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Continuum percolation in a nonstabilizing environment

Benedikt Jahnel[1,2], Sanjoy Kumar Jhawar[2], Anh Duc Vu[2]

[1] Technische Universität Braunschweig
Universitätsplatz 2
38106 Braunschweig
Germany

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: benedikt.jahnel@wias-berlin.de
sanjoykumar.jhawar@wias-berlin.de
anhduc.vu@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We prove nontrivial phase transitions for continuum percolation in a Boolean model based on a Cox point process with nonstabilizing directing measure. The directing measure, which can be seen as a stationary random environment for the classical Poisson–Boolean model, is given by a planar rectangular Poisson line process. This Manhattan grid type construction features long-range dependencies in the environment, leading to absence of a sharp phase transition for the associated Cox–Boolean model. Our proofs rest on discretization arguments and a comparison to percolation on randomly stretched lattices established in [Hof05].

Introduction

In continuum percolation, one is interested in the clustering behavior of point clouds in $\mathbb{R}^d$ in which any pair of points is connected by an edge depending on their mutual distance. The prototypical example is the Poisson–Boolean model, first introduced in [Gil61], in which the point cloud is given by a homogeneous Poisson point processes $X = \{X_i\}_{i\in I}$ with intensity $\lambda > 0$ and any pair of points $X_i, X_j \in X$ is connected iff $|X_i - X_j| < r$. The celebrated phase transition of continuum percolation is then expressed by the existence of a nontrivial critical threshold $0 < \lambda_{\mathrm{c}}(r) < \infty$ such that for $\lambda < \lambda_{\mathrm{c}}(r)$, the network contains almost surely no infinite connected component, and for $\lambda > \lambda_{\mathrm{c}}(r)$, this is no longer the case.

The analysis of spatial models with respect to continuum percolation has flourished ever since and critical behavior has been established in a multitude of generalizations of the Poisson–Boolean model. For example, instead of a fixed connectivity threshold $r > 0$, random radii can be used to define connected components [MR96; Gou08]. Also in this direction, other local geometries have been used to define edges, see for example [Roy91; BT16; Bro22]. Another line of research is concerned with generalizations towards using other stationary point processes as the underlying set of vertices in the network. Let us mention for example the continuum-percolation results for Gibbs point processes in [Mür75; Stu13; Jan16; CD14; Mag18], for repelling point processes in [GKP16; BY14], for negatively associated point processes in [BY13], or general stationary point process [MR96; Gou09].

A particularly interesting class of stationary point process, for which continuum percolation can be investigated, is given by Cox point processes. These can be seen as Poisson point processes

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in random environments, where the environment enters the definition via the (random) intensity measure. Recently, continuum percolation and associated properties have been studied for the Cox–Boolean model with fixed and random connectivity thresholds in [HJC19; JTC22]. Here, the key ingredient for the proofs of nontrivial critical percolation behavior is a spatial mixing property of the random environments called stabilization [SY13]. In short, stabilizing environments have the feature that – with high probability and in sufficiently distant large boxes – the environment behaves independently. This property is crucial in order to couple the system with finite-range dependent Bernoulli percolation models as well as for the use of multiscale arguments.

However, while still covering a large family of environments, such as Poisson–Voronoi tessellations [Oka+00] or Poisson–Boolean models, the stabilization assumption also excludes many natural examples, such as the Poisson line tessellation or infinite-range shot-noise fields, see [JTC22] for more details. Another prominent example for which stabilization fails is the rectangular Poisson line tessellation. Here, we consider two independent homogeneous Poisson point processes on the axes of $\mathbb{R}^2$. We attach to any point on the x-axis an infinite vertical line, and correspondingly we attach horizontal lines to the points on the y-axis. The resulting environment resembles a random rectangular street system and hence is often called a Manhattan grid. The fact that infinite lines are used creates long-range correlations, for example in the horizontal direction, and in turn, standard stabilization-based methods can not be used for the analysis.

The existence of long-range dependencies has serious consequences for percolation in the Cox–Boolean model based on Manhattan grids. For example, there is no sharp-threshold phenomenon [AB87; DT16; HJM22]. More precisely, in the subcritical percolation regime, the probability of the event that the cluster of the origin reaches beyond a large ball does not decrease exponentially, see Proposition 5. However, the existence of nontrivial sub- and supercritical regimes can be established via different means, namely via couplings to discrete bond-percolation models with long-range dependencies.

More precisely, the results in [Hof05] provide nontrivial thresholds for percolation in a planar Bernoulli bond percolation model based on a randomly stretched lattice. In this model, each column of horizontal edges $\big((i,j),(i+1,j)\big)_{j\in\mathbb{Z}}$ in the standard $\mathbb{Z}^2$-lattice gets assigned an independent random variable $N_i^{(x)}$ and the same is done for each row of vertical edges $\big((i,j),(i,j+1)\big)_{i\in\mathbb{Z}}$ with independent random variables $N_j^{(y)}$. Then, for some fixed $p\in(0,1)$, conditioned on $(N_i^{(x)},N_j^{(y)})_{i,j\in\mathbb{Z}}$, the horizontal edge $\big((i,j),(i+1,j)\big)$ is open independently with probability $p^{N_i^{(x)}}$, respectively any vertical edge $\big((i,j),(i,j+1)\big)$ is open independently with probability $p^{N_j^{(y)}}$. Now, for sufficiently light-tailed variables $(N_i^{(x)},N_j^{(y)})_{i,j\in\mathbb{Z}}$, [Hof05] states the existence of a critical $p_{\rm c}\in(0,1)$ for percolation. Let us mention that an associated result on $\mathbb{Z}^3$ was obtained earlier in [JMP00] and other related lattice systems have been studied in [MW68; MW69; BBS00].

Finally, we note that continuum percolation models have a natural application in the rigorous probabilistic analysis of wireless networks, where randomly positioned network components can exchange messages whenever they are sufficiently close to each other [JK20; BB10a; BB10b].

In view of this, existence of a supercritical percolation regime of the Cox–Boolean model based on Manhattan grids can be seen as a rough indication for the existence of a regime in which sufficiently many network participants enable global connectivity in an urban street system of Manhattan type.

The paper is organized as follows:

- In Section 1, we define our model of interest: the Manhattan grid model. Furthermore, we state the main results, that is, sub- and supercriticality of this Cox–Boolean model in certain parameter regimes. We also introduce the randomly stretched lattice, a discrete auxiliary model that we heavily rely on.
- In Section 2, the existence of an infinite component of the Manhattan grid model is shown for different choices of parameters.
- The second half of this paper deals with the complementary case: Under which assumptions is the absence of infinite components guaranteed? We discretize our model in Section 3 and show in Section 4 that we find blocking circuits in its dual for a suitable choice of parameters.
- Section 4 builds up on [Hof05]. Due to its technicality, the finer details that are independent of our proof idea are given in the appendix.

1 Setting and main results

1.1 The Manhattan grid model and main results

We introduce our model of interest. In short, the Manhattan grid model is a Boolean model based on a Poisson point process defined on a rectangular Poisson line process, see Figure 1.

Definition 1 (Manhattan grid model)**.** Let $r, \mu_x, \mu_y, \lambda > 0$ and consider independent homogeneous Poisson point processes $\Phi^{(x)}, \Phi^{(y)} \subset \mathbb{R}$ on $\mathbb{R}$ with intensities μ_x and μ_y. Define the random measure Λ, where for $A \subset \mathbb{R}^2$ Borel-measurable

$$\Lambda(A) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_A(x, y) \mathrm{d}y \, \Phi^{(x)}(\mathrm{d}x) + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_A(x, y) \mathrm{d}x \, \Phi^{(y)}(\mathrm{d}y) \, .$$

Let $\Psi \subset \mathbb{R}^2$ be a Poisson point process on $\mathbb{R}^2$ with intensity measure $\lambda\Lambda$. Then, the **Manhattan grid model** (MGM) is defined as

$$\Xi(r, \mu_x, \mu_y, \lambda) := \{x \in \mathbb{R}^2 \,|\, \exists P \in \Psi : \, \|x - P\| < r\} \, .$$

Note that the MGM can be seen as a Boolean model based on a stationary Cox point process with intensity $\lambda(\mu_x + \mu_y)$.

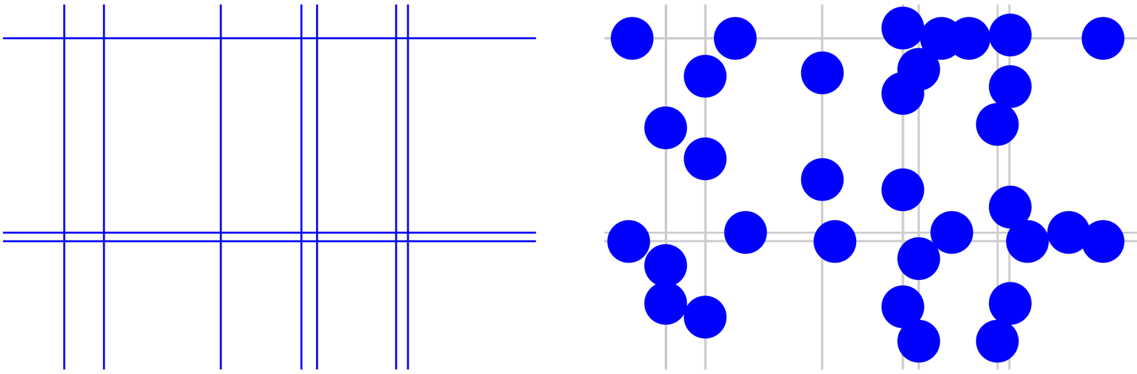

Figure 1: Construction of the Manhattan grid model: First, generate a random Manhattan grid (left). Second, place r balls with random centers on the grid.

We now concern ourselves with the question whether the MGM percolates, that is, whether $\Xi(r, \mu_x, \mu_y, \lambda)$ contains an infinite connected component. Let us first mention the following scaling relation:

Lemma 2 (Scaling relations)**.** *For all $\alpha > 0$, the Manhattan grid model $\Xi(r, \mu_x, \mu_y, \lambda)$ has the same distribution as $\alpha\Xi(\frac{r}{\alpha}, \alpha\mu_x, \alpha\mu_y, \alpha\lambda)$.*

Proof. One verifies that $\frac{1}{\alpha}\Xi(r, \mu_x, \mu_y, \lambda)$ is a MGM with parameters $(\frac{r}{\alpha}, \alpha\mu_x, \alpha\mu_y, \alpha\lambda)$, e.g., via coupling. □

A direct consequence is that $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates if and only if $\Xi(\frac{r}{\alpha}, \alpha\mu_x, \alpha\mu_y, \alpha\lambda)$ does so. Let us now present our main results.

Theorem 3 (Existence of supercritical regimes)**.** *Let $r > 0$ be arbitrary.*

1. *For every $\lambda > 0$, there exists $\mu_c(r, \lambda) > 0$, such that for all $\mu_x, \mu_y \geq \mu_c(r, \lambda)$: $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates almost surely.*
2. *For every μ_x, μ_y, there exists $\lambda_c(r, \mu_x, \mu_y) > 0$, such that for all $\lambda \geq \lambda_c(r, \mu_x, \mu_y)$: $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates almost surely.*
3. *For every $\lambda, \mu_x > 0$, there exists $\mu_{y,c}(r, \mu_x, \lambda) > 0$, such that for all $\mu_y \geq \mu_{y,c}(r, \mu_x, \lambda)$: $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates almost surely.*

Proof. The statements are represented in Proposition 12, respectively Proposition 14 and Proposition 17. □

Let us mention that Statement 1 in Theorem 3 is a direct consequence of Statement 3. We will prove Theorem 3 by showing that $\Xi(r, \mu_x, \mu_y, \lambda)$ is lower bounded by some discrete bond percolation model, which we introduce in the next section. Then, the supercriticality of this discrete model finishes the claim. Now we give a complementing result on absence of percolation:

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Theorem 4 (Existence of a subcritical regime). *For every $r > 0$ and every $\mu_x, \mu_y > 0$, there exists a $\lambda_c(r, \mu_x, \mu_y) > 0$ such that for all $\lambda \leq \lambda_c(r)$, we have: $\Xi(r, \mu_x, \mu_y, \lambda)$ does not percolate almost surely.*

The proof of Theorem 4 is more complicated and requires us to deal with finer details of the discrete bond percolation model that we introduce in the next section. Roughly, we first find another discrete upper bound model. This model features a dual relation with the previously mentioned bond percolation model. Employing Peierls' argument, we therefore need to find circuits in this dual.

Let us note that Theorem 4 only provides the corresponding statement to Theorem 3 Part 2. Our current discretization scheme does not allow us to derive a corresponding statement to Theorem 3 Part 3 where λ is fixed.

1.2 No sharp thresholds

Before we present details on the discrete models, let us show a result on the slow decay of the percolation function of the MGM. For this, we consider the so called percolation function θ_n of $\Xi(r, \mu_x, \mu_y, \lambda)$, that is,

$$\theta_n := \theta_n(r, \mu_x, \mu_y, \lambda) := \mathbb{P}(\mathcal{C}_o \cap \partial\big([-n, n]^2\big) \neq \emptyset)\,,$$

where $\mathcal{C}_o \subset \mathbb{R}^2$ is the connected component of $\Xi(r, \mu_x, \mu_y, \lambda)$ containing the origin $o \in \mathbb{R}^2$. We only state the result for $r > \sqrt{2}$. The general case can be obtained using Lemma 2.

Proposition 5 (No exponential decay). *Let $r > \sqrt{2}$, $\mu_x, \mu_y > 0$ and $k \in \mathbb{N}$. Then,*

$$\liminf_{n\to\infty} \lfloor \lambda^{-1} \log n \rfloor! \cdot n^{\lambda^{-1}\log(\min\{\mu_x,\mu_y\})} \cdot \theta_n > 0\,, \tag{1}$$

in particular, for every $\varepsilon > 0$

$$\liminf_{n\to\infty} n^{(1+\varepsilon)\lambda^{-1}\log(\log n)} \cdot \theta_n = \infty\,.$$

Proof. Assume that $\mu_y = \min\{\mu_x, \mu_y\} > 0$, otherwise exchange the roles. We show the claim by considering clusters that only grow to the right. We have that the event

$$\{\forall 0 \leq k \leq n :\ \Psi\big([k, k+1) \times [0, 1)\big) > 0\}$$

implies the event

$$E_n := \{\mathcal{C}_o \cap \partial\big([-n, n]^2\big) \neq \emptyset\}\,.$$

Disregarding vertical lines, the probability of at least one Poisson point lying in one such cube $[k, k+1) \times [0, 1)$, given that there are s horizontal lines, is

$$1 - \exp(-\lambda)^s\,,$$

while the probability of having these s horizontal lines in $[0,1)$ is

$$\mathbb{P}\big(\Phi^{(y)}([0,1)) = s\big) = \exp(-\mu_y) \cdot \frac{\mu_y^s}{s!}\,.$$

The idea is that we only need to “pay” once to generate many streets, but all n cubes benefit from that. Therefore,

$$\theta_n \geq \mathbb{P}(E_n) \geq \exp(-\mu_y) \sum_{s=0}^{\infty} \frac{\mu_y^s}{s!} \,\cdot\, \big(1 - \exp(-\lambda)^s\big)^n\,.$$

Now, consider for some arbitrary $c > 0$

$$f_c(n) := \lceil \lambda^{-1} \log \frac{n}{c} \rceil\,.$$

Then, with $\sigma := 1$ if $\mu_y < 1$ and $\sigma := -1$ else:

$$\begin{aligned}\theta_n &\geq \exp(-\mu_y) \frac{\mu_y^{f_c(n)}}{f_c(n)!} \,\cdot\, \big(1 - \exp(-\lambda)^{f_c(n)}\big)^n \\ &\geq \exp(-\mu_y) \frac{\mu_y^{(\lambda^{-1}\log\frac{n}{c}+\sigma)}}{f_c(n)!} \,\cdot\, \big(1 - \frac{c}{n}\big)^n\,.\end{aligned}$$

Choosing c large enough such that $\lambda^{-1}\log c \geq 2$, we have for some constant $C > 0$

$$\theta_n \geq C n^{-\lambda^{-1}\log \mu_y} \cdot \lfloor \lambda^{-1} \log n \rfloor !^{-1} \,\cdot\, \big(1 - \frac{c}{n}\big)^n\,.$$

Therefore,

$$\liminf_{n\to\infty} \lfloor \lambda^{-1} \log n \rfloor ! \,\cdot\, n^{\lambda^{-1}\log\mu_y} \,\cdot\, \theta_n \geq \liminf_{n\to\infty} C\big(1 - \frac{c}{n}\big)^n = Ce^{-c} > 0\,,$$

which proves Inequality (1). The second statement follows from Stirling’s formula. □

1.3 The randomly stretched lattice

The randomly stretched lattice is a bond percolation model on $\mathbb{Z}^2$ with the usual neighborhood structure. There is a fixed parameter $p \in (0,1)$, which says how likely it is for a simple bond to be open. In more precise terms:

Definition 6 (Randomly stretched lattice)**.** Let $N^{(x)} := (N_i^{(x)})_{i\in\mathbb{Z}}$ and $N^{(y)} := (N_j^{(y)})_{j\in\mathbb{Z}}$ be families of mutually independent positive random variables and fix $p \in (0,1)$. Given a realization of $N^{(x)}$ and $N^{(y)}$, all the bonds in $\mathbb{Z}^2$ are open independently with probabilities

$$\mathbb{P}\Big((i,j) \leftrightarrow (i+1,j) \text{ is open} \mid N^{(x)},\, N^{(y)}\Big) := p^{N_i^{(x)}}$$

and

$$\mathbb{P}\Big((i,j) \leftrightarrow (i,j+1) \text{ is open} \mid N^{(x)},\, N^{(y)}\Big) := p^{N_j^{(y)}}\,.$$

This model is called the **randomly stretched lattice** (RSL).

