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**Stochastic homogenization on perforated domains III – General
estimates for stationary ergodic random connected Lipschitz
domains**

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Stochastic homogenization on perforated domains III – General estimates for stationary ergodic random connected Lipschitz domains

Martin Heida

Abstract

This is Part III of a series on the existence of uniformly bounded extension operators on randomly perforated domains in the context of homogenization theory. Recalling that randomly perforated domains are typically not John and hence extension is possible only from $W^{1,p}$ to $W^{1,r}$, $r < p$, we will show that the existence of such extension operators can be guaranteed if the weighted expectations of four geometric characterizing parameters are bounded: The local Lipschitz constant M , the local Lipschitz radius δ , the mesoscopic Voronoi diameter \mathfrak{d} and the local connectivity radius \mathcal{R} .

1 Introduction

Let $\mathbf{p}(\omega) \subset \mathbb{R}^d$ be a stationary ergodic connected random open set with random variable ω and let $\varepsilon > 0$ be the smallness parameter. The concept of stationary ergodic random open sets was introduced in detail in Part I [5], and we will give a simplified version below, which focuses on the properties used in the present Part III.

For a bounded open domain \mathbf{Q} , we then consider $\mathbf{p}^\varepsilon(\omega) = \varepsilon\mathbf{p}(\omega)$, $\mathbf{Q}_\mathbf{p}^\varepsilon(\omega) := \mathbf{Q} \cap \mathbf{p}^\varepsilon(\omega)$ and $\Gamma^\varepsilon(\omega) := \mathbf{Q} \cap \partial\mathbf{p}^\varepsilon(\omega)$ with outer normal $\nu_{\Gamma^\varepsilon(\omega)}$. In order to simplify notation, we keep in mind that \mathbf{p} and $\mathbf{Q}_\mathbf{p}^\varepsilon$ are random variables and drop the explicit writing of ω .

Denoting $W_{0,\partial\mathbf{Q}}^{1,p}(\mathbf{Q}_\mathbf{p}^\varepsilon) := \{u \in W^{1,p}(\mathbf{Q}_\mathbf{p}^\varepsilon) : u|_{\partial\mathbf{Q}} \equiv 0\}$ one would classically be interested in a family of extension operators $\mathcal{U}_\varepsilon : W_{0,\partial\mathbf{Q}}^{1,p}(\mathbf{Q}_\mathbf{p}^\varepsilon) \rightarrow W^{1,p}(\mathbf{Q})$ such that for some C independent from ε it holds

$$\|\nabla\mathcal{U}_\varepsilon u\|_{L^p(\mathbf{Q})} \leq C \|\nabla u\|_{L^p(\mathbf{Q}_\mathbf{p}^\varepsilon)}, \quad \|\mathcal{U}_\varepsilon u\|_{L^p(\mathbf{Q})} \leq C \|u\|_{L^p(\mathbf{Q}_\mathbf{p}^\varepsilon)}. \quad (1.1)$$

However, estimates of the form (1.1) are known to exist only for (global) John domains but from Part I we know that even random Lipschitz domains are mostly not (globally) John.

On the other hand Part I [5] gives rise to the hope that we can find $1 \leq r < p$ and a family of extension operators $\mathcal{U}_\varepsilon : W_{0,\partial\mathbf{Q}}^{1,p}(\mathbf{Q}_\mathbf{p}^\varepsilon) \rightarrow W^{1,r}(\mathbf{Q})$ for scalar valued functions resp. $\mathcal{U}_\varepsilon : \mathbf{W}_{0,\partial\mathbf{Q}}^{1,p}(\mathbf{Q}_\mathbf{p}^\varepsilon) \rightarrow \mathbf{W}^{1,r}(\mathbb{R}^d)$ for vector valued functions such that

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbb{R}^d} |\nabla\mathcal{U}_\varepsilon u|^r \leq C \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \cap \varepsilon\mathbf{p}} |\nabla u|^p \right)^{\frac{r}{p}}, \quad \frac{1}{|\mathbf{Q}|} \int_{\mathbb{R}^d} |\mathcal{U}_\varepsilon u|^r \leq C \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \cap \varepsilon\mathbf{p}} |u|^p \right)^{\frac{r}{p}}, \quad (1.2)$$

where the full support of $\mathcal{U}_\varepsilon u$ lies within $\mathbb{B}_{\varepsilon\beta}(\mathbf{Q})$ for ε small enough and some fixed $\beta \in (0, 1)$ depending on \mathbf{p} .

In Part I we have established a general abstract framework for the derivation of uniform bounds on extension operators and except for two special examples, the results in Part I are rather vague, missing a general theory to deal with the connectivity of the domain. The connectivity for general geometries will be the main topic of the present work. We note at this point that connectivity is also the major issue for other former works to restrict to inclusions of an absolutely bounded diameter [3, 10]. Our method of proof, based on Part I, is different from other proofs in the literature, particularly the literature for periodic [7] or John [8, 2] domains, even though some patterns recur such as the construction of suitable paths along overlapping sets of an open covering. For a further overview over the history and the literature, the reader is referred to Parts I and II [6].

Let us finally note that replacing (1.1) by (1.2) also affects the analysis in the homogenization process and we refer to Part II [6] where this has been discussed.

1.1 The setting

Throughout this work, we use $(e_i)_{i=1,\dots,d}$ for the Euclidean basis of \mathbb{R}^d . Given a metric space (M, d) we denote $\mathbb{B}_r(x)$ the open ball around $x \in M$ with radius $r > 0$. The surface of the unit ball in \mathbb{R}^d is \mathbb{S}^{d-1} . Furthermore, we denote

$$\forall A \subset \mathbb{R}^d : \quad \mathbb{B}_r(A) := \bigcup_{x \in A} \mathbb{B}_r(x).$$

A sequence of points will be labeled by $\mathbf{x} := (x_i)_{i \in \mathbb{N}}$.

In what follows, we will assume that $\mathbf{p} = \mathbf{p}(\omega)$ is also a random connected domain, that is Lipschitz for almost every realization. We formally introduce the concepts of stationarity and ergodicity of stochastic processes in Section 2.4. If no confusion occurs, we drop (ω) in the notation wherever possible in order to improve readability.

According to Part I Chapter 3 for every stationary ergodic random open set \mathbf{p} the following can be established.

Lemma 1.1. *Let \mathbf{p} be a stationary ergodic random open set. Then there exists $\tau > 0$ and a positive, monotonically decreasing functions $f_{\mathbf{P}}$ with $f_{\mathbf{P}}(R) \rightarrow 0$ as $R \rightarrow \infty$ and a random point process $\mathbf{x}_{\tau} = (x_a)_{a \in \mathbb{N}}$ jointly stationary with \mathbf{p} such that*

- $\mathbb{B}_{\frac{\tau}{2}}(\mathbf{x}_{\tau}) \subset \mathbf{p}$,
- for all $a, b \in \mathbb{N}$, $a \neq b$, it holds $|x_a - x_b| > 2\tau$,
- $\mathbb{P}(\mathbb{B}_R(0) \cap \mathbf{x}_{\tau} = \emptyset) \leq f_{\mathbf{P}}(R)$.

Jointly stationary in the sense of Part I means that either both the joint distributions of \mathbf{x}_{τ} and \mathbf{p} are invariant over all shifts $x \in \mathbb{R}^d$ or over all shifts $x \in 2\tau\mathbb{Z}^d$. Constructing from $\mathbf{x}_{\tau} = (x_a)_{a \in \mathbb{N}}$ a Voronoi tessellation of cells $(G_a)_{a \in \mathbb{N}}$ with diameter $\mathfrak{d}_a = \mathfrak{d}(x_a) := \sup_{x, y \in G_a} |x - y|$, then according to Part I for some constant $C \geq 1$

$$\mathbb{P}(\mathfrak{d}(x_a) > D) < f_{\mathfrak{d}}(D) := C f_{\mathbf{P}}(C^{-1}D). \quad (1.3)$$

Furthermore, for any $x \in \mathbf{x}_{\tau}$ and $y \in \mathbf{p}$ let

$$\Upsilon(x, y) := \{\gamma : [0, 1] \rightarrow \mathbf{p} \mid \gamma \in C([0, 1]; \mathbf{p}), \gamma(0) = x, \gamma(1) = y\}$$

denote the set of all continuous paths from x to y inside \mathbf{p} . Given $x \in \mathbf{x}_\tau$ we further denote

$$\mathcal{R}(x) := \tau + \inf \left\{ R > \tau : \forall y \in \mathbb{B}_{5\delta(x)}(x) \exists \gamma \in \Upsilon(x, y) : \gamma([0, 1]) \subset \mathbb{B}_R(x) \right\}. \quad (1.4)$$

Connectedness ensures $\mathcal{R}(x) < \infty$ for every $x \in \mathbf{x}_\tau$. Denoting $\mathcal{S}(x) := \mathcal{R}(x)/\delta(x)$ we consider monotonically decreasing functions $f_{\mathcal{S}}, f_{\mathcal{R}} : [0, \infty) \rightarrow \mathbb{R}$ given through

$$f_{\mathcal{R}}(R) := \mathbb{P}(\mathcal{R}(x_a) > R), \quad f_{\mathcal{S}}(S) := \mathbb{P}(\mathcal{S}(x_a) > S). \quad (1.5)$$

We call \mathcal{R} the connectivity radius and \mathcal{S} the stretch factor.

Definition 1.2 (Local (δ, M) -Regularity). The domain $\mathbf{p} \subset \mathbb{R}^d$ is called (δ, M) -regular in $p_0 \in \partial\mathbf{p}$ if there exists an open set $U \subset \mathbb{R}^{d-1}$ and a Lipschitz continuous function $\phi : U \rightarrow \mathbb{R}$ with Lipschitz constant greater or equal to M such that $\partial\mathbf{p} \cap \mathbb{B}_\delta(p_0)$ is subset of the graph of the function $\varphi : U \rightarrow \mathbb{R}^d$, $\tilde{x} \mapsto (\tilde{x}, \phi(\tilde{x}))$ in some suitable coordinate system.

Definition 1.3. For a stationary random Lipschitz domain $\mathbf{p} \subset \mathbb{R}^d$ with τ from Lemma 1.1 and for every $p \in \partial\mathbf{p}$ and $n \in \mathbb{N} \cup \{0\}$

$$\Delta_\tau(p) := \sup_{\delta < 2\tau} \{ \exists M > 0 : \mathbf{p} \text{ is } (\delta, M) \text{-regular in } p \}, \quad \delta_{\Delta_\tau}(p) := \frac{\Delta_\tau(p)}{2}, \quad (1.6)$$

$$M_R(p) := \inf_{\eta > R} \inf \{ M : \mathbf{p} \text{ is } (\eta, M) \text{-regular in } p \}, \quad (1.7)$$

$$\rho_n(p) := \sup_{R < \delta(p)} r \left(4M_R(p)^2 + 2 \right)^{-\frac{n}{2}}, \quad (1.8)$$

For every $p \in \partial\mathbf{p}$ it holds that

$$R_2 > R_1 \quad \text{implies} \quad M_{R_2}(p) \geq M_{R_1}(p).$$

Since no confusion occurs, we write $\delta = \delta_{\Delta_\tau}$ for simplicity.

Definition 1.4 (Extension order). The geometry is of extension order $n \in \mathbb{N} \cup \{0\}$ if there exists $C > 0$ such that for almost every $p \in \partial\mathbf{p}$ there exists a local extension operator

$$\begin{aligned} \mathcal{U}_n : W^{1,p}(\mathbb{B}_{\frac{1}{8}\delta(p)}(p) \cap \mathbf{p}) &\rightarrow W^{1,p}(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p)), \\ \|\nabla \mathcal{U}_n u\|_{L^p(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p))} &\leq C \left(1 + M_{\frac{1}{8}\delta(p)}(p) \right) \|\nabla u\|_{L^p(\mathbb{B}_{\frac{1}{8}\delta(p)}(p))}. \end{aligned} \quad (1.9)$$

Part I shows that every locally Lipschitz geometry is of extension order $n = 1$, though better values (i.e. $n=0$) for n are possible for some geometries.

