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A cascadic multigrid algorithm in the finite element method for the plane elasticity problem

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Abstract

For the plane elasticity problem a standard scheme of the finite element method with the use of piecewise linear elements on triangles is discussed. For its solution on a sequence of embedded triangulations, a cascadic arrangement of two iterative algorithms is used, which gives the simplest version of multigrid methods without preconditioning and restriction onto a coarser grid. The cascadic algorithm begins on the coarsest grid where the grid problem is solved by direct method. To obtain approximate solutions on finer grids, the iterative method is used; interpolation of the approximate solution from the preceding coarser grid is taken as the initial guess. It is proved that the convergence rate of this algorithm does not depend on the number of unknowns and grids.

1. Introduction

Originally the cascadic conjugate-gradient method was presented by P.Deuflhard in the papers [1] and [2] where high convergence was shown computationally. The papers [3], [4] prove optimal computational complexity of this algorithm for elliptic equation of the second order with smooth solution. The papers [5] and [6] establish optimal complexity for the elliptic equation, the solution of which is not sufficiently smooth since the domain has angles in excess of π. The paper [7] extends the proof to the case of this problem in the domain with curvilinear boundary, where the embedded Galerkin subspaces are used without strict embedding of sequence of triangulations.

Besides, in the paper [6] F. A. Bornemann set up estimates of convergence rate of other iterative smoothing operators in two- and three-dimensional boundary-value problems for elliptic equation of the second order. In the two-dimensional case of the smoothing operators he analyzed, only the conjugate-gradient method gave optimal arithmetic complexity. From the papers [3], [4] this optimality follows also for Jacobi-type method with the Chebyshev iterative parameters. In the three-dimensional case, several more smoothing operators ensure optimal complexity.

The present paper investigates application of cascadic conjugate-gradient method and Jacobi-type one with the Chebyshev parameters to a sequence of grid problems for the system of second-order elliptic equations arising in the plane elasticity theory.

2. Formulation of the differential problem

Consider the plane elasticity problem on the bounded convex polygon $\Omega \subset R^2$ with the boundary Γ:

$$-\mu\Delta \boldsymbol{u} - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma, \tag{2.2}$$

where $\lambda, \mu > 0$ are the Lamé coefficients, $\boldsymbol{u}$ is the desired vector-function of displacement

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

f is the known vector-function of mass forces with two components

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Introduce the inner product and the norm for the vector-function from $(L_2(\Omega))^2$:

$$(\boldsymbol{u}, \boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} dx, \quad \|\boldsymbol{u}\|_{0,\Omega} = (\boldsymbol{u}, \boldsymbol{u})_\Omega^{1/2}.$$

Let

$$f_1, f_2 \in L_2(\Omega), \tag{2.3}$$

then on the base of [9] there exists an unique solution of problem (2.1)–(2.2) for which

$$u_1, u_2 \in W_2^2(\Omega). \tag{2.4}$$

Introduce norms for the vector-functions from $(W_2^m(\Omega))^2$ by the formula

$$\|\boldsymbol{u}\|_{m,\Omega} = \left(\sum_{i=1}^{2} \|u_i\|_{m,\Omega}^2 \right)^{1/2},$$

using standard notation for the norms of scalar functions $v \in W_2^m(\Omega)$:

$$\|v\|_{m,\Omega}^2 = \sum_{0 \le i+j \le m} \left\| \frac{\partial^{i+j} v}{\partial x_1^i \partial x_2^j} \right\|_{0,\Omega}^2.$$

With the constraint (2.3), problem (2.1)–(2.2) obeys the estimate

$$\|\boldsymbol{u}\|_{2,\Omega} \le c_1 \|\boldsymbol{f}\|_{0,\Omega}. \tag{2.5}$$

In accordance with [9] we formulate for (2.1)–(2.2) the generalized problem:

$$\begin{gathered} \textit{find } \boldsymbol{u} \in (\overset{\circ}{W}{}_2^1(\Omega))^2, \textit{ satisfying the equality} \\ \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \forall\, \boldsymbol{v} \in (\overset{\circ}{W}{}_2^1(\Omega))^2 \end{gathered} \tag{2.6}$$

where the bilinear form $\mathcal{L}$ is defined by the relation

$$\begin{aligned} \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Bigg\{ 2\mu \left(\frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\ + \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) + \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \Bigg\} dx. \end{aligned} \tag{2.7}$$

For $\boldsymbol{f} \in (L_2(\Omega))^2$ problem (2.6) also has an unique solution [8], [9].

