1. INTRODUCTION

The multifractal analysis, i.e., the analysis of invariant sets and measures with multifractal structure, has been recently developed as a powerful tool for numerical study of dynamical systems. These spectra capture information about various dimensions associated with the dynamics. Among them are the well-known Hausdorff dimension, correlation dimension, and information dimension of invariant measures.

Another example of multifractal spectra is entropy spectra introduced in [3]. They provide an integrated information on the distribution of topological entropy associated with local entropies.

In [3, 4, 5] it is demonstrated that multifractal spectra can be used in a sense to "restore" the dynamics – the phenomenon that we call *multifractal rigidity*.

The multifractal analysis is essentially measuring the "size" (in the sense of Hausdorff dimension or topological entropy) of special discontinuous measurable functions (such as the local entropy, the pointwise dimension, etc.) on geometrically complicated objects (supports of invariant measures). This is expressed in a function f which is called the *spectrum*.

This analysis was investigated in several different situations and there is a huge amount of literature on this subject (see [9] for references and more details). One of the main results of this theory is that there is an interval (α_1, α_2) on which the spectrum $f(\alpha)$ (for definition see below) is analytic, convex, and can be continuously extended to the boundary provided it is not a point spectrum (see section 4).

One of the questions which arise is what happens outside this interval? We will see that for expanding conformal repellers the closed interval $[\alpha_1, \alpha_2]$ coincides with the domain of definition of the spectrum (*completeness of the spectrum*).

The other question we are interested in is which intervals occur as the domain of definition of spectra, and which values does it take on the boundary of $[\alpha_1, \alpha_2]$. We will give a complete answer to this question by proving that for a given expanding conformal repeller $\mathcal{J}$ any interval in $\mathbb{R}^+$ containing the point $\dim_H \mathcal{J}$ is the domain of definition of the dimension spectrum of some Gibbs measure. We call those intervals *admissible*.

This is supported by experimental and numerical observations of chaotic systems. Although the majority of these studies seem to indicate that the spectrum vanishes at its boundary there is no rigorous result in this direction known to the authors besides some results on geometric constructions with Bernoulli measures supported on their limit sets (see for example [6, 15]). In fact we will show that this is not true in general. On the contrary, for a given admissible interval (α_1, α_2) and a given pair of *admissible* boundary values on f_1, f_2 we construct a Hölder continuous potential for which the spectrum takes these values at the endpoints of its domain of definition $[\alpha_1, \alpha_2]$. Hereby the admissability of a pair of boundary values is determined by canonical restrictions given by the general shape of the graph of the spectrum. We

will call spectra with at least one strictly positive boundary value *degenerate.* Note that point spectra are degenerated in this sense.

On the other hand, in section 7 we prove that degenerate spectra are not very likely. Namely, for a typical (in the sense of Baire) Hölder continuous potential the dimension or entropy spectrum of its corresponding Gibbs measure is *non-degenerated.* This justifies the experimental and numerical observations. It also shows that despite the spectrum being defined in terms of the analytic pressure functional the graph of the spectrum does not depend continuously on the potential.

Throughout the paper we use some standard notations which are explained in the appendix.

2. Examples of multifractal spectra

In this section we illustrate the general concept of multifractal spectra. See [3] for more details.

2.1. Dimension and entropy spectra.
Let X be a complete separable metric space and $F \colon X \to X$ a continuous map. There are two "natural" set functions on X. The first one is generated by the metric structure on X. Namely, given a subset $Z \subset X$, we set

$$G_D(Z) = \dim_H Z, \tag{1}$$

where $\dim_H Z$ is the Hausdorff dimension of Z (see Appendix).

The second function is generated by the dynamical system f acting on X and the metric on X. Namely,

$$G_E(Z) = h(f|_Z), \tag{2}$$

where $h(F|_Z)$ is the topological entropy of F on Z (see Appendix; notice that Z need not be compact nor F-invariant). We call the multifractal spectra generated by the function G_D *dimension spectra*, and the multifractal spectra generated by the function G_E *entropy spectra*. We give a precise description below.

2.2. Multifractal spectra for pointwise dimensions.
Let m be a Borel finite measure on X. Consider the subset $Y \subset X$ consisting of all points $x \in X$ for which the limit

$$d_m(x) = \lim_{r \to 0} \frac{\log m(B(x,r))}{\log r}$$

exists, where $B(x,r)$ denotes the ball of radius r centered at x. The number $d_m(x)$ is called the *pointwise dimension* of m at x. Whenever $x \in Y$ we say that the pointwise dimension of m exists at the point x. Although the pointwise dimension may not

exist for all points in an expanding conformal repeller $\mathcal{J}$ with respect to an invariant measure $m \in M_{\mathrm{inv}}(\mathcal{J})$ we can always define

$$\underline{d}_m(x) = \varliminf_{r \to 0} \frac{\log m(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_m(x) = \varlimsup_{r \to 0} \frac{\log m(B(x,r))}{\log r}$$

We define the function g_D on Y by

$$g_D(x) = d_m(x).$$

The corresponding multifractal decomposition consists of the sets

$$D_\alpha = \{x : d_m(x) = \alpha\}.$$

We also consider the sets

$$D_\alpha^- = \{x : \underline{d}_m(x) = \alpha\} \quad \text{and} \quad D_\alpha^+ = \{x : \overline{d}_m(x) = \alpha\}$$

We obtain the multifractal spectrum $f^D(\alpha) = f_m^D(\alpha) = G_D(D_\alpha) = \dim_H D_\alpha$ specified by the pair of functions (g_D, G_D). The spectrum f^D is known in the literature as the *dimension spectrum* or *$f(\alpha)$-spectrum for dimensions*. We will omit the subscript m if it will cause no confusion. The concept of a multifractal analysis was suggested by a group of physicists in [8] (see [9] for more references and details).

In [7], Eckmann and Ruelle discussed the pointwise dimension of hyperbolic measures (that is, measures with non-zero Lyapunov exponents almost everywhere), invariant under diffeomorphisms. They conjectured that the pointwise dimension exists almost everywhere, that is, $m(X \setminus Y) = 0$. This claim has been known as the *Eckmann–Ruelle conjecture* and has become a celebrated problem in the dimension theory of dynamical systems. In [2], we establish the affirmative solution of this conjecture for $C^{1+\varepsilon}$ diffeomorphisms (an announcement appeared in [1]).

2.3. Multifractal spectra for local entropies.

Let X be a complete separable metric space and $F \colon X \to X$ a continuous map preserving a Borel probability measure μ. Consider a finite measurable partition ξ of X. For every $n > 0$, we write $\xi_n = \xi \vee F^{-1}\xi \vee \cdots \vee F^{-n}\xi$, and denote by $\xi_n(x)$ the element of the partition ξ_n that contains the point x. Consider the set $Y = Y_\xi \subset X$ consisting of all points $x \in X$ for which the limit

$$h_\mu(F, \xi, x) = \lim_{n \to \infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists. We call $h_\mu(F, \xi, x)$ the *μ-local entropy* of F at the point x (with respect to ξ). Clearly, Y is F-invariant and $h_\mu(F, \xi, Fx) = h_\mu(F, \xi, x)$ for every $x \in Y$. By the Shannon–McMillan–Breiman theorem, $\mu(X \setminus Y) = 0$. In addition, if ξ is a *generating partition* and μ is ergodic, then

$$h_\mu(F) = h_\mu(F, \xi, x)$$

for μ-almost all $x \in X$, where $h_\mu(F)$ is the measure-theoretic entropy of F (with respect to μ). We define the function g_E on Y by

$$g_E(x) = h_\mu(F, \xi, x).$$

Let us stress that g_E may depend on ξ. The corresponding multifractal decomposition consists of the sets

$$E_\alpha = \{x : h_\mu(F, \xi, x) = \alpha\}.$$

We obtain the multifractal spectrum $f^E = f_\mu^E$ specified by the pair of functions (g_E, G_E). We call it the *multifractal spectra for (local) entropies* or simply *entropy spectrum*. In Sections 3 and 4 below we will observe that in some situations these spectra, in fact, do not dependent on ξ for a broad class of partitions.

We remark that in the study of the multifractal spectra for local entropies, the Shannon–McMillan–Breiman theorem plays the same role as the Eckmann–Ruelle conjecture in the study of the multifractal spectra for pointwise dimensions.

3. Multifractal Spectra of Gibbs measures for Subshifts of Finite Type

The results described in the next two sections are proved in [11, 3]. Let A be a $p \times p$ matrix whose entries are either 0 or 1. The topological Markov chain Σ_A^+ consists of the sequences $\underline{x} = (i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbb{N}}$ such that $a_{i_k i_{k+1}} = 1$ for every $k \geq 1$. Let $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$ be the shift map on Σ_A^+. We assume that A is *transitive*, i.e., there exists a positive integer M such that all entries of A^M are positive (this holds if and only if $\sigma|_{\Sigma_A^+}$ is topologically mixing).

Fix $a > 1$ and define a metric on Σ_A^+ by

$$d(\underline{x}, \underline{x}') = \sum_{k=1}^{\infty} a^{-k} |i_k - i_k'|.$$

Notice that $d(\sigma\underline{x}, \sigma\underline{x}') = a \cdot d(\underline{x}, \underline{x}')$ for all $\underline{x}, \underline{x}' \in \Sigma_A^+$ with $d(\underline{x}, \underline{x}') < a^{-1}$.

Given a continuous function φ on Σ_A^+, a measure μ on Σ_A^+ is said to be a *Gibbs measure* for φ if there exist constants $C_1, C_2 > 0$, such that for every $\underline{x} = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$

$$C_1 \leq \frac{\mu(C_{i_1 \cdots i_n})}{\exp(-nP(\varphi) + \sum_{k=0}^{n-1} \varphi(\sigma^k \underline{x}))} \leq C_2,$$

where $C_{i_1 \cdots i_n} = C_n(\underline{x}) = \{\underline{x}' \in \Sigma_A^+ : i_k = i_k' \text{ for } 1 \leq k \leq n\}$ is the cylinder set of length n containing $\underline{x}$, and P is the topological pressure with respect to σ (see Appendix).

