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On the existence of generalized solutions to a spatio-temporal predator-prey system

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Abstract

In this paper we consider a pair of coupled non-linear partial differential equations describing the interaction of a predator-prey pair. We introduce a concept of generalized solutions and show the existence of such solutions in all space dimension with the aid of a regularizing term, that is motivated by overcrowding phenomena. Additionally, we prove the weak-strong uniqueness of these generalized solutions and the existence of strong solutions at least locally-in-time for space dimension two and three.

Contents

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1 Introduction

In this paper, we consider a pair of coupled partial differential equations modelling the spatio-temporal interaction of a predator-prey pair. For Ω being a bounded $\mathcal{C}^2$-domain in $\mathbb{R}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$ and $T > 0$, we are going to consider the following model

$$\partial_t u - \nu\Delta u + \kappa\nabla\cdot(u\nabla w) = (\alpha w - \beta)u \qquad \text{in } (0,T)\times\Omega, \tag{1a}$$
$$\partial_t w - \mu\Delta w = (\gamma - \delta u)w \qquad \text{in } (0,T)\times\Omega, \tag{1b}$$
$$u = u_0 \qquad \text{on } \{0\}\times\Omega, \tag{1c}$$
$$w = w_0 \qquad \text{on } \{0\}\times\Omega, \tag{1d}$$
$$\nabla u\cdot n = 0 \qquad \text{on } [0,T]\times\partial\Omega, \tag{1e}$$
$$\nabla w\cdot n = 0 \qquad \text{on } [0,T]\times\partial\Omega. \tag{1f}$$

The variable $u : [0,T]\times\Omega \to \mathbb{R}$ represents the number of predators, $w : [0,T]\times\Omega \to \mathbb{R}$ the number of prey and n denotes the outer normal vector of Ω. All other appearing variables are positive constants.

The system considered is inspired by an application in biological pest control. In the production of ornamental plants, as for example roses, it is desirable to reduce the use of chemical pesticides. This can be achieved by releasing natural enemies of the pest involved, which do not have a damaging effect on the plants. A typical example of such a predator-prey pair is the two-spotted spider mite (Tetranychus urticae) and the predatory mite (Phytoseiulus persimilis), see [37] for a detailed discussion of this predator-prey pair.

In order to describe the interaction of these two populations over a bounded domain and on a finite time horizon, we consider a typical Lotka–Volterra system coupled with diffusive movement of both populations over the whole domain, modelling the random movement of the mite populations. The Lotka–Volterra model was first introduced in form of an ordinary differential equation in 1920 by Alfred J. Lotka [29] and in 1926 by Vito Volterra [40] describing the evolution of the number of predators and prey in time. It makes the following assumptions. The prey population is assumed to grow exponentially with rate γ if there are no predators present and decline by predation with a rate proportional to the number of predators $-\delta u$. The predator population is assumed to decline exponentially in the absence of prey with rate $-\beta$ and has a natality rate proportional to the number of prey available αw. Even though the model has some drawbacks, as the lack of capturing saturation effects, it is a good starting point for the investigation of population dynamics, see [33, Ch. 3.1].

The considered model (1) is very close to the model introduced in [8] and identical to it except for the non-linear higher-order coupling term $\kappa\nabla\cdot(u\nabla w)$, we include this cross-diffusion term in order to model the predator's hunting behaviour as some directed movement towards higher concentrations of prey. This so called chemotaxis, or in the case of a predator-prey model also referred to as prey-taxis, was first introduced by Evelyn F. Keller and Lee A. Segel in 1970 and 1971, see [20] and [21], where the movement of one-celled organisms under the influence of some chemical attractant was considered.

Similar models to the one above have been of interest to researchers in the mathematical and numerical analysis of partial differential equations over the last decades [5]. Replacing the prey-taxis coefficient κ by a function $\chi(u)$ dependent on the predator density u and presuming various conditions on this function, the existence of weak or even classical solutions to models similar to (1) is known. Assuming that $\chi(u)$ vanishes for large values of the predator density u and considering a different response function on the right-hand side, classical solutions are known to exists in dimensions $d = 1,2,3$, see [31, 39] and weak solutions exist in all space dimensions [6]. Presuming some smallness condition for the prey-taxis coefficient κ, the existence of classical solution is also proven in all space dimensions in [42].

Renormalized solutions were considered in [41]. Here, the equation constitutive for the solution is a weak formulation for some smooth function of the solution u. In [41] the global existence of these solutions was shown for a chemotaxis model including the non-linear coupling term $\kappa\nabla\cdot(u\nabla w)$ with the prey-taxis coefficient $\kappa = 1$ in all space dimension $d \geq 4$.

Another generalized solution concept for a Keller–Segel model was considered in [24], with a weak formulation for the prey and some weak inequalities for the coupled quantity $u^p w^q$ for some $p, q \in (0,1)$. Here it was still necessary to impose some smallness conditions on the prey-taxis coefficient.

To the best of the authors knowledge model (1) was not yet considered in this generality, with no constraints, apart from the positivity, on the prey-taxis coefficient κ.

Our definition of generalized solution consists of a weak formulation for the prey equation for w and two inequalities for the predator u, see Definition 2.1 below. These two inequalities bear some resemblance to mass conservation (in)equalities and entropy inequalities, as they include the total number of predators $\int_\Omega u \,\mathrm{d}\boldsymbol{x}$ and the term $-\int_\Omega \ln u \,\mathrm{d}\boldsymbol{x}$ commonly associated to the entropy of a physical system. Since reasonable *a priori* estimates seem to be out of reach for the function u itself, we rather formulate the solvability concept for a non-linear function of the predator variable, namely $\ln u$. With this, we follow the landmark paper [12], where renormalized solutions were introduced for the first time in the context of Boltzmann equations. Similar concepts with a similar non-linear transformation of u have been considered for different versions of the Keller–Segel model [23, 11].

Our main motivation to make these inequalities constitutive in our definition of generalized solutions is purely mathematical. With these inequalities we are able to prove a relative energy inequality, an estimate for the so-called relative energy,

$$\mathcal{R}(u, w|\tilde{u}, \tilde{w}) = \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu}\tilde{u}|w - \tilde{w}|^2 \,\mathrm{d}\boldsymbol{x}$$

for u and $\tilde{u}$ two different predator and w and $\tilde{w}$ two different prey populations, which is inspired by the Kullback–Leibler divergence. This distance measure has many names, one of which is relative entropy, and many application areas. In its original form it measures the difference of two probability densities and is commonly used in information theory but also finds its application in biological system as the Lotka–Volterra model, see [2]. The relative energy serves nowadays as a general tool in the analysis of PDEs and is used to consider aside from the weak-strong uniqueness of solutions also long-time behaviour, singular limits, convergence of numerical schemes [4] or comparison with reduced models and even optimal control [25].

The deviation from the solution concept of the commonly used weak solutions allowed us to tackle the higher-order non-linear coupling introduced by the Keller–Segel prey-taxis term in (1). The meaningfulness of this solution concept is further supported by the fact that weak-strong uniqueness holds, which is a consequence of the above mentioned relative energy inequality.

In the numerical simulations, we performed to visualize the influence of the prey-taxis, the bias of the random motion of the predator u modelled by the diffusion towards higher concentration of prey is clearly visible. This suggests that model (1) is suitable to model the populations of a predator-prey pair which includes some hunting behaviour, as it is needed for certain applications in biological pest control.

Plan of the paper: The paper is structured as follows. In the next section we collect the main results. In Section 3, we present some numerical simulations motivating the chosen model and illustrating the effect of the prey-taxis term. In Section 4, we proof the existence of generalized solutions using the special regularization of adding a term modelling overcrowding. The weak-strong uniqueness proof is conducted in Section 5, whereas the existence of strong solutions locally-in-time in dimension two and three is proven in Section 6. The Appendix contains certain technical lemmata.

Notation: Before we begin with the main part of this work, we make some remarks on our notation. By $\Omega \subseteq \mathbb{R}^d$, we denote a bounded $\mathcal{C}^2$-domain with $d \geq 2$. The variable $T \in (0,\infty)$ denotes the finite time horizon. By n, we denote the outer-normal vector of the domain Ω. For any Banach space V we denote the dual pairing between V^* and V by $\langle \cdot,\cdot\rangle_V$. In the remainder of this paper we will drop the subscript V for the sake of readability as it will be clear from the context which space is meant. Additionally, we will sometimes use the shorthand notation $L^r(V)$ for the Bochner space $L^r(0,T;V)$. Furthermore, we denote the space of abstract functions of bounded variation with values in V by $\mathrm{BV}([0,T];V)$. The space of weakly continuous functions with values in V is denoted by $C_w([0,T];V)$ and the space of V-valued regular measures on $[0,T]$ by $\mathcal{M}(0,T;V)$, see for example [10] for an introduction. We generally use $C>0$ for constant upper bounds, where the exact value of C may change throughout a calculation without this being indicated in the notation.

2 Main results

We start off by defining the appropriate spaces for our solutions. We first define the regularity space of the solution $\mathcal{X}$. We say $(u,w) \in \mathcal{X}$ if

$$\begin{aligned}
u &\in L^\infty(0,T;L^1(\Omega)),\\
\ln u &\in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,p}(\Omega)^*),\\
u &> 0 \text{ a.e. in } (0,T)\times\Omega,\\
w &\in L^\infty((0,T)\times\Omega) \cap L^2(0,T;W^{1,2}(\Omega)),\\
\partial_t w &\in L^2(0,T;W^{1,2}(\Omega)^* + L^1(\Omega)),\\
w &\geq 0,
\end{aligned}$$

where $p>d$ and the sum $X = X_1 + X_0$ of two Banach spaces X_0 and X_1 continuously embedded into a Hausdorff topological vector space $\mathcal{H}$ is the set of all elements $x \in \mathcal{H}$ such that there are $x_0 \in X_0$ and $x_1 \in X_1$ with $x = x_0 + x_1$, see [7, p. 97]. The here given sum of Banach spaces is well-defined since both $L^1(\Omega)$ and $W^{1,2}(\Omega)^*$ are continuously embedded into the space of distributions $\mathcal{D}'(\Omega)$. We now define the generalized solutions as follows.

Definition 2.1 (Generalized solution)**.** *We say $(u,w) \in \mathcal{X}$ is a generalized solution to (1) if the population inequality for the predator*

$$\int_\Omega u \,\mathrm{d}\boldsymbol{x}\Big|_0^t + \beta \int_0^t \int_\Omega u \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leq \alpha \int_0^t \int_\Omega w u \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \tag{2}$$

and the logarithmic inequality for the predator

$$-\int_\Omega \ln u \vartheta \,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t \int_\Omega \nu|\nabla \ln u|^2\vartheta - \nu\nabla\ln u\cdot\nabla\vartheta - \kappa\vartheta\nabla w\cdot\nabla\ln u + \kappa\nabla w\cdot\nabla\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ \leq \int_0^t\int_\Omega (\beta - \alpha w)\vartheta - \ln u\,\partial_t\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \tag{3}$$

hold for all $\vartheta \in C^1([0,T];L^\infty(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ *non-negative and all* $t \in [0,T]$. *Additionally, the prey equation is fulfilled in the weak sense, that is*

$$\int_\Omega w\varphi \,\mathrm{d}\boldsymbol{x}\Big|_0^t - \int_0^t \int_\Omega w\partial_t\varphi - \mu\nabla w \cdot \nabla\varphi - \delta uw\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \int_0^t \int_\Omega \gamma w\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \tag{4}$$

holds for all $\varphi \in C^1([0,T];L^\infty(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ *and all* $t \in [0,T]$. *Moreover, the initial conditions for the prey* w *are fulfilled in* $L^2(\Omega)$ *and the initial conditions for the predator* u *are fulfilled in* $L^1(\Omega)$.

Remark 2.2. *The energy inequality* (2) *is formally derived by testing the predator equation* (1a) *with the test function* $\varphi \equiv 1$ *and relaxing the equality to an inequality. The logarithmic inequality* (3) *is formally derived by testing the predator equation* (1a) *with* $\varphi = -\frac{\vartheta}{u}$ *and again relaxing the equality to an inequality.*

Remark 2.3. *The values of the solution* (u,w) *at time zero are well-defined in the given spaces as along as the initial conditions* (u_0,w_0) *live in the appropriate spaces, i.e.* $\ln u_0, u_0 \in L^1(\Omega)$ *and* $w_0 \in L^2(\Omega)$, *since by the definition of* $\mathcal{X}$ *we have* $\ln u \in \mathrm{BV}([0,T];W^{1,p}(\Omega)^*)$ *and*

$$w \in L^\infty(0,T;L^2(\Omega)) \cap C_w([0,T];W^{1,p}(\Omega)^*) \hookrightarrow C_w([0,T];L^2(\Omega)).$$

The main result of this work is the proof, that such solutions exist under certain assumptions on the initial data, which are formulated in the following theorem.

Theorem 2.4 (Existence of generalized solution). *Let* $\Omega \subseteq \mathbb{R}^d$ *be a smooth and bounded domain with* $d \in \mathbb{N}, d \geq 2$. *Additionally assume* $u_0 \in L^1(\Omega)$ *with* $u_0 > 0$ *almost everywhere in* Ω *as well as* $\ln u_0 \in L^1(\Omega)$ *and* $w_0 \in L^\infty(\Omega)$ *with* $w_0 \geq 0$ *almost everywhere in* Ω. *Then there exists a generalized solution* $(u,w) \in \mathcal{X}$ *to* (1) *in the sense of Definition 2.1.*

To see that our generalized solution concept is meaningful, we show weak-strong uniqueness. To do that we need the notion of a strong solution. We define strong solutions in the following way.

Definition 2.5 (Strong solution). *We call the pair* $(\tilde{u},\tilde{w})$ *strong solution to the system* (1) *on* $[0,\tilde{T}]$ *for some* $\tilde{T} > 0$ *to the initial data* $u_0, w_0 \in C^3(\overline{\Omega})$ *non-negative, if*

$$\tilde{u},\tilde{w} \in C^1([0,\tilde{T}] \times \overline{\Omega}) \text{ and } \Delta\tilde{u}, \Delta\tilde{w} \in C([0,\tilde{T}] \times \overline{\Omega}),$$

$\tilde{w}$ *and* $\tilde{u}$ *are non-negative and the equations* (1a)–(1f) *are fulfilled pointwise.*

Theorem 2.6 (Weak-strong uniqueness). *Let* $(\tilde{u},\tilde{w})$ *be a strong solution according to Definition 2.5 for the initial conditions* $u_0, w_0 \in C^3(\overline{\Omega})$ *non-negative, with* u_0 *bounded away from zero. Then every generalized solution* $(u,w) \in \mathcal{X}$ *emanating from the same initial values coincides with the strong solution and thus the generalized solution is unique.*

In the final part of this paper we show that under stronger assumption on the initial conditions, we indeed have at least local-in-time existence of strong solution.

Theorem 2.7 (Local existence of weak solution). *For* $d \in \{2,3\}$, $u_0 \in W^{1,2}(\Omega)$ *and* $w_0 \in W^{2,6}(\Omega)$ *both fulfilling zero Neumann boundary conditions there is a* $T^* > 0$ *such that* (1) *has a weak solution* (u,w) *with*

$$u \in W^{1,2}(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;W^{2,2}(\Omega)) \cap L^\infty(0,T^*;W^{1,2}(\Omega)),$$
$$w \in W^{1,6}(0,T^*;L^6(\Omega)) \cap L^6(0,T^*;W^{2,6}(\Omega)) \cap L^\infty(0,T^*;W^{2,2}(\Omega)).$$

The necessary *a priori* estimates for this result are given at the end of the paper. Given smoother initial data we can deduce even more regularity of the local solution so that we obtain strong solution.

Proposition 2.8 (Local existence of strong solution)**.** *For $d \in \{2,3\}$ and $w_0, u_0 \in C^3(\overline{\Omega})$ non-negative and both fulfilling zero Neumann boundary conditions, we find that the solutions from Theorem 2.7 are strong solutions.*

3 Numerical simulations

With the application of model (1) in biological pest control in mind, we performed some numerical simulations illustrating the influence of the prey-taxis term in the predator equation, using the finite element method and the python package FEniCS. The non-linear, higher-order coupling prey-taxis term made the existence proof of solutions quite challenging, but it also made the modelling of a certain hunting behaviour possible as can be seen, when keeping the diffusion coefficients constant and increasing the coefficient of the prey-taxis term.