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Example 7. A version of a RSL can be obtained by a random thinning of a Bernoulli bond percolation model on the $\mathbb{Z}^2$-lattice with parameter p, see for illustration Figure 2. The simplest

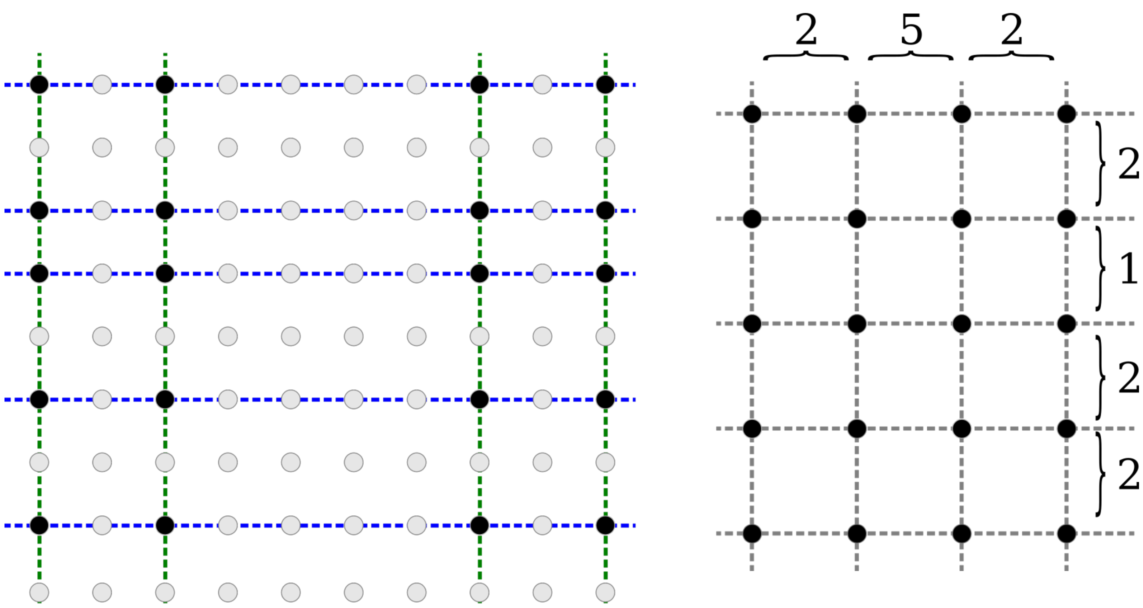

Figure 2: Left: Randomly delete rows/columns of edges. Right: Collapsing everything back into the standard $\mathbb{Z}^2$-lattice with now random distances yields a RSL.

way to do so is to delete rows and columns of bonds (while keeping the vertices) with some probability q_y, respectively q_x. As a consequence, to pass from one remaining 4-way crossing to the next 4-way crossing to the right, it is no longer sufficient to walk distance 1, but a random distance $N_i^{(x)}$. In particular, the probability that the whole path between the 4-way crossings is open is given by $p^{N_i^{(x)}}$. With this thinning procedure, we would have that the random distances $N_i^{(x)}$ are iid geometric random variables with $\mathbb{P}(N_i^{(x)} \geq l+1) = q_x^l$.

We say that the RSL percolates if there exists an infinite self-avoiding path of open bonds. The following result is due to [Hof05, Theorem 4.1]:

Theorem 8 (Existence of supercritical regime in the RSL). *Consider a RSL as in Definition 6. If for all $i, j \in \mathbb{Z}$ and all $l \in \mathbb{N}$*

$$\mathbb{P}(N_i^{(x)} \geq l+1) \leq 2^{-1000\cdot l} \quad \text{and} \quad \mathbb{P}(N_j^{(y)} \geq l+1) \leq 2^{-1000\cdot l}, \tag{2}$$

then there exists $p_c \in (0,1)$ such that the RSL percolates almost surely for every $p \geq p_c$.

The intuition behind Theorem 8 is fairly simple in sharp contrast to its proof: Columns with large distances (i.e., large $N_i^{(x)}$) in the RSL are rare. In absence of these, we have huge open clusters. In order to connect two neighboring clusters, they have to overcome a column with large distance. However, they have a lot of trials to do so due to their size. Therefore, these clusters connect with high probability and we obtain an infinite open cluster.

We conclude this section with the following observation:

Lemma 9 (RSL scaling relation)**.** *Let* $N^{(x)} := (N_i^{(x)})_{i\in\mathbb{Z}}$ *and* $N^{(y)} := (N_j^{(y)})_{j\in\mathbb{Z}}$ *be families of mutually independent positive random variables and* $p \in (0,1)$*. Then for all* $\alpha > 0$*, the* RSL *with parameters* $N^{(x)}, N^{(y)}$ *and* p *has the same distribution as the RSL for* $\alpha N^{(x)}$*,* $\alpha N^{(y)}$ *and* $p^{\frac{1}{\alpha}}$*.*

As a consequence, we notice that the tail condition (2) can be guaranteed in the case of geometric random variables:

Corollary 10 (Compensating heavy geometric tails)**.** *Let* $\tilde{N}^{(x)} := (\tilde{N}_i^{(x)})_{i\in\mathbb{Z}}$ *and* $\tilde{N}^{(y)} := (\tilde{N}_j^{(y)})_{j\in\mathbb{Z}}$ *be families of mutually independent geometric random variables. There exists a* $\tilde{p} \in (0,1)$ *such that the RSL with parameters (*$\tilde{N}^{(x)}$*,* $\tilde{N}^{(y)}$*,* $\tilde{p}$*) has the same distribution as a RSL satisfying the conditions of Theorem 8.*

Proof. Since $\tilde{N}^{(x)}$ and $\tilde{N}^{(y)}$ are families of geometric random variables, we find a $q \in (0,1)$ such that for all $i \in \mathbb{Z}$ and $l \in \mathbb{N}$

$$\mathbb{P}\big(\tilde{N}_i^{(x)} \geq l+1\big) \leq q^l \qquad \text{and} \qquad \mathbb{P}\big(\tilde{N}_i^{(y)} \geq l+1\big) \leq q^l.$$

Then, using Lemma 9 with

$$\alpha := -1000\log(2)/\log(q) > 0,$$

finishes the proof. □

2 Existences of supercritical regimes

Depending on which parameters we want to fix, we need different discretizations. The goal is to arrive at an RSL dominated by a MGM. Due to Lemma 2, we may always fix some $r > \sqrt{2}$. This has the benefit that, if a Cox point lies inside some unit square, $\Xi(r, \mu_x, \mu_y, \lambda)$ will cover the whole square.

2.1 Fixed intensity of Poisson points, variable street intensities

We may compensate for the low intensity λ of Poisson points by simply having an overwhelming amount of streets. We will choose our parameters as follows.

Assumption 11 (Parameters (Theorem 3 Part 1))**.** *Let $r > \sqrt{2}$ and $\lambda > 0$ arbitrary. Let $n_\lambda \in \mathbb{N}$ such that*

$$1 - \exp(-n_\lambda \cdot \lambda) \geq p_c\,,$$

with p_c as in Theorem 8. Let $\mu_c := \mu_c(r, \lambda) > 0$ such that

$$1 - \exp(-\mu_c) \sum_{k=0}^{n_\lambda} \frac{\mu_c^k}{k!} \geq 1 - 2^{-1000}\,.$$

In words: n_λ can be understood as the minimum number of streets in order to percolate and μ_c as the minimal parameter that ensures this number with high probability.

Proposition 12 (Existence of supercritical regime (Theorem 3 Part 1))**.** *With parameters as in Assumption 11, we have that the MGM $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates almost surely for every $\mu_x, \mu_y \geq \mu_c(r, \lambda)$.*

Proof. The strategy of proof is also visualized in Figure 3. Let $i, j \in \mathbb{Z}$. We draw an edge

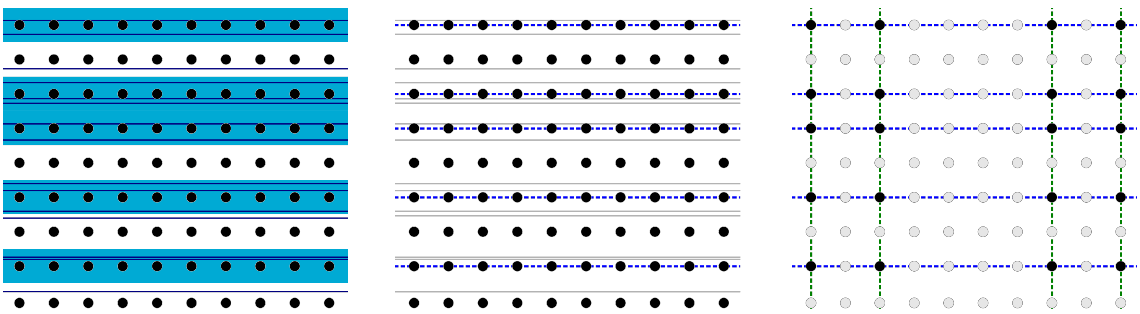

Figure 3: Procedure for $n_\lambda = 2$. Black circles correspond to points $(i, j) \in \mathbb{R}^2$. Left: We check if there are enough horizontal streets in each horizontal strip. Middle: If so, we draw horizontal edges between points in these rectangles. Right: Doing the same for vertical streets yields a RSL.

between (i, j) and $(i + 1, j)$ if there are at least n_λ horizontal streets in $[j - \frac{1}{2}, j + \frac{1}{2})$, i.e., if

$$\Phi^{(y)}\Big([j - \tfrac{1}{2}, j + \tfrac{1}{2})\Big) \geq n_\lambda\,.$$

Since $\mu_y \geq \mu_c$, for all $j \in \mathbb{Z}$, these are independent events (of drawing edges) happening with probability at least $1 - 2^{-1000}$. We call such an edge *open* if there exists a Poisson point P on one of these horizontal streets inside the square $[i, i + 1) \times [j - \frac{1}{2}, j + \frac{1}{2})$. Since $r > \sqrt{2}$, this implies that the ball of radius r around P contains both vertices (i, j) and $(i + 1, j)$, in particular, it connects them. Conditioned that we have at least n_λ streets, the event that there is a Poisson point P on one of these streets happens with probability at least

$$1 - \exp(-n_\lambda \cdot \lambda) \geq p_c\,.$$

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Analogously, we draw an edge between (i, j) and $(i, j+1)$ if there are at least n_λ vertical streets in $[i-\frac{1}{2}, i+\frac{1}{2})$. This way, the distance from one 4-way crossing to the next in horizontal direction (that is, the number of times we consecutively did not draw a vertical edge plus 1) is a geometric random variable $N^{(x)}_{i'}$ with

$$\mathbb{P}(N^{(x)}_{i'} \geq l+1) \leq 2^{-1000\cdot l}\,.$$

The same holds for the vertical direction. This is now a RSL with parameters as in Theorem 8. Since this RSL percolates almost surely, then so does the MGM. □

2.2 Fixed intensities of streets, variable Poisson-point intensity

Assumption 13 (Parameters (Theorem 3 Part 2)). *Let $r > \sqrt{2}$ and $\mu_x, \mu_y > 0$ be arbitrary. Write $\mu := \min(\mu_x, \mu_y)$. Let $n_\mu > 2$ such that*

$$e^{-(n_\mu-2)\cdot\mu} \leq 2^{-1000}\,.$$

Let $\lambda_c := \lambda_c(r, \mu) > 0$ large enough such that for any $D \in [2,\, 2n_\mu - 2]$

$$\mathbb{P}\big(\forall x \in [0, D]\, \exists P_x \in \Phi_{\lambda_c} \cap [0, D] :\, \|x - P_x\| < r\big) \geq p_c\,, \tag{3}$$

where Φ_{λ_c} is a Poisson point process on $\mathbb{R}$ of intensity λ_c.

The quantity n_μ can be understood as the minimum size of an interval that ensures the existence of a street with high probability. The intensity λ_c is then the minimum intensity such that crossing a distance up to n_μ is very likely. The proof of Inequality (3) is given in Lemma 15.

Proposition 14 (Existence of supercritical regime (Theorem 3 Part 2)). *With parameters as in Assumption 13, the MGM $\Xi(r, \mu_x, \mu_y, \lambda)$ percolates almost surely for every $\lambda \geq \lambda_c(r, \min\{\mu_x, \mu_y\})$.*

Proof. The discretization scheme is quite different from the one in Section 2.1 and sketched in Figure 4. We divide $\mathbb{R}^2$ into squares $n_\mu \cdot ([i, i+1) \times [j, j+1))$ of side length n_μ and identify each such square as a vertex $(i, j) \in \mathbb{Z}^2$. We draw an edge between (i, j) and $(i+1, j)$ if $\Phi^{(y)}\Big(\big[n_\mu \cdot j + 1, n_\mu \cdot (j+1) - 1\big)\Big) \geq 1$, that is, if there is a street with distance at least 1 from the boundary. By Assumption 13, this happens with probability at least $1 - 2^{-1000}$. Analogously, we do the same for vertical streets. Now, let (i, j) and $(i, j+s)$ be vertices in $\mathbb{Z}^2$ which have 4 neighbors (these exist almost surely). That means that there is a crossing $\mathfrak{c}_0 \in \mathbb{R}^2$ of a horizontal with a vertical street inside $\big[n_\mu \cdot i + 1, n_\mu \cdot (i+1) - 1\big) \times \big[n_\mu \cdot j + 1, n_\mu \cdot (j+1) - 1\big)$, respectively a crossing $\mathfrak{c}_s$ in $\big[n_\mu \cdot i + 1, n_\mu \cdot (i+1) - 1\big) \times \big[n_\mu \cdot (j+s) + 1, n_\mu \cdot (j+s+1) - 1\big)$. Furthermore, we may assume that the vertical streets of the crossings are the same, e.g., by picking the leftmost one. We now want to see that these two intersection points

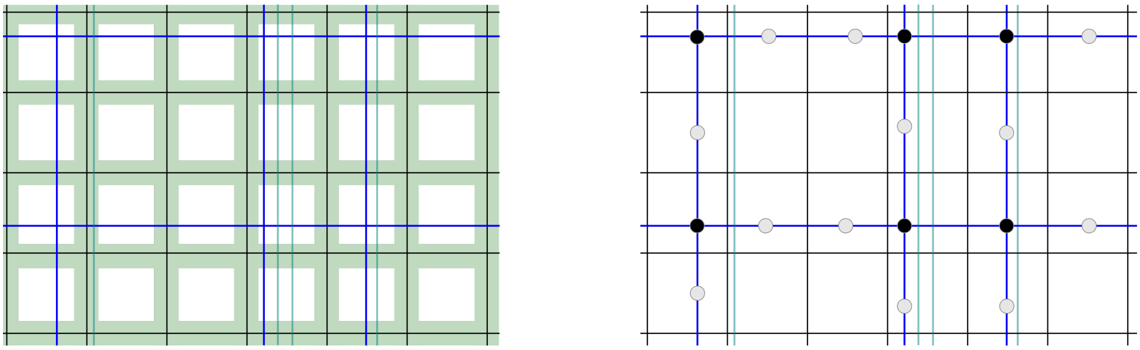

Figure 4: $\mathbb{R}^2$ is discretized into squares. Left: Streets that are too close to parallel black lines are discarded (green zones). We choose the lowest/leftmost remaining street (dark blue) per square and discard all the others. Right: Circles indicate crossings (black) and intermediate breakpoints (gray) in $\mathbb{R}^2$. The black circles correspond to vertices in the RSL with gray circles indicating the distances.