Let $\mathcal{F}_\theta$ be the space of Hölder continuous functions on Σ_A^+ with Hölder exponent θ. We can decompose $\mathcal{F}_\theta$ into

$$\mathcal{F}_\theta = \bigcup_{K>0} \mathcal{F}_\theta^K$$

where

$$\mathcal{F}_\theta^K = \{\varphi \in C^0(\Sigma_A^+) : |\varphi(\underline{x}) - \varphi(\underline{x}')| \leq K d(\underline{x}, \underline{x}')^\theta \text{ for all } \underline{x}, \underline{x}' \in \Sigma_A^+\}.$$

For a Hölder continuous function $\varphi \in \mathcal{F}_\theta$ on Σ_A^+ we define its norm $\|\varphi\|_\theta$ by

$$\|\varphi\|_\theta = \sup |\varphi| + \inf\{K : \varphi \in \mathcal{F}_\theta^K\}.$$

If $\varphi \in \mathcal{F}_\theta$ and $\underline{x}$, $\underline{x}'$ are contained in the same cylinder C_n of length n then

$$\left|\sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) - \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}')\right| \leq \sum_{j=0}^{n-1} \|\varphi\|_\theta d(\sigma^j \underline{x}, \sigma^j \underline{x}')^\theta$$

$$\leq \|\varphi\|_\theta \sum_{j=0}^{\infty} a^{-\theta j} = \|\varphi\|_\theta \frac{1}{1 - a^{-\theta}}$$

Let φ be a Hölder continuous function on Σ_A^+ and μ the corresponding Gibbs measure; it exists and is unique (because $\sigma|_{\Sigma_A^+}$ is topologically mixing). It is more convenient to work with the "normalized" function $\log \psi$ on Σ_A^+ defined by $\log \psi = \varphi - P(\varphi)$. Note that μ is also the Gibbs measure for $\log \psi$.

For each $q \in \mathbb{R}$ let us consider the function

$$\varphi_q^E = -T(q) + q \log \psi,$$

and the corresponding Gibbs measure ν_q^E where the number $T(q)$ is chosen in such a way that $P(\varphi_q^E) = 0$. Clearly,

$$T(q) = P(q \log \psi). \tag{3}$$

Let h be the spectral radius of A (which is also the topological entropy of $\sigma|_{\Sigma_A^+}$).

Proposition 3.1. *The function T is real analytic on $\mathbb{R}$, and satisfies $T'(q) \leq 0$ and $T''(q) \geq 0$ for every $q \in \mathbb{R}$. Moreover, $T(0) = h/\log a$ and $T(1) = 0$.*

Denote by $\mathfrak{P}$ the class of finite partitions of Σ_A^+ into disjoint cylinder sets (not necessarily of the same length). Clearly, each $\xi \in \mathfrak{P}$ is a generating partition. We use it to define the entropy spectrum.

Let μ_E be the measure of maximal entropy. We set $\alpha(q) = \alpha^E(q) = -T'(q)$. The range of the function $\alpha(q)$ is the interval $[\alpha_1, \alpha_2]$, where $\alpha_1 = \alpha(+\infty)$ and $\alpha_2 = \alpha(-\infty)$.

Theorem 3.2.

1. *There exists a set $S \subset \Sigma_A^+$ with $\mu(S) = 1$ such that for every partition $\xi \in \mathfrak{P}$ and every $x \in S$, the local entropy of μ at x exists, has the same value for every ξ, and*

$$g_E(x) = h_\mu(\sigma, \xi, x) = -\int_{\Sigma_A^+} \log \psi \, d\mu.$$

2. *For ν_q^E-a.e. point $\underline{x} \in \Sigma_A^+$,*

$$\alpha(q) = -\int_{\Sigma_A^+} \log \psi \, d\nu_q^E = -\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \log \psi(\sigma^j \underline{x}).$$

3. *The domain of the function $\alpha \mapsto f^E(\alpha)$ contains a closed interval $[\alpha_1, \alpha_2] \in (0, \infty)$ which is the range of the function $\alpha(q)$. For every $q \in \mathbb{R}$, we have*

$$f^E(\alpha(q)) = T(q) + q\alpha(q).$$

4. *If $\mu \neq \mu_E$, then f^E is an analytic strictly convex function on (α_1, α_2), and hence, $(f^E(\cdot), T)$ are Legendre pairs with respect to the variables α, q.*

Remarks. Let HP_μ and R_μ be respectively the Hentschel–Procaccia and Rényi spectra for dimensions (see [9]). In [11], Pesin and Weiss proved that for every $q \in \mathbb{R}$,

$$T(q) = (1 - q)HP_\mu(q) = (1 - q)R_\mu(q) = \lim_{n\to\infty} \frac{1}{n \log a} \log \sum_C \mu(C)^q,$$

where the sum is taken over all cylinder sets of length n.

4. MULTIFRACTAL SPECTRA OF GIBBS MEASURES FOR CONFORMAL REPELLERS

We consider Gibbs measures invariant under conformal expanding maps, and describe the associated multifractal spectra for dimensions and entropies.

4.1. Coding of expanding repellers.
Let M be a smooth Riemannian manifold and $F \colon M \to M$ a C^1 map. Consider a compact subset $\mathcal{J}$ of M. We say that F is *expanding* and $\mathcal{J}$ is a *repeller* of F if:

1. there are constants $C > 0$ and $\beta > 1$ such that $\|D_x F^n u\| \geq C\beta^n \|u\|$ for all $x \in \mathcal{J}$, $u \in T_x M$, and $n \geq 1$;
2. $\mathcal{J} = \bigcap_{n \geq 0} F^{-n} V$ for some open neighborhood V of $\mathcal{J}$.

One can easily show that $F\mathcal{J} = \mathcal{J}$.

We recall that a finite cover $\{R_1, \ldots, R_p\}$ of X by closed sets is called a *Markov partition* if:

1. $\overline{\operatorname{int} R_i} = R_i$ for each $i = 1, \ldots, p$;
2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ if $i \neq j$;
3. each FR_i is a union of sets R_j.

It is well known that repellers admit Markov partitions of arbitrarily small diameter. Markov partitions are used to build symbolic models of repellers by subshifts of finite type (see Section 3).

Let $\mathcal{J}$ be a repeller of an expanding map F, and $\xi = \{R_1, \ldots, R_p\}$ a Markov partition of $\mathcal{J}$ with respect to F. We define a $p \times p$ *transfer* matrix $A = (a_{ij})$ by setting $a_{ij} = 1$ if $R_i \cap F^{-1}R_j \neq \varnothing$, and $a_{ij} = 0$ otherwise. Consider the associated subshift of finite type (Σ_A^+, σ). For each $\underline{x} = (i_1 i_2 \cdots) \in \Sigma_A^+$, we set

$$\chi(\underline{x}) = \left\{ x \in X : F^{k-1}x \in R_{i_k} \text{ for every } k \geq 1 \right\}.$$

The set $\chi(\underline{x})$ consists of a single point $x \in \mathcal{J}$, and we obtain the *coding map* $\chi \colon \Sigma_A^+ \to \mathcal{J}$ for the repeller. The map χ is continuous, onto, and the following diagram is commutative:

$$
\begin{array}{ccc}
\Sigma_A^+ & \xrightarrow{\ \sigma\ } & \Sigma_A^+ \\
\chi \downarrow & & \downarrow \chi \\
\mathcal{J} & \xrightarrow{\ F\ } & \mathcal{J}
\end{array}
$$

We assume that the matrix A is transitive (and thus, F is topologically mixing).

It is clear that any Markov partition ξ is a generating partition. The same is true for any partition of $\mathcal{J}$ by rectangles obtained from a Markov partition (not necessarily all of the same level) and corresponding to disjoint cylinder sets in Σ_A^+. We denote the class of such partitions by $\mathfrak{P}_F$. It is easy to see that for every partition $\xi \in \mathfrak{P}_F$, there is a partition $\eta \in \mathfrak{P}$ such that $\chi\eta = \xi$.

A smooth map $F \colon M \to M$ is called *conformal* if $D_x F$ is a multiple of an isometry at every point $x \in M$. Well-known examples of conformal expanding maps include one-dimensional Markov maps, and holomorphic maps. We write $\hat{a}(x) = \|D_x F\|$ for each $x \in M$ and denote by a some lift of $\hat{a}$ to Σ_A^+ via χ, i.e., $\hat{a}(x) = a(\chi(\underline{x}))$ for $x \in \mathcal{J} \setminus \bigcup_{i,j} (R_i \cap R_j)$. The multifractal spectra for $\mathcal{J}$ do not depend on the ambiguity at the boundaries of the rectangles and we have the set $\{x \in \mathcal{J} : F^n x \notin \bigcup_{i,j} (R_i \cap R_j) \text{ for all } n\}$ in mind when we write $\mathcal{J}$. On this set the function χ is invertible and the point $\underline{x} = \chi^{-1}(x)$ is uniquely defined.

A *Moran cover of depth n* associated to a positive number B and a set $\mathcal{X} \subset \mathcal{J}$ is a finite set of points $\underline{x}_i \in \Sigma_A^+$ such that $\mathcal{X} \subset \bigcup_i \chi(C_{n_i}(\underline{x}_i))$ where $C_{n_i}(\underline{x}_i)$ are the cylinder sets of length $n_i \geq n$ containing $\underline{x}_i$ and

$$B^{-1} \prod_{j=0}^{n_i - 1} a(\sigma^j \underline{x}_i) \leq \operatorname{diam} \chi(C_{n_i}(\underline{x}_i)) \leq B \prod_{j=0}^{n_i - 1} a(\sigma^j \underline{x}_i).$$

Sometimes we will also view the Moran cover as a set of cylinder sets $\{C_{n_i}(\underline{x}_i)\}$. An expanding conformal repeller $\mathcal{J}$ admits Moran covers of arbitrary depths and multiplicity Q for any subset $\mathcal{X} \subset \mathcal{J}$ and some positive real numbers B and Q not depending on the depth or the subset (see [11] for details).