Simulations of another Keller–Segel model were conducted in [17]. As described there, blow-up solution are known to exists for certain initial values and as we have seen in Section 1 also the value of the prey-taxis coefficient played a crucial role in various existence proofs. With these difficulties in mind, we would like to point out that the simulations performed here are only used as a visualization tool and we do not claim any accuracy.

Numerical simulations were performed over the domain $\overline{\Omega} = [-1,1]^2$, discretized by a regular triangulation with $n_x = n_y = 200$ and choosing the final time $T = 10$ and continuous $P1$-Lagrange elements for u and continuous $P2$-Lagrange elements for w.

Additionally, we chose the coefficients of system (1) to be

$$\nu = 0.1,\ \mu = 0.01,\ \alpha = 2.0,\ \beta = 0.8,\ \gamma = 0.8,\ \delta = 2.0,\ \kappa \in \{0,1,2,3\}$$

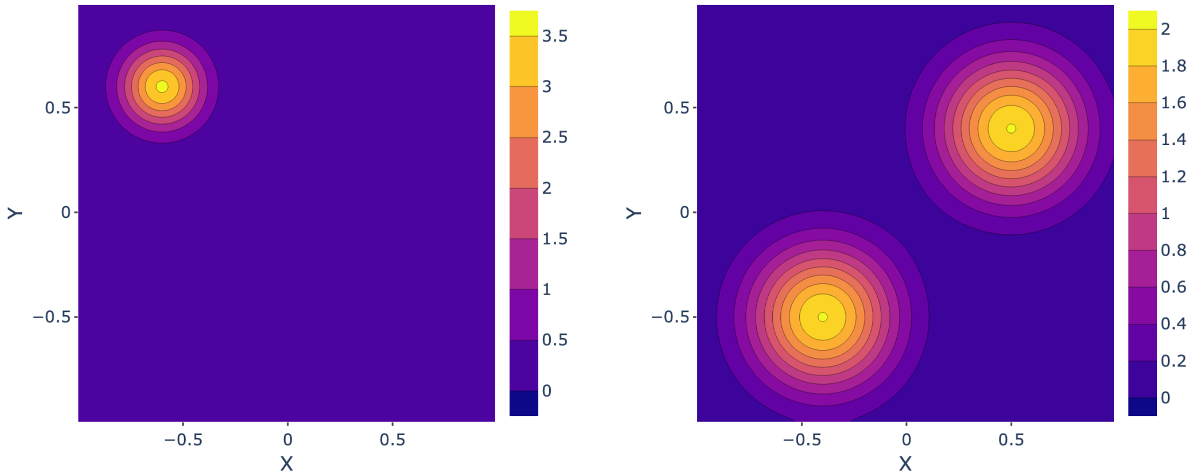
and the initial conditions

$$\begin{aligned} u_0 &= 4\exp(-30(x+0.6)^2 - 30(y-0.6)^2),\\ w_0 &= 2\exp(-9(x+0.4)^2 - 9(y+0.5)^2) + 2\exp(-9(x-0.5)^2 - 9(y-0.4)^2). \end{aligned}$$

Note that the case $\kappa = 0$ is not covered by our analysis, but was part of our numerical simulations for illustration purposes. In this case the difficult prey-taxis term in (1a) vanishes and the existence of weak solution should follow by standard means. The initial conditions are visualized in Figure 1.

For $\kappa = 0$ the densities at time $t = 0.4$ are shown in Figure 2. The initial concentration of predators and prey in Gaussians has already spread out slightly as one would expect from diffusive movement. Here no predators have accumulated at the peaks of the prey density yet.

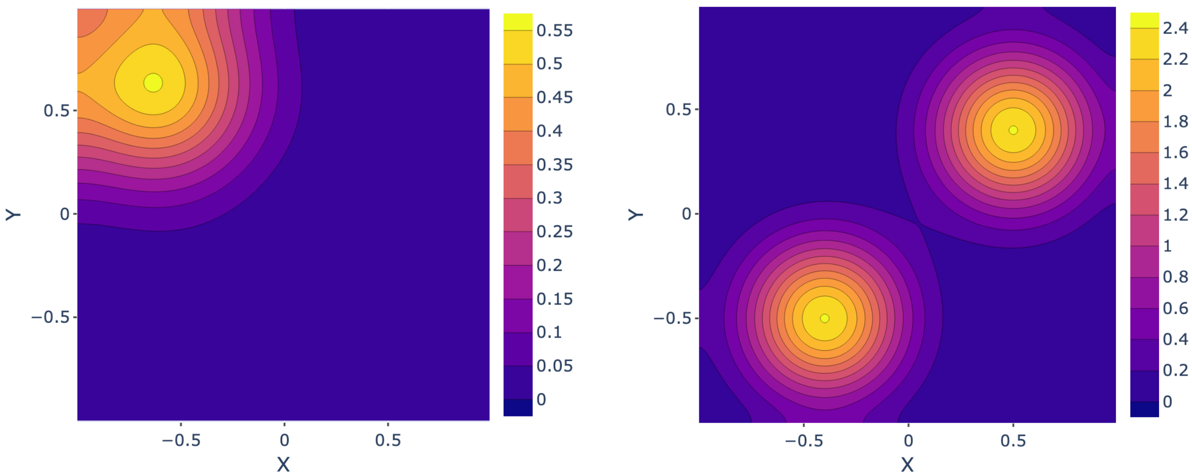
In contrast for $\kappa = 1$ and $t = 0.4$ we have a higher concentration of predators were prey is plenty, see Figure 3. Here the biased movement towards higher concentration of prey, modelled by the prey-taxis term is nicely visible. To further illustrate this accumulation of predators, a surface plot of the densities at $t = 0.4$ for $\kappa = 1$ is depicted in Figure 4.



(a) Initial condition u_0 for the predator.

(b) Initial condition w_0 for the prey.

Figure 1: Initial conditions.



(a) Predator density u.

(b) Prey density w.

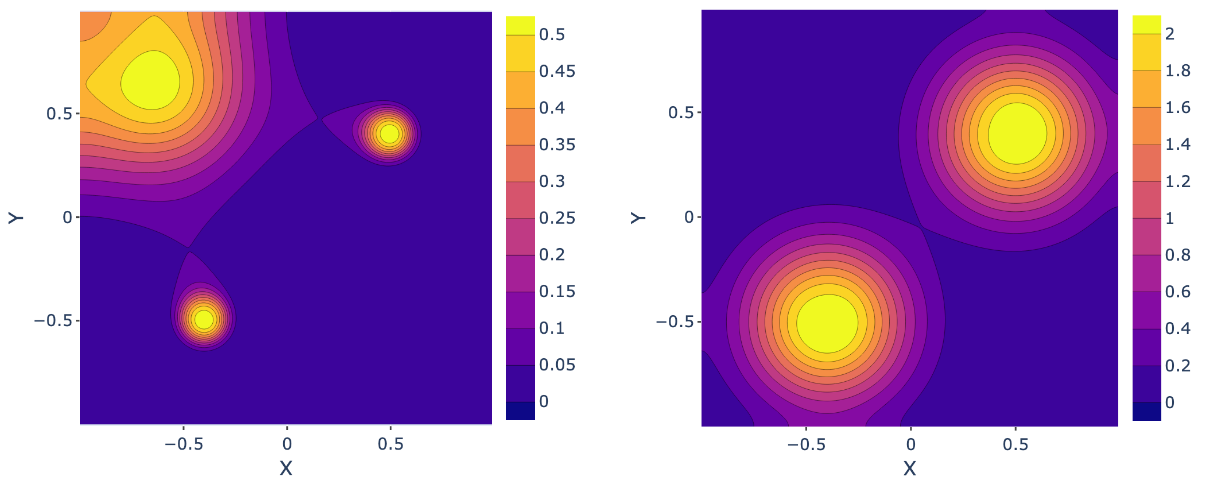
Figure 2: Mite densities for $\kappa = 0$ at $t = 0.4$.

4 Existence of generalized solutions

4.1 The regularized system

Now we introduce the regularizing term $-\varepsilon\, u|u|^{p-1}$ on the right-hand side of the predator equation (1a), where $\varepsilon > 0$ is the regularizing coefficient, which we will take to zero in the proof of the existence of generalized solutions, and $p > \max\{d, 4\}$. So the system we are considering in this section is identical to the system (1) with (1a) replaced by

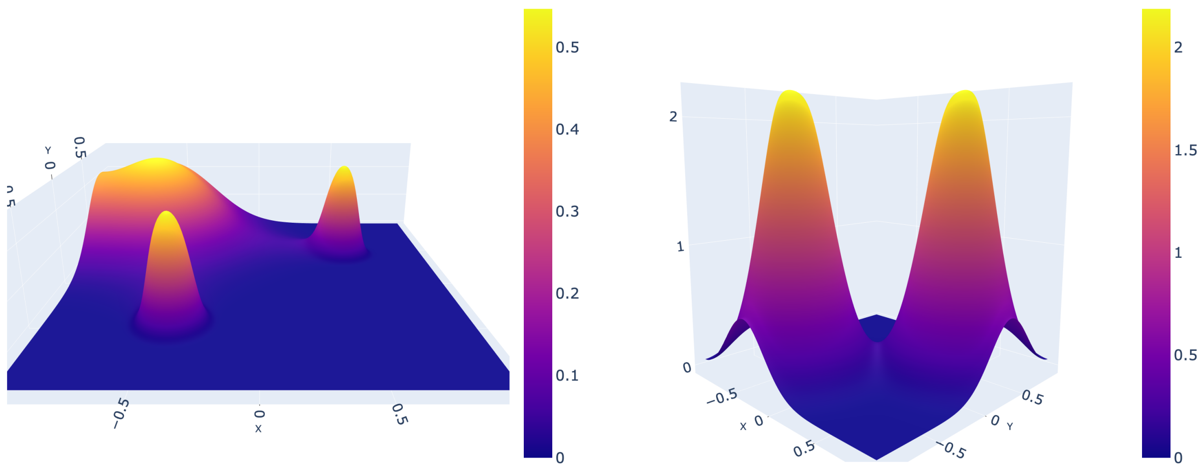
$$\partial_t u - \nu\Delta u + \kappa\nabla\cdot(u\nabla w) = (\alpha w - \beta)u - \varepsilon\, u|u|^{p-1} \quad \text{in } (0,T)\times\Omega. \tag{5a}$$



(a) Predator density u.

(b) Prey density w.

Figure 3: Mite densities for $\kappa = 1$ at $t = 0.4$.



(a) Predator density u.

(b) Prey density w.

Figure 4: Surface plot of the mite densities for $\kappa = 1$ at $t = 0.4$.

4.2 Existence of weak solution to the regularized system

The chosen regularization term allows for an easy attainable estimate of u in $L^{p+1}(0,T;L^{p+1}(\Omega))$ and with the use of maximal L^p-regularity in the prey equation (1b) we get rather strong estimates for ∇w, which we need to estimate the prey-taxis term in (5a), which is the term with the prefactor κ.

Lets start by defining our notion of weak solutions. We first define our solution space. We say $w \in \mathbb{W}$ if

$$w \in L^2(0,T;W^{1,2}(\Omega)) \text{ and } \partial_t w \in L^2(0,T;W^{1,2}(\Omega)^*)$$

and we say that $u \in \mathbb{U}$ if

$$\begin{aligned} u &\in L^2(0,T;W^{1,2}(\Omega)) \cap L^{p+1}(0,T;L^{p+1}(\Omega)) =: \mathcal{U}, \\ \partial_t u &\in L^2(0,T;W^{1,2}(\Omega)^*) + L^q(0,T;L^q(\Omega)) = \mathcal{U}^*. \end{aligned}$$

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Here, $q > 1$ denotes the conjugate exponent of $p+1$. The time derivative of w and u are to be understood in the distributional sense, that is $u_t \in \mathcal{U}^*$ is the weak time derivative of $u \in \mathcal{U}$ if

$$\int_0^T \langle u(t), \varphi \rangle \, \phi'(t) \, \mathrm{d}t = -\int_0^T \langle u_t(t), \varphi \rangle \, \phi(t) \, \mathrm{d}t \tag{6}$$

holds for all $\varphi \in L^{p+1}(\Omega) \cap W^{1,2}(\Omega)$ and $\phi \in C_0^\infty(0,T)$. We then write $\partial_t u = u_t$. The time derivative of w is defined similarly with test functions $\varphi \in W^{1,2}(\Omega)$.

Definition 4.1 (Weak solution to the regularized system)**.** *We say that a pair* $(u, w) \in \mathbb{U} \times \mathbb{W}$ *is a weak solution to problem* (5)*, if*

- $u, w \geq 0$,
- $w \in L^\infty((0,T) \times \Omega)$,
- $\nabla w \in L^p(0,T; L^\infty(\Omega))$,

the integral equalities

$$\int_0^t \langle \partial_t u, \psi \rangle \, \mathrm{d}s + \int_0^t \int_\Omega \nu \nabla u \cdot \nabla \psi - \kappa u \nabla w \cdot \nabla \psi - (\alpha w - \beta) u \psi + \varepsilon u |u|^{p-1} \psi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s = 0, \tag{7}$$

$$\int_0^t \langle \partial_t w, \varphi \rangle \, \mathrm{d}s + \int_0^t \int_\Omega \mu \nabla w \cdot \nabla \varphi - (\gamma - \delta u) w \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s = 0 \tag{8}$$

are fulfilled for all $\varphi \in L^2(0,T; W^{1,2}(\Omega))$ *and all* $\psi \in \mathcal{U}$ *and almost all* $t \in (0,T)$ *and the initial conditions are fulfilled in the sense of traces in* $L^2(\Omega)$.

Theorem 4.2 (Existence of weak solutions to the regularized system)**.** *For every pair of initial data* (u_0, w_0) *with non-negative* $u_0 \in L^p(\Omega)$ *and non-negative and bounded*

$$w_0 \in \left\{ w \in W^{2,p}(\Omega) \mid \nabla w \cdot n = 0 \text{ on } \partial\Omega \right\}$$

there exists a weak solution according to Definition 4.1 to the regularized system (5).

Proof. We only give a rough outline of the proof since it mainly follows the standard procedure of decoupling the system, using a Galerkin approximation to tackle the single equations and Schauder's fixed point theorem to show the existence of a solution to the coupled equations. A detailed proof for $d = 2$ can be found in [34]. The existence and uniqueness of a weak solution w to the prey equation (1b) for fixed non-negative $\bar{u} \in L^p(L^p)$ follows from Theorem 8.30 and Theorem 8.34 in [36]. The non-negativity and boundedness of w follows from a comparison principle, see Lemma 4.3 below. The additional regularity of w follows from the maximal L^p-regularity of the Laplace operator, see [9, Thm. 8.2]. Here the assumptions that Ω has a $\mathcal{C}^2$-boundary and the initial condition w_0 fulfills some compatibility condition are needed.

For fixed $\bar{w} \in \mathbb{W}$ non-negative and bounded with $\nabla \bar{w} \in L^p(L^\infty)$, the existence of a weak solution $u \in \mathbb{U}$ to (5a) with w replaced by $\bar{w}$ can be shown via a standard Galerkin discretization using the Gelfand triple $W^{1,2}(\Omega) \cap L^{p+1}(\Omega) = V \subseteq L^2(\Omega) \subseteq V^*$ and Minty's trick to handle the monotone regularization term. Here we needed the integration by parts rule for elements of $\mathbb{U}$ and the continuous embedding of $\mathbb{U}$ into $C([0,T]; L^2(\Omega))$, which can be proven analogously to the well-known versions of these results for the space $\mathbb{W}$. We obtained the non-negativity of u by testing equation (5a) with u^- and applying Gronwall's inequality. Additionally we obtained a constant $L^p(L^p)$-bound for u by using 1 as a test function.

