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Abstract

We analyze a phase field model for tumor growth consisting of a Cahn–Hilliard–Brinkman system, ruling the evolution of the tumor mass, coupled with an advection-reaction-diffusion equation for a chemical species acting as a nutrient. The main novelty of the paper concerns the discussion of the existence of weak solutions to the system covering all the meaningful cases for the nonlinear potentials; in particular, the typical choices given by the regular, the logarithmic, and the double obstacle potentials are admitted in our treatise. Compared to previous results related to similar models, we suggest, instead of the classical no-flux condition, a Dirichlet boundary condition for the chemical potential appearing in the Cahn–Hilliard-type equation. Besides, abstract growth conditions for the source terms that may depend on the solution variables are postulated.

1 Introduction

Cancer is nowadays still one of the main diseases causing death worldwide. Beyond doubt, the understanding of the development of solid tumor growth is one of the major challenges scientists have to face in the current century. Moreover, it is now, more than ever, apparent that only interdisciplinary efforts may enable us to gain deeper insights into cancer development mechanisms. In this scenario, mathematics could play a crucial role, since multiscale mathematical modeling provides a quantitative tool that may help in diagnostic and prognostic applications (see, e.g., the seminal book [7]). Among others, mathematics has two decisive advantages: the first one is that of being able to select particular mechanisms that may be more relevant than others, while the second one is that of being able to foresee and make predictions that may be precious for medical practitioners, without inflicting any harm to the patients. Furthermore, the extremely fast development of computational methods for the solution of nonlinear PDEs opens the doors for a direct interaction between the experimental methods used by physicians and the more theoretical mathematical ones: indeed, advanced numerical solvers may be implemented as a supporting tool in clinical therapies. Recently, lots of phase field models modeling tumor growth have been proposed: a brief description of the state of the art will be provided later on. As biological materials like tumor agglomerates exhibit viscoelastic properties, we prescribe a velocity equation of Brinkman type.

Let $\Omega \subset \mathbb{R}^3$ be a spatial domain in which the tumor is located, $T > 0$ be a fixed final time, and set $Q := \Omega \times (0,T)$, and $\Sigma := \partial\Omega \times (0,T)$. Then the system under investigation in this paper is a

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Cahn–Hilliard–Brinkman model related to tumor growth and reads as follows:

$$-\operatorname{div}\mathbb{T}(\boldsymbol{v},p)+\nu\boldsymbol{v}=\mu\nabla\varphi+\big(\sigma+\chi(1-\varphi)\big)\nabla\sigma \qquad \text{in } Q, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{v}=g \qquad \text{in } Q, \tag{1.2}$$

$$\partial_t\varphi+\operatorname{div}(\varphi\boldsymbol{v})-\operatorname{div}(m(\varphi)\nabla\mu)=\mathcal{S}_\varphi(\varphi,\sigma) \qquad \text{in } Q, \tag{1.3}$$

$$\mu=-\epsilon\Delta\varphi+\epsilon^{-1}F'(\varphi)-\chi\sigma \qquad \text{in } Q, \tag{1.4}$$

$$\partial_t\sigma+\operatorname{div}\big(\sigma\boldsymbol{v}\big)-\operatorname{div}(n(\varphi)\nabla(\sigma+\chi(1-\varphi)))\big)=\mathcal{S}_\sigma(\varphi,\sigma) \qquad \text{in } Q, \tag{1.5}$$

where the coefficients m and n are positive functions and the *viscous Brinkman stress tensor* $\mathbb{T}$ is defined as

$$\mathbb{T}(\boldsymbol{v},p)=2\eta D\boldsymbol{v}+\lambda(\operatorname{div}\boldsymbol{v})\mathbb{I}-p\mathbb{I}\,. \tag{1.6}$$

Here, the standard notation

$$D\boldsymbol{v}:=\frac{1}{2}\big(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\big) \tag{1.7}$$

is used for the symmetrized gradient of the velocity field $\boldsymbol{v}$ which represents the *volume-averaged velocity field* of the mixture with *permeability* ν. Moreover, p is the pressure, η and λ are nonnegative constants denoting the *shear viscosity* and the *bulk viscosity*, respectively, and $\mathbb{I}\in\mathbb{R}^{3\times3}$ is the identity matrix. Although several choices are possible, we endow the above system with the following boundary and initial conditions:

$$\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}=\mathbf{0} \qquad \text{on } \Sigma, \tag{1.8}$$

$$\partial_{\boldsymbol{n}}\varphi=0, \quad \mu=\mu_\Sigma, \quad \partial_{\boldsymbol{n}}\sigma=\kappa(\sigma_\Sigma-\sigma) \qquad \text{on } \Sigma, \tag{1.9}$$

$$\varphi(0)=\varphi_0, \quad \sigma(0)=\sigma_0 \qquad \text{in } \Omega, \tag{1.10}$$

where $\boldsymbol{n}$ denotes the outer unit normal vector to $\partial\Omega$, and $\partial_{\boldsymbol{n}}$ the associated outward normal derivative. The other variables of the system are φ, μ, and σ. In relevant cases, the phase variable φ is an order parameter taking values in $[-1,1]$ that represents the difference between the volume fractions of tumor cells and healthy cells. It allows us to keep track of the evolution of the boundary of the tumor, since the level sets $\{\varphi=1\}:=\{x\in\Omega:\varphi(x)=1\}$ and $\{\varphi=-1\}$ describe the region of pure phases: the tumorous phase and the healthy phase, respectively. The second variable μ denotes the chemical potential related to φ as in the framework of the Cahn–Hilliard equation. We postulate the growth and proliferation of the tumor to be driven by the absorption and consumption of some nutrient σ (usually oxygen). Whenever $0\leq\sigma\leq1$, $\sigma\simeq1$ represents a rich nutrient concentration, whereas $\sigma\simeq0$ a poor one. The functions $m(\varphi)$ and $n(\varphi)$ are nonnegative mobility functions related to the phase and to the nutrient variables, respectively. The positive physical constants ϵ and ν are related to the interfacial thickness and surface tension, while the nonnegative constant χ represents the chemotactic sensitivity. Finally, $\mathcal{S}_\varphi$ and $\mathcal{S}_\sigma$ denote nonlinearities representing some source terms that account for the mutual interplay between tumor, healthy cells, and nutrients. For further details concerning the modeling, we refer to [15] and the references therein.

Concerning the boundary conditions, we point out that (1.8) can be understood as a *no-friction* boundary condition for the velocity field $\boldsymbol{v}$. This is a common request for similar systems, see, e.g., [8, 9, 22], as it does not enforce any compatibility condition on the velocity source term g in (1.2), which is otherwise needed if one assumes a no-slip boundary condition like $\boldsymbol{v}=\mathbf{0}$ on Σ or the no-penetration boundary condition $\boldsymbol{v}\cdot\boldsymbol{n}=\mathbf{0}$ on Σ. Both these conditions entail that $\int_\Omega g=0$, which is not ideal from the modeling perspective. Let us also notice that the vector field $\boldsymbol{v}$ is not solenoidal (as typically in fluid-type problems), which entails challenges from the mathematical viewpoint.

As a common denominator of a more general Cahn–Hilliard equation, in (1.4) F' represents the (generalized) derivative of a double-well shaped nonlinearity F. Prototypical examples are the regular, the logarithmic, and the double obstacle potentials. These read, in the order, as

$$F_{reg}(r) := \frac{1}{4}(r^2-1)^2\,, \quad r \in \mathbb{R}\,, \tag{1.11}$$

$$F_{log}(r) := \begin{cases} \frac{\theta}{2}\left[(1+r)\ln(1+r) + (1-r)\ln(1-r)\right] - \frac{\theta_0}{2}r^2 & \text{if } r \in (-1,1)\,, \\ \theta\,\ln(2) - \frac{\theta_0}{2} & \text{if } r \in \{-1,1\}\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.12}$$

$$F_{dob}(r) := \begin{cases} c(1-r^2) & \text{if } r \in [-1,1]\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.13}$$

for some positive constant c and $0 < \theta < \theta_0$. Besides, in the case of nonregular potentials like the double obstacle (1.13), the second equation (1.4) has to be read as a differential inclusion.

The main novelty of this paper is the prescription of the Dirichlet boundary condition $\mu = \mu_\Sigma$ for the chemical potential on Σ, in contrast to the standard homogeneous Neumann (no-flux) boundary condition $\partial_{\boldsymbol{n}}\mu = 0$ on Σ (see, e.g., [8,9,22]). The limitation behind the latter choice regards the nonlinear potentials that can be considered: in all of the aforementioned papers, the authors were forced to restrict the analysis to regular potentials, possibly of just quadratic growth at infinity; singular potentials like (1.12) or (1.13) were excluded from the analysis. These, however, are actually more relevant physically, since, if solutions to the system exist, then the condition $\varphi \in [-1,1]$ is automatically fulfilled. The restriction of the admitted potentials originates from the presence of the source term $\mathcal{S}_\varphi$ in the Cahn–Hilliard equation (1.1). Roughly speaking, for proving that $\mu \in L^2(0,T;H^1(\Omega))$, the energetic approach provides just a control on $\nabla\mu$ in $L^2(Q)$. The classical approach then requires the employment of the Poincaré–Wirtinger inequality along with a control of the spatial mean of μ in $L^2(0,T)$. This can be achieved by comparison in equation (1.4), provided that F is regular and its derivative F' possesses a prescribed growth, which leads to the choice of potentials of polynomial type. Therefore, the novelty of our work revolves around the different boundary condition, which allows us to apply another version of Poincaré's inequality to establish immediate control of $\mu \in L^2(0,T;H^1(\Omega))$ from the bound $\nabla\mu \in L^2(Q)$ without the need of an additional control of the spatial average of μ; in this way, unpleasant growth restrictions for the potential can be avoided (see also [13]).

Without claiming to be exhaustive, let us now review some literature connected with the system (1.1)–(1.10). It is well known that the Brinkman law interpolates between the Stokes and Darcy paradigms, and it has become rather popular in recent times, see [8,9,22]. For tumor growth models both descriptions seem reasonable, because the associated Reynolds number is very small. Formally, we recover the *Darcy limit* when $\eta \equiv \lambda \equiv 0$, and $\nu > 0$, where the boundary condition (1.8) yields that $p = 0$ on Σ; similarly, the *Stokes* limit is obtained when $\eta, \lambda > 0$, and $\nu = 0$. The Stokes equation was suggested, e.g., in [5, 10], by approximating the tumor as a viscous fluid, while Darcy's law describes a viscous fluid permeating a porous medium represented by the extracellular matrix and accounts for the inclination of cells to move away from regions of high compression, see, e.g., [14, 15]. Stationary approximations for system (1.1)–(1.10) are popular as well, and we mention [12, 13, 17] and the references therein, where just polynomial-type potentials were considered. To cope with the case of singular potential, some authors (see [6]) suggested to include suitable relaxations. Besides, we refer to [18, 19, 27] and the references therein, for related *nonlocal* versions, to [1, 16, 23] for the additional coupling with elasticity, and to [26] for the coupling with the Keller–Segel equation.

1.1 Biological Examples and Modeling Considerations

Before diving into the mathematical details, let us outline some physically meaningful choices for the source terms $\mathcal{S}_\varphi$ and $\mathcal{S}_\sigma$ introduced above.

Linear Kinetic. A first typical form for $\mathcal{S}_\varphi$ and $\mathcal{S}_\sigma$ was proposed by Cristini et al. in [7] motivated by linear kinetic:

$$\mathcal{S}_\varphi(\varphi,\sigma) := (\mathcal{P}\sigma - \mathcal{A})\mathbb{h}(\varphi), \quad \mathcal{S}_\sigma(\varphi,\sigma) := \mathcal{B}(\sigma_c - \sigma) - \mathcal{C}\sigma\mathbb{h}(\varphi). \tag{1.14}$$

In the above expressions, $\mathcal{P}, \mathcal{A}, \mathcal{B},$ and $\mathcal{C}$ denote nonnegative constants related to biological quantities that are, in the order, the proliferation rate of tumoral cells by consumption of nutrient, the apoptosis rate, the consumption rate of the nutrient with respect to a preexisting concentration σ_c, and the nutrient consumption rate. As for the function $\mathbb{h}$, it denotes an interpolation function between -1 and 1 with the property that $\mathbb{h}(-1) = 0$ and $\mathbb{h}(1) = 1$. Roughly speaking, $\mathbb{h}$ weights the corresponding mechanism compared to the amount of cancer located in that region and “turns off” the associated mechanism when the tumor is not present.

Linear phenomenological laws for chemical reactions. Another approach was proposed by accounting for linear phenomenological laws for chemical reactions by A. Hawkins-Daarud et al. in [21], where the following form was suggested:

$$\mathcal{S}_\varphi(\varphi,\sigma,\mu) = -\mathcal{S}_\sigma(\varphi,\sigma,\mu) := P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \tag{1.15}$$

where P stands for a suitable nonnegative proliferation function.

In the forthcoming analysis, we will proceed in an abstract fashion without prescribing explicit structures for the source terms, but just postulating suitable growth conditions. Those are straightforwardly fulfilled by the above cases but this last one. Namely, we cannot allow a linear growth of the sources with respect to μ. Nevertheless, let us claim that the special form (1.15) can still be considered provided to adjust some estimates accordingly (cf. [6]).

The above system (1.1)–(1.10) (see, e.g., [15]) is naturally associated to a free energy $\mathcal{E}$ with following the structure:

$$\mathcal{E}(\varphi,\sigma) = \frac{\epsilon}{2}\int_\Omega |\nabla\varphi|^2 + \frac{1}{\epsilon}\int_\Omega F(\varphi) + \int_\Omega N(\varphi,\sigma), \tag{1.16}$$

$$N(\varphi,\sigma) = \frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi), \tag{1.17}$$

where the first two terms in $\mathcal{E}$ yield the well-known Ginzburg–Landau energy modeling phase segregation and adhesion effects, while N stands for the chemical free energy density, respectively. For convenience, let us immediately set a specific notation to denote the partial derivatives of the free energy N with respect to the variables φ and σ:

$$N_\varphi(\varphi,\sigma) := \partial_\varphi N(\varphi,\sigma) = -\chi\sigma \quad \text{and} \quad N_\sigma(\varphi,\sigma) := \partial_\sigma N(\varphi,\sigma) = \sigma + \chi(1-\varphi), \tag{1.18}$$

as a direct computation shows. This notation will turn very convenient for the estimation procedure later on.

As they will play any role from the mathematical viewpoint, we will set $m(\cdot) \equiv n(\cdot) \equiv 1$, and $\epsilon = 1$ in our discussion. However, let us claim that our well-posedness result might be proven also for suitable nonconstant, albeit non-degenerate, mobilities (see [8]).

