Institut für Angewandte Analysis
und Stochastik

im Forschungsverbund Berlin e.V.

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10th June 1992

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Preprint No. 3
Berlin 1992

Key words and phrases. Multigrid methods, Algebraic multigrid, Schar complement iterative methods, Iterative methods for convection–diffusion equations.

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ON THE CONVERGENCE OF ALGEBRAICALLY DEFINED MULTIGRID METHODS

JÜRGEN FUHRMANN

Abstract. Based on the theory for multigrid methods with nonnested spaces and noninherited quadratic forms, a V-cycle convergence proof for an algebraically defined multigrid method using the approximation and smoothing property from the theory of algebraic multigrid is given. The estimation of the approximation property is carried out by means of strengthened Cauchy inequalities.

Further, a method is suggested which allows to construct multigrid algorithms for special nonsymmetric problems.

The ideas of the paper are illustrated by some examples of multigrid methods for problems with strongly varying coefficients in two- and three-dimensional rectangular domains.

1. INTRODUCTION

This paper contains a multigrid V-cycle convergence proof which combines the convergence theory of Bramble, Pasciak and Xu ([BP87], [BPX91]) which has historical connections to that of Braess and Hackbusch [BH83], with the algebraically defined approximation and smoothing assumptions of Ruge and Stüben [RS87] which can be estimated numerically. Here, the estimation of this approximation assumption in terms of cosines of angles between some subspaces has been done using techniques of Haase, Langer and Meyer [HLM91a],[HLM91b]. The language of this paper is algebraic, using features developed by Axelsson and Vassilevski [AV90], Dahmen and Elsner [DE89] and others. The connection of this language with that of Ruge and Stüben one can find in the paper of Popa [Pop91], too.

The aim of the presented theory is the ability to prove the convergence of multigrid methods for diffusion and convection diffusion problems with strongly varying coefficients discretized by inverse averaging and exponential fitting schemes, respectively, where the usual finite element estimates cannot be used. The main features of these methods are operator dependent intergrid transfer operators and harmonical means to define the coarse grid operator coefficients. Such methods have been presented in [ABDJP81], [DJ87], [Kuz90], [Sto91] and others and by the author in [FG91b], [FG91a].

Section 2 contains an algebraic framework for the definition of such multigrid methods, based on that in [BPX91].

The convergence result is obtained in section 3. Together with the estimation of the approximation property in section 4 and 5 it comes out that the multigrid convergence rate depends on a number of reasonable factors:

- some spectral equivalences in the space obtained by the fine grid nodes, which can be expected to be $O(1)$,
- the cosine of the angle between the fine grid space and the coarse grid space in the energy scalar product, which can be estimated locally on each coarse grid cell,
- the spectral equivalence of the coarse grid operator and the Galerkin coarse grid operator,
- the smoothing property of the smoothers which for a properly chosen reference norm holds for a large class of matrices.

In section 6 one can find a proposal to carry over the theory to a class of nonsymmetric problems by means of the change of the scalar product.

In section 7 the formulation of the multigrid method described in [FG91b], [FG91a] in the language developed in this paper is given.

Finally, section 8 consists of some numerical examples for two- and threedimensional multigrid preconditioners..

The appendix contains some remarks on base changes and the notations used in this paper. Further, a short collection of basic results on preconditioning is provided.

2. An algebraically defined multigrid method

The aim of this section is the algebraic definition of a multigrid method using the terminology of Bramble, Pasciak and Xu [BPX91] together with the Schur complement formulations of Axelsson and Vassilevski [AV90] who consider algebraically defined multilevel preconditioners or Dahmen and Elsner [DE89]. The basic parameter of the method are the interpolations between the level spaces and the operators in the level spaces which are assumed to be given.

2.1. The spaces and the scalar products. So let

$$\mathcal{M}_0, \ldots \mathcal{M}_j$$

be a sequence of finite dimensional Euclidean vector spaces with the scalar products

$$(\cdot,\cdot)_k : \mathcal{M}_k \times \mathcal{M}_k \to \mathcal{R}$$

and the symmetric, positive definite with respect to them operators

$$A_k : \mathcal{M}_k \to \mathcal{M}_k$$

which define the energy scalar products

$$a_k(\cdot,\cdot) := (A_k \cdot, \cdot)_k$$

on $\mathcal{M}_k$.

2.2. The problem. The following problem will be considered: For $f \in \mathcal{M}_j$ find $u \in \mathcal{M}_j$ with

$$A_j u = f \tag{2.1}$$

which is equivalent to the finite dimensional variational problem: find $u \in \mathcal{M}_j$ with

$$a_j(u,v) = (f,v)_j \quad \forall v \in \mathcal{M}_j \tag{2.2}$$

2.3. The prolongation operators. For $k = 1, \ldots, j$, assume that there exists a "nodal embedding"

$$e_k : \mathcal{M}_{k-1} \hookrightarrow \mathcal{M}_k$$

and a splitting

$$\mathcal{M}_k = \mathcal{M}_k^{(1)} \oplus \mathcal{M}_k^{(2)}$$

where

$$\mathcal{M}_k^{(2)} = \text{Im} \quad e_k$$

is isomorphic to $\mathcal{M}_{k-1}$ which thus can be viewed as a subspace of $\mathcal{M}_k$. So one can assume a block partition

$$A_k = \begin{pmatrix} A_{k,11} & A_{k,12} \\ A_{k,21} & A_{k,22} \end{pmatrix}.$$

Let

$$U_k = \begin{pmatrix} F_k & G_k \\ 0 & I \end{pmatrix}$$

with F_k and G_k having full rank be the transformation matrix to an approximate harmonical basis [HLM91a]. The terminology of the following lemma is described in the appendix (subsection A.2).

LEMMA 2.1

$$A_k = U_k^T \begin{pmatrix} A_{k,11[F_k]} & \Delta_k \\ \Delta_k^T & S_{k-1} \end{pmatrix} U_k = U_k^T A_{k[U]} U_k$$

with

$$\Delta_k = F_k^{-T}(A_{k,12} - A_{k,11}F_k^{-1}G_k)$$

and

$$\begin{aligned} S_{k-1} &= \hat{S}_{k-1} + \Delta_k^T A_{k,11[F_k]}^{-1} \Delta_k & (2.3) \\ &= A_{k,22} - A_{k,21}F_k^{-1}G_k - G_k^T F_k^{-T} A_{k,12} + G_k^T A_{k,11[F_k]} G_k & (2.4) \end{aligned}$$

where

$$\hat{S}_{k-1} = A_{k,22} - A_{k,21}(A_{k,11})^{-1} A_{k,12} \qquad (2.5)$$

is the Schur complement from the block factorization

$$A_k = \hat{U}_k^T \begin{pmatrix} A_{k,11}^{-1} & 0 \\ 0 & \hat{S}_{k-1} \end{pmatrix} \hat{U}_k$$

with

$$\hat{U}_k = \begin{pmatrix} A_{k,11} & A_{k,12} \\ 0 & I \end{pmatrix}$$

Proof. Straightforward calculations. □

It follows that

$$A_k^{-1} = \hat{U}_k^{-1} \begin{pmatrix} A_{k,11} & 0 \\ 0 & \hat{S}_{k-1}^{-1} \end{pmatrix} \hat{U}_k^{-T}$$

and that

$$\begin{aligned} \hat{T}_k &:= I - \hat{U}_k{}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}_{k-1}^{-1} \end{pmatrix} \hat{U}_k{}^{-T} A_k \\ &= I - \begin{pmatrix} -A_{k,11}^{-1}A_{k,12} \\ I \end{pmatrix} \hat{S}_{k-1}^{-1} \begin{pmatrix} -A_{k,21}A_{k,11}^{-1} & I \end{pmatrix} A_k \end{aligned}$$

can be interpreted as a coarse grid correction error propagation operator which consists in projecting the error onto $\hat{U}_k \mathcal{M}_k^1$. Then, viewing U_k as an approximation to $\hat{U}_k$, define the prolongation operator

$$I_k = \begin{pmatrix} -F_k^{-1}G_k \\ I \end{pmatrix} : \mathcal{M}_{k-1} \to \mathcal{M}_k. \qquad (2.6)$$

2.4. The Galerkin and the approximate Galerkin coarse grid corrector. After all, the operator

$$T_k^G := I - U_k^{-1} \begin{pmatrix} 0 & 0 \\ 0 & S_{k-1}^{-1} \end{pmatrix} U_k^{-T} A_k \tag{2.7}$$

could be described as an coarse grid corrector which has been derived from $\hat{T}_k$ by some perturbation of $\hat{U}_k$. A further perturbation consisting of in using A_{k-1} instead of S_{k-1} leads to the definition of T_k which will serve as the actual coarse grid corrector for the multigrid method defined in the next section.