3. Formulation of the discrete problem

To construct the Bubnov-Galerkin scheme, we triangulate the polygon Ω. We first divide Ω into a small number of closed triangles, in so doing the resulting triangulation $\mathcal{T}_0$ to be consistent, *i.e.*, each of the triangle couples either has a common side or has a common vertex or does not have common points. Denote the maximum side length of all triangles by h_0. Put $N_i = 2^i, h_i = h_0/N_i$. For $i = 1, \dots, l$ we divide every initial triangle into N_i^2 equal triangles. Denote the set of all vertices of the resulting consistent triangulation $\mathcal{T}_i$ by $\overline{\Omega}_i$. Introduce $\Omega_i = \overline{\Omega}_i \cap \Omega$ and denote by n_i the number of points of set Ω_i. For every node $y \in \Omega_i$ we form the basis function $\varphi_y^i \in \overset{\circ}{W}{}_2^1(\Omega)$ which equals 1 at node y, equals 0 at all the other nodes from $\overline{\Omega}_i$ and is linear on each elementary triangle of triangulation $\mathcal{T}_i$. Denote by H^i the linear span of functions $\varphi_y^i, \ \ y \in \Omega_i$.

Consider problem (2.6) on the subspace $\boldsymbol{H}^i = (H^i)^2 \in (\overset{\circ}{W}{}_2^1(\Omega))^2$. We obtain the discrete problem:

$$\begin{aligned} &\textit{find } \boldsymbol{v}_i \in \boldsymbol{H}^i, \textit{ satisfying the equality} \\ &\mathcal{L}(\boldsymbol{v}_i, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^i. \end{aligned} \tag{3.8}$$

Assume M_i to be the $2n_i$-space consisting of vectors W with n_i components $W(x) = \begin{bmatrix} W_1\ (x) \\ W_2\ (x) \end{bmatrix}$, $x \in \Omega_i$. Problem (3.8) is then equivalent to the block system of linear algebraic equations

$$L_i V_i = F_i \tag{3.9}$$

where $V_i \in M_i$ is the vector of unknowns with n_i components $V_i(x) = \begin{bmatrix} V_{i,1}\ (x) \\ V_{i,2}\ (x) \end{bmatrix}$, $x \in \Omega_i$; $F_i \in M_i$ has n_i components $F_i(x) = \begin{bmatrix} F_{i,1}\ (x) \\ F_{i,2}\ (x) \end{bmatrix} = \begin{bmatrix} (f_1,\ \varphi_x^i) \\ (f_2,\ \varphi_x^i) \end{bmatrix}$, $\quad x \in \Omega_i$;
L_i is $2n_i \times 2n_i$-block matrix with 2×2-blocks

$$L_{i,x,y} = \begin{pmatrix} \mathcal{L}\ (\varphi_x^i,\ 0;\ \varphi_y^i,\ 0) & \mathcal{L}\ (0,\ \varphi_x^i,\ \varphi_y^i,\ 0) \\ \mathcal{L}\ (\varphi_x^i,\ 0;\ 0,\ \varphi_y^i) & \mathcal{L}\ (0,\ \varphi_x^i;\ 0,\ \varphi_y^i) \end{pmatrix},\ x, y \in \Omega_i. \tag{3.10}$$

From (2.7) it is evident that matrix L_i is symmetric. Using the bilinearity of functional $\mathcal{L}$ and its positive definiteness [3], the matrix L_i can be shown to be positive definite and, hence, nonsingular.

To the vector $V \in M_i$ we associate the interpolant in $\boldsymbol{H}^i$:

$$\boldsymbol{v}(x) = \sum_{y \in \Omega_i} \begin{bmatrix} V_1 & (y) \\ V_2 & (y) \end{bmatrix} \varphi_y^i(x),\ x \in \overline{\Omega}. \tag{3.11}$$

It is obvious that

$$V(y) = \boldsymbol{v}(y),\ y \in \Omega_i.$$

Thus we determined an isomorphism between vectors $V \in M_i$ and vector-functions $\boldsymbol{v} \in \boldsymbol{H}^i$.