2 Notation, Assumptions and Main Results

Throughout the paper, Ω is a bounded and connected open subset of $\mathbb{R}^3$ (the two-dimensional case can be treated in the same way) having a smooth boundary $\Gamma := \partial\Omega$. In the following, $|\Omega|$ denotes the Lebesgue measure of Ω. Similarly, we write $|\Gamma|$ for the two-dimensional Hausdorff measure of Γ. Given a final time $T > 0$, we set, for every $t \in (0, T]$,

$$Q_t := \Omega \times (0,t), \quad \Sigma_t := \Gamma \times (0,t), \quad Q := Q_T, \quad \Sigma := \Sigma_T. \tag{2.1}$$

Given a Banach space X, we denote by $\|\cdot\|_X$, X^*, and $\langle\cdot,\cdot\rangle_X$, its norm, its dual space, and the associated duality pairing, respectively. As for the notation of norms, some exceptions will be utilized in the sequel. Moreover, the symbol used for the norm in some space X is adopted for the one in any power of X. Another standard notation that we employ concerns vectors, or vector-valued functions, which are denoted by bold symbols; for instance, $\mathbf{0}$ stands for the zero vector in $\mathbb{R}^3$, and $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. For $1 \le q \le \infty$ and $k \ge 0$, we indicate the usual Lebesgue and Sobolev spaces on Ω by $L^q(\Omega)$ and $W^{k,q}(\Omega)$, with the standard abbreviation $H^k(\Omega) := W^{k,2}(\Omega)$. The norm in $L^q(\Omega)$ is simply denoted by $\|\cdot\|_q$, and the same symbol is used for the norms in the analogous spaces constructed on Q, Γ and Σ, if no confusion can arise. Then, we introduce the shorthands

$$\begin{aligned} &H := L^2(\Omega), \quad V := H^1(\Omega), \quad V_0 := H^1_0(\Omega), \\ &\text{and} \quad W := \{v \in H^2(\Omega) : \partial_{\boldsymbol{n}} v = 0 \text{ a.e. on } \Gamma\}, \end{aligned} \tag{2.2}$$

and endow these spaces with their natural norms. For simplicity, we write $\|\cdot\|$ instead of $\|\cdot\|_H$. Besides, the space H will be identified with its dual, so that we have the following continuous, dense, and compact embeddings:

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V^*,$$

yielding that (V, H, V^*) is a Hilbert triplet. Similarly, (V_0, H, V_0^*) is a Hilbert triplet that will be used as well. We observe at once the compatible embeddings

$$\begin{aligned} &V \hookrightarrow H^{s_2}(\Omega) \hookrightarrow H^{s_1}(\Omega) \hookrightarrow H \hookrightarrow (H^{s_1}(\Omega))^* \hookrightarrow (H^{s_2}(\Omega))^* \hookrightarrow V^* \\ &\text{for } 0 < s_1 < s_2 < 1, \end{aligned} \tag{2.3}$$

and we recall that

$$(H^s(\Omega))^* = (H^s_0(\Omega))^* = H^{-s}(\Omega) \quad \text{for } 0 \le s \le 1/2\,. \tag{2.4}$$

Finally, the same symbols written with boldface characters denote the corresponding spaces of vector-valued functions. So, we have for instance that $\boldsymbol{L}^p(\Omega) = (L^p(\Omega))^3$, $\boldsymbol{H} = H^3$ and $\boldsymbol{V} = V^3$. However, since $\boldsymbol{H}$ and $\boldsymbol{V}$ are powers of H and V, we simply write $\|\cdot\|$ and $\|\cdot\|_V$ instead of $\|\cdot\|_{\boldsymbol{H}}$ and $\|\cdot\|_{\boldsymbol{V}}$, respectively. Moreover, for given matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3\times3}$, we define the scalar product

$$\boldsymbol{A} : \boldsymbol{B} := \sum_{i=1}^{3}\sum_{j=1}^{3} [\boldsymbol{A}]_{ij}\,[\boldsymbol{B}]_{ij}\,.$$

The following structural assumptions will be in order in our analysis. The double-well potential F introduced in the examples (1.11)–(1.13) is replaced by a more general one. Indeed, we can make the following assumptions:

$$F : \mathbb{R} \to (-\infty, +\infty] \quad \text{admits the decomposition} \quad F = \widehat{\beta} + \widehat{\pi}, \quad \text{where} \tag{2.5}$$

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex and l.s.c. with subdifferential} \quad \beta := \partial\widehat{\beta}$$
$$\text{and fulfills} \quad \widehat{\beta}(0) = 0 \quad \text{and} \quad \lim_{r\to+\infty} \widehat{\beta}(r)|r|^{-2} = +\infty\,. \tag{2.6}$$

$$\widehat{\pi} \in C^1(\mathbb{R}) \quad \text{with Lipschitz continuous derivative } \pi := \widehat{\pi}'. \tag{2.7}$$

For the source terms we assume:

$$\mathcal{S}_\varphi, \mathcal{S}_\sigma : \mathbb{R}^2 \to \mathbb{R} \quad \text{are Lipschitz continuous functions satisfying} \tag{2.8}$$

$$|\mathcal{S}_\varphi(r,s)| + |\mathcal{S}_\sigma(r,s)| \le \Theta(|r| + |s| + 1)$$
$$\text{for some constant } \Theta > 0 \text{ and every } r, s \in \mathbb{R}\,. \tag{2.9}$$

$$g \in L^\infty(Q). \tag{2.10}$$

Finally, the permeability, viscosity and sensitivity constants ν, η, λ and χ are requested to satisfy

$$\nu,\ \eta \in (0, +\infty) \quad \text{and} \quad \lambda,\ \chi \in [0, +\infty). \tag{2.11}$$

It is well known that β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with corresponding domain $D(\beta)$, and that $0 \in \beta(0)$. It is worth noticing that in the case $\widehat{\beta} \in C^1(\mathbb{R})$ it follows that it is single-valued, and we can write $\beta = \widehat{\beta}'$ as well as $F' = \beta + \pi$. Here, we immediately observe that all of the standard potentials (1.11)–(1.13) fulfill (2.5)–(2.7), as well as that the biologically relevant examples given above for the source terms satisfy (2.8)–(2.10).

As for the initial and boundary data, we assume:

$$\varphi_0 \in V \quad \text{and} \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega)\,. \tag{2.12}$$

$$\mu_\Sigma \in H^1(0,T; L^2(\Gamma)) \cap L^2(0,T; H^{1/2}(\Gamma))\,. \tag{2.13}$$

$$\sigma_\Sigma \in L^2(\Sigma)\,. \tag{2.14}$$

However, it is convenient to transform the Dirichlet inhomogeneous boundary condition for μ into a homogeneous one by performing a change of variable for the chemical potential. To this end, we introduce the harmonic extension of the boundary datum μ_Σ, i.e., the function $h : Q \to \mathbb{R}$ defined as follows:

$$h(t) \in V, \quad -\Delta h(t) = 0 \quad \text{and} \quad h(t)|_\Gamma = \mu_\Sigma(t) \quad \text{for a.a. } t \in (0,T)\,. \tag{2.15}$$

We notice at once that (2.13) ensures that h enjoys at least the regularity

$$h \in H^1(0,T; H) \cap L^2(0,T; V)\,. \tag{2.16}$$

Thus, upon setting $\widetilde{\mu} := \mu - h$, the boundary condition for μ in (1.9) now becomes the homogeneous Dirichlet condition $\widetilde{\mu} = 0$ on Σ. However, for the sake of simplicity, we proceed with abuse of notation and still denote by μ the above difference $\widetilde{\mu}$ between the chemical potential and h. This change of notation obviously slightly modifies the equations: the right-hand side of (1.1) has to be adapted, and we have to rewrite (1.4) as

$$\mu \in -\epsilon\Delta\varphi + \epsilon^{-1}\partial F(\varphi) - \chi\sigma - h,$$

where the differential inclusion arises from the possible multi-valued nature of the nonlinearity F, in accordance with (2.5)–(2.7). For this reason, and in order to clarify the meaning of the equation, we state the new problem to be dealt with in a precise form. In particular, the equations (1.1), (1.3) and (1.5), and the corresponding boundary conditions, are replaced by variational equations (also owing to the Leibniz rule for the divergence and (1.2)), and the homogeneous Dirichlet condition for μ is enforced by the forthcoming (2.19).

By recalling (1.6)–(1.7) and (1.17)–(1.18), and setting $m(\cdot) \equiv n(\cdot) \equiv 1$ and $\epsilon = 1$ as announced in the Introduction, we look for a sextuple $(\boldsymbol{v}, p, \mu, \varphi, \xi, \sigma)$ enjoying the regularity properties

$$\boldsymbol{v} \in L^2(0,T;\boldsymbol{V})\,, \tag{2.17}$$

$$p \in L^{4/3}(0,T;H)\,, \tag{2.18}$$

$$\mu \in L^2(0,T;V_0)\,, \tag{2.19}$$

$$\varphi \in H^1(0,T;V_0^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{2.20}$$

$$\xi \in L^2(0,T;H)\,, \tag{2.21}$$

$$\sigma \in W^{1,4/3}(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,, \tag{2.22}$$

that satisfies

$$\int_\Omega \mathbb{T}(\boldsymbol{v},p) : \nabla \boldsymbol{\zeta} + \nu \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\zeta} = \int_\Omega (\mu + h) \nabla\varphi \cdot \boldsymbol{\zeta} + \int_\Omega N_\sigma(\varphi,\sigma) \nabla\sigma \cdot \boldsymbol{\zeta}$$
$$\text{for every } \boldsymbol{\zeta} \in \boldsymbol{V} \text{ and a.e. in } (0,T)\,, \tag{2.23}$$

$$\operatorname{div} \boldsymbol{v} = g \quad \text{a.e. in } Q\,, \tag{2.24}$$

$$\langle \partial_t \varphi, \phi \rangle_{V_0} + \int_\Omega \nabla\mu \cdot \nabla\phi = \int_\Omega \mathcal{S}_\varphi(\varphi,\sigma)\phi - \int_\Omega (\nabla\varphi \cdot \boldsymbol{v} + \varphi g)\,\phi$$
$$\text{for every } \phi \in V_0 \text{ and a.e. in } (0,T)\,, \tag{2.25}$$

$$\int_\Omega \nabla\varphi \cdot \nabla z + \int_\Omega (\xi + \pi(\varphi)) z = \int_\Omega \big(\mu + h - N_\varphi(\varphi,\sigma)\big) z$$
$$\text{for every } z \in V \text{ and a.e. in } (0,T)\,, \tag{2.26}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q\,, \tag{2.27}$$

$$\langle \partial_t \sigma, \zeta \rangle_V + \int_\Omega \nabla N_\sigma(\varphi,\sigma) \cdot \nabla\zeta$$
$$= \int_\Omega \mathcal{S}_\sigma(\varphi,\sigma)\zeta - \int_\Omega (\nabla\sigma \cdot \boldsymbol{v} + \sigma g)\zeta + \kappa \int_\Gamma (\sigma_\Sigma - \sigma)\zeta$$
$$\text{for every } \zeta \in V \text{ and a.e. in } (0,T)\,, \tag{2.28}$$

as well as the initial conditions

$$\varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega\,, \quad \text{and} \quad \langle \sigma(0), \zeta \rangle_V = \langle \sigma_0, \zeta \rangle_V \quad \text{for every } \zeta \in V\,. \tag{2.29}$$

Here is our main result:

Theorem 2.1. *Assume* (2.5)–(2.11)*, and let the notations* (1.6)–(1.7) *and* (1.17)–(1.18) *be in force. Moreover, let* (2.12)–(2.14) *be fulfilled, and let h be defined by* (2.15)*. Then, the weak formulation* (2.23)–(2.29) *of the Cahn–Hilliard–Brinkman system admits at least one solution $(\boldsymbol{v}, p, \mu, \varphi, \xi, \sigma)$ with the regularity specified by* (2.17)–(2.22)*.*

Remark 2.2. It is worth pointing out that equation (2.26) has been formulated in a weak form just for convenience. Indeed, thanks to (2.27), along with the regularity properties (2.20) and (2.21), the variational equation (2.26) is equivalent to the boundary value problem

$$-\Delta\varphi + \xi + \pi(\varphi) = \mu + h - N_\varphi(\varphi,\sigma) \text{ a.e. in } Q \quad \text{and} \quad \partial_{\boldsymbol{n}}\varphi = 0 \text{ on } \Sigma. \tag{2.30}$$

As further regularity is concerned, the components φ and ξ of every solution satisfy

$$\varphi \in L^2(0,T;W^{2,6}(\Omega)) \cap L^4(0,T;H^2(\Omega)) \quad \text{and} \quad \xi \in L^2(0,T;L^6(\Omega)) \tag{2.31}$$

(in the two-dimensional case the summability exponent 6 can be replaced by any $q \geq 1$), as shown in the forthcoming Remark 4.1. Finally, we notice that, in view of the very low regularity at disposal for the nutrient variable σ and the velocity field $\boldsymbol{v}$, the uniqueness of weak solutions is not to be expected.

We continue this section by listing some tools that will be useful later on. We first recall Young's inequality

$$a\,b \leq \frac{\delta}{q}\,a^q + \frac{(\delta)^{-q'/q}}{q'}\,b^{q'} \quad \text{for all } a,b \in [0,+\infty),\, q \in (1,+\infty) \text{ and } \delta > 0, \tag{2.32}$$

where q' denotes the conjugate exponent of q given by the identity $(1/q)+(1/q') = 1$. We repeatedly use it, mainly with $q = q' = 2$. We also account for Hölder's inequality, as well as for the following Sobolev, compactness, Poincaré, and Korn inequalities:

$$\|v\|_q \leq C_S\,\|v\|_V \qquad \text{for every } v \in V \text{ and } q \in [1,6], \tag{2.33}$$

$$\|v\|_4^2 \leq \delta\,\|\nabla v\|^2 + C_\delta\,\|v\|^2 \qquad \text{for every } v \in V \text{ and } \delta > 0, \tag{2.34}$$

$$\|v\|_V \leq C_P\,\|\nabla v\| \qquad \text{for every } v \in V_0, \tag{2.35}$$

$$\|\boldsymbol{v}\|_V^2 \leq C_K\big(\|\boldsymbol{v}\|^2 + \|D\boldsymbol{v}\|^2\big) \quad \text{for every } \boldsymbol{v} \in \boldsymbol{V}. \tag{2.36}$$

Here, the constants C_S, C_P, and C_K, depend only on Ω, while C_δ depends on δ, in addition. In (2.36), the notation (1.7) is used.