Defining the solution operator $\mathcal{T} : L^p(L^p) \to L^p(L^p)$, which maps a non-negative $\bar{u} \in L^p(L^p)$ to the solution of (5a) with $w = \bar{w}$, where $\bar{w}$ is the solution of (1b) with u replaced by $\bar{u}$. Using these existence results and *a priori* estimates, we find that $\mathcal{T}$ is a well-defined self-map on a bounded, convex and closed subset of the Banach space $L^p(L^p)$. The continuity can be proven by testing appropriately, using L^p-interpolation inequalities and maximal L^p-regularity of the heat equation. For any bounded sequence $(\bar{u}_n)_n \subseteq L^p(L^p)$ the sequence and $(\mathcal{T}(\bar{u}_n))_n$ is bounded in $L^{p+1}(L^{p+1})$, were the bound depends on ε, and the strong convergence of a subsequence of $(\mathcal{T}(\bar{u}_n))_n$ in $L^p(L^p)$ follows by Vitali's theorem, see [14, Thm. 5.6]. This implies the compactness of $\mathcal{T}$ and Schauder's fixed point theorem then implies the existence of a weak solution to (5). □

4.3 *A priori* estimates

In order to show the existence of generalized solutions according to Definition 2.1, we show some *a priori* estimates for solutions $(u_\varepsilon, w_\varepsilon)$ of the regularized system. These *a priori* estimates will allow us to extract a convergent subsequence, whose limit fulfills our notion of generalized solutions.

At first, we proved the following comparison principle for the prey variable.

Lemma 4.3 (Comparison principle for w)**.** *Let $u \in L^1(0,T;L^1(\Omega))$ be non-negative and assume that $\underline{w}$ and $\overline{w}$ are a sub- and a super-solution of* (1b)*, i.e. $\underline{w}$ and $\overline{w}$ fulfill equation* (1b) *in the weak sense with the equality sign replaced by $\leq$ and $\geq$, respectively, and*

$$\begin{gathered}
\underline{w}, \overline{w} \in L^2(0,T;L^\infty(\Omega) \cap W^{1,2}(\Omega)),\\
\partial_t \underline{w}, \partial_t \overline{w} \in L^2(0,T;L^1(\Omega) + W^{1,2}(\Omega)^*),\\
0 \leq \operatorname{ess\,inf}_{(0,T)\times\Omega} \underline{w} \leq \operatorname{ess\,sup}_{(0,T)\times\Omega} \underline{w} < \infty,\\
0 \leq \operatorname{ess\,inf}_{(0,T)\times\Omega} \overline{w} \leq \operatorname{ess\,sup}_{(0,T)\times\Omega} \overline{w} < \infty
\end{gathered}$$

hold as well as $\underline{w}(0,\boldsymbol{x}) \leq \overline{w}(0,\boldsymbol{x})$ a.e. in Ω. Then

$$\underline{w}(t,\boldsymbol{x}) \leq \overline{w}(t,\boldsymbol{x}) \text{ a.e. in } (0,T)\times\Omega \tag{9}$$

holds.

Proof. Subtracting the equation for the sub-solution $\underline{w}$ and the super-solution $\overline{w}$ and testing the resulting inequality with $(\underline{w} - \overline{w})_+ := \max\{0, \underline{w} - \overline{w}\}$, we obtain

$$\begin{aligned}
\frac{1}{2}\|(\underline{w}(t) - \overline{w}(t))_+\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \mu |\nabla(\underline{w} - \overline{w})_+|^2 + \delta u |(\underline{w} - \overline{w})_+|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s&\\
= \frac{1}{2}\|(\underline{w}(0) - \overline{w}(0))_+\|^2_{L^2(\Omega)} + \gamma \int_0^t \int_\Omega |(\underline{w} - \overline{w})_+|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s&,
\end{aligned}$$

where we integrated the first term by parts. The standard integration by parts rule can be generalized to this case by an approximation with smooth functions, cf. Corollary 7.4 in the Appendix. An application of Gronwall's inequality yields

$$\|(\underline{w}(t) - \overline{w}(t))_+\|^2_{L^2(\Omega)} \leq \|(\underline{w}(0) - \overline{w}(0))_+\|^2_{L^2(\Omega)} e^{2\gamma t} = 0,$$

by the assumption for the initial values and our proof is complete. □

Corollary 4.4 (Boundedness of w). *Let $u \in L^1(0,T;L^1(\Omega))$ and $w_0 \in L^\infty(\Omega)$ be non-negative. Then the solution w of (1b) fulfills*

$$0 \leq w \leq \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} w_0(\boldsymbol{x}) e^{\gamma t}. \tag{10}$$

Proof. One easily checks that 0 is a sub- and $\operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} w_0(\boldsymbol{x}) e^{\gamma t}$ is a super-solution to (1b). Then the assertion follows from Lemma 4.3. □

Proposition 4.5 (*A priori* estimates). *Assume that the initial values can be bounded independently of ε. That is $\ln u_{0\varepsilon}$ and $u_{0\varepsilon}$ are bounded in $L^1(\Omega)$ independently of ε and $w_{0\varepsilon}$ is bounded in $L^\infty(\Omega)$ independently of ε. For a solution $(u_\varepsilon, w_\varepsilon)$ to the approximate system (5) we have the following a priori estimates*

$$\|u_\varepsilon\|_{L^\infty(L^1)} + \varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(L^p)} + \|\ln u_\varepsilon\|_{L^\infty(L^1)} + \|\nabla \ln u_\varepsilon\|_{L^2(L^2)} + \|\partial_t \ln u_\varepsilon\|_{L^1(W^{1,p}(\Omega)^*)} \leq C, \tag{11a}$$

$$\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)} + \|\nabla w_\varepsilon\|_{L^2(L^2)} + \|w_\varepsilon u_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1)\|_{L^1(L^1)} + \|\partial_t w_\varepsilon\|_{L^2(W^{1,p}(\Omega)^*)} \leq C, \tag{11b}$$

where the constant C is independent of ε.

Remark 4.6. *A sequence of initial values fulfilling the boundedness assumptions and the appropriate regularity criteria is constructed in the proof of Theorem 2.4 below.*

Proof. The $L^\infty((0,T)\times\Omega)$-bound of w_ε follows from Corollary 4.4. Note that this bound is independent of ε as long as $w_{0\varepsilon}$ is bounded independently of ε. Testing equation (1b) with w_ε, we get

$$\frac{1}{2}\|w_\varepsilon(T)\|^2_{L^2(\Omega)} + \mu\|\nabla w_\varepsilon\|^2_{L^2(L^2)} + \delta\int_0^T\int_\Omega u_\varepsilon w_\varepsilon^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \gamma\|w_\varepsilon\|^2_{L^\infty((0,T)\times\Omega)}|\Omega|T + \frac{1}{2}\|w_{0\varepsilon}\|^2_{L^2(\Omega)},$$

which gives the $L^2(L^2)$-bound for ∇w_ε, since u_ε is non-negative. Using $\varphi \equiv 1$ as a test function in the predator equation (5a), we get

$$\int_0^t \langle \partial_t u_\varepsilon(\tau), 1\rangle \,\mathrm{d}\tau + \int_0^t\int_\Omega \varepsilon u_\varepsilon^p + \beta u_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau = \alpha\int_0^t\int_\Omega w_\varepsilon u_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau. \tag{12}$$

By the non-negativity of u_ε, we obtain

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} \leq \|u_{0\varepsilon}\|_{L^1(\Omega)} + \alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}\int_0^t \|u_\varepsilon(\tau)\|_{L^1(\Omega)}\,\mathrm{d}\tau,$$

for almost all $t \in (0,T)$. An application of Gronwall's inequality then yields

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} \leq e^{\alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}}\|u_{0\varepsilon}\|_{L^1(\Omega)} \tag{13}$$

for almost all $t \in (0,T)$. Additionally, (12) implies

$$\varepsilon\|u_\varepsilon\|^p_{L^p(L^p)} \leq \alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)} T e^{\alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}}\|u_{0\varepsilon}\|_{L^1(\Omega)}.$$

Now we have shown that the first two terms of (11a) and (11b) have constant upper bounds independent of ε. To derive an estimate for the third and fourth term in (11a) we use $-\frac{1}{u_\varepsilon+\lambda}$ as a test function in the predator equation (5a) for some $\lambda \in (0,1)$. This yields

$$\begin{aligned}\int_0^t \left\langle \partial_t u_\varepsilon(s), \frac{-1}{u_\varepsilon(s)+\lambda}\right\rangle \mathrm{d}s + \int_0^t\int_\Omega \nu\nabla u_\varepsilon \nabla\left(\frac{-1}{u_\varepsilon+\lambda}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ = \int_0^t\int_\Omega \kappa u_\varepsilon\nabla w_\varepsilon\nabla\left(\frac{-1}{u_\varepsilon+\lambda}\right) + \varepsilon u_\varepsilon^{p-1}\frac{u_\varepsilon}{u_\varepsilon+\lambda} + (\alpha w_\varepsilon - \beta)\frac{-u_\varepsilon}{u_\varepsilon+\lambda}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.\end{aligned}$$

We chose this test function, as we have only shown that $u_\varepsilon \geq 0$ holds and therefore we are not allowed to test with $1/u_\varepsilon$, Using integration by parts and Young's inequality we can estimate

$$-\int_\Omega \ln(u_\varepsilon+\lambda)\Big|_0^t \,\mathrm{d}\boldsymbol{x} + \frac{\nu}{2}\int_0^t\int_\Omega |\nabla \ln(u_\varepsilon+\lambda)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leq \frac{\kappa^2}{2\nu}\int_0^t\int_\Omega |\nabla w_\varepsilon|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ + \varepsilon\int_0^t\int_\Omega \frac{p-1}{p}u_\varepsilon^p + \frac{1}{p}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \left(\beta + \alpha\,\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}\right)T|\Omega|. \tag{14}$$

By the already derived estimates for the first two terms of (11a) and (11b), this gives a constant upper bound for $-\int_\Omega \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x}$ as long as $\int_\Omega \ln(u_{0\varepsilon}+\lambda)\,\mathrm{d}\boldsymbol{x}$ is bounded, which is the case since $u_{0\varepsilon}$ is assumed to be bounded in $L^1(\Omega)$, as can be seen from the following estimate

$$\int_\Omega \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} = \int_{\{u_\varepsilon(t)+\lambda\leq 1\}} \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} \\ + \int_{\{u_\varepsilon(t)+\lambda> 1\}} \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} \leq \|u_\varepsilon(t)\|_{L^1(\Omega)} + \lambda|\Omega| \leq C \tag{15}$$

for some $C>0$, where we use $\ln(u_\varepsilon(t)+\lambda)\leq 0$ for $u_\varepsilon(t)+\lambda\leq 1$ and $\ln(u_\varepsilon(t)+\lambda)\leq u_\varepsilon(t)+\lambda$ for $u_\varepsilon(t)+\lambda>1$. The constant upper bound follows from the fact, that we already know that u_ε is bounded in the $L^\infty(L^1)$-norm, cf. equation (13). We can now derive a constant upper bound for the L^1-norm of $\ln u_\varepsilon(t)$. We estimate

$$\|\ln(u_\varepsilon(t)+\lambda)\|_{L^1(\Omega)} = -\int_{\{u_\varepsilon(t)+\lambda<1\}} \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} + \int_{\{u_\varepsilon(t)+\lambda\geq 1\}} \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} \\ = -\int_\Omega \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} + 2\int_{\{u_\varepsilon(t)+\lambda\geq 1\}} \ln(u_\varepsilon(t)+\lambda)\,\mathrm{d}\boldsymbol{x} \leq C, \tag{16}$$

where we use (14) and (15). To get the $L^\infty(L^1)$-bound for $\ln u_\varepsilon$, we take λ to zero from above. Since the logarithm is continuous, we have

$$\liminf_{\lambda\downarrow 0} |\ln(u_\varepsilon(t)+\lambda)| = |\ln(u_\varepsilon(t))|,$$

where we define $\ln(0) := -\infty$. By Fatou's lemma we can estimate

$$\int_\Omega |\ln u_\varepsilon(t)|\,\mathrm{d}\boldsymbol{x} = \int_\Omega \liminf_{\lambda\downarrow 0} |\ln(u_\varepsilon(t)+\lambda)|\,\mathrm{d}\boldsymbol{x} \leq \liminf_{\lambda\downarrow 0}\int_\Omega |\ln(u_\varepsilon(t)+\lambda)|\,\mathrm{d}\boldsymbol{x} \leq C,$$

where the constant upper bound follows from (16). Thus we get the $L^\infty(L^1)$-bound of $\ln u_\varepsilon$ and we have $u_\varepsilon(t)>0$ almost everywhere. Similarly, we get the bound for $\nabla \ln u_\varepsilon$. We already have that $\nabla\ln(u_\varepsilon+\lambda)$ is bounded in $L^2(L^2)$ by equation (14). Again using Fatou's lemma we get

$$\|\nabla \ln u_\varepsilon\|_{L^2(L^2)} = \int_0^T\int_\Omega \frac{1}{u_\varepsilon^2}|\nabla u_\varepsilon|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T\int_\Omega \liminf_{\lambda\downarrow 0}\frac{1}{(u_\varepsilon+\lambda)^2}|\nabla u_\varepsilon|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_0^T\int_\Omega \liminf_{\lambda\downarrow 0}|\nabla\ln(u_\varepsilon+\lambda)|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \liminf_{\lambda\downarrow 0}\int_0^T\int_\Omega |\nabla\ln(u_\varepsilon+\lambda)|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C.$$

In order to show the $L^1(L^1)$-bound of $u_\varepsilon w_\varepsilon\ln(u_\varepsilon w_\varepsilon+1)$, we note that

$$0 \leq u_\varepsilon w_\varepsilon \ln(u_\varepsilon w_\varepsilon+1) \leq u_\varepsilon w_\varepsilon\ln((u_\varepsilon+1)(w_\varepsilon+1)) = u_\varepsilon w_\varepsilon\ln(u_\varepsilon+1) + u_\varepsilon w_\varepsilon\ln(w_\varepsilon+1)$$

holds and it suffices to show a $L^1(L^1)$-bound for $u_\varepsilon w_\varepsilon \ln(u_\varepsilon+1)$ and $u_\varepsilon w_\varepsilon \ln(w_\varepsilon+1)$ separately. The upper bounds for u_ε and $\nabla \ln u_\varepsilon$ from (11a) can be transferred to $\ln(u_\varepsilon+1)$ and $\nabla\ln(u_\varepsilon+1)$. We test the prey equation (1b) by $\ln(u_\varepsilon+1)$. It can be shown by an approximation with smooth functions, see [43, Ex. 21.3c] that this is indeed an admissible test function since $u_\varepsilon \in L^2(W^{1,2})$ and $u_\varepsilon+1$ is bounded away from zero by 1. We then have

$$\nabla \ln(u_\varepsilon+1) = \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon.$$

Using $\partial_t w_\varepsilon \ln(u_\varepsilon+1) = -\partial_t \ln(u_\varepsilon+1) w_\varepsilon + \partial_t(\ln(u_\varepsilon+1)w_\varepsilon)$ and integration by parts we get

$$\begin{aligned}\delta\int_0^T\int_\Omega w_\varepsilon u_\varepsilon \ln(u_\varepsilon+1)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \int_0^T\int_\Omega \partial_t \ln(u_\varepsilon+1) w_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mu\left\|\nabla w_\varepsilon\right\|_{L^2(L^2)}\\ \left\|\nabla\ln(u_\varepsilon+1)\right\|_{L^2(L^2)} + (1+\gamma T)\left\|w_\varepsilon\right\|_{L^\infty((0,T)\times\Omega)}\left\|\ln(u_\varepsilon+1)\right\|_{L^\infty(L^1)}\\ + \left\|w_\varepsilon\right\|_{L^\infty((0,T)\times\Omega)}\left\|\ln(u_{0\varepsilon}+1)\right\|_{L^1(\Omega)}. \end{aligned} \tag{17}$$

All terms on the right-hand side except the first are already known to be bounded by a constant. To upper bound the first term on the right-hand side, we test (5a) with $\frac{w_\varepsilon}{u_\varepsilon+1}$, which is an admissible test function, since w_ε and u_ε lie in $L^2(0,T;W^{1,2}(\Omega))$ and w_ε is bounded,

$$\begin{aligned}\int_0^T\int_\Omega \partial_t u_\varepsilon \frac{w_\varepsilon}{u_\varepsilon+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T\int_\Omega \nu\nabla u_\varepsilon\cdot\nabla\left(\frac{w_\varepsilon}{u_\varepsilon+1}\right) - \kappa u_\varepsilon\nabla w_\varepsilon\cdot\nabla\left(\frac{w_\varepsilon}{u_\varepsilon+1}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ = \int_0^T\int_\Omega (\alpha w_\varepsilon-\beta)\,w_\varepsilon\frac{u_\varepsilon}{u_\varepsilon+1} - \varepsilon u_\varepsilon^{p-1}w_\varepsilon\frac{u_\varepsilon}{u_\varepsilon+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

Rearranging the terms and calculating the gradients yield

$$\begin{aligned}\int_0^T\int_\Omega \partial_t u_\varepsilon \frac{w_\varepsilon}{u_\varepsilon+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_0^T\int_\Omega \nu\nabla\ln(u_\varepsilon+1)\cdot\nabla w_\varepsilon - \nu w_\varepsilon|\nabla\ln(u_\varepsilon+1)|^2 - \kappa\frac{u_\varepsilon}{u_\varepsilon+1}|\nabla w_\varepsilon|^2\\ + \kappa\frac{u_\varepsilon}{u_\varepsilon+1}\nabla w_\varepsilon\cdot\nabla\ln(u_\varepsilon+1)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T\int_\Omega(\alpha w_\varepsilon-\beta)\,w_\varepsilon\frac{u_\varepsilon}{u_\varepsilon+1} - \varepsilon u_\varepsilon^{p-1}w_\varepsilon\frac{u_\varepsilon}{u_\varepsilon+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

Using $\partial_t\ln(u_\varepsilon+1)w_\varepsilon = \frac{w_\varepsilon}{u_\varepsilon+1}\partial_t u_\varepsilon$, we find a constant upper bound for $\int_0^T\int_\Omega \partial_t\ln(u_\varepsilon+1)w_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$, by estimating $\frac{u_\varepsilon}{u_\varepsilon+1}$ by one and using the already derived *a priori* estimates for the second and fourth term in (11a) and the first two terms in (11b). Now all terms on the right-hand side of (17) are known to be bounded and we find that $\left\|w_\varepsilon u_\varepsilon\ln(u_\varepsilon+1)\right\|_{L^1(L^1)}$ is bounded, since $u_\varepsilon w_\varepsilon\ln(u_\varepsilon+1)\ge 0$.