Introduce the energy norm for vector-functions

$$|\!|\!| \boldsymbol{v} |\!|\!|_\Omega = \mathcal{L}(\boldsymbol{v},\boldsymbol{v})^{1/2}, \;\; \boldsymbol{v} \in (\overset{\circ}{W}{}_2^1(\Omega))^2,$$

as well as the scalar product and norms of vectors

$$(V,W)_i = \sum_{x\in\Omega_i} V(x)W(x) = \sum_{x\in\Omega_i} V_1(x)W_1(x) + \sum_{x\in\Omega_i} V_2(x)W_2(x),$$

$$\|V\|_i = \left\{\sum_{x\in\Omega_i} |V(x)|^2\right\}^{1/2} = \left\{\sum_{x\in\Omega_i}(V_1(x))^2 + \sum_{x\in\Omega_i}(V_2(x))^2\right\}^{1/2},$$

$$|\!|\!| V |\!|\!|_i = (L_iV,V)_i^{1/2}, \quad V,W \in M_i.$$

Taking into account of (3.10), (3.11) and the bilinearity of functional $\mathcal{L}$, for isomorphic couple $V \in M_i$, $\boldsymbol{v} \in \boldsymbol{H}^i$ we have

$$(L_iV,V)_i = \left(\begin{bmatrix} \mathcal{L}(\sum\limits_{x\in\Omega_i} V_1(x)\varphi_x^i, \sum\limits_{x\in\Omega_i} V_2(x)\varphi_x^i;\ \varphi_y^i,\ 0) \\ \mathcal{L}(\sum\limits_{x\in\Omega_i} V_1(x)\varphi_x^i, \sum\limits_{x\in\Omega_i} V_2(x)\varphi_x^i;\ 0,\ \varphi_y^i) \end{bmatrix}, \begin{bmatrix} v^1(y) \\ v^2(y) \end{bmatrix}\right)_i$$

$$= \mathcal{L}(\sum_{x\in\Omega_i} V_1(x)\varphi_x^i, \sum_{x\in\Omega_i} V_2(x)\varphi_x^i; \sum_{y\in\Omega_i} V_1(y)\varphi_y^i, \sum_{y\in\Omega_i} V_2(y)\varphi_y^i) = \mathcal{L}(\boldsymbol{v},\boldsymbol{v}),$$

i.e.,

$$|\!|\!| V |\!|\!|_i = |\!|\!| \boldsymbol{v} |\!|\!|_\Omega. \tag{3.12}$$

When studying one elliptic equation for vectors Z of dimension n_i with elements $Z(x)$, $x \in \Omega_i$ we introduced in [4] the norm

$$\|Z\|_i = \left(\sum_{x\in\Omega_i}(Z(x))^2\right)^{1/2}.$$

It is equivalent with multiplier h_i to the norm $\|z\|_{0,\Omega}$ of interpolant z from H^i [3], *i.e.*,

$$d_1h_i\|Z\|_i \le \|z\|_{0,\Omega} \le d_2h_i\|Z\|_i.$$

From the fact that

$$\|\boldsymbol{v}\|_{0,\Omega}^2 = \sum_{j=1}^{2} |v_j|_{0,\Omega}^2,$$

we have that for the isomorphic couple $V \in M_i$ and $\boldsymbol{v} \in \boldsymbol{H}^i$ the norm $\|V\|_i$ is also equivalent to the norm $\|\boldsymbol{v}\|_{0,\Omega}$ with multiplier h_i:

$$c_2h_i\|V\|_i \le \|\boldsymbol{v}\|_{0,\Omega} \le c_3h_i\|V\|_i. \tag{3.13}$$

Introduce the prolongation operator $I_i : M_i \to M_{i+1}$ as follows. Let $x', x'' \in \Omega_i$ be two neighbouring nodes of the *ith* triangulation $\mathcal{T}_i$. The prolongation $W = I_i V,\ V \in M_i$ is then defined by the formulae

$$W(x') = V(x'), \quad W(x'') = V(x''),$$
$$W\left(\frac{x' + x''}{2}\right) = \frac{V(x') + V(x'')}{2}.$$

Note that the interpolants of vectors V and W coincide, *i.e.*, $\boldsymbol{v} = \boldsymbol{w}$. Thus the operator I_i corresponds to the identical operator on subspace $\boldsymbol{H}^i$ with respect to the isomorphism defined above.