Next, we present three auxiliary results that will be used in the sequel. The first one is stated in a more general setting as an exercise in [11, Ex. III.3.5], but it readily follows as a corollary from [11, Thm. III.3.1]. The second one is related to the Stokes resolvent operator and is a particular case of [2, Thm. 3] (which is an extension of the results in [20]). Finally, the last one regards the trace operator.

Lemma 2.3. *There exists a constant C that depends only on Ω such that for every f and $\boldsymbol{a}$ satisfying*

$$f \in H, \quad \boldsymbol{a} \in \boldsymbol{H}^{1/2}(\Gamma), \quad \textit{and} \quad \int_\Omega f = \int_\Gamma \boldsymbol{a}\cdot\boldsymbol{n}, \tag{2.37}$$

there exists some $\boldsymbol{u} \in \boldsymbol{V}$ satisfying

$$\operatorname{div}\boldsymbol{u} = f \quad \textit{a.e. in } \Omega, \quad \boldsymbol{u}|_\Gamma = \boldsymbol{a}, \quad \textit{and} \quad \|\boldsymbol{u}\|_V \leq C\big(\|f\| + \|\boldsymbol{a}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big). \tag{2.38}$$

Lemma 2.4. *Assume that $\boldsymbol{f} \in \boldsymbol{H}$ and $f \in V$. Then, there exists a unique pair $(\boldsymbol{v},p)$ satisfying*

$$\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \quad \textit{and} \quad p \in V,$$

$$-\operatorname{div}\mathbb{T}(\boldsymbol{v},p) + \nu\boldsymbol{v} = \boldsymbol{f} \quad \textit{and} \quad \operatorname{div}\boldsymbol{v} = f \quad \textit{in } \Omega,$$

$$\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{0} \quad \textit{on } \Gamma.$$

Moreover, the mapping $\Psi = (\Psi_1, \Psi_2) : (\boldsymbol{f}, f) \mapsto (\boldsymbol{v}, p)$ *is linear and continuous from* $\boldsymbol{H} \times V$ *into* $\boldsymbol{H}^2(\Omega) \times V$.

Lemma 2.5. *The trace operator maps* $L^\infty(0,T;H) \cap L^2(0,T;V)$ *into* $L^4(0,T;L^2(\Gamma))$*, and it holds the estimate*

$$\int_0^T \|v(t)\|_{L^2(\Gamma)}^4 \, dt \le C \, \|v\|_{L^\infty(0,T;H)}^2 \, \|v\|_{L^2(0,T;V)}^2 \tag{2.39}$$

for every $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$*, where the constant* C *depends only on* Ω*, and where* v *also denotes the trace of* v *on* Γ.

Proof. As we did not find a precise reference, we provide a sketch of the proof. We denote by C_1, C_2,... constants that depend only on Ω. We introduce the real interpolation space (by the way, the Besov space $B^{1/2}_{2,1}(\Omega)$)

$$B := (V, H)_{1/2,1} \,. \tag{2.40}$$

The trace operator $v \mapsto v|_\Gamma$ maps B into $L^2(\Gamma)$ and is linear and continuous. In the half-space case $\Omega = \mathbb{R}^3_+$, this can be deduced, e.g., from formula (I.17) (with the notation $Y(1, \mathbb{R}^3_+)$ for (2.40)) in the paper [29], where it is also shown that the operator maps B onto $L^2(\Gamma)$. As usual, the result is then extended to the general case by using local charts and a partition of unity. This leads to the estimate

$$\|v\|_{L^2(\Gamma)} \le C_1 \, \|v\|_B \quad \text{for every } v \in B.$$

On the other hand, the interpolation inequality

$$\|v\|_B \le C_2 \, \|v\|_V^{1/2} \, \|v\|^{1/2}$$

holds true for every $v \in V$. Now, letting $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$, we have for a.a. $t \in (0,T)$ that

$$\|v(t)\|_{L^2(\Gamma)} \le C_1 \, \|v(t)\|_B \le C_3 \, \|v(t)\|_V^{1/2} \, \|v(t)\|^{1/2} \le C_3 \, \|v(t)\|_V^{1/2} \, \|v\|_{L^\infty(0,T;H)}^{1/2} \,,$$

and (2.39) directly follows by taking the $4th$ powers and integrating over $(0,T)$. □

Remark 2.6. We notice that the application of the trace estimate in [11, Thm. II.4.1] (whose proof is left to the reader), with the parameters therein being chosen as $r = q = 2, m = 1, n = d = 3, \lambda = 0$, produces a similar trace inequality, namely,

$$\|v\|_{L^2(\Gamma)} \le c \, (\|v\| + \|v\|^{1/2} \|v\|_V^{1/2}) \quad \text{for every } v \in V.$$

Besides, let us state a general rule concerning the constants that appear in the estimates to be performed in the following. The small-case symbol c stands for a generic constant whose actual value may change from line to line, and even within the same line, and depends only on Ω, the shape of the nonlinearities, and the constants and the norms of the functions involved in the assumptions of the statements. In particular, the values of c do not depend on the parameters ε and k that will be introduced in the next section. A small-case symbol with a subscript like c_δ (specifically, with $\delta = \varepsilon$) indicates that the constant may depend on the parameter δ, in addition. On the contrary, we mark precise constants that we can refer to by using different symbols (see, e.g., (2.33) and (2.38)).

The next sections aim to rigorously prove Theorem 2.1. A standard approach to guarantee the existence of solutions to similar Cahn–Hilliard type systems is based on suitable approximation procedures that can be schematized as follows: first, one has to regularize the possibly singular nonlinearity F, with the classical choice being the Yosida regularization that depends on a parameter, say, $\varepsilon > 0$. Then, for every fixed $\varepsilon > 0$, one further discretizes the system in space via the Galerkin method and solves a family of finite-dimensional problems that depend on a parameter $k \in \mathbb{N}$. Next, one has to provide some rigorous estimates, which are independent of both ε and k. From these estimates, using weak and weak star compactness arguments, one can find a suitable subsequence and eventually pass to the limit as $k \to \infty$ and as $\varepsilon \to 0$, thus showing that the obtained limits yield a solution to the original system.

The essence of the proof of Theorem 2.1 can be roughly schematized by the abovementioned steps, but we had to face a major obstacle arising from the different boundary conditions in (1.9). The latter prevent the solvability of the ODE system originating from the Galerkin scheme. In fact, due to (1.9), one is in the approximation naturally led to consider Schauder bases (cf. (3.19) and (3.20)) for the Laplace operator with homogeneous Neumann and Dirichlet boundary conditions for $\varphi,\ \sigma$ and μ, respectively. Despite of being natural, this choice completely impedes the solvability of the discrete problem, because there occur inner products between the elements of the different two bases. To overcome this intrinsic difficulty, we introduce an intermediate approximation step, which consists in adding further regularizing terms at the level $\varepsilon > 0$ that somehow dominate the mixed terms and enable us to solve the Galerkin system (cf. (3.24)–(3.29)). Finally, we pass to the limit as $\varepsilon \searrow 0$ as anticipated above, thus proving the theorem.

3 Approximation

In this section, we introduce and solve a proper approximating problem depending on the parameter $\varepsilon > 0$. First of all, we replace the functional $\widehat{\beta}$ and the graph β by their Moreau–Yosida regularizations $\widehat{\beta}_\varepsilon$ and β_ε, respectively (see, e.g., [4, pp. 28 and 39]). Then, we set

$$F_\varepsilon := \widehat{\beta}_\varepsilon + \widehat{\pi} \tag{3.1}$$

and recall that classical theory of convex analysis entails the following facts:

$$\beta_\varepsilon \text{ is monotone and Lipschitz continuous with } \beta_\varepsilon(0) = 0\,, \tag{3.2}$$

$$0 \le \widehat{\beta}_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\,ds \le \widehat{\beta}(r) \quad \text{for every } r \in \mathbb{R}\,, \tag{3.3}$$

$$\text{for every } M > 0 \text{ there exist } C_M > 0 \text{ and } \varepsilon_M > 0 \text{ such that}$$
$$F_\varepsilon(r) \ge M\,r^2 - C_M \quad \text{for every } r \in \mathbb{R} \text{ and every } \varepsilon \in (0, \varepsilon_M)\,. \tag{3.4}$$

The only nonobvious fact is the coercivity property in (3.4). In this direction, let us fix $M > 0$ and observe that our assumption (2.6) on $\widehat{\beta}$ implies that

$$\widehat{\beta}(r) \ge 2M\,r^2 - C_M \quad \text{for every } r \in \mathbb{R} \text{ and some constant } C_M > 0.$$

It then follows that

$$\begin{aligned}
&\widehat{\beta}_\varepsilon(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|s-r|^2 + \widehat{\beta}(s) \right\} \\
&\ge \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|s-r|^2 + 2M\,s^2 - C_M \right\} = \frac{1}{2\varepsilon}|s_*-r|^2 + 2M\,s_*^2 - C_M\,,
\end{aligned}$$

where s_* is the minimum point, namely, $s_* = r/(1+4M\varepsilon)$. Hence, we have that

$$\widehat{\beta}_\varepsilon(r) \geq 2M\, s_*^2 - C_M = \frac{2M}{(1+4M\varepsilon)^2}\, r^2 - C_M \geq M\, r^2 - C_M$$

for every $r \in \mathbb{R}$, whenever $(1+4M\varepsilon)^2 \leq 2$, i.e., (3.4) with F_ε replaced by $\widehat{\beta}_\varepsilon$ (with an obvious choice of ε_M). Then, (3.4) itself follows, since π is Lipschitz continuous.

Besides this regularization, we replace g and h by smoother functions g_ε and h_ε satisfying

$$g_\varepsilon \in C^0(\overline{Q}), \quad \|g_\varepsilon\|_\infty \leq c\,, \quad \text{and} \quad g_\varepsilon \to g \quad \text{a.e. in } Q \text{ as } \varepsilon \searrow 0\,, \tag{3.5}$$

$$h_\varepsilon \in H^1(0,T;H) \cap L^2(0,T;V) \cap C^0(\overline{Q})\,, \quad \|h_\varepsilon\|_{H^1(0,T;H)\cap L^2(0,T;V)} \leq c\,, \\ \text{and } h_\varepsilon \to h \text{ strongly in } L^2(0,T;V) \text{ as } \varepsilon \searrow 0. \tag{3.6}$$

For simplicity, we do not enter in the details concerning the construction of the regularizations above. However, let us mention that standard mollification arguments are enough to get the desired properties prescribed by (3.5) and (3.6), respectively. Moreover, as anticipated, we introduce artificial viscous terms in some of the equations. We prefer to present all of them in their variational form. The approximating problem thus consists in finding a quintuple $(\boldsymbol{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon)$ that satisfies the regularity properties

$$\boldsymbol{v}_\varepsilon \in L^2(0,T;\boldsymbol{V})\,, \tag{3.7}$$

$$p_\varepsilon \in L^{4/3}(0,T;H)\,, \tag{3.8}$$

$$\mu_\varepsilon \in H^1(0,T;V_0^*) \cap L^2(0,T;V_0)\,, \tag{3.9}$$

$$\varphi_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{3.10}$$

$$\sigma_\varepsilon \in W^{1,4/3}(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,, \tag{3.11}$$

and solves the variational equations

$$\int_\Omega \mathbb{T}(\boldsymbol{v}_\varepsilon, p_\varepsilon) : \nabla\boldsymbol{\zeta} + \nu \int_\Omega \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\zeta} = \int_\Omega (\mu_\varepsilon + h_\varepsilon)\nabla\varphi_\varepsilon \cdot \boldsymbol{\zeta} + \int_\Omega N_\sigma(\varphi_\varepsilon, \sigma_\varepsilon)\nabla\sigma_\varepsilon \cdot \boldsymbol{\zeta} \\ \text{for every } \boldsymbol{\zeta} \in \boldsymbol{V} \text{ and a.e. in } (0,T)\,, \tag{3.12}$$

$$\operatorname{div} \boldsymbol{v}_\varepsilon = g_\varepsilon \quad \text{a.e. in } Q \tag{3.13}$$

$$\langle \partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon), \phi\rangle_{V_0} + \int_\Omega \nabla\mu_\varepsilon \cdot \nabla\phi = \int_\Omega \mathcal{S}_\varphi(\varphi_\varepsilon, \sigma_\varepsilon)\phi - \int_\Omega (\nabla\varphi_\varepsilon \cdot \boldsymbol{v}_\varepsilon + \varphi_\varepsilon g_\varepsilon)\phi \\ \text{for every } \phi \in V_0 \text{ and a.e. in } (0,T)\,, \tag{3.14}$$

$$\varepsilon \int_\Omega \partial_t\varphi_\varepsilon\, z + \int_\Omega \nabla\varphi_\varepsilon \cdot \nabla z + \int_\Omega F_\varepsilon'(\varphi_\varepsilon)\, z = \int_\Omega \big(\mu_\varepsilon + h_\varepsilon - N_\varphi(\varphi_\varepsilon, \sigma_\varepsilon)\big) z \\ \text{for every } z \in V \text{ and a.e. in } (0,T)\,, \tag{3.15}$$

$$\langle \partial_t\sigma_\varepsilon, \zeta\rangle_V + \int_\Omega \nabla N_\sigma(\varphi_\varepsilon, \sigma_\varepsilon) \cdot \nabla\zeta \\ = \int_\Omega \mathcal{S}_\sigma(\varphi_\varepsilon, \sigma_\varepsilon)\zeta - \int_\Omega (\nabla\sigma_\varepsilon \cdot \boldsymbol{v}_\varepsilon + \sigma_\varepsilon g_\varepsilon)\zeta + \kappa \int_\Gamma (\sigma_\Sigma - \sigma_\varepsilon)\zeta \\ \text{for every } \zeta \in V \text{ and a.e. in } (0,T)\,, \tag{3.16}$$

as well as the initial conditions

$$\varphi_\varepsilon(0) = \varphi_0\,, \quad \mu_\varepsilon(0) = 0\,, \quad \text{a.e. in } \Omega\,, \tag{3.17}$$

$$\langle \sigma_\varepsilon(0), \zeta\rangle_V = \langle \sigma_0, \zeta\rangle_V \quad \text{for every } \zeta \in V. \tag{3.18}$$

Let us incidentally notice that the presence of a selection ξ (cf. (2.27)) is no longer needed in the approximating ε-problem due to the regularity at disposal for F_ε defined by (3.1). Moreover, we can simply write $F'_\varepsilon(\varphi)$ in place of $\beta_\varepsilon(\varphi) + \pi(\varphi)$ in (3.15).

Remark 3.1. We notice that, due to (3.10) and the initial condition for φ_ε, the regularity for $\partial_t\mu_\varepsilon$ given in (3.9) and the initial conditions for μ_ε are equivalent to

$$\partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon) \in L^2(0,T;V_0^*) \quad \text{and} \quad (\varepsilon\mu_\varepsilon + \varphi_\varepsilon)(0) = \varphi_0\,,$$

respectively.