4.2. Multifractal spectra. Let $\mathcal{J}$ be a repeller of a conformal $C^{1+\varepsilon}$ expanding map F, for some $\varepsilon > 0$. Let also m_D be the unique Gibbs measure corresponding to the function $x \mapsto -\dim_H \mathcal{J} \cdot \log \hat{a}(x)$ on $\mathcal{J}$ and μ_D its lift to Σ_A^+. It is known that m_D is a measure of maximal dimension, i.e., $\dim_H \mathcal{J} = \dim_H m_D$ (see [14]). We denote by m_E the measure of maximal entropy for $f : \mathcal{J} \to \mathcal{J}$, by μ_E its lift to Σ_A^+ and by h the topological entropy of F on $\mathcal{J}$.

Let φ be a Hölder continuous function on Σ_A^+ and $m = \mu \circ \chi^{-1}$ the corresponding Gibbs measure. Write $\log \psi = \varphi - P(\varphi)$.

For each q, $p \in \mathbb{R}$, consider the functions

$$\varphi_{D,q} = -T_D(q) \log a + q \log \psi$$

and the corresponding Gibbs states ν_q^D on Σ_A^+ where the numbers $T_D(q)$ are chosen such that

$$P(\varphi_{D,q}) = 0.$$

Proposition 4.1. *The function T_D is real analytic and satisfies $T_D'(q) \leq 0$ and $T_D''(q) \geq 0$ for every $q \in \mathbb{R}$. We have $T_D(0) = \dim_H J$ and $T_D(1) = 0$.*

Note that the equality $T_D(0) = \dim_H J$ follows the formula for the dimension of a conformal repeller established by Ruelle in [14]. Set

$$\alpha(q) = \alpha^D(q) = -T_D'(q).$$

The range of the function $\alpha(q)$ is the interval $[\alpha_1, \alpha_2]$ where $\alpha_1 = \alpha(+\infty)$ and $\alpha_2 = \alpha(-\infty)$.

We now give a full description of the multifractal spectrum f^D, for Gibbs measures supported on repellers, i.e., measures projected from Gibbs measures on Σ_A^+ – of conformal smooth expanding maps.

The following theorem shows (with minor exceptions) that f^D is defined on an interval, is analytic, and strictly convex. It also establishes a relationship between the functions $f^D(\alpha)$ and $T_D(q)$; namely, they form a Legendre pair.

Theorem 4.2.

1. *For m-almost every $x \in \mathcal{J}$, the pointwise dimension of m at x exists and*

$$g_D(x) = d_m(x) = -\frac{\int_{\Sigma_A^+} \log \psi \, d\mu}{\int_{\Sigma_A^+} \log a \, d\mu}.$$

2. *For ν_q^D-a.e. point $\underline{x} \in \Sigma_A^+$,*

$$\alpha^D(q) = -\frac{\int_{\Sigma_A^+} \log \psi \, d\nu_q^D}{\int_{\Sigma_A^+} \log a \, d\nu_q^D} = -\lim_{n \to \infty} \frac{\sum_{j=0}^n \log \psi(\sigma^j \underline{x})}{\sum_{j=0}^n \log a(\sigma^j \underline{x})};$$

3. *The domain of the function* $\alpha \mapsto f^D(\alpha)$ *contains a closed interval* $(\alpha_1^D, \alpha_2^D) \in (0, +\infty)$ *which coincides with the range of the function* $\alpha^D(q)$. *For every* $q \in \mathbb{R}$, *we have*

$$f^D(\alpha^D(q)) = T_D(q) + q\alpha^D(q). \tag{4}$$

4. *If* $m \neq m_D$, *then* f^D *and* T_D *are analytic strictly convex functions, and hence,* (f^D, T_D) *is a Legendre pair with respect to the variables* α, q.

5. *If* $m = m_D$, *then* f^D *is the delta function*

$$f^D(\alpha) = \begin{cases} \dim_H J & \text{if } \alpha = \dim_H \mathcal{J} \\ 0 & \text{if } \alpha \neq \dim_H \mathcal{J} \end{cases}.$$

The identity (4) is a consequence of property 1 in Theorem 4.3 below.

We now describe the full measures for the spectra f^D and f^E. It turns out that these are the unique Gibbs measures ν_q^D and ν_p^E for the (Hölder continuous) functions $\varphi_{D,q}$ and $\varphi_{E,p}$, respectively.

Theorem 4.3. *The following properties hold:*

1. *For every* $q \in \mathbb{R}$, *we have* $m_q^D(D_{\alpha^D(q)}) = 1$ *and*

$$d_{m_q^D}(x) = T_D(q) + q\alpha^D(q)$$

for $m_q^D = \nu_q^D \circ \chi^{-1}$-*a.e.* $x \in D_{\alpha^D(q)}$.

2. *For every* $p \in \mathbb{R}$, *we have* $m_p^E(E_{\alpha^E(p)}) = 1$ *and*

$$h_{m_p^E}(f, \chi\xi, x) = h_{\nu_p^E}(\sigma, \xi, x) = T_E(p) + p\alpha^E(p)$$

for $m_p^E = \nu_p^E \circ \chi^{-1}$-*a.e.* $x \in E_{\alpha^E(p)}$ *and every* $\xi \in \mathfrak{P}_f$.

In the next three sections we are going to establish the main results for the dimension spectrum $f(\alpha) = f^D(\alpha)$ on $\mathcal{J}$. The corresponding statements for the entropy spectrum can be obtained from the dimension spectrum results by setting $\log a \equiv 1$.

Some of the notations used in the following sections are explained in the appendix.

5. MULTIFRACTAL SPECTRA ARE COMPLETE

In this section we are going to prove that $D_\alpha = \varnothing$ if $\alpha \notin [\alpha_1, \alpha_2]$ where the numbers α_1 and α_2 are defined as $\alpha_1 = \lim_{q \to \infty} \alpha(q)$ and $\alpha_2 = \lim_{q \to -\infty} \alpha(q)$ (see section 4). In fact we show that there is no limit point of $\frac{\log \mu(B(x,r))}{\log r}$ as $r \to 0$ outside the interval $[\alpha_1, \alpha_2]$.

Let φ be a Hölder continuous function on Σ_A^+, μ its Gibbs measure on Σ_A^+ and $m = \mu \circ \chi^{-1}$ the corresponding Gibbs state on $\mathcal{J}$.

Lemma 5.1.

$$\inf_{\rho \in \mathcal{M}_e(\mathcal{J})} -\frac{\int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho} = \alpha_1 \quad and \quad \sup_{\rho \in \mathcal{M}_e(\mathcal{J})} -\frac{\int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho} = \alpha_2.$$

Proof. We present the proof of the first equality. In view of theorem 4.3 we only have to prove that $\inf_{\rho \in \mathcal{M}_e(\mathcal{J})} -\frac{\int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho} \leq \alpha_1$. Simpelaere proved in [16] that

$$T(q) = \inf_{\rho \in \mathcal{M}_e(\mathcal{J})} \frac{h_\rho - q \int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho} = \frac{h_{\nu_q^D} - q \int_{\Sigma_A^+} \log \psi \, d\nu_q^D}{\int_{\Sigma_A^+} \log a \, d\nu_q^D}.$$

This gives for $q \geq 0$ that

$$q\alpha_1 \leq -q\frac{\int \log \psi \, d\nu_q}{\int \log a \, d\nu_q} \leq \dim_H \nu_q - q\frac{\int \log \psi \, d\nu_q}{\int \log a \, d\nu_q} = T(q)$$

$$= \inf_{\rho \in \mathcal{M}_e(\mathcal{J})} \left(\dim_H \rho - q\frac{\int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho} \right)$$

$$\leq \dim_H \mathcal{J} - q \inf_{\rho \in \mathcal{M}_e(\mathcal{J})} \frac{\int \log \psi \circ \chi \, d\rho}{\int \log \hat{a} \, d\rho}.$$

Dividing by q and letting $q \to +\infty$ gives the first assertion of the lemma. The other equality can be proved in a similar fashion. $\square$

We are now ready to prove the following

Theorem 5.2. *We have* $\alpha_1 = \inf_{x \in \mathcal{J}} \underline{d}_m(x)$ *and* $\alpha_2 = \sup_{x \in \mathcal{J}} \overline{d}_m(x)$. *Hence,*

$$\mathcal{D}_\alpha = \varnothing \quad \text{iff } \alpha \notin [\alpha_1, \alpha_2].$$

Proof. In view of Theorem 4.3 we only have to prove that $\alpha_1 \leq \inf_{x \in \mathcal{J}} \underline{d}_m(x)$. Let us assume that this is not the case. By theorem 4.2 there is a $\Delta > 0$, a point $\underline{x} \in \Sigma_A^+$ with $\chi(\underline{x}) = x \in \mathcal{J}$ and a subsequence n_k of the natural numbers such that

$$-\frac{\sum_{j=0}^{n_k} \log \psi(\sigma^j \underline{x})}{\sum_{j=0}^{n_k} \log a(\sigma^j \underline{x})} < \alpha_1 - \Delta$$

Let ρ be an accumulation point of the sequence of measures $\rho_k = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\sigma^j \underline{x}}$. Obviously,

$$-\frac{\int \log \psi \, d\rho_k}{\int \log a \, d\rho_k} \leq \alpha_1 - \Delta.$$

Hence,

$$-\frac{\int \log \psi \, d\rho}{\int \log a \, d\rho} \leq \alpha_1 - \Delta$$

what contradicts lemma 5.1. This proves the first statement. The proof of the second assertion goes along the same lines. $\square$

6. There are degenerate multifractal spectra

Let us now fix an expanding repeller $\mathcal{J}$ together with its metric. This gives rise to the symbolic coding space Σ_A^+ with transition matrix A and to the Hölder continuous potential $\log a = \log \|DF\|$ as explained in section 4. In this section we show that for arbitrary "allowed" boundary values $(\alpha_1, f(\alpha_1), \alpha_2, f(\alpha_2))$ we can find a Hölder continuous potential whose multifractal dimension spectrum attains these values at its boundary. Since the spectrum $f(\alpha) = f^D(\alpha)$ is a convex function and is not above the diagonal–i.e., $f(\alpha) \leq \alpha$ for all $\alpha \in (\alpha_1, \alpha_2)$ – , its maximum value is equal to $\dim_H \mathcal{J}$ and its graph touches the diagonal to the left of its maximum at $\alpha = \alpha(1) = f(\alpha(1)) = \dim_H m$ we get the obvious restrictions:

$$(\alpha_1, f(\alpha_1), \alpha_2, f(\alpha_2)) \in \mathbb{B} \in \mathbb{R}^4$$

with

$$\mathbb{B} = \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : y_1 \leq x_1 \leq \dim_H \mathcal{J}, y_2 \leq \dim_H \mathcal{J} \leq x_2 \right\}$$