Lemma 1. *For* $\boldsymbol{f} \in (L_2(\Omega))^2$ *the problem* (3.8) *has an unique solution. It obeys the estimate*

$$|||\boldsymbol{u} - \boldsymbol{v}_i|||_\Omega \leq c_4 h_i \|\boldsymbol{f}\|_{0,\Omega}. \tag{3.14}$$

Proof. Since $\boldsymbol{H}^i \in (\overset{\circ}{W}{}_2^1(\Omega))^2$, from (2.6) it follows that

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^i.$$

From (3.8) we have

$$\mathcal{L}(\boldsymbol{u} - \boldsymbol{v}_i, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^i.$$

Putting $\boldsymbol{v} = \boldsymbol{v}_i - \boldsymbol{w}$ where $\boldsymbol{w}$ is an arbitrary function from $\boldsymbol{H}^i$, the last relation can be recasted as

$$\mathcal{L}(\boldsymbol{u} - \boldsymbol{v}_i, \boldsymbol{v}_i - \boldsymbol{w}) = 0 \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^i.$$

Hence it follows that

$$\mathcal{L}(\boldsymbol{u} - \boldsymbol{v}_i, \boldsymbol{u} - \boldsymbol{v}_i) = \mathcal{L}(\boldsymbol{u} - \boldsymbol{v}_i, \boldsymbol{u} - \boldsymbol{w}) \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^i.$$

Using the Cauchy-Bunyakovski inequality, we have

$$|||\boldsymbol{u} - \boldsymbol{v}_i|||^2_\Omega \leq |||\boldsymbol{u} - \boldsymbol{v}_i|||_\Omega \cdot |||\boldsymbol{u} - \boldsymbol{w}|||_\Omega \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^i$$

or

$$|||\boldsymbol{u} - \boldsymbol{v}_i|||_\Omega \leq |||\boldsymbol{u} - \boldsymbol{w}|||_\Omega \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^i. \tag{3.15}$$

Show the equivalence of norms $|||\cdot|||_\Omega$ and $\|\cdot\|_{1,\Omega}$. In [8] it has been shown that the bilinear form $\mathcal{L}$ is positive definite, *i.e.*,

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{v}) \geq d_5 \|\boldsymbol{v}\|^2_{1,\Omega}$$

and bounded, *i.e.*,

$$|\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})| \leq d_6 \|\boldsymbol{v}\|_{1,\Omega} \|\boldsymbol{w}\|_{1,\Omega}.$$

It is obvious that

$$d_7\|\boldsymbol{v}\|_{1,\Omega} \leq \|\!|\boldsymbol{v}|\!\|_{\Omega} \leq d_8\|\boldsymbol{v}\|_{1,\Omega},$$

i.e., these norms are equivalent. Then from (3.15) it follows that

$$\|\!|\boldsymbol{u} - \boldsymbol{v}_i|\!\|_{\Omega} \leq d_9 \inf_{\boldsymbol{w}\in \boldsymbol{H}^i} \|\boldsymbol{u} - \boldsymbol{w}\|_{1,\Omega}. \tag{3.16}$$

In [10] it has been proved that there exist functions $w_j \in H^i$, $j = 1, 2$, which obey the estimates

$$\|u_j - w_j\|_{1,\Omega} \leq d_{10}h_i\|u_j\|_{2,\Omega}, \quad j = 1, 2,$$

where u_j, $j = 1, 2$, are the components of vector-function $\boldsymbol{u}$. Then, obviously, the estimate

$$\|\boldsymbol{u} - \boldsymbol{w}\|_{1,\Omega} \leq d_{11}h_i\|\boldsymbol{u}\|_{2,\Omega} \tag{3.17}$$

is valid, where $\boldsymbol{w} \in \boldsymbol{H}^i$ is the vector-function with the components w_j, $j = 1, 2$.

From (3.16), (3.17), and (3.15) we have

$$\|\!|\boldsymbol{u} - \boldsymbol{v}_i|\!\|_{\Omega} \leq d_{12}h_i\|\boldsymbol{u}\|_{2,\Omega} \leq d_{13}h_i\|\boldsymbol{f}\|_{0,\Omega},$$

i.e., the validity of estimate (3.14) has been proved. □

Note that the eigenvalues of matrix L_i obey the estimate

$$0 < \lambda_i^* \leq c_6 \tag{3.18}$$

where λ_i^* is the maximum eigenvalue of matrix L_i [11].