Theorem 3.2. *Under the assumptions of Theorem 2.1 and with the above notation, the approximating problem* (3.12)–(3.18) *has at least one solution* $(\boldsymbol{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon)$ *which fulfills* (3.7)–(3.11).

The rest of this section is devoted to the proof of this theorem. The method we use starts from a discrete problem based on a Faedo–Galerkin scheme. To this end, we introduce the nondecreasing sequences $\{\lambda_j\}$ and $\{\lambda_j^0\}$ of eigenvalues and the corresponding complete orthonormal sequences $\{e_j\}$ and $\{e_j^0\}$ of eigenfunctions of the eigenvalue problems for the Laplace operator with homogeneous Neumann and Dirichlet boundary conditions, respectively. Namely, we have that

$$-\Delta e_j = \lambda_j e_j \quad \text{in } \Omega \quad \text{and} \quad \partial_{\boldsymbol{n}} e_j = 0 \quad \text{on } \Gamma\,, \tag{3.19}$$

$$-\Delta e_j^0 = \lambda_j^0 e_j^0 \quad \text{in } \Omega \quad \text{and} \quad e_j^0 = 0 \qquad \text{on } \Gamma\,, \tag{3.20}$$

for $j = 1, 2, \dots$, as well as the normalization conditions

$$\int_\Omega e_i e_j = \int_\Omega e_i^0 e_j^0 = \delta_{ij} \quad \text{for every } i \text{ and } j\,, \tag{3.21}$$

with the standard Kronecker symbol δ_{ij}. Moreover, if we set, for $k = 1, 2, \dots$,

$$V_k := \operatorname{span}\{e_j :\ 1 \le j \le k\} \quad \text{and} \quad V_k^0 := \operatorname{span}\{e_j^0 :\ 1 \le j \le k\}\,, \tag{3.22}$$

then the unions of the these spaces are dense in V and V_0, respectively, and both are dense in H as well. We notice at once that all of the above eigenfunctions are smooth since Ω is smooth, that $\lambda_1 = 0$, and that $e_1 = |\Omega|^{-1/2}$. Then, the discrete problem related to k consists in finding a quintuple $(\boldsymbol{v}_k, p_k, \mu_k, \varphi_k, \sigma_k)$ with the regularity specified by

$$\begin{aligned} &\boldsymbol{v}_k \in L^2(0,T;\boldsymbol{H}^2(\Omega)), \quad p_k \in L^2(0,T;V), \quad \varphi_k, \sigma_k \in C^1([0,T];V_k) \cap L^\infty(Q), \\ &\text{and} \quad \mu_k \in C^1([0,T];V_k^0) \cap L^\infty(Q), \end{aligned} \tag{3.23}$$

that solves the system

$$\int_\Omega \mathbb{T}(\boldsymbol{v}_k, p_k) : \nabla\boldsymbol{\zeta} + \nu \int_\Omega \boldsymbol{v}_k \cdot \boldsymbol{\zeta} = \int_\Omega (\mu_k + h_\varepsilon)\nabla\varphi_k \cdot \boldsymbol{\zeta} + \int_\Omega N_\sigma(\varphi_k, \sigma_k)\nabla\sigma_k \cdot \boldsymbol{\zeta}\,, \tag{3.24}$$

$$\operatorname{div} \boldsymbol{v}_k = g_\varepsilon \quad \text{a.e. in } Q\,, \tag{3.25}$$

$$\varepsilon \int_\Omega \partial_t\mu_k\, \phi + \int_\Omega \nabla\mu_k \cdot \nabla\phi = - \int_\Omega \partial_t\varphi_k\, \phi + \int_\Omega \mathcal{S}_\varphi(\varphi_k, \sigma_k)\phi - \int_\Omega (\nabla\varphi_k \cdot \boldsymbol{v}_k + \varphi_k g_\varepsilon)\phi\,, \tag{3.26}$$

$$\varepsilon \int_\Omega \partial_t\varphi_k\, z + \int_\Omega \nabla\varphi_k \cdot \nabla z + \int_\Omega F'_\varepsilon(\varphi_k)\, z = \int_\Omega \bigl(\mu_k + h_\varepsilon - N_\varphi(\varphi_k, \sigma_k)\bigr) z\,, \tag{3.27}$$

$$\int_\Omega \partial_t\sigma_k\, \zeta + \int_\Omega \nabla N_\sigma(\varphi_k, \sigma_k) \cdot \nabla\zeta = \int_\Omega \mathcal{S}_\sigma(\varphi_k, \sigma_k)\zeta - \int_\Omega (\nabla\sigma_k \cdot \boldsymbol{v}_k + \sigma_k g_\varepsilon)\zeta + \kappa \int_\Gamma (\sigma_\Sigma - \sigma_k)\zeta\,, \tag{3.28}$$

for every $\boldsymbol{\zeta} \in \boldsymbol{V}$, $\phi \in V_k^0$, z, $\zeta \in V_k$, and $t \in [0, T)$, and that fulfills the initial conditions

$$\int_\Omega \varphi_k(0)\phi = \int_\Omega \varphi_0\phi\,, \ \ \mu_k(0) = 0 \quad \text{and} \quad \int_\Omega \sigma_k(0)\zeta = \int_\Omega \sigma_0\zeta \tag{3.29}$$

for every $\phi \in V_k^0$ and $\zeta \in V_k$.

Existence for the discrete problem. The first aim of ours is to show the existence of at least one solution (we do not care about uniqueness, since it is not needed). The method relies on a proper application of Lemma 2.4. Besides, let us point out that the idea of employing the Stokes resolvent to express the velocity field $\boldsymbol{v}_k$ in terms of the other variables φ_k, μ_k, and σ_k is largely inspired by [8] (see also [9, 22]). For a while, the symbols φ_k, μ_k and σ_k denote independent variables. To every triplet $(\varphi_k, \mu_k, \sigma_k) \in V_k \times V_k^0 \times V_k$, we associate the vector-valued function

$$\boldsymbol{f}_k := (\mu_k + h_\varepsilon)\nabla\varphi_k + (\sigma_k + \chi(1 - \varphi_k))\nabla\sigma_k\,, \tag{3.30}$$

and we notice that $\boldsymbol{f}_k$ only depends on time through the continuous function h_ε, while the dependence on space occurs just through the eigenfunctions and their gradients, which are smooth. In particular, on the one hand, if we read $\boldsymbol{f}_k$ as a function of the coefficients of φ_k, μ_k, σ_k (with respect to the bases just chosen), and t, then we see that it is continuous. On the other hand, for every $t \in [0, T]$, we are allowed to apply Lemma 2.4 with $\boldsymbol{f} = \boldsymbol{f}_k(t)$ and $f = g_\varepsilon(t)$. Since the mapping Ψ is linear, continuous, and time independent, and since g_ε and h_ε are continuous, this yields a pair of functions that are continuous with respect to the coefficients of φ_k, μ_k, σ_k, and t. This observation is made to ensure the continuity of the functions that rule the system of ODE's we are going to introduce.

Now, we let φ_k, μ_k and σ_k depend on time. To every triplet $(\varphi_k, \mu_k, \sigma_k) \in L^\infty(0, T; V_k \times V_k^0 \times V_k)$ we associate the function $\boldsymbol{f}_k$ still given by (3.30) and, for every $t \in [0, T]$, we apply Lemma 2.4 as before. We obtain two functions, which we still term $\Psi_1(\boldsymbol{f}_k, g_\varepsilon)$ and $\Psi_2(\boldsymbol{f}_k, g_\varepsilon)$ with an abuse of notation, that, according to the lemma, belong to $L^\infty(0, T; \boldsymbol{H}^2(\Omega))$ and $L^\infty(0, T; \boldsymbol{V})$, respectively. By construction, the pair of functions $(\boldsymbol{v}_k, p_k) := (\Psi_1(\boldsymbol{f}_k, g_\varepsilon), \Psi_2(\boldsymbol{f}_k, g_\varepsilon))$ solves the equations (3.24)–(3.25) corresponding to the given triplet $(\varphi_k, \mu_k, \sigma_k)$. Therefore, the whole problem (3.24)–(3.29) is equivalent to the problem of finding a triplet $(\varphi_k, \mu_k, \sigma_k)$ with the regularity specified in (3.23) that solves

$$\varepsilon\langle\partial_t\mu_k,\phi\rangle_{V_0}+\int_\Omega\nabla\mu_k\cdot\nabla\phi$$
$$=-\langle\partial_t\varphi_k,\phi\rangle_{V_0}+\int_\Omega\mathcal{S}_\varphi(\varphi_k,\sigma_k)\phi-\int_\Omega(\nabla\varphi_k\cdot\Psi_1(\boldsymbol{f}_k,g_\varepsilon)+\varphi_k g_\varepsilon)\phi\,, \tag{3.31}$$
$$\varepsilon\langle\partial_t\varphi_k,z\rangle_V+\int_\Omega\nabla\varphi_k\cdot\nabla z+\int_\Omega F'_\varepsilon(\varphi_k)\,z=\int_\Omega\bigl(\mu_k+h_\varepsilon-N_\varphi(\varphi_k,\sigma_k)\bigr)z\,, \tag{3.32}$$
$$\langle\partial_t\sigma_k,\zeta\rangle_V+\int_\Omega\nabla N_\sigma(\varphi_k,\sigma_k)\cdot\nabla\zeta$$
$$=\int_\Omega\mathcal{S}_\sigma(\varphi_k,\sigma_k)\zeta-\int_\Omega(\nabla\sigma_k\cdot\Psi_1(\boldsymbol{f}_k,g_\varepsilon)+\sigma_k g_\varepsilon)\zeta+\kappa\int_\Gamma(\sigma_\Sigma-\sigma_k)\zeta\,, \tag{3.33}$$

for every $\phi\in V_k^0$, $z,\,\zeta\in V_k$ and $t\in[0,T)$, with $\boldsymbol{f}_k$ given by (3.30), and satisfies the initial conditions (3.29). We show that this problem has at least one solution. To this end, we represent the unknowns in terms of the bases of the spaces V_k and V_k^0, i.e.,

$$\varphi_k(t)=\sum_{j=1}^k\varphi_{kj}(t)e_j,\quad \mu_k(t)=\sum_{j=1}^k\mu_{kj}(t)e_j^0\,,\quad\text{and}\quad\sigma_k(t)=\sum_{j=1}^k\sigma_{kj}(t)e_j\,,$$

and introduce the $\mathbb{R}^k$-valued functions

$$\widehat{\boldsymbol{\varphi}}_k:=(\varphi_{kj})_{j=1}^k\,,\quad\widehat{\boldsymbol{\mu}}_k:=(\mu_{kj})_{j=1}^k\,,\quad\text{and}\quad\widehat{\boldsymbol{\sigma}}_k:=(\sigma_{kj})_{j=1}^k\,,$$

which are the true unknowns. In terms of these coefficients, the discrete problem takes the form

$$\varepsilon\widehat{\boldsymbol{\mu}}_k'(t)=\boldsymbol{A}_{\varepsilon,k}(\widehat{\boldsymbol{\varphi}}_k(t),\widehat{\boldsymbol{\mu}}_k(t),\widehat{\boldsymbol{\sigma}}_k(t),t)-\widehat{\boldsymbol{\varphi}}_k'(t)\,, \tag{3.34}$$
$$\varepsilon\widehat{\boldsymbol{\varphi}}_k'(t)=\boldsymbol{B}_{\varepsilon,k}(\widehat{\boldsymbol{\varphi}}_k(t),\widehat{\boldsymbol{\mu}}_k(t),\widehat{\boldsymbol{\sigma}}_k(t),t)\,, \tag{3.35}$$
$$\widehat{\boldsymbol{\sigma}}_k'(t)=\boldsymbol{C}_{\varepsilon,k}(\widehat{\boldsymbol{\varphi}}_k(t),\widehat{\boldsymbol{\mu}}_k(t),\widehat{\boldsymbol{\sigma}}_k(t),t)\,, \tag{3.36}$$

with some continuous functions $\boldsymbol{A}_{\varepsilon,k},\boldsymbol{B}_{\varepsilon,k},\boldsymbol{C}_{\varepsilon,k}:\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^k\times[0,T]\to\mathbb{R}^k$, and the initial conditions for $\widehat{\boldsymbol{\varphi}}_k$, $\widehat{\boldsymbol{\mu}}_k$ and $\widehat{\boldsymbol{\sigma}}_k$ are trivially derived from (3.29). By replacing $\widehat{\boldsymbol{\varphi}}_k'$ in (3.34) using (3.35) (recall that now $\varepsilon>0$ is fixed), we obtain a standard Cauchy problem for a $3k$-dimensional nonlinear ODE system ruled by a continuous function. This allows us to apply the Cauchy–Peano theorem, which ensures the existence of at least one local solution. This local solution can be extended to a maximal solution, which provides a maximal solution $(\varphi_k,\mu_k,\sigma_k)$ to (3.24)–(3.29) defined in the interval $[0,T_k)$ for some $T_k\in(0,T]$. We claim that this solution is bounded (as required) and global, i.e., that $T_k=T$. The proof relies on the estimate

$$\|\varphi_k\|_{L^\infty(0,T_k;H)}+\|\mu_k\|_{L^\infty(0,T_k;H)}+\|\sigma_k\|_{L^\infty(0,T_k;H)}\le c_\varepsilon\,, \tag{3.37}$$

which we are going to prove in the next lines. Since (3.21) implies that

$$\|\varphi_k(t)\|^2=\sum_{j=1}^k|\varphi_{kj}(t)|^2=|\widehat{\boldsymbol{\varphi}}_k(t)|^2\quad\text{for every } t\in[0,T_k)\,,$$

and similarly for the other two components, (3.37) shows that the $\mathbb{R}^{3k}$-valued function $(\widehat{\boldsymbol{\varphi}}_k,\widehat{\boldsymbol{\mu}}_k,\widehat{\boldsymbol{\sigma}}_k)$ is bounded. Then, maximality also implies that the solution is global.

Next, we perform a number of a priori estimates that will allow us to let k tend to infinity and to show that the approximating problem (3.12)–(3.18) has at least one solution. The first of these estimates proves the validity of (3.37), in particular.

