mathbb{T}, X) := \{u \in \mathrm{L}_p(\mathbb{T}, X) \mid \partial_t u \in \mathrm{L}_p(\mathbb{T}, X)\}, \;\; \|u\|_{\mathrm{H}^1_p(\mathbb{T},X)} := \|u\|_{\mathrm{L}_p(\mathbb{T},X)} + \|\partial_t u\|_{\mathrm{L}_p(\mathbb{T},X)}.$$

The velocity field $\mathbf{u}$ of a solution will be identified in the classical parabolic space, at least near the boundary, that is, $\mathbf{u} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega_b)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega_b)^3)$ for some $p, q \in (1, \infty)$. To shorten notation, we introduce the corresponding class of boundary traces on $\Gamma \times \mathbb{T}$ via

$$\mathrm{T}_{p,q}(\Gamma \times \mathbb{T}) := \big\{\mathbf{h} = \mathbf{u}|_{\Gamma\times\mathbb{T}} \;\big|\; \mathbf{u} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3)\big\} \tag{2.1}$$

with corresponding norm

$$\|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})} := \inf\big\{\|\mathbf{u}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\partial_t \mathbf{u}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \;\big|\; \mathbf{h} = \mathbf{u}|_{\Gamma\times\mathbb{T}}\big\}.$$

Note that $\mathrm{T}_{p,q}(\Gamma \times \mathbb{T})$ can be identified with a real interpolation space, but we shall not make use of this property in what follows.

We often decompose time-periodic functions $f\colon \Omega \times \mathbb{T} \to \mathbb{R}$ into a steady-state part f_S and a purely oscillatory part $f_\perp$ defined by

$$f_S(x) = \int_{\mathbb{T}} f(x,t)\,\mathrm{d}t, \qquad f_\perp(x,t) = f(x,t) - f_S(x). \tag{2.2}$$

To quantify decay rates of these two parts, we use the norms

$$< f_S >_\alpha = \sup_{x\in\Omega} |f_S(x)|(1+|x|)^\alpha, \qquad < f_S >^w_{\alpha,\beta} = \sup_{x\in\Omega} |f_S(x)|(1+|x|)^\alpha (1+|x|-x_1)^\beta$$

for the steady-state parts and

$$< f_\perp >_{p,\alpha} = \sup_{x\in\Omega} \|f_\perp(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})}(1+|x|)^\alpha$$

for the purely oscillatory part, where $\alpha, \beta \geq 0$ and $p \in (1,\infty)$.

We equip $\Omega\times\mathbb{T}$ with the product measure and denote the associated Lebesgue spaces by $\mathrm{L}_p(\Omega\times\mathbb{T})$. If $\Omega = \mathbb{R}^3$, then $\mathbb{R}^3 \times \mathbb{T}$ is a locally compact abelian group, and we can define generalized Schwartz spaces and spaces of tempered distributions on $\mathbb{T}$ and $\mathbb{R}^3 \times \mathbb{T}$ and the respective dual groups $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{R}^3$. see [3, 6]. Moreover, there is an associated notion of Fourier transform, which can be defined via

$$\mathscr{F}_{\mathbb{T}}[u](k) := \int_{\mathbb{T}} u(t)\,\mathrm{e}^{-i\frac{2\pi}{\mathcal{T}}kt}\,\mathrm{d}t, \qquad \mathscr{F}_{\mathbb{T}}^{-1}[w](t) := \sum_{k\in\mathbb{Z}} w(k)\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt},$$

and we set $\mathscr{F}_{\mathbb{R}^3\times\mathbb{T}} = \mathscr{F}_{\mathbb{R}^3} \otimes \mathscr{F}_{\mathbb{T}}$ and $\mathscr{F}^{-1}_{\mathbb{R}^3\times\mathbb{T}} = \mathscr{F}^{-1}_{\mathbb{R}^3} \otimes \mathscr{F}^{-1}_{\mathbb{T}}$, where the Fourier transform in the Euclidean setting and its inverse are given by

$$\mathscr{F}_{\mathbb{R}^3}[u](\xi) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} u(x)\,\mathrm{e}^{-i\xi\cdot x}\,\mathrm{d}x, \qquad \mathscr{F}^{-1}_{\mathbb{R}^3}[w](x) := \int_{\mathbb{R}^3} w(\xi)\,\mathrm{e}^{i\xi\cdot x}\,\mathrm{d}\xi.$$

To study operator-valued Fourier multipliers, we further need the notion of UMD spaces, which are Banach spaces X such that the Hilbert transform H, defined by

$$Hf(t) := \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{f(t-s)}{s}\,\mathrm{d}s,$$

is a bounded linear operator on $\mathrm{L}_p(\mathbb{R}, X)$ some $p \in (1,\infty)$. Moreover, we say that a family of operators $\mathcal{T} \subset \mathscr{L}(X,Y)$ is $\mathscr{R}$-bounded in $\mathscr{L}(X,Y)$ if there is $C > 0$ such that

$$\Big\|\sum_{k=1}^n r_k T_k f_k\Big\|_{\mathrm{L}_1((0,1),Y)} \leq C\Big\|\sum_{k=1}^n r_k f_k\Big\|_{\mathrm{L}_1((0,1),X)} \tag{2.3}$$

for all $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \in \mathcal{T}^n$, and $\{f_j\}_{j=1}^n \in X^n$. Here $r_k\colon [0,1] \to \{-1,1\}$, $t \mapsto \operatorname{sign}(\sin 2^k\pi t)$, denote Rademacher functions. Moreover, $\mathscr{R}_{\mathscr{L}(X,Y)}\mathcal{T}$ denotes the smallest constant C such that (2.3) holds.

For $\varepsilon \in (0,\pi/2)$ and $\delta > 0$ we define the perturbed sector

$$\Sigma_{\varepsilon,\delta} := \{\lambda \in \mathbb{C} \mid |\lambda| > \delta,\ |\arg\lambda| < \pi - \varepsilon\}.$$

Moreover, $\operatorname{Hol}(\Sigma_{\varepsilon,\delta}, X)$ denotes the class of X-valued holomorphic functions on $\Sigma_{\varepsilon,\delta}$.

3 Formulation on a reference domain

To reformulate the system (1.1) as a problem in a time-independent spatial domain, we describe the motion of the body and its boundary by suitable functions.

Let Ω be the exterior domain in $\mathbb{R}^3$. Let $\mathbf{v}_{\mathcal{B}} \in C^0(\mathbb{R})^3$ with $\mathbf{v}_{\mathcal{B}}(t+\mathcal{T}) = \mathbf{v}_{\mathcal{B}}(t)$ for all $t \in \mathbb{R}$, and let

$$\phi \in C^0(\mathbb{R}; C^3(\Omega)^3) \cap C^1(\mathbb{R}; C^1(\Omega)^3) \tag{3.1}$$

such that $\phi(y,0) = 0$ and $\phi(y,t+\mathcal{T}) = \phi(y,t)$ for each $t \in \mathbb{R}$ and $y \in \Omega$, and such that $\phi(y,t) = 0$ for $y \notin B_{2b}$. Then the fluid domain $\Omega_t \subset \mathbb{R}^3$ shall be given by

$$\Omega_t = \Big\{ x = y + \phi(y,t) + \int_0^t \mathbf{v}_{\mathcal{B}}(\tau)\,\mathrm{d}\tau \,\Big|\, y \in \Omega \Big\} \quad (t \in \mathbb{R}). \tag{3.2}$$

By rotating the coordinate frame, we may assume that the mean velocity over one time period is directed along the x_1-axis such that there is $\kappa \in \mathbb{R}$ with

$$\kappa e_1 = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \mathbf{v}_{\mathcal{B}}(\tau)\,\mathrm{d}\tau.$$

Then we can redefine ϕ in such a way that

$$\Omega_t = \Big\{ x = y + \phi(y,t) + t\kappa e_1 \,\Big|\, y \in \Omega \Big\} \quad (t \in \mathbb{R}) \tag{3.3}$$

instead of (3.2). Indeed, since

$$\phi(y,t) + \int_0^t \mathbf{v}_{\mathcal{B}}(\tau)\,\mathrm{d}\tau = \Big(\phi(y,t) + \int_0^t (\mathbf{v}_{\mathcal{B}}(\tau) - \kappa e_1)\,\mathrm{d}\tau \Big) + t\kappa e_1,$$

we may assume that $\mathbf{v}_{\mathcal{B}}$ is constant in time and replace ϕ with the term in parenthesis, which defines a time-periodic function. Notice that to preserve the condition $\phi(y,t) = 0$ for $|y| > 2b$, it might be necessary to multiply the term by a suitable cut-off function, which would not change the set Ω_t.

Given κ and ϕ, the domain Ω_t is the image of the transformation $\Phi_t\colon \Omega \to \mathbb{R}^3$, $\Phi_t(y) = y + \phi(y,t) + t\kappa e_1$, for $t \in \mathbb{R}$, and the boundary $\Gamma_t = \partial\Omega_t$ is given by $\Gamma_t = \{x = y + \phi(y,t) + t\kappa e_1 \mid y \in \Gamma\}$. To reduce system (1.1) to a problem in the reference domain $\Omega = \Omega_0$, we assume that

$$\sup_{t\in\mathbb{R}} \|\phi(\cdot,t)\|_{\mathrm{H}^3_\infty(\Omega)} + \sup_{t\in\mathbb{R}} \|\partial_t\phi(\cdot,t)\|_{\mathrm{H}^1_\infty(\Omega)} \leq \varepsilon_0 \tag{3.4}$$

with some small number $\varepsilon_0 > 0$, and we use the change of variables induced by Φ_t, namely $x = y + \phi(y,t) + t\kappa e_1$. By the smallness assumption (3.4), we may assume the existence of the inverse transformation, which has the form $y = x + \psi(x,t) - t\kappa e_1$. The associated Jacobi matrix $\partial(t,y)/\partial(t,x)$ is given by the formulas:

$$\frac{\partial t}{\partial t} = 1, \quad \frac{\partial t}{\partial x_j} = 0, \quad \frac{\partial y_\ell}{\partial t} = \frac{\partial \psi_\ell}{\partial t} - \kappa e_1, \quad \frac{\partial y_\ell}{\partial x_j} = \delta_{\ell j} + \frac{\partial \psi_\ell}{\partial x_j}$$

for $j, \ell = 1,2,3$. Set

$$a_{\ell 0}(y,t) = (\partial\psi_\ell/\partial t)(y + \phi(y,t) + t\kappa e_1, t), \qquad a_{\ell j}(y,t) = (\partial\psi_\ell/\partial x_j)(y + \phi(y,t) + t\kappa e_1, t).$$

Then partial derivatives transform as

$$\frac{\partial f}{\partial t}=\frac{\partial g}{\partial t}+\sum_{\ell=1}^{3}a_{\ell 0}(y,t)\frac{\partial g}{\partial y_\ell}-\kappa\frac{\partial g}{\partial y_1},\qquad \frac{\partial f}{\partial x_j}=\frac{\partial g}{\partial y_j}+\sum_{\ell=1}^{3}a_{\ell j}(y,t)\frac{\partial g}{\partial y_\ell} \tag{3.5}$$

for $f(x,t)=g(y,t)$. Let $\mathrm{J}=\det(\partial x/\partial y)=1+\mathrm{J}_0(y,t)$ be the Jacobian of Φ_t. From (3.4) we obtain $C>0$ such that

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\|a_{\ell j}(\cdot,t)\|_{\mathrm{H}^2_\infty(\Omega)}+\sup_{t\in\mathbb{R}}\|\partial_t a_{\ell j}(\cdot,t)\|_{\mathrm{L}_\infty(\Omega)}+\sup_{t\in\mathbb{R}}\|a_{0j}(\cdot,t)\|_{\mathrm{L}_\infty(\Omega)}\\&\quad+\sup_{t\in\mathbb{R}}\|\mathrm{J}_0(\cdot,t)\|_{\mathrm{H}^2_\infty(\Omega)}+\sup_{t\in\mathbb{R}}\|\partial_t\mathrm{J}_0(\cdot,t)\|_{\mathrm{L}_\infty(\Omega)}\le C\varepsilon_0\end{aligned} \tag{3.6}$$

for $j,\ell=1,2,3$. For $\mathbf{v}(y,t)=(v_1,v_2,v_3)^\top=\mathbf{u}(x,t)$, and $\mathfrak{q}(y,t)=\mathfrak{p}(x,t)$ we then have

$$\begin{aligned}&\partial_t\mathbf{u}=\partial_t\mathbf{v}+\sum_{\ell=1}^{3}a_{\ell 0}\frac{\partial\mathbf{v}}{\partial y_\ell}-\kappa\frac{\partial\mathbf{v}}{\partial y_1},\quad \mathbf{u}\cdot\nabla\mathbf{u}=\mathbf{v}\cdot(\mathrm{I}+\mathrm{A})\nabla\mathbf{v},\\
&\Delta\mathbf{u}=\Delta\mathbf{v}+\sum_{\ell=1}^{3}(a_{\ell j}+a_{j\ell})\frac{\partial^2\mathbf{v}}{\partial y_\ell\partial y_j}+\sum_{j,\ell,m=1}^{3}a_{\ell j}a_{mj}\frac{\partial^2\mathbf{v}}{\partial y_\ell\partial y_m}\\
&\qquad\qquad+\sum_{\ell,m=1}^{3}\left(\frac{\partial a_{m\ell}}{\partial y_\ell}+\sum_{j=1}^{3}a_{\ell j}\frac{\partial a_{mj}}{\partial y_\ell}\right)\frac{\partial\mathbf{v}}{\partial y_m},\\
&\operatorname{div}\mathbf{u}=\mathrm{J}^{-1}\Big(\operatorname{div}\mathbf{v}+\operatorname{div}(\mathrm{J}_0\mathbf{v})+\sum_{j,\ell=1}^{3}\frac{\partial}{\partial y_\ell}(a_{\ell j}\mathrm{J}v_j)\Big),\quad \nabla\mathfrak{p}=(\mathrm{I}+\mathrm{A})\nabla\mathfrak{q},\end{aligned}$$

where A is a (3×3)-matrix whose (j,k)-th component is a_{jk}. Setting $w_\ell=v_\ell+\mathrm{J}_0v_\ell+\sum_{j=1}^{3}a_{\ell j}\mathrm{J}v_j$, we have $\mathrm{J}\operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{w}$ with $\mathbf{w}=(w_1,w_2,w_3)^\top$. Notice that $\mathbf{w}=(\mathrm{I}+\mathrm{J}_0\mathrm{I}+\mathrm{A}^\top\mathrm{J})\mathbf{v}$. In view of (3.6), choosing $\varepsilon_0>0$ sufficiently small, we see that there exists a (3×3)-matrix B_{-1} such that $(\mathrm{I}+\mathrm{J}_0\mathrm{I}+\mathrm{A}^\top\mathrm{J})^{-1}=\mathrm{I}+\mathrm{B}_{-1}$ and

$$\sup_{t\in\mathbb{R}}\|\mathrm{B}_{-1}(\cdot,t)\|_{\mathrm{H}^2_\infty(\Omega)}\le C\varepsilon_0,\quad \sup_{t\in\mathbb{R}}\|\partial_t\mathrm{B}_{-1}(\cdot,t)\|_{\mathrm{L}_\infty(\Omega)}\le C\varepsilon_0. \tag{3.7}$$