First testing (1b) with $\ln(w_\varepsilon+1)$ and then with $\frac{w_\varepsilon}{w_\varepsilon+1}$ we find an $L^1(L^1)$-bound of $u_\varepsilon w_\varepsilon\ln(w_\varepsilon+1)$ in a similar manner. This shows the upper bound for the third term in (11b).

Using equation (1b) tested with $\varphi\in W^{1,p}(\Omega)$, yields

$$\begin{aligned}\int_\Omega \partial_t w_\varepsilon\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega(\gamma-\delta u_\varepsilon)w_\varepsilon\varphi - \mu\nabla w_\varepsilon\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\\ \le C\left\|\varphi\right\|_{W^{1,p}}\Big(\left\|w_\varepsilon\right\|_{L^\infty((0,T)\times\Omega)}(\gamma+\delta\left\|u_\varepsilon\right\|_{L^1(\Omega)}) + \mu\left\|\nabla w_\varepsilon\right\|_{L^2(\Omega)}\Big),\end{aligned}$$

where we used $p>d$ such that $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$. Noting that the right-hand side is bounded in $L^2(0,T)$ independently of ε, we obtain the upper bound for $\partial_t w_\varepsilon$ in $L^2(W^{1,p}(\Omega)^*)$.

To obtain the upper bound for $\partial_t \ln u_\varepsilon$ in $L^1(W^{1,p}(\Omega)^*)$, we first note that a regularized form of the logarithmic inequality (3) holds with an equality sign on the approximate level. To see that we use $-\frac{\vartheta}{u_\varepsilon+\lambda}$ as a test function in (5a) for an arbitrary $\vartheta \in C^1([0,T];L^\infty(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ and $\lambda \in (0,1)$. This yields,

$$\begin{aligned} -\int_\Omega \partial_t \ln(u_\varepsilon+\lambda)\vartheta \,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu|\nabla \ln(u_\varepsilon+\lambda)|^2\vartheta - \nu\nabla\ln(u_\varepsilon+\lambda)\cdot\nabla\vartheta \\ + \kappa\frac{u_\varepsilon}{u_\varepsilon+\lambda}\nabla w_\varepsilon\cdot\nabla\vartheta - \kappa\frac{u_\varepsilon}{u_\varepsilon+\lambda}\vartheta\,\nabla w_\varepsilon\cdot\nabla\ln(u_\varepsilon+\lambda)\,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega(\beta-\alpha w_\varepsilon)\frac{u_\varepsilon}{u_\varepsilon+\lambda}\,\vartheta + \varepsilon|u_\varepsilon|^{p-1}\frac{u_\varepsilon}{u_\varepsilon+\lambda}\,\vartheta\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

We take $\lambda \downarrow 0$ and pull the limit into the integral, which we are allowed to do by the following reasoning. Note that $-\partial_t \ln(u_\varepsilon+\lambda)$ is monotonically increasing for $\lambda \downarrow 0$, thus, by the monotone convergence theorem, we can pull the limit into the integral in the first term. For the other integrals we note that we have pointwise convergence in λ and that we can find an integrable dominating function by the estimates from the first three terms of (11a) and the first two terms of (11b). The dominating functions of the terms including λ are given by

$$|\nabla\ln(u_\varepsilon+\lambda)|^2 = \frac{1}{(u_\varepsilon+\lambda)^2}|\nabla u_\varepsilon|^2 \le \frac{1}{u_\varepsilon^2}|\nabla u_\varepsilon|^2 = |\nabla\ln u_\varepsilon| \quad\text{and}\quad \frac{u_\varepsilon}{u_\varepsilon+\lambda}\le 1.$$

Thus by Lebesgue's dominated convergence theorem we can also pull the limit $\lambda\downarrow 0$ into the remaining integrals with equality and we get

$$\begin{aligned} -\int_\Omega \partial_t \ln u_\varepsilon\vartheta\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu|\nabla\ln u_\varepsilon|^2\vartheta - \nu\nabla\ln u_\varepsilon\cdot\nabla\vartheta + \kappa\nabla w_\varepsilon\cdot\nabla\vartheta \\ - \kappa\vartheta\,\nabla w_\varepsilon\cdot\nabla\ln u_\varepsilon\,\mathrm{d}\boldsymbol{x} = \int_\Omega(\beta-\alpha w_\varepsilon)\vartheta + \varepsilon|u_\varepsilon|^{p-1}\,\vartheta\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{18}$$

where we used the convergence of $\frac{u_\varepsilon}{u_\varepsilon+\lambda}\to 1$ for $\lambda\downarrow 0$ pointwise almost everywhere as we have $u_\varepsilon > 0$ almost everywhere due to the fact that $\ln u_\varepsilon$ is bounded in $L^\infty(L^1)$, cf. equation (13).

Using the test function $\vartheta = -\varphi$ for some $\varphi \in W^{1,p}(\Omega)$ we can derive an estimate for $\partial_t \ln u_\varepsilon$ in $L^1(W^{1,p}(\Omega)^*)$, by estimating

$$\begin{aligned} \int_\Omega \partial_t\ln u_\varepsilon\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nu|\nabla\ln u_\varepsilon|^2\varphi - \nu\nabla\ln u_\varepsilon\cdot\nabla\varphi + \kappa\nabla w_\varepsilon\cdot\nabla\varphi \\ - \kappa\varphi\,\nabla w_\varepsilon\cdot\nabla\ln u_\varepsilon + (\beta-\alpha w_\varepsilon)\varphi + \varepsilon|u_\varepsilon|^{p-1}\,\varphi\,\mathrm{d}\boldsymbol{x} \\ \le C\,\|\varphi\|_{W^{1,p}(\Omega)}\left(1+\|\nabla\ln u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla w_\varepsilon\|^2_{L^2(\Omega)} + \|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}\right). \end{aligned}$$

Note that this time the right-hand side is only in $L^1(0,T)$ and thus we get the weaker bound for $\partial_t \ln u_\varepsilon$. □

4.4 Convergence of approximate solutions

We now tend the regularization coefficient to zero and show the existence of generalized solutions for vanishing regularization.

Proposition 4.7 (Convergence of solutions). *Let $\{(u_\varepsilon, w_\varepsilon)\}$ be a sequence of solutions to the approximate system (5). Then there exists a pair $(u, w) \in \mathcal{X}$, such that the following convergences hold for a subsequence $\varepsilon \downarrow 0$,*

$$
\begin{aligned}
w_\varepsilon &\rightharpoonup w && \text{in } L^2(W^{1,2}), && (19)\\
w_\varepsilon &\to w && \text{in } L^2(L^2), && (20)\\
w_\varepsilon &\to w && \text{pointwise a.e. in } (0,T)\times\Omega, && (21)\\
w_\varepsilon &\rightharpoonup^* w && \text{in } L^\infty((0,T)\times\Omega) && (22)\\
\ln u_\varepsilon &\rightharpoonup \ln u && \text{in } L^2(W^{1,2}), && (23)\\
\ln u_\varepsilon &\to \ln u && \text{in } L^2(L^2), && (24)\\
\ln u_\varepsilon &\to \ln u && \text{pointwise a.e. in } (0,T)\times\Omega, && (25)\\
u_\varepsilon &\to u && \text{pointwise a.e. in } (0,T)\times\Omega, && (26)\\
w_\varepsilon u_\varepsilon &\to wu && \text{in } L^1(L^1), && (27)\\
\ln u_\varepsilon &\rightharpoonup^* \ln u && \text{in } \mathrm{BV}([0,T];W^{1,p}(\Omega)^*), && (28)\\
\partial_t w_\varepsilon &\rightharpoonup \partial_t w && \text{in } L^2(0,T;W^{1,p}(\Omega)^*), && (29)\\
w_\varepsilon &\to w && \text{in } C_w([0,T];L^2(\Omega)). && (30)
\end{aligned}
$$

Proof. We will not relabel subsequences throughout this proof. The boundedness of $\{w_\varepsilon\}$ and $\{\nabla w_\varepsilon\}$ in $L^2(L^2)$ from Proposition 4.5 imply that there exists a subsequence weakly convergent in $L^2(W^{1,2})$ to some $w \in L^2(W^{1,2})$, since $L^2(L^2)$ is reflexive. Additionally, due to the boundedness of $\{\partial_t w_\varepsilon\}$ in $L^2(W^{1,p}(\Omega)^*)$ and Aubin–Lions Lemma, see for example [36, Lem. 7.7], there exists a subsequence $\{w_\varepsilon\}$ strongly convergent to w in $L^2(L^2)$ and thus we find another subsequence which converges pointwise almost everywhere to w in $(0,T)\times\Omega$. The uniform boundedness of $\{w_\varepsilon\}$ in $L^\infty((0,T)\times\Omega)$ additionally implies the weak*-convergence of a subsequence to w in $L^\infty((0,T)\times\Omega)$, see for example [15, Thm. A.2.18].

Since $\{\nabla \ln u_\varepsilon\}$ is bounded in $L^2(L^2)$ there is a subsequence, which is weakly convergent in $L^2(L^2)$. By Poincaré's inequality, see Theorem 13.27 in [27, p. 432], we have

$$
\|\ln u_\varepsilon(t) - (\ln u_\varepsilon)^{\text{ave}}(t)\|_{L^2(\Omega)} \le C\,\|\nabla \ln u_\varepsilon(t)\|_{L^2(\Omega)} \le C \tag{31}
$$

for almost all $t \in (0,T)$, where for $g \in L^1(\Omega)$ we define g^{ave} by

$$
g^{\text{ave}} := \frac{1}{|\Omega|}\int_\Omega g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.
$$

We find the constant upper bound for the $L^2(L^2)$-norm of $\ln u_\varepsilon$ by the *a priori* estimates from Proposition 4.5. Using the $L^\infty(L^1)$ *a priori* bound for $\ln u_\varepsilon$, cf. the third term of (11a), we can estimate

$$
(\ln u_\varepsilon)^{\text{ave}}(t) = \frac{1}{|\Omega|}\int_\Omega \ln u_\varepsilon(t)\,\mathrm{d}\boldsymbol{x} \le \frac{1}{|\Omega|}\|\ln u_\varepsilon\|_{L^\infty(L^1)} \le \frac{1}{|\Omega|}C.
$$

Plugging this into (31) and using the reverse triangle inequality of the norm, we obtain

$$
\int_0^T \|\ln u_\varepsilon(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \int_0^T C^2\left(\|\nabla \ln u_\varepsilon(t)\|_{L^2(\Omega)} + 1\right)^2\mathrm{d}t \le C,
$$

by the *a priori* estimate from the fourth term of (11a). Again we find a subsequence $\{\ln u_\varepsilon\}$, which is weakly convergent to some $\xi \in L^2(W^{1,2})$. We define $u := e^\xi$. This implies the convergence from (23).

Using the *a priori* bound on $\{\partial_t \ln u_\varepsilon\}$ and the compact embedding from Aubin–Lions Lemma, see [36, Lem. 7.7], we find, as above, another subsequence $\{\ln u_\varepsilon\}$ converging strongly in $L^2(L^2)$ to $\ln u$ and pointwise almost everywhere. This implies that also $u_\varepsilon \to u$ pointwise almost everywhere and we have shown the convergences from (23)–(26).

The product $\{u_\varepsilon w_\varepsilon\}$ is bounded in $L^1(L^1)$ and equi-integrable by Proposition 4.5. The equi-integrability follows from the fact that $u_\varepsilon w_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1)$ is bounded in $L^1(L^1)$ by an application of the de la Vallée Poussin theorem, see for example [27, p. 675], which is applicable since $G : [0,\infty) \to [0,\infty]$ with $x \mapsto x\ln(x+1)$ is an increasing, convex function and fulfills $\frac{G(|x|)}{x} \to \infty$ as $x \to \infty$. By Vitali's theorem, see [14, Thm. 5.6], and the pointwise convergence of $\{u_\varepsilon w_\varepsilon\}$ to uw, cf. convergences (21) and (26), we can deduce the strong convergence of $\{u_\varepsilon w_\varepsilon\}$ to uw in $L^1(L^1)$.

By the reflexivity of $L^2(W^{1,p}(\Omega)^*)$ we find a weakly convergent subsequence of $\{\partial_t w_\varepsilon\}$ convergent to some $\eta \in L^2(W^{1,p}(\Omega)^*)$. By the convergence of $\{w_\varepsilon\}$ to w in $L^2(L^2)$ we find $\eta = \partial_t w$. Thus the limit w is an absolutely continuous function with values in $W^{1,p}(\Omega)^*$, *i.e.* $w \in \mathrm{AC}([0,T]; W^{1,p}(\Omega)^*)$, and consequently

$$w \in L^\infty(0,T;L^2(\Omega)) \cap C_w([0,T];W^{1,p}(\Omega)^*) \hookrightarrow C_w([0,T];L^2(\Omega)) \tag{32}$$

by Lemma 8.1 in [28, p. 275]. We can additionally infer the convergence of a subsequence $\{w_\varepsilon\}$ in $C_w([0,T];L^2(\Omega))$ as a consequence of the boundedness in the intersection space of (32), see Proposition 4.9 in [13]. This gives the convergence in (30).