Let us sum up. On a sequence of grids Ω_i, $i = 0, \ldots, l$, we obtained the sequence of problems:

for given $F_i \in M_i$ find $V_i \in M_i$ so that

$$L_iV_i = F_i. \tag{3.19}$$

For their sequential solving, we use the cascadic iterative method and prove its convergence.

4. Formulation of the cascadic algorithm

We first formulate the cascadic algorithm with some abstract iterative process S_i (smoothing operator).

The cascadic algorithm:

1. $U_0 = L_0^{-1}F_0$;

2. *for* $i = 1, 2, \ldots, l$ *do*

begin

2.1. $W_i = I_{i-1}U_{i-1}$;

2.2. *put* $U_i = S_i(L_i, W_i, F_i)$;

end.

We consider two iterative processes as smoothing operators.

The conjugate-gradient method (m_i iterations);

procedure $S_i(L_i, W_i, F_i)$:

3. $Y_0 = W_i$; $P_0 = R_0 = F_i - L_iY_0$; $\sigma_0 = (R_0, R_0)_i$;
4. *for* $k = 1, 2, \dots, m_i$ *do*

begin

$$\begin{aligned}
&\textit{if } \sigma_{k-1} = 0 \textit{ then } Y_{m_i} = Y_{k-1} \textit{ and go to } 5;\\
&\alpha_{k-1} = \sigma_{k-1}/(P_{k-1}, L_iP_{k-1})_i;\\
&Y_k = Y_{k-1} + \alpha_{k-1}P_{k-1};\\
&R_k = R_{k-1} - \alpha_{k-1}L_iP_{k-1};\\
&\sigma_k = (r_k, r_k)_i;\ \beta_k = \sigma_k/\sigma_{k-1};\\
&P_k = R_k + \beta_kP_{k-1};
\end{aligned} \tag{4.20}$$

end;

5. *put* $S_i = Y_{m_i}$.

The Jacobi-type method (m_i iterations);

procedure $S_i(L_i, W_i, F_i)$:

3. $Y_0 = W_i$;
4. *for* $k = 1, 2, \dots, m_i$ *do*

begin

$$\tau_{k-1} = \frac{1}{\Lambda_i^*}\cos^{-2}\frac{\pi(2k-1)}{2(2m_i+1)}; \tag{4.21}$$

$$Y_k = Y_{k-1} - \tau_{k-1}(L_iY_{k-1} - F_i);$$

end;

5. *put* $S_i = Y_{m_i}$.

Here Λ_i^* is an upper estimate of eigenvalues λ of operator L_i in space $M_i : L_i\Phi = \lambda\Phi$. In Jacobi-type method this value is required to be in explicit form and so it is found subject to Gershgorin's Lemma [12] and satisfies the inequality

$$\lambda_i^* = \max_{\lambda\in Sp(L_i)} \lambda \le \Lambda_i^* \le c_1 \max_{\lambda\in Sp(L_i)} \lambda = c_1\lambda_i^*. \tag{4.22}$$

In the conjugate-gradient method it is supposed to equal λ_i^*, *i.e.*, (4.22) is fulfilled with the constant $c_1 = 1$.

5. Proof of the convergence of cascadic algorithm

In [3] the convergence criterion has been formulated as follows:

there exists a constant $c^* > 0$ such that $\forall\ i = 1, \dots, l$

$$\|V_i - I_{i-1}V_{i-1}\|_i \leq \frac{c^*}{\sqrt{\Lambda_i^*}} \|\!| V_i - I_{i-1}V_{i-1} |\!\|_i \tag{5.23}$$

where Λ_i^* is the upper estimate of the maximum eigenvalue of operator L_i.