We further replace the time axis with the torus group $\mathbb{T}:=\mathbb{R}/\mathcal{T}\mathbb{Z}$ associated to the period $\mathcal{T}$. In total, system (1.1) is transformed to

$$\partial_t\mathbf{w}-\mu\Delta\mathbf{w}-\kappa\partial_1\mathbf{w}+\nabla\mathfrak{q}=\mathbf{f}+\mathcal{L}(\mathbf{w},\mathfrak{q})+\mathcal{N}(\mathbf{w}),\quad \operatorname{div}\mathbf{w}=0\quad\text{in }\Omega\times\mathbb{T},\qquad \mathbf{w}|_{\Gamma\times\mathbb{T}}=\mathbf{h}, \tag{3.8}$$

where the data $\mathbf{f}$ and $\mathbf{h}$ are now prescribed with respect to the reference domain Ω, and

$$\begin{aligned}\mathcal{L}(\mathbf{w},\mathfrak{q})&=-\partial_t(\mathrm{B}_{-1}\mathbf{w})-\sum_{\ell=1}^{3}a_{\ell 0}\frac{\partial}{\partial y_\ell}((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w})+\mu\Delta(\mathrm{B}_{-1}\mathbf{w})\\
&\quad+\sum_{\ell=1}^{3}(a_{\ell j}+a_{\ell j})\frac{\partial^2}{\partial y_\ell\partial y_j}((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w})+\sum_{j,\ell,m=1}^{3}a_{\ell j}a_{mj}\frac{\partial^2}{\partial y_\ell\partial y_m}((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w})\\
&\quad+\sum_{\ell,m=1}^{3}\left(\frac{\partial a_{m\ell}}{\partial y_\ell}+\sum_{j=1}^{3}a_{\ell j}\frac{\partial a_{mj}}{\partial y_\ell}\right)\frac{\partial}{\partial y_m}((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w})\\
&\quad-\kappa\{\partial_1(\mathrm{B}_{-1}\mathbf{w})+\sum_{\ell=1}^{3}a_{1\ell}\frac{\partial}{\partial y_\ell}((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w}\}-A\nabla\mathfrak{q},\\
\mathcal{N}(\mathbf{w})&=((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w})\cdot(\mathrm{I}+\mathrm{A})\nabla((\mathrm{I}+\mathrm{B}_{-1})\mathbf{w}).\end{aligned} \tag{3.9}$$

Notice that in (3.8) we omitted that $\mathbf{w}$ vanishes at infinity. This condition will later be included in a suitable sense in the definition of the function spaces.

Remark 3.1. In both formulations (1.1) and (3.8), we consider a general class of boundary data in the from of an inhomogeneous Dirichlet condition. The most classical choice would be given by no-slip conditions such that the fluid velocity coincides with the boundary velocity. With the notation from above, this means to assume that

$$\mathbf{u}(x,t) = \mathbf{v}(y,t) = \partial_t \Phi_t(y) = \partial_t \phi(t,y) + \kappa e_1$$

for $y \in \Gamma$ and $x = \Phi_t(y) = y + \phi(y,t) + t\kappa e_1 \in \Gamma_t$. Therefore, no-slip conditions correspond to the choice

$$\mathbf{h} = (\mathrm{I} + \mathrm{J}_0 \mathrm{I} + \mathrm{A}^\top \mathrm{J})(\partial_t \phi + \kappa e_1) \tag{3.10}$$

in system (3.8). In particular, the prescribed boundary data $\mathbf{h}$ depend on the translational velocity κ, which also appears as a parameter in the linearization of (3.8). Therefore, for the treatment of no-slip conditions, the dependence of smallness conditions on κ has to be taken into account. In Remark 4.4 we clarify in how far no-slip conditions can be handled in the frameworks proposed here.

4 Main results

We first state the results on existence of time-periodic solutions to (3.8). Their proofs will be based on the study of a suitable linearization, which is given by

$$\partial_t \mathbf{v} - \mu\Delta\mathbf{v} - \kappa\partial_1\mathbf{v} + \nabla\mathfrak{p} = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega\times\mathbb{T}, \qquad \mathbf{v}|_{\Gamma\times\mathbb{T}} = \mathbf{h}. \tag{4.1}$$

We obtain a time-periodic Stokes system for $\kappa = 0$, and a time-periodic Oseen problem for $\kappa \neq 0$, which have different mathematical properties. The results on unique existence of solutions to the linear problem (4.1) are collected in Subsection 4.2.

4.1 Solutions to the nonlinear problem

In the theorems on existence of solutions to problem (3.8), we always assume

$$2 < p < \infty, \quad 3 < q < \infty, \quad h \in \mathrm{T}_{p,q}(\Gamma\times\mathbb{T}), \quad \phi \in C^0(\mathbb{R}; C^3(\Omega)^3) \cap C^1(\mathbb{R}; C^1(\Omega)^3), \tag{4.2}$$

where $\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})$ is the space from (2.1). We begin with the case without translation, that is, where $\kappa = 0$ in (3.8). To quantify the pointwise decay of functions, we use the weighted norms introduced in Section 2.

Theorem 4.1. *Assume* (4.2) *and let* $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ *with* $\mathbf{f}_S = \operatorname{div}\mathbf{F}_S$ *and* $\mathbf{f}_\perp = \operatorname{div}\mathbf{F}_\perp$. *There exist constants* $\varepsilon, \varepsilon_0 > 0$ *such that if the smallness conditions* (3.4) *and*

$$< \mathbf{f}_S >_3 + < \mathbf{F}_S >_2 + < \mathbf{f}_\perp >_{p,2} + < \mathbf{F}_\perp >_{p,1} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})} < \varepsilon^2 \tag{4.3}$$

are satisfied, then problem (3.8) *with* $\kappa = 0$ *admits a unique solution* $(\mathbf{w}, \mathfrak{p})$ *with*

$$\mathbf{w} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega))$$

satisfying the estimate

$$< \mathbf{w} >_{p,1} + < \nabla\mathbf{w} >_{p,2} + \|\mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\partial_t\mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\nabla\mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \leq \varepsilon.$$

In the case with translation, that is, where $\kappa \neq 0$ in (3.8), we obtain existence of a time-periodic solution with anisotropic pointwise decay.

Theorem 4.2. *Let $\kappa_0 > 0$ and $\delta \in (0, 1/4)$, and assume* (4.2). *Let $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ with $\mathbf{f}_\perp = \operatorname{div} \mathbf{F}_\perp$. Then there exist $\varepsilon, \varepsilon_0 > 0$ such that if the smallness conditions* (3.4) *and*

$$< \mathbf{f}_S >^w_{5/2,1/2+2\delta} + < \mathbf{f}_\perp >_{p,2+\delta} + < \mathbf{F}_\perp >_{p,1+\delta} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})} < \varepsilon^2 |\kappa|^{2\delta} \tag{4.4}$$

are satisfied, then problem (3.8) *with $\kappa \neq 0$ admits a unique solution $(\mathbf{w}, \mathfrak{p})$ with*

$$\mathbf{w} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega))$$

satisfying the estimate

$$\begin{aligned} < \mathbf{w}_S >^w_{1,\delta} + < \nabla \mathbf{w}_S >^w_{3/2,1/2+\delta} + < \mathbf{w}_\perp >_{p,1+\delta} + < \nabla \mathbf{w}_\perp >_{p,2+\delta} \\ + \|\mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\partial_t \mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \leq \varepsilon |\kappa|^{2\delta}. \end{aligned}$$

Alternatively, the case $\kappa \neq 0$ allows to avoid spaces of functions with suitable pointwise decay, such that the spatial asymptotics are merely quantified in terms of integrability. However, the steady-state part of the velocity field only belongs to suitable homogeneous Sobolev spaces.

Theorem 4.3. *Let $\kappa_0 > 0$ and $\delta \in (0, 1)$, and assume* (4.2). *Let $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ and $1 < s < 4/3$. Then there exist $\varepsilon, \varepsilon_0 > 0$ such that if the smallness conditions* (3.4) *and*

$$\|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})} \leq \varepsilon^2 |\kappa|^{1/(1+\delta)} \tag{4.5}$$

are satisfied, then problem (3.8) *with $\kappa \neq 0$ admits a unique solution $(\mathbf{w}, \mathfrak{p})$ with $\mathbf{w} = \mathbf{w}_S + \mathbf{w}_\perp$ and*

$$\mathbf{w}_S \in \hat{\mathrm{H}}^2(\Omega)^3, \quad \mathbf{w}_\perp \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega))$$

satisfying the estimate

$$\begin{aligned} \|\nabla^2 \mathbf{w}_S\|_{\mathrm{L}_s(\Omega)} + |\kappa|^{1/4} \|\nabla \mathbf{w}_S\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} + |\kappa|^{1/2} \|\mathbf{w}_S\|_{\mathrm{L}_{2s/(2-s)}(\Omega)} + |\kappa| \, \|\partial_1 \mathbf{w}_S\|_{\mathrm{L}_s(\Omega)} \\ + \|\nabla^2 \mathbf{w}_S\|_{\mathrm{L}_q(\Omega)} + \|\partial_t \mathbf{w}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathbf{w}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_s(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} \\ + \|\partial_t \mathbf{w}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{w}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \leq \varepsilon \, |\kappa|^{1/2}. \end{aligned}$$

Theorem 4.1, Theorem 4.2 and Theorem 4.3 will be proved in Section 6.

Remark 4.4. While Theorem 4.1 deals with the case $\kappa = 0$ of vanishing translational velocity, Theorem 4.2 and Theorem 4.3 yield existence of time-periodic solutions for arbitrary large $\kappa \neq 0$ if the data are sufficiently small. However, the treatment of no-slip boundary conditions requires to take $|\kappa|$ small. Indeed, as explained in Remark 3.1, no-slip conditions are expressed by boundary data $\mathbf{h}$ of the form (3.10). In virtue of (3.4), (3.6) and (3.7), we can then estimate

$$\|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})} \leq C(1 + \varepsilon_0)(\varepsilon_0 + |\kappa|).$$

Therefore, the smallness conditions (4.4) and (4.5) can be satisfied by fixing $\kappa_0 > 0$ and choosing $|\kappa|$ and $\varepsilon_0 > 0$ sufficiently small.

4.2 The associated linear problems

In the case $\kappa = 0$, system (4.1) reduces to the time-periodic Stokes equations in an exterior domain. The following existence theorem was shown in [9]. We set $\mathrm{L}_{q,3b}(\Omega) = \{f \in \mathrm{L}_q(\Omega) \mid \operatorname{supp} f \subset B_{3b}\}$ to shorten the notation.

Theorem 4.5. *Let $\kappa = 0$. Let $1 < p < \infty$, $3 < q < \infty$ and $\ell \in (0,3]$. For all $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ such that $\mathbf{f}_S = \operatorname{div} \mathbf{F}_S + \mathbf{g}_S$ and $\mathbf{f}_\perp = \operatorname{div} \mathbf{F}_\perp + \mathbf{g}_\perp$ with $\mathbf{g} = \mathbf{g}_S + \mathbf{g}_\perp \in \mathrm{L}_p(\mathbb{T}, \mathrm{L}_{q,3b}(\Omega)^3)$ and*

$$< \mathbf{F}_S >_2 + < \operatorname{div} \mathbf{F}_S >_3 + < \mathbf{F}_\perp >_{p,\ell} + < \operatorname{div} \mathbf{F}_\perp >_{p,\ell+1} < \infty,$$

and for all $\mathbf{h} \in \mathrm{T}_{p,q}(\Gamma \times \mathbb{T})$, problem (4.1) with $\kappa = 0$ admits a unique solution $(\mathbf{v}, \mathfrak{p})$ with

$$\mathbf{v} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega)),$$

possessing the estimate

$$\begin{aligned} &\|\partial_t \mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\ &\quad + < \mathbf{v}_S >_1 + < \nabla \mathbf{v}_S >_2 + < \mathbf{v}_\perp >_{p,\ell} + < \nabla \mathbf{v}_\perp >_{p,\ell+1} \\ &\qquad \leq C\big(< \operatorname{div} \mathbf{F}_S >_3 + < \mathbf{F}_S >_2 + < \operatorname{div} \mathbf{F}_\perp >_{p,\ell+1} + < \mathbf{F}_\perp >_{p,\ell} \\ &\qquad\qquad + \|\mathbf{g}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big). \end{aligned} \tag{4.6}$$

Here the constant $C > 0$ only depends on Ω, $\mathcal{T}$, μ, p, q, and ℓ.

If $\kappa \neq 0$, then (4.1) is a time-periodic Oseen problem, and we have to take into account the anisotropic spatial behavior of solutions. In this case we shall derive the following result on existence of solutions with suitable pointwise decay.

Theorem 4.6. *Let $0 < |\kappa| \leq \kappa_0$. Let $1 < p < \infty$, $3 < q < \infty$, $\delta \in (0, \frac{1}{4})$ and $\ell \in (0,3]$. For all $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ such that $\mathbf{f}_S = \tilde{\mathbf{f}}_S + \mathbf{g}_S$ and $\mathbf{f}_\perp = \operatorname{div} \mathbf{F}_\perp + \mathbf{g}_\perp$ with $\mathbf{g} = \mathbf{g}_S + \mathbf{g}_\perp \in \mathrm{L}_p(\mathbb{T}, \mathrm{L}_{q,3b}(\Omega)^3)$ and*

$$< \tilde{\mathbf{f}}_S >^w_{5/2,1/2+2\delta} + < \operatorname{div} \mathbf{F}_\perp >_{p,1+\ell} + < \mathbf{F}_\perp >_{p,\ell} < \infty,$$

and for all $\mathbf{h} \in \mathrm{T}_{p,q}(\Gamma \times \mathbb{T})$, problem (4.1) admits a unique solution $(\mathbf{v}, \mathfrak{p})$ with

$$\mathbf{v} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega)),$$

possessing the estimate

$$\begin{aligned} &\|\partial_t \mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\ &\quad + |\kappa|^\delta < \mathbf{v}_S >^w_{1,\delta} + |\kappa|^\delta < \nabla \mathbf{v}_S >^w_{3/2,1/2+\delta} + < \mathbf{v}_\perp >_{p,\ell} + < \nabla \mathbf{v}_\perp >_{p,\ell+1} \\ &\qquad \leq C\big(< \tilde{\mathbf{f}}_S >^w_{5/2,1/2+2\delta} + < \operatorname{div} \mathbf{F}_\perp >_{p,\ell+1} + < \mathbf{F}_\perp >_{p,\ell} \\ &\qquad\qquad + \|\mathbf{g}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big). \end{aligned} \tag{4.7}$$

Here the constant $C > 0$ only depends on Ω, $\mathcal{T}$, μ, p, q, δ, ℓ and κ_0.

Alternatively, the following well-posedness result for $\kappa \neq 0$ does not quantify the decay of the data and the solutions in a pointwise sense, but merely uses (homogeneous) Sobolev spaces.