For the convergence of $\{\partial_t \ln u_\varepsilon\}$, we consider $\{\ln u_\varepsilon\}$ as abstract functions of bounded variation, more precisely in the space $\mathrm{BV}([0,T];W^{1,p}(\Omega)^*)$. We first show that $\{\ln u_\varepsilon\}$ is bounded in that space. To that aim we show that $\{\ln u_\varepsilon\}$ is bounded in $W^{1,1}(0,T;W^{1,p}(\Omega)^*)$ and use the continuous embedding

$$W^{1,1}(0,T;W^{1,p}(\Omega)^*) \hookrightarrow \mathrm{BV}([0,T];W^{1,p}(\Omega)^*).$$

The $L^1(0,T;W^{1,p}(\Omega)^*)$-norm of $\{\ln u_\varepsilon\}$ is bounded since

$$L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;W^{1,p}(\Omega)^*) \hookrightarrow L^1(0,T;W^{1,p}(\Omega)^*)$$

holds and the $L^1(0,T;W^{1,p}(\Omega)^*)$-norm of $\{\partial_t \ln u_\varepsilon\}$ is bounded by the *a priori* estimate from (11a). Thus we get

$$\|\ln u_\varepsilon\|_{\mathrm{BV}} \le C$$

for some $C > 0$ independent of ε. By Corollary 3.11 from [18] we have that $\{\ln u_\varepsilon\}$ is relatively compact in $\mathrm{BV}([0,T];W^{1,p}(\Omega)^*)$ and we can find a subsequence, which converges weak* in this space to some l. By the convergence of $\ln u_\varepsilon \to \ln u$ in $L^2(L^2)$ we can deduce $l = \ln u$ by possible redefining u on a set of measure zero. □

Proposition 4.8. *Let $\{(u_\varepsilon, w_\varepsilon)\}$ be the sequence of approximate solutions and (u,w) the limit from Proposition 4.7. Under the assumption that $w_\varepsilon(0) \to w_0$ strongly in $L^2(\Omega)$, we have $\nabla w_\varepsilon \to \nabla w$ in $L^2(L^2)$ as $\varepsilon \downarrow 0$.*

Proof. To show this convergence, we show that $\{\nabla w_\varepsilon\}$ is a Cauchy sequence in $L^2(L^2)$. This implies the strong $L^2(L^2)$-convergence to some $\eta \in L^2(L^2)$ and since $\{\nabla w_\varepsilon\}$ converges weakly to ∇w in $L^2(L^2)$ we can identify the limit with $\eta = \nabla w$.

For fixed $\varepsilon, \tilde\varepsilon > 0$ we can subtract the equations for w_ε and $w_{\tilde\varepsilon}$ and test the equation with $(w_\varepsilon - w_{\tilde\varepsilon})$. This yields

$$\frac{1}{2}\int_\Omega (w_\varepsilon - w_{\tilde\varepsilon})^2 \,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t \int_\Omega \mu |\nabla (w_\varepsilon - w_{\tilde\varepsilon})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \int_0^t \int_\Omega \gamma (w_\varepsilon - w_{\tilde\varepsilon})^2 - \delta (u_\varepsilon w_\varepsilon - u_{\tilde\varepsilon} w_{\tilde\varepsilon})(w_\varepsilon - w_{\tilde\varepsilon}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,$$

which in turn implies

$$\mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w_{\tilde\varepsilon})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \frac{1}{2}\int_\Omega (w_\varepsilon - w_{\tilde\varepsilon})^2(0)\,\mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega \gamma(w_\varepsilon - w_{\tilde\varepsilon})^2 - \delta(u_\varepsilon w_\varepsilon - u_{\tilde\varepsilon} w_{\tilde\varepsilon})(w_\varepsilon - w_{\tilde\varepsilon})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Since $\{w_\varepsilon\}$ is strongly convergent in $L^2(L^2)$ it is also a Cauchy sequence. Let $\xi > 0$ be arbitrary. Then we find $\varepsilon_1 > 0$, such that

$$\int_0^t \int_\Omega \gamma(w_\varepsilon - w_{\tilde\varepsilon})^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \frac{\xi}{8}$$

holds for all $0 < \varepsilon, \tilde\varepsilon < \varepsilon_1$. Since $\{w_\varepsilon\}$ is uniformly bounded in $L^\infty((0,T)\times\Omega)$ and pointwise almost everywhere convergent to w and $\{u_\varepsilon w_\varepsilon\}$ is pointwise almost everywhere convergent to uw and equi-integrable in $L^1(L^1)$, cf. Proposition 4.5, Vitali's theorem, see [14, Thm. 5.6] yields

$$\int_0^t \int_\Omega -\delta u_\varepsilon w_\varepsilon w_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \to \int_0^t \int_\Omega -\delta u w^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Thus we find $\varepsilon_2 > 0$, such that

$$\int_0^t \int_\Omega -\delta u_\varepsilon w_\varepsilon w_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \int_0^t \int_\Omega -\delta u w^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{8}$$

for all $\varepsilon < \varepsilon_2$. By the strong convergence of $\{u_\varepsilon w_\varepsilon\}$, cf. (27), we find an $\varepsilon_3 > 0$ such that

$$\int_0^t \int_\Omega \delta u_\varepsilon w_\varepsilon \vartheta \le \int_0^t \int_\Omega \delta u w \vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{8}$$

for all $\varepsilon < \varepsilon_3$ and $\vartheta \in L^\infty((0,T)\times\Omega)$. Since $w_{\tilde\varepsilon} \in L^\infty((0,T)\times\Omega)$ this inequality holds for $\vartheta = w_{\tilde\varepsilon}$. By the assumption on the strong convergence of $\{w_\varepsilon(0)\}$, we find $\varepsilon_4 > 0$ such that

$$\frac{1}{2}\int_\Omega (w_\varepsilon - w_{\tilde\varepsilon})^2(0)\,\mathrm{d}\boldsymbol{x} \le \frac{\xi}{8}$$

for all $\varepsilon, \tilde\varepsilon < \varepsilon_4$. Thus for all $\varepsilon, \tilde\varepsilon < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ we get

$$\mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w_{\tilde\varepsilon})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \frac{\xi}{4} - 2\delta \int_0^t \int_\Omega u w^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{4} + \delta \int_0^t \int_\Omega u w w_{\tilde\varepsilon} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{8} + \delta \int_0^t \int_\Omega u w w_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{8}. \quad (33)$$

By the convergence from (22), we find $0 < \varepsilon_5 < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, such that we have

$$\delta \int_0^t \int_\Omega u w w_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \delta \int_0^t \int_\Omega u w^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\xi}{8}$$

for all $\varepsilon < \varepsilon_5$. Plugging this estimate into the inequality (33) we get

$$\mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w_{\tilde\varepsilon})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \xi.$$

This shows that $\{\nabla w_\varepsilon\}$ is indeed a Cauchy sequence in $L^2(L^2)$ and our proof is complete. □

Proof (of Theorem 2.4). For every (u_0, w_0), fulfilling the conditions from Theorem 2.4, we find a sequence $\{(u_{0\varepsilon}, w_{0\varepsilon})\}$ of initial values fulfilling the assumptions from Theorem 4.2 and Proposition 4.5, such that $u_{0\varepsilon} \rightharpoonup u_0$ in $L^1(\Omega)$ and $w_{0\varepsilon} \to w_0$ in $L^2(\Omega)$. Using mollifiers we find a sequence $\{w_{0\varepsilon}\} \subseteq C_0^\infty(\Omega)$ such that $w_{0\varepsilon} \geq 0$, $w_{0\varepsilon} \rightharpoonup^* w_0$ in $L^\infty(\Omega)$ and

$$\|w_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)}$$

holds. Thus $\{w_{0\varepsilon}\}$ is uniformly bounded in ε and fulfills all the required assumptions. Applying an L^p-interpolation theorem, we get the strong convergence $w_{0\varepsilon} \to w_0$ in $L^r(\Omega)$ for all $r \in [1, \infty)$. Defining

$$u_{0\varepsilon} := \begin{cases} u_0 & \text{if } u_0 \leq n_\varepsilon, \\ n_\varepsilon & \text{if } u_0 > n_\varepsilon, \end{cases}$$

for a sequence $\{n_\varepsilon\} \subseteq \mathbb{N}$ with $n_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$, we have constructed a sequence of approximate initial conditions for the predator fulfilling the required assumptions.

The weak convergence (19) of $\{w_\varepsilon\}$ from Proposition 4.7 implies

$$\int_0^t \int_\Omega w_\varepsilon g \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \to \int_0^t \int_\Omega w g \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$

for all $g \in L^2(L^2)$ and all $t \in (0, T)$. Since we have $\vartheta \in L^2(L^2)$, $\nabla\vartheta \in L^2(L^2)$ and $\partial_t\vartheta \in L^2(L^2)$, we get

$$\int_0^t \int_\Omega w_\varepsilon \partial_t\vartheta - \mu\nabla w_\varepsilon \cdot \nabla\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \to \int_0^t \int_\Omega w\partial_t\vartheta - \mu\nabla w \cdot \nabla\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,$$
$$\int_0^t \int_\Omega \gamma w_\varepsilon \vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \to \int_0^t \int_\Omega \gamma w \vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,$$

for all $t \in (0, T)$ and all $\vartheta \in C^1([0, T]; L^\infty(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$. The strong convergence of $\{w_\varepsilon u_\varepsilon\}$ in $L^1(L^1)$ implies

$$\int_0^t \int_\Omega \delta u_\varepsilon w_\varepsilon \vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \to \int_0^t \int_\Omega \delta u w \vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$

for all $t \in (0, T)$, since $\vartheta \in C^1([0, T]; L^\infty(\Omega)) \hookrightarrow L^\infty((0, T) \times \Omega)$. Making use of the the convergence of $\{w_\varepsilon\}$ in $C_w([0, T]; L^2(\Omega))$, cf. (32), we find

$$\int_\Omega w_\varepsilon(t)\vartheta \,\mathrm{d}\boldsymbol{x} \to \int_\Omega w(t)\vartheta \,\mathrm{d}\boldsymbol{x}$$

for *all* $t \in [0, T]$ and all $\vartheta \in C^1([0, T]; L^\infty(\Omega))$. By the convergence of the initial data, we have

$$\int_\Omega w_\varepsilon(0) \,\mathrm{d}\boldsymbol{x} = \int_\Omega w_{0\varepsilon} \,\mathrm{d}\boldsymbol{x} \to \int_\Omega w_0 \,\mathrm{d}\boldsymbol{x}.$$

This shows that the limit identified in Proposition 4.7 fulfills the variational formulation (4). The equality of $w(0) = w_0$ in $L^2(\Omega)$ follows by the convergence of the initial data $\{w_{0\varepsilon}\}$ in $L^2(\Omega)$ and the convergence of $\{w_\varepsilon\}$ in $C_w([0, T]; L^2(\Omega))$, cf. (30). Since $\{w_\varepsilon\}$ is pointwise almost everywhere convergent and uniformly bounded in $L^\infty((0, T) \times \Omega)$ this bound transfers to the limit and we get $w \in L^\infty((0, T) \times \Omega)$.

The Laplacian is a bounded linear map between $W^{1,2}(\Omega)$ and its dual $W^{1,2}(\Omega)^*$. Since we know w to be in $L^2(0,T;W^{1,2}(\Omega))$, we find $\Delta w \in L^2(0,T;W^{1,2}(\Omega)^*)$ and equation (1b) now implies

$$\partial_t w \in L^2(0,T;W^{1,2}(\Omega)^* + L^1(\Omega)).$$

Next, we show that the population and the logarithmic inequality for the predator, cf. equations (2) and (3) respectively, are fulfilled. On the approximate level the population inequality (2) holds with equality, cf. equation (12). By the strong convergence of $\{u_\varepsilon w_\varepsilon\}$ in $L^1(L^1)$ we get

$$\alpha \int_0^t \int_\Omega u_\varepsilon w_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \to \alpha \int_0^t \int_\Omega uw \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Using the weak lower semicontinuity of the convex function $x \mapsto e^x$ and the weak convergence of $\{\ln u_\varepsilon\}$ we get

$$\beta \int_0^t \int_\Omega u \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \beta \int_0^t \int_\Omega e^{\ln u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \liminf_{\varepsilon\downarrow 0} \beta \int_0^t \int_\Omega e^{\ln u_\varepsilon} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Using the boundedness of $\{\ln u_\varepsilon\}$ in $L^2(0,T;L^2(\Omega)) \cap BV([0,T];W^{1,p}(\Omega)^*)$ and applying Theorem A.5 from [35], we find a subsequence of $\{\ln u_\varepsilon\}$ and an $l \in L^2(L^2) \cap L^\infty(W^{1,p}(\Omega)^*)$, such that

$$\ln u_\varepsilon \rightharpoonup^* l \text{ in } L^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega)^*), \tag{34}$$

$$\ln u_\varepsilon(t) \rightharpoonup l(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0,T). \tag{35}$$

Since, we already have the strong convergence of $\{\ln u_\varepsilon\}$ in $L^2(L^2)$ to $\ln u$, we can identify the limit l with $\ln u$. The uniform boundedness of $\{\ln u_{0\varepsilon}\}$ in $W^{1,p}(\Omega)^*$, which is needed to apply Theorem A.5, follows from the boundedness of $\{\ln u_{0\varepsilon}\}$ in $L^1(\Omega)$ and the continuous embedding of $L^1(\Omega)$ into $W^{1,p}(\Omega)^*$. Using Lemma 7.2 from [32], we find that

$$\ln u_\varepsilon(t) \rightharpoonup^* \bar{l}(t) \text{ in } W^{1,p}(\Omega)^* \text{ for } \underline{\textit{all}}\ t \in [0,T], \tag{36}$$

for some $\bar{l} \in \mathrm{BV}([0,T];W^{1,p}(\Omega)^*)$. We find $\ln u = \bar{l}$ everywhere in $[0,T]$, after possibly redefining u on a set of measure zero. Again using the weak lower semicontinuity of $x \mapsto e^x$, we get

$$\int_\Omega u(t)\,\mathrm{d}\boldsymbol{x} = \int_\Omega e^{\ln u(t)}\,\mathrm{d}\boldsymbol{x} \le \liminf_{\varepsilon\downarrow 0} \int_\Omega e^{\ln u_\varepsilon(t)}\,\mathrm{d}\boldsymbol{x} = \liminf_{\varepsilon\downarrow 0} \int_\Omega u_\varepsilon(t)\,\mathrm{d}\boldsymbol{x}$$

for all $t \in [0,T]$. The weak convergence of the initial values implies

$$\int_\Omega u_0 \,\mathrm{d}\boldsymbol{x} = \lim_{\varepsilon\downarrow 0} \int_\Omega u_{0\varepsilon}\,\mathrm{d}\boldsymbol{x}.$$

Putting these results together we get

$$\begin{aligned}\int_\Omega u \,\mathrm{d}\boldsymbol{x}\Big|_0^t + \beta \int_0^t \int_\Omega u \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s &\le \liminf_{\varepsilon\downarrow 0} \int_\Omega u_\varepsilon(t)\,\mathrm{d}\boldsymbol{x} \\ &\quad - \lim_{\varepsilon\downarrow 0} \int_\Omega u_{0\varepsilon}\,\mathrm{d}\boldsymbol{x} + \liminf_{\varepsilon\downarrow 0} \beta \int_0^t \int_\Omega u_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \liminf_{\varepsilon\downarrow 0} \int_0^t \int_\Omega \varepsilon u_\varepsilon |u_\varepsilon|^{p-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &= \liminf_{\varepsilon\downarrow 0} \alpha \int_0^t \int_\Omega w_\varepsilon u_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \alpha \int_0^t \int_\Omega wu \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,\end{aligned}$$

for all $t \in [0,T]$. This shows, that the population inequality (2) holds.