Lemma 2. *For $\boldsymbol{f} \in (L_2(\Omega))^2$ criterion (5.23) is fulfilled with the constant $c^* = 2c_4\sqrt{c_6}/c_2$.*

Proof. Consider the auxiliary problem:

find a vector-function $\boldsymbol{w} \in (\overset{\circ}{W}{}_2^1(\Omega))^2$ such that

$$\mathcal{L}(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{v}_i - \boldsymbol{v}_{i-1}, \boldsymbol{v})_\Omega \quad \forall\ \boldsymbol{v} \in (\overset{\circ}{W}{}_2^1(\Omega))^2. \tag{5.24}$$

According to (2.5) we have

$$\|\boldsymbol{w}\|_{2,\Omega} \leq c_1 \|\boldsymbol{v}_i - \boldsymbol{v}_{i-1}\|_{0,\Omega}.$$

Use the Bubnov-Galerkin method:

$$\mathcal{L}(\boldsymbol{w}_{i-1}, \boldsymbol{v}) = (\boldsymbol{v}_i - \boldsymbol{v}_{i-1}, \boldsymbol{v})_\Omega \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^{i-1}.$$

From Lemma 1 there follows the estimate

$$\|\!| \boldsymbol{w} - \boldsymbol{w}_{i-1} |\!\|_\Omega \leq c_4 h_{i-1} \|\boldsymbol{v}_i - \boldsymbol{v}_{i-1}\|_{0,\Omega}. \tag{5.25}$$

Putting in (5.24) $\boldsymbol{v} = \boldsymbol{v}_i - \boldsymbol{v}_{i-1}$ gives

$$\mathcal{L}(\boldsymbol{w}, \boldsymbol{v}_i - \boldsymbol{v}_{i-1}) = \|\boldsymbol{v}_i - \boldsymbol{v}_{i-1}\|_{0,\Omega}^2. \tag{5.26}$$

Considering that any function from subspace $\boldsymbol{H}^{i-1}$ is contained in subspace $\boldsymbol{H}^i$, from (3.8) we have

$$\mathcal{L}(\boldsymbol{v}_i, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^{i-1}.$$

Subtracting from here the identity

$$\mathcal{L}(\boldsymbol{v}_{i-1}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^{i-1},$$

we have

$$\mathcal{L}(\boldsymbol{v}_i - \boldsymbol{v}_{i-1}, \boldsymbol{v}) = 0 \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^{i-1}. \tag{5.27}$$

Putting $\boldsymbol{v} = \boldsymbol{w}_{i-1}$ and taking account of the symmetry of $\mathcal{L}$, we subtract this equality from (5.26):

$$\mathcal{L}(\boldsymbol{w} - \boldsymbol{w}_{i-1}, \boldsymbol{v}_i - \boldsymbol{v}_{i-1}) = \|\boldsymbol{v}_i - \boldsymbol{v}_{i-1}\|_{0,\Omega}^2.$$

Using the Cauchy-Bunyakovski inequality and (5.25), we get

$$\begin{aligned} \|v_i - v_{i-1}\|_{0,\Omega}^2 &\le \|\!|w - w_{i-1}|\!\|_\Omega \cdot \|\!|v_i - v_{i-1}|\!\|_\Omega \\ &\le c_4 h_{i-1} \|v_i - v_{i-1}\|_{0,\Omega} \cdot \|\!|v_i - v_{i-1}|\!\|_\Omega. \end{aligned}$$

Hence, from (3.12) and (3.13), it follows that

$$\begin{aligned} c_2 h_i \|V_i - I_{i-1} V_{i-1}\|_i &\le \|v_i - v_{i-1}\|_{0,\Omega} \le c_4 h_{i-1} \|\!|v_i - v_{i-1}|\!\|_\Omega \\ &= c_4 h_{i-1} \|\!|V_i - I_{i-1} V_{i-1}|\!\|_i. \end{aligned}$$

Using (4.22) or (5.23) together with (3.18) yields the inequality

$$\|V_i - I_{i-1} V_{i-1}\|_i \le 2 \frac{c_4}{c_2} \sqrt{\frac{c_6}{\Lambda_i^*}} \|\!|V_i - I_{i-1} V_{i-1}|\!\|_i. \qquad \square$$

Thus from [3] (Theorem 4.1) it is evident that the estimate

$$\|\!|V_i - U_i|\!\|_i \le c^* \sum_{j=1}^{i} \frac{1}{2m_j + 1} \|\!|V_j - I_{j-1} V_{j-1}|\!\|_j, \tag{5.28}$$

is fulfilled, where V_i is the exact solution of algebraic problem (3.9); U_i is its approximation obtained by the cascadic algorithm; m_j is the number of iterations in the conjugate-gradient algorithm (4.20) or in the Jacobi-type one (4.21) on the jth step. These two methods differ from each other only by different constants c^*.