$\mathfrak{c}_0 = (x', y_0), \mathfrak{c}_s = (x', y_s)$ are connected in the MGM with probability at least p_c^s. This mainly follows from Inequality (3):

Indeed, if $s = 1$, this follows immediately. Otherwise, for any $k < s$, we pick an arbitrary $\mathfrak{c}_k = (x', y_k) \in \mathbb{R}^2$ such that

$$y_k \in \big[n_\mu \cdot (j + k) + 1, n_\mu \cdot (j + k + 1) - 1\big) .$$

For all $0 \le k \le s$, the $\mathfrak{c}_k$ lie on the same line. The probability that $\mathfrak{c}_k, \mathfrak{c}_{k+1}$ are connected is at least p_c by Inequality (3) and all these events are independent. Therefore, we obtain a RSL with parameters as in Theorem 8. Furthermore, since the RSL percolates almost surely, then so does the MGM. □

We have to work with intervals of the form $\big[n_\mu \cdot i + 1, n_\mu \cdot (i + 1) - 1\big)$ to make sure that there is always a minimal distance of 2 between two crossings in $\mathbb{R}^2$. Otherwise we would not be able to establish Inequality (3):

Lemma 15 (Probability of line coverings)**.** *Let $r, a, b > 0$ with $a \le b$ and $\tilde{p} \in (0, 1)$. Let Φ_λ be a Poisson point process on $\mathbb{R}$ with intensity λ. There exists $\lambda_c > 0$ such that for every $\lambda \ge \lambda_c$ and every $D \in [a, b]$*

$$\mathbb{P}\big(\forall x \in [0, D]\, \exists P_x \in \Phi_\lambda \cap [0, D] : \, \|x - P_x\| < r\big) \ge \tilde{p} .$$

Proof. By monotonicity, we may assume $r \le a$. Define the event

$$E_D := \{\forall x \in [0, D]\, \exists P_x \in \Phi_\lambda \cap [0, D] : \, \|x - P_x\| < r\} .$$

We will show that

$$\mathbb{P}(E_D) \ge \big(1 - \exp(-\tfrac{r}{2}\lambda_c)\big)^{\lfloor 2\frac{b}{r} \rfloor} ,$$

which proves the claim for sufficiently large λ_c. Let

$$n_D := \lfloor 2\tfrac{D}{r} \rfloor \leq \lfloor 2\tfrac{b}{r} \rfloor\,.$$

We see that E_D is implied by the event

$$\Big\{\forall 0 \leq i < n_D :\ \Phi_\lambda\big([i\cdot\tfrac{r}{2},\,(i+1)\cdot\tfrac{r}{2})\big) \geq 1\Big\}\,.$$

Since all of these intervals are disjoint, we have

$$\begin{aligned}\mathbb{P}(E_D) &\geq \prod_{i=0}^{n_D-1}\mathbb{P}\big(\Phi_\lambda[i\cdot\tfrac{r}{2},\,(i+1)\cdot\tfrac{r}{2}) \geq 1\big) = \mathbb{P}\big(\Phi_\lambda[0,\,\tfrac{r}{2}) \geq 1\big)^{n_D}\\ &= \big(1-\exp(-\tfrac{r}{2}\lambda)\big)^{n_D} \geq \big(1-\exp(-\tfrac{r}{2}\lambda_c)\big)^{\lfloor 2\frac{b}{r}\rfloor}\,,\end{aligned}$$

as desired. □

2.3 Fixed intensity of Poisson points, fixed intensity of horizontal streets, variable vertical street intensity

Assumption 16 (Parameters (Theorem 3 Part 3)). *Let $r > \sqrt{2}$, $\lambda > 0$ and $\mu_x > 0$ be arbitrary. Let $n_{\lambda,x} \in \mathbb{N}$ such that*

$$1 - e^{-n_{\lambda,x}\cdot\lambda} \geq p_c\,.$$

Let $n_\mu \in \mathbb{N}$ such that

$$\left[e^{-\mu_x}\cdot\sum_{k=0}^{n_{\lambda,x}}\frac{\mu_x^k}{k!}\right]^{n_\mu} \leq 2^{-1000}\,.$$

Let $n_{\lambda,y} \in \mathbb{N}$ such that

$$\left[1 - e^{-n_{\lambda,y}\cdot\lambda}\right]^{2n_\mu} \geq p_c$$

and finally $\mu_c := \mu_c(r,\mu_x,\lambda) > 0$ large enough such that

$$e^{-\mu_c}\cdot\sum_{k=0}^{2n_{\lambda,y}}\frac{\mu_c^k}{k!} \leq 2^{-1000}\,.$$

Proposition 17 (Existence of supercritical regime (Theorem 3 Part 3)). *With parameters as in Assumption 16, the MGM $\Xi(r,\mu_x,\mu_y,\lambda)$ percolates almost surely for every $\mu_y \geq \mu_c(r,\mu_x,\lambda)$.*

Proof. The discretization scheme is sketched in Figure 5. $n_{\lambda,x}$ is the minimal number of vertical streets that we need inside the unit square such that an edge is open with probability at least p_c. However, having this many streets is rather rare. After n_μ trials, there will be one such square with probability at least $1-2^{-1000}$. This takes care of the vertical edges. The problem

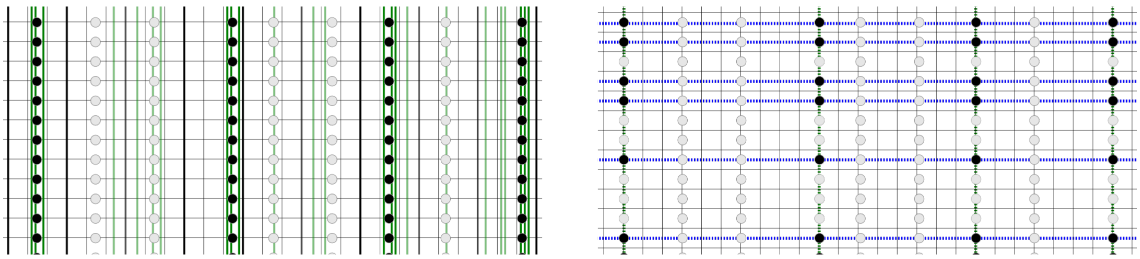

Figure 5: Procedure for $n_{\lambda,\mu} = 3$ and $n_\mu = 3$. We need 3 lines inside a box for it to be useful. To achieve this, we need 3 trials, which are grouped by the bold black lines (left). Afterwards (right), we draw the horizontal edges in boxes where we have sufficiently many horizontal streets. The horizontal distance between circles is now not 1 but up to 6.

with requiring n_μ trials instead of 1 is that now horizontal edges are distance up to $2n_\mu$ apart instead of 1. To cross up to $2n_\mu$ squares of side length 1, we need $n_{\lambda,y}$ many horizontal streets. Choosing μ_y large enough, having this many streets happens with probability at least $1-2^{-1000}$. We again obtain a RSL with parameters as in Theorem 8. Since the RSL percolates almost surely, so does our MGM and we conclude. □

3 Existence of a subcritical regime

3.1 The random highway model

We introduce another discrete model: the random highway model.

Definition 18 (Random highway model)**.** Let $N^{(x)} := (N_i^{(x)})_{i\in\mathbb{Z}}$ and $N^{(y)} := (N_j^{(y)})_{j\in\mathbb{Z}}$ be families of mutually independent positive random variables and fix $p \in (0,1)$. Given a realization of $N^{(x)}$ and $N^{(y)}$, all the bonds in $\mathbb{Z}^2$ are closed independently with probabilities

$$\mathbb{P}((i,j) \leftrightarrow (i+1,j) \text{ is closed} \mid N^{(x)}, N^{(y)}) = p^{N_j^{(y)}}$$

and

$$\mathbb{P}((i,j) \leftrightarrow (i,j+1) \text{ is closed} \mid N^{(x)}, N^{(y)}) = p^{N_i^{(x)}}.$$

This model is called the **random highway model** (RHM).

Remark. The interpretation is as follows: At height j, there are $N_j^{(y)}$ many infinite horizontal streets. In each segment, that is between (i,j) and $(i+1,j)$, each street has a probability $1-p$ of being intact. Then, (i,j) is connected to $(i+1,j)$ if at least one of the $N_j^{(y)}$ street segments is intact.

Proposition 19 (MGM upper bounded by RHM)**.** *The RHM with parameters*

$$p = e^{-2r\lambda} \qquad \text{and} \qquad \mathbb{P}(N_i^{(x)} \geq l+1) = \mathbb{P}(N_i^{(y)} \geq l+1) = \big(1 - e^{-2r\max\{\mu_x,\mu_y\}}\big)^l \ \forall l \in \mathbb{N} \tag{4}$$

percolates if the MGM does.

The proof is given in Section 3.2. The RSL and the RHM share the following dual relation: We obtain the RSL by making all open edges of the RHM's dual lattice closed and vice versa. Therefore, circuits in the RSL are of particular interest.

Proposition 20 (Existence of arbitrarily large circuits)**.** *With $p_c \in (0,1)$ and $N^{(x)}, N^{(y)}$ as in Theorem 8, the following holds almost surely: For every $p \geq p_c$ and every finite $V \subset \mathbb{Z}^2$, there exists an open circuit in the RSL such that V lies inside that circuit.*

All of Section 4 is dedicated to the proof of Proposition 20. By Peierls' argument, both these propositions yield the absence of an infinite cluster – that is Theorem 4 – in the following way:

Proof of Theorem 4. By Proposition 19, the MGM is upper bounded by a RHM with parameters as in Equation (4). Due to the dual relation between the RHM and the RSL, Peierls' argument tells us that the RHM does not percolate if we find an open circuit surrounding the $[-1,1]^2$ box in the RSL. Using Corollary 10, there exists a $\lambda_c(r,\mu)$ such that this RSL has the same distribution as a RSL with parameters as in Theorem 8 for every $\lambda \leq \lambda_c(r,\mu)$. Due to Proposition 20, we always find such an open circuit and conclude that the RHM does not percolate almost surely. Therefore, the MGM does not percolate either. □

3.2 Discretizing the Manhattan grid model

We discretize the MGM in a way that yields a RHM with parameters as in Proposition 19. The procedure relies on grouping streets to clusters:

Definition 21 (Enumeration of r-clusters)**.** Let $r > 0$ and $\phi \subset \mathbb{R}$. Then, $C \subset \phi$ is called an r**-cluster** of ϕ if there exists a connected component $A \subset \mathbb{R}$ of

$$\bigcup_{x \in \phi} (x - r, x + r)\,,$$

such that $C = A \cap \phi$. Given $x \in \phi$, we write $C(x,\phi)$ for the cluster containing x. Now, assume that $\phi \subset \mathbb{R}$ is locally finite and unbounded in both directions. We can enumerate the clusters in the following way: Let $C_0(\phi) := C(x_0,\phi)$ where

$$x_0 := \min\{x \in \phi \,|\, x > 0,\ x = \max C(x,\phi)\}\,.$$

Given $C_0(\phi), \dots, C_i(\phi)$, let $C_{i+1}(\phi) := C(x_{i+1}, \phi)$ where

$$x_{i+1} = \min\{x \in \phi \,|\, x > 0,\, x \notin C_k(\phi)\, \forall 0 \le k \le i\}\,.$$

In this way, we have defined $C_i(\phi)$ for all $i \in \mathbb{N}$. In a similar way, we can define $C_{-i}(\phi)$. We let $C_{-i-1}(\phi) := C(x_{-i-1}, \phi)$ for

$$x_{-i-1} := \max\{x \in \phi \,|\, x \notin C_k(\phi)\, \forall k \ge -i\}\,.$$

Proof of Proposition 19. We look at clusters of streets in the MGM as in Definition 21. Let $C_i^{(x)} := C_i(\Phi^{(x)})$ and $C_j^{(y)} := C_j(\Phi^{(y)})$, sketched in Figure 6. Since $\Phi^{(x)}$ is a Poisson point

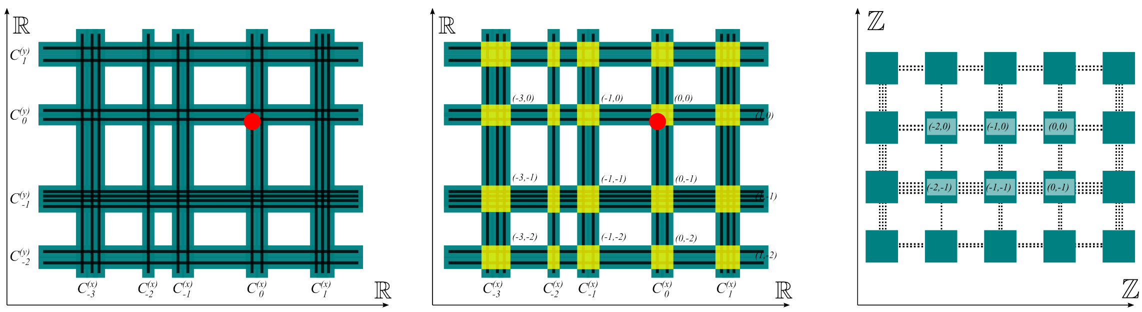

Figure 6: Left: Realization of the street system. Around each street, we consider its r neighborhood (blue) and distinguish the streets by clusters and enumerate them $(\dots, C_{-1}^{(x)}, C_0^{(x)}, C_1^{(x)}, \dots)$. The red disk indicates the origin $(0,0)$. Middle: We identify the crossings of vertical street clusters with horizontal street clusters as vertices (yellow rectangles). Right: This results in a discretized model with multi-edges.

process of intensity $\mu_x \le \mu$, we have that $N_i^{(x)} := \#C_i^{(x)}$ is a geometric random variable with

$$\mathbb{P}(N_i^{(x)} \ge l+1) = (1 - e^{-2r\mu_x})^l \le (1 - e^{-2r\mu})^l \;\forall l \in \mathbb{N}$$

and all the $N_i^{(x)}$ are independent from each other. The same holds for $\Phi^{(y)}$. For each $(i,j) \in \mathbb{Z}^2$, we may look at the rectangle

$$C_{i,j} := [\min C_i^{(x)}, \max C_i^{(x)}) \times [\min C_j^{(y)}, \max C_j^{(y)})\,.$$

Each such rectangle $C_{i,j}$ directly connects only to its neighbors $C_{i',j'}$, i.e., $|i-i'|+|j-j'| = 1$. Let us consider $C_{i,j}$ and $C_{i+1,j}$ for now. We know that

$$\inf_{x_1 \in C_i^{(x)}, x_2 \in C_{i+1}^{(x)}} \|x_1 - x_2\| \ge 2r\,,$$

otherwise they would have combined. Therefore, if $C_{i,j}$ connects to $C_{i+1,j}$ in the MGM, it has to do so via one of the $N_j^{(y)} = \#C_j^{(y)}$ horizontal streets. In particular, there needs to be a Cox point $P = (x_z, y_z) \in \Psi$ of the MGM such that

$$x_z \in (\max C_i^{(x)}, \max C_i^{(x)} + 2r) \qquad \text{and} \qquad y_z \in C_j^{(y)}\,.$$

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The probability that such a P exists under a realization of $N^{(x)}$ and $N^{(y)}$ is therefore

$$1 - p^{N_j^{(y)}} = 1 - (e^{-2r\lambda})^{N_j^{(y)}} .$$

If we collapse the rectangles $C_{i,j}$ into nodes, we obtain a RHM on $\mathbb{Z}^2$ with parameters as in Equation (4). Moreover, percolation of the MGM implies percolation of the RHM. □

4 Existence of arbitrarily large blocking circuits

As said before, the existence of circuits in the RSL will heavily depend on the framework developed in [Hof05]. Therefore, we recapitulate the most relevant objects and results for the reader's convenience.