Definition 1.5 (Inner microscopic regularity). Given $n \in \mathbb{N}$ and $\tilde{\rho} := 2^{-5}\rho_n$, the inner microscopic regularity $\alpha \in [0, 1]$ is

$$\alpha := \inf \left\{ \tilde{\alpha} \geq 0 : \forall p \in \partial\mathbf{p} \exists y \in \mathbf{p} : \mathbb{B}_{\tilde{\rho}(p)/32(1+M_{\tilde{\rho}(p)}(p)^{\tilde{\alpha}})}(y) \subset \mathbb{B}_{\tilde{\rho}(p)/8}(p) \right\}.$$

As demonstrated in Part I, the values of α and n as well as the distribution of M and ρ_n are crucial for the validity of (1.2) for a given pair (r, p) .

1.2 Main Result: Uniform extension estimates for stationary ergodic random sets $\mathbf{p}(\omega)$

We find the following main result.

Theorem 1.6. *Let $\mathbf{Q} \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\mathbf{p}(\omega)$ be a stationary ergodic random connected open set in \mathbb{R}^d of extension order n and with inner microscopic regularity α . Furthermore let \mathbf{x}_τ be a jointly stationary point process satisfying Assumption 1.1. Given constants $1 \leq r < s < p$ and $q, \tilde{q} \in [1, \infty)$ with $\frac{s}{p} + \frac{1}{q} + \frac{s}{r\tilde{q}} = 1$ and writing*

$$\mathbb{P}_{k,R} := \mathbb{P}(\text{for } x \in \mathbf{x}_\tau : \mathfrak{d}(x) \in [k, k+1), \mathcal{R}(x) \in [R, R+1))$$

let the following hold:

$$\sum_{k,R=1}^{\infty} (k+1)^{d(q+1)+3drq+r(q-1)} (R+1)^{d(q+1)+\frac{s(r+1)-r}{r}q+r(q-1)} \mathbb{P}_{k,R} < \infty, \quad (1.10)$$

$$\mathbb{E} \left(\delta \left(1 + M_{\frac{3\delta}{4}} \right)^{\frac{p}{p-r}[(n+\alpha)(d-1)+r]+d-2} \right) < \infty, \quad (1.11)$$

$$\mathbb{E} \left(\tilde{\rho}_n^{(1-d)(r\tilde{q}-1)+2+d} \tilde{M}_{4\tilde{\rho}_n}^{\alpha d+d-2} \right) < \infty. \quad (1.12)$$

Alternatively let \mathfrak{d} and \mathcal{S} be independent and writing

$$\mathbb{P}_{\mathfrak{d},k} := \mathbb{P}(\text{for } x \in \mathbf{x}_\tau : \mathfrak{d}(x) \in [k, k+1))$$

$$\mathbb{P}_{\mathcal{S},S} := \mathbb{P}(\text{for } x \in \mathbf{x}_\tau : \mathcal{S}(x) \in [S, S+1))$$

replace condition (1.10) with

$$\sum_{k,S=1}^{\infty} (k+1)^{d(q+1)+d(3r+\frac{s(r+1)-r}{r})q+r(q-1)} (S+1)^{d(q+1)+d\frac{s(r+1)-r}{r}q+r(q-1)} < \infty. \quad (1.13)$$

Then there exists $\beta_0 \in (0, 1)$ not depending on ω such that for almost every ω there exists an extension operator $\mathcal{U}_\omega : W_{\text{loc}}^{1,p}(\mathbf{p}(\omega)) \rightarrow W_{\text{loc}}^{1,r}(\mathbb{R}^d)$ and a constant C_ω and $N_0 \geq 1$ such that for every $N > N_0$ it holds

$$\frac{1}{|N\mathbf{Q}|} \int_{N\mathbf{Q}} |\nabla \mathcal{U}_\omega u|^r \leq C_\omega \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbf{p}(\omega) \cap \mathbb{B}_{N\beta_0}(N\mathbf{Q})} |\nabla u|^p \right)^{\frac{r}{p}}, \quad (1.14)$$

$$\frac{1}{|N\mathbf{Q}|} \int_{N\mathbf{Q}} |\mathcal{U}_\omega u|^r \leq C_\omega \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbf{p}(\omega) \cap \mathbb{B}_{N\beta_0}(N\mathbf{Q})} |u|^p \right)^{\frac{r}{p}}. \quad (1.15)$$

1.3 Discussion

We may apply a rescaling $N = \varepsilon^{-1}$ for some $\varepsilon > 0$. Writing

$$[\mathcal{U}_\omega^\varepsilon u](x) := [\mathcal{U}_\omega u(\varepsilon \cdot)] \left(\frac{x}{\varepsilon} \right)$$

inequality (1.14) reads

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} |\mathcal{U}_\omega^\varepsilon u|^r \leq C_\omega \left(\frac{1}{|\mathbf{Q}|} \int_{[\varepsilon\mathbf{p}(\omega)] \cap \mathbb{B}_{\varepsilon^{-1}\beta_0}(\mathbf{Q})} |\nabla u|^p \right)^{\frac{r}{p}}.$$

The important insight is that $\chi_{\mathbb{B}_{\varepsilon^{-1}\beta_0}(\mathbf{Q})} \rightarrow \chi_{\mathbf{Q}}$ in $L^p(\mathbb{R}^d)$ for any $1 \leq p < \infty$ and hence in the limit $\mathcal{U}_\omega^\varepsilon u$ is determined mostly by the values of $u(x)$ for $x \in \mathbf{Q}$. Moreover it was shown in Part I that $u|_{(\varepsilon\mathbf{p}) \cap \partial\mathbf{Q}} \equiv 0$ implies that the support of $\mathcal{U}_\omega^\varepsilon u$ will ultimately reduce to \mathbf{Q} in the limit.

1.4 Structure of the article

In Section 2 we collect some results from Part I and modify the Voronoi integration lemma from there including a new and shortened proof. In Section 3 we prove Theorem 1.6 based on one of the main results from Part I. An outline of the proof is provided at the beginning of Section 3.

2 Preliminaries from Part I

The constant C on the right hand side of (1.14) depends on averaged weights of δ , M , \mathfrak{d}_a , \mathcal{S}_a and \mathcal{R}_a related to (1.10)–(1.13). In order to judge whether these averages are bounded as $n \rightarrow \infty$, we will rely on the integration theory that is recalled below. In particular, this theory is connected to the ergodic theorems and the Palm measure. We start by briefly explaining how the following results will be applied later on.

In Section 2.1 we recall η -regularity introduced in Part I. This concept allows us to cover any closed sets by a suitable family of open balls such that the covering is locally finite and uniformly bounded by a constant. While in Part I this was used to cover only the boundary of \mathbf{p} in terms of ρ_n , we will later in Section 3.3 use this result to extend the covering to the interior full domain.

In Section 2.2 we construct from (δ, M) (notably only defined on $\partial\mathbf{p}$) various integrable functions on \mathbb{R}^d which are denoted e.g. $\rho_{[\dots], \mathbb{R}^d}$, $\delta_{[\dots], \mathbb{R}^d}$, $M_{[\dots], \mathbb{R}^d}$. However, we emphasize at this point that the distribution of $\rho_n(x)$, $\delta(x)$ or $M(x)$ are w.r.t. the condition that $x \in \partial\mathbf{p}(\omega)$. Hence, it is necessary to control integrals over the functions $\rho_{[\dots], \mathbb{R}^d}$, $\delta_{[\dots], \mathbb{R}^d}$, $M_{[\dots], \mathbb{R}^d}$ by integrals over the functions $\rho_n(x)$, $\delta(x)$ or $M(x)$, which leads to Lemma 2.6.

Section 2.3 provides a frequently used Poincaré inequality and in Section 2.4 we introduce the ergodic theorems on \mathbf{p} and $\partial\mathbf{p}$ which will ensure that all the above mentioned averaging integrals converge to their expectation as the support grows infinitely large.

Finally in Section 2.5 we study functions

$$\mathfrak{b}(y) := \sum_{x \in \mathbb{X}_\tau} \chi_{\mathbb{B}_{\mathcal{R}(x)}(x)}(x) \mathfrak{d}(x)^\eta \mathcal{R}(x)^\xi,$$

and provide an estimate on the expectation of \mathfrak{b}^q , $q \in [1, \infty)$. This will help us to control integrals that enter the constant C from the mesoscopic geometric properties.

2.1 Local η -Regularity

We summarize the concept of η -regularity and its consequences from Part I. Note that Lemma 2.2 was proved in Part I only for $\Gamma = \partial\mathbf{p}$. However, the only property of $\partial\mathbf{p}$ used for the proof is its closedness.

Definition 2.1 (η -regularity). Let Γ be a closed set. For a function $\eta : \Gamma \rightarrow (0, r]$ we call Γ η -regular if

$$\forall p \in \Gamma, \varepsilon \in \left(0, \frac{1}{2}\right), \tilde{p} \in \mathbb{B}_{\varepsilon\eta(p)}(p) \cap \Gamma : \eta(\tilde{p}) > (1 - \varepsilon)\eta(p). \quad (2.1)$$

Lemma 2.2. Let Γ be a locally η -regular set for $\eta : \Gamma \rightarrow (0, \tau)$. Then $\eta : \mathbf{p} \rightarrow \mathbb{R}$ is locally Lipschitz continuous with Lipschitz constant 1 and for every $\varepsilon \in (0, \frac{1}{2})$ and $\tilde{p} \in \mathbb{B}_{\varepsilon\eta(p)}(p) \cap \Gamma$ it holds

$$\frac{1 - \varepsilon}{1 - 2\varepsilon} \eta(p) > \eta(\tilde{p}) > \eta(p) - |p - \tilde{p}| > (1 - \varepsilon) \eta(p), \quad (2.2)$$

$$|p - \tilde{p}| \leq \varepsilon \max\{\eta(p), \eta(\tilde{p})\} \Rightarrow |p - \tilde{p}| \leq \frac{\varepsilon}{1 - \varepsilon} \min\{\eta(p), \eta(\tilde{p})\}. \quad (2.3)$$

Theorem 2.3. Let $\Gamma \subset \mathbb{R}^d$ be a closed set and let $\eta(\cdot) \in C(\Gamma)$ be bounded and satisfy for every $\varepsilon \in (0, \frac{1}{2})$ and for $|p - \tilde{p}| < \varepsilon\eta(p)$

$$\frac{1 - \varepsilon}{1 - 2\varepsilon} \eta(p) > \eta(\tilde{p}) > \eta(p) - |p - \tilde{p}| > (1 - \varepsilon) \eta(p). \quad (2.4)$$

and define $\tilde{\eta}(p) = 2^{-K}\eta(p)$, $K \geq 2$. Then for every $C \in (0, 1)$ there exists a locally finite covering of Γ with balls $\mathbb{B}_{\tilde{\eta}(p_k)}(p_k)$ for a countable number of points $(p_k)_{k \in \mathbb{N}} \subset \Gamma$ such that for every $i \neq k$ with $\mathbb{B}_{\tilde{\eta}(p_i)}(p_i) \cap \mathbb{B}_{\tilde{\eta}(p_k)}(p_k) \neq \emptyset$ it holds

$$\begin{aligned} \frac{2^{K-1} - 1}{2^{K-1}} \tilde{\eta}(p_i) &\leq \tilde{\eta}(p_k) \leq \frac{2^{K-1}}{2^{K-1} - 1} \tilde{\eta}(p_i) \\ \text{and } \frac{2^K - 1}{2^{K-1} - 1} \min \{ \tilde{\eta}(p_i), \tilde{\eta}(p_k) \} &\geq |p_i - p_k| \geq C \max \{ \tilde{\eta}(p_i), \tilde{\eta}(p_k) \} \end{aligned} \quad (2.5)$$

In Part I the last lead immediately to the following corollary.