Remark 3.3. We note that the initial values $\varphi_k(0)$ and $\sigma_k(0)$ are the H-projections of φ_0 and σ_0 onto V_k. To deal with them and for other purposes, it is worth noting a property of the H-projection v_k of a generic element $v \in V$ onto V_k. We have that $\|v_k\| \le \|v\|$, and we derive a similar inequality for the gradients. By definition, for $j = 1, \dots, k$ we have that

$$\int_\Omega v_k e_j = \int_\Omega v e_j \,,$$

whence, using (3.19), also

$$\begin{aligned} \int_\Omega \nabla v_k \cdot \nabla e_j &= - \int_\Omega v_k \Delta e_j = -\lambda_j \int_\Omega v_k e_j \\ &= -\lambda_j \int_\Omega v e_j = - \int_\Omega v \Delta e_j = \int_\Omega \nabla v \cdot \nabla e_j \,. \end{aligned}$$

By linear combination, it follows that

$$\int_\Omega \nabla v_k \cdot \nabla w = \int_\Omega \nabla v \cdot \nabla w \quad \text{for every } w \in V_k \,,$$

and we conclude that

the H-projection v_k coincides with the V-projection of v.

By noting that the choice $w = v_k$ is admitted in the above identity, and collecting everything, we conclude that

$$\|v_k\| \le \|v\|, \quad \|v_k\|_V \le \|v\|_V, \quad \text{and} \quad \|\nabla v_k\| \le \|\nabla v\|, \quad \text{for every } v \in V. \tag{3.38}$$

Therefore, in particular, we have the inequalities

$$\|\varphi_k(0)\| \le \|\varphi_0\| \,, \quad \|\sigma_k(0)\| \le \|\sigma_0\| \,, \quad \text{and} \quad \|\nabla\varphi_k(0)\| \le \|\nabla\varphi_0\| \,. \tag{3.39}$$

Now let $q \in [1, +\infty]$ and $v \in L^q(0,T;V)$, and define $v_k : Q \to \mathbb{R}$ as follows: for a.a. $t \in (0,T)$, $v_k(t)$ is the H-projection of $v(t)$ onto V_k. Then,

$$v_k \in L^q(0,T;V_k) \,, \quad \text{and} \quad \|v_k\|_{L^q(0,T;V)} \le \|v\|_{L^q(0,T;V)} \,.$$

Moreover, if $q < +\infty$, then we also have that $v_k \to v$ strongly in $L^q(0,T;V)$. Indeed,

$$v_k(t) \to v(t) \quad \text{strongly in } V \,, \quad \text{and} \quad \|v_k(t) - v(t)\|_V^q \le 2^q \, \|v(t)\|^q \quad \text{for a.a. } t \in (0,T) \,,$$

so that one can apply the Lebesgue dominated convergence theorem. Clearly, everything can be repeated for the space V_0 and the projection on V_k^0.

First a priori estimate. We recall that we do not yet know that $T_k = T$. For every $t \in (0, T_k)$, we apply Lemma 2.3 with the choices

$$f = g_\varepsilon(t) \quad \text{and} \quad \boldsymbol{a} = \frac{1}{|\Gamma|} \Big(\textstyle\int_\Omega g_\varepsilon(t) \Big) \boldsymbol{n} \,,$$

by observing that the assumptions (2.37) are satisfied, and we term $\boldsymbol{u}_k(t)$ the function given by the lemma. Avoiding writing the time t for a while, and recalling (2.38) and (3.5), we have that

$$\|\boldsymbol{u}_k\|_{L^\infty(0,T_k;\boldsymbol{V})} \le c\,. \tag{3.40}$$

Then, we test (3.24) by $\boldsymbol{\zeta} = \boldsymbol{v}_k - \boldsymbol{u}_k$, with the aim of removing the pressure from the first estimate. This idea has been introduced in [8], and it relies on the identities $D\boldsymbol{v}_k : \nabla\boldsymbol{v}_k = |D\boldsymbol{v}_k|^2$ and $\mathbb{I} : \nabla(\boldsymbol{v}_k - \boldsymbol{u}_k) = \operatorname{div}(\boldsymbol{v}_k - \boldsymbol{u}_k) = 0$. Besides, by also integrating over $(0,t)$ with respect to time, where $t \in (0,T_k)$ is arbitrary, we obtain that

$$\begin{aligned}
&2\eta \int_{Q_t} |D\boldsymbol{v}_k|^2 + \nu \int_{Q_t} |\boldsymbol{v}_k|^2 \\
&= 2\eta \int_{Q_t} D\boldsymbol{v}_k : \nabla\boldsymbol{u}_k + \nu \int_{Q_t} \boldsymbol{v}_k \cdot \boldsymbol{u}_k \\
&\quad + \int_{Q_t} \mu_k \nabla\varphi_k \cdot \boldsymbol{v}_k + \int_{Q_t} N_\sigma(\varphi_k,\sigma_k)\nabla\sigma_k \cdot \boldsymbol{v}_k \\
&\quad - \int_{Q_t} \mu_k \nabla\varphi_k \cdot \boldsymbol{u}_k - \int_{Q_t} N_\sigma(\varphi_k,\sigma_k)\nabla\sigma_k \cdot \boldsymbol{u}_k + \int_Q h_\varepsilon \nabla\varphi_k \cdot (\boldsymbol{v}_k - \boldsymbol{u}_k)\,.
\end{aligned} \tag{3.41}$$

At the same time, we test (3.26) and (3.27) by μ_k and $\partial_t\varphi_k$, respectively, and integrate with respect to time. We obtain that

$$\begin{aligned}
&\frac{\varepsilon}{2} \int_\Omega |\mu_k(t)|^2 + \int_{Q_t} |\nabla\mu_k|^2 \\
&= - \int_{Q_t} \partial_t\varphi_k\, \mu_k + \int_{Q_t} \mathcal{S}_\varphi(\varphi_k,\sigma_k)\mu_k - \int_{Q_t} \nabla\varphi_k \cdot \boldsymbol{v}_k\, \mu_k - \int_{Q_t} \varphi_k g_\varepsilon \mu_k\,,
\end{aligned} \tag{3.42}$$

as well as

$$\begin{aligned}
&\varepsilon \int_{Q_t} |\partial_t\varphi_k|^2 + \frac{1}{2} \int_\Omega |\nabla\varphi_k(t)|^2 + \int_\Omega F_\varepsilon(\varphi_k(t)) + \int_{Q_t} N_\varphi(\varphi_k,\sigma_k)\partial_t\varphi_k \\
&= \frac{1}{2} \int_\Omega |\nabla\varphi_k(0)|^2 + \int_\Omega F_\varepsilon(\varphi_k(0)) + \int_{Q_t} \mu_k \partial_t\varphi_k + \int_{Q_t} h_\varepsilon \partial_t\varphi_k\,.
\end{aligned} \tag{3.43}$$

By recalling that $N_\sigma(\varphi_k,\sigma_k) = \sigma_k + \chi(1-\varphi_k)$ by (1.18), testing (3.28) by $N_\sigma(\varphi_k,\sigma_k)$ and integrating with respect to time, we also obtain that

$$\begin{aligned}
&\int_{Q_t} \partial_t\sigma_k\, N_\sigma(\varphi_k,\sigma_k) + \int_{Q_t} |\nabla N_\sigma(\varphi_k,\sigma_k)|^2 + \kappa \int_{\Sigma_t} |\sigma_k|^2 \\
&= \int_{Q_t} \mathcal{S}_\sigma(\varphi_k,\sigma_k) N_\sigma(\varphi_k,\sigma_k) - \int_{Q_t} \nabla\sigma_k \cdot \boldsymbol{v}_k\, N_\sigma(\varphi_k,\sigma_k) - \int_{Q_t} \sigma_k g_\varepsilon N_\sigma(\varphi_k,\sigma_k) \\
&\quad + \kappa \int_{\Sigma_t} \sigma_\Sigma\big(\sigma_k + \chi(1-\varphi_k)\big) - \kappa\chi \int_{\Sigma_t} \sigma_k(1-\varphi_k).
\end{aligned} \tag{3.44}$$

At this point, we add (3.41)–(3.44) to each other and notice that some cancellations occur. Moreover, we combine the term of (3.43) with the first one of (3.44). By recalling (1.18), we have that

$$\int_{Q_t} N_\varphi(\varphi_k,\sigma_k)\partial_t\varphi_k + \int_{Q_t} \partial_t\sigma_k\, N_\sigma(\varphi_k,\sigma_k) = \int_\Omega N(\varphi_k(t),\sigma_k(t)) - \int_\Omega N(\varphi_k(0),\sigma_k(0)).$$

Next, we observe that the Korn inequality (2.36) implies that

$$2\eta \int_{Q_t} |D\boldsymbol{v}_k|^2 + \nu \int_{Q_t} |\boldsymbol{v}_k|^2 \geq \alpha \int_0^t \|\boldsymbol{v}_k(s)\|_V^2 \, ds\,, \quad \text{where} \quad \alpha := \min\{2\eta, \nu\}/C_K\,.$$

Finally, as for the term involving F_ε, we apply (3.4) with $M = 1$, and obtain that

$$\int_\Omega F_\varepsilon(\varphi_k(t)) \geq \int_\Omega |\varphi_k(t)|^2 - c\,,$$

provided that ε is small enough (as in the lemma): from now on, it is understood that ε satisfies this smallness condition. Regarding the right-hand side, we start by treating the nontrivial terms coming from the identity (3.41). The symbol δ denotes a positive parameter whose value is chosen later on. By recalling (3.40), we have that

$$\begin{aligned}
&2\eta \int_{Q_t} D\boldsymbol{v}_k : \nabla \boldsymbol{u}_k + \nu \int_{Q_t} \boldsymbol{v}_k \cdot \boldsymbol{u}_k \\
&\leq \delta \int_0^t \|\boldsymbol{v}_k(s)\|_V^2 \, ds + c_\delta \int_0^t \|\boldsymbol{u}_k(s)\|_V^2 \, ds \leq \delta \int_0^t \|\boldsymbol{v}_k(s)\|_V^2 \, ds + c_\delta\,.
\end{aligned}$$

Next, with the help of the Hölder, Sobolev, Poincaré, and Young inequalities, we obtain that

$$\begin{aligned}
&- \int_{Q_t} \mu_k \nabla\varphi_k \cdot \boldsymbol{u}_k \leq \int_0^t \|\mu_k(s)\|_4 \, \|\nabla\varphi_k(s)\| \, \|\boldsymbol{u}_k(s)\|_4 \, ds \\
&\leq \delta \int_{Q_t} |\nabla\mu_k|^2 + c_\delta \int_0^t \|\nabla\varphi_k(s)\|^2 \, \|\boldsymbol{u}_k(s)\|_4^2 \, ds \leq \delta \int_{Q_t} |\nabla\mu_k|^2 + c_\delta \int_{Q_t} |\nabla\varphi_k|^2\,.
\end{aligned}$$

For the next term, we also apply the compactness inequality (2.34) to $N_\sigma(\varphi_k, \sigma_k)$ and have that

$$\begin{aligned}
&- \int_{Q_t} N_\sigma(\varphi_k, \sigma_k) \nabla\sigma_k \cdot \boldsymbol{u}_k = - \int_{Q_t} N_\sigma(\varphi_k, \sigma_k) \big(\nabla N_\sigma(\varphi_k, \sigma_k) + \chi \nabla\varphi_k\big) \cdot \boldsymbol{u}_k \\
&\leq \int_0^t \|N_\sigma(\varphi_k(s), \sigma_k(s))\|_4 \, \big(\|\nabla N_\sigma(\varphi_k(s), \sigma_k(s))\| + \chi \|\nabla\varphi_k(s)\|\big) \|\boldsymbol{u}_k(s)\|_4 \, ds \\
&\leq \delta \int_{Q_t} \big(|\nabla N_\sigma(\varphi_k, \sigma_k)|^2 + |\nabla\varphi_k|^2\big) + c_\delta \int_{Q_t} |N_\sigma(\varphi_k, \sigma_k)|^2\,.
\end{aligned}$$

Similarly, using (3.6) and the Sobolev inequality (2.33) for h_ε and $\boldsymbol{v}_k$, we have that

$$\int_{Q_t} h_\varepsilon \nabla\varphi_k \cdot (\boldsymbol{v}_k - \boldsymbol{u}_k) \leq \delta \int_0^t \|\boldsymbol{v}_k(s)\|_{\boldsymbol{V}}^2 \, ds + c_\delta \int_0^t \|h_\varepsilon(s)\|_V^2 \, \|\nabla\varphi_k(s)\|^2 \, ds + c_\delta\,.$$

We notice that the L^1 norm of the function $s \mapsto \|h_\varepsilon(s)\|_V^2$ is bounded by a constant independent of ε by (3.6). Now, among the volume integrals that come from (3.42)–(3.44) and should be estimated, just the last one on the right-hand side of (3.43) needs some treatment. Indeed, all the others can easily be dealt with by virtue of the Young inequality, possibly combined with other estimates, like (2.35) or (2.9), without any difficulty. We have that

$$\begin{aligned}
&\int_{Q_t} h_\varepsilon \partial_t \varphi_k = - \int_{Q_t} \partial_t h_\varepsilon \, \varphi_k + \int_\Omega h_\varepsilon(t) \varphi_k(t) - \int_\Omega h_\varepsilon(0) \varphi_k(0) \\
&\leq \int_{Q_t} (|\partial_t h_\varepsilon|^2 + |\varphi_k|^2) + \delta \int_\Omega |\varphi_k(t)|^2 + c_\delta \int_\Omega |h_\varepsilon(t)|^2 + \int_\Omega |\varphi_k(0)|^2 + \int_\Omega |h_\varepsilon(0)|^2 \\
&\leq \int_{Q_t} |\varphi_k|^2 + \delta \int_\Omega |\varphi_k(t)|^2 + \int_\Omega |\varphi_k(0)|^2 + c_\delta\,,
\end{aligned}$$

where we owe to (3.6) for the last inequality. Now, we move to the surface integrals. We have that

$$\kappa \int_{\Sigma_t} \sigma_\Sigma\big(\sigma_k + \chi(1-\varphi_k)\big) - \kappa\chi \int_{\Sigma_t} \sigma_k(1-\varphi_k) \le \frac{\kappa}{2}\int_{\Sigma_t} |\sigma_k|^2 + c\int_{\Sigma_t} |\varphi_k|^2 + c$$
$$\le \frac{\kappa}{2}\int_{\Sigma_t} |\sigma_k|^2 + c\int_0^t \|\varphi_k(s)\|_V^2\, ds + c$$
$$\le \frac{\kappa}{2}\int_{\Sigma_t} |\sigma_k|^2 + c\int_{Q_t} |\varphi_k|^2 + c\int_{Q_t} |\nabla\varphi_k|^2 + c\,.$$