Theorem 4.7. *Let $1 < p < \infty$, $1 < s < 2$. For all $\mathbf{f} \in \mathrm{L}_p(\mathbb{T}, \mathrm{L}_s(\Omega)^3)$ and $\mathbf{h} \in \mathrm{T}_{p,s}(\Gamma \times \mathbb{T})$ problem* (4.1) *with $\kappa \neq 0$ admits a unique solution $(\mathbf{v}, \mathfrak{p})$ with $\mathbf{v} = \mathbf{v}_S + \mathbf{v}_\perp$ satisfying*

$$\mathbf{v}_S \in \hat{\mathrm{H}}^2_s(\Omega)^3 \cap \mathrm{L}_{2s/(2-s)}(\Omega)^3, \quad \mathbf{v}_\perp \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_s(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_s(\Omega)^3), \quad \mathfrak{p} \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_s(\Omega)),$$

possessing the estimate

$$\begin{aligned} \|\nabla^2 \mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + |\kappa|^{1/4} \|\nabla \mathbf{v}_S\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} + |\kappa|^{1/2} \|\mathbf{v}_S\|_{\mathrm{L}_{2s/(2-s)}(\Omega)} + |\kappa| \, \|\partial_1 \mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} \\ + \|\partial_t \mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_s(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_s(\Omega))} + \|\nabla \mathfrak{p}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_s(\Omega))} \\ \leq C \big(\|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_s(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,s}(\Gamma \times \mathbb{T})} \big). \end{aligned} \tag{4.8}$$

If additionally $\mathbf{f} \in \mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3)$ and $\mathbf{h} \in \mathrm{T}_{p,q}(\Gamma \times \mathbb{T})$ for some $q \in (1, \infty)$, then

$$\|\nabla^2 \mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} + |\kappa| \, \|\partial_1 \mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} \leq C \big(\|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma \times \mathbb{T})} \big). \tag{4.9}$$

If $|\kappa| \leq \kappa_0$ and $1 < s < 3/2$, then the constant $C > 0$ only depends on Ω, $\mathcal{T}$, μ, p, q, s and κ_0.

For a proof of Theorem 4.5 we refer to [9, Theorem 5.2]. The derivation of Theorem 4.6 and Theorem 4.7 is the scope of Section 5.

5 The time-periodic Oseen problem

In this section we show existence of solutions to the time-periodic Oseen problem (4.1) for $\kappa \neq 0$ with suitable decay properties as stated in Theorem 4.6. To this end, we decompose all functions into a steady-state part and a purely oscillatory part according to (2.2). Due to the linearity of the system, this leads to two problems, which we examine in the case of homogeneous boundary conditions. Firstly, we obtain the steady-state problem

$$-\mu \Delta \mathbf{u} - \kappa \partial_1 \mathbf{u} + \nabla \mathfrak{p} = \mathbf{f}_S, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0, \tag{5.1}$$

that is, we study time-independent solutions to (4.1). Secondly, we consider purely oscillatory solutions to (4.1), which leads to the problem

$$\partial_t \mathbf{v}_\perp - \mu \Delta \mathbf{v}_\perp - \kappa \partial_1 \mathbf{v}_\perp + \nabla \mathfrak{p}_\perp = \mathbf{f}_\perp, \quad \operatorname{div} \mathbf{v}_\perp = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}_\perp|_{\Gamma \times \mathbb{T}} = 0. \tag{5.2}$$

Here the subscript $\perp$ means that all functions have vanishing time mean. We study these problems separately. A combination of the results will lead to a proofs of Theorem 4.6 and Theorem 4.7. As explained in Remark 4.4, it will be important to derive estimates where the constants are independent of κ for $|\kappa| \leq \kappa_0$.

5.1 Existence of time-independent solutions

For the steady-state Oseen problem (5.1), existence of solutions is guaranteed by the following result. It characterizes the solution in terms of integrability properties and pointwise decay for suitable forcing terms.

Theorem 5.1. *Let* $3 < q < \infty$, $0 < \delta < 1/4$ *and* $0 < |\kappa| \le \kappa_0$. *Let* $\mathbf{f}_S = \tilde{\mathbf{f}} + \mathbf{g}$, *where* $< \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} < \infty$ *and* $\mathbf{g} \in L_{q,3b}(\Omega)^3$. *Then, problem* (5.1) *admits a unique solution* $(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^2_q(\Omega)^3 \times \mathrm{H}^1_q(\Omega)$ *possessing the estimate*

$$\|\mathbf{u}\|_{\mathrm{H}^2_q(\Omega)} + |\kappa|^\delta < \mathbf{u} >^w_{1,\delta} + |\kappa|^\delta < \nabla\mathbf{u} >^w_{3/2,1/2+\delta} + \|\mathfrak{p}\|_{\mathrm{H}^1_q(\Omega)} \le C\big(< \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} + \|\mathbf{g}\|_{\mathrm{L}_q(\Omega)}\big) \tag{5.3}$$

for some constant $C > 0$ *depending solely on* Ω, μ, q, δ *and* κ_0.

Proof. We first show that $\mathbf{f}_S \in \mathrm{L}^1(\Omega)^3 \cap \mathrm{L}^q(\Omega)^3$ and

$$\|\tilde{\mathbf{f}}\|_{\mathrm{L}_1(\Omega)} + \|\tilde{\mathbf{f}}\|_{\mathrm{L}_q(\Omega)} \le C_\delta < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} . \tag{5.4}$$

In fact, using the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\varphi$, $x_3 = r\sin\theta\sin\varphi$ for $r > 0$, $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$ and a change of variables, we obtain

$$\begin{aligned}
\|\tilde{\mathbf{f}}\|_{\mathrm{L}_1(\Omega)} &\le < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} \int_{\mathbb{R}^3} (1+|x|)^{-5/2}(1+|x|-x_1)^{-1/2-2\delta}\,\mathrm{d}x \\
&\le 4\pi \; < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} \int_0^\infty \int_0^{\pi/2} (1+r)^{-5/2} \frac{r^2\sin\theta}{(1+r(1-\cos\theta))^{1/2+2\delta}}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\pi \; < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} \int_0^\infty \int_0^1 \frac{(1+r)^{-5/2}r^2}{(1+rt)^{1/2+2\delta}}\,\mathrm{d}r\mathrm{d}t \\
&\le C_\delta < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} \int_0^\infty (1+r)^{-5/2}(1+r)^{1/2-2\delta} r\,\mathrm{d}r \\
&\le C_\delta < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta} .
\end{aligned}$$

Since we clearly have $\|\tilde{\mathbf{f}}\|_{\mathrm{L}_\infty(\Omega)} \le < \tilde{\mathbf{f}} >^w_{5/2,1/2+2\delta}$, we thus conclude (5.4). Now the estimates of $\|\mathbf{u}\|_{\mathrm{H}^2_q(\Omega)}$ and $\|\mathfrak{p}\|_{\mathrm{H}^1_q(\Omega)}$ are a direct consequence of [30, Theorem 3.1], and the remaining estimates follow from [30, Theorem 4.1]. □

In contrast to the previous framework, where decay is specified in a pointwise sense, one can also show existence within homogeneous Sobolev spaces.

Theorem 5.2. *Let* $1 < s < 3/2$, $1 < q < \infty$ *and* $0 < |\kappa| \le \kappa_0$. *Let* $\mathbf{f}_S \in \mathrm{L}_q(\Omega)^3 \cap \mathrm{L}_s(\Omega)^3$. *Then, problem* (5.1) *admits a unique solution* $(\mathbf{u}, \mathfrak{p}) \in \hat{\mathrm{H}}^2_q(\Omega)^3 \times \hat{\mathrm{H}}^1_q(\Omega)$ *possessing the estimate*

$$\begin{aligned}
&\|\nabla^2\mathbf{u}\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{u}\|_{\mathrm{L}_q(\Omega)} + |\kappa|^{1/4}\|\nabla\mathbf{u}\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} + |\kappa|^{1/2}\|\mathbf{u}\|_{\mathrm{L}_{2s/(2-s)}(\Omega)} \\
&\quad + |\kappa|\,\|\partial_1\mathbf{u}\|_{\mathrm{L}_s(\Omega)} + \|\nabla\mathfrak{p}\|_{\mathrm{L}_s(\Omega)} + \|\nabla\mathfrak{p}\|_{\mathrm{L}_q(\Omega)} + \|\mathfrak{p}\|_{\mathrm{L}_{3s/(3-s)}(\Omega)} \le C\big(\|\mathbf{f}_S\|_{\mathrm{L}_s(\Omega)} + \|\mathbf{f}_S\|_{\mathrm{L}_q(\Omega)}\big)
\end{aligned} \tag{5.5}$$

for some constant $C > 0$ *depending solely on* Ω, μ, q, s *and* κ_0.

Proof. Let us assume $s \le q$. Otherwise, we reverse the role of s and q. Then [12, Theorem 2.1] implies the existence of a unique solution $(\mathbf{u}, \mathfrak{p})$ satisfying

$$\begin{aligned}
&\|\nabla^2\mathbf{u}\|_{\mathrm{L}_s(\Omega)} + |\kappa|^{1/4}\|\nabla\mathbf{u}\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} + |\kappa|^{1/2}\|\mathbf{u}\|_{\mathrm{L}_{2s/(2-s)}(\Omega)} \\
&\qquad + |\kappa|\,\|\partial_1\mathbf{u}\|_{\mathrm{L}_s(\Omega)} + \|\nabla\mathfrak{p}\|_{\mathrm{L}_s(\Omega)} + \|\mathfrak{p}\|_{\mathrm{L}_{3s/(3-s)}(\Omega)} \le C\|\mathbf{f}_S\|_{\mathrm{L}_s(\Omega)}
\end{aligned} \tag{5.6}$$

with $C > 0$ independent of κ. We now consider $(\mathbf{u}, \mathfrak{p})$ as a solution to the Stokes problem

$$-\mu\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f}_S + \kappa\partial_1\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0,$$

with right-hand side $\mathbf{f}+\kappa\partial_1\mathbf{u}\in \mathrm{L}_r(\Omega)$ with $r=\min\{q,\frac{4s}{4-s}\}$ since $\partial_1\mathbf{u}\in \mathrm{L}_s(\Omega)\cap \mathrm{L}_{4s/(4-s)}(\Omega)$ by the previous estimate. From [13, Theorem V.4.8] we conclude

$$\begin{aligned}\|\mathbf{u}\|_{\mathrm{L}_r(\Omega)}+\|\nabla\mathfrak{p}\|_{\mathrm{L}_r(\Omega)}&\leq C\big(\|\mathbf{f}_S\|_{\mathrm{L}_r(\Omega)}+|\kappa|\,\|\partial_1\mathbf{u}\|_{\mathrm{L}_r(\Omega)}\big)\\&\leq C\big(\|\mathbf{f}_S\|_{\mathrm{L}_r(\Omega)}+|\kappa|\,\|\partial_1\mathbf{u}\|_{\mathrm{L}_s(\Omega)}+|\kappa|\,\|\partial_1\mathbf{u}\|_{\mathrm{L}_{4s/(4-s)}(\Omega)}\big)\\&\leq C\big(\|\mathbf{f}_S\|_{\mathrm{L}_r(\Omega)}+\|\mathbf{f}_S\|_{\mathrm{L}_s(\Omega)}\big),\end{aligned}$$

where we used estimate (5.6) as well as $r\geq s$ and $|\kappa|\leq\kappa_0$. If $r=q$, this completes the proof. If $r<q$, then $r=4s/(4-s)$, and Sobolev's inequality and classical interpolation implies $\nabla\mathbf{u}\in\mathrm{L}_{\tilde r}(\Omega)$ for $\tilde r\in[r,3r/(3-r)]$ if $r<3$ and for $\tilde r\in[r,\infty)$ if $r>3$. By repeating the above argument with r replaced with $\tilde r$, an iteration will finally lead to the asserted estimate for $r=q$. □

5.2 Existence of purely oscillatory solutions

Existence of solutions to the purely oscillatory problem (5.2) is guaranteed by the following theorem. Observe that here it is not necessary to distinguish between the cases $\kappa=0$ and $\kappa\neq 0$.

Theorem 5.3. *Let $1<p,q<\infty$ and $\kappa\in\mathbb{R}$ with $|\kappa|\leq\kappa_0$. Then, for any $\mathbf{f}_\perp\in\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega)^3)$ with $\int_{\mathbb{T}}\mathbf{f}_\perp(\cdot,s)\,\mathrm{d}s=0$, problem* (5.2) *admits a solution $(\mathbf{v}_\perp,\mathfrak{p}_\perp)$ with*

$$\begin{aligned}\mathbf{v}_\perp\in\mathrm{H}^1_p(\mathbb{T},\mathrm{L}_q(\Omega)^3)\cap\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega)^3),&\quad\int_{\mathbb{T}}\mathbf{v}_\perp(\cdot,s)\,\mathrm{d}s=0,\\\mathfrak{p}_\perp\in\mathrm{L}_p(\mathbb{T},\hat{\mathrm{H}}^1_q(\Omega)),&\quad\int_{\mathbb{T}}\mathfrak{p}_\perp(\cdot,s)\,\mathrm{d}s=0,\end{aligned}$$

which satisfies the estimate

$$\|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))}+\|\nabla\mathfrak{p}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\tag{5.7}$$

for some constant $C>0$ only depending on Ω, $\mathcal{T}$, μ, p, q and κ_0.

If $(\tilde{\mathbf{v}}_\perp,\tilde{\mathfrak{p}}_\perp)$ is another solution to (5.2) *with*

$$\tilde{\mathbf{v}}_\perp\in\mathrm{H}^1_r(\mathbb{T},\mathrm{L}_s(\Omega)^3)\cap\mathrm{L}_r(\mathbb{T},\mathrm{H}^2_s(\Omega)^3),\ \ \tilde{\mathfrak{p}}_\perp\in\mathrm{L}_r(\mathbb{T},\hat{\mathrm{H}}^1_s(\Omega)),$$

for some $1<r,s<\infty$, then $\mathbf{v}_\perp=\tilde{\mathbf{v}}_\perp$ and $\nabla\mathfrak{p}_\perp=\nabla\tilde{\mathfrak{p}}_\perp$.

To prove Theorem 5.3, we use the method recently introduced in [9], which is based on the existence of suitable $\mathscr{R}$-bounds for solution operators to the corresponding resolvent problem

$$\lambda\mathbf{w}-\mu\Delta\mathbf{w}-\kappa\partial_1\mathbf{w}+\nabla\mathfrak{r}=\mathbf{f},\quad\operatorname{div}\mathbf{w}=0\quad\text{in }\Omega,\quad\mathbf{w}|_\Gamma=0.\tag{5.8}$$

For this problem, we have the following theorem.