Corollary 2.4. Let $\tau > 0$ and let $\mathbf{p} \subset \mathbb{R}^d$ be a locally (δ, M) -regular open set, where we restrict δ by $\delta(\cdot) \leq \frac{\tau}{4}$. Given $n \in \mathbb{N}$ there exists a countable number of points $(p_k)_{k \in \mathbb{N}} \subset \partial \mathbf{p}$ such that $\partial \mathbf{p}$ is completely covered by balls $\mathbb{B}_{\tilde{\rho}(p_k)}(p_k)$ where $\tilde{\rho}(p) := \tilde{\rho}_n(p) := 2^{-5}\rho_n(p)$. Writing

$$\tilde{\rho}_k := \tilde{\rho}_{n,k} := \tilde{\rho}_n(p_k), \quad \delta_k := \delta(p_k),$$

for two such balls with $\mathbb{B}_{\tilde{\rho}_k}(p_k) \cap \mathbb{B}_{\tilde{\rho}_i}(p_i) \neq \emptyset$ it holds

$$\begin{aligned} \frac{15}{16} \tilde{\rho}_i &\leq \tilde{\rho}_k \leq \frac{16}{15} \tilde{\rho}_i \\ \text{and } \frac{31}{15} \min \{ \tilde{\rho}_i, \tilde{\rho}_k \} &\geq |p_i - p_k| \geq \frac{1}{2} \max \{ \tilde{\rho}_i, \tilde{\rho}_k \}. \end{aligned} \quad (2.6)$$

Furthermore, depending on the inner microscopic regularity $\alpha \in [0, 1]$ there exists $y_{n,\alpha,k}$ and $\tau_{n,\alpha,k} \geq \frac{\tilde{\rho}_{n,k}}{32(1+M_{\tilde{\rho}_n(p_k)}(p_k))^\alpha}$ such that $\mathbb{B}_{\tau_{n,\alpha,k}}(y_{n,\alpha,k}) \subset \mathbb{B}_{\tilde{\rho}_k/8}(p_k) \cap \mathbf{p}$ and $\mathbb{B}_{2\tau_{n,\alpha,k}}(y_k) \cap \mathbb{B}_{2\tau_{n,\alpha,j}}(y_j) = \emptyset$ for $k \neq j$.

Remark 2.5. Given the covering from Corollary 2.4 Lemma 4.4 and Remark 4.5 from Part I imply

$$\# \left\{ j : x \in \overline{\mathbb{B}_{\tilde{\rho}_{n,j}}(p_j)} \right\} < C(1 + M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d}(x))^{n(d-1)}.$$

2.2 Integration of δ and M

Given $c \in (0, 1]$ let $\eta(p) = c\delta(p)$ or $\eta(p) = c\rho_n(p)$, $n \in \mathbb{N}$ and $r \in C^{0,1}(\partial \mathbf{p})$ and define the functions

$$\eta_{[r], \mathbb{R}^d}(x) := \inf \left\{ \eta(\tilde{x}) : \tilde{x} \in \partial \mathbf{p} \text{ s.t. } x \in \mathbb{B}_{r(\tilde{x})}(\tilde{x}) \right\}, \quad (2.7)$$

$$M_{[r,\eta], \mathbb{R}^d}(x) := \sup \left\{ M_{r(\tilde{x})}(\tilde{x}) : \tilde{x} \in \partial \mathbf{p} \text{ s.t. } x \in \overline{\mathbb{B}_{\eta(\tilde{x})}(\tilde{x})} \right\}, \quad (2.8)$$

where $\inf \emptyset = \sup \emptyset := 0$ for notational convenience. We also write $M_{[\eta], \mathbb{R}^d}(x) := M_{[\eta,\eta], \mathbb{R}^d}(x)$ and $\eta_{\mathbb{R}^d}(x) := \eta_{[\eta], \mathbb{R}^d}(x)$. The relations between η , $M : \partial \mathbf{p} \rightarrow \mathbb{R}$ and $\eta_{[r], \mathbb{R}^d}$, $M_{[r,\eta], \mathbb{R}^d} : \mathbb{R}^d \rightarrow \mathbb{R}$ as well as integrability and measurability are discussed in Part I. Furthermore, we define

$$\mathbf{p}_{[r], \mathbb{R}^d} := \mathbf{p} \cap \bigcup_{x \in \partial \mathbf{p}} \mathbb{B}_{r(x)}(x).$$

Lemma 2.6. *Let $\tau > 0$, let $\mathbf{P} \subset \mathbb{R}^d$ be a Lipschitz domain and let $\eta, r : \partial\mathbf{P} \rightarrow \mathbb{R}$ be continuous such that $\eta \leq \tau$ and \mathbf{P} is η - and r -regular. For $\varepsilon \in (0, 1]$ let $\eta(p) = \varepsilon\delta(p)$ or $\eta(p) = \varepsilon\rho_n(p)$, $n \in \mathbb{N}$. For $\tilde{\eta} := \eta|_{[\frac{\tau}{8}, \mathbb{R}^d]}$ there exists a constant $C > 0$ only depending on the dimension d such that for every bounded open domain \mathbf{Q} and $k \in [0, 4)$ it holds*

$$\int_{A_{\eta, \tau} \cap \mathbf{Q}} \chi_{\tilde{\eta} > 0} \tilde{\eta}^{-\alpha} \leq C \int_{\mathbb{B}_{\frac{\tau}{4}}(\mathbf{Q}) \cap \partial\mathbf{P}} \eta^{1-\alpha} M_{[\frac{\tau}{4}, \mathbb{R}^d]}^{d-2}, \quad (2.9)$$

$$\int_{A_{\eta, \tau} \cap \mathbf{Q}} \tilde{\eta}^{-\alpha} M_{[k\frac{\tau}{8}, \frac{\tau}{8}], \mathbb{R}^d}^r \leq C \int_{\mathbb{B}_{\frac{\tau}{4}}(\mathbf{Q}) \cap \partial\mathbf{P}} \eta^{1-\alpha} M_{[k\frac{\tau}{8}, \frac{\tau}{4}], \mathbb{R}^d}^r M_{[\frac{\tau}{4}, \mathbb{R}^d]}^{d-2}. \quad (2.10)$$

Finally, it holds

$$x \in \mathbb{B}_{\frac{1}{8}\eta(p)}(p) \quad \Rightarrow \quad \eta(p) > \tilde{\eta}(x) > \frac{3}{4}\eta(p). \quad (2.11)$$

2.3 A fundamental Poincaré inequality

We define for $a \in \mathbb{R}^d$ and $\delta > 0$ the operator

$$\mathcal{M}_a^\delta u := \int_{\mathbb{B}_\delta(a)} u. \quad (2.12)$$

The following two estimates are special cases of results already proved in Part I.

Lemma 2.7. *There exists $C > 0$ depending only on the dimension d such that for $a, b \in \mathbb{R}^d$ with $0 < \delta_a \leq \delta_b$ and for either $i \in \{a, b\}$*

$$|\mathcal{M}_a^{\delta_a} u - \mathcal{M}_b^{\delta_b} u| \leq C \left(\delta_b \left(\frac{\delta_a}{\delta_b} \right)^{1-d} + \delta_i^{1-d} \right) \int_{\mathbb{B}_{\delta_b}(b) \cup \text{conv}(\mathbb{B}_{\delta_i}(\{a, b\}))} |\nabla u|. \quad (2.13)$$

Proof. Inequality (2.13) follows from Part I Lemma 2.10 and Corollary 2.11. \square

2.4 Ergodic theorem and Palm measure

In order to make clear what we mean by a random stationary ergodic Lipschitz domain we briefly introduce the technical details which will be used for the averaging property given by the ergodic theorem [9, 11] below.

Definition 2.8. Throughout this work, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a dynamical system on Ω , i.e. a family $(\tau_x)_{x \in \mathbb{R}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$ satisfying (i)-(iii):

- (i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ (Group property)
- (ii) $\mathbb{P}(\tau_{-x}B) = \mathbb{P}(B) \quad \forall x \in \mathbb{R}^d, B \in \mathcal{F}$ (Measure preserving)
- (iii) $A : \mathbb{R}^d \times \Omega \rightarrow \Omega \quad (x, \omega) \mapsto \tau_x \omega$ is measurable (Measurability of evaluation)

We further assume that $(\tau_x)_{x \in \mathbb{R}^d}$ is ergodic, i.e. a \mathbb{P} -measurable function satisfies $f(\tau_x \cdot) = f(\cdot)$ if and only if f is constant.

Definition 2.9 (Stationary). Let X be a measurable space and let $f : \Omega \times \mathbb{R}^d \rightarrow X$. Then f is called (weakly) stationary if $f(\omega, x) = f(\tau_x \omega, 0)$ for (almost) every x .

Although the original definition is different, it is sufficient for this work (see [4] Section 2) to say that a random Lipschitz domain $\mathbf{p}(\omega)$ is stationary if $\chi_{\mathbf{p}(\omega)}(x)$ is stationary and there exists $\mathbf{P} \subset \Omega$ such that

$$\chi_{\mathbf{p}(\omega)}(x) = \chi_{\mathbf{P}}(\tau_x \omega).$$

A random measure is a measurable mapping

$$\mu_{\bullet} : \Omega \rightarrow \mathfrak{M}(\mathbb{R}^d), \quad \omega \mapsto \mu_{\omega}$$

which is equivalent to either one of the following two conditions

- 1 For every bounded Borel set $A \subset \mathbb{R}^d$ the map $\omega \mapsto \mu_{\omega}(A)$ is measurable
- 2 For every $f \in C_c(\mathbb{R}^d)$ the map $\omega \mapsto \int f d\mu_{\omega}$ is measurable.

A random measure is stationary if the distribution of $\mu_{\omega}(A)$ is invariant under translations of A that is $\mu_{\omega}(A)$ and $\mu_{\omega}(A+x)$ share the same distribution. The Palm measure is defined as

$$\mu_{\mathcal{P}}(A) = \int_{\Omega} \int_{[0,1]^d} \chi_A(\tau_s \omega) d\mu_{\omega}(s) d\mathbb{P}(\omega)$$

on the measurable space Ω and in case $\mu_{\omega} = \mathcal{L}$ we find $\mu_{\mathcal{P}} = \mathbb{P}$. By a deep theorem due to Mecke (see [9, 1]) every $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Omega)$ -measurable non negative or $\mu_{\mathcal{P}} \times \mathcal{L}$ -integrable functions f satisfies the Campbell formula

$$\int_{\Omega} \int_{\mathbb{R}^d} f(x, \tau_x \omega) d\mu_{\omega}(x) d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} \int_{\Omega} f(x, \omega) d\mu_{\mathcal{P}}(\omega) dx.$$

We denote by

$$\mathbb{E}_{\mu_{\mathcal{P}}}(f) := \int_{\Omega} f d\mu_{\mathcal{P}} \text{ the expectation of } f \text{ w.r.t. } \mu_{\mathcal{P}}. \quad (2.14)$$

For random measures we find the following.

Theorem 2.10 (Ergodic Theorem [1] 12.2.VIII). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, \mathbf{Q} be a bounded open domain with Lipschitz boundary and let $f : \Omega \rightarrow \mathbb{R}$ be measurable with $\int_{\Omega} |f| d\mu_{\mathcal{P}} < \infty$. Then for \mathbb{P} -almost all $\omega \in \Omega$*

$$\frac{1}{n^d |\mathbf{Q}|} \int_{n\mathbf{Q}} f(\tau_x \omega) d\mu_{\omega}(x) \rightarrow \mathbb{E}_{\mu_{\mathcal{P}}}(f). \quad (2.15)$$

In our setting, the above implies in total for any differentiable function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ that almost surely

$$\lim_{n \rightarrow \infty} \int_{\partial \mathbf{p}(\omega) \cap n\mathbf{Q}} f(\rho, \delta, M) = \mathbb{E}_{\mu_{\mathcal{P}}}(f(\rho, \delta, M)). \quad (2.16)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbf{p}(\omega) \cap n\mathbf{Q}} f(\mathfrak{d}, \mathcal{R}, \mathcal{S}) = \mathbb{E}(f(\mathfrak{d}, \mathcal{R}, \mathcal{S})) \mathbb{P}(\mathbf{P}). \quad (2.17)$$

Since the essential property of f in (2.15) is its stationarity, we infer that (2.17) also holds for “non-local” functions such as \mathfrak{b} in (2.18) in the following Lemma 2.11.

2.5 A Voronoi-integration lemma

We state and proof a variant of a Voronoi integration lemma that was proved in Section 4 of Part I.