Let

$$\mathbb{B}^0 = \mathbb{B} \cap \left\{ (x_1, y_1, x_2, y_2) : y_1 < x_1 < \dim_H \mathcal{J}, y_2 < \dim_H \mathcal{J} < x_2 \right\}$$

be the set of *admissible boundary values*. The main result of this section is the following

Theorem 6.1. *Let $\mathcal{J}$ be an expanding conformal repeller. Then for any quadruple $(x_1, y_1, x_2, y_2) \in \mathbb{B}^0$ there is a Hölder continuous potential $\log \psi$ with dimension spectrum $f(\alpha)$ satisfying*

$$\alpha_1 = x_1 \qquad\qquad f(\alpha_1) = y_1$$
$$\alpha_2 = x_2 \qquad\qquad f(\alpha_2) = y_2$$

Remark. If $(x_1, y_1, x_2, y_2) \in \mathbb{B}$ and $x_1 = y_1$ or $x_2 = y_2$ or $y_1 = \dim_H \mathcal{J}$ or $y_2 = \dim_H \mathcal{J}$ then the measure m is the measure of maximal dimension m_D and has a degenerate spectrum consisting of the point $(\dim_H \mathcal{J}, \dim_H \mathcal{J})$. We are not going to prove this explicitly but one can derive that no boundary values outside $\mathbb{B}^0$ can be attained from the proof of the above theorem 6.1 and the well-known fact that any proper subshift – i.e. closed shift-invariant proper subset of Σ_A^+ – has topological entropy strictly less then $h_{top}(\Sigma_A^+)$.

The proof of the theorem relies on a series of lemmas. Some of them are generalizations of well-known facts in symbolic dynamics. However, we choose to prove them with the help of the multifractal analysis in order to show that this analysis can be used to derive results in symbolic dynamics.

The next lemma is due to Simpelaere for repellers equipped with Gibbs measures. It has been proven by several other authors for "geometric constuctions"(see for example [6, 15]).

Lemma 6.2 ([16]). *Let $\log\psi$ be a Hölder continuous function on Σ_A^+, m the corresponding Gibbs measure on $\mathcal{J}$ and f its dimension spectrum. There exist probability measures ρ_1 and ρ_2 concentrated on D_{α_1} and D_{α_2}, respectively, such that*

$$\dim_H(D_{\alpha_1}) = \lim_{q\to+\infty} f(\alpha(q)) = \dim_H \rho_1 := f_1^D = f_1$$

and

$$\dim_H(D_{\alpha_2}) = \lim_{q\to-\infty} f(\alpha(q)) = \dim_H \rho_2 := f_2^D = f_2.$$

We also need the following fact.

Lemma 6.3. *Let $\log\psi$ be a Hölder continuous function on Σ_A^+, m the corresponding Gibbs measure on $\mathcal{J}$ and f its dimension spectrum. We have:*

  i) $\dim_H D_{\alpha(q)}^- \leq f(\alpha(q))$ *for $q \geq 0$;*
  ii) $\dim_H D_{\alpha(q)}^+ \leq f(\alpha(q))$ *for $q \leq 0$;*
In particular $\dim_H D_{\alpha_1}^- = f_1$ and $\dim_H D_{\alpha_2}^+ = f_2$.

Proof. fix $q \geq 0$. We are going to prove that the s-dimensional Hausdorff measure of $D_{\alpha(q)}^-$ is finite provided that $s > f(\alpha(q)) = T(q) + \alpha(q)q$.

We set $s = f(\alpha(q)) + \varepsilon = T(q) + \alpha(q)q + \varepsilon$ where $\varepsilon > 0$. If $x \in D_{\alpha(q)}^-$ then by theorem 4.2

$$-\frac{\sum_{j=0}^{n_k} \log\psi(\sigma^j \underline{x})}{\sum_{j=0}^{n_k} \log a(\sigma^j \underline{x})} \leq \alpha(q) + \frac{1}{k}$$

for some subsequence $n_k = n_k(x)$ of the naturals. Therefore, we can find a Moran cover (see section 4) $\mathfrak{C} = \{\underline{x}_1, \ldots, \underline{x}_l\}$ of $D_{\alpha(q)}^-$ with arbitrary large depth and multiplicity Q such that

$$\sum_{\underline{x}_i \in \mathfrak{C}} (\operatorname{diam} C_{n_k(\underline{x}_i)})^s \leq \sum_i QB \prod_{j=0}^{n_k(\underline{x}_i)-1} a(\sigma^j \underline{x}_i)^{-s}$$

$$\leq \sum_i QB \prod_{j=0}^{n_k(\underline{x}_i)-1} a(\sigma^j \underline{x}_i)^{-(T(q)+\alpha(q)q+\varepsilon)}$$

$$\leq QB \sum_i \prod_{j=0}^{n_k(\underline{x}_i)-1} \psi(\sigma^j \underline{x}_i)^q a(\sigma^j \underline{x}_i)^{-T(q)} \min(\psi)^{-\frac{q}{k}} \max a^\varepsilon$$

$$\leq QB \sum_{\underline{x}_i \in \mathfrak{C}} \mu_q(C_i) \leq QB < \infty$$

if k is sufficiently large. A similar proof works for the set $D_{\alpha(q)}^+$ when $q \geq 0$. $\qquad\square$

For our next sections we need estimates on the entropy – which equals the Hausdorff dimension in the symbolic space Σ_A^+ equipped with the constant metric corresponding to the function $\log a \equiv 1$ according to theorem 3.2 and theorem 4.2 – of larger sets then $D_{\alpha_i}^{\pm}$. These estimations are based on the variational principle for non-compact sets developed by Pesin and Pitskel' (see [10] and Appendix).

Define the *maximum* and *minimum* sets for φ by

$$\overline{\mathcal{M}}(\varphi) = \{\underline{x} \in \Sigma_A^+ : \varphi(\underline{x}) = \max \varphi\} \quad \text{and} \quad \underline{\mathcal{M}}(\varphi) = \{\underline{x} \in \Sigma_A^+ : \varphi(\underline{x}) = \min \varphi\}.$$

Lemma 6.4. *Let φ be a Hölder continuous function on Σ_A^+ and f^E the entropy spectrum for its Gibbs measure. Let us assume that $\overline{\mathcal{M}}(\varphi)$ and $\underline{\mathcal{M}}(\varphi)$ are compact invariant sets. Then*

$$f_1^E = h_{top}(\overline{\mathcal{M}}(\varphi)) \quad and \quad f_2^E = h_{top}(\underline{\mathcal{M}}(\varphi))$$

Proof. Since

$$\sup_{\underline{x} \in \Sigma_A^+} \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) \leq \max(\varphi) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{y})$$

for all $\underline{y} \in \overline{\mathcal{M}}(\varphi)$, the set $\overline{\mathcal{M}}(\varphi)$ is contained in the set D_{α_1}. Similary $\underline{\mathcal{M}}(\varphi) \in D_{\alpha_2}$. Hence,

$$f_1 = f_1^E \leq h_{top}(\overline{\mathcal{M}}(\varphi)) \quad \text{and} \quad f_2 = f_2^E \leq h_{top}(\underline{\mathcal{M}}(\varphi)).$$

For the reverse inequality we observe that

$$\underline{x} \in D_{\alpha_1}^- \quad \text{if and only if} \quad \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) = \max \varphi.$$

But this is only possible if $V_\sigma(\underline{x}) \cap M_{\text{inv}}(\overline{\mathcal{M}}(\varphi)) \neq \varnothing$ where $V_\sigma(\underline{x})$ is the set of accumulation points of the sequence of measures $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j \underline{x}}$ (see also appendix). By theorem 8.3 from the appendix this yields $f_1 = h_{top}(\overline{\mathcal{M}}(\varphi))$. The arguments for f_2 are analog. $\square$

Examples. Let $\Sigma_A^+ = \Sigma_N$ be the full shift on N symbols, $\log a \equiv \text{const.} = 1$ and μ the Bernoulli measure generated by the probability vector $(p_1, \ldots, p_N)$ – i.e. $\log \psi(\underline{x}) = \log \psi(x_1) = \log p_{x_1}$ where $\underline{x} = x_1 x_2 x_3 \ldots$ – then $f_i = 0$; $i = 1, 2$; if and only if there is a i_{max} and a i_{min} such that $p_{i_{min}} < p_j < p_{i_{max}}$ for all $j \notin \{i_{min}, i_{max}\}$. In this case $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ consist of the periodic point $\underline{x} = i_{min}, i_{min}, i_{min}, \ldots$ or $\underline{x} = i_{max}, i_{max}, i_{max}, \ldots$, respectively. Otherwise the sets $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ are subshifts of finite type generated by the sets of symbols $\{i : p_i = \min p_j\}$ or $\{i : p_i = \max p_j\}$, respectively. It is not hard to see that the extremal sets $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ for measures defined by potentials depending only on a finite number of coordinates - i.e. $\log \psi(\underline{x}) = \log \psi(x_0, \ldots, x_n)$ - are subshifts of finite type. This need not to be true for arbitrary Hölder potentials.