Finally, all of the terms involving the initial values can easily be estimated by accounting for (3.39) and recalling (3.3) and (1.17) to treat the convex part $\widehat{\beta}_\varepsilon$ of F_ε and the term involving N. At this point, by collecting everything, choosing δ small enough, and applying the Gronwall lemma, we obtain that

$$\|\boldsymbol{v}_k\|_{L^2(0,T_k;\boldsymbol{V})} + \|\nabla\mu_k\|_{L^2(0,T_k;\boldsymbol{H})} + \|\varphi_k\|_{L^\infty(0,T_k;V)} + \|F_\varepsilon(\varphi_k)\|_{L^\infty(0,T_k;L^1(\Omega))}$$
$$+ \|N(\varphi_k,\sigma_k)\|_{L^\infty(0,T_k;H)} + \|N_\sigma(\varphi_k,\sigma_k)\|_{L^2(0,T_k;V)}$$
$$+ \varepsilon^{1/2}\,\|\mu_k\|_{L^\infty(0,T_k;H)} + \varepsilon^{1/2}\,\|\partial_t\varphi_k\|_{L^2(0,T_k;H)} \le c\,.$$

In particular, this proves (3.37), so that $T_k = T$. Then, by recalling (1.17)–(1.18) and the Poincaré inequality once more and rearranging, we conclude that

$$\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{V})} + \|\mu_k\|_{L^2(0,T;V_0)} + \|\varphi_k\|_{L^\infty(0,T;V)} + \|\sigma_k\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$
$$+ \varepsilon^{1/2}\,\|\mu_k\|_{L^\infty(0,T;H)} + \varepsilon^{1/2}\,\|\partial_t\varphi_k\|_{L^2(0,T;H)} \le c\,. \tag{3.45}$$

Second a priori estimate. We now aim at recovering an estimate for the pressure p_k. Thus, we construct $\boldsymbol{q}_k \in L^2(0,T;\boldsymbol{V})$ such that

$$\operatorname{div}\boldsymbol{q}_k(t) = p_k(t) \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{q}_k(t)|_\Gamma = \frac{1}{|\Gamma|}\big(\textstyle\int_\Omega p_k(t)\big)\boldsymbol{n}\,,$$
$$\|\boldsymbol{q}(t)\|_V \le C\,\|p_k(t)\|\,,$$

for a.a. $t \in (0,T)$ and some constant $C > 0$. To this end, it suffices to apply Lemma 2.3 with an obvious choice of f and $\boldsymbol{a}$. Then, we test (3.24), written at the time t, by $\boldsymbol{q}_k(t)$. However, we avoid writing the time t for brevity. Recalling (1.6) and (3.25), we obtain that

$$\int_\Omega \big(2\eta D\boldsymbol{v}_k : \nabla\boldsymbol{q}_k + \lambda g_\varepsilon\ \operatorname{div}\boldsymbol{q}_k - p_k \operatorname{div}\boldsymbol{q}_k + \nu\boldsymbol{v}_k\cdot\boldsymbol{q}_k\big)$$
$$= \int_\Omega \big((\mu_k + h_\varepsilon)\nabla\varphi_k + N_\sigma(\varphi_k,\sigma_k)\nabla\sigma_k\big)\cdot\boldsymbol{q}_k\,,$$

and the definition of $\boldsymbol{q}_k$, as well as the Hölder, Sobolev, and Young inequalities, yield

$$\|p_k\|^2 \le c\big(\|\boldsymbol{v}_k\|_V + \|g_\varepsilon\|\big)\|\boldsymbol{q}_k\|_V$$
$$+ (\|\mu_k\|_4 + \|h_\varepsilon\|_4)\,\|\nabla\varphi_k\|\,\|\boldsymbol{q}_k\|_4 + \|N_\sigma(\varphi_k,\sigma_k)\|_3\,\|\nabla\sigma_k\|\,\|\boldsymbol{q}_k\|_6$$
$$\le c\big(\|\boldsymbol{v}_k\|_V + \|g_\varepsilon\|\big)\|p_k\|$$
$$+ c(\|\mu_k\|_4 + \|h_\varepsilon\|_4)\,\|\varphi_k\|_V\,\|p_k\| + c\,\|N_\sigma(\varphi_k,\sigma_k)\|_3\,\|\sigma_k\|_V\,\|p_k\|$$
$$\le \frac{1}{2}\,\|p_k\|^2 + c\big(\|\boldsymbol{v}_k\|_V^2 + \|g_\varepsilon\|^2\big)$$
$$+ c(\|\mu_k\|_{V_0}^2 + \|h_\varepsilon\|_V^2)\|\varphi_k\|_V^2 + c\,\|N_\sigma(\varphi_k,\sigma_k)\|_3^2\,\|\sigma_k\|_V^2\,.$$

Now, we rearrange, take the power of exponent $2/3$, and integrate over $(0,T)$. By accounting for (3.45) and the Young inequality once more, we deduce that

$$\begin{aligned}
&\int_0^T \|p_k\|^{4/3}\,dt \\
&\le c\int_0^T \bigl(\|\boldsymbol{v}_k\|_V^{4/3} + \|g_\varepsilon\|^{4/3} + (\|\mu_k\|_{V_0}^{4/3} + \|h_\varepsilon\|_V^{4/3})\|\varphi_k\|_V^{4/3} + \|N_\sigma(\varphi_k,\sigma_k)\|_3^{4/3}\,\|\sigma_k\|_V^{4/3}\bigr)\,dt \\
&\le c\int_0^T \|N_\sigma(\varphi_k,\sigma_k)\|_3^{4/3}\,\|\sigma_k\|_V^{4/3}\,dt + c \le c\int_0^T \bigl(\|N_\sigma(\varphi_k,\sigma_k)\|_3^4 + \|\sigma_k\|_V^2\bigr)\,dt + c \\
&\le c\int_0^T \|N_\sigma(\varphi_k,\sigma_k)\|_3^4\,dt + c\,,
\end{aligned}$$

and it remains to estimate the last integral. By interpolation, we have the continuous embedding

$$L^\infty(0,T;H)\cap L^2(0,T;L^6(\Omega)) \hookrightarrow L^4(0,T;L^3(\Omega))\,.$$

Since (3.45) implies that $N_\sigma(\varphi_k,\sigma_k)$ is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$ and the continuous embedding $V\hookrightarrow L^6(\Omega)$ holds, the integral at hand is uniformly bounded. Therefore, we have proved that

$$\|p_k\|_{L^{4/3}(0,T;H)} \le c\,. \tag{3.46}$$

By the way, the argument used for $N_\sigma(\varphi_k,\sigma_k)$ also applies to σ_k, so that

$$\|\sigma_k\|_{L^4(0,T;L^3(\Omega))} \le c\,. \tag{3.47}$$

Third a priori estimate. We test (3.27) by the admissible function $-\Delta\varphi_k$ and integrate in time. We obtain that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\nabla\varphi_k(t)|^2 + \int_{Q_t}|\Delta\varphi_k|^2 + \int_{Q_t}\beta'_\varepsilon(\varphi_k)|\nabla\varphi_k|^2 \\
&= \frac{\varepsilon}{2}\int_\Omega |\nabla\varphi_k(0)|^2 + \int_{Q_t} f(-\Delta\varphi_k) \quad \text{where} \quad f := \mu_k + h_\varepsilon - N_\varphi(\varphi_k,\sigma_k) - \pi(\varphi_k)\,.
\end{aligned}$$

As for the first term on the right-hand side, we recall (2.12) and Remark 3.3, to see that it is bounded. Since f is bounded in $L^2(0,T;H)$ by (3.45), we deduce that the same holds for $\Delta\varphi_k$. Then, elliptic regularity yields that

$$\|\varphi_k\|_{L^2(0,T;W)} \le c\,. \tag{3.48}$$

Fourth a priori estimate. We take any $\zeta\in L^4(0,T;V)$ and define ζ_k as follows: $\zeta_k(t)$ is the H-projection of $\zeta(t)$ onto V_k for a.a. $t\in(0,T)$. Then, for a.a. $t\in(0,T)$, we test (3.28), written at the time t, by $\zeta_k(t)$. However, we do not write the time t for simplicity. By also accounting for (3.25), we have that

$$\begin{aligned}
\int_\Omega \partial_t\sigma_k\,\zeta_k &= -\int_\Omega \nabla N_\sigma(\varphi_k,\sigma_k)\cdot\nabla\zeta_k + \int_\Omega \mathcal{S}_\sigma(\varphi_k,\sigma_k)\zeta_k \\
&\quad - \int_\Omega \bigl(\nabla\sigma_k\cdot\boldsymbol{v}_k + \sigma_k\,\mathrm{div}\,\boldsymbol{v}_k\bigr)\zeta_k + \kappa\int_\Gamma (\sigma_\Sigma - \sigma_k)\zeta_k\,.
\end{aligned}$$

Since $\partial_t\sigma_k(t) \in V_k$ and $\zeta_k(t)$ coincides with the V-projection of $\zeta(t)$ as explained in Remark 3.3, we can replace ζ_k by ζ (still omitting the time) on the left-hand side and obtain

$$\int_\Omega \partial_t\sigma_k \, \zeta_k = \int_\Omega \partial_t\sigma_k \, \zeta \, .$$

By the same remark, we have that $\|\zeta_k\|_V \le \|\zeta\|_V$ (see (3.38)). Hence, we infer that

$$- \int_\Omega \nabla N_\sigma(\varphi_k,\sigma_k)\cdot\nabla\zeta_k + \int_\Omega \mathcal{S}_\sigma(\varphi_k,\sigma_k)\zeta_k \le c\bigl(\|\varphi_k\|_V + \|\sigma_k\|_V + 1\bigr)\|\zeta\|_V \, .$$

To deal with the next term, we integrate by parts and obtain

$$\begin{aligned} &- \int_\Omega \bigl(\nabla\sigma_k\cdot\boldsymbol{v}_k + \sigma_k \operatorname{div}\boldsymbol{v}_k\bigr)\zeta_k = - \int_\Omega (\operatorname{div}(\sigma_k\boldsymbol{v}_k))\,\zeta_k \\ &= \int_\Omega \sigma_k\,\boldsymbol{v}_k\cdot\nabla\zeta_k - \int_\Gamma \sigma_k\,\zeta_k\,\boldsymbol{v}_k\cdot\boldsymbol{n} \\ &\le \|\sigma_k\|_3\,\|\boldsymbol{v}_k\|_6\,\|\nabla\zeta_k\| + \|\sigma_k|_\Gamma\|\,\|\boldsymbol{v}_k\cdot\boldsymbol{n}\|_4\,\|\zeta_k|_\Gamma\|_4 \, . \end{aligned} \tag{3.49}$$

We can replace the L^6 norm by the V norm since $V \hookrightarrow L^6(\Omega)$, and the L^2 norm of $\nabla\zeta_k$ by $\|\zeta\|_V$. Moreover, since the two-dimensional embedding $H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma)$ holds true and the trace operator is continuous from V to $H^{1/2}(\Gamma)$, we can estimate the last product as follows:

$$\|\sigma_k|_\Gamma\|\,\|\boldsymbol{v}_k\cdot\boldsymbol{n}\|_4\,\|\zeta_k|_\Gamma\|_4 \le c\,\|\sigma_k|_\Gamma\|\,\|\boldsymbol{v}_k\|_V\,\|\zeta\|_V \, . \tag{3.50}$$

Similarly, we have that

$$\kappa \int_\Gamma (\sigma_\Sigma - \sigma_k)\zeta_k \le c\bigl(\|\sigma_\Sigma\| + \|\sigma_k|_\Gamma\|\bigr)\,\|\zeta_k\|_V \le c\bigl(\|\sigma_\Sigma\| + \|\sigma_k\|_V\bigr)\,\|\zeta\|_V \, ,$$

where, for clarity, we point out that $\|\sigma_\Sigma\|$ here means the norm of $\sigma_\Sigma(t)$ in $L^2(\Gamma)$. At this point, we collect all these equalities and estimates and integrate over $(0,T)$. Omitting the integration variable t for brevity, we have that

$$\begin{aligned} \int_Q \partial_t\sigma_k\,\zeta &\le c\int_0^T \bigl(\|\varphi_k\|_V + \|\sigma_k\|_V + 1\bigr)\|\zeta\|_V\,dt + c\int_0^T \|\sigma_k\|_3\,\|\boldsymbol{v}_k\|_V\,\|\zeta\|_V\,dt \\ &\quad + c\int_0^T \|\sigma_k|_\Gamma\|\,\|\boldsymbol{v}_k\|_V\,\|\zeta\|_V\,dt + c\int_0^T \bigl(\|\sigma_\Sigma\| + \|\sigma_k\|_V\bigr)\,\|\zeta\|_V\,dt \, . \end{aligned}$$

Therefore, by using the Hölder inequality, we have that

$$\begin{aligned} \int_Q \partial_t\sigma_k\,\zeta &\le c\bigl(\|\varphi_k\|_{L^2(0,T;V)} + \|\sigma_k\|_{L^2(0,T;V)} + 1\bigr)\|\zeta\|_{L^2(0,T;V)} \\ &\quad + c\,\|\sigma_k\|_{L^4(0,T;L^3(\Omega))}\,\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{V})}\,\|\zeta\|_{L^4(0,T;V)} \\ &\quad + c\,\|\sigma_k|_\Gamma\|_{L^4(0,T;L^2(\Gamma))}\,\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{V})}\,\|\zeta\|_{L^4(0,T;V)} \\ &\quad + c\,\bigl(\|\sigma_\Sigma\|_{L^2(\Sigma)} + \|\sigma_k\|_{L^2(0,T;V)}\bigr)\|\zeta\|_{L^2(0,T;V)} \, . \end{aligned}$$

Finally, we account for (3.45), (3.47), and Lemma 2.5, to conclude that

$$\int_Q \partial_t\sigma_k\,\zeta \le c\,\|\zeta\|_{L^4(0,T;V)} \quad \text{for every } \zeta \in L^4(0,T;V).$$

This means that $\partial_t \sigma_k$ is bounded in the dual space of $L^4(0,T;V)$, i.e., that

$$\|\partial_t \sigma_k\|_{L^{4/3}(0,T;V^*)} \le c\,. \tag{3.51}$$

Fifth a priori estimate. We aim at estimating the time derivative $\partial_t(\varepsilon\mu_k + \varphi_k)$. To this end, we take any $\phi \in L^2(0,T;V_0)$ and define $\phi_k : Q \to \mathbb{R}$ by setting for a.a. $t \in (0,T)$: $\phi_k(t)$ is the H-projection of $\phi(t)$ onto V_k^0. Then, we rearrange (3.26), written at the time t, test it by $\phi_k(t)$, and integrate over $(0,T)$. We obtain that

$$\int_Q \partial_t(\varepsilon\mu_k + \varphi_k)\,\phi_k = -\int_Q \nabla\mu_k \cdot \nabla\phi_k + \int_Q \mathcal{S}_\varphi(\varphi_k,\sigma_k)\phi_k - \int_Q (\nabla\varphi_k \cdot \boldsymbol{v}_k + \varphi_k g_\varepsilon)\phi_k\,.$$