Lemma 2.11. *Let \mathbb{X}_τ be a stationary and ergodic random point process with minimal mutual distance 2τ for $\tau > 0$. Given fixed constants $\eta, \xi > 0$ let*

$$\mathfrak{b}(y) := \sum_{x \in \mathbb{X}_\tau} \chi_{\mathbb{B}_{\mathcal{R}(x)}(x)}(x) \mathfrak{d}(x)^\eta \mathcal{R}(x)^\xi, \quad (2.18)$$

and write $\mathbb{P}_{k,R} := \mathbb{P}(\mathfrak{d}(x) \in [k, k+1), \mathcal{R}(x) \in [R, R+1))$. Then there exists $C > 0$ depending only on d and τ such that for any $r > 1$ it holds

$$\mathbb{E}(\mathfrak{b}^p) \leq C \left(\sum_{k=1}^{\infty} k^{-r} \right)^2 \left(\sum_{k,R=1}^{\infty} (k+1)^{d(p+1)+\eta p+r(p-1)} (R+1)^{d(p+1)+\zeta p+r(p-1)} \mathbb{P}_{k,R} \right). \quad (2.19)$$

Proof. In what follows C is a varying constant depending only on d and τ . W.l.o.g let $\tau = 1$. We write $\mathfrak{d}_i = \mathfrak{d}(x_i)$, $R_i = \mathcal{R}(x_i)$, $\mathbb{B}_i := \mathbb{B}_{R_i}(x_i)$. Let

$$X_{k,R}(\omega) := \{x_i \in \mathbb{X}_\tau : \mathfrak{d}_i \in [k, k+1), R_i \in [R, R+1)\}, \quad A_{k,R} := \bigcup_{x_i \in X_{k,R}} \mathbb{B}_i$$

We observe that the mutual minimal distance of points in \mathbf{x}_τ implies

$$\forall x \in \mathbb{R}^d : \# \{x_i \in X_{k,R} : x \in \mathbb{B}_i\} \leq C (R+1)^d (k+1)^d, \quad (2.20)$$

which follows from the uniform boundedness of the \mathbb{B}_i for $x_i \in X_{k,R}$ and the minimal distance of $|x_i - x_j| > 2\tau$. Then for every $y \in \mathbb{R}^d$, $M > 0$ it holds by stationarity and the ergodic theorem for every $y \in \mathbb{R}^d$

$$\begin{aligned} \mathbb{P}(y \in A_{k,R}) &= \lim_{N \rightarrow \infty} |\mathbb{B}_N(0)|^{-1} |A_{k,R} \cap \mathbb{B}_N(0)| = \lim_{N \rightarrow \infty} |\mathbb{B}_N(0)|^{-1} \left| \mathbb{B}_N(0) \cap \bigcup_{x_i \in X_{k,R}} \mathbb{B}_i \right| \\ &\leq C \lim_{N \rightarrow \infty} |\mathbb{B}_N(0)|^{-1} \sum_{x_i \in X_{k,R} \cap \mathbb{B}_N(0)} (R+1)^d (k+1)^d \\ &\leq C \lim_{N \rightarrow \infty} \frac{\# \{x_i \in X_{k,R} \cap \mathbb{B}_N(0)\}}{\# \{x_i \in \mathbf{x}_\tau \cap \mathbb{B}_N(0)\}} (R+1)^d (k+1)^d \rightarrow C \mathbb{P}_{k,R} (R+1)^d (k+1)^d. \end{aligned} \quad (2.21)$$

In the last inequality we made use of the fact that every ball $\mathbb{B}_{R_i}(x_i)$, $x_i \in X_{k,N}$, has volume smaller than $C (R+1)^d (k+1)^d$ and $\# \{x_i \in \mathbf{x}_\tau \cap \mathbb{B}_N(0)\} < C |\mathbb{B}_N(0)|$. We note that for $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} \int_{\mathbb{Q}} \left(\sum_{x_i} \chi_{\mathbb{B}_i} \mathfrak{d}_i^\eta \mathcal{R}_i^\xi \right)^p &\leq \int_{\mathbb{Q}} \left(\sum_{k=1}^{\infty} \sum_{R=1}^{\infty} \left(\sum_{x_i \in X_{k,R}} \chi_{\mathbb{B}_i} (k+1)^\eta (R+1)^\xi \right) \right)^p \\ &\leq \int_{\mathbb{Q}} \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^q \right)^{\frac{p}{q}} \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^{-p} \left(\sum_{x_i \in X_{k,R}} \chi_{\mathbb{B}_i} (k+1)^\eta (R+1)^\xi \right)^p \right). \end{aligned}$$

Due to (2.20) we find

$$\sum_{x \in X_{k,R}} \chi_{\mathbb{B}_i} \leq \chi_{A_{k,R}} (R+1)^d (k+1)^d |\mathbb{S}^{d-1}|$$

and obtain for $q = \frac{p}{p-1}$ and $C_q := \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^q \right)^{\frac{p}{q}} |\mathbb{S}^{d-1}|^p$ due to (2.21):

$$\begin{aligned} & \frac{1}{|\mathbb{B}_N(0)|} \int_{\mathbb{B}_N(0)} \left(\sum_{x_i \in \mathbb{X}_t} \chi_{\mathbb{B}_i} \mathfrak{d}(x)^\eta \mathcal{R}(x)^\xi \right)^p \\ & \leq C_q \frac{1}{|\mathbb{B}_N(0)|} \int_{\mathbb{B}_N(0)} \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^{-p} \chi_{A_{k,R}} (R+1)^{dp+\zeta p} (k+1)^{dp+\eta p} \right) \\ & \rightarrow C_q \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^{-p} \mathbb{P}(A_{k,R}) (R+1)^{dp+\zeta p} (k+1)^{dp+\eta p} \right) \\ & \leq C_q \left(\sum_{k,R=1}^{\infty} \alpha_{k,R}^{-p} (k+1)^{d(p+1)+\eta p} (R+1)^{d(p+1)+\zeta p} \mathbb{P}_{k,R} \right) \end{aligned}$$

For the sum $\sum_{k,R=1}^{\infty} \alpha_{k,R}^q$ to converge, it is sufficient that $\alpha_{k,R}^q = (k+1)^{-r} (R+1)^{-r}$ for some $r > 1$. Hence, for such r it holds $\alpha_{k,R} = (k+1)^{-r/q} (R+1)^{-r/q}$ and thus (2.19). \square

3 Proof of Theorem 1.6

In this section, we will prove Theorem 1.6. The proof consists of 5 sections: In Section 3.1 we quote one of the main results from Part I. This is an estimate of the extended gradient field by the original gradient field and the difference of local averages. This makes it clear that one has to estimate differences of local averages by the gradient field “connecting” the two averaging regions. Since the geometry \mathfrak{p} is connected, we identify in Section 3.2 a constant $\beta \in (0, 1)$ such that for $M \in \mathbb{N}$ large enough the set $\mathbf{Q}_M := M\mathbf{Q}$ is connected through paths inside $\mathbb{B}_{M\beta}(\mathbf{Q}_M)$. In Section 3.3 we extend the covering Corollary 2.4 of $\partial\mathfrak{p}$ to a full covering of \mathfrak{p} using also the seeds \mathbf{x}_r . This covering will provide a basis to suitably integrate the gradient along paths connecting the averaging regions. In Section 3.5 we will finally prove the main theorem.

3.1 The Main Result from Part I

Based on the notation from Section 1.1 we use the Voronoi tessellation $(G_a)_{a \in \mathbb{N}}$ with seeds $(x_a)_{a \in \mathbb{N}} = \mathbf{x}_t$ and a partition of unity $(\Phi_a)_{a \in \mathbb{N}}$ with support $\mathbb{B}_{\frac{\tau}{2}}(G_a)$. The gradient of Φ_a is locally bounded by the number of sets $\mathbb{B}_{\frac{\tau}{2}}(G_a)$ interacting. Since the number of cells G_a interacting with $\mathbb{B}_\tau(G_a)$ is bounded by (Part I, Lemma 2.19) $(4\mathfrak{d}(x_a)\tau^{-1})^d$ we obtain

$$\forall x \in \mathbb{B}_{\frac{\tau}{2}}(G_a) : \quad |\nabla \Phi_a(x)| \leq 2 \left(4\mathfrak{d}(x_a)\tau^{-1} \right)^d. \quad (3.1)$$

Furthermore, there exists by Corollary 2.4 (cited from Part I) a complete covering of $\partial\mathfrak{p}$ by balls $A_i := \mathbb{B}_{\tilde{\rho}_n(p_i^n)}(p_i^n)$, $(p_i^n)_{i \in \mathbb{N}} \subset \partial\mathfrak{p}$, where $\tilde{\rho}_n(p) := 2^{-5}\rho_n(p)$ and where (2.6) holds for any two points p_i, p_k with $A_i \cap A_k \neq \emptyset$. Finally there exists a partition of unity $(\phi_i)_{i \in \mathbb{N} \setminus \{0\}}$ with support of ϕ_i in A_i and ϕ_0 with support in $\mathbb{R}^d \setminus \partial\mathfrak{p}$ such that $\sum_{i \in \mathbb{N}} \phi_i = 1$.

Given $n \in \{0, 1\}$ and $\alpha \in [0, 1]$ we chose

$$\mathfrak{r}_{n,\alpha,i} := \tilde{\rho}_{n,i}/32(1 + M_{\tilde{\rho}_{n,i}}(p_{n,i})^\alpha) \quad (3.2)$$

and some $y_{n,\alpha,i}$ such that

$$B_{n,\alpha,i} := \mathbb{B}_{\tau_{n,\alpha,i}}(y_{n,\alpha,i}) \subset \mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\bar{\rho}_{n,i}}(p_{n,i}). \tag{3.3}$$

and for every $p_i \in \partial\mathbf{p}$ and $x_a \in \mathbf{x}_\tau$, we define

$$\tau_{n,\alpha,i}u := \int_{B_{n,\alpha,i}} u, \quad \mathcal{M}_a u := \int_{\mathbb{B}_{\frac{\tau_a}{16}}(x_a)} u,$$

local averages close to $\partial\mathbf{p}$ and in x_a . We finally have to recall from Lemma 4.4 of Part I that

$$\#\{j : x \in \overline{\mathbb{B}_\delta(p_j)}\} < C(1 + M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d}(x))^{n(d-1)}. \tag{3.4}$$

Theorem 3.1. *Let $\mathbf{p} \subset \mathbb{R}^d$ be a stationary ergodic Lipschitz domain of extension order n with $\tau > 0$ from Lemma 1.1 and inner regularity $\alpha \in [0, 1]$. Then for every $1 \leq r < p$ there exists a linear extension operator*

$$\mathcal{U}_{n,\alpha} : W_{\text{loc}}^{1,p}(\mathbf{p}) \rightarrow W_{\text{loc}}^{1,r}(\mathbb{R}^d)$$

and $C > 0$ such that with

$$f_{\alpha,n}(M) := \left(\left(1 + M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d}\right)^{n(d-1)} \left(1 + M_{[\frac{1}{8}\delta], \mathbb{R}^d}\right)^r \left(1 + M_{[\bar{\rho}_n], \mathbb{R}^d}\right)^{\alpha(d-1)} \right)^{\frac{p}{p-r}}$$

for every bounded Lipschitz domain \mathbf{Q} the operator $\mathcal{U}_{n,\alpha}$ satisfies

$$\begin{aligned} \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} |\nabla(\mathcal{U}_{n,\alpha}u)|^r &\leq C \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{B}_\tau(\mathbf{Q})} f_{\alpha,n}(M) \right)^{r\frac{p-r}{p}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{B}_\tau(\mathbf{Q}) \cap \mathbf{p}} |\nabla u|^p \right)^{\frac{r}{p}} \\ &\quad + C \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \cap \mathbf{p}} \left| \sum_a \Phi_a \sum_{i \neq 0} \rho_{n,i}^{-1} \phi_i (\tau_{n,\alpha,i}^s u - \mathcal{M}_a^s u) \right|^r \\ &\quad + \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left| \sum_{l=1}^d \sum_{a: \partial_l \Phi_a > 0} \sum_{b: \partial_l \Phi_b < 0} \frac{\partial_l \Phi_a |\partial_l \Phi_b|}{D_{l+}^\Phi} (\mathcal{M}_a^s u - \mathcal{M}_b^s u) \right|^r, \end{aligned} \tag{3.5}$$

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} |\mathcal{U}_{n,\alpha}u|^r \leq C \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{B}_\tau(\mathbf{Q})} f_{\alpha,n}(M) \right)^{r\frac{p-r}{p}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{B}_\tau(\mathbf{Q}) \cap \mathbf{p}} |u|^p \right)^{\frac{r}{p}}. \tag{3.6}$$

where

$$D_{l+}^\Phi := \sum_{a \neq 0: \partial_l \Phi_a < 0} |\partial_l \Phi_a|. \tag{3.7}$$

3.2 The support lemma

Definition 3.2. Given a domain $\mathbf{Q} \subset \mathbb{R}^d$ and a stationary random domain \mathbf{p} with the jointly stationary point process \mathbf{x}_τ we define the sets

$$\begin{aligned} \mathbf{x}_\tau(\mathbf{Q}) &:= \{x_a \in \mathbf{x}_\tau : \mathbb{B}_{\delta(x_a)}(x_a) \cap \mathbf{Q} \neq \emptyset\}, \\ \mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau) &:= \bigcup_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \mathbb{B}_{\mathcal{R}(x_a)}(x_a). \end{aligned} \tag{3.8}$$

Remark 3.3. Since $\mathbb{B}_\tau(x_a) \subset G_a$ the last definition implies $x_a \in \mathbf{x}_\tau(\mathbf{Q})$ for every $x_a \in \mathbf{x}_\tau$ with $\chi_{\mathbf{Q}} \chi_{\mathbb{B}_\tau(G_a)} \neq 0$.