Theorem 5.4. *Let $1<q<\infty$ and $0\leq|\kappa|\leq\kappa_0$. Let*

$$\rho[\kappa]:=\begin{cases}\mathbb{C}\setminus(-\infty,0]&\text{if }\kappa=0,\\\big\{\lambda\in\mathbb{C}\mid|\kappa|\operatorname{Re}(\lambda)+\operatorname{Im}(\lambda)^2>0\big\}&\text{if }\kappa\neq0,\end{cases}\tag{5.9}$$

There are operator families $(\mathscr{S}_\kappa(\lambda)) \subset \mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}^2_q(\Omega)^3)$ *and* $(\mathscr{P}_\kappa(\lambda)) \subset \mathscr{L}(\mathrm{L}_q(\Omega)^3, \hat{\mathrm{H}}^1_q(\Omega)))$ *such that for every* $\lambda \in \rho[\kappa]$ *and every* $\mathbf{f} \in \mathrm{L}_q(\Omega)^3$ *the pair* $(\mathbf{w}, \mathfrak{r}) = (\mathscr{S}_\kappa(\lambda)\mathbf{f}, \mathscr{P}_\kappa(\lambda)\mathbf{f})$ *is the unique solution to* (5.8) *and satisfies the estimate*

$$|\lambda| \, \|\mathscr{S}_\kappa(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} + \|\nabla^2 \mathscr{S}_\kappa(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} + \|\nabla \mathscr{P}_\kappa(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C\|\mathbf{f}\|_{\mathrm{L}_q(\Omega)}. \tag{5.10}$$

If $0 < \varepsilon < \pi/2$ *and* $\delta > 0$ *such that* $\lambda \in \Sigma_{\varepsilon,\delta}$*, then* C *only depends on* Ω*,* q*,* ε*,* δ *and* κ_0*. Moreover, there exist constants* $\lambda_0, r_0 > 0$*, depending on* Ω*,* μ*,* q*,* ε *and* κ_0*, such that*

$$\mathscr{S}_\kappa \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}^2_q(\Omega)^3)), \quad \mathscr{P}_\kappa \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathscr{L}(\mathrm{L}_q(\Omega)^3, \hat{\mathrm{H}}^1_q(\Omega))),$$

and

$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}^{2-j}_q(\Omega)^3)}\big(\{(\lambda\partial_\lambda)^\ell(\lambda^{j/2}\mathscr{S}_\kappa(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}\big) \leq r_0,$$
$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{L}_q(\Omega)^3)}\big(\{(\lambda\partial_\lambda)^\ell(\nabla\mathscr{P}_\kappa(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}\big) \leq r_0$$

for $\ell = 0, 1$, $j = 0, 1, 2$.

Proof. For $\kappa = 0$, the result follows mainly from [31, Theorem 1.6] and [32, Theorem 9.1.4] as was shown in [9, Theorem 4.2]. If $\kappa \neq 0$, the existence of the solution operators $\mathscr{S}_\kappa(\lambda)$ and $\mathscr{P}_\kappa(\lambda)$ together with the estimate (5.10) for all $\lambda \in \Sigma_\varepsilon \setminus \{0\}$ was derived in [20, Theorem 4.4]. Since the term $\kappa\partial_1$ can be regarded as a perturbation of the Stokes operator, which is uniform for $|\kappa| \leq \kappa_0$, the asserted results on the analyticity and the $\mathscr{R}$-bounds follow from those for $\kappa = 0$ if λ_0 is taken sufficiently large. □

To show existence of solutions to (5.2), we combine the $\mathscr{R}$-bounds from Theorem 5.4 with the following multiplier theorem.

Theorem 5.5. *Let* X *and* Y *be UMD spaces, and let* $M \in \mathrm{L}_\infty(\mathbb{R}, \mathscr{L}(X,Y)) \cap C^1(\mathbb{R}, \mathscr{L}(X,Y))$ *satisfy*

$$\mathscr{R}_{\mathscr{L}(X,Y)}\big\{M(t) \mid t \in \mathbb{R}\setminus\{0\}\big\} \leq r_0, \qquad \mathscr{R}_{\mathscr{L}(X,Y)}\big\{tM'(t) \mid t \in \mathbb{R}\setminus\{0\}\big\} \leq r_0, \tag{5.11}$$

for some $r_0 > 0$*. Then* $M|_{\mathbb{Z}}$ *is an* $\mathrm{L}_p(\mathbb{T})$*-multiplier such that*

$$\forall f \in C^\infty(\mathbb{T}; X): \qquad \|\mathscr{F}_{\mathbb{T}}^{-1}[M|_{\mathbb{Z}}\, \mathscr{F}_{\mathbb{T}}[f]]\|_{\mathrm{L}_p(\mathbb{T};Y)} \leq C_p r_0 \|f\|_{\mathrm{L}_p(\mathbb{T};X)} \tag{5.12}$$

for some constant $C_p > 0$ *only depending on* p*.*

Proof. The result was derived in [9, Corollary 2.3] as a combination of an operator-valued transference principle for multipliers (see [19, Prop.5.7.1]) with the multiplier theorem due to Weis [34, Theorem 3.4]. □

For a proof of Theorem 5.4 we now follow the approach from [9].

Proof of Theorem 5.4. Let $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(\sigma) = 1$ for $\sigma \geq \lambda_0 + 1/2$ and $\varphi(\sigma) = 0$ for $\sigma \leq \lambda_0 + 1/4$. Set $\mathbf{f}_h = \mathscr{F}_{\mathbb{T}}^{-1}[\varphi(|\frac{2\pi}{\mathcal{T}}k|)\mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k)]$ and

$$\mathbf{v}_h = \mathscr{F}_{\mathbb{T}}^{-1}\big[\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\varphi(|\tfrac{2\pi}{\mathcal{T}}k|)\mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k)\big], \qquad \mathfrak{p}_h = \mathscr{F}_{\mathbb{T}}^{-1}\big[\mathscr{P}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\varphi(|\tfrac{2\pi}{\mathcal{T}}k|)\mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k)\big],$$

where λ_0, $\mathscr{S}_\kappa$ and $\mathscr{P}_\kappa$ are given in Theorem 5.4. Then $\mathbf{v}_h$ and $\mathfrak{p}_h$ satisfy the equations

$$\partial_t\mathbf{v}_h - \mu\Delta\mathbf{v}_h - \kappa\partial_1\mathbf{v}_h + \nabla\mathfrak{p}_h = \mathbf{f}_h, \quad \operatorname{div}\mathbf{v}_h = 0 \quad \text{in } \Omega\times\mathbb{T}, \quad \mathbf{v}_h|_{\Gamma\times\mathbb{T}} = 0.$$

Moreover, from the $\mathscr{R}$-bounds from Theorem 5.4 we derive

$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^3,\mathrm{H}_q^{2-j}(\Omega)^3)}\big(\big\{(\lambda\partial_\lambda)^\ell(\lambda^{j/2}\varphi(|\lambda|)\mathscr{S}_\kappa(\lambda)) \mid \lambda\in\Sigma_{\varepsilon,\lambda_0}\big\}\big)\leq C_\varphi r_0,$$
$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^3,\mathrm{L}_q(\Omega)^3)}\big(\big\{(\lambda\partial_\lambda)^\ell(\varphi(|\lambda|)\nabla\mathscr{P}_\kappa(\lambda)) \mid \lambda\in\Sigma_{\varepsilon,\lambda_0}\big\}\big)\leq C_\varphi r_0$$

for $\ell=0,1,\ j=0,1,2$, where C_φ is a constant only depending on φ. Using Theorem 5.5, we conclude

$$\|\partial_t\mathbf{v}_h\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}+\|\mathbf{v}_h\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}_q^2(\Omega))}+\|\nabla\mathfrak{p}_h\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\|\mathbf{f}_h\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}. \tag{5.13}$$

We now set

$$\mathbf{v}_\perp(t)=\mathbf{v}_h(t)+\sum_{0<|k|\leq\lambda_0}\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k),$$
$$\mathfrak{p}_\perp(t)=\mathfrak{p}_h(t)+\sum_{0<|k|\leq\lambda_0}\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}\mathscr{P}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k).$$

Then, $\mathbf{v}_\perp$ and $\mathfrak{p}_\perp$ satisfy (5.2), and from (5.10) and (5.13) we conclude estimate (5.7).

For the uniqueness statement, consider the difference $(\mathbf{u}_\perp,\mathfrak{q}_\perp)=(\mathbf{v}_\perp-\tilde{\mathbf{v}}_\perp,\mathfrak{p}_\perp-\tilde{\mathfrak{p}}_\perp)$, which is a solution to (5.2) with $\mathbf{f}_\perp=0$. Then, for each $k\in\mathbb{Z}\setminus\{0\}$, the functions $\hat{\mathbf{u}}_k=\mathscr{F}_\mathbb{T}[\mathbf{u}_\perp](k)$ and $\hat{\mathfrak{q}}_k=\mathscr{F}_\mathbb{T}[\mathfrak{q}_\perp](k)$ satisfy $\hat{\mathbf{u}}_k\in\mathrm{H}_q^2(\Omega)^3+\mathrm{H}_s^2(\Omega)^3$ and $\hat{\mathfrak{q}}_k\in\hat{\mathrm{H}}_q^1(\Omega)+\hat{\mathrm{H}}_s^1(\Omega)$ and solve the homogeneous equations

$$i\tfrac{2\pi}{\mathcal{T}}k\hat{\mathbf{u}}_k-\mu\Delta\hat{\mathbf{u}}_k-\kappa\partial_1\hat{\mathbf{u}}_k+\nabla\hat{\mathfrak{q}}_k=0,\quad \operatorname{div}\hat{\mathbf{u}}_k=0\quad\text{in }\Omega,\qquad \hat{\mathbf{u}}_k|_\Gamma=0.$$

Using elliptic regularity for the Stokes operator and Sobolev embeddings, similarly as in the proof of Theorem 5.2 we conclude $\hat{\mathbf{u}}_k=\nabla\hat{\mathfrak{q}}_k=0$ for any $k\in\mathbb{Z}$. This shows $\mathbf{v}=\nabla\mathfrak{p}=0$ and completes the proof. □

5.3 Pointwise decay of the oscillatory part

In this section, we study decay properties of solutions $(\mathbf{v}_\perp,\mathfrak{p}_\perp)$ to the purely oscillatory problem (5.2). We derive decay properties of $\|\mathbf{v}_\perp(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})}$ and $\|\nabla\mathbf{v}_\perp(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})}$ as $|x|\to\infty$ as stated in the following theorem.

Theorem 5.6. *Let $1<p<\infty$, $3<q<\infty$, $\ell\in(0,3]$ and $\kappa_0\geq0$. Let $\mathbf{f}_\perp=\operatorname{div}\mathbf{F}_\perp+\mathbf{g}_\perp$ with*

$$\begin{aligned}&\int_\mathbb{T}\mathbf{F}_\perp(x,t)\,\mathrm{d}t=0,\qquad <\mathbf{F}_\perp>_{p,\ell}+<\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1}<\infty,\\ &\int_\mathbb{T}\mathbf{g}_\perp(x,t)\,\mathrm{d}t=0,\qquad \mathbf{g}_\perp\in\mathrm{L}_p(\mathbb{T},\mathrm{L}_{q,3b}(\Omega)).\end{aligned} \tag{5.14}$$

Let $(\mathbf{v}_\perp,\mathfrak{p}_\perp)$ be the solution to (5.2) *according to Theorem 5.3. Then, $\mathbf{v}_\perp$ satisfies*

$$<\mathbf{v}_\perp>_{p,\ell}+<\nabla\mathbf{v}_\perp>_{p,\ell+1}\leq C(<\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1}+<\mathbf{F}_\perp>_{p,\ell}+\|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}) \tag{5.15}$$

with some constant $C>0$ only dependent on Ω, $\mathcal{T}$, μ, p, q, ℓ and κ_0.

Remark 5.7. Since $3<q<\infty$, we have $\|\operatorname{div}\mathbf{F}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C_{q,\ell}<\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1}$, so that $\mathbf{f}_\perp\in\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))$ and

$$\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\big(<\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1}+<\mathbf{F}_\perp>_{p,\ell}+\|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big).$$

Therefore, existence of $(\mathbf{v}_\perp,\mathfrak{p}_\perp)$ indeed follows from Theorem 5.3.

This pointwise estimate will be concluded by using the velocity fundamental solution to the purely oscillatory problem (5.2), which is a tensor field $\Gamma_\perp^\kappa$ such that $\mathbf{v}_\perp := \Gamma_\perp^\kappa * \mathbf{H}_\perp$ defines a solution to (5.2) for $\Omega = \mathbb{R}^3$. We use the following properties of $\Gamma_\perp^\kappa$, which were mainly derived in [7] in the general multidimensional case.

Theorem 5.8. *Let $\kappa \in \mathbb{R}$ and $\mu, \mathcal{T} > 0$. Let*

$$\Gamma_\perp^\kappa = \mathscr{F}^{-1}_{\mathbb{R}^3\times\mathbb{T}}\Big[\frac{1-\delta_\mathbb{Z}(k)}{\mu|\xi|^2 - i\kappa\xi_1 + i\frac{2\pi}{\mathcal{T}}k}\Big(\mathrm{I} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\Big]. \tag{5.16}$$

Then, it holds $\Gamma_\perp^\kappa \in \mathrm{L}_q(\mathbb{R}^3\times\mathbb{T})^{3\times3}$ for $q \in (1, 5/3)$, and $\partial_j\Gamma_\perp^\kappa \in \mathrm{L}_r(\mathbb{R}^3\times\mathbb{T})^{3\times3}$ for $r \in [1, 4/3)$, $j = 1, 2, 3$. If $\kappa_0 > 0$ such that $|\kappa| \le \kappa_0$, then

$$\|\Gamma_\perp^\kappa\|_{\mathrm{L}_q(\mathbb{R}^3\times\mathbb{T})} + \|\nabla\Gamma_\perp^\kappa\|_{\mathrm{L}_r(\mathbb{R}^3\times\mathbb{T})} \le C \tag{5.17}$$

for some constant $C > 0$ only dependent on $\mathcal{T}$, μ, q, r and κ_0. Moreover, for any $\alpha \in \mathbb{N}_0^3$, $\delta > 0$, $r \in [1,\infty)$ and $\theta > 0$ such that $\mathcal{T}\kappa^2 \le \theta$, there exists a constant $C > 0$, only dependent on μ, α, δ, r and θ, such that

$$\forall |x| \ge \delta : \quad \|\partial_x^\alpha \Gamma_\perp^\kappa(x,\cdot)\|_{\mathrm{L}_r(\mathbb{T})} \le C|x|^{-(3+|\alpha|)}. \tag{5.18}$$

Proof. The majority of the result was proved in [7]. However, it was not shown that the respective estimates are uniform in κ if κ is small. To show this property, we reconsider those parts of the proof in [7], where this assumption has an effect.