$$T_k := I - U_k^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A_{k-1}^{-1} \end{pmatrix} U_k^{-T} A_k \tag{2.8}$$

LEMMA 2.2 *The operator T_k^G in the approximate harmonical basis has the representation*

$$T_{k[U_k]}^G = \begin{pmatrix} I & 0 \\ -S_{k-1}^{-1}\Delta_k^T & 0 \end{pmatrix} \tag{2.9}$$

and is an A-orthoprojector. The operator T_k in the approximate harmonical basis has the representation

$$T_{k[U_k]} = \begin{pmatrix} I & 0 \\ -A_{k-1}^{-1}\Delta_k^T & I - A_{k-1}^{-1}S_{k-1} \end{pmatrix}. \tag{2.10}$$

If

$$\chi A_{k-1} \leq S_{k-1} \leq A_{k-1},$$

then

$$0 \leq a_k(T_k u, u) \leq \chi a_k(T_k^G u, u) + (1-\chi) a_k(u,u) \quad \forall u \in \mathcal{M}_k.$$

Proof. One according to subsection A.2 has to calculate

$$T_{k[U_k]} = U_k T_k U_k^{-1} = U_k U_k^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & A_{k-1}^{-1} \end{pmatrix} U_k^{-T} A_k U_k^{-1}.$$

Using S_{k-1} instead of A_{k-1} gives the formula for T_k. Now, let

$$V_k = \begin{pmatrix} I & 0 \\ S_{k-1}^{-1}\Delta_k^T & I \end{pmatrix}$$

be the basis transformation matrix which arises from the block LDU factorization of $A_{k[U_k]}$. Then calculating $A_{k[V_kU_k]} = V_k^{-T} A_{k[U_k]} V_k^{-1}$ and $T_{k[V_kU_k]} = V_k T_{k[U_k]} V_k^{-1}$ leads to

$$\begin{aligned} A_{k[V_kU_k]} &= \begin{pmatrix} A_{k,11[F_k]} - \Delta_k S_{k-1}^{-1}\Delta_k^T & 0 \\ 0 & S_{k-1} \end{pmatrix}, \\ T_{k[V_kU_k]}^G &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ T_{k[V_kU_k]} &= \begin{pmatrix} I & 0 \\ 0 & I - A_{k-1}^{-1}S_{k-1} \end{pmatrix} \end{aligned}$$

From this, the orthoprojector properties of T_k^G (definition in subsection A.4) are obvious.

To prove the last inequality, we have to show that

$$A_k T_k \leq \chi A_k T_k^G + (1-\chi) A_k.$$

We do this in the the V_k-basis, using theorem A.1:

$$\begin{aligned}
A_{k[V_kU_k]} T_{k[V_kU_k]} &= \begin{pmatrix} A_{k,11[F_k]} - \Delta_k S_{k-1}^{-1} \Delta_k^T & 0 \\ 0 & S_{k-1} - S_{k-1} A_{k-1}^{-1} S_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} A_{k,11[F_k]} - \Delta_k S_{k-1}^{-1} \Delta_k^T & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & S_{k-1} - S_{k-1} A_{k-1}^{-1} S_{k-1} \end{pmatrix} \\
&\leq A_{k[V_kU_k]} T^G_{k[V_kU_k]} \\
&+ (1-\chi) \begin{pmatrix} 0 & 0 \\ 0 & S_{k-1} \end{pmatrix} \\
&= A_{k[V_kU_k]} T^G_{k[V_kU_k]} \\
&+ (1-\chi)(A_{k[V_kU_k]} - A_{k[V_kU_k]} T^G_{k[V_kU_k]})
\end{aligned}$$

The fact that $T_k \geq 0$ comes from $A_{k,11[F_k]} - \Delta_k S_{k-1}^{-1} \Delta_k^T \geq 0$ as a Schur complement and $S_{k-1} - S_{k-1} A_{k-1}^{-1} S_{k-1} \geq 0$ because of $S_{k-1} \leq A_{k-1}$. □

LEMMA 2.3

$$a_k(I_k u, I_k v) = (S_{k-1} u, v) \quad \forall u, v \in \mathcal{M}_{k-1} \tag{2.11}$$

Proof. Straightforward calculations. □

So the perturbed Schur complement S_k is the Galerkin coarse grid operator for the given choice of the interpolation.

2.5. The restriction operators. The following operators are defined in [BPX91] in a variational way. $P_{k-1} : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is defined by

$$a_{k-1}(P_{k-1} u_k, u_{k-1}) = a_k(u_k, I_k u_{k-1}) \quad \forall u_{k-1} \in \mathcal{M}_{k-1},$$

and $P_{k-1}^0 : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is defined by

$$(P_{k-1}^0 u_k, u_{k-1})_{k-1} = (u_k, I_k u_{k-1})_k \quad \forall u_{k-1} \in \mathcal{M}_{k-1}.$$

For our purposes, we define yet another restriction: $P_{k-1}^G : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is defined by

$$(S_{k-1} P_{k-1}^G u_k, u_{k-1}) = a_k(u_k, I_k u_{k-1}) \quad \forall u_{k-1} \in \mathcal{M}_{k-1},$$

LEMMA 2.4 *In the algebraic language,*

$$P_{k-1}^0 = \begin{pmatrix} -G_k^T F_k^{-T} & I \end{pmatrix}, \tag{2.12}$$

$$P_{k-1}^G = S_{k-1}^{-1} P_{k-1}^0 A_k = \begin{pmatrix} S_{k-1}^{-1} \Delta_k^T F_k & I + S_{k-1}^{-1} \Delta_k^T G_k \end{pmatrix}. \tag{2.13}$$

and

$$P_{k-1} = A_{k-1}^{-1} P_{k-1}^0 A_k = \left(\begin{array}{cc} A_{k-1}^{-1} \Delta_k^T F_k & I + A_{k-1}^{-1} \Delta_k^T G_k \end{array} \right). \tag{2.14}$$

Further, for T_k, one has

$$T_k = I - I_k P_{k-1},$$

and T_k^G satisfies

$$T_k^G = I - I_k P_{k-1}^G.$$

Proof. Straightforward calculations. □

2.6. The multigrid operator. Assume to be given the smoothing preconditioners $R_k : \mathcal{M}_k \to \mathcal{M}_k$, $k = 1 \ldots j$ and define for $l \geq 0$

$$R_k^l = \begin{cases} R_k & , \quad l \quad \text{odd} \\ R_k^T & , \quad l \quad \text{even.} \end{cases}$$

Let $K_k = I - R_k A_k$ and define

$$\tilde{K}_k^{(m)} = \begin{cases} (K_k^* K_k)^{m/2} & , \quad m \quad \text{even} \\ (K_k^* K_k)^{(m-1)/2} K_k^* & , \quad m \quad \text{odd} \end{cases}$$

Here, $K_k^* = A_k^{-1} K_k^T A_k$ is the adjoint to K_k in the A_k- scalar product.

Now, let $m(k)$ and p be positive integers. Define after [BPX91] the multigrid operators $B_k : \mathcal{M}_k \to \mathcal{M}_k$ recursively by:

$$\begin{aligned} B_0 &= A_0^{-1} \\ I - B_k A_k &= (\tilde{K}_k^{(m(k))})^* \left(T_k + I_k (I - B_{k-1} A_{k-1})^p P_{k-1} \right) \tilde{K}_k^{(m(k))} \end{aligned}$$

For $m(k) = m$ and $p = 1, 2$ one has the usual multigrid V- and W-cycles with m pre- and m post-smoothing steps. One has to mention that the so defined B_k is symmetric according to the special choice of the combination of the smoothers and their transposed which allows to use it as a preconditioner for conjugate gradients [BPX91]. A similar choice has been taken in [JLM+89] where all post-smoothings are transposed to the pre-smoothings, but not all the features of the convergence proof stated below would go through.

3. Convergence

3.1. The assumptions. The first assumption which has been made in [BPX91] is the contraction property of the smoothers:

$$\text{spec } K_k^* K_k \subset [0, 1) \tag{3.1}$$

Further, the following spectral equivalences are assumed which actually measure the perturbation of the Galerkin property: There exists a constant χ such that

$$\chi A_k \leq S_k \leq A_k, \quad k = 0, \ldots, j-1 \tag{3.2}$$

independent of k. This includes condition (A.2) of [BPX91] which consists in the right part of the inequalities and leads to the nonnegativity of T_k and $I - B_k A_k$.