Notation. The $x = (x_i)_{i\in\mathbb{Z}}$ in [Hof05] corresponds to $N = (N_i^{(x)})_{i\in\mathbb{Z}}$ here. From now on, $[i, j]$ will be an interval of integers, i.e.,

$$[i, j] := \{i,\, i + 1, \ldots ,\, j - 1,\, j\} .$$

4.1 Bands and labels

The idea is to group columns into bands depending on how "bad" they are. A column i is bad if $N_i^{(x)}$ is large. Bad columns merge into bands which are even "worse". The procedure is done in a way that the resulting bands are exponentially far apart depending on their "badness". A key result is that the resulting bands are finite if $N_i^{(x)}$ is sufficiently light-tailed.

For now, let $N := (N_i)_{i\in\mathbb{Z}}$ be an arbitrary sequence with $N_i \in \mathbb{N}_{\geq 1}$. We will consecutively define the k bands of N, see Figure 7 for a rough illustration.

Definition 22 (k bands and k labels)**.** A 1 **band** is $\{i\}$ for $i \in \mathbb{Z}$. The 1 **label** of $\{i\}$ is

$$f_1(i) := N_i .$$

We now inductively define $k + 1$ bands and their $k + 1$ labels. Find $i \in \mathbb{Z}$ with the smallest $|i + 0.1|$ (i.e., $-i$ is preferred over i) such that there exists a $j \in \mathbb{Z}$ satisfying

1 j is not in the same k band as i,

2 $|j| \leq |i|$, and

3 $\min\big(f_k(i), f_k(j)\big) - \frac{1}{6}\log_2|1 + D_k(i, j)| > 1$ where

$$D_k(i, j) := \#\{k \text{ bands between } i \text{ and } j \text{ not containing either}\} .$$

As an example, we have $1 + D_1(i, j) = |i - j|$.

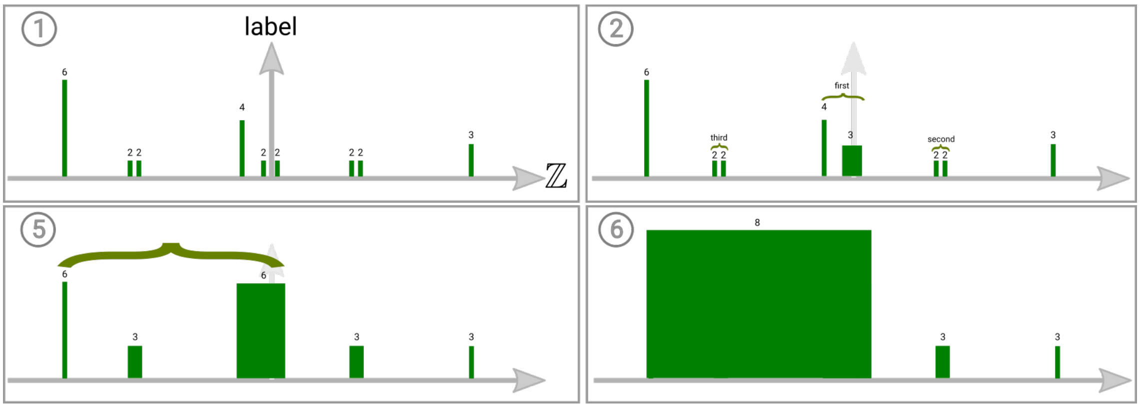

Figure 7: k bands and labels for $k = 1, 2, 5, 6$. The base height for labels in the diagrams is 1In picture 2, we indicate the order in which the k bands merge. After $k = 6$, nothing combines inside this observation window anymore.

If no such i exists, set $f_{k+1} := f_k$ and all the $k+1$ bands are the same as the k bands. Otherwise, define the $k+1$ **bands** in the following way:

1 If $[m, n]$ is a k band with $\{i, j\} \cap [m, n] = \emptyset$, then it is a $k+1$ band. In this case, all $s \in [m, n]$ have the $k+1$ **label** $f_{k+1}(s) := f_k(s)$.

2 Let $[m_i, n_i]$ be the k band containing i and $[m_j, n_j]$ the k band containing j. Then, $[\tilde{m}, \tilde{n}]$ is a $k+1$ band with $\tilde{m} := \min\{m_i, m_j\}$ and $\tilde{n} := \max\{n_i, n_j\}$. In this case, all $s \in [\tilde{m}, \tilde{n}]$ have the $k+1$ **label**

$$f_{k+1}(s) := f_k(i) + f_k(j) - \left\lfloor \tfrac{1}{18} \log_2 |1 + D_k(i, j)| \right\rfloor .$$

Note that $f_{k+1}(s) \geq \max\{f_k(i), f_k(j)\} + 1$.

Remark (Short summary)*.* In each step, two k bands and all bands in between will merge into a bigger $k+1$ band of higher label. All elements inside a k band have the same k label. Each k band will always consist of intervals of integers. Bands around the origin will be combined before others. Since we will consider N_i generated by nontrivial independent random variables, the merging procedure will never globally terminate.

Lemma 23 (Exponential decay of band labels, [Hof05, Lemma 3.4])**.** *If the N_i are independent random variables with $\mathbb{P}(N_i \geq l+1) \leq 2^{-1000 \cdot l}$ for all $i \in \mathbb{Z}$ and $l \in \mathbb{N}$, then for any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$*

$$\mathbb{P}\big(j \text{ lies in a } k \text{ band of label } \geq l\big) \leq 2^{-399 \cdot l} .$$

In particular, the following holds almost surely: For each $j \in \mathbb{Z}$, there exists a $K \in \mathbb{N}$ such that for all $k \geq K$, all the k bands containing j are identical.

The idea of the proof is to use the light-tailedness of the random variables to suppress certain combinatorial terms. This on the notion of (maximal) generators of k bands which will be introduced later. The second statement follows from Borel–Cantelli.

Definition 24 (Bands and labels)**.**

1 An (integer) interval $[m, n]$ is called a **band** (without k in front) if there exists some $K \in \mathbb{N}$ such that $[m, n]$ is a k band for all $k \geq K$. For $j \in \mathbb{Z}$, the label of j is $f(j) := \lim_k f_k(j)$. The label of a band $[m, n]$ is $f(m)$.

2 If $N = (N_i)_{i\in\mathbb{Z}}$ is such that $\mathbb{Z}$ decomposes into finite bands, then we call N **good**.

Note that bands and their labels are always finite, i.e., $f(m) < \infty$. From now on, we will only concern ourselves with good $N = (N_i)_{i\in\mathbb{Z}}$. The first thing we do is to enumerate k bands as well as bands similar to the enumeration of clusters in Section 3.

Definition 25 (Enumeration of bands)**.** Given $N = (N_i)_{i\in\mathbb{Z}}$, write $B^N_{0,k}$ for the k band containing 0. Set $B^N_{1,k}$ to be the k band containing $1+\max B^N_{0,k}$ and $B^N_{-1,k}$ to be the k band containing $-1 + \min B^N_{0,k}$. Inductively, this defines $B^N_{i,k}$ for all $i \in \mathbb{Z}$. Since $N = (N_i)_{i\in\mathbb{Z}}$ is good, we can analogously define B^N_i.

The “size” of a band is limited by its label and also, as indicated before, bands will be exponentially far apart depending on their labels:

Lemma 26 ([Hof05, Lemma 3.1, 3.6])**.**

1 *Let $[i, j] = B^N_{m,k}$ be a k band with label l. Then, $|j - i + 1| \leq 32^{l-1}$.*

2 *If B^N_m and $B^N_{m'}$ are bands with labels $\geq l$, then $|m - m'| \geq 64^{l-1} = (2^6)^{l-1}$.*

4.2 Regular bands

We will now make the first modification to the work of [Hof05].

Definition 27 (Neighboring bands and regularity)**.**

- Two bands B^N_m and $B^N_{m'}$ are called **neighboring bands with labels** $\geq l$ if they both have labels $\geq l$ and there is no band with label $\geq l$ in between.
- The good sequence $N = (N_i)_{i\in\mathbb{Z}}$ is called **regular** if for all l and all neighboring bands B^N_m and $B^N_{m'}$ with labels $\geq l$, we have $|m - m'| \in [64^{l-1}, 12 \cdot 64^{l-1}]$ and N is unbounded in both directions.

A regular sequence is “regular” in the sense that bands with certain label sizes regularly show up and are not spread too far apart. Next, we will show that a good sequence N can always be made regular by making it larger. N being unbounded guarantees the existence of bands of labels $\geq l$ for all $l \in \mathbb{N}$ and that each such band has exactly 2 neighbors.

Definition 28 ((Maximal) generators of bands)**.**

1 Let $B^N_{m,k} = [i,j]$ be a k band. Then, the k **generators** of $B^N_{m,k}$ are i and j. For $1 \leq \tilde{k} < k$, the $\tilde{k}$ generators of $B^N_{m,k}$ are the $\tilde{k}$ generators of the $\tilde{k}$ bands inside $[i,j]$ containing a $\tilde{k}+1$ generator of $B^N_{m,k}$. The 1 generators of $B^N_{m,k}$ are called the **generators** of $B^N_{m,k}$.

2 Let g be a generator of a band B^N_m. Then, g is called a **maximal generator** of B^N_m if the following holds. If $[i_1, j_1]$ and $[i_2, j_2]$ are two k bands that combine into the $k+1$ band $[i_1, j_2]$ with $g \in [i_1, j_2]$, then the label of k band containing g (i.e., either $[i_1, j_1]$ or $[i_2, j_2]$) has label greater or equal to the label of the other k band used to combine into $[i_1, j_2]$.

3 One verifies that a band always has at least one maximal generator. For each band B^N_m, we will pick its smallest maximal generator $g(B^N_m)$.

Lemma 29 (High labels near origin)**.** *Let $N^{(x)}$ and $N^{(y)}$ as in Theorem 8. Consider the event*

$$A_l := \left\{\forall \textit{bands } B^{N^{(x)}}_m, B^{N^{(y)}}_m \textit{ with } |m| \leq 12 \cdot 64^l, \textit{ their labels are } < l\right\}.$$

Then,

$$\lim_{l\to\infty} \mathbb{P}(A_l) = 1\,.$$

In particular, we have that almost-surely infinitely many of the A_l occur.

Proof. Let $C > 0$. Recall that $N^{(x)} = (N^{(x)}_i)_{i\in\mathbb{Z}}$ and $N^{(y)} = (N^{(y)}_i)_{i\in\mathbb{Z}}$ are families of mutually independent random variables with $\max\{\mathbb{P}(N^{(x)}_i \geq l+1), \mathbb{P}(N^{(y)}_i \geq l+1)\} \leq 2^{-1000\cdot l}$. By Lemma 23, we have

$$\begin{aligned}
\mathbb{P}\big(&\forall \text{bands } B^{N^{(x)}}_m \text{with } |m| \leq C\cdot 64^l, \text{ their labels are } < l\big) \\
&\geq 1 - \sum_{|m|=0}^{C\cdot 64^l} \mathbb{P}(B^{N^{(x)}}_m \text{ has label } \geq l) \\
&\geq 1 - 2\cdot C\cdot 64^l \cdot 2^{-399l} \geq 1 - C\cdot 2^{-350l}\,.
\end{aligned}$$

$N^{(x)}$ and $N^{(y)}$ are independent, so for the event

$$A_l(C) := \left\{\forall \text{bands } B^{N^{(x)}}_m, B^{N^{(y)}}_m \text{ with } |m| \leq C\cdot 64^l, \text{ their labels are } < l\right\},$$

we have that

$$\mathbb{P}(A_l(C)) \geq \left(1 - C\cdot 2^{-350l}\right)^2\,,$$

which proves $\lim_l \mathbb{P}(A_l(C)) = 1$. The last statement follows from Borel–Cantelli. □

Next, we want to find a regular $\tilde{N}^{(x)} \geq N^{(x)}$ such that it generates the same bands as $N^{(x)}$.

Lemma 30 (Raising labels of maximal generators, [Hof05, Lemma 3.7]). *Let $N = (N_i)_{i\in\mathbb{Z}}$ be good. Let B_m^N be a band of label l and $i' \in \mathbb{Z}$ be a maximal generator of B_m^N. If for all bands $B_{m'}^N$ of label $> l$, we have that $|m - m'| \geq 64^l$, then the sequence*

$$\tilde{N}_i = \begin{cases} N_i & i \neq i' \\ N_i + 1 & i = i' \end{cases}$$

satisfies the following properties:

1. $B_{n,k}^N = B_{n,k}^{\tilde{N}}\ \forall n \in \mathbb{Z}, k \in \mathbb{N}$, *i.e. all k bands are identical and $\tilde{N}$ is also good.*
2. *If the k label of $B_{n,k}^N$ is t, then the k label of $B_{n,k}^{\tilde{N}}$ is $t + \mathbb{1}\{i' \in B_{n,k}^N\}$.*

In particular, i' is still a maximal generator of $B_m^{\tilde{N}}$.

Lemma 31 (Making N more regular, [Hof05, Lemma 3.8]). *Let N be good. For each $L \geq 1$, there exists $N^L = (N_i^L)_{i\in\mathbb{Z}}$ such that*

1. $N \leq N^L \leq N^{L+1}$,
2. $B_{m,k}^N = B_{m,k}^{N^L}$ *for all $m \in \mathbb{Z}, k \in \mathbb{N}$, and*
3. *if $B_m^{N^L}$ and $B_{m'}^{N^L}$ are neighboring bands with label $\geq l$ and if $l \leq L$, then*

$$|m - m'| \in [64^{l-1}, 3 \cdot 64^{l-1}).$$

Furthermore, N^L can be chosen such that $(N_i^L)_{L\in\mathbb{N}}$ is unbounded for at most one i.

Due to its relevance in Lemma 35, we will give the proof again here.

Proof. We only consider the case of N being unbounded in both directions. The general case is proven similarly with slightly more technicalities. By Lemma 30, we may artificially raise the labels of bands to make N^L “more regular”. We show the claim via induction on L. For $L = 1$, set $N^1 := N$. Now, suppose the claim is true for L. Consider the sets of indices

$$\underline{S}(L) := \{m \in \mathbb{Z} \mid B_m^{N^L} \text{ has label } \geq L\} \qquad \text{and} \qquad \overline{S}(L) := \{m \in \mathbb{Z} \mid B_m^{N^L} \text{ has label } \geq L{+}1\}.$$

Clearly $\overline{S}(L) \subset \underline{S}(L)$. We now want to raise the labels of some bands in $\underline{S}(L)$ so that the regularity condition holds for $l = L$. More explicitly, we define an index set S such that

1. $\overline{S}(L) \subset S(L) \subset \underline{S}(L)$.

2 $|m - m'| \geq 64^l$ for all $m, m' \in S(L)$.