Proposition 6.5. *Let $\mathfrak{J}$ be an expanding conformal repeller. For any pair of numbers $d_i < \dim_H \mathfrak{J}$; $i = 1, 2$; there are disjoint closed invariant subsets $\mathcal{S}_i \subset \mathfrak{J}$ such that $\dim_H \mathcal{S}_i = d_i$ and it exists $\lim_{n \to \infty} \frac{1}{n} \log \|DF^n(x)\| = \lambda_i$ if $x \in \mathcal{S}_i$.*

Proof. We fix d_1 and d_2 according to the assumption of the proposition. The proof goes by constructing inductively closed invariant subsets $\mathcal{S}_i^{(n)}$ which are images of subshifts of finite type in Σ_A^+ under the projection χ and approximate the final sets $\mathcal{S}_i$. Let us denote by μ_D the pull back of the measure of maximal dimension on $\mathfrak{J}$ to Σ_A^+. Let $\lambda_D = \lambda_{\mu_D}$, $h_D = h_{\mu_D} = \lambda_D \dim_H \mathfrak{J}$ and $\overline{\lambda} = \max\{\log a, 1\}$; $\underline{\lambda} = \min \log a$. We also fix $\varepsilon < \min(\dim_H \mathfrak{J} - d_i)\lambda_D$. For $m \in \mathbb{N}$ let $L = 1/\min 4(d_i + 1)$,

$$\Lambda_\varepsilon^m = \left\{ \underline{x} \in \Sigma_A^+ : \left| \frac{1}{n} \sum_{j=0}^{n-1} \log a(\sigma^j \underline{x}) - \lambda_D \right| \leq L\varepsilon \text{ for all } n \geq m \right\}$$

and

$$H_\varepsilon^m = \left\{ \underline{x} \in \Sigma_A^+ : \left| h_D - \frac{1}{n} \log \mu_D(C_n \underline{x}) \right| \leq \varepsilon \text{ for all } n \geq m \right\}.$$

Finally we put

$$\Gamma_\varepsilon^m = \Lambda_\varepsilon^m \cap H_\varepsilon^m$$

and set

$$\hat{m}^{(1)} = \min\{m \in \mathbb{N} : \mu_D(\Gamma_\varepsilon^m) > 1/2\}$$

In view of the Birkhoff Ergodic Theorem and the Shannon-McMillan-Breiman Theorem the number $\hat{m}^{(1)}$ is finite and we can define

$$m^{(1)} = \max \left\{ \left\lfloor \frac{4(K+M)\overline{\lambda}(d_i+1)}{\varepsilon} \right\rfloor + 1, \left\lfloor \frac{1 + \log 4}{(\dim_H \mathfrak{J} - d_i)\lambda_D - \varepsilon} \right\rfloor, \hat{m}^{(1)} \right\}$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number, $K = \|\log a\|_\theta \frac{e^{\overline{\lambda}}}{1 - e^{\underline{\lambda}}}$ and M is the smallest number such that $A^M > 0$.

For $m \geq m^{(1)}$ we consider

$$\mathfrak{C}_m = \{C_m(\underline{x}) : \underline{x} \in \Gamma_\varepsilon^{m^{(1)}}\}$$

Then

$$\mu_D(C_m(\underline{x})) \leq e^{mh_D + m\varepsilon}$$

and

$$\text{Card}\{\mathfrak{C}_m\} \geq \frac{\mu_D(\Gamma_\varepsilon^m)}{\max \mu_D(C_m(\underline{x}))} \geq \frac{1}{2} e^{mh_D - m\varepsilon}$$

By the choice of ε and $m^{(1)}$ we can choose a subset $\mathfrak{C}_1^{(1)} \in \mathfrak{C}_{m^{(1)}}$ with cardinality $\lfloor \exp\{m^{(1)}(\lambda_D d_1 + d_1 \varepsilon/4)\} \rfloor + 1$. We define the subshift $S_1^{(1)}$ as the set of points $\underline{x} \in \Sigma_A^+$ with the property that there is a $l < m^{(1)} + M$ such that for all $j \in \mathbb{N}$ the

cylinders $C_{m^{(1)}}(\sigma^{l+j(m^{(1)}+M)}\underline{x})$ are from the set $\mathfrak{C}_1^{(1)}$. According to this construction the set $S_1^{(1)}$ is a subshift of finite type. We set $\mathbb{S}^{(i)} = \chi(S^{(i)})$. We note that M is the number that all entries in the transition matrix A are positive. This means that $S_1^{(1)} \neq \varnothing$ and

$$\lim_{k\to\infty} \frac{1}{k} \log \mathrm{Card}\{C_k(\underline{x}) : \underline{x} \in S_1^{(1)}\} \geq \lim_{k\to\infty} \frac{1}{k} \log \mathrm{Card}(\mathfrak{C}_1^{(1)})^{\left\lfloor \frac{k}{m^{(1)}+M}\right\rfloor - 1}$$

$$\geq \lambda_D d_1 + \frac{\varepsilon}{2}$$

and

$$\overline{\lim_{k\to\infty}} \frac{1}{k} \log \mathrm{Card}\{C_k(\underline{x}) : \underline{x} \in S_1^{(1)}\} \leq \overline{\lim_{k\to\infty}} \frac{1}{k} \log \mathrm{Card}(\mathfrak{C}_1^{(1)})^{\left\lfloor \frac{k}{m^{(1)}+M}\right\rfloor} \mathrm{rank}(A)^{\left\lfloor \frac{k}{m^{(1)}+M}\right\rfloor + l}$$

$$\leq \lambda_D d_1 + \frac{3}{2}\varepsilon$$

Hence,

$$\lambda_D d_1 + \frac{1}{2}\varepsilon \leq h_{top}(\mathbb{S}_1^{(1)}) \leq \lambda_D d_1 + \frac{3}{2}\varepsilon$$

For all $\underline{x} \in S_1^{(1)}$ we have

$$\overline{\lim_{k\to\infty}} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \leq \overline{\lim_{k\to\infty}} \sum_{l=0}^{\left\lfloor \frac{m^{(1)}+M}{k}\right\rfloor + 1} \frac{1}{m^{(1)}+M} \sum_{j=0}^{m^{(1)}-1} \log a(\sigma^{j+l(m^{(1)}+M)}\underline{x}) + M\overline{\lambda})$$

$$\leq \lambda_D + L\varepsilon + \frac{K}{m^{(1)}+M} + \frac{M}{m^{(1)}+M}$$

$$\leq \lambda_D + (L+1)\varepsilon$$

and

$$\underline{\lim_{k\to\infty}} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \geq \underline{\lim_{k\to\infty}} \sum_{l=0}^{\left\lfloor \frac{m^{(1)}+M}{k}\right\rfloor + 1} \frac{1}{m^{(1)}+M} \sum_{j=0}^{m^{(1)}-1} \log a(\sigma^{j+l(m^{(1)}+M)}\underline{x}) + M\overline{\lambda})$$

$$\geq \lambda_D - L\varepsilon - \frac{K}{m^{(1)}+M} - \frac{M}{m^{(1)}+M}$$

$$\geq \lambda_D - (L+1)\varepsilon$$

Combining the above estimates on the entropy and the Lyapunov exponents and setting $\mathbb{S}_1^{(1)} = \chi(S_1^{(1)})$ we can conclude that

$$d_1 + \frac{\varepsilon}{4} < \dim_H(\mathbb{S}_1^{(1)}) < d_1 + 2\varepsilon$$

Now we will construct the set $S_2^{(1)} \in \Sigma_A^+ \setminus S_1^{(1)}$. Since $\varepsilon < \min(\dim_H \mathcal{J} - d_i)\lambda_D$ we can choose a subset $\mathfrak{C}_2^{(1)} \in \mathfrak{C}_{m^{(1)}}$ with cardinality $\lfloor \exp\{m^{(1)}(\lambda_D d_2 + d_2\varepsilon/4)\} \rfloor + 1$. We note that there is a cylinder $\hat{C} \in \Sigma_A^+$ of length $m^{(1)}$ which has empty intersection with the subshift $S_1^{(1)}$. Let $m = 8m^{(1)}/\varepsilon$. Then we define the subshift $S_2^{(1)}$ as consisting of all points $\underline{x} \in \Sigma_A^+$ such that there is a $l < m + M$ such that for all $j \in \mathbb{N}$ the cylinders $C_{m+m^{(1)}}(\sigma^{l+j(m+m^{(1)}+M)}\underline{x})$ are of the form

$$C_{m^{(1)},1} * \ldots * C_{m^{(1)},m} * \hat{C}$$

where $C_{m^{(1)},i} \in \mathfrak{C}_2^{(1)}$; $i = 1, \ldots, m$; and $C_1 * C_2$ is the cylinder $C_1 \cap \sigma^{-|C_1|}C_2$. Obviously, $S_1^{(1)} \cap S_2^{(1)} = \varnothing$ and $S_2^{(1)}$ is of finite type.

Proceeding the same estimations for the entropy and the Lyapunov exponents for the subshift $S_2^{(1)}$ as for the subshift $S_1^{(1)}$ yields

$$\lambda_D d_2 + \frac{3}{8}\varepsilon \leq h_{top}(S_2^{(1)}) \leq \lambda_D d_2 + \frac{13}{8}\varepsilon$$

For all $\underline{x} \in S_2^{(1)}$ we have

$$\varlimsup_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \leq \lambda_D + \left(L + \frac{9}{8}\right)\varepsilon$$

and

$$\varliminf_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \geq \lambda_D - \left(L + \frac{9}{8}\right)\varepsilon$$

Combining the above estimates we see that for $S_2^{(1)} = \chi(S_2^{(1)})$

$$d_2 + \frac{\varepsilon}{8} < \dim_H(S_2^{(1)}) < d_2 + \frac{17}{8}\varepsilon$$

Let us now assume that for given sufficiently small δ and each $l \leq n$ we have constructed two subshifts of finite type $S_1^{(l)}$ and $S_2^{(l)}$ with the properties

(iii) $S_i^{(l)} \subset S_i^{(l-1)}$

(iv)

$$\lambda_D d_i + \frac{\delta}{2^{l+1}} \leq h_{top}(S_i^{(l)}) \leq \lambda_D d_i + \frac{\delta}{2^{l-2}}$$

(v) For all $\underline{x} \in S_i^{(l)}$ we have

$$\varlimsup_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \leq \lambda_i^{(l)} + \frac{\delta}{2^l}$$

and

$$\varliminf_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \geq \lambda_i^{(l)} - \frac{\delta}{2^l}$$

(vi) For $\mathcal{S}_i^{(l)} = \chi(S_i^{(l)})$

$$d_i + \frac{\delta}{2^{l+1}} < \dim_H \mathcal{S}_i^{(l)} < d_i + \frac{\delta}{2^{l-2}}$$

where $i = 1, 2$; $l = 1, \ldots, n$ and $\lambda_i^{(l)}$ are some real numbers. In particular, if we set $\delta = \varepsilon$ we can assume that $\lambda_i^{(1)} = \lambda_D$.