Third, in difference to the paper [BPX91], we will use the combined smoothing and approximation property which has been used by Ruge and Stüben [RS87]. Usually, it is splitted by means of a reference norm into two separate assumptions: a smoothing property which concerns the smoothers R_k, and a regularity and approximation property which concerns coarse grid correctors $T_k = I - I_k P_{k-1}$. We will, in a further difference to [BPX91], demand this property for the Galerkin coarse grid operator $T_k^G = I - I_k P_{k-1}^G$, only, as it has been done in [RS87], too. The approximation and smoothing assumption in the form used in [RS87] is the following: There exists a constant $M \geq 1$ so that independently of k,

$$a_k(K_k u, K_k u) \leq a_k(u,u) - \frac{1}{M} a_k(T_k^G K_k u, T_k^G K_k u) \quad \forall u \in \mathcal{M}_k \tag{3.3}$$

To obtain compatibility to [BPX91], obtain the equivalent formulation

$$a_k(T_k^G K_k u, T_k^G K_k u) \leq M a_k((I - K_k^* K_k)u, u) \quad \forall u \in \mathcal{M}_k$$

and, due to the fact that the T_k^G are A-orthoprojectors,

$$a_k(T_k^G K_k u, K_k u) \leq M a_k((I - K_k^* K_k)u, u) \quad \forall u \in \mathcal{M}_k. \tag{3.4}$$

We only remark that the combined smoothing and approximation property of [BPX91] would be

$$a_k(T_k K_k u, K_k u) \leq M a_k((I - K_k^* K_k)u, u) \quad \forall u \in \mathcal{M}_k, \tag{3.5}$$

which should have a worse constant than (3.4) because it hides in itself the perturbation of the Galerkin property.

It comes out that our assumption covers only the "H^2-regular" case, but intentions of this paper is the understanding of the algebraic nature of the multigrid algorithm, and the incorporation of lower regularity assumptions, which might be possible, would result in a loss of clarity of the situation.

3.2. The convergence theorem. The contraction property of the multigrid operator will be expressed by the following:

THEOREM 3.1 *Assume (3.1), (3.2) and the combined approximation and smoothing assumption (3.4) hold. Let $p \geq 1$ and $m(k) = m$. Then for $k = 0 \ldots j$*

$$a_k((I - B_k A_k)u, u) \leq \delta_k a_k(u,u) \quad \forall u \in \mathcal{M}_k \tag{3.6}$$

holds with

$$\delta_k = 1 - \frac{1}{\chi(1 + \frac{M}{m})}. \tag{3.7}$$

Proof. The proof follows the scheme in [BP87] and [BPX91]. For $k = 0$, (3.6) is obvious. Assume that (3.6) holds for $k-1$. Let for $u \in \mathcal{M}_k$, $\tilde{u} = \tilde{K}_k^{(m)} u$. One has, using $\delta_{k-1}^p \leq \delta_{k-1}$,

$$\begin{aligned}
&a_k((I - B_k A_k)u, u) = \\
&\quad = a_k(T_k \tilde{u}, \tilde{u}) + a_{k-1}((I - B_{k-1} A_{k-1}) P_{k-1} \tilde{u}, P_{k-1} \tilde{u}) \\
&\quad \leq a_k(T_k \tilde{u}, \tilde{u}) + \delta_{k-1}^p a_k(I_k P_{k-1} \tilde{u}, \tilde{u}) \\
&\quad \leq (1 - \delta_{k-1}) a_k(T_k \tilde{u}, \tilde{u}) + \delta_{k-1} a_k(\tilde{u}, \tilde{u})
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\delta_{k-1})\chi a_k(T_k^G\tilde{u},\tilde{u}) + (\delta_{k-1}+(1-\chi)(1-\delta_{k-1}))a_k(\tilde{u},\tilde{u}) \qquad \text{after lemma 2.2}\\
&\leq (1-\delta_{k-1})\chi M a_k((I-\bar{K}_k)\bar{K}_k^m u,u)\\
&\quad +(\delta_{k-1}+(1-\chi)(1-\delta_{k-1}))a_k(\tilde{u},\tilde{u}) \qquad \text{after (3.4)}\\
&\leq (1-\delta_{k-1})\chi\frac{M}{m}a_k((I-\bar{K}_k^m)u,u)\\
&\quad +(\delta_{k-1}+(1-\chi)(1-\delta_{k-1}))a_k(\bar{K}_k^m u,u) \qquad (3.8)\\
&= \frac{\frac{M}{m}}{1+\frac{M}{m}}a_k(u,u)\\
&\leq \frac{\chi(1+\frac{M}{m})-1}{\chi(1+\frac{M}{m})}a_k(u,u)
\end{aligned}
$$

In inequality (3.8),

$$\bar{K}_k = \begin{cases} K_k^*K_k & , \quad m \quad \text{even} \\ K_kK_k^* & , \quad m \quad \text{odd.} \end{cases}$$

Further, from [BP87] we used

$$
\begin{aligned}
a_k((I-\bar{K}_k)\bar{K}_k^m u,u) &\leq \frac{1}{m}\sum_{i=0}^{m-1} a_k((I-\bar{K}_k)\bar{K}_k^i u,u)\\
&= \frac{1}{m}a_k((I-\bar{K}_k^m)u,u)
\end{aligned}
$$

due to (3.1). □

The result coincides with that in [BPX91], in the case of full H^2- regularity and $\chi = 1$ where an improvement of the convergence rate for $p > 1$ takes place only in the case of lower regularity. The case $\chi < 1$ in [BPX91] is covered by the approximation property constant which should become worse in this case. Further results concerning variable V-cycles and the preconditioning properties of B_k could be obtained, too, but the focus of this paper lies in the discussion of the smoothing and approximation property.

4. Discussion of the combined approximation and smoothing assumption

The ideas of this section come from [RS87], [MMB87] and others.

4.1. The splitting of the combined approximation and smoothing assumption.

Lemma 4.1 *Let E_k be positive definite operators in $\mathcal{M}_k$. Define the bilinear form*

$$e_k(u,v) = (E_k^{-1}A_k u, A_k v)_k \quad \forall u,v \in \mathcal{M}_k$$

Then the E_k - approximation assumption

$$a_k(T_k^G u, T_k^G u) \leq \beta e_k(u,u) \quad \forall u \in \mathcal{M}_k \tag{4.1}$$

and the E_k - smoothing assumption

$$\sigma e_k(u,u) \leq a_k((I-K_k^*K_k)u,u) \quad \forall u \in \mathcal{M}_k \tag{4.2}$$

imply the combined approximation and smoothing assumption (3.4) with $M = \beta/\sigma$

Proof. The proof is obvious. □

4.2. The λ_k- approximation and smoothing assumptions. In [BP87] and [BPX91], the splitting of the approximation and smoothing assumption has been done by $E_k = \lambda_k I$ where λ_k is the largest eigenvalue of A_k. The arising smoothing property is equivalent to the fact that the smoother converges better than a Richardson method with some relaxation parameter $\omega \in (0, 2)$.

The λ_k- approximation property can be derived for H^2-regular problems discretized by conforming finite elements on quasi-uniform triangular meshes with bounded growth of the mesh size as k tends to zero [BP87],[BD81].

4.3. The D_k- approximation and smoothing assumptions. Let $E_k = D_k = \operatorname{diag} A_k$. The arising smoothing- and approximation assumptions have been used by Ruge and Stüben [RS87].

The D_k- smoothing assumption holds uniformly for a reasonable class of matrices.

THEOREM 4.1 *Let A_k be a symmetric M-matrix. Then for $E_k = D_k$, (4.1) holds with $\sigma = 1$ for the Gauß-Seidel iteration and with $\sigma = \frac{3}{\rho(D^{-1}A)}$ for the Jacobi iteration with parameter $\omega = \frac{3}{2\rho(D^{-1}A)}$.*

Proof. See [RS87]. □

In [RS87], the D_k- approximation assumption is subject of numerical estimates which are used to develop the coarsening strategy for AMG. Our objective is its derivation from a strengthened Cauchy inequality which is the contents of the next section.

5. A DERIVATION OF THE D_k- APPROXIMATION ASSUMPTION BY A STRENGTHENED CAUCHY INEQUALITY

Let k from $1 \ldots j$ be given and $\mathcal{M} = \mathcal{M}_k$. In the following calculations all subscripts will be omitted.

5.1. The approximation factor of T^G with respect to the approximate harmonical basis. The following theorem gives the D-approximation factor of T^G in terms of the spectral equivalence between D_{11} and A_{11} in $\mathcal{M}^{(1)}$ and the strengthened Cauchy inequality in the approximate harmonical basis.