Lemma 3.4. *Recalling (1.3) and (1.5) assume that*

- 1 *either there exist $C > 0$ and $\beta_{\mathfrak{d}}, \beta_{\mathfrak{d}} > d + 1$ such that for every $D > \mathfrak{r}$, $r > 1$ it holds $f_{\mathfrak{d}}(D) \leq CD^{-\beta_{\mathfrak{d}}}$ and $f_{\mathfrak{d}}(r) \leq Cr^{-\beta_{\mathfrak{d}}}$*
- 2 *or \mathfrak{d} and \mathcal{S} are independent and there exist $C > 0$ and $\beta_{\mathfrak{d}} > d + 2$, $\beta_{\mathcal{S}} > 1$ such that for every $D > \mathfrak{r}$, $S > 1$ it holds $f_{\mathfrak{d}}(D) \leq CD^{-\beta_{\mathfrak{d}}}$ and $f_{\mathcal{S}}(S) \leq CS^{-\beta_{\mathcal{S}}}$.*

Then there exists $\beta_0 \in (0, 1)$ such that the following holds: For every bounded open set \mathbf{Q} with $0 \in \mathbf{Q}$ there almost surely exists a constant $N_0 > 0$ such that for every $N > N_0$

$$\mathbb{C}(N\mathbf{Q}, \mathbf{x}_{\mathfrak{r}}) \subset \mathbb{B}_{N\beta_0}(N\mathbf{Q}).$$

Remark 3.5. The scaling $N\beta_0$ of the radius of $\mathbb{B}_{N\beta_0}(N\mathbf{Q})$ implies that the mass of $\mathbb{C}(N\mathbf{Q}, \mathbf{x}_{\mathfrak{r}}) \setminus N\mathbf{Q}$ becomes asymptotically negligible.

Proof. We consider two balls $\mathbb{B}_r(0) \subset \mathbf{Q} \subset \mathbb{B}_R(0)$ with $r > 0$. We write $\mathbf{Q}_N := N\mathbf{Q}$ and $\mathbb{B}_{N,\beta_0}^{k,\mathbf{Q}} := \mathbb{B}_{N\beta_0+k}(\mathbf{Q})$ and $\mathbb{S}_{N,\beta_0}^{k,\mathbf{Q}} := \mathbb{B}_{N,\beta_0}^{k,\mathbf{Q}} \setminus \mathbb{B}_{N,\beta_0}^{k-1,\mathbf{Q}}$ for $\beta_0 \in (0, 1)$. Our aim is to show that for the events $B_N := \left(\mathbb{C}(\mathbf{Q}, \mathbf{x}_{\mathfrak{r}}) \subset \mathbb{B}_{\frac{1}{2\beta_0}N, \beta_0}^{0, \mathbf{Q}_N} \right)$ it holds $\mathbb{P}(B_N) \rightarrow 1$ as $N \rightarrow \infty$, provided β_0 is chosen properly. For this we use

$$\mathbb{P}(\neg B_N) \leq \mathbb{P}(A_N \wedge \neg B_N) + \mathbb{P}(\neg A_N)$$

where $A_N := \left(\tilde{\mathbf{Q}}_N \subset \mathbb{B}_{N,\beta_0}^{0, \mathbf{Q}_N} \right)$, $\tilde{\mathbf{Q}}_N := \bigcup_{x_a \in \mathbf{x}_{\mathfrak{r}}(N\mathbf{Q})} \mathbb{B}_{\mathfrak{d}(x_a)}(x_a)$. (3.9)

Step 1: It holds $\mathbf{x}_{\mathfrak{r}}(N\mathbf{Q}) \subset \tilde{\mathbf{Q}}_N$ and we find

$$\begin{aligned} \mathbb{P}(\neg A_N) &\leq \sum_{k=0}^{\infty} \mathbb{P}\left(\exists x_a \in \left(\mathbb{B}_{N,\beta_0}^{k+1, \mathbf{Q}_N} \setminus \mathbb{B}_{N,\beta_0}^{k, \mathbf{Q}_N}\right) \cap \mathbf{x}_{\mathfrak{r}} : \mathbb{B}_{\mathfrak{d}}(x_a) \cap \mathbf{Q}_N \neq \emptyset\right) \\ &\leq \sum_{k=0}^{\infty} \mathbb{P}\left(\exists x_a \in \left(\mathbb{S}_{N,\beta_0}^{k+1, \mathbf{Q}_N}\right) \cap \mathbf{x}_{\mathfrak{r}} : \mathfrak{d}_a > N\beta_0 + k\right) \end{aligned}$$

We use the very rough estimate $\#\left(\mathbb{S}_{N,\beta_0}^{k+1, \mathbf{Q}_N}\right) \cap \mathbf{x}_{\mathfrak{r}} \leq (NR + N\beta_0 + k + 1)^d$ to find

$$\begin{aligned} \mathbb{P}(\neg A_N) &\leq \sum_k (RN + N\beta_0 + k + 1)^d f_{\mathfrak{d}}(N\beta_0 + k) \\ &\leq \sum_k (RN + N\beta_0 + k + 1)^d (N\beta_0 + k)^{-\beta_{\mathfrak{d}}} \\ &\leq C \int_1^{\infty} (2RN + N\beta_0 + x)^d (N\beta_0 + x)^{-\beta_{\mathfrak{d}}} dx \\ &\leq C \left(N^{d+1-\beta_{\mathfrak{d}}} + N^{\beta_0(d+1-\beta_{\mathfrak{d}})} \right), \end{aligned}$$

where in the last inequality we used $(d - 1)$ -times integration by parts and C depends on d , $\beta_{\mathfrak{d}}$ and R .

Step 2: We now we assume that A_N holds true. Since $\mathbb{C}(\mathbf{Q}_N, \mathbf{x}_\tau) = \bigcup_{x_a \in \mathbf{x}_\tau(N\mathbf{Q})} \mathbb{B}_{\mathcal{R}_a(x_a)}(x_a)$ and since $\mathbf{x}_\tau(N\mathbf{Q}) \subset \tilde{\mathbf{Q}}_N \subset \mathbb{B}_{N, \beta_0, 0, \mathbf{Q}_N}$ it holds

$$\begin{aligned} \mathbb{P}(A_N \wedge \neg B_N) &\leq \sum_{k=1}^{+\infty} \mathbb{P}\left(\exists x_a \in \left(\mathbb{S}_{N, \beta_0}^{-k+1, \mathbf{Q}_N}\right) \cap \mathbf{x}_\tau(\mathbf{Q}_N) : \mathbb{B}_{\mathcal{R}(x_a)}(x_a) \not\subset \mathbb{B}_{\frac{1}{2^{\beta_0}} M, \beta_0}^{0, \mathbf{Q}_N}\right) \\ &\leq \sum_k \mathbb{P}\left(\exists x_a \in \left(\mathbb{S}_{N, \beta_0}^{-k+1, \mathbf{Q}_N}\right) \cap \mathbf{x}_\tau : \mathbb{B}_{\mathcal{R}(x_a)}(x_a) \not\subset \mathbb{B}_{\frac{1}{2^{\beta_0}} M, \beta_0}^{0, \mathbf{Q}_N}\right) \\ &\leq \sum_k \sum_{x_a \in \mathbb{S}_{N, \beta_0}^{-k+1, \mathbf{Q}_N} \cap \mathbf{x}_\tau} \mathbb{P}(R_a(\mathbf{x}_\tau, (\Phi_i)_i) \geq N^{\beta_0} + k) \\ &\leq C \sum_{k=0}^{NR+1} (NR - k + 1)^d f_{\mathcal{R}}(N^{\beta_0} + k) \\ &\leq CN^d \int_0^\infty f_{\mathcal{R}}(N^{\beta_0} + x) dx \leq CN^{d-\beta_0\beta_{\mathcal{R}}+1}. \end{aligned}$$

If $\beta_{\mathcal{R}} > d + 1$ and $\beta_0 > d + 1$ it holds

$$(N^{d+1-\beta_0} + N^{\beta_0(d+1-\beta_0)}) + N^{d-\beta_0\beta_{\mathcal{R}}+1} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

and the first statement of the lemma almost surely holds due to (3.9).

Step 3: Alternatively we can assume that d_a and S_a are independent with $R_a \leq d_a S_a$. Then

$$\begin{aligned} \mathbb{P}(R_a \geq R) &\leq \int_\tau^\infty \mathbb{P}(d_a \geq D) \int_{\max\{1, R/D\}}^\infty \mathbb{P}(S_a \geq S) dS dD \\ &\leq C \int_\tau^\infty D^{-\beta_0} \int_{\max\{1, R/D\}}^\infty S^{-\beta_{\mathcal{S}}} dS dD \\ &\leq C \left(\int_\tau^R D^{-\beta_0} \int_{R/D}^\infty S^{-\beta_{\mathcal{S}}} dS dD + \int_R^\infty D^{-\beta_0} \int_1^\infty S^{-\beta_{\mathcal{S}}} dS dD \right) \\ &\leq C \left(\int_\tau^R D^{-\beta_0} \left(\frac{R}{D}\right)^{1-\beta_{\mathcal{S}}} dD + R^{1-\beta_0} \right) \leq CR^{1-\beta_0}. \end{aligned}$$

From here we conclude from the first part. □

3.3 An extended covering lemma

For $x \in \mathbf{p}$ let

$$\eta(x) := \min \left\{ \text{dist}(x, \partial \mathbf{p}), \frac{\mathbf{r}}{2} \right\} \quad \text{and} \quad \tilde{\eta} = \frac{1}{4} \eta. \quad (3.10)$$

Then we find the following:

Lemma 3.6. *Let \mathbf{p} be a connected open set which is locally (δ, M) -regular and has inner regularity $\alpha \in [0, 1]$. For $\tau > 0$ let $\mathbf{x}_\tau = (x_k)_{k \in \mathbb{N}}$ be a family of points with a mutual distance of at least 2τ satisfying $\text{dist}(x_k, \partial \mathbf{p}) > \frac{1}{2}\tau$ and let $n \in \mathbb{N}$ and $\partial \mathbf{x} := (p_k)_{k \in \mathbb{N}} \subset \partial \mathbf{p}$ with corresponding $(\tilde{\rho}_k)_{k \in \mathbb{N}} := (\tilde{\rho}_{n,k})_{k \in \mathbb{N}}, (\mathbf{r}_{n,\alpha,k})_{k \in \mathbb{N}} := (\mathbf{r}_{n,\alpha,k})_{k \in \mathbb{N}}$ and $\mathbf{y}_{\partial \mathbf{x}} := (y_k)_{k \in \mathbb{N}} := (y_{n,\alpha,k})_{k \in \mathbb{N}}$ like in Corollary 2.4. Then*

there exists a family of points $\hat{\mathbf{x}} = (\hat{p}_j)_{j \in \mathbb{N}} \subset \mathbf{p}$ with $\mathbf{x}_\tau \subset \hat{\mathbf{x}}$ such that with $\tilde{\eta}_k := \tilde{\eta}(\hat{p}_k)$, $\hat{B}_k := \mathbb{B}_{\tilde{\eta}_k}(\hat{p}_k)$ and $B_k := \mathbb{B}_{\tilde{\rho}_k}(p_k)$ the family $(B_k)_{k \in \mathbb{N}} \cup (\hat{B}_k)_{k \in \mathbb{N}}$ covers \mathbf{p} and

$$\hat{B}_k \cap \hat{B}_i \neq \emptyset \Rightarrow \begin{cases} \frac{1}{2} \tilde{\eta}_i \leq \tilde{\eta}_k \leq 2\tilde{\eta}_i \\ \text{and } 3 \min \{ \tilde{\eta}_i, \tilde{\eta}_k \} \geq |\hat{p}_i - \hat{p}_k| \geq \frac{1}{2} \max \{ \tilde{\eta}_i, \tilde{\eta}_k \}. \end{cases} \quad (3.11)$$