To derive the logarithmic inequality (3), we use the regularized logarithmic equality from (18). To take the limit $\varepsilon \downarrow 0$ we note the following. By the weak lower semicontinuity of the norm and the weak convergence (23), we get

$$\int_0^t \int_\Omega \vartheta |\nabla \ln u|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leq \liminf_{\varepsilon\downarrow 0} \int_0^t \int_\Omega \vartheta |\nabla \ln u_\varepsilon|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,$$

where we used the fact, that $\vartheta \geq 0$ to pull it out of the absolute value. Using the strong convergence of the sequence $\{\nabla w_\varepsilon\}$ in $L^2(L^2)$ to ∇w, which is shown in Proposition 4.8, we can deduce that

$$\nabla \ln u_\varepsilon \cdot \nabla w_\varepsilon \rightharpoonup \nabla \ln u \cdot \nabla w \text{ in } L^1(L^1).$$

By the convergence of the initial data, we get

$$\lim_{\varepsilon\downarrow 0} \int_\Omega \ln u_{0\varepsilon}(\boldsymbol{x})\vartheta(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \ln u_0 \vartheta(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},$$

for all $\vartheta \in C([0,T]; L^\infty(\Omega))$. Putting all this together, we see that the shown convergences are sufficient to pass to the limit with $\varepsilon \downarrow 0$ in the regularized logarithmic equality, where the equality becomes an inequality,

$$\begin{aligned}
-\int_\Omega \ln u\vartheta \,\mathrm{d}\boldsymbol{x}\Big|_0^t &+ \int_0^t \int_\Omega \nu|\nabla \ln u|^2\vartheta - \nu\nabla \ln u\cdot\nabla\vartheta - \kappa\vartheta\nabla w\cdot\nabla\ln u + \kappa\nabla w\cdot\nabla\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\leq \liminf_{\varepsilon\downarrow 0}\left[-\int_\Omega \ln u_\varepsilon \vartheta \,\mathrm{d}\boldsymbol{x}\Big|_0^t\right] + \liminf_{\varepsilon\downarrow 0}\int_0^t\int_\Omega \nu|\nabla\ln u_\varepsilon|^2\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad + \lim_{\varepsilon\downarrow 0}\int_0^t\int_\Omega -\nu\nabla\ln u_\varepsilon\cdot\nabla\vartheta - \kappa\vartheta\nabla w_\varepsilon\cdot\nabla\ln u_\varepsilon + \kappa\nabla w_\varepsilon\cdot\nabla\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&= \liminf_{\varepsilon\downarrow 0}\int_0^t\int_\Omega (\beta-\alpha w_\varepsilon)\vartheta - \ln u_\varepsilon\partial_t\vartheta + \varepsilon|u_\varepsilon|^{p-1}\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&= \int_0^t\int_\Omega (\beta-\alpha w)\vartheta - \ln u\partial_t\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$, where the regularizing term vanishes as ε goes to zero due to the boundedness of $\varepsilon^{1/p}u_\varepsilon$ in $L^p(L^p)$ from Proposition 4.5. The fact that u fulfills the initial condition u_0 in $L^1(\Omega)$ follows from the convergence for *all* $t \in [0,T]$ in $W^{1,p}(\Omega)^*$ cf. (36) and the convergence of the initial data. For all $\varphi \in W^{1,p}(\Omega)$ we have

$$\int_\Omega \ln u_0\varphi \,\mathrm{d}\boldsymbol{x} = \lim_{\varepsilon\downarrow 0}\int_\Omega \ln u_{0\varepsilon}\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \ln u(0)\varphi \,\mathrm{d}\boldsymbol{x}.$$

By a density argument we find $\ln u(0) = \ln u_0$ in $L^1(\Omega)$ and thus $u(0) = u_0$ in $L^1(\Omega)$. Using the population inequality, we find that u is bounded in $L^\infty(L^1)$, where we used the $L^1(L^1)$ integrability of uw and the $L^1(\Omega)$ integrability of u_0. Again using the fact, that $\ln u_\varepsilon(t) \rightharpoonup \ln u(t)$ in $L^2(\Omega)$ for almost all $t \in (0,T)$, we can transfer the $L^\infty(L^1)$-bound from $\{\ln u_\varepsilon\}$ to $\ln u$ by using the weak lower semicontinuity of the norm. Thus we have shown that $\ln u \in L^\infty(L^1)$, which implies $u > 0$ almost everywhere. The non-negativity of w follows from the comparison principle, cf. Lemma 4.3. □

5 Weak-strong uniqueness

In order to justify that our solution concept is meaningful, we prove that it fulfills the weak-strong uniqueness property. That is, if there exists a strong solution to the system (1) to some initial data, then all generalized solution emanating from the same initial data coincide with the strong solution and thus this solution is unique as long as it exists. In order to be able to prove such a property for the generalized solutions, some additional properties of strong solutions are needed.

5.1 Properties of strong solution

Later on, in the proof of the weak-strong uniqueness, we would like to test with $1/\tilde{u}$, for a strong solution $\tilde{u}$ to (1a). In order to justify that this is possible, we prove the following lemma.

Lemma 5.1. *Let the initial condition $u_0 \in C^3(\overline{\Omega})$ be bounded away from zero by some $\underline{l} > 0$. Then the strong solution $\tilde{u}$ from Definition 2.5 is bounded away from zero.*

To be able to prove this lemma, we first show the following comparison principle.

Proposition 5.2 (Comparison principle for $\tilde{u}$)**.** *Let $\tilde{w} \in C^1([0,T] \times \overline{\Omega})$ with $\Delta\tilde{w} \in C([0,T] \times \overline{\Omega})$. Assume that there exist a strong sub-solution $\underline{u}$ and a strong super-solution $\bar{u}$ to (1a) with $w = \tilde{w}$, fulfilling the non-negative initial data $\underline{u}_0, \bar{u}_0 \in C^3(\overline{\Omega})$ respectively. That is $\underline{u}$ and $\bar{u}$ fulfill*

$$\underline{u}, \bar{u} \in C^1([0,T] \times \overline{\Omega}) \text{ and } \Delta\underline{u}, \Delta\bar{u} \in C([0,T] \times \overline{\Omega}),$$

are non-negative and it holds,

$$\underline{u}_t - \nu\Delta\underline{u} + \kappa\nabla \cdot (\underline{u}\nabla\tilde{w}) \leq (\alpha\tilde{w} - \beta)\underline{u} \qquad \text{in } (0,T) \times \Omega, \tag{37}$$

$$\bar{u}_t - \nu\Delta\bar{u} + \kappa\nabla \cdot (\bar{u}\nabla\tilde{w}) \geq (\alpha\tilde{w} - \beta)\bar{u} \qquad \text{in } (0,T) \times \Omega, \tag{38}$$

$$\underline{u} = \underline{u}_0 \qquad \text{on } \{0\} \times \Omega, \tag{39}$$

$$\bar{u} = \bar{u}_0 \qquad \text{on } \{0\} \times \Omega, \tag{40}$$

$$\nabla\underline{u} \cdot n = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{41}$$

$$\nabla\bar{u} \cdot n = 0 \qquad \text{on } [0,T] \times \partial\Omega. \tag{42}$$

If additionally we have $\underline{u}_0 \leq \bar{u}_0$, then $\underline{u}(t,\boldsymbol{x}) \leq \bar{u}(t,\boldsymbol{x})$ holds everywhere in $[0,T] \times \overline{\Omega}$.

Proof. Subtracting (38) from (37) and testing the resulting inequality with $(\underline{u} - \bar{u})^+$, we find

$$\int_0^t \int_\Omega \partial_t(\underline{u} - \bar{u})(\underline{u} - \bar{u})^+ \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \int_0^t \int_\Omega \nu|\nabla(\underline{u} - \bar{u})^+|^2 - \kappa(\underline{u} - \bar{u})^+ \nabla\tilde{w} \cdot \nabla(\underline{u} - \bar{u})^+ \\ + \beta\left((\underline{u} - \bar{u})^+\right)^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \int_0^t \int_\Omega \alpha\tilde{w}\left((\underline{u} - \bar{u})^+\right)^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s. \tag{43}$$

Using Young's inequality, collecting alike terms and integrating by parts we get

$$\frac{1}{2}\left\|(\underline{u} - \bar{u})^+(t)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left(\alpha\left\|\tilde{w}\right\|_{L^\infty((0,T)\times\Omega))} + \frac{\kappa^2}{2\nu}\left\|\nabla\tilde{w}(s)\right\|_{L^\infty(\Omega)}^2\right)\left\|(\underline{u} - \bar{u})^+(s)\right\|_{L^2(\Omega)}^2 \,\mathrm{d}s.$$

An application of Gronwall's inequality yields $(\underline{u} - \bar{u})^+ = 0$ almost everywhere and with the continuity of $\underline{u}$ and $\bar{u}$ this equality extends to the whole domain $[0,T] \times \overline{\Omega}$. Thus we have $\underline{u}(t,\boldsymbol{x}) \leq \bar{u}(t,\boldsymbol{x})$ everywhere in $[0,T] \times \overline{\Omega}$ as claimed. □

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Now we prove Lemma 5.1.

Proof (of Lemma 5.1). We will construct a sub-solution $\underline{u}^*$ which is bounded away from zero and constant in space. The predator equation for such a sub-solution then reads

$$\partial_t \underline{u} + (\kappa\Delta\tilde{w} - \alpha\tilde{w} + \beta)\,\underline{u} \leq 0,$$

since all space derivatives of $\underline{u}$ vanish. We first consider the ordinary differential equation

$$\partial_t \underline{u} + \Big(\kappa \left\|\Delta\tilde{w}\right\|_{L^\infty((0,T)\times\Omega)} + \alpha \left\|\tilde{w}\right\|_{L^\infty((0,T)\times\Omega)} + \beta\Big)\,\underline{u} = 0$$

with the initial value $u(0) \equiv \underline{l}$. This ordinary differential equation can be solved explicitly by

$$\underline{u}^*(t) = \underline{l}\exp\Big(-\Big(\kappa \left\|\Delta\tilde{w}\right\|_{L^\infty((0,T)\times\Omega)} + \alpha \left\|\tilde{w}\right\|_{L^\infty((0,T)\times\Omega)} + \beta\Big)\,t\Big)\,.$$

The solution $\underline{u}^*$ is monotonically decreasing and continuous, thus it is bounded from below on $[0,T]$ by $\underline{u}^*(T)$ and $\underline{u}^*(T) > 0$ holds. Now,

$$\partial_t \underline{u}^* + (\kappa\Delta\tilde{w} - \alpha\tilde{w} + \beta)\,\underline{u}^* \leq \partial_t \underline{u}^* + \Big(\kappa \left\|\Delta\tilde{w}\right\|_{L^\infty((0,T)\times\Omega)} + \alpha \left\|\tilde{w}\right\|_{L^\infty((0,T)\times\Omega))} + \beta\Big)\,\underline{u}^* = 0$$

holds, since $\underline{u}^*$ is non-negative, and thus $\underline{u}^*$ is a sub-solution to (1a) and by the comparison principle from Proposition 5.2 we conclude that $\tilde{u}$ is bounded away from zero. □

Now we have finished the technical groundwork for our proof of weak-strong uniqueness.

5.2 Relative energy estimates

Using the integration by parts formula from the Appendix, cf. Lemma 7.3, we can prove a relative energy inequality, which serves as a strong tool when proving weak-strong uniqueness of generalized solutions. In order to formulate this inequality we need the following definitions. We say $(\tilde{u},\tilde{w}) \in \mathcal{Y}$ if

$$\tilde{u},\tilde{w} \in C^1([0,T]\times\overline{\Omega}) \text{ and } \Delta\tilde{u},\Delta\tilde{w} \in C([0,T]\times\overline{\Omega}),$$
$$\tilde{w},\tilde{u} \geq 0 \text{ and } \tilde{u} \geq l > 0 \text{ for some } l > 0$$

and $\tilde{u}$ and $\tilde{w}$ fulfill zero Neumann boundary conditions.

Definition 5.3. *For $(u,w) \in \mathcal{X}$ and a smooth test function $(\tilde{u},\tilde{w}) \in \mathcal{Y}$, we define the relative energy $\mathcal{R} : \mathcal{X}\times\mathcal{Y} \to L^\infty(0,T)$ by*

$$\mathcal{R}(u,w|\tilde{u},\tilde{w}) = \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln\tilde{u}) + \frac{\kappa^2}{\mu\nu}\tilde{u}|w-\tilde{w}|^2\,\mathrm{d}\boldsymbol{x}, \tag{44}$$

the relative dissipation $\mathcal{W} : \mathcal{X}\times\mathcal{Y} \to L^1(0,T)$

$$\begin{aligned}\mathcal{W}(u,w|\tilde{u},\tilde{w}) = \int_\Omega (\beta-\alpha\tilde{w})(u-\tilde{u}-\tilde{u}(\ln u - \ln\tilde{u}))\,\mathrm{d}\boldsymbol{x} + \frac{2\kappa^2}{\mu\nu}\int_\Omega (\delta\tilde{u}-\gamma)\tilde{u}|w-\tilde{w}|^2\,\mathrm{d}\boldsymbol{x} \\ + \frac{\nu}{2}\int_\Omega \tilde{u}|\nabla\ln u - \nabla\ln\tilde{u}|^2\,\mathrm{d}\boldsymbol{x} + \frac{\kappa^2}{2\nu}\int_\Omega \tilde{u}|\nabla w - \nabla\tilde{w}|^2\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{45}$$

the regularity weight $\mathcal{K} : \mathcal{Y} \to L^\infty(0,T)$

$$\begin{aligned}\mathcal{K}(\tilde u,\tilde w) = {}& \|\partial_t \ln \tilde u\|_{L^\infty(\Omega)} + \mu \|\nabla \ln \tilde u\|^2_{L^\infty(\Omega)} \\ & + \max\left\{\frac{1}{\|w\|_{L^\infty(\Omega)} + \|\tilde w\|_{L^\infty(\Omega)} + 1}, \|w\|_{L^\infty(\Omega)} + \|\tilde w\|_{L^\infty(\Omega)} + 1\right\} \\ & \left(\frac{\alpha\mu\nu}{\kappa^2} + \alpha + 2\delta \|\tilde u\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} \left(\frac{\kappa^2}{\mu\nu} + 1\right) + \frac{\kappa^2\delta}{2\mu\nu} \|\tilde u\|_{L^\infty(\Omega)}\right) \\ & + \max\{\alpha \|\tilde w\|_{L^\infty(\Omega)}, 2\gamma\}, \end{aligned} \tag{46}$$

and the system operator $\mathcal{A} : \mathcal{Y} \to (L^\infty((0,T)\times\Omega))^2$

$$\mathcal{A}(\tilde u,\tilde w) = \begin{pmatrix} \partial_t \tilde u - \nu\Delta\tilde u + \kappa\nabla\cdot(\tilde u\nabla\tilde w) + (\beta - \alpha\tilde w)\tilde u \\ \partial_t\tilde w - \mu\Delta\tilde w + \delta\tilde u\tilde w - \gamma\tilde w \end{pmatrix}. \tag{47}$$

Note, that the regularity weight $\mathcal{K}$ can be considered as independent of w for fixed and bounded initial data, since the $L^\infty((0,T)\times\Omega)$-bound of w only depends on the initial value w_0 and the finial time T, cf. Lemma 4.3.