3 For any $m \in S(L)$, there exists a $m' \in S(L)$ with $|m - m'| \leq 3 \cdot 64^l$.

We do so in the following way. $\overline{S}(L) \neq \emptyset$ since N is unbounded, so consider $m \in \overline{S}(L)$. Let $m' := \min\{\tilde{m} \in \overline{S}(L) \,|\, \tilde{m} > m\}$. Let $m^{(0)} := m$. If $m' - m^{(0)} > 3 \cdot 64^l$, we choose a $m^{(1)} \in \underline{S}(L)$ such that $m^{(0)} + 64^l \leq m^{(1)} \leq m^{(0)} + 64^l + 3 \cdot 64^{l-1}$. We can do so by the induction hypothesis. We check that

$$m' - m^{(1)} \geq \left(m^{(0)} + 3 \cdot 64^l\right) - \left(m^{(0)} + 64^l + 3 \cdot 64^{l-1}\right) > 64^l \,.$$

If $m' - m^{(1)} > 3 \cdot 64^l$, we define again $m^{(2)}$ and proceed until we find $m^{(s)}$ such that $m' - m^{(s)} < 3 \cdot 64^l$. Define $S_m(L) := \{m^{(0)}, \dots, m^{(s)}\}$ and finally

$$S(L) := \bigcup_{m \in \overline{S}(L)} S_m(L) \,.$$

This $S(L)$ satisfies all of our 3 conditions. We now define N^{L+1} in the following way:

$$N_i^{L+1} = \begin{cases} N_i^L + 1 & \text{if } i = g(B_m^N) \text{ for some } m \in S(L) \backslash \overline{S}(L) \\ N_i^L & \text{else.} \end{cases}$$

By Lemma 30, N_i^{L+1} is as desired. The unboundedness part is proven in the next lemma. □

Lemma 32 (Making sequences regular, [Hof05, Lemma 3.9])**.** *Let N be good. There exists a sequence $\tilde{N} \geq N$ such that all the k bands for $\tilde{N}$ are identical to the k bands for N and such that for neighboring bands $B_m,\ B_{m'}$ of label $\geq l$, we have*

$$|m - m'| \in [64^{l-1},\, 6 \cdot 64^{l-1}) \,.$$

In particular, $\tilde{N}$ is regular. (The labels may differ.)

Proof. With N^L from Lemma 31, we consider

$$N_i^\infty := \lim_{L \to \infty} N_i^L \in \mathbb{N} \cup \{\infty\} \,.$$

We make the following observations:

1 If $N_i^\infty = \infty$, then i must be the maximal generator of some band B_m^N.

2 $N_i^\infty = \infty$ for at most one i. Otherwise, we would find two separate bands $B_m^N \ni i$ and $B_{m'}^N \ni i'$. The label of $B_m^{N^L}$ is bounded from below by N_i^L, respectively $N_{i'}^L$ for $B_{m'}^{N^L}$. So for $l > 0$ such that $|m - m'| < 64^l$ and L such that $\min(N_i^L,\, N_{i'}^L) \geq l$, we would violate Lemma 31 Condition 3, on the minimal distance between bands.

Let i^∞ be the value with $N^\infty_{i^\infty} = \infty$. We set

$$\tilde{N}_i = \begin{cases} \lim_{L\to\infty} N_i^L & i \neq i^\infty \\ N_i & i = i^\infty \end{cases}.$$

By construction, we have that neighboring bands $B^{\tilde{N}}_m,\ B^{\tilde{N}}_{m'}$ always satisfy

$$|m - m'| \in [64^{l-1}, 6 \cdot 64^{l-1})$$

which shows the claim. □

4.3 l segments and their interior

The following additions enable us to prove the existence of circuits in the RSL which have not been a focus in the original work. We make the following observation: Let N be regular and $B^N_m,\ B^N_{m'}$ be two neighboring bands of label $\geq\ l+1$. Let $\{m_0, \dots, m_k\} = \{\tilde{m} \in [m, m'] \,|\, B^N_{\tilde{m}}$ has label $\geq l\}$. Then, $k \geq 6$ since $m_i - m_{i-1} < 12 \cdot 64^{l-1}$ and $m' - m = m_k - m_0 \geq 64^l$. With the same reasoning, we have $k \leq 12 \cdot 64 = 768$. Our next object of interest is "the space between neighboring bands":

Definition 33 (l segments)**.** Let N be good and $[i_1,\ i_2],\ [i_3,\ i_4]$ be two neighboring bands of label $\geq l$ (for N). Then we call

$$[i_2 + 1, i_3]$$

an l **segment**. We will also call $[i_2 + 1, i_3]$ an l segment if there is a good sequence M such that

$$M_i = N_i \quad \forall i \in [i_2 + 1, i_3]$$

and $[i_2 + 1, i_3]$ is an l segment for M.

Definition 34 (Inside of an l segment)**.** Let N be good. We say that i **lies on the inside of an** $l+1$ **segment** $\mathcal{S}$ if all of the following hold.

1 i lies in a band B^N_m of label $< l$.

2 There are 2 different bands of label l denoted by $B^N_{m_1^+},\ B^N_{m_2^+}$ inside $\mathcal{S}$ with $m_1^+, m_2^+ > m$.

3 There are 2 different bands of label l denoted by $B^N_{m_1^-},\ B^N_{m_2^-}$ inside $\mathcal{S}$ with $m_1^-, m_2^- < m$.

See Figure 8 for illustrations of l segments and their interior.

We will now use the proof of Lemmas 31, 32 and 29 to make sure that almost surely, the origin will lie on the inside of an $l+1$ segment for both $N^{(x)}$ and $N^{(y)}$ for infinitely many l. The downside is that the sequence becomes less regular.

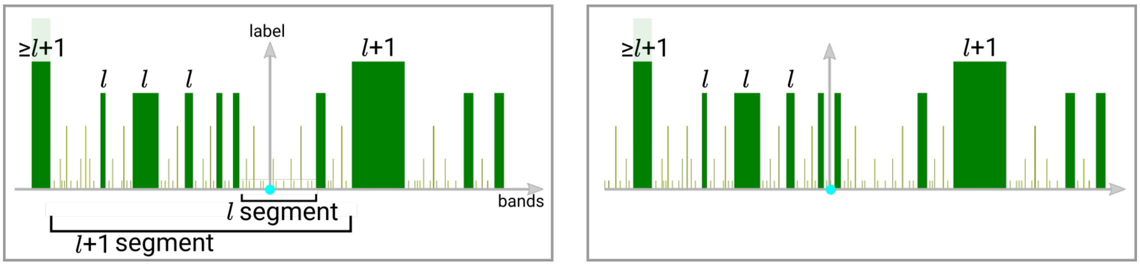

Figure 8: $l+1$ bands and their $l+1$ segments in between. The blue ball indicates the origin 0. 0 is not on the inside of its $l+1$ segment in the left picture, but it is on the inside in the right picture.

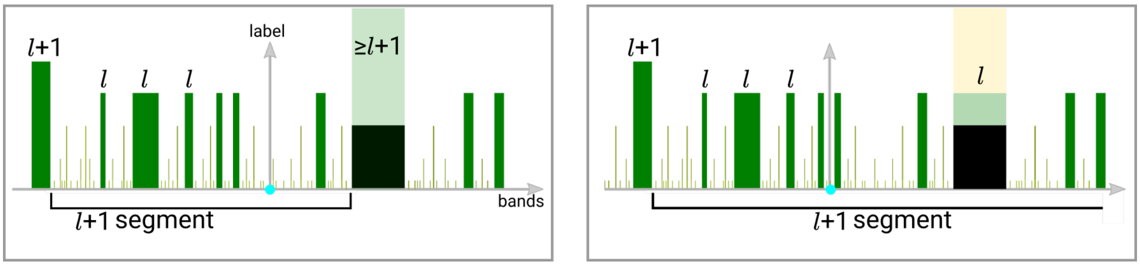

Figure 9: Left: The origin does not lie on the inside of its $l+1$ segment since it is too close to the right band of label $\geq l+1$ (under the sequence $\tilde{N}^{(x)}$). The label of that band has been raised in the construction of Lemma 31 (indicated in light green). However, Lemma 29 tells us that for infinitely many l, the label under $N^{(x)}$ is lower than that (black portion). Right: We can lower the value of the band to l without changing the band structure but still being larger than $N^{(\bullet)}$. Then, the $l+1$ segment containing 0 becomes larger and 0 is now on the inside of its $l+1$ segment.

Lemma 35 (Regular sequence with origin inside segments (see Figure 9))**.** *Almost surely, there exist regular $\hat{N}^{(x)}, \hat{N}^{(y)}$ with $N^{(x)} \leq \hat{N}^{(x)} \leq \tilde{N}^{(x)}$ with $\tilde{N}^{(x)}$ from Lemma 32 (respectively for $\hat{N}^{(y)}$) such that for infinitely many l, 0* lies on the inside of an l segment for both $\hat{N}^{(x)}$ and $\hat{N}^{(y)}$.

Proof. All the constructions for $\hat{N}^{(x)}$, we also do for $\hat{N}^{(y)}$ with the symbols (x) and (y) exchanged. First, set $\hat{N}^{(0,x)} := \tilde{N}^{(x)}$. From Lemma 32, we know that neighboring bands of label $\geq l$ are at most $6 \cdot 64^{l-1}$ bands apart. By Lemma 29, we know that almost surely

$$A_l = \{\forall \text{bands } B_m^{N^{(x)}}, B_m^{N^{(y)}} \text{ with } |m| \leq 12 \cdot 64^{l-1}, \text{ their labels are } \leq l\}$$

happens for infinitely many l. Let $\tilde{l}_0^{(x)}$ be the label of $B_0^{\tilde{N}^{(0,x)}}$, $\bar{l}_0 := \max\{\tilde{l}_0^{(x)}, \tilde{l}_0^{(y)}\}$ and

$$l_1 := \min\{l \geq \bar{l}_0 + 1 \,|\, A_l \text{ happens}\}\,.$$

We will now find $N^{(x)} \leq \hat{N}^{(1,x)} \leq \hat{N}^{(0,x)}$ such that 0 lies on the inside of the $l_1 + 1$ segment for $\hat{N}^{(1,x)}$. Assume that 0 does not lie on the inside of its $l_1 + 1$ segment under $\hat{N}^{(0,x)}$. Since $l_1 \geq \bar{l}_0 + 1$, either condition 2 or 3 are violated. Assume that is is Condition 2; the other case follows analogously. Let

$$m_1^+ := \min\{m > 0 \,|\, B_m^{\hat{N}^{(0,x)}} \text{ has label } > l_1\}\,,$$

$\tilde{l}_1^{(x)}$ be the label of $B_{m_1^+}^{\hat{N}^{(0,x)}}$ and $i_1 = g(B_{m_1^+}^{N^{(x)}})$ the chosen generator. By the construction in Lemma 31 and violation of Condition 2, we have that $\left|m_1^+\right| \leq 2 \cdot 6 \cdot 64^l$. We then set

$$\hat{N}^{(1,x)} = \begin{cases} \hat{N}_i^{(0,x)} - \left[\tilde{l}_1^{(x)} - l_1\right] & \text{if } i = i_1 \\ \hat{N}_i^{(0,x)} & \text{else.} \end{cases}$$

By this construction and the fact that A_{l_1} occurs, we first verify that $N^{(x)} \leq \hat{N}^{(1,x)} \leq \hat{N}^{(0,x)}$. Furthermore, the label of $B_{m_1^+}^{\hat{N}^{(1,x)}}$ is l_1. Since the label of $B_{m_1^+}^{\hat{N}^{(1,x)}}$ is now only l_1, it no longer separates the $l_1 + 1$ segments to its left and right, so they merge together. Therefore, this new l_1+1 segment for $\hat{N}^{(1,x)}$ has at least 6 or more l_1-1 bands to the right side of 0, i.e. Condition 2 is now satisfied. We have established that for $\hat{N}^{(1,x)}$, 0 lies on the inside the interior of the $l_1 + 1$ segment. One easily verifies for neighboring bands $B_m^{\hat{N}^{(0,x)}}$, $B_{m'}^{\hat{N}^{(0,x)}}$ of label $\geq l$ that for all $l \leq \tilde{l}_1^{(x)}$, we have

$$|m - m'| \in [64^{l-1}, 12 \cdot 64^{l-1}]$$

and for all $l > \tilde{l}_1^{(x)}$

$$|m - m'| \in [64^{l-1}, 6 \cdot 64^{l-1})\,.$$

Now, set $\bar{l}_1 := \max\{\tilde{l}_1^{(x)}, \tilde{l}_1^{(y)}\}$,

$$l_2 := \min\{l \geq \bar{l}_1 + 1 \,|\, A_l \text{ happens}\}$$

and inductively continue the whole procedure. After setting $\hat{N}^{(x)}$ to be the monotone limit of $\hat{N}^{(L,x)}$, the claim holds for all $l \in \{l_1, l_2, \dots\}$. □

We conclude this section with some final remarks. Regularity alone is unfortunately insufficient to utilize the framework of [Hof05]. There, the notion of “very regular” is used to estimate the crossing probabilities along rectangular strips. The main statement we need here is the following lemma. Since no new ideas come up in our setting, the definition of “very regular” and the proof are moved to the appendix.

Lemma 36 (Very regular sequences)**.** *Let N be good and regular. Then, there exists $\overline{N} \geq N$ such that $\overline{N}$ is very regular (in particular regular) and such that all bands and their labels are identical under both $\overline{N}$ and N. In particular, we may always replace a regular sequence with a very regular sequence without changing its band structure.*

4.4 Good n boxes and conclusion

Due to the considerations made in this section so far, we may assume that $N^{(x)}$ and $N^{(y)}$ are almost surely (very) regular. Next, we will finally introduce our central objects in the $\mathbb{Z}^2$ lattice.

Definition 37 (n boxes)**.** Suppose $[i_2 + 1,\, i_3]$ is a vertical n segment and $[j_2 + 1,\, j_3]$ is a horizontal n segment. An n **box** is a product of these two segments, i.e. it is the graph with vertices

$$V = [i_2 + 1,\, i_3] \times [j_2 + 1,\, j_3]$$

and edges

$$E = \{\text{edges between two vertices in } V \text{ with at most one edge in } \partial V\}\,.$$

(This definition of an n box also applies to the generalized definition of an n segment.) We say that $o = (0, 0)$ **lies on the inside of an** n **box** if 0 lies on the inside of both generating n segments.

We will inductively define the notion of a **good** n box. For this, recall from the definition of an RSL (Definition 6) that an edge $(i, j) \leftrightarrow (i + 1, j)$ is open with probability $p^{N_i^{(x)}}$ independent from all other edges. A cluster C in a subgraph G is a maximal connected subgraph of G whose edges are all open.

Definition 38 (Good n boxes)**.** Let $\mathbb{K}_n$ be an n box.

1 A **crossing (cluster)** is an open cluster inside $\mathbb{K}_n$ which contains vertices on all four faces of $\mathbb{K}_n$.

2 For $n \leq 200$, $\mathbb{K}_n$ is called **good** if all edges in $\mathbb{K}_n$ are open.

3 For $n > 200$, $\mathbb{K}_n$ is called **good** if $a_1 + a_2 \leq 1$, where

$$\begin{aligned} a_1 &:= \#\{\text{bad } n-1 \text{ boxes inside } \mathbb{K}_n\} \\ a_2 &:= \#\{\text{pairs of neighboring good } n-1 \text{ boxes s.t. the } (n-1, n-1) \text{ strip inbetween} \\ &\qquad \text{does not have a crossing intersecting the crossing clusters of the } n-1 \text{ boxes}\}. \end{aligned}$$

The definition of a (k, k) strip is given in Definition 43 in the appendix. It can be understood as the space between two adjacent n boxes, i.e. a product of an n segment with a band of label n. For illustration, see Figure 10.