To have this estimate in an easier form, we prove the inequality

$$\|\!|v_j - v_{j-1}|\!\|_\Omega \le \|\!|u - v_{j-1}|\!\|_\Omega. \tag{5.29}$$

Subject to (2.6) and (3.8) at $v = v_i - v_{i-1}$, we have

$$\mathcal{L}(u, v_i - v_{i-1}) = \mathcal{L}(v_i, v - v_{i-1}.$$

Taking into account of the symmetry of $\mathcal{L}$ and putting in (5.27) $v = v_{i-1}$, we subtract (5.27) from both sides of the last equality

$$\mathcal{L}(u - v_{i-1}, v_i - v_{i-1}) = \mathcal{L}(v - v_{i-1}, v_i - v_{i-1}).$$

Apply the Cauchy-Bunyakovski inequality to the left-hand side:

$$\|\!|v_i - v_{i-1}|\!\|_\Omega^2 \le \|\!|u - v_{i-1}|\!\|_\Omega \cdot \|\!|v_i - v_{i-1}|\!\|_\Omega.$$

Hence there follows the inequality (5.29). (5.28) can therefore be recast as

$$\|\!|V_i - U_i|\!\|_i \le c^* \sum_{j=1}^{i} \frac{1}{2m_j + 1} \|\!|u - v_{j-1}|\!\|_\Omega$$

or, from (3.14), as

$$\|\!|V_i - U_i|\!\|_i \le c^* c_4 \sum_{j=1}^{i} \frac{1}{2m_j + 1} h_{j-1} \|f\|_{0,\Omega}. \tag{5.30}$$

Among these inequalities for $i = 1, \ldots, l$, the estimate relating to the finest triangulation $\mathcal{T}_l$ is most useful. We express it as

Theorem 1. *Assume for the problem* (2.1) – (2.2) *on the bounded convex polygon* Ω *that condition* (2.3) *holds. Then for the solution* U_l *of the cascadic algorithm with one of the iterative smoothers* (4.20) *or* (4.21) *on every level* $j = 1, \ldots, l$, *we have the estimate*

$$\|V_l - U_l\|_l \leq d_1 \sum_{j=1}^{l} \frac{h_{j-1}}{2m_j + 1} \quad \textit{where} \quad d_1 = c^* c_4 \|f\|_{0,\Omega}. \tag{5.31}$$

6. Optimization of the number of iterations

By analyzing the sequence of computations in view of the sparsity of matrices L_i, the upper estimate of the number of arithmetic operations in the cascadic algorithm is established as follows:

$$S_l = d_2 \sum_{j=1}^{l} (m_j + d_3) n_j + d_4. \tag{6.32}$$

Here the constants d_2, d_3, d_4 are independent of n_j and m_j but different for iterative processes (4.20) and (4.21). It is obvious that these constants are smaller for the latter process.

We now propose to choose the number of iterations $m_1, \ldots, m_{l-1}$ to minimize S_l as a function of $m_1, \ldots, m_{l-1}$ when the right-hand side of inequality (5.31) is fixed. Applying the Lagrange multiplier method gives

$$2m_i + 1 = (2m_l + 1)\sqrt{n_l h_{i-1}/n_i h_{l-1}}.$$

Following this equality gives non-integer m_i. Therefore we put $m_l = m$ and choose m_i on the succeeding levels as the least integer satisfying the inequality

$$2m_i + 1 \geq (2m + 1)\sqrt{n_l h_{i-1}/n_i h_{l-1}}. \tag{6.33}$$

Theorem 2. *When the conditions of Theorem 1 are satisfied, the error of the cascadic algorithm with conjugate-gradient iterations* (4.20) *or Jacobi-type smoother* (4.21) *is estimated as*

$$\|U_l - V_l\|_l \leq \frac{c_3 h_l}{2m + 1} \|f\|_{0,\Omega}. \tag{6.34}$$

The piecewise linear interpolant $\boldsymbol{u}_l \in \boldsymbol{H}^l$ *of vector* U_l *obeys the estimate*

$$\|\boldsymbol{u}_l - \boldsymbol{u}\|_\Omega \leq h_l \left(c_4 + \frac{c_3}{2m + 1} \right) \|f\|_{0,\Omega}. \tag{6.35}$$

The number of arithmetic operations is estimated above by the value

$$S_l \leq (c_5 m + c_6) n_l \tag{6.36}$$

with the constants $c_3 - c_6$ *independent of* m *and* n_l.