Furthermore, $B_k \cap \hat{B}_j \neq \emptyset$ implies

$$\frac{1}{4} \tilde{\rho}_k \leq \tilde{\eta}_j \leq \frac{1}{3} \tilde{\rho}_k, \quad 4\tilde{\eta}_j \leq |\hat{p}_j - p_k| \leq \frac{4}{3} \tilde{\rho}_k, \quad (3.12)$$

i.e. $\mathbb{B}_{\tau_k}(y_k) \cap \mathbb{B}_{\frac{1}{8}\tilde{\eta}_j}(\hat{p}_j) = \emptyset$ and $x \in \hat{B}_i$ for some i implies

$$\forall p \in \partial \mathbf{p}: \quad \text{dist}(x, \partial \mathbf{p}) > \frac{4}{5} \tilde{\rho}_n(p). \quad (3.13)$$

Finally, there exists $C > 0$ such that for every $x \in \mathbf{p}$

$$\# \{j \in \mathbb{N}: x \in \mathbb{B}_{\tilde{\eta}_j}(\hat{p}_j)\} + \# \{k \in \mathbb{N}: x \in \mathbb{B}_{\tilde{\rho}_k}(p_k)\} \leq C. \quad (3.14)$$

Proof of Lemma 3.6. We recall $\tilde{\rho}_k := \tilde{\rho}(p_k) := 2^{-5} \rho(p_k)$ and $\tau_k = \frac{\tilde{\rho}_k}{32(1+M_k)}$ and that (2.6) holds. Furthermore, $\mathbb{B}_{\tau_k}(y_k) \subset \mathbb{B}_{\tilde{\rho}_k/8}(p_k) \cap \mathbf{p}$ and hence $\mathbb{B}_{\tau_k}(y_k) \cap \mathbb{B}_{\tau_j}(y_j) = \emptyset$ for $k \neq j$.

If we define $\mathbf{p}_B := \overline{\mathbf{p} \setminus \bigcup_k B_k}$ and observe that \mathbf{p}_B is η -regular (for η defined in (3.10)). Then Lemma 2.2 and Theorem 2.3 yield a cover of \mathbf{p}_B by a locally finite family of balls $\hat{B}_k = \mathbb{B}_{\tilde{\eta}_k}(\hat{p}_k)$, where $(\hat{p}_k)_{k \in \mathbb{N}} \subset \mathbf{p}_B$, and where (3.11) holds. Looking into the proof of Theorem 2.3 we can assume w.l.o.g. that $(x_k)_{k \in \mathbb{N}} \subset (\hat{p}_k)_{k \in \mathbb{N}}$ by suitably bounding η .

Furthermore, we find for $B_k \cap \hat{B}_j \neq \emptyset$ that on one hand

$$\tilde{\eta}_j + \tilde{\rho}_k \geq |\hat{p}_j - p_k| \geq 4\tilde{\eta}_j \quad \Rightarrow \quad \tilde{\eta}_j \leq \frac{1}{3} \tilde{\rho}_k \quad \text{and} \quad |\hat{p}_j - p_k| \leq \frac{4}{3} \tilde{\rho}_k.$$

On the other hand $\hat{p}_j \notin B_k$ by construction of $(\hat{B}_i)_{i \in \mathbb{N}}$. Hence $\tilde{\eta}_j \geq \frac{1}{4} \tilde{\rho}_k$. Finally, $\mathbb{B}_{\tau_k}(y_k) \cap \mathbb{B}_{\frac{1}{8}\tilde{\eta}_j}(\hat{p}_j) = \emptyset$ follows from $\tilde{\rho}_k \leq 4\tilde{\eta}_j \leq |\hat{p}_j - p_k|$.

If $x \in \hat{B}_i$ let $p_x \in \partial \mathbf{p}$ with $|p_x - x| = \text{dist}(x, \partial \mathbf{p})$ and chose some p_k with $p_x \in B_k$. Then the above implies

$$|p_x - x| = \text{dist}(x, \partial \mathbf{p}) > 3\tilde{\eta}_i > \frac{3}{4} \tilde{\rho}_k > \frac{4}{5} \tilde{\rho}_n(p_x).$$

To see (3.14) let $x \in \mathbf{p}$ and let \hat{p}_j such that $\tilde{\eta}_j$ is maximal among all \hat{B}_j with $x \in \hat{B}_j$. Let \hat{p}_i with $x \in \hat{B}_i \cap \hat{B}_j$ and observe that both $|\hat{p}_i - \hat{p}_j|$ and $\tilde{\eta}_i$ are bounded from below and above by a multiple of $\tilde{\eta}_j$. If $x \in \hat{B}_i \cap \hat{B}_k \cap \hat{B}_j$, $|\hat{p}_i - \hat{p}_k|$ is bounded from above and below by $\tilde{\eta}_i$, hence by $\tilde{\eta}_j$. This provides a uniform bound on $\# \{j \in \mathbb{N}: x \in \mathbb{B}_{\tilde{\eta}_j}(\hat{p}_j)\}$. The second part of (3.14) follows in an analogue way. \square

3.4 Set-paths

Lemma 3.7. *There exists a constant $C > 0$ such that the following holds:*

Let \mathbf{p} be a connected open set which is locally Lipschitz regular and has inner regularity $\alpha \in [0, 1]$ and extension order $n \in \mathbb{N} \cup \{0\}$. For $\tau > 0$ let $\mathbf{x}_\tau = (x_k)_{k \in \mathbb{N}}$ be a family of points with a mutual distance of at least 2τ satisfying $\text{dist}(x_k, \partial \mathbf{p}) > \frac{1}{2}\tau$ and $\partial \mathbf{x} := (p_k)_{k \in \mathbb{N}} \subset \partial \mathbf{p}$ with corresponding $(\tilde{\rho}_k)_{k \in \mathbb{N}} := (\tilde{\rho}_{n,k})_{k \in \mathbb{N}}$, $(\mathbf{r}_{n,\alpha,k})_{k \in \mathbb{N}} := (\mathbf{r}_{n,\alpha,k})_{k \in \mathbb{N}}$ and $\mathbf{y}_{\partial \mathbf{x}} := (y_k)_{k \in \mathbb{N}} := (y_{n,\alpha,k})_{k \in \mathbb{N}}$ like in Corollary 2.4.

If $x \in \mathbf{x}_\tau$ with $b_x := \mathbb{B}_{\frac{\tau}{64}}(x)$ and either $y \in \mathbf{y}_{\partial \mathbf{x}} \cap \mathbb{B}_{4\delta}(x)$ with $b_y = \mathbb{B}_{\frac{1}{8}\tilde{\eta}(y)}(y)$ or $y \in \mathbf{x}_\tau \cap \mathbb{B}_{4\delta}(x)$ with $b_y = \mathbb{B}_{\frac{1}{64}\tau}(y)$ then there exists an open set $\gamma(x, y) \subset (\mathbf{p} \cap \mathbb{B}_{\mathcal{R}(x)}(x))$ with $b_x \cup b_y \subset \gamma(x, y)$ and such that for C independent of $u \in L^1_{\text{loc}}(\mathbf{p})$, x , y and \mathbf{p}

$$\left| \int_{b_x} u - \int_{b_y} u \right| \leq C \int_{\gamma(x,y)} \mathfrak{z} |\nabla u| \quad (3.15)$$

where

$$\mathfrak{z}(\xi) := \chi_{\mathbf{p}_{3\delta/8}}(\xi) (\tilde{\rho}_n)_{[\frac{3}{8}\delta], \mathbb{R}^d}^{1-d}(\xi) M_{[3\tilde{\rho}_n, \frac{3}{8}\delta], \mathbb{R}^d}^{\alpha(d-1)}(\xi) M_{[\frac{1}{8}\delta, \frac{3}{8}\delta], \mathbb{R}^d}^{n(d-1)}(\xi) + \chi_{\mathbb{R}^d \setminus \mathbf{p}_{\frac{1}{5}\tilde{\rho}_n}}(\xi) \text{dist}(\xi, \partial \mathbf{p})^{d-1}. \quad (3.16)$$

Proof. We cover \mathbf{p} by a set of balls given by Lemma 3.6 and write for simplicity $\tilde{\rho} = \tilde{\rho}_n$. Given $x \in \mathbf{x}_\tau$ and $y \in \mathbf{y}_{\partial \mathbf{x}} \cup \mathbf{x}_\tau \cap \mathbb{B}_{4\delta}(x)$ let then $\gamma : [0, 1] \rightarrow \mathbf{p} \cap \mathbb{B}_{\mathcal{R}(x) - \frac{\tau}{2}}(x)$ be a continuous path with $\gamma(0) = x$ and $\gamma(1) = y$.

Step 1: We chose a finite sequence of points $(Y_i)_i$ as a discrete equivalent of γ using the following algorithm:

- 1 Set $Y_0 := x$ and $b_0 := \mathbb{B}_{\frac{1}{4}\tilde{\eta}(x)}(x) = \mathbb{B}_{\frac{\tau}{8}}(x)$, $t_0 = 0$.
- 2 For $i \in \mathbb{N} \cup \{0\}$: If $\gamma(t) \in b_i$ for every $t > t_i$ cancel loop. Otherwise define

$$t_{i+1} := \sup \{T > t_0 : \forall t \in (t_0, T) : \gamma(t) \in b_i\}$$

and chose $\varepsilon > 0$ and

- either $Y_{i+1} \in \partial \mathbf{x}$ with $b_{i+1} = \mathbb{B}_{\tilde{\rho}(Y_{i+1})}(Y_{i+1})$
 - or $Y_{i+1} \in \mathring{\mathbf{x}}$ with $b_{i+1} = \mathbb{B}_{\tilde{\eta}(Y_{i+1})}(Y_{i+1})$
- such that it holds $\gamma(t_{i+1}) \in b_{i+1}$.

We have thus constructed a sequence of points $(Y_i)_{i=0, \dots, I}$ with $Y_0 = x$ and $y \in b_I$. Furthermore, it holds $b_i \cap b_{i+1} \neq \emptyset$ for every $i \in \{0, \dots, I-1\}$ and $\gamma([0, 1]) \subset \cup_i b_i$.

Step 2: For two points $\hat{p}_1, \hat{p}_2 \in \mathring{\mathbf{x}}$ with $\tilde{\eta}_i := \tilde{\eta}(p_i)$ and $\mathbb{B}_{\tilde{\eta}_2}(\hat{p}_2) \cap \mathbb{B}_{\tilde{\eta}_1}(\hat{p}_1) \neq \emptyset$ and $\eta_1 > \eta_2$ we find due to (3.11) that $\mathbb{B}_{\frac{1}{8}\tilde{\eta}_2}(\hat{p}_2) \subset \mathbb{B}_{\tilde{\eta}_1}(\hat{p}_1)$. Hence for the convex hull holds $\text{conv}(\mathbb{B}_{\frac{1}{8}\tilde{\eta}_2}(\hat{p}_2) \cup \mathbb{B}_{\frac{1}{8}\tilde{\eta}_1}(\hat{p}_1)) \subset \mathbb{B}_{\tilde{\eta}_1}(\hat{p}_1)$ and according to (3.11) together with Lemma 2.7 we find

$$\left| \mathcal{M}_{\frac{1}{8}\tilde{\eta}_2}^{\hat{p}_2} u - \mathcal{M}_{\frac{1}{8}\tilde{\eta}_1}^{\hat{p}_1} u \right| \leq C \eta_1^{1-d} \int_{\mathbb{B}_{\tilde{\eta}_1}(\hat{p}_1)} |\nabla u| \quad .$$

We define $\tilde{\gamma}(\hat{p}_1, \hat{p}_2) = \tilde{\gamma}(\hat{p}_2, \hat{p}_1) := \mathbb{B}_{\tilde{\eta}_1}(\hat{p}_1)$.