Now we repeat the construction of $S_1^{(1)}$ in Σ_A^+ by substituting Σ_A^+ by $S_1^{(n)}$ and setting $\varepsilon = \delta/2^n$. The role of the measure of maximal dimension μ_D is played by the lift $\mu_n^{(1)}$ to Σ_A^+ of the measure of maximal dimension of $\mathcal{S}_1^{(n)}$. This means we construct subshifts of finite type $S_1^{(n+1)}$ inside the subshifts of finite type $S_1^{(n)}$ in the same manner as we constructed $S_1^{(1)}$ inside Σ_A^+. We then get a subshift $S_1^{(n+1)}$ which satisfies the inequalities (iv) – (vi) for $l = n + 1$.

The subshift $S_2^{(n+1)}$ is constructed in a similar manner by substituting Σ_A^+ by $S_2^{(n)}$ and proceeding as above. We define $\mathcal{S}_i^{(n+1)} = \chi(S^{(n+1)})$.

This way for each natural number we have two subshifts of finite type with properties (iv) – (vi) and which form two nested sequences. We observe that the numbers $\lambda_i^{(n)}$ converge to a number $\lambda_i = \lambda_i^{(\infty)}$ as n tends to infinity. Let

$$\mathcal{S}_i = \bigcap_{n\geq 1} \mathcal{S}_i^{(n)} \quad \text{and} \quad S_i = \bigcap_{n\geq 1} S_i^{(n)} \qquad i = 1, 2$$

Then for all $\underline{x} \in S_i$

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) = \lambda_i^{(\infty)}$$

and for $\mathcal{S}_i = \chi(S_i)$

$$\dim_H \mathcal{S}_i \leq \inf_n \dim_H \mathcal{S}_i^{(n)} = d_i$$

Moreover, $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ since $\mathcal{S}_1^{(1)} \cap \mathcal{S}_2^{(1)} = \varnothing$. But on the other hand any accumulation point $\mu_\infty^{(i)}$ of the sequence of measures $\{\mu_n^{(i)}\}$ sits on S_i. By the upper semicontinuity of the metric entropies we derive

$$\dim_H \mathcal{S}_i \geq \dim_H \mu_\infty^{(i)} \circ \chi^{-1} \geq \varliminf \dim_H \mu_n^{(i)} \circ \chi^{-1} = \varliminf \dim_H \mathcal{S}_i^{(n)} = d_i$$

This completes the proof of the proposition. □

Corollary. $h_{top}(\mathcal{S}_i) = \lambda_i d_i$

We are going to define a potential φ on Σ_A^+ which achives its maximum on S_1 and its minimum on S_2. We will show that the corresponding Gibbs measure has an entropy spectrum with $f_1 = d_1$ and $f_2 = d_2$.

Let d_1, d_2, S_1 and S_2 be as in the proof of the proposition – i.e. $S_i = \chi(S_i)$. For $z_1 > z_2$ we consider the following set of functions

$$\mathcal{F}(z_1, z_2) = \left\{ \varphi \in \mathcal{F}_\theta : \varphi(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \in S_1 \\ z_2 & \text{if } \underline{x} \in S_2 \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else} \end{cases} \right\}$$

Lemma 6.6. *If $d_1 = \dim_H S_1 < -z_1 < \dim_H \mathcal{J}$ and $-z_2 > \dim_H \mathcal{J}$ then there is a function $\varphi_0 \in \mathcal{F}(z_1, z_2)$ with $P(\varphi_0 \log a) = 0$.*

Proof. Let $U_n^{(1)}$ and $U_n^{(2)}$ be two nested sequences of open sets converging to S_1 and S_2, respectively. We specify for $n \in \mathbb{N}$ two functions

$$\varphi_n^{(1)}(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \notin U_n^{(2)} \\ z_2 & \text{if } \underline{x} \in S_2 \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else} \end{cases}$$

and

$$\varphi_n^{(2)}(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \in S_1 \\ z_2 & \text{if } \underline{x} \notin U_n^{(1)} \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else} \end{cases}$$

We note that we can find such functions because the sets S_i are closed. The functions $\varphi_n^{(i)}$ are in the closure of the set $\mathcal{F}(z_1, z_2)$. If we denote by $P_n^{(i)}$; $i = 1, 2$; the topological pressure of the Gibbs measures corresponding to $\varphi_n^{(i)} \log a$, respectively we have

$$P_n^{(1)} = \max_{\mu \in M_e(\Sigma_A^+)} \left\{ h_\mu + \int \varphi_n^{(1)} \log a \, d\mu \right\}$$

$$\geq h_{\mu_D} + \int \varphi_n^{(1)} \log a \, d\mu_D$$

$$\geq \dim_H \mathcal{J} + z_2 \mu_D(U_n^{(2)}) + z_1 \lambda_D (1 - \mu_D(U_n^{(2)}))$$

Because $\dim_H S_2 < \dim_H \mathcal{J}$ and consequently, $m_D(S_2) = 0$ and the fact that μ_D is a Borel measure we see that $\lim_{n\to\infty} \mu_D(U_n^{(2)}) = 0$. Hence, by the assumptions of the lemma

$$\lim_{n\to\infty} P_n^{(1)} \geq \lambda_D \dim_H \mathcal{J} + z_1 > 0$$

On the other hand

$$P_n^{(2)} = \max_{\mu \in M_e(\Sigma_A^+)} \{h_\mu + \int \varphi_n^{(2)} \log a \, d\mu\}$$

$$= \max \left\{ \max_{\mu \in M_1} \{h_\mu + \int \varphi_n^{(2)} \log a \, d\mu\}; \max_{\mu \in M_2} \{h_\mu + \int \varphi_n^{(2)} \log a \, d\mu\} \right\}$$

$$\leq \max \left\{ h_{\mu_D} + \int \varphi_n^{(2)} \log a \, d\mu_D; h_{top}(S_1) + \max_{\mu \in M_2} \{\int \varphi_n^{(2)} \log a \, d\mu\} \right\}$$

$$\leq \max \left\{ \dim_H \mathcal{J} + z_2 + z_1 \mu_D(U_n^{(1)}); d_1\lambda_1 + z_1\lambda_1 + \max_{\mu \in M_2} \{z_2 \mu(U_n^{(1)})\} \right\}$$

where $M_1 = \{\mu \in M_e(\Sigma_A^+) : \mu(S_1) = 0\}$ and $M_2 = \{\mu \in M_e(\Sigma_A^+) : \mu(S_1) = 1\}$. Hence, by the assumptions of the lemma and the definition of the sets $U_n^{(1)}$ and $U_n^{(2)}$

$$\lim_{n \to \infty} P_n^{(1)} \leq \max\{\dim_H \mathcal{J} + z_2; (d_1 + z_1)\lambda_1\} < 0$$

Since the topological pressure is a continuous functional on the connected set $\mathcal{F}(z_1, z_2)$ the assertion of the lemma follows. $\square$

The next lemma gives us information on the boundary values of the entropy spectrum for the above considered potentials. It will also be used in the next section.

Lemma 6.7. *Let d_i, S_i, z_i ; $i = 1, 2$; be as above and $\varphi_0 \in \mathcal{F}(z_1, z_2)$ be a potential with $P_\sigma(\varphi_0 \log a) = 0$ according to the previous lemma. If $f^D(\alpha)$ is the dimension spectrum of the Gibbs measure corresponding to $\varphi_0 \log a$ then $f_i^D = d_i$ and $\alpha_i = -z_i$.*

Proof. We are going to show that $\underline{x} \in D_{\alpha_i}$ then $V_\sigma(\underline{x}) \cap M_{inv}(S_i) \neq \varnothing$. Since each $\underline{x} \in S_i$ has the property that $V_\sigma(\underline{x}) \cap M_{inv} S_i \neq \varnothing$ the corollary 8 of the variational principle yields the assertion, because, $\dim_H(S_i = d_i$ and

$$\dim_{\mu_{\varphi_0 \log a}}(x) = \frac{\int_{S_i} \varphi_0 \log a \, d\rho}{\int_{S_i} \log d\rho} = z_i$$

for $x \in S_i$ where $\rho \in V_\sigma(\underline{x})$ and $\mu_{\varphi_0 \log a}$ is the Gibbs measure for the potential $\varphi_0 \log a$. For the latter equality we used that the pressure of $\varphi_0 \log a$ is zero, $\varphi_0 = z_1$ is constant on S_i and theorem 4.2.