THEOREM 5.1 *Assume*

$$D_{11} \leq \omega A_{11} \tag{5.1}$$

and let γ_U be the cosine of the angle between the subspaces $U\mathcal{M}^{(1)}$ and $U\mathcal{M}^{(2)}$ in the A- energy scalar product. Then the approximation factor of the Galerkin coarse grid corrector T can be estimated by

$$\beta \leq \frac{\omega}{(1-\gamma_U)^2}. \tag{5.2}$$

Proof. From theorem A.2 it follows that

$$\begin{pmatrix} A_{11[F]} & 0 \\ 0 & S \end{pmatrix} \leq \frac{1}{1-\gamma_U} A_{[U]}.$$

Further,

$$0 \leq A_{11[F]} - \Delta S^{-1} \Delta^T \leq A_{11[F]}$$

as a Schur complement of $A_{[U]}$. It has to be shown that

$$(T^G)^T A T^G \leq \beta A D^{-1} A$$

which is equivalent to

$$(T^G_{[U]})^T A_{[U]} T^G_{[U]} \leq \beta A_{[U]} (D^{-1})_{[U-T]} A_{[U]}.$$

One has

$$\begin{aligned}
(T^G_{[U]})^T A_{[U]} T^G_{[U]} &= \begin{pmatrix} A_{11[F]} - \Delta S^{-1} \Delta^T & 0 \\ 0 & 0 \end{pmatrix} \\
&\leq \begin{pmatrix} A_{11[F]} & 0 \\ 0 & 0 \end{pmatrix} \\
&\leq \omega \begin{pmatrix} A_{11[F]} F D_1^{-1} F^T A_{11[F]} & 0 \\ 0 & 0 \end{pmatrix} \\
&= \omega \begin{pmatrix} A_{11[F]} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} F D_1^{-1} F^T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11[F]} & 0 \\ 0 & S \end{pmatrix} \\
&\leq \frac{\omega}{(1-\gamma_U)^2} A_{[U]} \begin{pmatrix} F D_1^{-1} F^T & 0 \\ 0 & 0 \end{pmatrix} A_{[U]} \\
&= \frac{\omega}{(1-\gamma_U)^2} A_{[U]} U \begin{pmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T A_{[U]} \\
&\leq \frac{\omega}{(1-\gamma_U)^2} A_{[U]} U \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} U^T A_{[U]} \\
&= \frac{\omega}{(1-\gamma_U)^2} A_{[U]} (D^{-1})_{[U-T]} A_{[U]}.
\end{aligned}$$

□

COROLLARY 5.1 *The approximation factor of* $\hat{T}$ *is* $\rho(A_{11}^{-1} D_{11})$.
Proof. Indeed, $\gamma_{\hat{U}} = 0$. □

5.2. The approximation factor of T^G with respect to the nodal basis. The following theorem in the case of $G = A_{12}$ establishes a connection between γ_U and γ, the cosine of the angle between $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ in the A- energy scalar product which can be calculated locally on each coarse grid cell [Sch86]. They are connected based on a factor which expresses how close F is to A_{11}.
THEOREM 5.2 *Assume* $G = A_{12}$, *so that*

$$U = \begin{pmatrix} F & A_{12} \\ 0 & I \end{pmatrix}$$

Let

$$\delta_F = \left\| I - F^{-1} A_{11} \right\|_{A_{11}} \tag{5.3}$$

and

$$\gamma = \sup_{0 \neq u_1 \in \mathcal{M}^{(1)}, 0 \neq u_2 \in \mathcal{M}^{(2)}} \frac{|(Au_1, u_2)|}{\|u_1\|_A \|u_2\|_A}.$$

Then

$$\gamma_U^2 \leq \frac{1}{1 + \frac{1-\gamma^2}{\delta_F^2 \gamma^2}} \leq 1. \tag{5.4}$$

Proof. Equation (5.3) means that

$$(I - A_{11}F^{-T})A_{11}(I - F^{-1}A_{11}) \leq \delta_F^2 A_{11}.$$

This is equivalent to

$$(I - F^{-T}A_{11})A_{11}^{-1}(I - A_{11}F^{-1}) \leq \delta_F^2 A_{11}^{-1}$$

and implies

$$A_{21}(I - F^{-T}A_{11})A_{11}^{-1}(I - A_{11}F^{-1})A_{12} \leq \delta_F^2 A_{21}A_{11}^{-1}A_{12}$$

which by the definition of Δ and $G = A_{12}$ leads to

$$\begin{aligned} \Delta^T A_{11}^{-1}\Delta &\leq \delta_F^2 A_{21}A_{11}^{-1}A_{12} \\ &\leq \delta_F^2 \gamma^2 A_{22} \\ &\leq \delta_F^2 \frac{\gamma^2}{1-\gamma^2}\hat{S}, \end{aligned}$$

using theorem A.2 twice. The following is essentially lemma 4.2 from [HLM91a]. Let

$$\mu = \delta_F^2 \frac{\gamma^2}{1-\gamma^2}$$

Then adding in the last inequality $\mu\Delta^T A_{11}^{-1}\Delta$ at both sides yields

$$(1+\mu)\Delta^T A_{11}^{-1}\Delta \leq \mu(\hat{S} + \Delta^T A_{11}^{-1}\Delta)$$

which by the definition of S is equivalent to

$$\Delta^T A_{11}^{-1}\Delta \leq \frac{\mu}{1+\mu}S$$

Theorem A.2 gives then that

$$\gamma_U^2 \leq \frac{\mu}{1+\mu}.$$

Inserting the definition of μ yields the desired result. □

COROLLARY 5.2 *If $\delta_F \leq 1$, then $\gamma_U \leq \gamma$.*

□

COROLLARY 5.3 *Under the conditions of theorem 5.2,*

$$\beta \leq \omega \frac{(1-\gamma^2) + \delta^2\gamma^2}{(1-\gamma^2) + \delta\gamma\sqrt{(1-\gamma^2) + \delta^2\gamma^2}}$$

□

Again, the case of $F = A_{11}$ yields $\beta \leq \omega$.

6. MULTIGRID METHODS FOR OPERATORS SYMMETRIZABLE BY DIAGONAL TRANSFORMATION

An important class of partial differential equations are the convection-diffusion equations. If the convection vector field is independent of the solution and is a gradient

of some potential, there exists a variable transformation which results in a selfadjoint elliptic boundary value problem. Box discretizations with exponential fitting and some kinds of mixed finite element methods carry over this property to the discretized equation ([BMP87] [FG91b] [FG91a]). This fact leads to a recipe to design iterative methods for the class of convection - diffusion equations with convection potential based on the theory of preconditioned iterative methods for selfadjoint problems [Gär90]. Especially, this can be carried out for multigrid methods within the framework described above.

DEFINITION 6.1 *An operator $A : \mathcal{M} \to \mathcal{M}^*$ is called symmetrizable (by right diagonal transformation) if there exists a diagonal matrix E such that $A = CE$ with C symmetric and positive definite in the Euclidian scalar product.*

Sometimes it is not possible to do calculations in the transformed variables because of the bad condition number of C. So, introduce the scalar product

$$((u, v)) = (Eu, v)\,.$$

LEMMA 6.1 *If A is symmetrizable in the sense of the previous definition, then A is selfadjoint with respect to the scalar product $((\cdot,\cdot))$.*

Proof.

$$((u, Av)) = (Eu, CEv) = (ECEu, v) = ((Au, v))\,.$$

□

Now one can construct multigrid preconditioners as above for symmetrizable operators. For this, one has to mention that the scalar products $(\cdot,\cdot)_k$ can be understood as "parameters" of the multigrid method. So assume to be given E_j in $\mathcal{M}_j$, define recursively

$$E_{k-1} = e_k^T E_k e_k$$

and substitute $(\cdot,\cdot)_k$ by $(E_k\cdot,\cdot)_k = ((\cdot,\cdot))_k$ whenever it occurs. This, especially, affects the symmetry of the operators, which is replaced by the selfadjointness with respect to the new scalar products. So, for instance, the exact coarse grid correction will look as

$$\hat{T}_k = I - E_k^{-1}\hat{U}_{C,k}^{-1}E_k \begin{pmatrix} 0 & 0 \\ 0 & E_{k-1}^{-1}\hat{S}_{C,k-1}^{-1} \end{pmatrix} \hat{U}_{C,k}^{-T}A_k,$$

where $\hat{U}_{C,k}$ and $\hat{S}_{C,k-1}$ are derived from $C_k = A_k E_k^{-1}$ in the same manner as in (2.5). One has to mention that $E_k^{-1}\hat{U}_k^{-1}E_k$ and $\hat{U}^{-T}$ are adjoint one to another with respect to the scalar product $((\cdot,\cdot))_k$ and that the coarse grid operator $\hat{S}_{k-1}E_{k-1}$ is of the same type as A_k, that is, selfadjoint in the E_{k-1}-scalar product.

The same procedure can be used to derive a multigrid method for symmetrizable problems from a given one for symmetric problems in the case, when $U_{C,k} \neq \hat{U}_{C,k}$

This leads to the fact that all results of the previous sections carry over to this case, if the basic assumptions hold with respect to the new scalar product.

Multigrid algorithms based on coarse grid corrections of this type can be used solely or as preconditioners in a conjugate gradient method in the E_j-scalar product. Both possibilities have been successfully tested by the author. In section 8 numerical experiments show exactly the same behaviour of the base method for the symmetric problem and the derived method for the nonsymmetric one.