Let $p_1, p_2 \in \mathbf{x}_\partial$, with $\tilde{\rho}_i := \tilde{\rho}(p_i)$ and $\mathbb{B}_{\tilde{\rho}_2}(p_2) \cap \mathbb{B}_{\tilde{\rho}_1}(p_1) \neq \emptyset$. We find for \mathfrak{r}_i and y_i given by Corollary 2.4 w.l.o.g. $\mathbb{B}_{\mathfrak{r}_2}(y_2) \subset \mathbb{B}_{3\tilde{\rho}_1}(p_1)$ and $\mathfrak{r}_1 < \mathfrak{r}_2$. Furthermore, there exists a connected set $\tilde{\gamma}(y_1, y_2)$ consisting of $\mathbb{B}_{\mathfrak{r}_2}(y_2)$ and of two cylinders inside $\mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\delta(p_1)}(p_1)$ of radius \mathfrak{r}_1 and length smaller than $\tilde{\rho}(p_1) (1+M)^n(p_1)$ such that $\mathbb{B}_{\mathfrak{r}_1}(y_1) \subset \tilde{\gamma}(y_1, y_2)$ and $\mathbb{B}_{\mathfrak{r}_2}(y_2) \subset \tilde{\gamma}(y_1, y_2)$. Together this implies with Lemma 2.7

$$|\mathcal{M}_{y_2}^{\mathfrak{r}_2} u - \mathcal{M}_{y_1}^{\mathfrak{r}_1} u| \leq C M_{3\tilde{\rho}_1}^{\alpha(d-1)}(p_1) \tilde{\rho}_1^{1-d} \int_{\mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\delta(p_1)}(p_1)} |\nabla u| .$$

We define $\tilde{\gamma}(p_1, p_2) = \tilde{\gamma}(p_2, p_1) := \mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\delta(p_1)}(p_1)$.

Let $p_1 \in \mathbf{x}_\partial$, $\hat{p}_2 \in \dot{\mathbf{x}}$ with $\tilde{\rho}_1 := \tilde{\rho}(p_1)$, $\tilde{\eta}_2 := \tilde{\eta}(p_2)$ and $\mathbb{B}_{\tilde{\eta}_2}(\hat{p}_2) \cap \mathbb{B}_{\tilde{\rho}_1}(p_1) \neq \emptyset$. According to (3.12) we find $\mathbb{B}_{\frac{1}{8}\tilde{\eta}_2}(\hat{p}_2) \subset \mathbb{B}_{2\tilde{\rho}_1}(p_1)$ and from here we conclude similar to the previous case

$$\left| \mathcal{M}_{\hat{p}_2}^{\frac{1}{8}\tilde{\eta}_2} u - \mathcal{M}_{y_1}^{\mathfrak{r}_1} u \right| \leq C M_{3\tilde{\rho}_1}^{\alpha(d-1)}(p_1) \tilde{\rho}_1^{1-d} \int_{\mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\delta(p_1)}(p_1)} |\nabla u| .$$

We define $\tilde{\gamma}(p_1, \hat{p}_2) = \tilde{\gamma}(\hat{p}_2, p_1) := \mathbf{p} \cap \mathbb{B}_{\frac{1}{8}\delta(p_1)}(p_1)$.

Step 3: Let $(Y_i)_{i=0, \dots, I}$ be the sequence of points constructed in Step 1 and we assume w.l.o.g. that every point appears only once in the sequence (otherwise the path may be shortened). Let $\gamma(x, y) := \bigcup_{i=0}^{I-1} \tilde{\gamma}(Y_i, Y_{i+1})$. Then $\gamma([0, 1]) \subset \gamma(x, y)$ and by Step 2, the total bound on the number of local overlaps (3.14) of \mathbb{B}_{η_i} and estimate (3.4) on the local bound on the number of overlapping $\mathbb{B}_{\delta_i}(p_i)$, the condition (3.13), Remark 2.5 and the triangle inequality we find $C > 0$ such that (3.15)–(3.16) holds. \square

3.5 Proof of Theorem 1.6

Proof. Throughout the proof, $C > 0$ is a varying constant depending on $s, r, q, \tilde{q}, \mathfrak{r}, d, \mathbf{Q}$ but not on \mathbf{p} or N .

Step 1: For simplicity of notation, set $N = 1$ during Steps 1 and 2 but keep in mind that the constant C below does not depend on \mathbf{Q} unless this is state explicitly. In view of Theorem 3.1 it remains to derive estimates on the terms

$$I_1 := \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \setminus \mathbf{p}} \left| \sum_a \Phi_a \sum_{i \neq 0} \rho_{n,i}^{-1} \phi_i (\tau_{n,\alpha,i}^s u - \mathcal{M}_a^s u) \right|^r , \quad (3.17)$$

$$I_{2,l} := \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left| \sum_{a: \partial_l \Phi_a > 0} \sum_{b: \partial_l \Phi_b < 0} \frac{\partial_l \Phi_a |\partial_l \Phi_b|}{D_{l+}^\Phi} (\mathcal{M}_a^s u - \mathcal{M}_b^s u) \right|^r , \quad (3.18)$$

in terms of $C(\mathbf{Q}, \mathbf{p}) \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{p}(\omega) \cap \mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} |\nabla u|^p \right)^{\frac{r}{p}}$.

Denoting $c_i := \rho_{n,i}^{-1}$ and $\tilde{c} = \rho_{n, [\tilde{\rho}_n]}^{-1}$ observe $c_i \leq \tilde{c}$ and apply Lemma 3.7 and Jensens inequality:

$$\begin{aligned} I_1 &\leq \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \setminus \mathbf{p}} \left| \sum_a \Phi_a \sum_{i \neq 0} c_i \phi_i \int_{\gamma(x_a, y_i)} \mathfrak{z} |\nabla u| \right|^r \\ &\leq \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q} \setminus \mathbf{p}} dx \int_{\gamma(x_a, y_i)} dy \sum_a \Phi_a(x) \sum_{i \neq 0} |\gamma(x_a, y_i)|^{r-1} \tilde{c}^r(x) \phi_i(x) \mathfrak{z}^r(y) |\nabla u|^r(y) . \end{aligned}$$

We write $\mathbb{B}_a := \mathbb{B}_{\mathcal{R}(x_a)}(x_a)$ and make use of $\Phi_a \phi_i |\gamma(x_a, y_i)|^{r-1} \leq \Phi_a \phi_i |\mathbb{B}_a|^{r-1}$, $\gamma(x_a, y_i) \subset \mathbb{B}_a$ and $\sum_{i \neq 0} \phi_i \leq 1$ to find for $s \in (r, p)$ from Hölder's inequality

$$\begin{aligned} I_1 &\leq C \sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{1}{|\mathbf{Q}|} \left(\int_{\mathbb{B}_a} \mathfrak{d}_a^d \mathfrak{z}^r(y) |\mathbb{B}_a|^{r-1} |\nabla u|^r(y) dy \right) \left(\int_{\mathbf{Q} \setminus \mathbf{P}} \Phi_a \tilde{c}^r \right) \\ &\leq C \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{1}{|\mathbf{Q}|} \left(\int_{\mathbb{B}_a} \mathfrak{d}_a^{d\frac{r}{s}} |\mathbb{B}_a|^{r-1} \mathfrak{z}^r(y) |\nabla u|^r(y) dy \right)^{\frac{s}{r}} \right)^{\frac{r}{s}} \\ &\quad \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{1}{|\mathbf{Q}|} \left(\frac{1}{\mathfrak{d}_a^d} \int_{\mathbf{Q} \setminus \mathbf{P}} \mathfrak{d}_a^{d\frac{s-r}{s}} \Phi_a \tilde{c}^r \right)^{\frac{s-r}{s}} \right)^{\frac{s}{s-r}} \end{aligned} \tag{3.19}$$

From Jensen's inequality and the fact that $|\text{supp} \Phi_a| \leq \mathfrak{d}_a^d$ and $\sum \Phi_a^{\frac{s}{s-r}} \leq 1$ we find

$$\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{1}{|\mathbf{Q}|} \left(\frac{1}{\mathfrak{d}_a^d} \int_{\mathbf{Q} \setminus \mathbf{P}} \mathfrak{d}_a^{d\frac{s-r}{s}} \Phi_a \tilde{c}^r \right)^{\frac{s}{s-r}} \leq \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}(\mathbf{x}_\tau) \setminus \mathbf{P}} \tilde{c}^{\frac{rs}{s-r}}. \tag{3.20}$$

Next, we simplify the notation and write $f_{\mathbf{C}} f := \frac{1}{|\mathbf{Q}|} \int_{\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)} f$. For q and \tilde{q} with $\frac{s}{p} + \frac{1}{q} + \frac{s}{r\tilde{q}} = 1$ it then holds

$$\begin{aligned} &\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{1}{|\mathbf{Q}|} \left(\int_{\mathbb{B}_a} \mathfrak{d}_a^{d\frac{r}{s}} |\mathbb{B}_a|^r \mathfrak{z}^r(y) |\nabla u|^r(y) dy \right)^{\frac{s}{r}} \\ &\leq C \frac{1}{|\mathbf{Q}|} \sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \int_{\mathbb{B}_a} \mathfrak{d}_a^d |\mathbb{B}_a|^{\frac{s(r+1)-r}{r}} \mathfrak{z}^s(y) |\nabla u|^s(y) dy^{\frac{r}{s}} \\ &\leq C \left(f_{\mathbf{C}} \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \chi_{\mathbb{B}_a} \mathfrak{d}_a^d |\mathbb{B}_a|^{\frac{s(r+1)-r}{r}} \right)^q \right)^{\frac{1}{q}} \left(f_{\mathbf{C}} \mathfrak{z}^{r\tilde{q}} \right)^{\frac{s}{r\tilde{q}}} \left(f_{\mathbf{C}} |\nabla u|^p \right)^{\frac{r}{p}} \end{aligned} \tag{3.21}$$

Now define $\tilde{\Phi}_{a,l} := \frac{\partial_l \Phi_a}{D_{l+}^\Phi}$. Since the number of cells interacting with the support of Φ_a is limited by $(4\mathfrak{d}(x_a)\mathfrak{r}^{-1})^2$ and since (3.1) holds we observe $D_{l+}^\Phi \leq \sum_a \mathfrak{d}(x_a)^{2d} \chi_{G_a}(x)$. Hence by a similar calculation to the estimate of I_1

$$\begin{aligned} I_{2,l} &\leq \frac{C}{|\mathbf{Q}|} \int_{\mathbf{Q} \setminus \mathbf{P}} \left| \sum_a \tilde{\Phi}_{a,l} \sum_b D_{l+}^\Phi \tilde{\Phi}_{b,l} \int_{\gamma(x_a, x_b)} \mathfrak{z} |\nabla u| \right|^r \\ &\leq \frac{C}{|\mathbf{Q}|} \int_{\mathbf{Q} \setminus \mathbf{P}} dx \int_{\gamma(x_a, x_b)} dy \sum_a \Phi_a(x) \mathfrak{d}(x_a)^{2rd} \sum_b |\gamma(x_a, x_b)|^{r-1} \tilde{\Phi}_{b,l}(x) \mathfrak{z}^r(y) |\nabla u|^r(y) \end{aligned}$$

We make use of $\Phi_a \tilde{\Phi}_{b,l} |\gamma(x_a, x_b)|^r \leq \Phi_a \tilde{\Phi}_{b,l} |\mathbb{B}_a|^r$, $\gamma(x_a, x_b) \subset \mathbb{B}_a$ and $\sum_b \tilde{\Phi}_{b,l} \leq 1$ as well as the definition of $\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)$ to find that

$$\begin{aligned} I_{2,l} &\leq \sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \frac{C}{|\mathbf{Q}|} \left(\int_{\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)} \chi_{\mathbb{B}_a} \mathfrak{d}_a^{3rd} |\mathbb{B}_a|^{r-1} \mathfrak{z}^r(y) |\nabla u|^r(y) dy \right) \\ &\leq C \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)} \left(\sum_{x_a} \chi_{\mathbb{B}_a} \mathfrak{d}_a^{3rd} |\mathbb{B}_a|^{r-1} \right)^q \right)^{\frac{1}{q}} \\ &\quad \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)} \mathfrak{z}^{r\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{C}(\mathbf{Q}, \mathbf{x}_\tau)} |\nabla u|^p \right)^{\frac{r}{p}} \end{aligned} \tag{3.22}$$

Step 2: We continue deriving an estimate on $\frac{1}{|\mathbf{Q}|} \int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \mathfrak{z}^{r\tilde{q}}$ in terms of (δ, M) .