Since there are at least 6 rows and columns of $n - 1$ boxes, each good n box has a crossing cluster, so the definition above makes sense. We have the following estimate on the probability of an n box to be good:

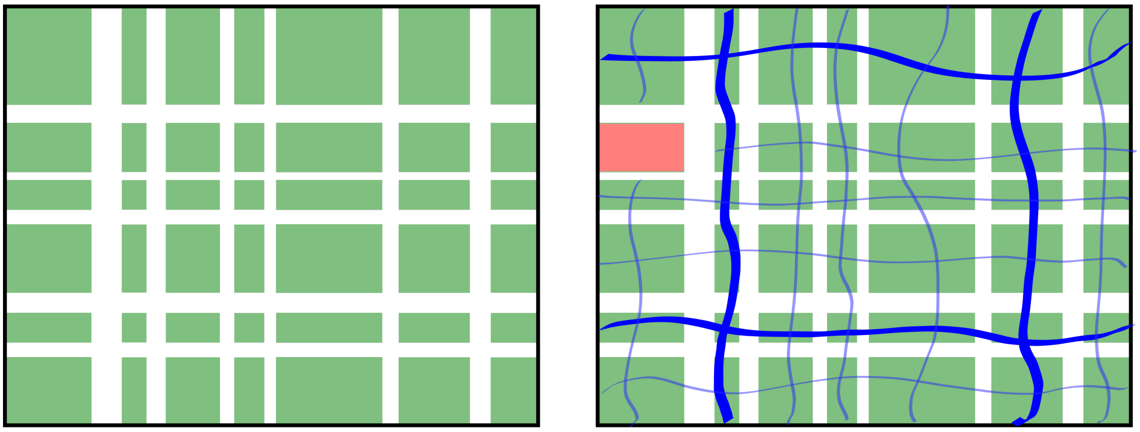

Figure 10: n boxes (green rectangles) inside a $n+1$ box (black boundary). Since each good n box has an open crossing cluster, a good $n+1$ box will also have one. Furthermore, n boxes on the inside of the good $n+1$ box are encircled by an open circuit (blue) even if there is a bad n box or a missing connection (red rectangle).

Lemma 39 (Probability of good n boxes)**.** *There exists a $p_c \in (0,1)$ such that for every $p \geq p_c$, every very regular environment $N^{(x)}, N^{(y)}$, every $n \in \mathbb{N}$ and every n box $\mathbb{K}_n$:*

$$\mathbb{P}(\mathbb{K}_n \text{ is good} \,|\, N^{(x)}, N^{(y)}) \geq 1 - 4^{-n}\,.$$

This follows immediately from Lemma 50 in the appendix. The proof of that is quite complicated and identical to the one in [Hof05, Lemma 4.3] except for different numerical values coming from the fact that our notion of regularity is weaker.

Lemma 40 (Helpful lemma)**.** *The following statements are true:*

1. *If $v \in \mathbb{Z}^2$ lies in a good n box for every $n \leq N$, then it lies in the crossing cluster of its N box. In particular, v lies in an infinite cluster if v lies in a good n box for every $n \in \mathbb{N}$.*

2. *Let $\mathbb{K}_n$ be the n box containing the origin $o = (0,0) \in \mathbb{Z}^2$. Then,*

$$\mathbb{K}_n \nearrow \mathbb{Z}^2 \quad \text{as } n \to \infty\,.$$

3. *If $V \subset \mathbb{Z}^2$ with $\#V < \infty$, then there exists an $N_0 \in \mathbb{N}$ such that V lies completely in an n box for all $n \geq N_0$.*

4. *If $V \subset \mathbb{Z}^2$ such that V lies on the inside of a good $n+1$ box, then there exists a circuit in the $n+1$ box with V lying inside that circuit.*

Proof. Part 1 follows inductively from the definition of good n boxes. Due to regularity, the labels of bands are finite but unbounded. Therefore, Part 2 follows. Since V is finite, Part 3 follows from Part 2. If $V \subset \mathbb{K}_n$ and $\mathbb{K}_{n+1}$ is a good $n+1$ box, then there exists either a circuit along the outermost n boxes or the second outermost n boxes. This gives Part 4. □

Let us now collect the previous considerations into the main statement about good n boxes.

Corollary 41 (Finitely many bad n boxes)**.** *Under the conditions of Theorem 8, there exists a $p_c \in (0,1)$ such that for every $p \geq p_c$ and almost every realization of $N^{(x)}, N^{(y)}$, it holds almost surely that the origin o only lies in finitely many bad n boxes.*

Proof. By the considerations made in Section 4, we may assume that $N^{(x)}$and $N^{(y)}$ are almost surely very regular. Then, the claim follows from Lemma 39 and Borel–Cantelli. □

We finally have all the tools needed to prove Proposition 20, that is, the almost sure existence of arbitrarily large circuits.

Proof of Proposition 20. By Lemma 35, o lies on the inside of its n box $\mathbb{K}_n$ for infinitely many $n \in \mathbb{N}$. By Lemma 40 Part 3, there is an $N_0 \in \mathbb{N}$ such that $V \subset \mathbb{K}_n$ for all $n \geq N_0$. By Corollary 41, only finitely many $\mathbb{K}_n$ are bad. Let $N_1 \in \mathbb{N}$ such that all $\mathbb{K}_n$ with $n \geq N_1$ are good. Finally, choose $N \geq \max\{N_0 + 1,\ N_1\}$ such that o lies on the inside of $\mathbb{K}_N$. Then, by Lemma 40 Part 4, there exists an open circuit surrounding $\mathbb{K}_{N-1}$, in particular V. □

A Appendix

The appendix deals with the proof of Lemma 50. Let us note that the following section is to some extent a detailed reproduction of the results in [Hof05]. Due to our manual construction of circuits, we lose out on regularity which in turn gives us weaker estimates. Coupled with the technical challenges of the framework considered in [Hof05], we decided to reformulate the procedure with greater detail and with additional illustration.

A.1 Definitions: vertical/horizontal bands, boxes and strips

We first give the remaining definitions. They differ slightly from the original paper due to the additional definition of a “segment” (see Definition 33).

Definition 42 (Vertical/horizontal bands and segments)**.**

1. A **vertical band** is a band that is generated from $N^{(x)} = (N_i^{(x)})_{i\in\mathbb{Z}}$. A **vertical segment** is a segment generated from vertical bands.
2. A **horizontal band** is a band that is generated from $N^{(y)} = (N_i^{(y)})_{i\in\mathbb{Z}}$. A **horizontal segment** is a segment generated from horizontal bands.

Next, we deal with more rectangular objects.

Definition 43 (Vertical/horizontal strips)**.** Let $N^{(x)}$ and $N^{(y)}$ be regular. Suppose $[j_5,\ j_6]$ is a horizontal band with label n and $[i_2+1,\ i_3]$ is a vertical m segment. Then, we say that

$$V = [i_2+1, i_3] \times [j_5,\ j_6+1]$$

is a **horizontal** (m,n) **strip** (first argument for the segment, second argument for the band). This graph with vertices V has edges

$$E = \{\text{edges between two vertices in } V \text{ with at most one edge in } \partial V\}\,.$$

We will also call V a **horizontal** (m,n) **strip** if there are $M^{(x)}$ and $M^{(y)}$ which are regular such that

$$M_i^{(x)} = N_i^{(x)} \qquad \text{and} \qquad M_j^{(y)} = N_j^{(y)} \quad \forall (i,j) \in V$$

and V is a horizontal (m,n) strip for $M^{(x)}$ and $M^{(y)}$. Analogously, we define the notion of a **vertical strip** by exchanging the roles of horizontal and vertical.

A.2 Very regular bands and segments

Lastly, we need a bit more information about the internal structure of bands. This is needed to obtain crossing probabilities of strips.

Definition 44 (Very regular k bands and n segments)**.** Let a regular sequence N be given.

1 Any k band that is a singleton $[i,i]$ is very regular.

2 Any 1 segment is very regular.

3 Any 2 segment $[a,\ b]$ is very regular if $a-b \in [6, 768]$.

4 Let $[a,d]$ be a k band with label l which was formed by combining the $\tilde{k}$ bands $[a,\ b]$ and $[c,\ d]$ into the $\tilde{k}{+}1$ band $[a,\ b]$. $[a,\ b]$ is called very regular if there are $b_1 = b, b_2, \dots, b_m$ as well as $c_1, c_2 \dots, c_{m-1}, c_m = c$ with $m \le 768$ as well as a $q > 0$ such that

   4.1 All $\tilde{k}$ bands inside the interval $[a,\ b]$ are very regular $\tilde{k}$ bands.

   4.2 For all s, we have that $[b_s,\ c_s]$ is a very regular q segment.

   4.3 For all $s < m$, we have that $[c_s,\ b_{s+1}-1]$ is a very regular $\tilde{k}$ band with label q.

5 An n segment $\mathcal{S}$ is called **N** if

   5.1 All k bands with labels $n-1$ inside $\mathcal{S}$ are very regular.

   5.2 All $l-1$ segments inside $\mathcal{S}$ are very regular.

6 A band is called very regular if it is a **very regular** k band for some k.

7 A regular sequence N is called **very regular** if all the bands generated by N are very regular.

Proof of Lemma 36. This is [Hof05, Lemma 3.12] which is an analogon to Lemma 31 and is proven similarly. For that, one establishes the analogon of Lemma 30. The labels of the final bands being unchanged follows from the construction: To make bands very regular, one only needs to change the labels of the k bands "inside". But these labels do not contribute to the label of the final combined band. □

We conclude this section with the following lemma which covers both the main aspect from [Hof05] (being able to reduce the random sequence to a very regular one) and our own priority (making sure that the origin is always on the inside of an n box):

Lemma 45 (Very regular N with origin on the inside)**.** *For almost every realization of $N^{(x)}$ and $N^{(y)}$ as in Theorem 8, there exists $\overline{N}^{(x)} \geq N^{(x)}$ such that $\overline{N}^{(x)}$ is very regular (analogously for $N^{(y)}$) and such that for infinitely many l, 0 lies on the inside of an l segment for both $\overline{N}^{(x)}$ and $\overline{N}^{(y)}$.*

This is a direct consequence of Lemma 35 and Lemma 36.

A.3 $(4, m)$ trees and setting

Fix some very regular $N^{(x)}$ and $N^{(y)}$. Recall the definitions of (good) n boxes and their crossing clusters (Definition 37). A crossing of a horizontal (m, n) strip $[a, b] \times [c, d]$ is a cluster in the (m, n) strip which contains at least one vertex in $[a, b]\times[c]$ and at least one vertex in $[a, b]\times[d]$.

We define the notion of a $(4, m)$ tree in a horizontal (m, n) strip inductively. In the end, a $(4, m)$ tree will be a set of vertices on the vertical ends of the (m, n) strip.

Definition 46 ($(4, m)$ trees)**.** Consider horizontal (m, n) strips.

- Let $n \in \mathbb{N}$. Let $[a, b]\times[c, d]$ be a $(2, n)$ strip (i.e., $[a, b]$ is the 2-segment and $[c, d]$ the n band) and $I \subset [a, b]$ with $\#I = 4$. We then define two $(4, 2)$ **trees** T and T' in a $(2, n)$ strip by
$$T := I \times \{c\} \qquad \text{and} \qquad T' := I \times \{d\}\,.$$
- Each (m, n) strip contains at least six disjoint $(m - 1, n)$ strips. A $(4, m)$ **tree** in an (m, n) strip is a union of 4 of the $(4, m - 1)$ trees within the (m, n) strip.

Thus we see that each $(4, m)$ tree in an (m, n) strip consists of 4^{m-1} vertices. Furthermore, for any $m' < m$, we have that the 4^{m-1} vertices lie in $4^{m-m'}$ different $(4, m')$ trees in disjoint (m', n) strips.

The following lemma shows why we need to work with 4 even though each n segment contains at least six $n-1$ segments:

Lemma 47 ($(4,n)$ trees given by pairs of good n boxes)**.** *Every pair of good n boxes separated by an (n,n) strip defines at least one $(4,n)$ tree on each side of the (n,n) strip with all its vertices lying inside the crossing cluster of the n box. Furthermore, the tree is the same except for the side it is located.*

Proof. The proof is by induction. Every pair of good 2 boxes separated by a $(2,2)$ strip has at least 4 pairs of vertices, one in each of the 2 boxes, such that every pair is separated by one edge. This forms a $(4,2)$ tree in the $(2,2)$ strip. Every pair of good n boxes separated by an (n,n) strip has at least 6 pairs of $n-1$ boxes, one in each of the n boxes, such that every pair is separated by an $(n-1,n-1)$ strip. In each of the good n boxes, at least 5 of these six $n-1$ boxes are good. Thus every pair of good n boxes separated by an (n,n) strip has at least 4 pairs of good $n-1$ boxes, one in each of the n boxes, such that every pair is separated by an $(n-1,n-1)$ strip. With the induction hypothesis, this forms a $(4,n)$ tree in the (n,n) strip with all its vertices lying in good n' boxes, $n' \le n$. Therefore, these vertices lie in the crossing cluster of the n box. □

Many calculations will require the following lemma:

Lemma 48 ([Hof05, Lemma 4.2])**.** *For any $c, p_1, \dots, p_n$ with $0 < p_i < 1$ and $a := \sum_1^n p_i$, we have*

$$1 - \prod_{i=1}^{n}(1-p_i) \ge \min\left\{1-e^{-c},\, \tfrac{a}{c}(1-e^{-c})\right\}\,.$$

Notation. For any set $T \subset \mathbb{Z}^2$, we write for the rows spanned by T

$$R(T) := \{(x,y) \in \mathbb{Z}^2 \,|\, \exists x' \in \mathbb{Z} :\ (x',y) \in T\}\,.$$

For any set $V \subset \mathbb{Z}$, we also write for the rows spanned by V

$$R(V) := \{(x,y) \in \mathbb{Z}^2 \mid y \in V\}\,.$$

Analogously, consider $C(T)$ and $C(V)$ for columns.

Assumption 49. *For the rest of this section, we fix the following (see Figure 11):*

- *Let B be any n box.*
- *Let $\bar{S} = [a,b] \times [\tilde{c}, \tilde{d}]$ be a horizontal (n,n) strip between two good n boxes.*
- *Let $R_1 \subset R(\tilde{d})$ and $R_2 \subset R(\tilde{c})$ be two $(4,n)$ trees defined by the crossing clusters of these boxes (Lemma 47).*
- *Let $S = [\tilde{a}, \tilde{b}] \times [\tilde{c}, \tilde{d}]$ be a horizontal $(\lfloor 2n/3 \rfloor, n)$ strip inside $\bar{S}$, $T \subset R_1$ be a $(4, \lfloor 2n/3 \rfloor)$ tree in S and $T' \subset C(T) \cap R(\tilde{c})$ be a collection of $(4,k)$ trees in S with $k \le \lfloor 2n/3 \rfloor$.*

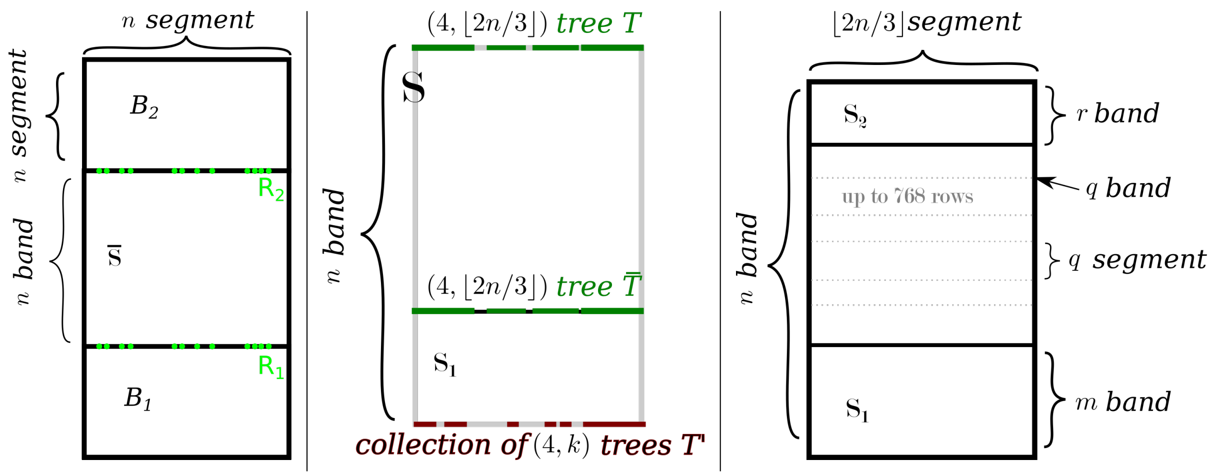

Figure 11: On the left, we have the two good n boxes B_1 and B_2 together with the (n, n) strip $\bar{S}$ between them. In the middle, we have the $(\lfloor 2n/3 \rfloor, n)$ strip S (gray box). On the right side, we divide the strip S even further into 3 parts. The main idea is to connect R_1 and R_2 in $\bar{S}$. For this, we split $\bar{S}$ into $4^{n-\lfloor 2n/3 \rfloor}$ many $(\lfloor 2n/3 \rfloor, n)$ strips S and try to find a crossing in each of these.