Lemma 5.4 (Relative energy inequality)**.** *Let* $(u,w) \in \mathcal{X}$ *be a generalized solution to system* (1) *according to Definition 2.1 and* $(\tilde u,\tilde w) \in \mathcal{Y}$ *be a smooth test function. Then, the following relative energy inequality holds,*

$$\begin{aligned}&\mathcal{R}(u,w|\tilde u,\tilde w)(t) + \int_0^t \Big(\mathcal{W}(u,w|\tilde u,\tilde w) + \max\{\alpha\|\tilde w\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}(u,w|\tilde u,\tilde w)\Big) e^{\int_s^t \mathcal{K}(\tilde u,\tilde w)\,\mathrm{d}\tau}\,\mathrm{d}s \\ &\quad\le \mathcal{R}(u,w|\tilde u,\tilde w)(0)e^{\int_0^t \mathcal{K}(\tilde u,\tilde w)\,\mathrm{d}s} - \int_0^t\int_\Omega \mathcal{A}(\tilde u,\tilde w)\cdot\begin{pmatrix}\ln u - \ln\tilde u \\ \frac{2\kappa}{\mu\nu}\tilde u(w-\tilde w)\end{pmatrix} e^{\int_s^t \mathcal{K}(\tilde u,\tilde w)\,\mathrm{d}\tau}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \end{aligned} \tag{48}$$

for almost all $t \in [0,T]$*, where* $\mathcal{R}, \mathcal{W}, \mathcal{A}$ *and* $\mathcal{K}$ *are defined in Definition 5.3.*

Proof. Adding the population inequality (2) for u and the logarithmic inequality (3) for u tested with $\tilde u$ and adding and subtracting the system operator $\mathcal{A}(\tilde u,\tilde w)$ tested with $(\ln u - \ln\tilde u, 0)^T$ yields

$$\begin{aligned}&\int_\Omega u - \tilde u - \tilde u(\ln u - \ln\tilde u)\Big|_0^t \,\mathrm{d}\boldsymbol{x} + \beta\int_0^t\int_\Omega u - \tilde u - \tilde u(\ln u - \ln\tilde u)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &\quad + \nu\int_0^t\int_\Omega \tilde u|\nabla\ln u|^2 - \nabla\ln u\cdot\nabla\tilde u - \nabla\tilde u\cdot\nabla(\ln u - \ln\tilde u)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &\quad + \kappa\int_0^t\int_\Omega \nabla w\cdot\nabla\tilde u - \tilde u\nabla w\cdot\nabla\ln u + \nabla\tilde w\cdot\nabla(\ln u - \ln\tilde u)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &\qquad \le \alpha\int_0^t\int_\Omega uw - \tilde u\tilde w - \tilde u(w-\tilde w) - \tilde w\tilde u\,(\ln u - \ln\tilde u)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\ &\qquad\quad - \int_0^t\int_\Omega \mathcal{A}(\tilde u,\tilde w)\cdot\begin{pmatrix}\ln u - \ln\tilde u \\ 0\end{pmatrix}\mathrm{d}\boldsymbol{x}\,\mathrm{d}s, \end{aligned} \tag{49}$$

where we used

$$-\partial_t\tilde u\,(\ln u - \ln\tilde u) + (\partial_t\tilde u)\ln u = (\partial_t\tilde u)\;\ln\tilde u = \partial_t(\tilde u\ln\tilde u) - \tilde u\,\partial_t\ln\tilde u = \partial_t(\tilde u\ln\tilde u) - \partial_t\tilde u.$$

We observe

$$\nu \int_\Omega \tilde{u}|\nabla \ln u|^2 - \nabla \ln u \cdot \nabla \tilde{u} - \nabla \tilde{u} \cdot \nabla(\ln u - \ln \tilde{u})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nu \tilde{u} |\nabla \ln u - \nabla \ln \tilde{u}|^2\,\mathrm{d}\boldsymbol{x},$$

which is a reformulation of the terms in the second line of (49). Using similar transformations for terms of the third line of (49), yields

$$\kappa \int_\Omega \nabla w \cdot \nabla \tilde{u} - \tilde{u}\nabla w \cdot \nabla \ln u + \nabla \tilde{w} \cdot \nabla(\ln u - \ln \tilde{u})\,\mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega -\kappa\, \tilde{u}(\nabla \ln u - \nabla \ln \tilde{u}) \cdot (\nabla w - \nabla \tilde{w})\,\mathrm{d}\boldsymbol{x}.$$

Now, inequality (49) may be rewritten as

$$\begin{aligned}
&\int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})\Big|_0^t \,\mathrm{d}\boldsymbol{x} + \beta \int_0^t \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad + \int_0^t \int_\Omega \nu \tilde{u}|\nabla \ln u - \nabla \ln \tilde{u}|^2 + \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ 0 \end{pmatrix} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\le \alpha \int_0^t \int_\Omega uw - \tilde{u}\tilde{w} - \tilde{u}(w - \tilde{w}) - \tilde{w}\tilde{u}\,(\ln u - \ln \tilde{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad + \int_0^t \int_\Omega \kappa\, \tilde{u}(\nabla w - \nabla \tilde{w}) \cdot (\nabla \ln u - \nabla \ln \tilde{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\le \frac{\nu}{2} \int_0^t \int_\Omega \tilde{u}|\nabla \ln u - \nabla \ln \tilde{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{\kappa^2}{2\nu} \int_0^t \int_\Omega \tilde{u}|\nabla w - \nabla \tilde{w}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad + \alpha \int_0^t \int_\Omega \tilde{w}(u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})) + (w - \tilde{w})(u - \tilde{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s,
\end{aligned} \tag{50}$$

where we have used Young's inequality. Note, that the first term on the right-hand side can be absorbed into the left. Now we test the weak formulation for the prey (4) with $\tilde{u}(w - \tilde{w})$. We are allowed to do so, since the weak formulation holds in particular for all $\vartheta \in C^1([0,T]; L^\infty(\Omega) \cap W^{1,2}(\Omega))$ and this space is dense in the solution space of the prey w, cf. Lemma 7.5. Passing to the limit, we find that the weak formulation (4) remains true with equality. Setting $\overline{w} = (w - \tilde{w})$ and adding and subtracting the system operator tested with $(0, \tilde{u}\overline{w})^T$, we obtain

$$\begin{aligned}
&\int_0^t \langle \partial_t \overline{w}, \tilde{u}\overline{w}\rangle\,\mathrm{d}s + \int_0^t \int_\Omega \mu \nabla w \cdot \nabla(\tilde{u}\overline{w}) + \delta u w \tilde{u}\overline{w} + \mu \Delta \tilde{w}\tilde{u}\overline{w} - \delta \tilde{u}\tilde{w}\tilde{u}\overline{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad = \int_0^t \int_\Omega \gamma w \tilde{u}\overline{w} - \gamma \tilde{w}\tilde{u}\overline{w} - \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} 0 \\ \tilde{u}\overline{w} \end{pmatrix} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned} \tag{51}$$

Using the integration by parts rule from Lemma 7.3 in the Appendix, we get

$$\begin{aligned}
\int_0^t \langle \partial_t \overline{w}, \tilde{u}\overline{w}\rangle\,\mathrm{d}s &= -\int_0^t \langle \partial_t(\tilde{u}\overline{w}), \overline{w}\rangle\,\mathrm{d}s + \int_\Omega \tilde{u}|\overline{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t \\
&= -\int_0^t \int_\Omega \partial_t \tilde{u}|\overline{w}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s - \int_0^t \langle \partial_t \overline{w}, \tilde{u}\overline{w}\rangle\,\mathrm{d}s + \int_\Omega \tilde{u}|\overline{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t,
\end{aligned}$$

where we also used the product rule from Lemma 7.6 in the Appendix. Pulling the second term on the right-hand side into the left-hand side and dividing by 2, we obtain

$$\int_0^t \langle \partial_t \overline{w}, \tilde{u}\overline{w}\rangle\,\mathrm{d}s = \frac{1}{2}\int_\Omega \tilde{u}|\overline{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t - \frac{1}{2}\int_0^t \int_\Omega \partial_t \tilde{u}|\overline{w}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Plugging this into (51), gives

$$\frac{1}{2}\int_\Omega \tilde{u}|\bar{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega \mu\bar{w}\nabla\bar{w}\cdot\nabla\tilde{u} + \mu\tilde{u}|\nabla\bar{w}|^2 + \delta\tilde{u}(uw-\tilde{u}\tilde{w})\bar{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$= \int_0^t\int_\Omega \gamma\tilde{u}|\bar{w}|^2 + \frac{1}{2}\partial_t\tilde{u}|\bar{w}|^2 - \mathcal{A}(\tilde{u},\tilde{w})\cdot\begin{pmatrix}0\\ \tilde{u}\bar{w}\end{pmatrix}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Adding $\delta\tilde{u}^2|\bar{w}|^2$ on both sides in the integral and using the equality,

$$\delta\tilde{u}^2|\bar{w}|^2 - \delta\tilde{u}(uw-\tilde{u}\tilde{w})\bar{w} = -\delta\tilde{u}w(u-\tilde{u})\bar{w},$$

some reordering of the terms yields

$$\frac{1}{2}\int_\Omega \tilde{u}|\bar{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega \mu\tilde{u}|\nabla\bar{w}|^2 + (\delta\tilde{u}-\gamma)\tilde{u}|\bar{w}|^2 + \mathcal{A}(\tilde{u},\tilde{w})\cdot\begin{pmatrix}0\\ \tilde{u}\bar{w}\end{pmatrix}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$= \int_0^t\int_\Omega -\delta\tilde{u}w(u-\tilde{u})\bar{w} + \frac{1}{2}\partial_t\tilde{u}|\bar{w}|^2 - \mu\bar{w}\nabla\bar{w}\cdot\nabla\tilde{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Further estimating the right-hand side of this equality by an application of Young's inequality, we arrive at

$$\frac{1}{2}\int_\Omega \tilde{u}|\bar{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega \mu\tilde{u}|\nabla\bar{w}|^2 + (\delta\tilde{u}-\gamma)\tilde{u}|\bar{w}|^2 + \mathcal{A}(\tilde{u},\tilde{w})\cdot\begin{pmatrix}0\\ \tilde{u}\bar{w}\end{pmatrix}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$\leq \int_0^t \left(\frac{1}{2}\|\partial_t\ln\tilde{u}\|_{L^\infty(\Omega)} + \frac{\mu}{2}\|\nabla\ln\tilde{u}\|^2_{L^\infty(\Omega)}\right)\int_\Omega \tilde{u}|\bar{w}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$+ \int_0^t\int_\Omega \frac{\mu}{2}\tilde{u}|\nabla\bar{w}|^2 - \delta\tilde{u}w(u-\tilde{u})\bar{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Absorbing the second to last term from the right into the left-hand side, multiplying by $\frac{2\kappa^2}{\mu\nu}$ and adding (50) leaves us with

$$\int_\Omega u-\tilde{u}-\tilde{u}(\ln u-\ln\tilde{u})\Big|_0^t\,\mathrm{d}\boldsymbol{x} + \frac{\kappa^2}{\mu\nu}\int_\Omega \tilde{u}|w-\tilde{w}|^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t$$
$$+ \int_0^t\int_\Omega (\beta-\alpha\tilde{w})(u-\tilde{u}-\tilde{u}(\ln u-\ln\tilde{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \int_0^t\int_\Omega \frac{\kappa^2}{2\nu}\tilde{u}|\nabla w-\nabla\tilde{w}|^2$$
$$+ \frac{2\kappa^2}{\mu\nu}(\delta\tilde{u}-\gamma)\tilde{u}|w-\tilde{w}|^2 + \frac{\nu}{2}\tilde{u}|\nabla\ln u-\nabla\ln\tilde{u}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$+ \int_0^t\int_\Omega \mathcal{A}(\tilde{u},\tilde{w})\cdot\begin{pmatrix}\ln u-\ln\tilde{u}\\ \frac{2\kappa^2}{\mu\nu}\tilde{u}(w-\tilde{w})\end{pmatrix}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$\leq \int_0^t\int_\Omega \alpha(w-\tilde{w})(u-\tilde{u}) + \frac{2\kappa^2}{\mu\nu}\delta\tilde{u}w(\tilde{w}-w)(u-\tilde{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$$
$$+ \int_0^t\left(\frac{\kappa^2}{\mu\nu}\|\partial_t\ln\tilde{u}\|_{L^\infty(\Omega)} + \frac{\kappa^2}{\nu}\|\nabla\ln\tilde{u}\|^2_{L^\infty(\Omega)}\right)\int_\Omega \tilde{u}|w-\tilde{w}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s. \quad (52)$$

Since w and $\tilde{w}$ are non-negative and bounded by $K := \|w\|_{L^\infty(\Omega)} + \|\tilde{w}\|_{L^\infty(\Omega)} + 1$ and $\tilde{u}, u > 0$ almost everywhere, we can apply Lemma 7.7 from the Appendix, which is an application of the Fenchel–Young inequality, to get the pointwise almost everywhere estimate,

$$(w-\tilde{w})(u-\tilde{u}) \leq \max\left\{\frac{1}{4K}, 4K\right\}\left(\tilde{u}|w-\tilde{w}|^2 + u-\tilde{u}-\tilde{u}(\ln u-\ln\tilde{u})\right).$$

Note that this estimate also holds for exchanged roles of w and $\tilde{w}$ and that we know that $\tilde{u}, w \geq 0$ holds, so that the inequality remains true when multiplied with the product $\tilde{u}w$. Inserting this estimate into (52) and introducing the operators defined in Definition 5.3, yields

$$
\begin{aligned}
&\mathcal{R}(u,w|\tilde{u},\tilde{w})\Big|_0^t + \int_0^t \mathcal{W}(u,w|\tilde{u},\tilde{w}) + \int_\Omega \mathcal{A}(\tilde{u},\tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ \frac{2\kappa^2}{\mu\nu}\tilde{u}(w-\tilde{w}) \end{pmatrix} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\leq \int_0^t \int_\Omega \left(\alpha + \frac{2\kappa^2}{\mu\nu}\delta\tilde{u}w\right) \max\left\{\frac{1}{4K}, 4K\right\} \left(\tilde{u}|w-\tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad + \int_0^t \left(\frac{\kappa^2}{\mu\nu}\|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \frac{\kappa^2}{\nu}\|\nabla \ln \tilde{u}\|^2_{L^\infty(\Omega)}\right) \int_\Omega \tilde{u}|w-\tilde{w}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad \leq \int_0^t \Bigg(\|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \mu \|\nabla \ln \tilde{u}\|^2_{L^\infty(\Omega)} + \max\left\{\frac{1}{4K}, 4K\right\} \\
&\qquad \left(\frac{\alpha\mu\nu}{\kappa^2} + \alpha + 2\delta\|\tilde{u}\|_{L^\infty(\Omega)}\|w\|_{L^\infty(\Omega)} + \frac{2\kappa^2\delta}{\mu\nu}\|\tilde{u}\|_{L^\infty(\Omega)}\|w\|_{L^\infty(\Omega)}\right)\Bigg) \\
&\qquad\qquad \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu}\tilde{u}|w-\tilde{w}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned}
$$

Adding $\max\{\alpha\|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}(u,w|\tilde{u},\tilde{w})$ on both sides of the inequality we find, with the definition of the regularity potential $\mathcal{K}$, cf. Definition 5.3,

$$
\begin{aligned}
&\mathcal{R}(u,w|\tilde{u},\tilde{w})\Big|_0^t + \int_0^t \mathcal{W}(u,w|\tilde{u},\tilde{w}) + \max\{\alpha\|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}(u,w|\tilde{u},\tilde{w}) \\
&\qquad + \int_\Omega \mathcal{A}(\tilde{u},\tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ \frac{2\kappa^2}{\mu\nu}\tilde{u}(w-\tilde{w}) \end{pmatrix} \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leq \int_0^t \mathcal{K}(\tilde{u},\tilde{w})\mathcal{R}(u,w|\tilde{u},\tilde{w})\,\mathrm{d}s. \qquad (53)
\end{aligned}
$$

Applying Gronwall's inequality, see for example Lemma 7.3.1 in [15, p. 180], (48) follows and our proof of the lemma is finished. □

Using the relative energy inequality from the previous lemma, makes the proof of the weak-strong uniqueness, cf. Theorem 2.6, quite simple.

Proof (of Theorem 2.6). The relative energy inequality (48) from Lemma 5.4 holds for the generalized solution $(u,w) \in \mathcal{X}$ and the strong solution $(\tilde{u},\tilde{w})$. We indeed have $\tilde{u},\tilde{w} \in \mathcal{Y}$, since the initial value u_0 is bounded away from zero and thus also $\tilde{u}$ is bounded away from zero, by Lemma 5.1. Since $(\tilde{u},\tilde{w})$ is a strong solution to (1) the term including the system operator $\mathcal{A}$ vanishes. By the non-negativity of $\mathcal{W} + \max\{\alpha\|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}$, we estimate

$$
\mathcal{R}(u,w|\tilde{u},\tilde{w})(t) \leq \mathcal{R}(u,w|\tilde{u},\tilde{w})(0)e^{\int_0^t \mathcal{K}(\tilde{u},\tilde{w})\,\mathrm{d}s} = 0
$$

for almost all $t \in [0,T]$. Thus we have $w(t) = \tilde{w}(t)$ and $u(t) = \tilde{u}(t)$ almost everywhere in Ω by the definition of the relative energy $\mathcal{R}$, see (44), where we used that $x - y - y(\ln x - \ln y) = 0$ implies $x = y$ for all $x, y > 0$. □

6 Local existence of strong solution

We only show the *a priori* estimates needed for the local-in-time existence of strong solutions for system (1) in a formal way, working directly with the equations from (1). To make this rigorous one would need to choose an appropriate Galerkin basis of eigenfunctions, which fulfill $\nabla(\Delta\varphi)\cdot n = 0$ on $\partial\Omega$ and perform these estimates on the discrete level. The structure of the proof follows the proof of Theorem 2.5 in [26].

For all functions $\varphi \in W^{2,2}(\Omega)$ fulfilling zero Neumann boundary conditions we can consider the norm

$$\|\varphi\|^2_{\overline{W}^{2,2}(\Omega)} := \|\varphi\|^2_{L^2(\Omega)} + \|\Delta\varphi\|^2_{L^2(\Omega)}\,,$$

which defines an equivalent norm to the standard $W^{2,2}$-norm.