We first observe that

$$\begin{aligned} \int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \mathfrak{z}^{r\tilde{q}} &\leq C \int_{\mathbf{p}_{3\delta/8} \cap \mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} (\tilde{\rho}_n)_{[\frac{3}{8}\delta], \mathbb{R}^d}^{(1-d)r\tilde{q}}(\xi) M_{[3\tilde{\rho}_n, \frac{3}{8}\delta], \mathbb{R}^d}^{\alpha(d-1)r\tilde{q}} M_{[\frac{1}{8}\delta, \frac{3}{8}\delta], \mathbb{R}^d}^{n(d-1)r\tilde{q}}(\xi) \\ &\quad + C \int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau) \setminus \mathbf{p}_{\frac{4}{5}\tilde{\rho}_n}} (\text{dist}(\xi, \partial\mathbf{p})^{1-d})^{r\tilde{q}} \end{aligned} \tag{3.23}$$

Since the first integral on the right hand side can be estimated using Lemma 2.6, we focus on the second integral. Because of Lemma 2.2 it holds for the support

$$\mathbf{p}_{\frac{4}{5}\tilde{\rho}_n} \supset \mathbf{p} \cap \bigcup_k \mathbb{B}_k, \quad \text{where} \quad \mathbb{B}_k := \mathbb{B}_{\frac{1}{2}\tilde{\rho}_n}(p_k)$$

for the family of points p_k given by Corollary 2.4 resp. Lemma 3.6. Using that the covering with \mathbb{B}_k is absolutely locally bounded it holds

$$\begin{aligned} \int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \chi_{\mathbb{R}^d \setminus \mathbf{p}_{\frac{4}{5}\tilde{\rho}_n}}(\xi) (\text{dist}(\xi, \partial\mathbf{p})^{1-d})^{r\tilde{q}} d\xi \\ \leq C_q \left(\int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \mathfrak{r}^{r\tilde{q}(1-d)} + \sum_k \int_{\mathbf{p} \cap (\mathbb{B}_\tau(p_k) \setminus \mathbb{B}_k)} (\text{dist}(\xi, \partial\mathbf{p})^{1-d})^{r\tilde{q}} \right), \end{aligned}$$

and using

$$\begin{aligned} \int_{\mathbf{p} \cap (\mathbb{B}_\tau(p_k) \setminus \mathbb{B}_k)} (\text{dist}(\xi, \partial\mathbf{p})^{1-d})^{r\tilde{q}} &\leq C \int_{\frac{1}{2}\tilde{\rho}_n(p_k)}^\tau r^{(1-d)r\tilde{q}} r^{d-1} dr \\ &\leq C_q \tilde{\rho}_n(p_k)^{(1-d)(r\tilde{q}-1)+1} \\ &\leq C_q \tilde{\rho}_n(p_k)^{(1-d)(r\tilde{q}-1)+1+d} \tilde{M}_{\tilde{\rho}_n}^{\alpha d}(p_k) |\mathbb{B}_{\tau_k}(y_k)| \\ &\leq C_q \int_{\mathbf{p} \cap \mathbb{B}_k} \tilde{\rho}_n(p_k)^{(1-d)(r\tilde{q}-1)+1+d} \tilde{M}_{\tilde{\rho}_n}^{\alpha d}(p_k) \end{aligned}$$

we find

$$\begin{aligned} \int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \chi_{\mathbb{R}^d \setminus \mathbf{p}_{\frac{4}{5}\tilde{\rho}_n}}(\xi) (\text{dist}(\xi, \partial\mathbf{p})^{1-d})^{r\tilde{q}} d\xi \\ \leq C_q \left(\int_{\mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \mathfrak{r}^{r\tilde{q}(1-d)} + \int_{\mathbf{p} \cap \mathbb{C}(\mathbf{Q}, \mathbf{x}_\tau)} \tilde{\rho}_{n, [\tilde{\rho}_n]}^{(1-d)(r\tilde{q}-1)+1+d} \tilde{M}_{[\tilde{\rho}_n, \tilde{\rho}_n]}^{\alpha d} \right) \end{aligned} \tag{3.24}$$

Step 3: Let now $N > 1$, i.e. replace \mathbf{Q} by $N\mathbf{Q}$ in the above calculations. We observe from Lemma 3.4 for sufficiently large N_0 and every $N > N_0$ that

$$\mathbb{C}(N\mathbf{Q}, \mathbf{x}_\tau) \subset \mathbb{B}_{N\beta_0}(N\mathbf{Q}) \subset 2N\mathbf{Q}. \tag{3.25}$$

Given Theorem 3.1, the definition of I_1 and $I_{2,l}$ as well as (3.19)–(3.24) we find

$$\frac{1}{|N\mathbf{Q}|} \int_{N\mathbf{Q}} |\nabla \mathcal{U}u|^r \leq C_0 (C_{1,N} + C_{2,N}(C_{00} + C_{3,N})) \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbf{p}(\omega) \cap \mathbb{B}_{N\beta_0}(N\mathbf{Q})} |\nabla u|^p \right)^{\frac{r}{p}}$$

where the finite positive constants C_0, C_{00} depend only on r, s, p and q, \tilde{q} as well as d, τ and \mathbf{Q} but not on N and where

$$C_{1,N} = \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbb{B}_\tau(N\mathbf{Q})} f_{\alpha,n} \right)^{\frac{p}{p-r}}, \quad C_{2,N} = \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbf{p} \cap 2N\mathbf{Q}} f_{\text{mes}} \right)^{\frac{1}{q}},$$

$$C_{3,N} = \left(\frac{1}{|N\mathbf{Q}|} \int_{\mathbf{p} \cap 2N\mathbf{Q}} f_{\text{mic}} \right)^{\frac{1}{\tilde{q}}}$$

with $f_{\alpha,n}$ given by Theorem 3.1 and

$$f_{\text{mes}} := \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \chi_{\mathbb{B}_a} \mathfrak{d}_a^d |\mathbb{B}_a|^{\frac{s(r+1)-r}{r}} \right)^q + \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \chi_{\mathbb{B}_a} \mathfrak{d}_a^{3rd} |\mathbb{B}_a|^{r-1} \right)^q,$$

$$f_{\text{mic}} := \tilde{\rho}_{n, [\tilde{\rho}_n]}^{(1-d)(r\tilde{q}-1)+1+d} \tilde{M}_{[\tilde{\rho}_n, \tilde{\rho}_n]}^{\alpha d}.$$

It remains to show that $C_{i,N}, i = 1, 2, 3$, are bounded independently from N . Due to the ergodic theorem, this is guaranteed if

$$\lim_{N \rightarrow \infty} C_{1,N} + C_{2,N} + C_{3,N} = \mathbb{E} f_{\alpha,n} + \mathbb{E} f_{\text{mes}} + \mathbb{E} f_{\text{mic}} < \infty. \tag{3.26}$$

Step 4: Using Lemma 2.6 and $M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d} > M_{[\frac{1}{8}\delta], \mathbb{R}^d} > M_{[\tilde{\rho}_n], \mathbb{R}^d}$ as well as $M_{\frac{3\delta}{4}} > M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d}$ on $\partial \mathbf{p}$ we infer

$$C_{1,N}^{\frac{p-r}{p}} \leq \frac{1}{|N\mathbf{Q}|} \int_{\mathbb{B}_\tau(N\mathbf{Q})} \left(1 + M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d} \right)^{\frac{p}{p-r} [(n+\alpha)(d-1)+r]}$$

$$\leq \frac{1}{|N\mathbf{Q}|} \int_{\mathbb{B}_{2\tau}(N\mathbf{Q}) \cap \partial \mathbf{p}} \delta \left(1 + M_{[\frac{3\delta}{8}, \frac{\delta}{8}], \mathbb{R}^d} \right)^{\frac{p}{p-r} [(n+\alpha)(d-1)+r]+d-2}$$

$$\leq \frac{1}{|N\mathbf{Q}|} \int_{\mathbb{B}_{2\tau}(N\mathbf{Q}) \cap \partial \mathbf{p}} \delta \left(1 + M_{\frac{3\delta}{4}} \right)^{\frac{p}{p-r} [(n+\alpha)(d-1)+r]+d-2}$$

Taking the limit $N \rightarrow \infty$ and using the ergodic theorem in its form (2.16) we obtain the condition

$$\lim_{N \rightarrow \infty} C_{1,N}^{\frac{p-r}{p}} \leq \mathbb{E} \left(\delta \left(1 + M_{\frac{3\delta}{4}} \right)^{\frac{p}{p-r} [(n+\alpha)(d-1)+r]+d-2} \right).$$

Similarly we can show that

$$\lim_{N \rightarrow \infty} C_{3,N}^{\tilde{q}} \leq \mathbb{E} \left(\tilde{\rho}_n^{(1-d)(r\tilde{q}-1)+2+d} \tilde{M}_{4\tilde{\rho}_n}^{\alpha d+d-2} \right).$$

Step 5: We observe from the lower bound on ∂ and \mathcal{R} that

$$f_{\text{mes}} \leq \tilde{f} := C \left(\sum_{x_a \in \mathbf{x}_\tau(\mathbf{Q})} \chi_{\mathbb{B}_a} \mathfrak{d}_a^{3dr} |\mathbb{B}_a|^{\frac{s(r+1)-r}{r}} \right)^q$$

Lemma 2.11 now shows that

$$\lim_{N \rightarrow \infty} C_{2,N}^q \leq \mathbb{E} f \leq \mathbb{E} \tilde{f}$$

$$\leq \sum_{k,R=1}^{\infty} (k+1)^{d(q+1)+3drq+r(q-1)} (R+1)^{d(q+1)+\frac{s(r+1)-r}{r}q+r(q-1)} \mathbb{P}_{k,R}.$$

Step 6: Steps 4 and 5 imply (3.26) and the theorem is thus proved in the first case. In the second case, if \mathcal{S} and ∂ are independent, we can proceed in a similar way except that $\mathbb{B}_a := \mathbb{B}_{\mathcal{S}(x_a)\partial(x_a)}(x_a)$ and we use Part I Lemma 3.18 and thus

$$\mathbb{E} \tilde{f} \leq \sum_{k,S=1}^{\infty} (k+1)^{d(q+1)+d(3r+\frac{s(r+1)-r}{r})q+r(q-1)} (S+1)^{d(q+1)+d\frac{s(r+1)-r}{r}q+r(q-1)} \mathbb{P}_{\partial,k} \mathbb{P}_{\mathcal{S},S}.$$

□

References

- [1] D. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer-Verlag New York, 1988.
- [2] R. G. Durán and M. A. Muschietti. The korn inequality for jones domains. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 2004:Paper–No, 2004.
- [3] N. Guillen and I. Kim. Quasistatic droplets in randomly perforated domains. *Archive for Rational Mechanics and Analysis*, 215(1):211–281, 2015.
- [4] M. Heida. An extension of the stochastic two-scale convergence method and application. *Asymptotic Analysis*, 72(1):1–30, 2011.
- [5] M. Heida. Stochastic homogenization on perforated domains i: Extension operators. *WIAS Preprint*, (2849), 2021.
- [6] M. Heida. Stochastic homogenization on perforated domains ii – application to nonlinear elasticity models. *WIAS Preprint*, (2865), 2021.
- [7] M. Höpker. *Extension Operators for Sobolev Spaces on Periodic Domains, Their Applications, and Homogenization of a Phase Field Model for Phase Transitions in Porous Media*. University Bremen, 2016.
- [8] P. W. Jones. Quasiconformal mappings and extendability of functions in sobolev spaces. *Acta Mathematica*, 147(1):71–88, 1981.
- [9] J. Mecke. Stationäre zufällige Maße auf lokalkompakten abelschen Gruppen. *Probability Theory and Related Fields*, 9(1):36–58, 1967.
- [10] A. Piatnitski and M. Ptashnyk. Homogenization of biomechanical models of plant tissues with randomly distributed cells. *Nonlinearity*, 33(10):5510, 2020.
- [11] A. Tempel'man. Ergodic theorems for general dynamical systems. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 26:95–132, 1972.