A.4 Proof of main lemma of the randomly stretched lattice

The rest of the paper deals with the proof of the following lemma:

Lemma 50 (Main lemma, [Hof05, Lemma 4.3]). *There exists $p_c \in (0, 1)$ such that in the RSL*

1. $\mathbb{P}(B$ *is good*$) \geq 1 - 4^{-n}$
2. $\mathbb{P}(\exists$ *a crossing of* S *intersecting both* T *and* $T') \geq \frac{\#T'}{4^{\lfloor 2n/3 \rfloor}}$.
3. $\mathbb{P}(\exists$ *a crossing of* $\bar{S}$ *intersecting both* R_1 *and* $R_2) \geq 1 - 4^{-n}$.

As said before, we paraphrase [Hof05] and verify that all results hold with modified values. The main intuition comes from the regularly stretched lattice, see [Hof05, Chapter 2].

The proof is by induction with base case $n = 200$. Statement 2 is introduced because it is possible to induct on this statement. Statement 3 then follows easily from Statement 2. When the height of S is one ($\tilde{c} = \tilde{d}$), the proof of Statement 2 is a simple calculation. The proof of Statement 2 when the height of S is greater than one is the most complicated part of the proof of Lemma 50.

Consider now the case that the height of S is greater than one. Because $N^{(x)}$ and $N^{(y)}$ are very regular, S has the following structure: We can break S up into 3 parts. On the bottom, we have a $(\lfloor 2n/3 \rfloor, m)$ strip $S_1 = [\tilde{a}, \tilde{b}] \times [\tilde{c}, \tilde{c}']$. On the top, we have a $(\lfloor 2n/3 \rfloor, r)$ strip $S_2 = [\tilde{a}, \tilde{b}] \times [\tilde{d}', \tilde{d}]$. In the middle are up to 768 rows of q boxes separated by l bands with labels q (see Figure 11). We will use the following relation:

Lemma 51. *The parameters m, r, q, n of the $(\lfloor 2n/3 \rfloor, n)$ strip S satisfy $m, r > q$ and*

$$\lfloor 2n/3 \rfloor \geq \lfloor 2m/3 \rfloor + 1\,. \tag{5}$$

Furthermore, for $q > 100$

$$\lfloor 2n/3 \rfloor > \lfloor 2m/3 \rfloor + \lfloor 2r/3 \rfloor - q + 30\,. \tag{6}$$

Proof. By the definition of very regular and the way labels were assigned to bands, we have that $m, r > q$ and

$$n = m + r - \lfloor \tfrac{1}{18} \log_2(L) \rfloor\,,$$

where

$$L := \#\{l \text{ bands between the label } m \text{ and } r \text{ bands}\}\,.$$

There exist at most $768 = 12 \cdot 64$ many very regular q segments (and at least 1), so there are at most 768 bands with label q. One q segment contains between 64^{q-1} and $12 \cdot 64^{q-1}$ many bands due to regularity. Therefore, for the number of bands in between, we have the following chain of implications

$$\begin{aligned}
64^{q-1} &\leq L \leq 12 \cdot 64^{q-1} \cdot 12 \cdot 64 \\
\tfrac{1}{3}(q-1) \leq \tfrac{1}{18}\log_2(L) &\leq \tfrac{1}{18}\log_2(2^4 \cdot 2^4 \cdot 2^{6(q-1)} \cdot 2^6) \\
\tfrac{1}{3}(q-1) \leq \tfrac{1}{18}\log_2(L) &\leq \tfrac{1}{3}(q-1) + 1\,,
\end{aligned}$$

Therefore

$$\lfloor q/3 \rfloor = \lfloor \tfrac{1}{18} \log_2\big(L\big) \rfloor + \text{either } 0 \text{ or } 1\,,$$

which is equivalent to

$$m + r - \lfloor q/3 \rfloor - n \in \{0, 1\}\,. \tag{7}$$

Using $r \geq q + 1$, we obtain $r \geq 2 + \lfloor q/3 \rfloor$. Then, Equation (7) implies Inequality (5). For $q > 100$, Equation (7) directly implies Inequality (6). □

The outline of the proof of Lemma 50 is as follows. If

1. there are “enough” (Equation (8)) crossings of S_1 which intersect T',
2. there is at least one crossing of S_2 which intersects T,
3. there exists a v contained in a crossing of S_1 which intersects T' and w contained in a crossing of S_2 which intersects T such that v and w are contained in a column of q boxes, and
4. v and w are connected,

then there exists a crossing of S intersecting T' and T.

In Lemma 52, we bound from below the probability that there is at least one crossing of S_1 intersecting T'. The probability of Event 1 in the list above is estimated in Lemma 53. Then, we use Lemma 52 to bound the probability that Event 2 and 3 are satisfied conditioned on Event 1 occurring. The probability of Event 4 being satisfied is estimated in Lemma 54. Finally, the proof of Lemma 50 Part 2 is done by combining all of the previous calculations.

Let $S' = [a', b'] \times [c', d']$ be a (J, j) strip with $j \leq n$ and $J \geq \lfloor 2j/3 \rfloor$. Let $\hat{S} := \cup \hat{S}_i$ be a union of $(\lfloor 2j/3 \rfloor, j)$ strips in S', $\hat{S}_i = [f_i, g_i] \times [c', d']$. Let $T^* \subset R(d')$ be a $(4, J)$ tree in S' (same nodes as in $\hat{S}$) which intersects each $\hat{S}_i$ in a $(4, \lfloor 2j/3 \rfloor)$ tree. Let $\hat{T} \subset C(T^*) \cap R(c')$ be a union of $(4, l)$ trees in disjoint (l, j) strips in $\hat{S}$ where $l \leq \lfloor 2j/3 \rfloor$ (see Figure 12).

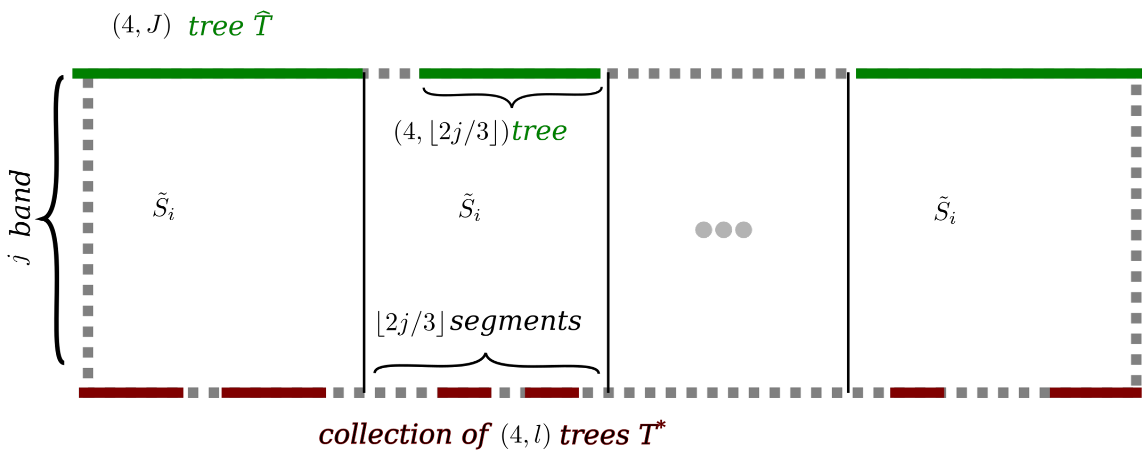

Figure 12: Situation of Lemma 52. The dotted rectangle is S'.

Lemma 52 (Long strip crossing). *Suppose Lemma 50 holds for $j \leq n-1$. Then,*

$$\mathbb{P}(\exists \text{ a crossing of } \hat{S} \text{ intersecting } \hat{T} \text{ and } T^*) \geq \min\left\{0.9, \frac{\#\hat{T}}{3 \cdot 4^{\lfloor 2j/3 \rfloor}}\right\}.$$

Each such crossing is confined to its respective $(\lfloor 2j/3 \rfloor, j)$ strip.

Proof. $\hat{T}$ is a union of $(4, l)$ trees. Let $\hat{T} = \cup \hat{T}_i$ where $\hat{T}_i$ consists of the $(4, l)$ trees belonging to $\hat{T}$ that lie inside the $(\lfloor 2j/3 \rfloor, j)$ strip $\hat{S}_i$ (recall $l \leq \lfloor 2j/3 \rfloor$). By the induction hypothesis, we have

$$\mathbb{P}(\exists \text{ a crossing of } \hat{S}_i \text{ intersecting } \hat{T}_i \text{ and } T^*) \geq \frac{\#\hat{T}_i}{4^{\lfloor 2j/3 \rfloor}}.$$

These are independent events since all the $\hat{S}_i$ are disjoint. Lemma 48 with $c = 3$ yields

$$\begin{aligned}\mathbb{P}(\exists \text{ a crossing of } \hat{S} \text{ intersecting } \hat{T} \text{ and } T^*)\\ &\geq 1 - \prod_i (1 - \mathbb{P}(\exists \text{ a crossing of } \hat{S}_i \text{ intersecting } \hat{T}_i \text{ and } T^*))\\ &\geq \min\left\{1 - e^{-3}, \frac{\sum_i \#\hat{T}_i}{3 \cdot 4^{\lfloor 2j/3 \rfloor}}\right\} \geq \min\left\{0.9, \frac{\#\hat{T}}{3 \cdot 4^{\lfloor 2j/3 \rfloor}}\right\}\end{aligned}$$

which shows the claim. Furthermore, the crossing happens in one of the $\hat{S}_i$. □

In Assumption 49, we have defined a $(\lfloor 2n/3 \rfloor, n)$ strip S, a $(4, \lfloor 2n/3 \rfloor)$ tree T, a union of $(4, k)$ trees T' with $T' \subset C(T)$ and constants q, m, k. Let

$$q^* := \max\{100,\, q\} \qquad M := \max\{\lfloor 2m/3 \rfloor,\, 100,\, q\} \qquad k' := \min\{k,\, \lfloor 2m/3 \rfloor\}\ ,$$

and

$$\bar{T} := R(\tilde{c}') \cap C(T)\,.$$

Define $\tilde{T}$ to be the union of the $(4, q^*)$ trees in $\bar{T}$ satisfying the following: Let $\tilde{T}_i \subset \tilde{T}$ be a union of $(4, q^*)$ trees inside a (M, m) strip. Then $\tilde{T}_i \subset \bar{T}$ if there are $\tilde{v}_i \in \tilde{T}_i$, $v_i \in T' \cap C(\tilde{T}_i)$ and a crossing of S_1 containing $\tilde{v}_i$ and v_i. Define the event

$$\mathfrak{X} := \left\{\#\tilde{T} \geq \max\left\{\frac{4^{q^*-1} \cdot \#T'}{1000 \cdot 4^M},\, 4^{q^*-1}\right\}\right\}\,. \tag{8}$$

Lemma 53 (Probability of “sufficiently many” crossings)**.** *Suppose Lemma 50 holds for* $j \leq n - 1$*. Then*

$$\mathbb{P}(\mathfrak{X}) \geq \min\left\{0.9,\, \frac{\#T'}{100 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\}\,.$$

Proof. Since $\tilde{T}$ consists of $(4, q^*)$ trees and each such tree has 4^{q^*-1} many vertices, we have $\#\tilde{T} \geq 4^{q^*-1}$ if and only if $\tilde{T} \neq \emptyset$. We also have $C(T) = C(\bar{T}) \supset C(T')$, so it suffices to have that T' and $\bar{T}$ are connected inside a $(\lfloor 2m/3 \rfloor, m)$ strip in order to show $\#\tilde{T} \geq 4^{q^*-1}$. This will be used multiple times. The proof is broken up into cases based on the size of $\#T'$ and the value of M.

1 $\#T' \leq 1000 \cdot 4^{\lfloor 2m/3 \rfloor}$ and $M = \lfloor 2m/3 \rfloor$. In particular, $\lfloor 2m/3 \rfloor \geq q^*$. Therefore, by Lemma 52 with $S' = S_1$, $\hat{S}$ to be a union of $(\lfloor 2m/3 \rfloor, m)$ strips, $T^* = \bar{T}$ and $\hat{T} = T'$

$$\mathbb{P}(\mathfrak{X}) \geq \mathbb{P}(\#\tilde{T} \geq 4^{q^*-1}) \geq \mathbb{P}(\exists \text{ a crossing } T' \leftrightarrow \bar{T} \text{ inside } S_1) \geq \min\left\{0.9,\, \frac{\#T'}{3 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\}.$$

2 $\#T' \leq 1000 \cdot 4^M$ and $M = q^*$. Again

$$\mathbb{P}(\mathfrak{X}) \geq \mathbb{P}(\#\tilde{T} \geq 4^{M-1}) = \mathbb{P}(\#\tilde{T} \geq 4^{q^*-1}) .$$

Write $T' = \cup_{i=1}^N T_i$ where each T_i is a union of $(4, k')$ trees in a $(\lfloor 2m/3 \rfloor, m)$ strip. Then, for all i by Lemma 52

$$\mathbb{P}(\exists \text{ a crossing intersecting } T_i \text{ and } \bar{T} \text{ in a } (\lfloor 2m/3 \rfloor, m) \text{ strip}) \geq \min\left\{0.9,\ \frac{\#T_i}{3 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\} .$$

We are done if the minimum for one of the i is 0.9. Otherwise, Lemma 48 concludes

$$\mathbb{P}(\exists \text{ a crossing intersecting } T' \text{ and } \bar{T} \text{ in a } (\lfloor 2m/3 \rfloor, m) \text{ strip}) \geq \min\left\{0.9,\ \frac{\#T'}{3 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\} .$$

3 $\#T' > 1000 \cdot 4^M$. Write $T' = \cup_{i=1}^{N'} T_i'$ where each T_i' is now a union of $(4, k')$ trees that belong to a union of (M, m) strips $\tilde{S}_i$. Do this in a way such that for each i

$$3 \cdot 4^M \leq \#T_i' \leq 4 \cdot 4^M$$

and such that for $i \neq j$, the corresponding unions of (M, m) strips $\tilde{S}_i$ and $\tilde{S}_j$ are disjoint. This is possible since each $(4, k')$ tree has $4^{k'-1}$ vertices and $M \geq \lfloor 2m/3 \rfloor \geq k'$. Thus, N' satisfies

$$N' \geq \frac{\#T'}{4 \cdot 4^M} \geq \frac{1000 \cdot 4^M}{4 \cdot 4^M} = 250 \geq 100 .$$

By Lemma 52, we have with $\#T_i' \geq 3 \cdot 4^M$

$$\mathbb{P}(\exists \text{ a crossing } T_i' \text{ to } \bar{T} \text{ in a } (M, m) \text{ strip}) \geq \min\left\{0.9,\ \frac{\#T_i'}{3 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\} = 0.9 .$$

Therefore, we have $N' \geq 100$ independent events with probability greater or equal to 0.9. The probability of at least $\lceil N'/10 \rceil$ of these happening is greater than the probability that at least 11 events happen with $N' = 100$. The latter probability is > 0.9. Each such event gives us a contribution of 4^{q^*-1} to $\#\tilde{T}$, so we see that under the event of at least $\lceil N'/10 \rceil$ crossings happening

$$\#\tilde{T} \geq \frac{N'}{10} \cdot 4^{q^*-1} \geq \frac{\#T' \cdot 4^{q^*-1}}{40 \cdot 4^M} .$$

Therefore

$$\mathbb{P}(\mathfrak{X}) \geq \mathbb{P}(\#\tilde{T} \geq \frac{\#T' \cdot 4^{q^*-1}}{40 \cdot 4^M}) \geq 0.9 = \min\left\{0.9,\ \frac{\#T'}{100 \cdot 4^{\lfloor 2m/3 \rfloor}}\right\} .$$

4 Finally, if $M = 100$ and $\#T' \leq 1000 \cdot 4^M$, then $q \leq 100$ and $m \leq 150$. The probability that a straight vertical line in S_1 is open is then bounded from below by $1 - 4^{-200}$ (the probability of a 200 box being good) and we conclude: $\mathbb{P}(\mathfrak{X}) \geq \mathbb{P}(\#\tilde{T} \geq 4^{q^*-1}) \geq 1 - 4^{-200}$.