For the first two terms on the right-hand side, we invoke the Lipschitz continuity of $\mathcal{S}_\varphi$ and (3.45) to immediately obtain that

$$\begin{aligned}
&-\int_Q \nabla\mu_k \cdot \nabla\phi_k + \int_Q \mathcal{S}_\varphi(\varphi_k,\sigma_k)\phi_k \\
&\le c\,\|\mu_k\|_{L^2(0,T;V_0)}\,\|\phi_k\|_{L^2(0,T;V_0)} + c\big(\|\varphi_k\|_{L^2(0,T;H)} + \|\sigma_k\|_{L^2(0,T;H)} + 1\big)\|\phi_k\|_{L^2(0,T;H)} \\
&\le c\,\|\phi_k\|_{L^2(0,T;V_0)},
\end{aligned}$$

while the last one needs some care. By the Hölder and Sobolev inequalities, (3.5) and (3.45) once more, we obtain that

$$\begin{aligned}
&-\int_Q (\nabla\varphi_k \cdot \boldsymbol{v}_k + \varphi_k g_\varepsilon)\phi_k \\
&\le \int_0^T \|\nabla\varphi_k(t)\|\,\|\boldsymbol{v}_k(t)\|_4\,\|\phi_k(t)\|_4\,dt + \|\varphi_k\|_{L^2(0,T;V)}\,\|g_\varepsilon\|_\infty\,\|\phi_k\|_{L^2(0,T;V_0)} \\
&\le \|\varphi_k\|_{L^\infty(0,T;V)}\,\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{V})}\,\|\phi_k\|_{L^2(0,T;V_0)} + c\,\|\varphi_k\|_{L^2(0,T;V)}\,\|\phi_k\|_{L^2(0,T;V_0)} \\
&\le c\,\|\phi_k\|_{L^2(0,T;V_0)}\,.
\end{aligned}$$

Since $\|\phi_k\|_{L^2(0,T;V_0)} \le \|\phi\|_{L^2(0,T;V_0)}$ by Remark 3.3 and we can replace ϕ by $-\phi$, we infer that

$$\Big|\int_Q \partial_t(\varepsilon\mu_k + \varphi_k)\,\phi_k\Big| \le c\,\|\phi\|_{L^2(0,T;V_0)}\,. \tag{3.52}$$

Unfortunately, just $\partial_t\mu_k$ is V_k^0-valued while $\partial_t\varphi_k$ is not, so that we only have that

$$\int_Q \partial_t(\varepsilon\mu_k + \varphi_k)\,\phi = \int_Q \partial_t(\varepsilon\mu_k + \varphi_k)\,\phi_k + \int_Q \partial_t\varphi_k(\phi - \phi_k).$$

However, on account of (3.45) and Remark 3.3 once more, we can write

$$\Big|\int_Q \partial_t\varphi_k(\phi - \phi_k)\Big| \le \|\partial_t\varphi_k\|_{L^2(0,T;H)}\big(\|\phi\|_{L^2(0,T;H)} + \|\phi_k\|_{L^2(0,T;H)}\big) \le c_\varepsilon\,\|\phi\|_{L^2(0,T;V_0)}\,.$$

Combining this with (3.52) yields

$$\|\partial_t(\varepsilon\mu_k + \varphi_k)\|_{L^2(0,T;V_0^*)} \le c_\varepsilon\,. \tag{3.53}$$

Passage to the limit as $k \to \infty$. We collect the basic estimates we have proved, namely, (3.45), (3.46), (3.48), (3.51), and (3.53), and apply well-known weak and weak star compactness results. Since ε is fixed, for some (not relabeled) subsequence and suitable limit functions, we have, as $k \to \infty$,

$$\boldsymbol{v}_k \to \boldsymbol{v}_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{V}) \hookrightarrow L^2(0,T;\boldsymbol{L}^4(\Omega)), \tag{3.54}$$

$$p_k \to p_\varepsilon \quad \text{weakly in } L^{4/3}(0,T;H), \tag{3.55}$$

$$\mu_k \to \mu_\varepsilon \quad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V_0), \tag{3.56}$$

$$\varphi_k \to \varphi_\varepsilon \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.57}$$

$$\sigma_k \to \sigma_\varepsilon \quad \text{weakly star in } W^{1,4/3}(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.58}$$

$$\varepsilon\mu_k + \varphi_k \to \varepsilon\mu_\varepsilon + \varphi_\varepsilon \quad \text{weakly in } H^1(0,T;V_0^*). \tag{3.59}$$

Then, the quintuple $(\boldsymbol{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon)$ satisfies (3.7)–(3.11), as well as the initial conditions (3.17)–(3.18) (recall the Remarks 3.1 and 3.3), and we aim at proving that it solves the whole approximating problem. By linearity (cf. (1.18)), we deduce that, as $k \to \infty$,

$$N_\varphi(\varphi_k,\sigma_k) \to N_\varphi(\varphi_\varepsilon,\sigma_\varepsilon) \quad \text{and} \quad N_\sigma(\varphi_k,\sigma_k) \to N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)$$
$$\text{weakly star in } W^{1,4/3}(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V).$$

Moreover, by strong compactness (see, e.g., [24, Thm. 5.1, p. 58] and [28, Sect. 8, Cor. 4]) we may without loss of generality assume that

$$\varphi_k \to \varphi_\varepsilon \quad \text{strongly in } L^2(0,T;W^{1,4}(\Omega)) \text{ and a.e. in } Q, \tag{3.60}$$

$$\sigma_k \to \sigma_\varepsilon \quad \text{strongly in } L^2(0,T;L^4(\Omega)) \text{ and a.e. in } Q, \tag{3.61}$$

$$\varepsilon\mu_k + \varphi_k \to \varepsilon\mu_\varepsilon + \varphi_\varepsilon \quad \text{strongly in } L^2(0,T;L^4(\Omega)). \tag{3.62}$$

Notice that combining (3.54) and (3.56) with (3.60) yields that

$$\nabla\varphi_k \cdot \boldsymbol{v}_k \to \nabla\varphi_\varepsilon \cdot \boldsymbol{v}_\varepsilon \quad \text{weakly in } L^1(0,T;H), \tag{3.63}$$

$$\mu_k \nabla\varphi_k \to \mu_\varepsilon \nabla\varphi_\varepsilon \quad \text{weakly in } L^1(0,T;\boldsymbol{H}). \tag{3.64}$$

On the other hand, the above convergence properties and the Lipschitz continuity imply that, as $k \to \infty$,

$$\mathcal{S}_\sigma(\varphi_k,\sigma_k) \to \mathcal{S}_\sigma(\varphi_\varepsilon,\sigma_\varepsilon) \quad \text{and} \quad F'_\varepsilon(\varphi_k) \to F'_\varepsilon(\varphi_\varepsilon), \quad \text{strongly in } L^2(0,T;L^4(\Omega)). \tag{3.65}$$

Now, we claim that

$$\zeta_k \to \zeta \text{ strongly in } L^\infty(0,T;V) \quad \text{implies that} \quad \int_Q (\nabla\sigma_k \cdot \boldsymbol{v}_k)\zeta_k \to \int_Q (\nabla\sigma_\varepsilon \cdot \boldsymbol{v}_\varepsilon)\zeta. \tag{3.66}$$

Assume ζ_k and ζ as said. From (3.58) and the strong compactness results already quoted, we deduce that, as $k \to \infty$,

$$\sigma_k \to \sigma_\varepsilon \quad \text{strongly in } L^2(0,T;H^{3/4}(\Omega)).$$

From this, we deduce that, as $k \to \infty$,

$$\nabla\sigma_k \to \nabla\sigma_\varepsilon \quad \text{strongly in } L^2(0,T;\boldsymbol{H}^{-1/2}(\Omega)), \tag{3.67}$$

owing to the interpolation theory in Hilbert spaces (see, e.g., [25, Ch. I]). We know that ∇ is continuous from V into $\boldsymbol{H}$ and from H into $\boldsymbol{H}^{-1}(\Omega) = (\boldsymbol{H}_0^1(\Omega))^*$. Then, it is continuous from $H^{3/4}(\Omega) = (V,H)_{1/4}$ into $(\boldsymbol{H},(\boldsymbol{H}_0^1(\Omega))^*)_{1/4}$. Since $1/4 < 1/2$, the last space is $(\boldsymbol{H}_0^{1/4}(\Omega))^* = (\boldsymbol{H}^{1/4}(\Omega))^*$, which is embedded in $\boldsymbol{H}^{-1/2}(\Omega)$ (see (2.3)–(2.4)). Then, (3.67) follows. On the other hand, we have that

$$\begin{aligned}
&\|\nabla(\zeta_k \boldsymbol{v}_k)\|_{L^2(0,T;\boldsymbol{L}^{3/2}(\Omega))} \\
&\leq \|\nabla\zeta_k\|_{L^\infty(0,T;\boldsymbol{H})}\,\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} + \|\zeta_k\|_{L^\infty(0,T;L^6(\Omega))}\,\|\nabla\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{H})} \\
&\leq c\,\|\zeta_k\|_{L^\infty(0,T;V)}\,\|\boldsymbol{v}_k\|_{L^2(0,T;\boldsymbol{V})} \leq c\,,
\end{aligned}$$

so that $\{\zeta_k \boldsymbol{v}_k\}$ has a weak limit in $L^2(0,T;\boldsymbol{W}^{1,3/2}(\Omega))$. Since $\zeta_k\boldsymbol{v}_k \to \zeta\boldsymbol{v}_\varepsilon$ weakly in $L^2(0,T;\boldsymbol{H})$, we deduce that

$$\zeta_k\boldsymbol{v}_k \to \zeta\boldsymbol{v}_\varepsilon \quad \text{weakly in } L^2(0,T;\boldsymbol{W}^{1,3/2}(\Omega)) \hookrightarrow L^2(0,T;\boldsymbol{H}^{1/2}(\Omega))\,,$$

where the last (three-dimensional) embedding follows from a particular case of the embedding

$$W^{1,q}(\Omega) \hookrightarrow H^s(\Omega) = W^{s,2}(\Omega), \quad \text{where} \quad s\in(0,1), \quad q\in[1,+\infty), \quad \text{and} \quad s - \frac{3}{2} \leq 1 - \frac{3}{q}\,.$$

Therefore, (3.66) holds true since $L^2(0,T;\boldsymbol{H}^{-1/2}(\Omega)) = (L^2(0,T;\boldsymbol{H}^{1/2}(\Omega)))^*$. By the same argument (by replacing the velocities by the N_σ terms, essentially), one obtains that

$$\int_Q N_\sigma(\varphi_k,\sigma_k)\nabla\sigma_k\cdot\boldsymbol{\zeta} \to \int_Q N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)\nabla\sigma_\varepsilon\cdot\boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta}\in L^\infty(0,T;\boldsymbol{V})\,. \tag{3.68}$$

Conclusion of the proof of Theorem 3.2. At this point, we can verify that the quintuple just found solves the equations of problem (3.12)–(3.18). Clearly, (3.13) follows from (3.25) and (3.54). As for (3.12), it suffices to check that some equivalent formulation is satisfied. This is the case if we take

$$\int_Q \mathbb{T}(\boldsymbol{v}_\varepsilon,p_\varepsilon):\nabla\boldsymbol{\zeta} + \nu\int_Q \boldsymbol{v}_\varepsilon\cdot\boldsymbol{\zeta} = \int_Q(\mu_\varepsilon+h_\varepsilon)\nabla\varphi_\varepsilon\cdot\boldsymbol{\zeta} + \int_Q N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)\nabla\sigma_\varepsilon\cdot\boldsymbol{\zeta}$$
$$\text{for every } \boldsymbol{\zeta}\in L^\infty(0,T;\boldsymbol{V}). \tag{3.69}$$

Indeed, this equation actually is satisfied due to the convergence properties just mentioned. Similarly, instead of considering (3.14)–(3.16), we deal with some time integrated version of theirs (analogous to (3.69)), but with step functions as test functions (this is convenient at least for some of the equations, and for simplicity we choose step functions for all of them). So, we assume that ϕ is a V_0-valued step function and that z and ζ are V-valued step functions. However, we have to be careful, since we are starting from the discrete setting. So, given ϕ, z and ζ as said, we introduce ϕ_k as we did in proving our fifth estimate, i.e., by defining $\phi_k(t)$ as the H-projection of $\phi(t)$ onto V_k^0, for a.a. $t\in(0,T)$. Similarly, we define z_k and ζ_k starting from z and ζ, now with V_k instead of V_k^0. By accounting for Remark 3.3, we point out at once that, as $k\to\infty$,

$$\phi_k\to\phi, \quad z_k\to z, \quad \text{and} \quad \zeta_k\to\zeta, \quad \text{strongly in } L^\infty(0,T;V), \tag{3.70}$$

since ϕ, z and ζ have a finite number of values. Next, we test (3.26),written at the time t, by $\phi_k(t)$ and integrate over $(0,T)$. Upon rearranging, we obtain that

$$\int_Q \partial_t(\varepsilon\mu_k+\varphi_k)\,\phi_k + \int_Q\nabla\mu_k\cdot\nabla\phi_k = \int_Q \mathcal{S}_\varphi(\varphi_k,\sigma_k)\phi_k - \int_Q\big(\nabla\varphi_k\cdot\boldsymbol{v}_k + \varphi_k g_\varepsilon\big)\phi_k\,.$$

On account of the strong convergence in (3.70), recalling (3.59) and (3.63), and letting k tend to infinity, we conclude that

$$\int_0^T \langle \partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon)(t), \phi(t)\rangle_{V_0} + \int_Q \nabla\mu_\varepsilon \cdot \nabla\phi = \int_Q \mathcal{S}_\varphi(\varphi_\varepsilon, \sigma_\varepsilon)\phi - \int_Q \big(\nabla\varphi_\varepsilon \cdot \boldsymbol{v}_\varepsilon + \varphi_\varepsilon g_\varepsilon\big)\phi\,. \tag{3.71}$$

What we have obtained is an integrated version of (3.14), which is equivalent to (3.14) itself since it holds for every V_0-valued step function ϕ. Similarly, we test (3.27) and (3.28) written at the time t by $z_k(t)$ and $\zeta_k(t)$, respectively, and integrate over $(0,T)$. We obtain that

$$\varepsilon \int_Q \partial_t\varphi_k\, z_k + \int_Q \nabla\varphi_k \cdot \nabla z_k + \int_Q F'_\varepsilon(\varphi_k)\, z_k = \int_Q \big(\mu_k + h_\varepsilon - N_\varphi(\varphi_k, \sigma_k)\big) z_k$$

$$\int_Q \partial_t\sigma_k\, \zeta_k + \int_Q \nabla N_\sigma(\varphi_k,\sigma_k)\cdot\nabla\zeta_k = \int_Q \mathcal{S}_\sigma(\varphi_k,\sigma_k)\zeta_k - \int_\Omega (\nabla\sigma_k \cdot \boldsymbol{v}_k + \sigma_k g_\varepsilon)\zeta_k + \kappa\int_\Sigma (\sigma_\Sigma - \sigma_k)\zeta_k\,.$$

By accounting for (3.70) and (3.66), we conclude that, as $k \to \infty$,

$$\varepsilon \int_Q \partial_t\varphi_\varepsilon\, z + \int_Q \nabla\varphi_\varepsilon \cdot \nabla z + \int_Q F'_\varepsilon(\varphi_\varepsilon)\, z = \int_Q \big(\mu_\varepsilon + h_\varepsilon - N_\varphi(\varphi_\varepsilon, \sigma_\varepsilon)\big) z\,, \tag{3.72}$$

$$\int_Q \partial_t\sigma_\varepsilon\, \zeta + \int_Q \nabla N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)\cdot\nabla\zeta = \int_Q \mathcal{S}_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)\zeta - \int_\Omega (\nabla\sigma_\varepsilon \cdot \boldsymbol{v}_\varepsilon + \sigma_\varepsilon g_\varepsilon)\zeta + \kappa\int_\Sigma (\sigma_\Sigma - \sigma_\varepsilon)\zeta\,, \tag{3.73}$$

for all V-valued step functions z and ζ. Thus, (3.15)–(3.16) hold as well, and the proof of Theorem 3.2 is complete.