To study integrability properties of $\Gamma_\perp^\kappa$, the components of $\Gamma_\perp^\kappa$ were expressed as

$$(\Gamma_\perp^\kappa)_{j\ell} = \big[\delta_{j\ell}\mathfrak{R}_m\mathfrak{R}_m - \mathfrak{R}_j\mathfrak{R}_\ell\big] \circ \mathscr{F}^{-1}_{\mathbb{R}^3\times\mathbb{T}}\big[M_{\kappa,\mathcal{T}}\mathscr{F}_{\mathbb{R}^3\times\mathbb{T}}[\Psi]\big],$$

where $\mathfrak{R}_j$ denotes a Riesz transform, which is a continuous operator on $\mathrm{L}_q(\mathbb{R}^3\times\mathbb{T})$. Moreover, $M_{\kappa,\mathcal{T}}$ is given by

$$M_{\kappa,\mathcal{T}}(k,\xi) := \frac{(1-\delta_\mathbb{Z}(k))\,|\frac{2\pi}{\mathcal{T}}k|^{\frac{2}{5}}(1+|\xi|^2)^{\frac{3}{5}}}{\mu|\xi|^2 + i\kappa\xi_1 + i\frac{2\pi}{\mathcal{T}}k},$$

and $\Psi\colon \mathbb{R}^3\times\mathbb{T} \to \mathbb{R}$ is given as the product $\Psi(x,t) = \psi(x)\chi(t)$ with $\mathscr{F}_{\mathbb{R}^3}[\psi](\xi) = (1+|\xi|^2)^{-\frac{3}{5}}$ and $\mathscr{F}_\mathbb{T}[\chi](k) = (1-\delta_\mathbb{Z}(k))|\frac{2\pi}{\mathcal{T}}k|^{-\frac{2}{5}}$. Then $M_{\kappa,\mathcal{T}}$ was shown to be an L^q-multiplier in $\mathbb{R}^3\times\mathbb{T}$ such that

$$\|\Gamma_\perp^\kappa\|_{\mathrm{L}_q(\mathbb{R}^3\times\mathbb{T})} \le C\|\Psi\|_{\mathrm{L}_q(\mathbb{R}^3\times\mathbb{T})}.$$

Going through the proof in [7], one readily verifies that the multiplier norm of $M_{\kappa,\mathcal{T}}$ and thus the constant $C > 0$ in this estimate can be chosen uniformly in κ if $|\kappa| \le \kappa_0$. It was also shown that $\Psi \in \mathrm{L}^q(\mathbb{R}^3\times\mathbb{T})$ for all $q \in (1, 5/3)$, the norm of which is clearly independent of κ. This leads to the asserted estimate for $\Gamma_\perp^\kappa$, and arguing in the same way for $\nabla\Gamma_\perp^\kappa$, we obtain the uniform estimate (5.17).

To derive the pointwise estimate (5.18), a central term in the proof from [7] given by

$$\mu(\kappa,k) := \Big(\frac{\kappa}{2}\Big)^2 + i\frac{2\pi}{\mathcal{T}}k$$

for $k \in \mathbb{Z}$, and its square root $\sqrt{-\mu(\kappa,k)}$, where $\sqrt{z}$ denotes the square root of z with nonnegative imaginary part. In particular, we need a constant $C_\theta > 0$, only depending on θ, such that

$$\mathrm{Im}\sqrt{-\mu(\kappa,k)} - \frac{|\kappa|}{2} \ge C_\theta\Big|\frac{2\pi}{\mathcal{T}}k\Big|^{\frac{1}{2}} \tag{5.19}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Repeating the calculations in [7, Lemma 3.2], we obtain

$$\mathrm{Im}\sqrt{-\mu(\kappa,k)} - \frac{|\kappa|}{2} = \big|\frac{2\pi}{\mathcal{T}}k\big|\;\Phi\Big(\frac{|\kappa|/2}{|\frac{2\pi}{\mathcal{T}}k|^{1/2}}\Big)$$

for Φ given by

$$\Phi(s) := s\Big(\frac{1}{\sqrt{2}}\Big(1+\big(1+s^{-4}\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}} - 1\Big).$$

Since $\lim_{s\to 0}\Phi(s) = \frac{1}{\sqrt{2}}$ and $\Phi(s) > 0$ for all $s > 0$, estimate (5.19) follows with

$$C_\theta = \min\big\{\Phi(s) \mid 0 < s \leq \frac{\sqrt{\theta}}{2\sqrt{2\pi}}\big\} > 0.$$

Moreover, we have

$$\big|\frac{2\pi}{\mathcal{T}}k\big| \leq |\mu(\kappa,k)| = \big|\frac{2\pi}{\mathcal{T}}k\big|\sqrt{1+\Big(\frac{|\kappa|^2/4}{|\frac{2\pi}{\mathcal{T}}k|}\Big)^2} \leq \tilde{C}_\theta\,\big|\frac{2\pi}{\mathcal{T}}k\big|$$

with $\tilde{C}_\theta^2 = 1 + (\theta/8\pi)^2$, so that μ is comparable with $\big|\frac{2\pi}{\mathcal{T}}k\big|$ with constants only depending on θ. Based on these observations, one can now repeat the proof of the pointwise estimate (5.18) given in [7] and see that all constants can be chosen uniformly in κ and $\mathcal{T}$ as long as $\mathcal{T}\kappa^2 \leq \theta$. □

We can now show the statements of Theorem 5.6.

Proof of Theorem 5.6. We proceed as in the proof of [9, Theorem 5.6], where the result was proved for $\kappa = 0$. In order to clarify that the constant C in (5.15) can be chosen independently of κ for $|\kappa| \leq \kappa_0$, we repeat the arguments here.

Since we assume $3 < q < \infty$, Sobolev embeddings and estimate (5.7) imply

$$\sup_{|x|\leq 4b}\|\mathbf{v}_\perp(\cdot,x)\|_{\mathrm{L}_p(\mathbb{T})} + \sup_{|x|\leq 4b}\|(\nabla\mathbf{v}_\perp)(\cdot,x)\|_{\mathrm{L}_p(\mathbb{T})} \leq C\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} \leq C\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}.$$

In virtue of Remark 5.7, it thus remains to estimate $\mathbf{v}_\perp$ for $|x| > 4b$. As seen in the proof of Theorem 5.3, we have $\mathbf{v}_\perp = \mathscr{F}_\mathbb{T}^{-1}[\mathscr{S}_\kappa(i\frac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k)]$ and $\mathfrak{p}_\perp = \mathscr{F}_\mathbb{T}^{-1}[\mathscr{P}_\kappa(i\frac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k)]$, where $\mathscr{S}_\kappa$ and $\mathscr{P}_\kappa$ are the families of solution operators given in Theorem 5.4.

We first derive a representation formula of $\mathscr{S}_\kappa(i\frac{2\pi}{\mathcal{T}}k)$ for $k \in \mathbb{Z}\setminus\{0\}$ and $|x| > 4b$. Notice that $\mathscr{S}_\kappa(i\frac{2\pi}{\mathcal{T}}k) \in \mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}^2_q(\Omega)^3)$ and $\mathscr{P}_\kappa(i\frac{2\pi}{\mathcal{T}}k) \in \mathscr{L}(\mathrm{L}_q(\Omega)^3, \hat{\mathrm{H}}^1_q(\Omega))$ satisfy the estimate

$$\|\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}\|_{\mathrm{H}^2_q(\Omega)} + \|\nabla\mathscr{P}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C\|\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \tag{5.20}$$

for $\mathbf{f} \in \mathrm{L}_q(\Omega)^3$, where C depends solely on Ω, μ, q and κ_0. Moreover, $\mathbf{u} = \mathscr{F}_\mathbb{T}[\mathbf{v}_\perp](k) = \mathscr{S}_\kappa(i\frac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k)$ and $\mathfrak{q} = \mathscr{F}_\mathbb{T}[\mathfrak{p}_\perp](k) = \mathscr{P}_\kappa(i\frac{2\pi}{\mathcal{T}}k)\mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k)$ satisfy the equations

$$i\tfrac{2\pi}{\mathcal{T}}k\mathbf{u} - \mu\Delta\mathbf{u} - \kappa\partial_1\mathbf{u} + \nabla\mathfrak{q} = \mathbf{f}_k, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0, \tag{5.21}$$

where $\mathbf{f}_k = \mathscr{F}_\mathbb{T}[\mathbf{f}_\perp](k)$. Let φ be a function in $C_0^\infty(\mathbb{R}^3)$ that equals 1 for $|x| < 2b$ and 0 for $|x| > 3b$. Let

$$\mathbf{w} = (1-\varphi)\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k + \mathbb{B}[(\nabla\varphi)\cdot\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k], \quad \mathfrak{r} = (1-\varphi)\mathscr{P}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k, \tag{5.22}$$

where $\mathbb{B}$ denotes the Bogovskiĭ operator [1, 2]. Then $\mathbf{w} \in \mathrm{H}^2_q(\mathbb{R}^3)^3$ and $\mathfrak{r} \in \hat{\mathrm{H}}^1_q(\mathbb{R}^3)$, and the functions $\mathbf{w}$ and $\mathfrak{r}$ satisfy the equations

$$i\tfrac{2\pi}{\mathcal{T}}k\mathbf{w} - \mu\Delta\mathbf{w} - \kappa\partial_1\mathbf{w} + \nabla\mathfrak{r} = (1-\varphi)\mathbf{f}_k + \mathcal{R}_{1,\kappa}(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k, \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \mathbb{R}^3,$$

where we have set

$$\begin{aligned}\mathcal{R}_{1,\kappa}(\lambda)\mathbf{f} = 2\mu(\nabla\varphi)\cdot\nabla\mathscr{S}_\kappa(\lambda)\mathbf{f} + (\mu\Delta\varphi + \kappa\partial_1\varphi)\mathscr{S}_\kappa(\lambda)\mathbf{f} - (\nabla\varphi)\mathscr{P}_\kappa(\lambda)\mathbf{f} \\ + (\lambda - \mu\Delta - \kappa\partial_1)\mathbb{B}[(\nabla\varphi)\cdot\mathscr{S}_\kappa(\lambda)\mathbf{f}].\end{aligned}$$

By the uniqueness of solutions to the Oseen resolvent problem in $\mathbb{R}^3$, we have $\mathbf{w} = \mathcal{T}_\kappa(i\frac{2\pi}{\mathcal{T}}k)((1-\varphi)\mathbf{f}_k + \mathcal{R}_{1,\kappa}(i\frac{2\pi}{\mathcal{T}}k)\mathbf{f}_k)$, where

$$\mathcal{T}_\kappa(\lambda)\mathbf{f} = \mathscr{F}^{-1}_{\mathbb{R}^3}\Big[\frac{1}{\mu|\xi|^2 - i\kappa\xi_1 + \lambda}\Big(\mathrm{I} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_{\mathbb{R}^3}[\mathbf{f}]\Big]. \tag{5.23}$$

Since $1-\varphi(x) = 1$ and $\mathbb{B}[(\nabla\varphi)\cdot\mathscr{S}_\kappa(i\frac{2\pi}{\mathcal{T}}k)\mathbf{f}_k](x,t) = 0$ for $|x| > 4b$, by (5.22) we thus obtain

$$\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k(x) = \mathcal{T}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)((1-\varphi)\mathbf{f}_k)(x) + \mathcal{T}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)(\mathcal{R}_{1,\kappa}(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k)(x)$$

for $|x| > 4b$ and any $k \in \mathbb{Z}\setminus\{0\}$.

This representation formula implies

$$\begin{aligned}\mathbf{v}_\perp &= \mathscr{F}^{-1}_{\mathbb{T}}[(1-\delta_{\mathbb{Z}}(k))\mathscr{S}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k)] \\ &= \mathscr{F}^{-1}_{\mathbb{T}}[(1-\delta_{\mathbb{Z}}(k))\mathcal{T}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)\mathscr{F}_{\mathbb{T}}[(1-\varphi)\mathbf{f}_\perp](k))] \\ &\quad + \mathscr{F}^{-1}_{\mathbb{T}}[(1-\delta_{\mathbb{Z}}(k))\mathcal{T}_\kappa(i\tfrac{2\pi}{\mathcal{T}}k)(\mathcal{R}_{1,\kappa}(i\tfrac{2\pi}{\mathcal{T}}k)\mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k))]\end{aligned} \tag{5.24}$$

for $|x| > 4b$. Moreover, from Theorem 5.4 we conclude

$$\begin{aligned}&\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}^1_q(\mathbb{R}^3)^3)}(\{(\lambda\partial_\lambda)^\ell\mathcal{R}_{1,\kappa}(\lambda) \mid \lambda \in \mathbb{R}\setminus[-\lambda_0,\lambda_0]\}) \le r_0 \quad (\ell = 0,1), \\ &\|\mathcal{R}_{1,\kappa}(i\tfrac{2\pi}{\mathcal{T}}k)\mathbf{f}_k\|_{\mathrm{H}^1_q(\mathbb{R}^3)} \le r_0\|\mathbf{f}_k\|_{\mathrm{L}_q(\Omega)}\end{aligned}$$

for any $k \in \mathbb{Z}\setminus\{0\}$ with some constant r_0 independent of κ. We can thus define $\mathcal{R}_{2,\kappa}\mathbf{f}_\perp$ by setting $\mathcal{R}_{2,\kappa}\mathbf{f}_\perp = \mathscr{F}^{-1}_{\mathbb{T}}[(1-\delta_{\mathbb{Z}}(k))\mathcal{R}_{1,\kappa}(i\frac{2\pi}{\mathcal{T}}k)\mathbf{f}_k]$ and obtain that

$$\begin{aligned}&\operatorname{supp}\mathcal{R}_{2,\kappa}\mathbf{f}_\perp \subset D_{2b,3b} := \{(x,t) \in \mathbb{R}^3\times\mathbb{R} \mid 2b < |x| < 3b\}, \\ &\|\mathcal{R}_{2,\kappa}\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \le C\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\end{aligned} \tag{5.25}$$

by employing Theorem 5.5 in the same way as in the proof of Theorem 5.3. Recalling that $\mathbf{f}_\perp = \operatorname{div}\mathbf{F}_\perp + \mathbf{g}_\perp$, we set $\mathbf{G} = (1-\varphi)\mathbf{F}_\perp$ and $\mathbf{h} = (\nabla\varphi)\mathbf{F}_\perp + (1-\varphi)\mathbf{g}_\perp + \mathcal{R}_{2,\kappa}\mathbf{f}_\perp$. In virtue of (5.16), (5.23) and (5.24), we then have

$$\mathbf{v}_\perp(x,t) = \Gamma^\kappa_\perp * (\operatorname{div}\mathbf{G})(x,t) + \Gamma^\kappa_\perp * \mathbf{h}(x,t)$$

for $t \in \mathbb{T}$ and $|x| > 4b$.