7. Algebraically defined multigrid on rectangular grids

In the following, the indices k are omitted again. The matrix A is assumed to be a symmetric, irreducibly diagonally dominant with nonnegative main diagonal entries and nonpositive off diagonal entries. Then A is a Stieltjes matrix, and thus positive definite, so it meets our basic demands. The terminus "grid" will mean the graph $\Gamma(A) = (V(A), E(A))$ of this matrix in the sense of [Var62] which can be assumed to be non-directed, as A is symmetric.

What follows, are examples of multigrid methods in the two- and three- dimensional case will be given which use 5-point or 7-point stencils on each grid and fit into the framework described in the previous sections. The following suggestion, which should be regarded in a another paper, is assumed in all cases: ω and γ from section 5 can be estimated independently of k, locally on each coarse grid cell if the grids are not too coarse. In [Sch86] and [Szi91] ways are proposed to verify this.

In section 8, the performance of these methods is described.

7.1. The matrix partition in the two-dimensional case. For a two- dimensional, rectangular grid, one has a partition of the vertex set

$$V(A) = V_F \cup V_E \cup V_N$$

into the sets of coarse grid face midpoints, coarse grid edge midpoints and coarse grid node points, respectively. This in the 5-point case implies the matrix partitioning

$$\begin{aligned} A &= \begin{pmatrix} A_F & B_{FE} & 0 \\ B_{EF} & A_E & B_{EN} \\ 0 & B_{NE} & A_N \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} A_F & B_{FE} \\ B_{EF} & A_E \end{pmatrix} & \begin{pmatrix} 0 \\ B_{EN} \end{pmatrix} \\ \begin{pmatrix} 0 & B_{NE} \end{pmatrix} & A_N \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned}$$

where the off diagonal blocks are (element by element) nonpositive and the diagonal blocks

$$\begin{aligned} A_F &= A_{FE} + M_F \\ A_E &= A_{EF} + A_{EN} + M_E \\ A_N &= A_{NE} + M_N \end{aligned}$$

are positive diagonal matrices which consist of the sum of the off diagonal row entries and a nonnegative "mass" term. The assumptions made on A imply that at least one entry of the M_* is positive.

Under these conditions, the operator $e_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ is the straight injection of the coarse grid node values corresponding to V_N.

7.1.1. Multigrid: method A_{2D}. Choose

$$F = \begin{pmatrix} A_F & B_{FE} \\ 0 & A_E \end{pmatrix} \quad \text{and} \quad G = A_{12}.$$

Then $\delta_F < 1$ because $A_{11} = F - (A_{11} - F)$ is a regular splitting. One has after some calculations using (2.6)

$$I_k = \begin{pmatrix} -F^{-1}G \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A_F^{-1}B_{FE}A_E^{-1}B_{EN} \\ -A_E^{-1}B_{EN} \end{pmatrix} \\ I \end{pmatrix}$$

which results in a three stage prolongation: First perform a straight injection into the coarse grid nodes, then by means of $-A_E^{-1}B_{EN}$ interpolate the edge midpoint values, and at last, use $-A_F^{-1}B_{FE}$ to get the face midpoint values from the edge midpoint values. The Galerkin coarse grid operator after (2.4) is

$$\begin{aligned} S &= A_N - 2B_{NE}A_E^{-1}B_{EN} + B_{NE}A_E^{-1}(A_E - B_{EF}A_F^{-1}B_{FE})A_E^{-1}B_{EN} \\ &= A_N - B_{NE}A_E^{-1}B_{EN} - B_{NE}A_E^{-1}B_{EF}A_F^{-1}B_{FE}A_E^{-1}B_{EN} \\ &\leq A_N - B_{NE}A_E^{-1}B_{EN} \end{aligned}$$

where the last inequality is fulfilled because of $A_F \geq 0$. So one can take

$$A_{k-1} = A_N - B_{NE}A_E^{-1}B_{EN}$$

as the actual coarse grid operator. It is the Schur complement of the positive definite matrix $\begin{pmatrix} A_E & B_{EN} \\ B_{NE} & A_N \end{pmatrix}$ and has the same properties as A. The perturbed Galerkin property (3.2) in one direction is fulfilled, so that the whole coarse grid correction operator is nonnegative definite. The lower spectral equivalence constant χ seems to be very low, so that the algorithm should converge, but slowly.

7.1.2. Multigrid: method B_{2D}. The following method is exactly the two-dimensional case of the method described in [FG91b],[FG91a]. Let

$$\tilde{A}_E = A_{EN} + M_E$$

Choose

$$F = \begin{pmatrix} A_F & B_{FE} \\ 0 & \tilde{A}_E \end{pmatrix} \quad \text{and} \quad G = A_{12}.$$

One has after some calculations using (2.6)

$$I_k = \begin{pmatrix} -F^{-1}G \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A_F^{-1}B_{FE}\tilde{A}_E^{-1}B_{EN} \\ -\tilde{A}_E^{-1}B_{EN} \end{pmatrix} \\ I \end{pmatrix}$$

which results in a similar three stage prolongation as in method A_{2D}: First perform a straight injection into the coarse grid nodes, then by means of $-\tilde{A}_E^{-1}B_{EN}$ interpolate the edge midpoint values, and at last, use $-A_F^{-1}B_{FE}$ to get the face midpoint values from the edge midpoint values. The Galerkin coarse grid operator after (2.4) is

$$\begin{aligned} S &= A_N - 2B_{NE}\tilde{A}_E^{-1}B_{EN} + B_{NE}\tilde{A}_E^{-1}(A_E - B_{EF}A_F^{-1}B_{FE})\tilde{A}_E^{-1}B_{EN} \\ &= A_N - B_{NE}\tilde{A}_E^{-1}B_{EN} + B_{NE}\tilde{A}_E^{-1}(A_{EF} - B_{EF}A_F^{-1}B_{FE})\tilde{A}_E^{-1}B_{EN}. \end{aligned}$$

The last term in the second equation is the projection of a "skew Laplacian" with varying coefficients residing on the coarse grid edge midpoints onto the coarse grid. So the suggestion

$$B_{NE}\tilde{A}_E^{-1}(A_{EF} - B_{EF}A_F^{-1}B_{FE})\tilde{A}_E^{-1}B_{EN} \approx A_N - B_{NE}\tilde{A}_E^{-1}B_{EN}$$

for coefficients varying not too strongly is not too far away. So one might take

$$A_{k-1} = \kappa(A_N - B_{NE} A_E^{-1} B_{EN})$$

with $\kappa = 2$ as the actual coarse grid operator. It is twice the Schur complement of the positive definite matrix $\begin{pmatrix} A_{EN} + M_E & B_{EN} \\ B_{NE} & A_N \end{pmatrix}$ and has the same properties as A. The spectral bounds of A_{k-1} with respect to S_{k-1} seem to be better than for algorithm A_{2D}. This suggestion and the evaluation of δ_F are outstanding. The experiments show a very good convergence rates for this algorithm.

7.2. The matrix partition in the three-dimensional case. For a three-dimensional, rectangular grid, one has a partition of the vertex set

$$V(A) = V_C \cup V_F \cup V_E \cup V_N$$

into the sets of coarse grid cell midpoints, coarse grid cell face midpoints, coarse grid cell edge midpoints and coarse grid node points, respectively. This in the 7-point case implies the matrix partitioning

$$\begin{aligned} A &= \begin{pmatrix} A_C & B_{CF} & 0 & 0 \\ B_{FC} & A_F & B_{FE} & 0 \\ 0 & B_{EF} & A_E & B_{EN} \\ 0 & 0 & B_{NE} & A_N \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} A_C & B_{CF} & 0 \\ B_{FC} & A_F & B_{FE} \\ 0 & B_{EF} & A_E \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ B_{EN} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & B_{NE} \end{pmatrix} & A_N \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned}$$

where the off diagonal blocks are (element by element) nonpositive and the diagonal blocks

$$\begin{aligned} A_C &= A_{CF} + M_C \\ A_F &= A_{FC} + A_{FE} + M_F \\ A_E &= A_{EF} + A_{EN} + M_E \\ A_N &= A_{NE} + M_N \end{aligned}$$

are positive diagonal matrices which consist of the sum of the off diagonal row entries and a nonnegative "mass" term. The assumptions made on A imply that at least one entry of the M_* is positive.

Then the operator $e_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ is the straight injection of the coarse grid node values.