4 Existence of a Weak Solution

In this section, we prove Theorem 2.1. We start from the approximating problem analyzed in the previous section and let ε tend to zero. Since we did not prove uniqueness for the approximating solution, we take a particular one, namely, the solution we have constructed above. This ensures a number of bounds. Indeed, by the estimates established for the discrete solution and the semicontinuity of the norms, it is clear that

$$\begin{aligned} &\|\boldsymbol{v}_\varepsilon\|_{L^2(0,T;\boldsymbol{V})} + \|p_\varepsilon\|_{L^{4/3}(0,T;H)} \\ &\quad + \|\mu_\varepsilon\|_{L^2(0,T;V_0)} + \|\varphi_\varepsilon\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\sigma_\varepsilon\|_{W^{1,4/3}(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad + \varepsilon^{1/2}\,\|\mu_\varepsilon\|_{L^\infty(0,T;H)} + \varepsilon^{1/2}\,\|\partial_t\varphi_\varepsilon\|_{L^2(0,T;H)} \le c\,, \end{aligned} \tag{4.1}$$

for a positive constant c independent of ε. However, we need some additional estimates.

Sixth a priori estimate. We take any $\phi \in L^2(0,T;V_0)$, and (for a.a. $t \in (0,T)$) we test (3.14), written at the time t, by $\phi(t)$. Then, we integrate over $(0,T)$. With a procedure that is completely similar to the one used to prove (3.52), we see that

$$\Big|\int_0^T \langle \partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon)(t), \phi(t)\rangle_{V_0}\Big| \le c\,\|\phi\|_{L^2(0,T;V_0)}\,,$$

meaning that

$$\|\partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon)\|_{L^2(0,T;V_0^*)} \le c\,. \tag{4.2}$$

Seventh a priori estimate. We recall that $F'_\varepsilon = \beta_\varepsilon + \pi$ and test (3.15) written at the time t by $\beta_\varepsilon(\varphi_\varepsilon(t))$. Then, we integrate over $(0,T)$. By also accounting for (4.1), (3.3), (2.12) and (3.6), we obtain that

$$\begin{aligned}
&\varepsilon\int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(T)) + \int_Q \beta'_\varepsilon(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2 + \int_Q |\beta_\varepsilon(\varphi_\varepsilon)|^2 \\
&= \varepsilon\int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) + \int_Q \big(\mu_\varepsilon + h_\varepsilon - N_\varphi(\varphi_\varepsilon,\sigma_\varepsilon) - \pi(\varphi_\varepsilon)\big)\beta_\varepsilon(\varphi_\varepsilon) \\
&\le \frac{1}{2}\int_Q |\beta_\varepsilon(\varphi_\varepsilon)|^2 + c\,.
\end{aligned}$$

Since all of the terms on the left-hand side are nonnegative, we conclude that

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(0,T;H)} \le c\,. \tag{4.3}$$

Conclusion of the proof of Theorem 2.1. By recalling (4.1)–(4.3), we see that, as $\varepsilon \to 0$,

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \quad \text{weakly in } L^2(0,T;\boldsymbol{V}) \hookrightarrow L^2(0,T;\boldsymbol{L}^4(\Omega))\,, \tag{4.4}$$
$$p_\varepsilon \to p \quad \text{weakly in } L^{4/3}(0,T;H)\,, \tag{4.5}$$
$$\mu_\varepsilon \to \mu \quad \text{weakly in } L^2(0,T;V_0)\,, \tag{4.6}$$
$$\varphi_\varepsilon \to \varphi \quad \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{4.7}$$
$$\beta_\varepsilon(\varphi_\varepsilon) \to \xi \quad \text{weakly in } L^2(0,T;H)\,, \tag{4.8}$$
$$\sigma_\varepsilon \to \sigma \quad \text{weakly star in } W^{1,4/3}(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,, \tag{4.9}$$
$$\varepsilon\mu_\varepsilon \to 0 \quad \text{strongly in } L^\infty(0,T;H) \cap L^2(0,T;V_0)\,, \tag{4.10}$$
$$\varepsilon\partial_t\varphi_\varepsilon \to 0 \quad \text{strongly in } L^2(0,T;H)\,, \tag{4.11}$$
$$\varepsilon\mu_\varepsilon + \varphi_\varepsilon \to \varphi \quad \text{weakly in } H^1(0,T;V_0^*) \cap L^2(0,T;V)\,, \tag{4.12}$$

for suitable limit functions $(\boldsymbol{v}, p, \mu, \varphi, \xi, \sigma)$. More precisely, this holds for some subsequence $\varepsilon_n \searrow 0$. Nevertheless, here and in the sequel, we write just ε for simplicity. Then, the sextuple $(\boldsymbol{v}, p, \mu, \varphi, \xi, \sigma)$ satisfies the regularity properties in (2.17)–(2.22). By noting that $(\varepsilon\mu_\varepsilon + \varphi_\varepsilon)(0)$ converges to $\varphi(0)$ weakly in V_0^*, we infer that the initial conditions (2.29) are satisfied as well, and we now prove that the sextuple we have found yields in fact a solution to the original system (2.23)–(2.29). To this end, we try to follow the lines used to solve the approximating problem at the end of the previous section. However, the analogues of some of those convergence properties require some further work.

First of all, we clearly have that, as $\varepsilon \to 0$,

$$N_\varphi(\varphi_\varepsilon,\sigma_\varepsilon) \to N_\varphi(\varphi,\sigma) \quad \text{and} \quad N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon) \to N_\sigma(\varphi,\sigma) \quad \text{weakly in } L^2(0,T;V).$$

Now, we account for the strong compactness results [24, Thm. 5.1, p. 58] and [28, Sect. 8, Cor. 4] to deduce that $\varepsilon\mu_\varepsilon + \varphi_\varepsilon$ converges to φ strongly in $L^2(0,T;H)$. By combining this with (4.10), we infer that, as $\varepsilon \to 0$,

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^2(0,T;H). \tag{4.13}$$

On the other hand, we have the interpolation identity

$$(H^2(\Omega), H)_{1/8} = H^{7/4}(\Omega)$$

and the associated interpolation inequality

$$\|v\|_{H^{7/4}(\Omega)} \le c\,\|v\|_{H^2(\Omega)}^{1/8}\,\|v\|^{7/8} \quad \text{for every } v \in H^2(\Omega)\,.$$

Therefore, by applying the Hölder inequality, we deduce, for every $v \in L^2(0,T;H^2(\Omega))$, that

$$\int_0^T \|v(t)\|^2_{H^{7/4}(\Omega)}\,dt \le \int_0^T \|v(t)\|^{1/4}_{H^2(\Omega)}\,\|v(t)\|^{7/4}\,dt$$
$$\le \Bigl(\int_0^T \|v(t)\|^2_{H^2(\Omega)}\,dt\Bigr)^{1/8}\Bigl(\int_0^T \|v(t)\|^2\,dt\Bigr)^{7/8}.$$

By applying this inequality to $\varphi_\varepsilon - \varphi$ and owing to the boundedness in $L^2(0,T;W)$ and to strong convergence in $L^2(0,T;H)$, we readily deduce that

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^2(0,T;H^{7/4}(\Omega)).$$

Now, we recall the (three-dimensional) continuous embedding

$$H^s(\Omega) \hookrightarrow W^{1,q}(\Omega), \quad \text{where} \quad s > 1, \quad q \ge 2, \quad \text{and} \quad 1 - \frac{3}{q} \le s - \frac{3}{2},$$

and apply it with $s = 7/4$ and $q = 4$. We conclude that, as $\varepsilon \searrow 0$,

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^2(0,T;W^{1,4}(\Omega)).$$

This is the analogue of (3.60). As in the previous proof, we derive the analogues of (3.61), (3.63)–(3.64), and the first of (3.65), namely,

$$\sigma_\varepsilon \to \sigma \quad \text{strongly in } L^2(0,T;L^4(\Omega)) \text{ and a.e. in } Q\,,$$
$$\nabla\varphi_\varepsilon \cdot \boldsymbol{v}_\varepsilon \to \nabla\varphi \cdot \boldsymbol{v} \quad \text{weakly in } L^1(0,T;H)\,,$$
$$\mu_\varepsilon \nabla\varphi_\varepsilon \to \mu\nabla\varphi \quad \text{weakly in } L^1(0,T;\boldsymbol{H})\,,$$
$$\mathcal{S}_\sigma(\varphi_\varepsilon,\sigma_\varepsilon) \to \mathcal{S}_\sigma(\varphi,\sigma) \quad \text{strongly in } L^2(0,T;L^4(\Omega)).$$

Moreover, by the same argument, we obtain the analogue of (3.66), which now sounds

$$\int_Q (\nabla\sigma_\varepsilon \cdot \boldsymbol{v}_\varepsilon)\zeta \to \int_Q (\nabla\sigma \cdot \boldsymbol{v})\zeta \quad \text{for every } \zeta \in L^\infty(0,T;V)\,.$$

In a similar fashion, dealing with the N_σ terms in place of the velocities, one obtains the analogue of (3.68), i.e.,

$$\int_Q N_\sigma(\varphi_\varepsilon,\sigma_\varepsilon)\nabla\sigma_\varepsilon \cdot \boldsymbol{\zeta} \to \int_Q N_\sigma(\varphi,\sigma)\nabla\sigma \cdot \boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta} \in L^\infty(0,T;\boldsymbol{V})\,.$$

At this point, we recall that the approximating solution we are considering also satisfies (3.13) and the four variational equations (3.69) and (3.71)–(3.73), the first of which being satisfied for every $\boldsymbol{\zeta} \in L^\infty(0,T;\boldsymbol{V})$ and the others for arbitrary step functions ϕ, z and ζ taking values in V_0, V and V, respectively. At this point, we can let ε tend to zero. By recalling (3.5) and (3.6), we obtain (2.24) and the integrated version of the variational equations (2.23) and (2.25)–(2.28) with the same type of test functions. Thus, these equations are satisfied as they are written in the original problem. It remains to verify (2.27). To this end, it suffices to observe that the weak convergence (4.8) coupled with the strong convergence (4.13) allows us to apply a well-known property of the Yosida approximation (see, e.g., [3, Prop. 2.2, p. 38]) which yields the inclusion $\xi \in \beta(\varphi)$, as desired. Thus, the proof of Theorem 2.1 is complete.

Remark 4.1. We further justify the regularity properties claimed in (2.31). The rigorous argument should involve a regularization of β and truncation in the choice of the test function. However, we proceed formally, for brevity. We write (2.26) in the form

$$\int_\Omega \nabla\varphi \cdot \nabla z + \int_\Omega \beta(\varphi) z = \int_\Omega f z \quad \text{for every } z \in V \tag{4.14}$$

where $f := \mu + h - N_\varphi(\varphi,\sigma) - \pi(\varphi)$. Then, we test the above equation by $z = (\beta(\varphi))^5$ (a.e. in $(0,T)$) and owe to the Young inequality (2.32) on the right-hand side. We obtain (a.e. in $(0,T)$) that

$$\int_\Omega 5|\beta(\varphi)|^4|\nabla\varphi|^2 + \int_\Omega |\beta(\varphi)|^6 = \int_\Omega |f|\,|\beta(\varphi)|^5 \le \frac{5}{6}\int_\Omega |\beta(\varphi)|^6 + \frac{1}{6}\int_\Omega |f|^6$$

whence immediately $\|\beta(\varphi)\|_6 \le \|f\|_6$. Since $f \in L^2(0,T;V) \subset L^2(0,T;L^6(\Omega))$, we deduce that $\beta(\varphi) \in L^2(0,T;L^6(\Omega))$. Then, elliptic regularity theory yields $\varphi \in L^2(0,T;W^{2,6}(\Omega))$. To show the remaining regularity property, we test the variational equation (4.14) by $-\Delta\varphi$ and recall that $N_\varphi(\varphi,\sigma) = -\chi\sigma$. We thus have that

$$\begin{aligned}
&\|\Delta\varphi\|^2 + \int_\Omega \beta'(\varphi)|\nabla\varphi|^2 \le \|\nabla\mu\|\,\|\nabla\varphi\| + \|h + \chi\sigma - \pi(\varphi)\|\,\|\Delta\varphi\| \\
&\le \|\nabla\mu\|\,\|\nabla\varphi\| + \frac{1}{2}\,\|\Delta\varphi\|^2 + c\,\|h + \chi\sigma - \pi(\varphi)\|^2 .
\end{aligned}$$

Since $\|\nabla\varphi\|$ and the last norm are bounded over $(0,T)$ by (2.16), (2.22) and (2.20), we deduce that

$$\frac{1}{2}\,\|\Delta\varphi\|^2 \le c\,\|\nabla\mu\| + c \quad \text{whence also} \quad \|\Delta\varphi\|^4 \le c\,\|\nabla\mu\|^2 + c\,.$$

Then, (2.19) implies that $\Delta\varphi \in L^4(0,T;H)$, whence the elliptic regularity theory ensures that $\varphi \in L^4(0,T;H^2(\Omega))$.

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