We decompose this formula into two parts and set $\mathbf{v}_1 = \Gamma^\kappa_\perp * (\operatorname{div}\mathbf{G})$ and $\mathbf{v}_2 = \Gamma^\kappa_\perp * \mathbf{h}$. By the divergence theorem, we obtain

$$\begin{aligned}&\mathbf{v}_1(x,t) = \nabla\Gamma^\kappa_\perp * \mathbf{G}(x,t) \\ &= \int_{\mathbb{T}}\int_{|y|\le 1}\nabla\Gamma^\kappa_\perp(y,s)\mathbf{G}(x-y,t-s)\,\mathrm{d}y\mathrm{d}s + \int_{\mathbb{T}}\int_{1\le|y|\le|x|/2}\nabla\Gamma^\kappa_\perp(y,s)\mathbf{G}(x-y,t-s)\,\mathrm{d}y\mathrm{d}s \\ &+ \int_{\mathbb{T}}\int_{|x|/2\le|y|\le 2|x|}\nabla\Gamma^\kappa_\perp(y,s)\mathbf{G}(x-y,t-s)\,\mathrm{d}y\mathrm{d}s + \int_{\mathbb{T}}\int_{|y|\ge 2|x|}\nabla\Gamma^\kappa_\perp(y,s)\mathbf{G}(x-y,t-s)\,\mathrm{d}y\mathrm{d}s.\end{aligned}$$

We set $\gamma_\ell =< \mathbf{G} >_{p,\ell}$ and consider $p < r_0 < \infty$, $r_1 \in (1, 5/4)$ such that $1 + 1/r_0 = 1/r_1 + 1/p$. From Young's inequality and Theorem 5.8, we thus conclude

$$\|\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_{r_0}(\mathbb{T})} \le \gamma_\ell \|\nabla\Gamma_\perp^\kappa\|_{\mathrm{L}_{r_1}(B_1\times\mathbb{T})}(1+|x|)^{-\ell} + C\gamma_\ell(1+|x|)^{-\ell}\int_{1\le|y|\le|x|/2}|y|^{-4}\,\mathrm{d}y$$
$$+ C\gamma_\ell(|x|/2)^{-4}\int_{|z|\le 3|x|}(1+|z|)^{-\ell}\,\mathrm{d}z + C\gamma_\ell\int_{|y|\ge 2|x|}|y|^{-4-\ell}\,\mathrm{d}y.$$

Noting that $p \le r_0$ and $\gamma_\ell \le < \mathbf{F}_\perp >_{p,\ell}$, we infer

$$\|\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le C_b|x|^{-\min\{\ell,4\}} < \mathbf{F}_\perp >_{p,\ell} \quad \text{for } |x| \ge 4b.$$

In the same way we decompose $\nabla\mathbf{v}_1 = \nabla\Gamma_\perp^\kappa * \operatorname{div}\mathbf{G}$ and use Theorem 5.8 and Young's inequality to obtain

$$\|\nabla\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_{r_0}(\mathbb{T})}$$
$$\le \gamma_{\ell+1}\|\nabla\Gamma_\ell\|_{\mathrm{L}_{r_1}(B_1\times\mathbb{T})}(1+|x|)^{-\ell-1} + C\gamma_{\ell+1}(1+|x|)^{-\ell-1}\int_{1\le|y|\le|x|/2}|y|^{-4}\,\mathrm{d}y$$
$$+ C\gamma_{\ell+1}(|x|/2)^{-4}\int_{|z|\le 3|x|}(1+|z|)^{-\ell-1}\,\mathrm{d}z + C\gamma_{\ell+1}\int_{|y|\ge 2|x|}|y|^{-5-\ell}\,\mathrm{d}y,$$

where $\gamma_{\ell+1} =< \operatorname{div}\mathbf{G} >_{p,\ell+1}$. Since we have

$$< \operatorname{div}\mathbf{G} >_{p,\ell+1} \le < \operatorname{div}\mathbf{F}_\perp >_{p,\ell+1} + < (\nabla\varphi)\mathbf{F}_\perp >_{p,\ell+1}$$
$$\le < \operatorname{div}\mathbf{F} >_{p,\ell+1} + \|\nabla\varphi\|_{\mathrm{L}_\infty(\mathbb{R}^3)} 3b < \mathbf{F} >_{p,\ell}$$

and $p \le r_0$, we thus obtain

$$\|\nabla\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le C_b|x|^{-\min\{\ell+1,4\}}(< \operatorname{div}\mathbf{F}_\perp >_{p,\ell+1} + < \mathbf{F}_\perp >_{p,\ell}) \quad \text{for } |x| \ge 4b.$$

Using that $\mathbf{h}(y,s)$ vanishes for $|y| \ge 3b$, we obtain have

$$\nabla^m\mathbf{v}_2(x,t) = \int_{\mathbb{T}}\int_{|x-y|\le 3b}\nabla^m\Gamma_\perp^\kappa(y,s)\mathbf{h}(x-y,t-s)\,\mathrm{d}y\mathrm{d}s$$

for $m = 0, 1$. Since $|x| \ge 4b$ and $|x-y| \le 3b$ implies $|y| \ge |x|/4 \ge b$, by Theorem 5.8 and Young's inequality, we deduce

$$\|\nabla^m\mathbf{v}_2(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le \int_{|x-y|\le 3b}\|\nabla^m\Gamma_\perp^\kappa(y,\cdot)\|_{\mathrm{L}_p(\mathbb{T})}\|\mathbf{h}(x-y,\cdot)\|_{\mathrm{L}_1(\mathbb{T})}\,\mathrm{d}y$$
$$\le C_m|x|^{-3-m}\|\mathbf{h}\|_{\mathrm{L}_1(B_{3b}\times\mathbb{T})}.$$

Noting (5.25), we can estimate the last term as

$$\|\mathbf{h}\|_{\mathrm{L}_1(B_{3b}\times\mathbb{T})} \le C\|\mathbf{h}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(B_{3b}))} \le C\big(< \mathbf{F}_\perp >_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{R}_{2,\kappa}\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big)$$
$$\le C\big(< \mathbf{F}_\perp >_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big).$$

For $|x| \ge 4b$ we now conclude

$$\|\nabla^m\mathbf{v}_2(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le C|x|^{-3-m}\big(< \mathbf{F}_\perp >_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + < \operatorname{div}\mathbf{F}_\perp >_{p,\ell+1}\big)$$

in virtue of Remark 5.7. Since $\mathbf{v}(x,t) = \mathbf{v}_1(x,t) + \mathbf{v}_2(x,t)$ for $|x| \ge 4b$, we conclude (5.15). □

5.4 The full time-periodic problem

To conclude the existence result for time-periodic solutions to the Oseen problem as stated in Theorem 4.6, we combine Theorem 5.1 and Theorem 5.6 with a lifting argument for inhomogeneous boundary conditions.

Proof of Theorem 4.6. We first reduce the problem to the case of homogeneous boundary conditions. For this purpose, let

$$\mathbf{v}_1 \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \mathfrak{p}_1 \in \mathrm{L}_p(\mathbb{T}, \hat{\mathrm{H}}^1_q(\Omega)),$$

be a solution to the time-periodic Stokes problem

$$\partial_t \mathbf{v}_1 - \mu \Delta \mathbf{v}_1 + \nabla \mathfrak{p}_1 = 0, \quad \operatorname{div} \mathbf{v}_1 = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}_1|_{\Gamma \times \mathbb{T}} = \mathbf{h}|_{\Gamma \times \mathbb{T}},$$

which exists due to Theorem 4.5. Let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi \equiv 1$ in B_{2b} and $\varphi \equiv 0$ in $\mathbb{R}^3 \setminus B_{3b}$. Let $D_{2b,3b} = \{x \in \mathbb{R}^3 \mid 2b < |x| < 3b\}$ and

$$\mathrm{H}^2_{q,0,a}(D_{2b,3b}) = \Big\{ f \in \mathrm{H}^2_q(D_{2b,3b}) \mid \partial_x^\alpha f|_{S_L} = 0 \text{ for } L = 2b, 3b \text{ and } |\alpha| \leq 1, \int_{D_{2b,3b}} f(x) \, \mathrm{d}x = 0 \Big\}.$$

According to [33, Lemma 5], we know that $(\nabla \varphi) \cdot \mathbf{v}_1(t) \in \mathrm{H}^2_{q,0,a}(D_{2b,3b})$ for a.a. $t \in \mathbb{R}$, and setting $\tilde{\mathbf{v}} = \varphi \mathbf{v}_1 - \mathbb{B}[(\nabla \varphi) \cdot \mathbf{v}_1]$, we see that

$$\begin{aligned} &\tilde{\mathbf{v}} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \quad \operatorname{supp} \tilde{\mathbf{v}} \subset B_{3b} \cap \overline{\Omega}, \quad \operatorname{div} \tilde{\mathbf{v}} = 0, \quad \tilde{\mathbf{v}}|_\Gamma = \mathbf{h}, \\ &\|\partial_t \tilde{\mathbf{v}}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))} + \|\tilde{\mathbf{v}}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega))} \leq C(\|\partial_t \mathbf{h}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega))}). \end{aligned} \tag{5.26}$$

Now let $(\mathbf{w}_S, \mathfrak{q}_S)$ and $(\mathbf{w}_\perp, \mathfrak{q}_\perp)$ be the unique solutions to the equations

$$-\Delta \mathbf{w}_S - \kappa \partial_1 \mathbf{w}_S + \nabla \mathfrak{q}_S = \mathbf{f}_S + \kappa \partial_1 \tilde{\mathbf{v}}_S, \quad \operatorname{div} \mathbf{w}_S = 0 \quad \text{in } \Omega, \qquad \mathbf{w}_S|_\Gamma = 0, \tag{5.27}$$

$$\partial_t \mathbf{w}_\perp - \Delta \mathbf{w}_\perp - \kappa \partial_1 \mathbf{w}_\perp + \nabla \mathfrak{q}_\perp = \mathbf{f}_\perp + \kappa \partial_1 \tilde{\mathbf{v}}_\perp, \quad \operatorname{div} \mathbf{w}_\perp = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{w}_\perp|_{\Gamma \times \mathbb{T}} = 0, \tag{5.28}$$

which exist due to Theorem 5.1 and Theorem 5.3. Also invoking Theorem 5.6, we see that $(\mathbf{v}, \mathfrak{p}) = (\tilde{\mathbf{v}} + \mathbf{w}_S + \mathbf{w}_\perp, \mathfrak{q}_S + \mathfrak{q}_\perp)$ is a solution to the original problem (4.1) and belongs to the asserted function class since $\tilde{\mathbf{v}}$ vanishes in $\mathbb{R}^3 \setminus B_{3b}$. Estimate (4.7) follows by combining the estimates from (5.3), (5.7), (5.15) and (5.26).

The uniqueness statement is a direct consequence of decomposing a solution $(\mathbf{v}, \mathfrak{p})$ into a stationary and an oscillatory part by means of (2.2) and using the respective statements from Theorem 5.1 and Theorem 5.3. ☐

Proof of Theorem 4.7. We proceed as in the proof of Theorem 4.6 and first reduce the problem to the case of homogeneous boundary conditions. We construct the function $\tilde{\mathbf{v}}$ as before, and let $(\mathbf{w}_S, \mathfrak{q}_S)$ and $(\mathbf{w}_\perp, \mathfrak{q}_\perp)$ be the solutions to (5.27) and (5.28), which exist due to Theorem 5.2 and Theorem 5.3. Then $(\mathbf{v}, \mathfrak{p}) = (\tilde{\mathbf{v}} + \mathbf{w}_S + \mathbf{w}_\perp, \mathfrak{q}_S + \mathfrak{q}_\perp)$ is a solution to (4.1) and the estimates (4.8) and (4.9) follow from (5.5), (5.7) and (5.26). As before, the uniqueness statement follows from the respective statements of Theorem 5.2 and Theorem 5.3. ☐

6 Existence of solutions to the nonlinear problem

To show existence of time-periodic solutions to the nonlinear problem (3.8), we combine the linear theory from Theorem 4.5 Theorem 4.6 and Theorem 4.7 with suitable estimates for the linear perturbation term $\mathcal{L}$ and the nonlinear term $\mathcal{N}$ given in (3.9). Then Banach's fixed-point theorem will finally lead to the proofs of Theorem 4.1, Theorem 4.2 and Theorem 4.3. More precisely, we consider the set

$$\mathcal{I}_{\kappa,\rho} = \big\{(\mathbf{v},\mathfrak{q}) \mid \mathbf{v}\in \mathrm{L}_{1,\mathrm{loc}}(\Omega\times\mathbb{T})^3, \quad \mathfrak{q}\in \mathrm{L}_{1,\mathrm{loc}}(\Omega\times\mathbb{T}), \quad \|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa}\leq\rho\big\} \tag{6.1}$$

for a suitable norm $\|\cdot\|_{\mathcal{I}_\kappa}$ that is defined in the respective proofs and is suggested by the associated linear theory. For given $(\mathbf{v},\mathfrak{q})\in\mathcal{I}_{\kappa,\rho}$, we then consider the solution $(\mathbf{u},\mathfrak{p})$ to the linear system

$$\partial_t\mathbf{u}-\mu\Delta\mathbf{u}-\kappa\partial_1\mathbf{u}+\nabla\mathfrak{p} = \mathbf{f}+\mathcal{L}(\mathbf{v},\mathfrak{q})+\mathcal{N}(\mathbf{v}), \quad \operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega\times\mathbb{T}, \qquad \mathbf{u}|_{\Gamma\times\mathbb{T}}=\mathbf{h}|_{\Gamma\times\mathbb{T}}, \tag{6.2}$$

where $\mathcal{L}$ and $\mathcal{N}$ are defined in (3.9). We show that in the respective settings and for a suitable choice of ρ, this leads to a well-defined solution mapping $\Xi_\kappa\colon \mathcal{I}_{\kappa,\rho}\to\mathcal{I}_{\kappa,\rho}$, $(\mathbf{v},\mathfrak{q})\mapsto(\mathbf{u},\mathfrak{p})$, which is contractive. Therefore, Banach's fixed-point theorem yields the existence of an element $(\mathbf{w},\mathfrak{q})$ such that $\Xi_\kappa(\mathbf{w},\mathfrak{q})=(\mathbf{w},\mathfrak{q})$, that is, $(\mathbf{w},\mathfrak{q})$ is a solution to the nonlinear problem (3.8).

To derive suitable estimates, we write $\mathcal{L}(\mathbf{v},\mathfrak{q})$ and $\mathcal{N}(\mathbf{v})$ as

$$\begin{aligned}\mathcal{L}(\mathbf{v},\mathfrak{q}) &= \mathcal{L}_1\partial_t\mathbf{v}+\sum_{|\alpha|\leq2}\mathcal{L}_{2,\alpha}D^\alpha\mathbf{v}+\sum_{|\alpha|=1}\mathcal{L}_{3,\alpha}D^\alpha\mathfrak{q},\\ \mathcal{N}(\mathbf{v}) &= \mathcal{N}^1(\mathbf{v})+\mathcal{N}^2(\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{v}+\sum_{|\alpha|\leq1}\mathcal{L}_{4,\alpha}\mathbf{v}\cdot D^\alpha\mathbf{v}.\end{aligned} \tag{6.3}$$

Here, $\mathcal{L}_1$, $\mathcal{L}_{2,\alpha}$, $\mathcal{L}_{3,\alpha}$ and $\mathcal{L}_{4,\alpha}$ correspond to time-periodic continuous functions on $\Omega\times\mathbb{T}$, which vanish on $\mathbb{R}^3\setminus B_{2b}\times\mathbb{T}$ and satisfy the estimate

$$\|(\mathcal{L}_1,\mathcal{L}_{2,\alpha},\mathcal{L}_{3,\alpha},\mathcal{L}_{4,\alpha})\|_{\mathrm{L}_\infty(\Omega\times\mathbb{T})}\leq C\varepsilon_0 \tag{6.4}$$

if $|\kappa|\leq\kappa_0$, due to (3.6) and (3.7). We define $\Omega_{2b}=\Omega\cap B_{2b}$. The terms that vanish outside of $\Omega_{2b}\times\mathbb{T}$, where $\Omega_{2b}=\Omega\cap B_{2b}$, can be estimated in the following manner.