7.2.1. Multigrid: method A_{3D}. Choose

$$F = \begin{pmatrix} A_C & B_{CF} & 0 \\ 0 & A_F & B_{FE} \\ 0 & 0 & A_E \end{pmatrix} \quad \text{and} \quad G = A_{12}.$$

Then $\delta_F < 1$ because $A_{11} = F - (A_{11} - F)$ is a regular splitting. One has after some calculations using (2.6)

$$I_k = \begin{pmatrix} -F^{-1}G \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -A_C^{-1}B_{CF}A_F^{-1}B_{FE}A_E^{-1}B_{EN} \\ A_F^{-1}B_{FE}A_E^{-1}B_{EN} \\ -A_E^{-1}B_{EN} \end{pmatrix} \\ I \end{pmatrix}$$

which results in a four stage prolongation: First perform a straight injection into the coarse grid nodes, then by means of $-A_E^{-1}B_{EN}$ interpolate the edge midpoint values, then, use $-A_F^{-1}B_{FE}$ to get the face midpoint values from the edge midpoint values and, at last, take $-A_C^{-1}B_{CF}$ to get the cell midpoint values. Let

$$S_F = A_F - B_{FC}A_F^{-1}B_{CF}$$

The Galerkin coarse grid operator after (2.4) is then

$$\begin{aligned} S &= A_N - 2B_{NE}A_E^{-1}B_{EN} + B_{NE}A_E^{-1}(A_E - B_{EF}S_F^{-1}B_{FE})A_E^{-1}B_{EN} \\ &= A_N - B_{NE}A_E^{-1}B_{EN} - B_{NE}A_E^{-1}B_{EF}S_F^{-1}B_{FE}A_E^{-1}B_{EN} \\ &\leq A_N - B_{NE}A_E^{-1}B_{EN} \end{aligned}$$

where the last inequality is fulfilled because of $S_F \geq 0$ as a Schur complement of the nonnegative definite matrix $\begin{pmatrix} A_C & B_{CF} \\ B_{FC} & A_F \end{pmatrix}$. So one can take

$$A_{k-1} = A_N - B_{NE}A_E^{-1}B_{EN}$$

as the actual coarse grid operator. It is the Schur complement of the positive definite matrix $\begin{pmatrix} A_E & B_{EN} \\ B_{NE} & A_N \end{pmatrix}$ and has the same properties as A. The perturbed Galerkin property (3.2) in one direction is fulfilled, so that the whole coarse grid correction operator is nonnegative definite and the algorithm should converge. The lower spectral equivalence constant χ, however, seems to be very low, so that the convergence rate may be not very good. This conjecture is covered by the results of the experiments.

7.2.2. Multigrid: method B_{3D}. The following method is exactly the three-dimensional case of the method described in [FG91b],[FG91a]. Let

$$\begin{aligned} \tilde{A}_F &= A_{FE} + M_F, \\ \tilde{A}_E &= A_{EN} + M_E. \end{aligned}$$

Choose

$$F = \begin{pmatrix} A_C & B_{CF} & 0 \\ 0 & \tilde{A}_F & B_{FE} \\ 0 & 0 & \tilde{A}_E \end{pmatrix} \quad \text{and} \quad G = A_{12}.$$

One has after some calculations using (2.6)

$$I_k = \begin{pmatrix} -F^{-1}G \\ I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -A_C^{-1}B_{CF}\tilde{A}_F^{-1}B_{FE}\tilde{A}_E^{-1}B_{EN} \\ \tilde{A}_F^{-1}B_{FE}\tilde{A}_E^{-1}B_{EN} \\ -\tilde{A}_E^{-1}B_{EN} \end{pmatrix} \\ I \end{pmatrix}$$

which results in a similar four stage prolongation as in method A_{3D}: First perform a straight injection into the coarse grid nodes, then by means of $-\tilde{A}_E^{-1}B_{EN}$ interpolate the edge midpoint values, then, use $-\tilde{A}_F^{-1}B_{FE}$ to get the face midpoint values from the edge midpoint values and, at last, take $-A_C^{-1}B_{CF}$ to get the cell midpoint values. Let

$$\begin{aligned}\tilde{S}_E &= A_E - 2B_{EF}\tilde{A}_F^{-1}B_{FE} + B_{EF}\tilde{A}_F^{-1}(A_F - B_{FC}A_C^{-1}B_{CF})\tilde{A}_F^{-1}B_{FE} \\ &= A_E - B_{EF}\tilde{A}_F^{-1}B_{FE} + B_{EF}\tilde{A}_F^{-1}(A_{FC} - B_{FC}A_C^{-1}B_{CF})\tilde{A}_F^{-1}B_{FE}\end{aligned}$$

The Galerkin coarse grid operator after (2.4) is then

$$\begin{aligned}S &= A_N - 2B_{NE}\tilde{A}_E^{-1}B_{EN} + B_{NE}\tilde{A}_E^{-1}\tilde{S}_E\tilde{A}_E^{-1}B_{EN} \\ &= (A_N - B_{NE}\tilde{A}_E^{-1}B_{EN}) + B_{NE}A_E^{-1}(A_{EF} - B_{EF}\tilde{A}_F^{-1}B_{FE})\tilde{A}_E^{-1}B_{EN} \\ &\quad + B_{NE}A_E^{-1}B_{EF}\tilde{A}_F^{-1}(A_{FC} - B_{FC}A_C^{-1}B_{CF})\tilde{A}_F^{-1}B_{FE}\tilde{A}_E^{-1}B_{EN}.\end{aligned}$$

The second term in the second equation is the projection of some "skew Laplacians" residing on the coarse grid hyperplanes which thus possibly can be estimates by $2(A_N - B_{NE}\tilde{A}_E^{-1}B_{EN})$, the third term resides on the coarse grid face midpoints and should be smaller than $(A_N - B_{NE}\tilde{A}_E^{-1}B_{EN})$ This corresponds to the numerical experiments which suggest that

$$S \approx 4(A_N - B_{NE}\tilde{A}_E^{-1}B_{EN})$$

for coefficients varying not too strongly. So one might take

$$A_{k-1} = \kappa(A_N - B_{NE}A_E^{-1}B_{EN})$$

with $\kappa = 4$ as the actual coarse grid operator. It is four times the Schur complement of the positive definite matrix $\begin{pmatrix} A_{EN} + M_E & B_{EN} \\ B_{NE} & A_N \end{pmatrix}$ and has the same properties as A. The spectral bounds of A_{k-1} with respect to S_{k-1} seem to be better then for algorithm A_{3D}. This suggestion and the evaluation of δ_F are outstanding. The experiments show nearly the same convergence rates for this algorithm as for algorithm B_{2D}.

8. Numerical experiments

The algorithms described in the previous section have been implemented within a multigrid program written in ANSI-C which works under UNIX, VMS and MS-DOS systems. It is capable to solve both symmetric and symmetrizable in the sense of section 6 problems in two- or threedimensional rectangular domains using multigrid solely or as a preconditioner within conjugate gradient or GMRES methods. According to the described algorithm, the generation of the coarse grid problems and the intergrid transfer operators are performed in an algebraic manner, so that that only the size of the finest grid and the matrix diagonals have to be given as input. This, especially, implies that the fine grids needn't to be the results of any coarse grid refinement by stepsize halving. This way, curvilinear discretizations resulting in 5- or 7-point stencils, are covered, too. The method uses ILU smoothing and preconditioned conjugate gradients as a coarse grid solver.

All calculations have been done on a CONVEX-C220 with one processor. In all the tables, the contraction stated is the average contraction per step or per work unit which results from iterating until a residual contraction of 10^{-10}, or a residual explosion of $1/\epsilon_{mach}$ or maximally 2000 work units, where one work unit is the time for one vector jacobi step which had a speed of about 9.5 MFLOPS.

8.1. A test problem. The following problem will be considered here in a symmetric and a nonsymmetric version: Let $\Omega = [0,1] \times [0,1] \times [0.1]$. Let $\psi : \Omega \to [-1, 10]$ be a given potential function as shown in figure 1. The symmetric version of the prob-

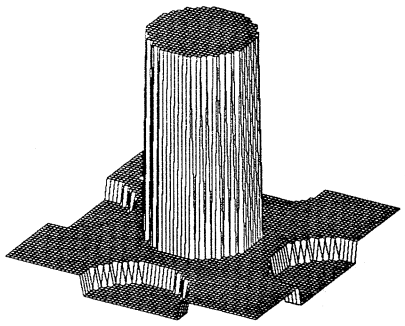

FIG. 1. *ψ for $x = 0.5$, $y = 0.5$ or $z = 0.5$, respectively*

lem is

$$-\nabla e^{\psi} \nabla u = f. \tag{8.1}$$

The nonsymmetric version of the problem is

$$-\nabla(\nabla u - u \nabla \psi) = f. \tag{8.2}$$

The following Dirichlet boundary conditions are given:

$$\begin{aligned} u &= 100 \quad \text{on} \quad 0 \times [0,1] \times [0,1] \\ u &= 0 \quad \text{on} \quad 1 \times [0,1] \times [0,1]. \end{aligned}$$

On all other parts of the boundary, homogeneous Neumann boundary conditions are assumed. The twodimensional test problem consists of the data for $z = 0.5$. Its solutions are shown in figure 2.