All cases have been covered, so the claim is proven. □

Next, we will work with the part inside the strip S between S_1 and S_2. Assume that $q \geq 200$. Consider a column $G := [e, f] \times [g_1, h_l]$ of alternating q boxes and (q, q) strips where

- there are $l \leq 768$ many q boxes $[e, f] \times [g_i, h_i]$, $i = 1, \dots, l$, and
- each $[e, f] \times [h_i, g_{i+1}]$ is a horizontal (q, q) strip.

Let $v \in R(g_1)$, $w \in R(h_l)$ be vertices on the top respectively bottom of G. We say that G is normal for v and w if there is an open cluster in G connecting v and w.

Lemma 54 (Probability of normal columns)**.** *Suppose Lemma 50 holds for $q \leq n$. Then,*

$$\mathbb{P}(G \textit{ is normal for } v \textit{ and } w) \geq 0.99\,.$$

Proof. A sufficient condition for G to be normal for v and w is:

1 All of the q boxes inside G are good.

2 v and w lie in the crossing clusters of their respective q boxes.

3 All of the (q, q) strips in G have a cluster which connects the crossing clusters of the good q boxes on the top / bottom of the (q, q) strip.

By the induction hypothesis

$$\mathbb{P}(\text{all of the } q \text{ boxes are good}) \geq (1 - 4^{-q})^{768} \geq 1 - 768 \cdot 4^{-200} \geq 1 - 2^{-100}\,.$$

If the j box containing v is good for all j with $200 \leq j \leq q$, then v is in the crossing cluster of the q box (Lemma 40). The same holds for w. Thus,

$$\mathbb{P}(\text{Condition } 2 \text{ is satisfied}) \geq 1 - 2 \sum_{j \geq 200} 4^{-j} \geq 1 - 4^{-199}\,.$$

By Lemma 50 Part 3,

$$\mathbb{P}(\text{Condition } 3 \text{ is satisfied}) \geq (1 - 4^{-q})^{768} \geq 1 - 2^{-100}\,.$$

Therefore,

$$\mathbb{P}(G \text{ is normal for } v \text{ and } w) \geq 1 - 3 \cdot 2^{-100} \geq 0.99\,,$$

which shows the claim. □

We are now able to proof Lemma 50. As indicated before, we do so by induction.

Proof of Lemma 50. Choose $p_c \in (0,1)$ such that Lemma 50 holds for every $p \geq p_c$ and $n \leq 200$. This is the base case. Now assume that the lemma is true for all $j < n$.

Part 1: Since $N^{(x)}$ and $N^{(y)}$ are regular, there are at most 768^2 many $n-1$ boxes inside an n box. Therefore, we have that

$$\mathbb{P}(a_1 = 1) \leq \mathbb{P}_p(a_1 \geq 1) \leq 768^2 \cdot 4^{-n+1} \leq 4^{11} \cdot 4^{-n}$$

and

$$\mathbb{P}(a_1 \geq 2) \leq (768^2)^2 \cdot (4^{-n+1})^2 \leq 4^{50} \cdot 4^{-2n} \leq 4^{-50} \cdot 4^{-n}\,.$$

There are at most $2 \cdot (768)^2$ many $(n-1, n-1)$ strips in an n box. If such a strip lies between two good $n-1$ boxes, then the probability that there exists a crossing which connects both crossing clusters of the good n boxes is calculated as follows: By the induction hypothesis (statement 3)

$$\mathbb{P}(a_2 \geq 2) \leq \left(2 \cdot (768)^2\right)^2 \cdot (4^{-n+1})^2 \leq 4^{50} \cdot 4^{-2n} \leq 4^{-50} \cdot 4^{-n}$$

and

$$\mathbb{P}(a_2 \geq 1 \,|\, a_1 = 1) \leq 2 \cdot 768^2 \cdot (4^{-n+1}) \leq 4^{25} \cdot 4^{-n}\,.$$

Combining everything yields

$$\begin{aligned}\mathbb{P}(a_1 + a_2 \geq 2) &\leq \mathbb{P}(a_1 \geq 2) + \mathbb{P}(a_2 \geq 2) + \mathbb{P}(a_1 = 1) \cdot \mathbb{P}(a_2 = 1 \,|\, a_1 = 1)\\ &\leq 4^{-n} \cdot \left[4^{-50} + 4^{-50} + 4^{11} \cdot 4^{-n} \cdot 4^{25}\right] \leq 4^{-n}\,.\end{aligned}$$

Part 2: First assume that the height of the $(\lfloor 2n/3 \rfloor, n)$ strip S is 1. In this case, there are $\#T'$ edges would form an appropriate crossing if they were open. Thus, using Lemma 48 and $n \geq 200$

$$\begin{aligned}\mathbb{P}(\exists \text{ a cluster in } S \text{ connecting } T \text{ and } T') &\geq 1 - (1-p^n)^{\#T'}\\ &\geq \min\left\{1 - e^{-1},\, \#T' \cdot p^n(1-e^{-1})\right\} \geq \frac{\#T'}{4^{\lfloor 2n/3 \rfloor}}\,.\end{aligned}$$

Next, one checks that if either $m < 200$ or $r < 200$, then also $q < 200$ and that the induction hypothesis is then easily proven: WLOG, assume that it is $r < 200$. Recall that $\bar{T} = C(T) \cap R(\tilde{c}')$. By Lemma 52, we have

$$\mathbb{P}(\exists \text{ crossing of } S_1 \text{ intersecting } T' \text{ and } \bar{T}) \geq \min\{0.9,\, \frac{\#T'}{3 \cdot 4^{\lfloor 2m/3 \rfloor}}\}\,.$$

If this crossing exists and contains a $v \in C(T) \cap R(\tilde{c}')$, then the probability of the $q+1$ box containing v inside S to be good is at least $1 - 4^{-200}$. Since $q + 1 \leq 200$, being good means that all the edges are open. The same holds for the corresponding $m+1$ box inside S_2 that connects to the $q+1$ box containing v. Therefore,

$$\begin{aligned}\mathbb{P}(\exists\text{a cluster in } S \text{ connecting } T \text{ and T'})\\ &\geq \mathbb{P}(\exists \text{ crossing of } S_1 \text{ intersecting } T' \text{ and } \bar{T}) \cdot \big(1-4^{-200}\big)^2\\ &\geq \min\{0.8, \frac{\#T'}{4^{\lfloor 2m/3\rfloor+1}}\} \geq \min\{0.8, \frac{\#T'}{4^{\lfloor 2n/3\rfloor}}\} = \frac{\#T'}{4^{\lfloor 2n/3\rfloor}}.\end{aligned}$$

where we used Equation (5) and $\#T' \leq \#T \leq 4^{\lfloor 2n/3\rfloor-1}$ in the last line.
Now, consider the case when $m, r \geq 200$ and the height of S is greater than 1. Furthermore, assume $q \geq 100$ (in particular, $q = q^*$) and consider the following events:

0.1 $\mathfrak{X}$ happens on S_1. This event gives us a collection of $(4, q)$ trees $\tilde{T} \subset C(T) \cap R(\tilde{c}')$. Set $T^* := C(\tilde{T}) \cap R(\tilde{d}')$.

0.2 There exists a crossing of S_2 intersecting T^* and T. This event gives us some $v \in \tilde{T}$ and $w \in T^*$. These are separated by a column of q boxes.

0.3 The column of q boxes separating v and w is normal for v and w.

If all these events hold, then there exists a crossing of S that intersects T and T'. By Lemma 53

$$\mathbb{P}(\text{event } [a]) = \mathbb{P}(\mathfrak{X}) \geq \min\left\{0.9, \frac{\#T'}{100\cdot 4^{\lfloor 2m/3\rfloor}}\right\}.$$

Under $\mathfrak{X}$, we have

$$\#\tilde{T} \geq \max\left\{4^{q^*-1}, \frac{4^{q^*-1}\cdot \#T'}{1000\cdot 4^M}\right\}.$$

If now $\#T' \leq 1000\cdot 4^M$, then $\#T^* = \#\tilde{T} \geq 4^{q^*-1}$ and by Lemma 53, Lemma 54

$$\begin{aligned}\mathbb{P}(\text{event } [b]\,\&\,[c] \mid [a]) &\geq \mathbb{P}(\exists \text{ a crossing of } S_2 \text{ intersecting } T^* \text{ and } T \mid \#T^* = 4^{q^*-1})\cdot 0.99\\ &\geq 0.99\cdot\min\left\{0.9, \frac{4^{q^*-1}}{3\cdot 4^{\lfloor 2r/3\rfloor}}\right\} \geq \min\left\{0.8, \frac{4^{q^*-1}}{4\cdot 4^{\lfloor 2r/3\rfloor}}\right\}.\end{aligned}$$

If additionally $\#T' < 100\cdot 4^{\lfloor 2m/3\rfloor}$, then using $n \geq m \geq 200$, $q \geq 100$ as well as Equation (6)

$$\begin{aligned}\mathbb{P}(\exists \text{ a cluster in } S \text{ connecting } T \text{ and } T') &\geq \mathbb{P}([a])\cdot\mathbb{P}([b]\,\&\,[c] \mid [a])\\ &\geq 0.9\cdot\frac{\#T'}{100\cdot 4^{\lfloor 2m/3\rfloor}}\cdot\min\left\{0.8, \frac{4^{q^*-1}}{4\cdot 4^{\lfloor 2r/3\rfloor}}\right\}\\ &\geq \min\left\{\frac{\#T'}{200\cdot 4^{\lfloor 2m/3\rfloor}}, \frac{\#T'\cdot 4^{q^*-1}}{500\cdot 4^{\lfloor 2n/3\rfloor+q-30}}\right\} \geq \frac{\#T'}{4^{\lfloor 2n/3\rfloor}}.\end{aligned}$$

If instead $100 \cdot 4^{\lfloor 2m/3 \rfloor} \le \#T' \le 1000 \cdot 4^M$, then

$$\begin{aligned} m + r - \lfloor q/3 \rfloor &\le n + 1 \\ m + r - q/3 &\le n + 1 \\ 2m/3 + 2r/3 - 2q/9 &\le 2n/3 + 1 \\ \lfloor 2m/3 \rfloor + \lfloor 2r/3 \rfloor - 2q/9 &\le \lfloor 2n/3 \rfloor + 2 \end{aligned}$$

Since $q < m$, we have

$$M = \max\{\lfloor 2m/3 \rfloor, q\} \le \lfloor 2m/3 \rfloor + 1 + \tfrac{1}{3}q$$

which yields

$$\begin{aligned} \mathbb{P}(\exists \text{ a cluster in } S \text{ connecting } T \text{ and } T') &\ge \mathbb{P}([a]) \cdot \mathbb{P}([b] \,\&\, [c] \mid [a]) \\ &\ge 0.9 \cdot \min\left\{0.8,\ \frac{4^{q^*-1}}{4 \cdot 4^{\lfloor 2r/3 \rfloor}}\right\} \ge \min\left\{0.5,\ \frac{4^{q^*-1} \cdot 4^{\lfloor 2m/3 \rfloor}}{4^2 \cdot 4^{\lfloor 2m/3 \rfloor + \lfloor 2r/3 \rfloor}}\right\} \\ &\ge \min\left\{0.5,\ \frac{4^{q^*-1} \cdot 4^{\lfloor 2m/3 \rfloor}}{4^{\lfloor 2n/3 \rfloor + 2q/9}}\right\} \ge \min\left\{0.5,\ \frac{4^{\lfloor 2m/3 \rfloor + q/3 + 1} \cdot 1000}{4^{\lfloor 2n/3 \rfloor}}\right\} \\ &\ge \min\left\{0.5,\ \frac{4^M \cdot 1000}{4^{\lfloor 2n/3 \rfloor}}\right\} \ge \frac{\#T'}{4^{\lfloor 2n/3 \rfloor}} \end{aligned}$$

where the last inequality follows from $\#T' \le \#T \le 4^{\lfloor 2n/3 \rfloor - 1}$.
If instead $\#T' \ge 1000 \cdot 4^M$, then using

$$\#\tilde{T} = \#T^* \ge \frac{\#T' \cdot 4^{q^*-1}}{1000 \cdot 4^M}$$

and Lemma 52 gives

$$\begin{aligned} \mathbb{P}([b] \mid [a]) &\ge \mathbb{P}\Big(\exists \text{ a crossing of } S_2 \text{ intersecting } T^* \text{ and } T \mid \#T^* \ge \frac{\#T' \cdot 4^{q^*-1}}{1000 \cdot 4^M}\Big) \\ &\ge \min\left\{0.9,\ \frac{\#T' \cdot 4^{q^*-1}}{1000 \cdot 4^M} \cdot \frac{1}{3 \cdot 4^{\lfloor 2r/3 \rfloor}}\right\} \ge \min\left\{0.9,\ \frac{\#T'}{3000 \cdot 4^{\lfloor 2m/3 \rfloor + \lfloor 2r/3 \rfloor - 2q/3 + 2}}\right\} \\ &\ge \min\left\{0.9,\ \frac{\#T'}{3000 \cdot 4^{\lfloor 2n/3 \rfloor + 2q/9 - 2q/3 + 4}}\right\} \ge 2\frac{\#T'}{4^{\lfloor 2n/3 \rfloor}}\,, \end{aligned}$$

where the minimum disappears again from $\#T' \le \#T \le 4^{\lfloor 2n/3 \rfloor - 1}$. Lemma 54 yields

$$\mathbb{P}([c] \mid [a] \,\&\, [b]) \ge 0.99\,.$$

Putting everything together, we conclude the $\#T' \ge 1000 \cdot 4^M$ case:

$$\begin{aligned} \mathbb{P}(\exists \text{ a cluster in } S \text{ connecting } T \text{ and } T') &\ge \mathbb{P}([a]) \cdot \mathbb{P}([b] \mid [a]) \cdot \mathbb{P}([c] \mid [a] \,\&\, [b]) \\ &\ge 0.9 \cdot 2 \cdot \frac{\#T'}{4^{\lfloor 2n/3 \rfloor}} \cdot 0.99 \ge \frac{\#T'}{4^{\lfloor 2n/3 \rfloor}}\,. \end{aligned}$$

The only thing left to prove in statement 2 is the case where $m, r \geq 200$ and $q < 100$. Here, replace the event $[c]$ with

$[c']$: "All edges in the rectangle spanned by v and w are open."

Then, the proof works as before since any such rectangle is a portion of a 200 box.

Part 3: We finally prove the last statement. The $(4, n)$ trees R and R' define a set of $4^{n-\lfloor 2n/3 \rfloor}$ many $(\lfloor 2n/3 \rfloor, n)$ strips in $\bar{S}$. Let $\tilde{S}$ be one of those strips. By the induction hypothesis, the probability of having a cluster in $\tilde{S}$ that intersects R and R' is at the very least $\frac{1}{4}$. There are $4^{n-\lfloor 2n/3 \rfloor}$ many of those strips and all these events are independent. Therefore, we have (using Lemma 48 again)

$$\begin{aligned}
&\mathbb{P}(\exists \text{ a crossing of } \bar{S} \text{ intersecting both } R_1 \text{ and } R_2) \\
&\qquad \geq 1 - \left(\tfrac{3}{4}\right)^{4^{n-\lfloor 2n/3 \rfloor}} \geq \min\left\{1 - e^{-2n}, \frac{4^{n-\lfloor 2n/3 \rfloor}}{2n}\left(1 - e^{-2n}\right)\right\} \\
&\qquad \geq 1 - e^{-2n} \geq 1 - 4^{-n}.
\end{aligned}$$

This finishes the proof. □

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