Lemma 6.1. *Let $2<p<\infty$ and $3<q<\infty$, and let*

$$\mathbf{v}\in\mathrm{H}^1_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b})^3)\cap\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega_{2b})^3), \qquad \nabla\mathfrak{q}\in\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b})^3).$$

It holds

$$\|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\varepsilon_0\big(\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))}+\|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega_{2b}))}+\|\nabla\mathfrak{q}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))}\big), \tag{6.5}$$

$$\|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\varepsilon_0\big(\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))}+\|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega_{2b}))}\big)^2. \tag{6.6}$$

Proof. Estimate (6.5) is a direct consequence of (6.3) and (6.4). For estimate (6.6), we first use Hölder's inequality and (6.4) to obtain

$$\|\mathcal{N}^2(\mathbf{v})_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\leq C\varepsilon_0\|\mathbf{v}\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(\Omega_{2b}))}\|\nabla\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))}$$

Now choose $\sigma>0$ so small that $\sigma+3/q<2(1-1/p)$, which is possible due to $2/p+3/q<2$. Using Sobolev's inequality and real interpolation, we then have

$$\begin{aligned}\|\mathbf{w}\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(D))} &\leq C\|\mathbf{w}\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{W}^{\sigma+3/q}_q(D))}\leq C\|\mathbf{w}\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{B}^{2(1-1/p)}_{q,p}(D))}\\ &\leq C(\|\partial_t\mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(D))}+\|\mathbf{w}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(D))})\end{aligned} \tag{6.7}$$

for any Lipschitz domain $D \subset \mathbb{R}^3$. Using this estimate with $\mathbf{w} = \mathbf{v}$ and $D = \Omega_R$ leads to estimate (6.6). □

We shall use these estimates of the local terms in all proofs below. However, for estimates of the term $\mathcal{N}^1(\mathbf{v})$, which also gives a contribution far away from the boundary, the spatial decay of solutions has to be taken into account. Observe that $\operatorname{div}\mathbf{v} = 0$ implies $\mathcal{N}^1(\mathbf{v}) = \operatorname{div}\tilde{\mathcal{N}}^1(\mathbf{v})$ with $\tilde{\mathcal{N}}^1(\mathbf{v}) = \mathbf{v}\otimes\mathbf{v}$. Then

$$\begin{aligned}
\mathcal{N}^1(\mathbf{v})_S &= \mathbf{v}_S\cdot\nabla\mathbf{v}_S + \int_{\mathbb{T}}\mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp\,\mathrm{d}t\\
\tilde{\mathcal{N}}^1(\mathbf{v})_S &= \mathbf{v}_S\otimes\mathbf{v}_S + \int_{\mathbb{T}}\mathbf{v}_\perp\otimes\mathbf{v}_\perp\,\mathrm{d}t;\\
\mathcal{N}^1(\mathbf{v})_\perp &= \mathbf{v}_S\cdot\nabla\mathbf{v}_\perp + \mathbf{v}_\perp\cdot\nabla\mathbf{v}_S + \mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp - \int_{\mathbb{T}}\mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp\,\mathrm{d}t\\
\tilde{\mathcal{N}}^1(\mathbf{v})_\perp &= \mathbf{v}_S\otimes\mathbf{v}_\perp + \mathbf{v}_\perp\otimes\mathbf{v}_S + \mathbf{v}_\perp\otimes\mathbf{v}_\perp - \int_{\mathbb{T}}\mathbf{v}_\perp\otimes\mathbf{v}_\perp\,\mathrm{d}t.
\end{aligned}\tag{6.8}$$

and $\operatorname{div}\tilde{\mathcal{N}}(\mathbf{v})_S = \mathcal{N}^1(\mathbf{v})_S$ and $\operatorname{div}\tilde{\mathcal{N}}(\mathbf{v})_\perp = \mathcal{N}^1(\mathbf{v})_\perp$. Corresponding estimates that suit to the linear theory are derived in the following three lemmas.

Lemma 6.2. *Let $2 < p < \infty$ and $3 < q < \infty$, and let*

$$\mathbf{v} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3), \qquad <\mathbf{v}>_{p,1} + <\nabla\mathbf{v}>_{p,2} < \infty.$$

Then

$$\begin{aligned}
<\mathcal{N}^1(\mathbf{v})_S>_3 &\le C(<\mathbf{v}_S>_1<\nabla\mathbf{v}_S>_2 + <\mathbf{v}_\perp>_{p,1}<\nabla\mathbf{v}_\perp>_{p,2}),\\
<\tilde{\mathcal{N}}^1(\mathbf{v})_S>_2 &\le C(<\mathbf{v}_S>_1^2 + <\mathbf{v}_\perp>_{p,1}^2),\\
<\mathcal{N}^1(\mathbf{v})_\perp>_{p,2} &\le C\big(<\mathbf{v}_\perp>_{p,1}<\nabla\mathbf{v}_S>_2\\
&\qquad + (\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + <\mathbf{v}>_{p,1}) <\nabla\mathbf{v}_\perp>_{p,2}\big),\\
<\tilde{\mathcal{N}}^1(\mathbf{v})_\perp>_{p,1} &\le C\big(\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + <\mathbf{v}>_{p,1}\big) <\mathbf{v}_\perp>_{p,1}.
\end{aligned}$$

Proof. The estimates of $\mathcal{N}(\mathbf{v})_S$ and $\tilde{\mathcal{N}}(\mathbf{v})_S$ follow directly from using Hölder's inequality for the time integrals since $p > 2$. For the estimates of $\mathcal{N}(\mathbf{v})_\perp$ and $\tilde{\mathcal{N}}(\mathbf{v})_\perp$, Hölder's inequality leads to

$$\begin{aligned}
<\mathcal{N}^1(\mathbf{v})_\perp>_{p,2} &\le C(<\mathbf{v}_S>_1<\nabla\mathbf{v}_\perp>_{p,2} + <\mathbf{v}_\perp>_{p,1}<\nabla\mathbf{v}_S>_2\\
&\qquad + \|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(\Omega))} <\nabla\mathbf{v}_\perp>_{p,2} + <\mathbf{v}_\perp>_{p,1}<\nabla\mathbf{v}_\perp>_{p,2}),\\
<\tilde{\mathcal{N}}^1(\mathbf{v})_\perp>_{p,1} &\le C(<\mathbf{v}_S>_1<\mathbf{v}_\perp>_{p,1} + \|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(\Omega))} <\mathbf{v}_\perp>_{p,1} + <\mathbf{v}_\perp>_{p,1}^2).
\end{aligned}$$

Using now the interpolation inequality (6.7) completes the proof. □

Lemma 6.3. *Let $2 < p < \infty$, $3 < q < \infty$ and $\delta \in (0, \frac{1}{4})$, and let*

$$\begin{aligned}
&\mathbf{v} \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3),\\
&<\mathbf{v}_S>^w_{1,\delta} + <\nabla\mathbf{v}_S>^w_{3/2,1/2+\delta} + <\mathbf{v}_\perp>_{p,1+\delta} + <\nabla\mathbf{v}_\perp>_{p,2+\delta} < \infty.
\end{aligned}$$

Then

$$\begin{aligned}
&<\mathcal{N}^1(\mathbf{v})_S>^w_{5/2,1/2+2\delta} \le C\big(<\mathbf{v}_S>^w_{1,\delta}<\nabla\mathbf{v}_S>^w_{3/2,1/2+\delta} + <\mathbf{v}_\perp>_{p,1+\delta}<\nabla\mathbf{v}_\perp>_{p,2+\delta}\big),\\
&<\mathcal{N}^1(\mathbf{v})_\perp>_{p,2+\delta} \le C\big(<\mathbf{v}_\perp>_{p,1+\delta}<\nabla\mathbf{v}_S>^w_{3/2,1/2+\delta}\\
&\qquad + (\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + <\mathbf{v}_S>^w_{1,\delta} + <\mathbf{v}_\perp>_{p,1+\delta}) <\nabla\mathbf{v}_\perp>_{p,2+\delta}\big),\\
&<\tilde{\mathcal{N}}^1(\mathbf{v})_\perp>_{p,1+\delta}\\
&\qquad \le C(\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + <\mathbf{v}_S>^w_{1,\delta} + <\mathbf{v}_\perp>_{p,1+\delta}) <\mathbf{v}_\perp>_{p,1+\delta}.
\end{aligned}$$

Proof. As in the previous proof, the estimate of $\mathcal{N}^1(\mathbf{v})_S$ follows directly from Hölder's inequality. Moreover, we obtain

$$\begin{aligned}
< \mathcal{N}^1(\mathbf{v})_\perp >_{p,2+\delta} &\leq C(< \mathbf{v}_S >^w_{1,\delta}< \nabla\mathbf{v}_\perp >_{p,2+\delta} + < \mathbf{v}_\perp >_{p,1+\delta}< \nabla\mathbf{v}_S >^w_{3/2,1/2+\delta} \\
&\qquad + \|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(\Omega))} < \nabla\mathbf{v}_\perp >_{p,2+\delta} + < \mathbf{v}_\perp >_{p,1+\delta}< \nabla\mathbf{v}_\perp >_{p,2+\delta}), \\
< \tilde{\mathcal{N}}^1(\mathbf{v})_\perp >_{p,1+\delta} &\leq C(< \mathbf{v}_S >^w_{1,\delta}< \mathbf{v}_\perp >_{p,1+\delta} \\
&\qquad + \|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T},\mathrm{L}_\infty(\Omega))} < \mathbf{v}_\perp >_{p,1+\delta} + < \mathbf{v}_\perp >^2_{p,1+\delta}).
\end{aligned}$$

The asserted estimates now result from the interpolation inequality (6.7). □

Lemma 6.4. *Let $2 < p < \infty$, $3 < q < \infty$ and $1 < s \leq 4/3$, and let $\mathbf{v} = \mathbf{v}_S + \mathbf{v}_\perp$ be such that*

$$\begin{aligned}
&\mathbf{v}_S \in \mathrm{L}_{2s/(2-s)}(\Omega), \quad \nabla\mathbf{v}_S \in \mathrm{L}_{4s/(4-s)}(\Omega), \quad \nabla^2\mathbf{v}_S \in \mathrm{L}_s(\Omega) \cap \mathrm{L}_q(\Omega), \\
&\mathbf{v}_\perp \in \mathrm{H}^1_p(\mathbb{T}, \mathrm{L}_q(\Omega)^3 \cap \mathrm{L}_s(\Omega)^3) \cap \mathrm{L}_p(\mathbb{T}, \mathrm{H}^2_q(\Omega)^3 \cap \mathrm{H}^2_s(\Omega)^3).
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{N}^1(\mathbf{v})_S\|_{\mathrm{L}_s(\Omega)} &\leq C\big(\|\mathbf{v}_S\|^{1-\theta}_{\mathrm{L}_{2s/(2-s)}(\Omega)}\|\nabla^2\mathbf{v}_S\|^\theta_{\mathrm{L}_q(\Omega)}\|\nabla\mathbf{v}_S\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} \\
&\qquad + (\|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))})\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^1_s(\Omega))}\big), \\
\|\mathcal{N}^1(\mathbf{v})_S\|_{\mathrm{L}_q(\Omega)} &\leq C\big((\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)})^2 \\
&\qquad + (\|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))})^2\big), \\
\|\mathcal{N}^1(\mathbf{v})_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_s(\Omega))} &\leq C\big(\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))}\big) \\
&\qquad \times \big(\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^1_s(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^1_q(\Omega))}\big), \\
\|\mathcal{N}^1(\mathbf{v})_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_q(\Omega))} &\leq C\big(\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)}+\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} + \|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))}\big) \\
&\qquad \times \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^1_q(\Omega))},
\end{aligned}$$

where $\theta = (1/s - 3/4)/(1/s + 1/6 - 1/q)$.

Proof. Using the Gagliardo–Nirenberg inequality in exterior domains (see [4]) and Young's inequality, we obtain

$$\begin{aligned}
\|\mathbf{v}_S\|_{\mathrm{L}_4(\Omega)} &\leq C\|\mathbf{v}_S\|^{1-\theta}_{\mathrm{L}_{2s/(2-s)}(\Omega)}\|\nabla^2\mathbf{v}_S\|^\theta_{\mathrm{L}_q(\Omega)}, \\
\|\mathbf{v}_S\|_{\mathrm{L}_\infty(\Omega)} &\leq C\|\mathbf{v}_S\|^{1-\theta_1}_{\mathrm{L}_{3s/(3-2s)}(\Omega)}\|\nabla^2\mathbf{v}_S\|^{\theta_1}_{\mathrm{L}_q(\Omega)} \leq C\big(\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)}\big), \\
\|\nabla\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} &\leq C\|\nabla\mathbf{v}_S\|^{1-\theta_2}_{\mathrm{L}_{3s/(3-s)}(\Omega)}\|\nabla^2\mathbf{v}_S\|^{\theta_2}_{\mathrm{L}_q(\Omega)} \leq C\big(\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)}\big),
\end{aligned}$$

with θ as above, $\theta_1 = (1/s - 2/3)/(1/s - 1/q)$ and $\theta_2 = (1/s - 1/q - 1/3)/(1/s - 1/q)$. Combining these estimates with the interpolation inequality (6.7) for $\mathbf{w} = \mathbf{v}_\perp$ and $D = \Omega$, we obtain the asserted estimates of $\mathcal{N}(\mathbf{v})_S$ directly from Hölder's inequality. For the estimates of $\mathcal{N}(\mathbf{v})_\perp$, the Hölder inequality yields

$$\begin{aligned}
\|\mathcal{N}^1(\mathbf{v})_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_s(\Omega))} &\leq \|\mathbf{v}_S\|_{\mathrm{L}_{3s/(3-2s)}(\Omega)}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_{3/2}(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_3(\Omega))}\|\nabla\mathbf{v}_S\|_{\mathrm{L}_{3s/(3-s)}(\Omega)} \\
&\qquad +\|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T};\mathrm{L}_\infty(\Omega))}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_s(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T};\mathrm{L}_\infty(\Omega))}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_1(\mathbb{T};\mathrm{L}_s(\Omega))}, \\
\|\mathcal{N}^1(\mathbf{v})_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_q(\Omega))} &\leq \|\mathbf{v}_S\|_{\mathrm{L}_\infty(\Omega)}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_q(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_\infty(\Omega))}\|\nabla\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} \\
&\qquad +\|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T};\mathrm{L}_\infty(\Omega))}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T};\mathrm{L}_q(\Omega))}+\|\mathbf{v}_\perp\|_{\mathrm{L}_\infty(\mathbb{T};\mathrm{L}_\infty(\Omega))}\|\nabla\mathbf{v}_\perp\|_{\mathrm{L}_1(\mathbb{T};\mathrm{L}_q(\Omega))}.
\end{aligned}$$

Since $s \leq 4/3 < 3/2 < 3 < q$, the remaining estimates now follow by Sobolev embeddings and interpolation as before. □

After these preparations, we prove the main theorems of this article. We begin with the case $\kappa = 0$.