Problem (8.2) occurs in the field of semiconductor device simulation as the electron transport equation Its discretization is carried out by a Scharfetter - Gummel exponential fitting scheme using a box method. The discretization of problem (8.1) is exactly the discrete symmetrization according to section (6). It results in a harmonical averaging of the diffusion coefficient along the grid lines assuming ψ to be linear. This method can be interpreted as a mixed finite element scheme using piecewise bilinear (trilinear) u, piecewise constant fluxes on domains along the element edges for u, and piecewise bilinear (trilinear) ψ together with a special quadrature rule for integrals over the flux domains [FG91b]. The conjugate gradient method for the nonsymmetric problems uses the scalar product from section 6.

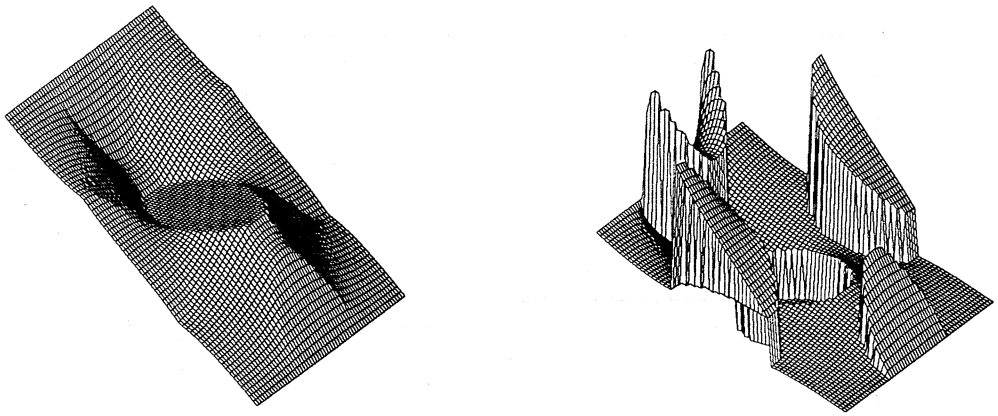

FIG. 2. *Solutions of problem (8.1) and problem (8.2) for* $z = 0.5$

The performance of the multigrid methods is shown in table 1 and table 2. The grid size means the number of grid lines in each direction. The coarse grid size is 8 in all cases. The suggestions of the previous section about the bad performance of

grid size	A_{2D}		B_{2D}		A_{3D}		B_{3D}	
	symm.	nons.	symm.	nons.	symm.	nons.	symm.	nons.
16	0.978	0.975	0.087	0.088	0.988	0.988	0.086	0.085
32	0.975	0.974	0.086	0.083	0.984	0.984	0.094	0.093
64	0.972	0.970	0.086	0.084	0.979	0.979	0.114	0.113

TABLE 1

Average contraction per iteration step for the multigrid iteration to solve problems (8.1) and (8.2)

grid size	A_{2D}		B_{2D}		A_{3D}		B_{3D}	
	symm.	nons.	symm.	nons.	symm.	nons.	symm.	nons.
16	0.270	0.278	0.040	0.049	0.581	0.579	0.052	0.049
32	0.511	0.532	0.047	0.046	0.757	0.759	0.049	0.052
64	0.970	0.708	0.065	0.064	0.875	0.877	0.081	0.080

TABLE 2

Average contraction per iteration step for the multigrid preconditioned conjugate gradient iteration to solve problem (8.1) and (8.2)

the A-methods and of the good performance of the B- methods are verified.

Figure 3 shows that no scaling of the coarse grid operator seems to be able to correct the bad performance of the A- methods. Figure 4 verifies the conjectures from the previous sections about the values of $\kappa = 2$ in the twodimensional case and $\kappa = 4$ in the threedimensional case for both preconditioned simple and conjugate gradient iteration.

8.2. The IAAS logo. Another interesting test problem which shows the limits of the proposed methods has been the design of the logo of the Institut für Angewandte Analysis und Stochastik which has been founded based on the applied departments of the former Karl-Weierstraß-Institut für Mathematik.

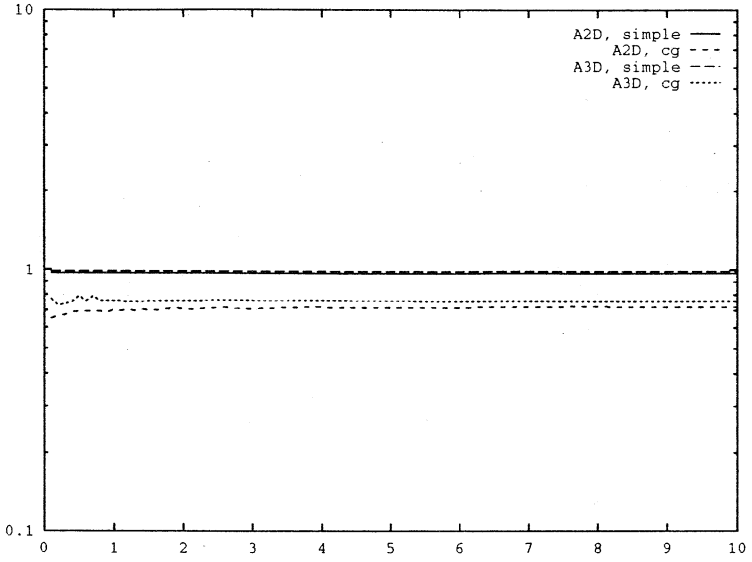

FIG. 3. *Average contraction per step in dependence of the coarse grid operator scaling by κ for methods A_{2D} and A_{3D}*

So consider problem 8.1 in two dimensions with homogeneous Dirichlet boundary conditions. Let $\Omega = [0, 200] \times [0, 100]$ Choose the potential function $\psi : \Omega \to [1, 10]$ according to figure 5. The right hand side $f : \Omega \to [-3.5 \cdot 10^2, 12.2 \cdot 10^4]$ (see figure 5) has been experimentally chosen so, that the heights of the letters are nearly the same. Behind this, an interesting parameter identification problem is hidden. The solution (see figure 6) is obtained on a 48×64 grid. The performance of different iteration schemes to solve the problem one finds in table 3. It comes out that the

preconditioner	simple iteration		conjugate gradients	
	cont/step	cont/WU	cont/step	cont/WU
mg,W-cycle, 4 coarse levels	$2.118 \cdot 10^{10}$	1.408	1.0067	1.00009
mg,W-cycle, 3 coarse levels	$4.395 \cdot 10^4$	1.112	1.001	1.00002
mg,W-cycle, 2 coarse levels	$5.120 \cdot 10^4$	1.08	1.004	1.00003
mg,W-cycle, 1 coarse level	0.422	0.9984	0.154	0.996
mg,V-cycle, 4 coarse levels	$2.167 \cdot 10^2$	1.358	0.397	0.951
mg,V-cycle, 3 coarse levels	$2.146 \cdot 10^2$	1.25	0.431	0.967
mg,V-cycle, 2 coarse levels	$2.167 \cdot 10^2$	1.12	0.456	0.984
mg,V-cycle, 1 coarse level	0.422	0.997	0.154	0.994
ilu	0.9995	0.9998	0.856	0.973

TABLE 3

Average contraction per iteration step and per work unit (1 vector jacobi) for different iteration schemes for the solution of the IAAS logo problem

fastest method are conjugate gradients preconditioned with multigrid V cycles with the maximum number of coarse levels. This multigrid method doesn't converge solely, an example for the theoretical prediction in [BPX91] that the violation of $S_k \leq A_k$

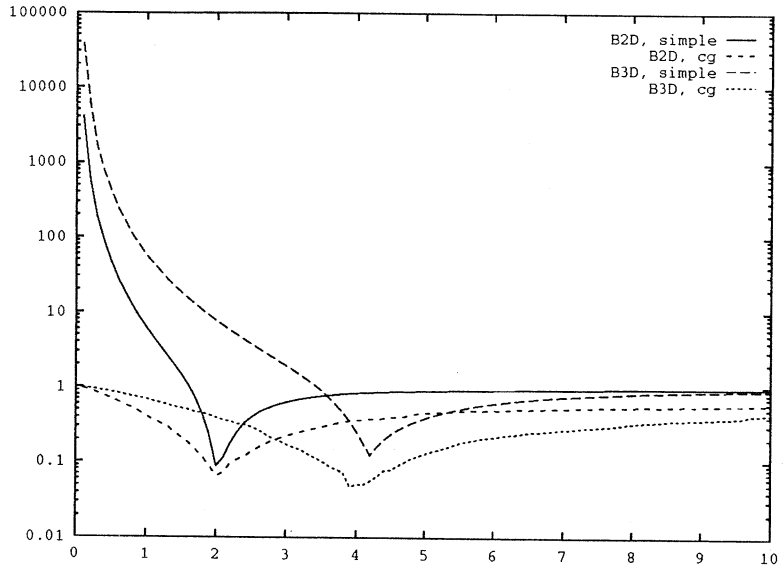

FIG. 4. *Average contraction per step in dependence of the coarse grid operator scaling by κ for methods B_{2D} and B_{3D}*

on the coarse grids need not to destroy the preconditioning properties.