Let us fix $\underline{x} \in \Sigma_A^+$ and assume that $V_\sigma(\underline{x}) \cap M_{inv} S_1 = \varnothing$. This means that

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_0(\sigma^j \underline{x}) = w < z_1$$

and

$$\overline{\lim_{n \to \infty}} A(n, \underline{x}) = \gamma < 1$$

where $A(n, \underline{x}) = \text{Card}\{G(n)\}$ and $G(n) = \{0 \leq j \leq n-1 : \sigma^j \geq w\}$. We can conclude that

$$\varlimsup_{n \to \infty} \frac{\sum_{j=0}^{n-1} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x})}{\sum_{j=0}^{n-1} \log a(\sigma^j \underline{x})}$$

$$\leq \varlimsup_{n \to \infty} \frac{\sum_{j \in G(n)} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x}) + \sum_{j \notin G(n)} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x})}{\sum_{j=0}^{n-1} \log a(\sigma^j \underline{x})}$$

$$\leq z_1 - (z_1 - w) \varlimsup_{n \to \infty} \frac{(1 - A(n, \underline{x})) \min(\log a)}{\max(\log a)}$$

$$< z_1$$

Therefore, $\underline{x} \notin D_{\alpha_1}$. The proof for α_2 is similar. $\qquad \square$

Proof of theorem 6.1. Let φ_0 be as in lemma 6.6 with $z_1 = x_1$, $z_2 = x_2$, $f_1 = y_1$ and $f_2 = y_2$. The proof of the theorem 6.1 is now a concatination of proposition 6.5 and lemma 6.7. $\qquad \square$

7. TYPICAL MULTIFRACTAL SPECTRA ARE NON-DEGENERATE

In this section we are going to show that a typical (in the sense of Baire category) Hölder continuous potential gives rise to a Gibbs measure with non-degenerate spectrum – i.e. $f_1 = f_2 = 0$. We note that the space of Hölder continuous functions on $\mathcal{J}$ as well as the space $\mathcal{F}_\theta$ are Baire spaces with the topology of uniform convergence. We first proof that the above situation holds for entropy spectra of Hölder continuous potentials on Σ_A^+ and then use the same approach as in the proof of theorem 6.1 to conclude the result for dimension spectra on $\mathcal{J}$. The result can be stated as

Theorem 7.1. *Let $\mathcal{J}$ be an expanding repeller for the map F. Then there is a residual subset $\mathcal{R} \subset \mathcal{F}_\theta$ such that every Gibbs state on $\mathcal{J}$ which corresponds to a potential in $\mathcal{R}$ has a non-degenerate spectrum.*

We will use the next lemma

Lemma 7.2. *For any invariant set S and any natural number m,*

$$\lim_{a \to 1} h_{top}\{\underline{x} \in \Sigma_A^+ : \lim_{N \to \infty} A(C_m(S), N, \underline{x}) \geq aN\} = h_{top}(U_m(S)).$$

Proof. Let $m \in \mathbb{N}$ and $S \in \Sigma_A^+$; $\sigma(S) = S$ be fixed. Since S is invariant the set $U_m(S) \neq \varnothing$. Moreover, $C_m(S)$ is a finite union of cylinder sets and, hence, closed. This implies that the characteristic function $\chi = \chi_{C_m(S)}$ of the set $C_m(S)$ is a Hölder continuous potential on Σ_A^+. It is easy to see that the multifractal decomposition D_α of the entropy spectrum of the Gibbs measure corresponding to χ has the following expression

$$D_\alpha = \left\{ \underline{x} \in \Sigma_A^+ : \lim_{N \to \infty} \frac{A(U_m, N, \underline{x})}{N} = -\alpha + P_\sigma(\chi) \right\}$$

and

$$D_\alpha^- = \left\{ \underline{x} \in \Sigma_A^+ : \varlimsup_{N \to \infty} \frac{A(U_m, N, \underline{x})}{N} \geq -\alpha + P_\sigma(\chi) \right\}$$

Because relative frequencies are bounded in between 0 and 1 we get $\alpha_1 = P_\sigma(\chi) - 1$, $\alpha_2 = P_\sigma(\chi)$ and

$$U_m(S) \subset D_{\alpha_1} \subset \bigcup_{\alpha \in (\alpha_1, \alpha_1 + (1-a))} D_\alpha^-$$
$$= D_{\alpha_1 + (1-a)}^- = \{ \underline{x} \in \Sigma_A^+ : \varlimsup_{N \to \infty} A(C_m(S), N, \underline{x}) \geq aN \}$$

This together with lemma 6.3 implies the assertion of the lemma. $\qquad\square$

The next proposition deals with the statements of theorem 7.1 in the case of the entropy spectrum of Gibbs measures on Σ_A^+.

Proposition 7.3. *There is a residual subset $\Theta \subset \mathcal{F}_\theta$ such that the entropy spectra for Gibbs measures corresponding to potentials in Θ are non-degenerated.*

Proof. The idea of the proof is to find a residual set $\Theta \subset \mathcal{F}_\theta$ such that any function $\varphi \in \Theta$ has $h_{top}(\overline{\mathcal{M}}(\varphi)) = h_{top}(\underline{\mathcal{M}}(\varphi)) = 0$. The assertion follows then from lemma 6.4. First we find for any $\varepsilon > 0$ an open dense subset $\Theta_1^\varepsilon \subset \mathcal{F}_\theta$.

Let us fix $\varepsilon > 0$, $r \in \mathbb{N}$ and $\varphi \in \mathcal{F}_\theta$. Then there is a periodic point $\underline{x}_0$ with some period n_0 such that

$$Max_\varphi - \frac{\varepsilon}{r} := \sup_{\underline{x} \in \Sigma_A^+} \varlimsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) - \frac{\varepsilon}{r} < \frac{1}{n_0} \sum_{j=0}^{n_0-1} \varphi(\sigma^j \underline{x}_0) < Max_\varphi$$

We consider the set $C = C_{n_0}(S_{\underline{x}_0}) = \bigcup_{j=0}^{n_0-1} C_{n_0}(\sigma^j \underline{x}_0)$ where $S_{\underline{x}_0}$ is the trajectory of the periodic point $\underline{x}_0$. We observe that $U = U_{n_0}(S_{\underline{x}_0}) = S_{\underline{x}_0}$ and hence has zero entropy. We define a new function close to φ by

$$\hat{\varphi}(\underline{x}) = \begin{cases} \varphi(\underline{x}) & \text{if } \underline{x} \in C \\ \varphi(\underline{x}) - \varepsilon & \text{else} \end{cases}$$

Clearly, $\hat{\varphi} \in \mathcal{F}_\theta$ and $\|\varphi - \hat{\varphi}\|_\theta < \varepsilon$. We will see that this function $\hat{\varphi}$ has the property that $h_{top}(\overline{\mathcal{M}}(\hat{\varphi})) < \varepsilon$. For $\underline{x} \in \Sigma_A^+$ and $n \in \mathbb{N}$ we estimate

$$\frac{1}{n}\sum_{j=0}^{n-1}\hat{\varphi}(\sigma^j\underline{x}) = \frac{1}{n}\left(\sum_{\sigma^j\underline{x}\in C}\hat{\varphi}(\sigma^j\underline{x}) + \sum_{\sigma^j\notin C}\hat{\varphi}(\sigma^j\underline{x})\right)$$

$$\leq \frac{1}{n}\left(\sum_{\sigma^j\underline{x}\in C}\varphi(\sigma^j\underline{x}) + \sum_{\sigma^j\notin C}\varphi(\sigma^j\underline{x}) - (n - A(C,n,\underline{x}))\varepsilon\right)$$

$$\leq \frac{1}{n}\sum_{j=1}^{n-1}\varphi(\sigma^j\underline{x}) - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon$$

$$\leq Max_\varphi - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon$$

$$\leq Max_{\hat{\varphi}} + \frac{\varepsilon}{r} - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon$$

because

$$Max_{\hat{\varphi}} \geq \frac{1}{n_0}\sum_{j=0}^{n_0-1}\varphi(\sigma^j\underline{x}_0) \geq Max_\varphi - \frac{\varepsilon}{r}$$

Hence,

$$\overline{\mathcal{M}}(\hat{\varphi}) \subset \left\{\underline{x} \in \Sigma_A^+ : \varlimsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(\sigma^j\underline{x}) = Max_{\hat{\varphi}}\right\}$$

$$\subset \left\{\underline{x} \in \Sigma_A^+ : \varlimsup_{n\to\infty}\frac{A(C,n,\underline{x})}{n} \geq 1 - \frac{1}{r}\right\}$$

In view of lemma 7.2 we can find a $\varepsilon > 0$ such that

$$\varepsilon > h_{top}\left\{\underline{x} \in \Sigma_A^+ : \varlimsup_{n\to\infty}\frac{A(C,n,\underline{x})}{n} \geq 1 - \frac{1}{r}\right\} \geq h_{top}(\overline{\mathcal{M}}(\hat{\varphi})).$$

We set

$$\Omega_1^\varepsilon(\varphi) := \{\psi \in \mathcal{F}_\theta : \|\hat{\varphi} - \psi\|_\theta < \varepsilon\}.$$

The above estimations still hold with $\varepsilon' = 2\varepsilon$ for all potentials $\psi \in \Omega_1^\varepsilon(\varphi)$. Therefore, for all $\varepsilon > 0$ the set

$$\Theta_1^\varepsilon := \bigcup_{\varphi\in\mathcal{F}_\theta}\Omega_1^\varepsilon(\varphi)$$

is an open and dense in $\mathcal{F}_\theta$ for which the entropy of the maximum sets $\overline{\mathcal{M}}(\varphi)$ of its elements φ has topological entropy less then 2ε. We repeat the same procedure

for the set $\underline{\mathcal{M}}$ and produce a set Θ_2^ε with the property that $h_{top}(\underline{\mathcal{M}}(\varphi)) < 2\varepsilon$ for all $\varphi \in \Theta_2^\varepsilon$. Let

$$\Theta := \bigcap_{N=0}^{\infty} \bigcup_{n \geq N} (\Theta_1^{\frac{1}{n}} \cap \Theta_2^{\frac{1}{n}}).$$

Now we observe that functions $\varphi \in \Theta$ have the property that

$$h_{top}(\overline{\mathcal{M}}(\varphi)) = h_{top}(\underline{\mathcal{M}}(\varphi)) = 0$$

Moreover, by the way of constructing the set Θ we immediately have that Θ is a residual subset of $\mathcal{F}_\theta$. $\qquad\square$

Proof of theorem 7.1. As we have seen in the proof of theorem 6.1 and f lemma 6.7 the boundary values of the dimension spectrum of the Gibbs measure $m = \mu \circ \chi^{-1}$ on $\mathcal{J}$ adjoint to the potential $\psi(\underline{x}) = \varphi(\underline{x}) \log a(\underline{x})$ coincide with those of the entropy spectrum of the Gibbs measure μ on Σ_A^+ corresponding to the potential φ. The proof of the theorem 7.1 concludes by observing that the operator $\mathcal{L}_a$ on $\mathcal{F}_\theta$ defined by $\mathcal{L}_a(\varphi(\underline{x})) = \varphi(\underline{x}) \log a(\underline{x})$ is in fact a homeomorphism of the space $\mathcal{F}_\theta$ – in particular it transforms residual sets into residual sets – because $\log a$ is bounded away from zero. $\qquad\square$

8. Concluding remarks

In the previous sections we proved results about the behavior of the dimension spectrum of a Gibbs state on an expanding conformal repeller at its boundary. We used an approach which is related to the entropy spectrum of the underlying symbolic space, but stated the results only in terms of the dimension spectrum. However, the corresponding results for the entropy spectrum can easily be derived from the dimension results by considering the symbolic space Σ_A^+ itself as an expanding conformal repeller equipped with the metric

$$d(\underline{x}, \underline{x}') = \sum_{k=1}^{\infty} e^{-k} |i_k - -'_k|.$$

This metric corresponds to the stretching rate by the constant factor e. Then the dimension spectrum for a Gibbs measure μ on σ_A^+ coincides with the entropy spectrum for the Gibbsmeasure $m = \mu \circ \chi$ on $\mathcal{J}$. Moreover, the same considerations are succesful in a more general situation. Namely, we can derive the same results for the entropy spectrum for *any* map that admits a finite Markov partion – i.e. admits a coding by some symbolic space Σ_A^+. Axiom A basic sets in any finite dimension belong to this class. The situation with the dimension spectrum is more complicated. The main problem in generalizing the results for expanding conformal repellers to non-conformal maps in higher dimensions is that there is no general theory of computing the dimension of an invariant set or measure. In particular, the Bowen–Ruelle dimension formula $T_D(0) = \dim_H \mathcal{J}$ (see proposition 4.1) need not hold. However, the

results on multifractal dimension spectra can be verified for two-dimensional hyperbolic horseshoes. This investigation was done in [16, 12, 4]. Using the same approach it is easy to see that the results in this paper also hold for two-dimensional hyperbolic horseshoes.