Proof of Theorem 4.1. Consider $\mathcal{I}_{0,\rho}$ as in (6.1) with

$$\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_0} := \|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla\mathfrak{q}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + <\mathbf{v}>_{p,1} + <\nabla\mathbf{v}>_{p,2}\,.$$

Let $(\mathbf{v},\mathfrak{q}) \in \mathcal{I}_{0,\rho}$. In virtue of Lemma 6.1 and Lemma 6.2, Theorem 4.5 implies the existence of a solution $(\mathbf{u},\mathfrak{p})$ to (6.2) such that

$$\begin{aligned}
\|(\mathbf{u},\mathfrak{p})\|_{\mathcal{I}_0} \leq C\big(&<\mathbf{f}_S>_3 + <\mathbf{F}_S>_2 + <\mathbf{f}_\perp>_{p,2} + <\mathbf{F}_\perp>_{p,1} \\
&+ <\mathcal{N}^1(\mathbf{v})_S>_3 + <\tilde{\mathcal{N}}^1(\mathbf{v})_S>_2 + <\mathcal{N}^1(\mathbf{v})_\perp>_{p,2} + <\tilde{\mathcal{N}}^1(\mathbf{v})_\perp>_{p,1} \\
&+ \|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big) \\
\leq C\big(&\varepsilon^2 + \|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_0} + \varepsilon_0\|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_0} + \varepsilon_0\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_0}\big) \\
\leq C(&\varepsilon^2\rho^{-1} + \rho + \varepsilon_0\rho + \varepsilon_0)\rho,
\end{aligned}$$

where $C > 0$ does not depend on the choice of $(\mathbf{v},\mathfrak{q})$. Choosing $\rho = \varepsilon$ and $\varepsilon, \varepsilon_0 > 0$ sufficiently small, we have $C(\varepsilon^2\rho^{-1} + \rho + \varepsilon_0\rho + \varepsilon_0) < 1$, so that the solution map $\Xi_0\colon (\mathbf{v},\mathfrak{q}) \mapsto (\mathbf{u},\mathfrak{p})$ is a well-defined self-mapping on $\mathcal{I}_{0,\rho}$.

Since $\mathcal{L}(\mathbf{v},\mathfrak{q})$ is linear and $\mathcal{N}(\mathbf{v})$ is quadratic in $(\mathbf{v},\mathfrak{q})$, similar arguments lead to a constant $C > 0$ such that

$$\begin{aligned}
&\|\Xi_0(\mathbf{v}_1,\mathfrak{q}_1) - \Xi_0(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_0} \\
&\quad\leq C\big(<\mathcal{N}^1(\mathbf{v}_1)_S - \mathcal{N}^1(\mathbf{v}_2)_S>_3 + <\tilde{\mathcal{N}}^1(\mathbf{v}_1)_S - \tilde{\mathcal{N}}^1(\mathbf{v}_2)_S>_2 \\
&\qquad + <\mathcal{N}^1(\mathbf{v}_1)_\perp - \mathcal{N}^1(\mathbf{v}_2)_\perp>_{p,2} + <\tilde{\mathcal{N}}^1(\mathbf{v}_1)_\perp - \tilde{\mathcal{N}}^1(\mathbf{v}_2)_\perp>_{p,1} \\
&\qquad + \|\mathcal{N}^2(\mathbf{v}_1) - \mathcal{N}^2(\mathbf{v}_2)\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{L}(\mathbf{v}_1,\mathfrak{q}_1) - \mathcal{L}(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big) \\
&\quad\leq C\big(\|(\mathbf{v}_1,\mathfrak{q}_1)\|_{\mathcal{I}_0} + \|(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_0} + \varepsilon_0\|(\mathbf{v}_1,\mathfrak{q}_1)\|_{\mathcal{I}_0} + \varepsilon_0\|(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_0} + \varepsilon_0\big) \\
&\qquad \times \|(\mathbf{v}_1 + \mathbf{v}_2, \mathfrak{q}_1 - \mathfrak{q}_2)\|_{\mathcal{I}_0} \\
&\quad\leq C(\rho + \varepsilon_0\rho + \varepsilon_0)\|(\mathbf{v}_1 - \mathbf{v}_2, \mathfrak{q}_1 - \mathfrak{q}_2)\|_{\mathcal{I}_0},
\end{aligned}$$

for all $(\mathbf{v}_1,\mathfrak{q}_1), (\mathbf{v}_2,\mathfrak{q}_2) \in \mathcal{I}_{0,\rho}$. Again we have $\rho = \varepsilon$, and choosing $\varepsilon, \varepsilon_0 > 0$ so small that $C(\varepsilon+\varepsilon_0\varepsilon+\varepsilon_0) < 1$, we see that Ξ_0 is also a contraction. Therefore, the contraction mapping principle yields existence of a fixed-point of Ξ_0, which is a solution to (3.8) with the asserted properties. □

We treat the case $\kappa \neq 0$ in a similar way, starting with the framework of functions with anisotropic pointwise decay.

Proof of Theorem 4.2. Consider $\mathcal{I}_{\kappa,\rho}$ as in (6.1) with

$$\begin{aligned}
\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa} := &\|\partial_t\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla\mathfrak{q}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\
&+ |\kappa|^\delta <\mathbf{v}_S>^w_{1,\delta} + |\kappa|^\delta <\nabla\mathbf{v}_S>^w_{3/2,1/2+\delta} + <\mathbf{v}_\perp>_{p,1+\delta} + <\nabla\mathbf{v}_\perp>_{p,2+\delta}\,.
\end{aligned}$$

Let $(\mathbf{v},\mathfrak{q}) \in \mathcal{I}_{\kappa,\rho}$. In virtue of Lemma 6.1 and Lemma 6.3, Theorem 4.6 implies the existence of a solution $(\mathbf{u},\mathfrak{p})$ to (6.2) such that

$$\begin{aligned}
\|(\mathbf{u},\mathfrak{p})\|_{\mathcal{I}_\kappa} \leq C\big(&<\mathbf{f}_S>^w_{5/2,1/2+2\delta} + <\mathbf{f}_\perp>_{p,2+\delta} + <\mathbf{F}_\perp>_{p,1+\delta} + <\mathcal{N}^1(\mathbf{v})_S>^w_{5/2,1/2+2\delta} \\
&+ <\mathcal{N}^1(\mathbf{v})_\perp>_{p,\ell+1} + <\tilde{\mathcal{N}}^1(\mathbf{v})_\perp>_{p,\ell} + \|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\
&+ \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big) \\
\leq C\big(&\varepsilon^2|\kappa|^{2\delta} + (1+|\kappa|^{-2\delta})\|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa} + \varepsilon_0\|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa} + \varepsilon_0\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa}\big) \\
\leq C\big(&\varepsilon^2|\kappa|^{2\delta}\rho^{-1} + (1+|\kappa|^{-2\delta})\rho + \varepsilon_0\rho + \varepsilon_0\big)\rho.
\end{aligned}$$

We further proceed similarly to the proof of Theorem 4.6. We choose $\rho = \varepsilon|\kappa|^{2\delta}$. Then the solution map $\Xi_\kappa\colon (\mathbf{v},\mathfrak{q}) \mapsto (\mathbf{u},\mathfrak{p})$ is a self-mapping on $\mathcal{I}_{\kappa,\rho}$ if $C(\varepsilon + \varepsilon(|\kappa|^{2\delta}+1) + \varepsilon_0(1+\varepsilon|\kappa|^{2\delta})) < 1$. This is the case for all $|\kappa| \leq \kappa_0$ if we choose $\varepsilon, \varepsilon_0 > 0$ sufficiently small. Arguing as in the previous proof, we further obtain

$$\begin{aligned}
&\|\Xi_\kappa(\mathbf{v}_1,\mathfrak{q}_1) - \Xi_\kappa(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_\kappa} \\
&\quad\leq C\big((1+|\kappa|^{-2\delta})(\|(\mathbf{v}_1,\mathfrak{q}_1)\|_{\mathcal{I}_\kappa} + \|(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_\kappa}) + \varepsilon_0(1+\|(\mathbf{v}_1,\mathfrak{q}_1)\|_{\mathcal{I}_\kappa} + \|(\mathbf{v}_2,\mathfrak{q}_2)\|_{\mathcal{I}_\kappa})\big) \\
&\qquad\qquad \times \|(\mathbf{v}_1-\mathbf{v}_2,\mathfrak{q}_1-\mathfrak{q}_2)\|_{\mathcal{I}_\kappa} \\
&\quad\leq C\big(\rho + |\kappa|^{-2\delta}\rho + \varepsilon_0 + \varepsilon_0\rho\big)\|(\mathbf{v}_1-\mathbf{v}_2,\mathfrak{q}_1-\mathfrak{q}_2)\|_{\mathcal{I}_\kappa}
\end{aligned}$$

for $(\mathbf{v}_1,\mathfrak{q}_1),(\mathbf{v}_2,\mathfrak{q}_2) \in \mathcal{I}_{\kappa,\rho}$. With the choice $\rho = \varepsilon|\kappa|^{2\delta}$ we see that Ξ_κ is also contractive for all $|\kappa| \leq \kappa_0$ for $\varepsilon, \varepsilon_0 > 0$ sufficiently small. Finally, Banach's fixed-point theorem yields an element $(\mathbf{w},\mathfrak{q}) \in \mathcal{I}_\kappa$ with $(\mathbf{w},\mathfrak{q}) = \Xi_\kappa(\mathbf{w},\mathfrak{q})$, which completes the proof. □

Finally, we treat the case $\kappa \neq 0$ in a framework of homogeneous Sobolev spaces.

Proof of Theorem 4.3. Consider $\mathcal{I}_{\kappa,\rho}$ as in (6.1) with

$$\begin{aligned}
\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa} := \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} &+ |\kappa|^{1/4}\|\nabla\mathbf{v}_S\|_{\mathrm{L}_{4s/(4-s)}(\Omega)} + |\kappa|^{1/2}\|\mathbf{v}_S\|_{\mathrm{L}_{2s/(2-s)}(\Omega)} \\
&+ |\kappa|\,\|\partial_1\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)} + \|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_s(\Omega))} \\
&+ \|\nabla\mathfrak{q}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\partial_t\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{H}^2_q(\Omega))} + \|\nabla\mathfrak{q}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}.
\end{aligned}$$

Let $(\mathbf{v},\mathfrak{q}) \in \mathcal{I}_{\kappa,\rho}$. In virtue of Lemma 6.1 and Lemma 6.4, Theorem 4.7 implies the existence of a solution $(\mathbf{u},\mathfrak{p})$ to (6.2) such that

$$\begin{aligned}
\|(\mathbf{u},\mathfrak{p})\|_{\mathcal{I}_\kappa} \leq C\big(&\|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathbf{f}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{N}^1(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathcal{N}^1(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\
&+ \|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} \\
&+ \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{h}\|_{\mathrm{T}_{p,s}(\Gamma\times\mathbb{T})} + \|\mathbf{h}\|_{\mathrm{T}_{p,q}(\Gamma\times\mathbb{T})}\big).
\end{aligned}$$

Observe that C is independent of $|\kappa| \leq 1$. Since $\mathcal{N}^2(\mathbf{v})$ and $\mathcal{L}(\mathbf{v})$ vanish outside Ω_{2b}, we can use Lemma 6.1 to estimate

$$\begin{aligned}
&\|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_s(\Omega))} + \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} \\
&\qquad\leq C\big(\|\mathcal{N}^2(\mathbf{v})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))} + \|\mathcal{L}(\mathbf{v},\mathfrak{q})\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega_{2b}))}\big) \leq C\varepsilon_0\big(\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa} + \|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa}\big),
\end{aligned}$$

where we used

$$\|\mathbf{v}_S\|_{\mathrm{H}^2_q(\Omega_{2b})} \leq C\big(\|\mathbf{v}_S\|_{\mathrm{L}_{3s/(3-2s)}(\Omega_{2b})} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega_{2b})}\big) \leq C\big(\|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_s(\Omega)} + \|\nabla^2\mathbf{v}_S\|_{\mathrm{L}_q(\Omega)}\big).$$

We further decompose $\mathcal{N}^1(\mathbf{v})$ into steady-state and oscillatory part and combine the estimates from Lemma 6.4 with the previous one to conclude

$$\begin{aligned}
\|(\mathbf{u},\mathfrak{p})\|_{\mathcal{I}_\kappa} &\leq C\big(\varepsilon^2|\kappa|^{(1+\delta)/2} + |\kappa|^{-(1-\theta)/2}\|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa} + \|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa} + \varepsilon_0(\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\kappa} + \|(\mathbf{v},\mathfrak{q})\|^2_{\mathcal{I}_\kappa})\big) \\
&\leq C\big(\varepsilon^2|\kappa|^{(1+\delta)/2}\rho^{-1} + |\kappa|^{-(1-\theta)/2}\rho + \rho + \varepsilon_0(1+\rho)\big)\rho
\end{aligned}$$

with θ as in Lemma 6.4, so that $\theta \in [0, 3/10)$. We now choose $\rho = \varepsilon|\kappa|^{1/2}$. Then the solution map $\Xi_\kappa\colon (\mathbf{v},\mathfrak{q}) \mapsto (\mathbf{u},\mathfrak{p})$ is a self-mapping on $\mathcal{I}_{\kappa,\rho}$ if

$$C\big(\varepsilon|\kappa|^{\delta/2} + \varepsilon|\kappa|^{\theta/2} + \varepsilon|\kappa|^{1/2} + \varepsilon_0(1+\varepsilon|\kappa|^{1/2})\big) < 1.$$

Since the constant C is independent of κ, we can take $\varepsilon, \varepsilon_0 > 0$ so small that this is satisfied for all $|\kappa| \leq \kappa_0$. Modifying the previous argument, we can further show

$$\begin{aligned}
&\|\Xi_\kappa(\mathbf{v}_1, \mathfrak{q}_1) - \Xi_\kappa(\mathbf{v}_2, \mathfrak{q}_2)\|_{\mathcal{I}_\kappa} \\
&\leq C\big((1+|\kappa|^{-(1-\theta)/2})(\|(\mathbf{v}_1, \mathfrak{q}_1)\|_{\mathcal{I}_\kappa} + \|(\mathbf{v}_2, \mathfrak{q}_2)\|_{\mathcal{I}_\kappa}) + \varepsilon_0(1 + \|(\mathbf{v}_1, \mathfrak{q}_1)\|_{\mathcal{I}_\kappa} + \|(\mathbf{v}_2, \mathfrak{q}_2)\|_{\mathcal{I}_\kappa})\big) \\
&\qquad \times \|(\mathbf{v}_1 - \mathbf{v}_2, \mathfrak{q}_1 - \mathfrak{q}_2)\|_{\mathcal{I}_\kappa} \\
&\leq C\big(\rho + |\kappa|^{-(1-\theta)/2}\rho + \varepsilon_0 + \varepsilon_0\rho\big)\|(\mathbf{v}_1 - \mathbf{v}_2, \mathfrak{q}_1 - \mathfrak{q}_2)\|_{\mathcal{I}_\kappa}
\end{aligned}$$

for some $C > 0$ independent of κ and for all $(\mathbf{v}_1, \mathfrak{q}_1)$, $(\mathbf{v}_2, \mathfrak{q}_2) \in \mathcal{I}_{\kappa,\rho}$. With $\rho = \varepsilon|\kappa|^{1/2}$, we can choose and $\varepsilon, \varepsilon_0 > 0$ so small that Ξ_κ is a contractive self-mapping for all $|\kappa| \leq \kappa_0$. Now Banach's fixed-point theorem yields existence of a unique fixed point in $\mathcal{I}_{\kappa,\rho}$, which is a solution to (3.8) as claimed. □

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