Proof. From the Euler formula for polygons, there follows the inequality $n_{j-1} \leq n_j/4$. Therefore

$$n_j \leq 4^{j-l} n_l. \tag{6.37}$$

Subject to construction of grids,

$$h_j = 2^{l-j} h_l. \tag{6.38}$$

Using these relations together with (6.33) in (5.31), we get a sequence of inequalities

$$|\!|\!| V_l - U_l |\!|\!|_l \leq 2d_1 h_l \sum_{j=1}^{l} \sqrt{\frac{n_j h_{j-1}}{n_l h_{l-1}}} \leq 2d_1 h_l \sum_{j=1}^{l} 2^{(j-l)/2} \leq h_l \frac{2\sqrt{2} d_1}{\sqrt{2}-1}.$$

Therefore the constant c_3 in (6.34) can be assumed to equal $c^* c_4 2\sqrt{2}/(\sqrt{2}-1)$.

From the triangle inequality, the equivalence of norms (3.12) and estimate (3.14), we arrive at the inequality

$$|\!|\!| \boldsymbol{u}_l - \boldsymbol{u} |\!|\!|_\Omega \leq |\!|\!| \boldsymbol{u}_l - \boldsymbol{v}_l |\!|\!|_\Omega + |\!|\!| \boldsymbol{v}_l - \boldsymbol{u} |\!|\!|_\Omega \leq |\!|\!| U_l - V_l |\!|\!|_l + c_4 h_l \|\boldsymbol{f}\|_{0,\Omega}.$$

Together with the inequality (6.34) already proved, this leads to (6.35).

To estimate the number of arithmetical operations, it will be remembered that m_i is chosen as the least integer satisfying the condition (6.33). Therefore

$$2(m_i - 1) + 1 \leq (2m+1)\sqrt{n_l h_{i-1}/n_i h_{l-1}}$$

whence

$$m_i \leq (m + 1/2)\sqrt{n_l h_{i-1}/n_i h_{l-1}} + 1/2. \tag{6.39}$$

Use this inequality in (6.32):

$$S_l \leq d_2 \sum_{j=1}^{l} ((m+1/2) n_l \sqrt{n_j h_{j-1}/n_l h_{l-1}} + (d_3 + 1/2) 4^{j-l} n_l) + d_4.$$

From relations (6.37), (6.38) we get

$$S_l \leq d_2 \sum_{j=1}^{l} ((m+1/2) n_l 2^{(j-l)/2} + (d_3 + 1/2) 4^{j-l} n_l) + d_4.$$

We replace the sums of two geometrical progressions with infinite series sums:

$$S_l \leq d_2 \left((m+1/2)\frac{\sqrt{2}}{\sqrt{2}-1} + (d_3 + 1/2)\frac{4}{3} \right) n_l + d_4.$$

Hence there follows (6.36) with the constants

$$c_5 = d_2\sqrt{2}/(\sqrt{2}-1) \quad \text{and} \quad c_6 = c_5/2 + 2(2d_3+1)/3 + d_4. \ \square$$

It is apparent that the number of iterations m on the highest level should be chosen subject to the condition $c_4 \approx c_3/(2m+1)$. Although the constants c_3, c_4 are unknown, it is seen that m is independent of the number of levels and unknowns. Therefore (6.34)–(6.36) characterize the property: under finite number of arithmetic operations per one unknown, the error of the iterative process is of the same order as the error of the discretization. When doing practical calculations, *e.g.* for the Poisson equation, m is small. However as the level index decreases, the number of iterations increases exponentially. Therefore some disadvantages of iterative processes (4.20) and (4.21) are seen [6]: a loss of orthogonality of sequences r_k, p_k in (4.20) and considerable accumulation of the error of intermediate iterations in (4.21). It should also be noted that the method (4.21) is less efficient than the method (4.20). All this leads to the following recommendation. *On the highest levels, m iterations of method* (4.20) *should be done; on the succeeding levels method* (4.21) *should be realised with the number of iterations* (6.39), *where $m = 3m_l$, and with the use of one of the mixing procedures of parameters τ_{k-1} to ensure stability [13], [14].*

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