<div align="center">APPENDIX</div>

Let (X, d) be a complete separable metric space. Consider a set $Z \subset X$ and a positive number δ. A cover of Z by sets of diameter at most δ is called a δ-*cover* of Z. For any $s > 0$, we define the *s-dimensional Hausdorff measure* of Z by

$$m_H(Z, s) = \lim_{\delta \to 0} \inf_{\mathfrak{U}} \sum_{U \in \mathfrak{U}} (\operatorname{diam} U)^s,$$

where the infimum is taken over all finite or countable δ-covers $\mathfrak{U}$ of Z. There exists a unique value of s at which $m_H(Z, s)$ jumps from $+\infty$ to 0. We call this value the *Hausdorff dimension* of Z and denote it by $\dim_H Z$. We have

$$\dim_H Z = \inf\{s : m_H(Z, s) = 0\} = \sup\{s : m_H(Z, s) = +\infty\}.$$

Let $f \colon X \to X$ be a continuous map. If $\mathfrak{U}$ is a finite open cover of X, for each integer $n \geq 1$ we denote by $S_n(\mathfrak{U})$ the collection of strings $\mathbf{U} = U_1 \cdots U_n$, where $U_1, \ldots, U_n \in \mathfrak{U}$. For each $\mathbf{U} \in S_n(\mathfrak{U})$, we write $n(\mathbf{U}) = n$ and define the open set

$$X(\mathbf{U}) = \left\{x \in X : f^{k-1}x \in U_k \text{ for } k = 1, \ldots, n\right\}.$$

Consider a set $Z \subset X$. We say that a collection of strings Γ *covers* Z if the union $\bigcup_{\mathbf{U} \in \Gamma} X(\mathbf{U}) \supset Z$. For every real number s, we define

$$M(Z, s, \mathfrak{U}) = \lim_{n \to \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp\left(-n(\mathbf{U})s\right),$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k \geq n} S_k(\mathfrak{U})$ covering Z. There exists a unique value of s at which $M(Z, s, \mathfrak{U})$ jumps from $+\infty$ to 0, given by

$$h(Z, \mathfrak{U}) = \inf\{s : M(Z, s, \mathfrak{U}) = 0\} = \sup\{s : M(Z, s, \mathfrak{U}) = +\infty\}.$$

We define the *topological entropy* of f on the set Z by

$$h(f|_Z) = \lim_{\operatorname{diam} \mathfrak{U} \to 0} h(Z, \mathfrak{U})$$

(one can show that the limit always exists). If Z is compact and f-invariant, then $h(f|_Z)$ coincides with the classical topological entropy (see, for example, [9]). However, the set Z need not be compact nor f-invariant for our definition.

The following simple statement follows from the special type of metric on Σ_A^+ introduced in Section 3.

Lemma 8.1. *For any subset $Z \subset \Sigma_A^+$ we have $h(f|_Z) = \dim_H Z \cdot \log a$.*

For each $n \in \mathbb{N}$, we define the metric d_n on X by
$$d_n(x, y) = \max\big\{d(f^k x, f^k y) : 0 \le k \le n - 1\big\}.$$
Given $\delta > 0$, we say that a finite set $E \subset X$ is a (n, δ)-*separated set* if $d_n(x, y) > \delta$ whenever $x, y \in E$ and $x \ne y$. We define the *topological pressure* of the continuous function $\varphi \colon X \to \mathbb{R}$ (with respect to f) by

$$P(\varphi) = \lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \, \frac{1}{n} \log \sup_E \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k x),$$

where the supremum is taken over all (n, δ)-separated sets E.

For a subset $U \in \Sigma_A^+$ we set $Z = Z(U) = \{\underline{x} \in U : V_\sigma(\underline{x}) \cap M_{\mathrm{inv}}(U) \ne \varnothing\}$ where $M_{\mathrm{inv}}(U)$ is the set of invariant measures concentrated on U. We also use the notation $M_e(U)$ for the set of ergodic measures on $U>$ Pesin and Pitskel' have proven the following variational principle.

Theorem 8.2 (Pesin and Pitskel' [10]).
$$h_{top}(Z) = \sup_{\mu \in M_{inv}(U)} h_\mu$$

We will use a slight modification of the above variational principle.

Theorem 8.3.
$$h_{top}(Z) = \sup_{\underline{x} \in Z} \inf\{h_\mu : \mu \in V_\sigma(\underline{x}) \cap M_e(U)\}$$

Proof. We denote by $\hat{Z}$ the set $\{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \text{ consists of a single measure}\}$. We consider an invariant measure μ on U. Let $\mu = \int \mu_\zeta \, dm(\zeta)$ be its ergodic decomposition. Then for $m - $ a.e. ζ $\mu_\zeta(U) = 1$ and μ_ζ is an ergodic measure. Moreover, since $h_\mu = \int h_{mu_\zeta} \, dm(\zeta)$ we can choose ζ in the way that $h_\mu \le h_{\mu_\zeta}$. If $\underline{x}(\zeta)$ is a generic point for the measure μ_ζ then $\underline{x}(\zeta) \in \hat{Z}$. It follows that

$$\sup_{\mu \in M_{\mathrm{inv}}(U)} h_\mu \le \sup_{\underline{x} \in \hat{Z}} h_{V_\sigma(\underline{x})}$$
$$\le \sup_{\underline{x} \in \hat{Z}} \inf\{h_\nu : \nu \in V_\sigma(\underline{x}) \cap M_e(U)\}$$
$$\le \sup_{\underline{x} \in Z} \inf\{h_\nu : \nu \in V_\sigma(\underline{x}) \cap M_e(U)\}$$

Obviously, we have

$$\sup_{\mu \in M_{\mathrm{inv}}(U)} h_\mu \ge \sup_{\underline{x} \in Z} \sup\{h_\mu : \mu \in V_\sigma(\underline{x}) \cap M_e(U)\}$$
$$\ge \sup_{\underline{x} \in Z} \inf\{h_\mu : \mu \in V_\sigma(\underline{x}) \cap M_e(U)\}$$

$\square$

We will state some simple consequence of this variational principle.

Corollary. Let S be a compact invariant set. Then
$$h_{top}\{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{inv}(S) \neq \varnothing\} = h_{top}(S)$$

Proof. Since $S \in \{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{inv}(S) \neq \varnothing\}$ the statement follows immediately. $\square$

The *Legendre transform* of the function T is the function $\mathcal{D}$ defined by $\mathcal{D}(\alpha) = \sup_q(\alpha q - T(q))$. We then say that the pair $(\mathcal{D}, T)$ is a *Legendre pair* with respect to the variables α, q. We say that a C^2 function T is *strictly convex* if $T'' > 0$ everywhere on its domain. Given two strictly convex C^2 functions $\mathcal{D}$ and T, one can show that the pair $(\mathcal{D}, T)$ is a Legendre pair with respect to the variables α, q if and only if $\mathcal{D}(\alpha) = T(q) + q\alpha$, where $\alpha = -T'(q)$ and $q = \mathcal{D}'(\alpha)$.

We will use the following additional notations. Let $\underline{x} \in \Sigma_A^+$. We consider the set $V_\sigma(\underline{x})$ of accumulation points of the sequence of measures $\left\{\frac{1}{n}\sum_{j=0}^{n-1}\right\}\delta_{\sigma^j\underline{x}}$. Clearly, any measure $\mu \in V_\sigma(\underline{x})$ is invariant.

For a given subset $S \in \Sigma_A^+$ we write $C_m(S)$ for the set $\bigcup_{\underline{x}\in S} C_m(\underline{x})$ and $U_m(S) = \{\underline{x} \in \Sigma_A^+ | \sigma^k \underline{x} \in C_m(S) \text{ for all } k \in \mathbb{N}\}$.

The number of times the trajectory $\{\sigma^k\underline{x}\}_{k=0}^N$ of the point $\underline{x}$ hits a given set U is denoted by $A(U, N, \underline{x}) = \text{Card}\{0 \leq k \leq N : \sigma^k\underline{x} \in U\}$.

For a given set X we denote the set of invariant, respectively ergodic, measures supported on X by $M_{inv}(X)$ and $M_e(X)$.

If $\mu \in M_e(\Sigma_A^+)$ then $m = \mu \circ \chi^{-1} \in M_e(\mathcal{J})$ and the limit $\lim_{n\to\infty}\frac{1}{n}\log\|DF^n(x)\| = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log a(\sigma^j\underline{x}) = \lambda(x)$ exists for m - a.e. $x \in \mathcal{J}$ or for μ - a.e. $\underline{x} = \chi^{-1}(x) \in \Sigma_A^+$, respectively. Since the measure μ is ergodic the number $\lambda(x)$ is constant almost everywhere and hence depends only on the measure. We will denote this number by λ_μ or λ_m.

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