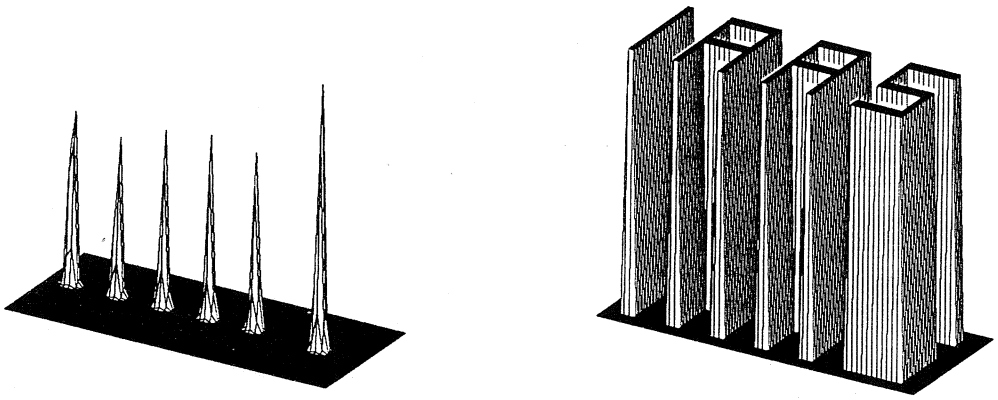

FIG. 5. *The right hand side and the diffusion coefficient for the IAAS logo*

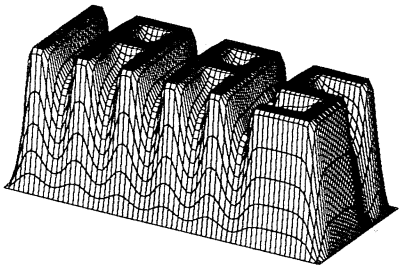

FIG. 6. *The IAAS logo*

A. Appendix: Basic results and notations

In the appendix there will be listed well known results and definitions which are used in this paper. Let $\mathcal{M}$ be a finite dimensional Euclidean vector space with the scalar product $(\cdot,\cdot)$ and the basis $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix}$.

A.1. Behaviour of vectors under base change. Let $U \in \mathrm{GL}(\mathcal{M})$ be any nonsingular matrix. Then to U corresponds a basis transformation which consists in expressing the vectors of the old basis through the vectors $\begin{pmatrix} f_1^U & f_2^U & \cdots & f_n^U \end{pmatrix}$ of the new one as the columns of U :

$$\begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} = \begin{pmatrix} f_1^U & f_2^U & \cdots & f_n^U \end{pmatrix} U.$$

Then for any

$$x = \sum_{i=1}^{n} e_i u_i = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} u \in \mathcal{M}$$

one has

$$x = \begin{pmatrix} f_1^U & f_2^U & \cdots & f_n^U \end{pmatrix} Uu = \begin{pmatrix} f_1^U & f_2^U & \cdots & f_n^U \end{pmatrix} u_{[U]},$$

A.2. Behaviour of matrices under base change. Let A be the symmetric positive semidefinite operator corresponding to an energy pairing $\mathrm{a}(\cdot,\cdot)$ in the basis $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix}$. Then it describes a linear mapping $A : \mathcal{M} \to \mathcal{M}^*$ into the space dual to $\mathcal{M}$. The representation $A_{[U]}$ of A in the basis defined by $U \in \mathrm{GL}(\mathcal{M})$ behaves as

$$\begin{aligned} A_{[U]} &= U^{-T} A U^{-1}, \\ (A_{[U]})^{-1} &= (A^{-1})_{[U^{-T}]}. \end{aligned}$$

Indeed,

$$\mathrm{a}(x,x) = (Au,u) = \left(U^{-T} A U^{-1} Uu, Uu\right) = \left(A_{[U]} u_{[U]}, u_{[U]}\right),$$

Let $X : \mathcal{M} \to \mathcal{M}$ be a linear mapping into $\mathcal{M}$ itself. Then one has

$$X_{[U]} := U X U^{-1},$$

This has to be used for operators of the type $X = I - B^{-1}A$, both A and B symmetric and positive definite.

A.3. Invariance of matrix inequalities under base changes.

Definition A.1 *For any two symmetric matrices A,B, there shall be said $A \leq B$ if $B - A$ is positive semidefinite.*

Lemma A.1 *If $A \leq B$ then $A_{[U]} \leq B_{[U]} \quad \forall U \in \mathrm{GL}(\mathcal{M})$.*

Proof. Let $U \in \mathrm{GL}(\mathcal{M})$. Then

$$\begin{aligned} (Au,u) \leq (Bu,u) \quad \forall u \in \mathcal{M} \quad &\Rightarrow \quad \left(A_{[U]} u_{[U]}, u_{[U]}\right) \leq \left(B_{[U]} u_{[U]}, u_{[U]}\right) \quad \forall u \in \mathcal{M} \\ &\Leftrightarrow \quad \left(A_{[U]} v, v\right) \leq \left(B_{[U]} v, v\right) \quad \forall v \in \mathcal{M}. \end{aligned}$$

□

The following fact is used extensively.

LEMMA A.2 *For A symmetric, positive semidefinite and $X \in \mathrm{M}(\mathcal{M})$ being any square matrix, $X^T A X$ is positive semidefinite and*

$$\|Xu\|_A^2 = (AXu, Xu) = \left(X^T AXu, u\right).$$

□

A.4. Orthoprojectors. Let $\mathcal{M}$ be a finite dimensional Euclidean vector space with the scalar product $(\cdot,\cdot)$.

DEFINITION A.2 *(see i.e.[KM86]) Let A be symmetric and positive definite. A linear operator $P : \mathcal{M} \to \mathcal{M}$ is called an A-orthoprojector if*

- $P^2 = P$
- $P^T A = AP$, *i.e. P is selfadjoint with respect to $(A\cdot,\cdot)$.*

COROLLARY A.1 *If P is an A-orthoprojector, then $\|u\|_A^2 = \|Pu\|_A^2 + \|(I-P)u\|_A^2$ and $\|P\|_A \leq 1$.*

A.5. The contraction number of the preconditioned Richardson iteration.

THEOREM A.1 *Let A be a symmetric, positive definite operator and B be symmetric satisfying*

$$\lambda_- B \leq A \leq \lambda_+ B.$$

Then

$$\left\|I - \omega B^{-1} A\right\|_A \leq \theta(\omega) = \max\{|1-\omega\lambda_-|, |1-\omega\lambda_+|\}$$

For $\omega = \frac{2}{\lambda_- + \lambda_+}$, $\theta(\omega)$ reaches its minimum $\frac{\frac{\lambda_+}{\lambda_-}-1}{\frac{\lambda_+}{\lambda_-}+1}$

Proof. i.e. [D'j89]. □

A.6. The angle between subspaces in the energy scalar product and block diagonal preconditioning. The following theorem is a collection of well known results:

THEOREM A.2 *Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and*

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \\ D_A &= \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \\ \hat{S} &= A_{22} - A_{21} A_{11}^{-1} A_{12}. \end{aligned}$$

Then the following inequalities are valid (i.e. [D'j89],[Man90]):

$$\begin{aligned} (1-\gamma) D_A \leq\ & A \ \leq (1+\gamma) D_A, \\ (1-\gamma^2) A_{22} \leq\ & \hat{S} \ \leq A_{22}. \end{aligned}$$

Here, γ is the cosine of angle between $\mathcal{M}_1$ and $\mathcal{M}_2$ in the A-energy scalar product and can be calculated in the following ways:

$$\gamma =_{\mathrm{def}} \sup_{0 \neq u_1 \in \mathcal{M}_1, 0 \neq u_2 \in \mathcal{M}_2} \frac{|(Au_1, u_2)|}{\|u_1\|_A \|u_2\|_A}$$

$$\gamma = \sup_{0\neq u_1\in\mathcal{M}_1, 0\neq u_2\in\mathcal{M}_2} \frac{2\,(Au_1,u_2)}{\|u_1\|_A^2+\|u_2\|_A^2} \qquad \text{[Axe82]}$$

$$\gamma^2 = \sup_{0\neq u_2\in\mathcal{M}_2} \frac{\left(A_{21}A_{11}^{-1}A_{12}u_2,u_2\right)}{(A_{22}u_2,u_2)} = \rho(A_{22}^{-1}A_{21}A_{11}^{-1}A_{12}) \qquad \text{[HLM91a]}$$

□

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