Proof (of Theorem 2.7). We show this by deriving appropriate *a priori* estimates and using an ODE comparison principle for

$$\xi(t) := \frac{1}{2}\left(2 + \|w(t)\|^2_{\overline{W}^{2,2}} + \|u(t)\|^2_{W^{1,2}}\right),$$

where we write L^2 instead of $L^2(\Omega)$ etc. throughout the proof. Note that for $d \le 3$ we have $W^{1,2} \hookrightarrow L^6$ and $W^{2,2} \hookrightarrow L^\infty$ by the Sobolev embedding theorem. Testing the prey equation (1b) with w itself we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|w\|^2_{L^2} + \mu\|\nabla w\|^2_{L^2} \le \left(\gamma + \frac{1}{2}\right)\|w\|^2_{L^2} + \frac{\delta^2}{2}\|u\|^2_{L^2} \le C\left(\|u\|^2_{W^{1,2}} + \|w\|^2_{\overline{W}^{2,2}}\right) \tag{54}$$

and testing (1b) with the bi-Laplacian $\Delta^2 w$ and applying Young's inequality yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta w\|^2_{L^2} + \mu\|\nabla(\Delta w)\|^2_{L^2} &= \int_\Omega \gamma|\Delta w|^2 + \delta u\nabla w\cdot\nabla(\Delta w) + \delta w\nabla u\cdot\nabla(\Delta w)\,\mathrm{d}\boldsymbol{x}\\
&\le \gamma\|\Delta w\|^2_{L^2} + \delta\|u\|_{L^4}\|\nabla w\|_{L^4}\|\nabla(\Delta w)\|_{L^2} + \delta\|w\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla(\Delta w)\|_{L^2}\\
&\le \gamma\|\Delta w\|^2_{L^2} + \frac{\mu}{2}\|\nabla(\Delta w)\|^2_{L^2} + \frac{\delta^2}{2\mu}\left(\|u\|^4_{L^4} + \|\nabla w\|^4_{L^4} + \|w\|^4_{L^\infty} + \|\nabla u\|^4_{L^2}\right),\end{aligned}$$

where the boundary terms vanished since $\nabla(\Delta w)\cdot n = 0$ on $\partial\Omega$ holds by our choice of Galerkin basis. Absorbing the term including the third partial derivatives of w from the right into the left-hand side and using the above mentioned Sobolev embeddings, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta w\|^2_{L^2} + \frac{\mu}{2}\|\nabla(\Delta w)\|^2_{L^2} \le C\left(1 + \|u\|^4_{W^{1,2}} + \|w\|^4_{\overline{W}^{2,2}}\right). \tag{55}$$

Testing the predator equation (1a) with u and applying Young's inequality gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{L^2} + \nu\|\nabla u\|^2_{L^2} + \beta\|u\|^2_{L^2} &= \int_\Omega \alpha w u^2 + \kappa u\nabla w\cdot\nabla u\,\mathrm{d}\boldsymbol{x}\\
&\le \frac{\alpha}{2}\|w\|^2_{L^\infty} + \frac{\alpha}{2}\|u\|^4_{L^2} + \frac{\kappa^2}{2\nu}\|\nabla w\|^2_{L^4}\|u\|^2_{L^4} + \frac{\nu}{2}\|\nabla u\|^2_{L^2}.\end{aligned}$$

Absorbing the last term on the right-hand side into the left-hand side and again using the above mentioned Sobolev embeddings, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{L^2} \le C\left(1 + \|u\|^4_{W^{1,2}} + \|w\|^4_{\overline{W}^{2,2}}\right). \tag{56}$$

Finally testing (1a) with $-\Delta u$ and integrating by parts we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla u\|^2_{L^2} + \nu\|\Delta u\|^2_{L^2} &= \int_\Omega \kappa u\Delta w\Delta u + \kappa\Delta u\nabla u\cdot\nabla w + (\beta - \alpha w)u\Delta u\,\mathrm{d}\boldsymbol{x}\\
&\le -\int_\Omega \kappa\Delta w|\nabla u|^2 + \kappa u\nabla(\Delta w)\cdot\nabla u\,\mathrm{d}\boldsymbol{x} - \int_\Omega \beta|\nabla u|^2\,\mathrm{d}\boldsymbol{x}\\
&\qquad + \kappa\|\nabla u\|_{L^3}\|\nabla w\|_{L^6}\|\Delta u\|_{L^2} + \alpha\|w\|_{L^\infty}\|u\|_{L^2}\|\Delta u\|_{L^2}\\
&\le \kappa\|\Delta w\|_{L^2}\|\nabla u\|^2_{L^4} + \kappa\|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla(\Delta w)\|_{L^2}\\
&\qquad + \kappa\|\nabla u\|_{L^3}\|\nabla w\|_{L^6}\|\Delta u\|_{L^2} + \alpha\|w\|_{L^\infty}\|u\|_{L^2}\|\Delta u\|_{L^2}.\end{aligned} \tag{57}$$

For $u \in W^{2,2}(\Omega)$ fulfilling zero Neumann boundary conditions we can estimate $\|\nabla u\|_{L^6} \le C\|\Delta u\|_{L^2}$ and we can deduce the following interpolation inequalities

$$\|\nabla u\|_{L^3} \le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}},$$
$$\|\nabla u\|_{L^4} \le C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\Delta u\|_{L^2}^{\frac{3}{4}}.$$

Using these inequalities and Young's inequality, we can further estimate (57) by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \\
&\qquad \le C\|\Delta w\|_{L^2}\|\Delta u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + C\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla(\Delta w)\|_{L^2} \\
&\qquad\quad + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^6}\|\Delta u\|_{L^2}^{\frac{3}{2}} + \alpha\|w\|_{L^\infty}\|u\|_{L^2}\|\Delta u\|_{L^2} \\
&\le \frac{\nu}{2}\|\Delta u\|_{L^2}^2 + C\left(\|\Delta w\|_{L^2}^6 + \|\nabla u\|_{L^2}^6\right) + \frac{\mu}{2}\|\nabla(\Delta w)\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^6 + \|u\|_{L^6}^6\right) \\
&\qquad + \frac{\nu}{2}\|\Delta u\|_{L^2}^2 + C\left(\|\nabla w\|_{L^6}^6 + \|\nabla u\|_{L^2}^6\right) + \frac{\alpha^2}{2\nu}\left(\|w\|_{L^\infty}^4 + \|u\|_{L^2}^4\right).
\end{aligned}$$

Absorbing the Δu terms from the right into the left-hand side, we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla u\|_{L^2}^2 \le C\left(1 + \|u\|_{W^{1,2}}^6 + \|w\|_{W^{2,2}}^6\right) + \frac{\mu}{2}\|\nabla(\Delta w)\|_{L^2}^2. \tag{58}$$

Adding equations (54), (55), (56) and (58) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) \le C\xi(t)^3. \tag{59}$$

For the initial values given in Theorem 2.7 the initial value $\xi(0)$ is bounded and thus we find a $T^* > 0$ and some $C^* > 0$ such that (59) has a solution on $[0, T^*)$ and

$$\|\xi\|_{L^\infty(0,T^*)} \le C^*$$

holds. This shows $u \in L^\infty(0, T^*; W^{1,2})$ and $w \in L^\infty(0, T^*; W^{2,2})$. By the maximal L^p-regularity of the heat equation, see [9, Thm. 8.2] we obtain the required regularity in time, where we proceeded as follows. First we find $w \in W^{1,6}(L^6) \cap L^6(W^{2,6}) \hookrightarrow C^1([0, T^*] \times \overline{\Omega})$, by the embedding from [22, Lem. 3.3] and maximal L^p-regularity for (1b). With this additional regularity for w and maximal L^p-regularity for (1a) we find $u \in W^{1,2}(L^2) \cap L^2(W^{2,2})$. □

Finally, we infer the additional regularity of strong solutions asserted in Proposition 2.8.

Proof (of Proposition 2.8). The proof is performed via a bootstrap argument. We apply the maximal L^p-regularity of the heat equation, see [9, Thm. 8.2], multiple times and use the following embeddings for parabolic Sobolev spaces

$$W^{1,q}(0, T^*; L^q(\Omega)) \cap L^q(0, T^*; W^{2,q}(\Omega)) \hookrightarrow L^p(0, T^*; W^{k,p}(\Omega)) \tag{60}$$

for $k = 0, 1$ with $p \ge q$ and $2 - k - (1/q - 1/p)(n+2) \ge 0$ and

$$W^{1,q}(0, T^*; L^q(\Omega)) \cap L^q(0, T^*; W^{2,q}(\Omega)) \hookrightarrow C^{(k+\alpha)/2, k+\alpha}([0, T^*] \times \overline{\Omega}) \tag{61}$$

for $q > (n+2)/(2-k)$, $k = 0, 1$ and $0 \leq \alpha < 2 - k - (n+2)/q$, see [22, Lem. 3.3]. The space $C^{\alpha/2,\alpha}([0,T^*] \times \overline{\Omega})$ is the parabolic Hölder space, see [30, p. 177] for a definition. We make use of the Hölder continuity of the right-hand side to deduce the regularity up to $t = 0$. Roughly speaking, we can say that when the right-hand side and initial condition of the heat equation are Hölder continuous, this continuity transfers to the time derivative and the Laplace of the solution and can be extended to $t = 0$, see Section 5.1.2 in [30]. This will be done more precisely later in the proof. We begin by noting that by the embedding (61) we find

$$w \in W^{1,6}(0,T^*; L^6(\Omega)) \cap L^6(0,T^*; W^{2,6}(\Omega)) \hookrightarrow C^{(1+\alpha)/2,1+\alpha}([0,T^*] \times \overline{\Omega}) \tag{62}$$

for all $\alpha \in (0, \frac{1}{6})$. Using the embedding from (60) we find

$$u \in W^{1,2}(0,T^*; L^2(\Omega)) \cap L^2(0,T^*; W^{2,2}(\Omega)) \hookrightarrow L^{10}(0,T^*; L^{10}(\Omega)),$$
$$u \in W^{1,2}(0,T^*; L^2(\Omega)) \cap L^2(0,T^*; W^{2,2}(\Omega)) \hookrightarrow L^{\frac{10}{3}}(0,T^*; W^{1,\frac{10}{3}}(\Omega)).$$

Using this additional regularity and considering equation (1a)

$$\partial_t u - \nu\Delta u = -\kappa u\Delta w - \kappa\nabla u \cdot \nabla w + (\alpha w - \beta)u, \tag{63}$$

we notice that the right-hand side of (63) is in $L^{\frac{10}{3}}(0,T^*; L^{\frac{10}{3}}(\Omega))$ and thus by the maximal L^p-regularity of the heat equation and again using embedding (60), we find

$$u \in W^{1,\frac{10}{3}}(0,T^*; L^{\frac{10}{3}}(\Omega)) \cap L^{\frac{10}{3}}(0,T^*; W^{2,\frac{10}{3}}(\Omega)) \hookrightarrow L^{10}(0,T^*; W^{1,10}(\Omega)).$$

An application of (61) yields

$$u \in W^{1,\frac{10}{3}}(0,T^*; L^{\frac{10}{3}}(\Omega)) \cap L^{\frac{10}{3}}(0,T^*; W^{2,\frac{10}{3}}(\Omega)) \hookrightarrow C^{\tilde{\alpha}/2,\tilde{\alpha}}([0,T^*] \times \overline{\Omega}),$$

for all $\tilde{\alpha} \in (0, \frac{1}{2})$. Now, we can deduce that the right-hand side of (63) is in $L^6(0,T^*; L^6(\Omega))$ and again applying maximal L^p-regularity we obtain

$$u \in W^{1,6}(0,T^*; L^6(\Omega)) \cap L^6(0,T^*; W^{2,6}(\Omega)) \hookrightarrow C^{(1+\alpha)/2,1+\alpha}([0,T^*] \times \overline{\Omega}) \tag{64}$$

for all $\alpha \in (0, \frac{1}{6})$ by embedding (61). Now we have that the right-hand side of (1b)

$$\partial_t w - \mu\Delta w = (\gamma - \delta u)w$$

is in the parabolic Hölder space $C^{(1+\alpha)/2,1+\alpha}([0,T^*] \times \overline{\Omega})$ for all $\alpha \in (0, \frac{1}{6})$, since the product of two Hölder continuous functions with Hölder exponents $\lambda_1 > 0$ and $\lambda_2 > 0$ is again Hölder continuous with Hölder exponent $\lambda = \min\{\lambda_1, \lambda_2\}$, see [16, Ch. 4.1]. Additionally, we have that the initial data fulfills $w_0 \in C^3(\overline{\Omega}) \hookrightarrow C^{2+2\alpha}(\overline{\Omega})$, the embedding holds by the Sobolev embedding theorem since Ω has a smooth enough boundary, see for example [1, Thm. 5.4], and w_0 fulfills the boundary conditions. Thus, by [30, Thm 5.1.18iii)], we find

$$w_t \in C^{\alpha,2\alpha}([0,T^*] \times \overline{\Omega}) \quad \text{and} \quad \Delta w \in C^{\alpha,0}([0,T^*] \times \overline{\Omega}).$$

For the regularity of u we find that the right-hand side of (63) is in $C^{\alpha,0}([0,T^*] \times \overline{\Omega})$ and with the properties of u_0, namely that $u_0 \in C^3(\overline{\Omega}) \hookrightarrow C^{2+2\alpha}(\overline{\Omega})$ and that it fulfills the zero Neumann boundary conditions, we find, again applying [30, Thm 5.1.18iii)], that

$$u_t \in C^{\alpha,2\alpha}([0,T^*] \times \overline{\Omega}) \quad \text{and} \quad \Delta u \in C^{\alpha,0}([0,T^*] \times \overline{\Omega}).$$

Combining these continuity results of the time derivatives and the Laplacians with the continuity of the first derivatives in space, cf. (62) and (64), we obtain

$$u, w \in C^1([0,T^*] \times \overline{\Omega}) \text{ and } \Delta u, \Delta w \in C([0,T^*] \times \overline{\Omega}).$$

The non-negativity of the initial values u_0 and w_0 can be transferred to the solutions u and w by first applying the comparison principle for strong solutions of the predator equation, cf. Proposition 5.2 and then the comparison principle for the prey w, cf. Lemma 4.3. This completes our proof. □

7 Appendix

7.1 A generalized integration by parts and product rule

By the definition of the solution space $\mathcal{X}$, cf. Section 2, we have $w \in L^2(0,T;W^{1,2}(\Omega)\cap L^\infty(\Omega))$ with $\partial_t w \in L^2(0,T;L^1(\Omega)+W^{1,2}(\Omega)^*)$. In this section we prove that the standard integration by parts and product rules are valid in this space.

We define $V_1 = W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $V_2 = L^1(\Omega)+W^{1,2}(\Omega)^*$. Both $L^1(\Omega)$ and $W^{1,2}(\Omega)^*$ are continuous embedded in $W^{1,p}(\Omega)^*$ for $p>d$ and thus the sum of these spaces is well-defined. Equipped with the norms

$$\|v_1\|_{V_1} := \max\left\{\|v_1\|_{W^{1,2}(\Omega)} + \|v_1\|_{L^\infty(\Omega)}\right\}$$
$$\|v_2\|_{V_2} := \inf\left\{\left\|v_2^1\right\|_{L^1(\Omega)} + \left\|v_2^2\right\|_{W^{1,2}(\Omega)^*} : v_2^1 \in L^1(\Omega), v_2^2 \in W^{1,2}(\Omega)^* \text{ s.t. } v_2^1+v_2^2 = v_2\right\}$$

for $v_1 \in V_1$ and $v_2 \in V_2$ these spaces are Banach, see [7, Thm. 1.3]. We will continue to denote the $L^1(\Omega)$ part of $v\in V_2$ by an upper index 1 and the $W^{1,2}(\Omega)^*$ by an upper index 2 throughout this section. Next we define the following space.

Definition 7.1. *We define* $\mathbb{Y} := \{w\in L^2(V_1) \mid \partial_t w \in L^2(V_2)\}$, *where the distributional time derivative is given as usual via*

$$-\int_0^T\int_\Omega w(t,\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\phi'(t)\,\mathrm{d}t = \int_0^T \phi(t)\langle\partial_t w(t),v\rangle\,\mathrm{d}t \tag{65}$$

for all $\phi\in C_0^\infty(0,T)$ *and all* $v\in L^\infty(\Omega)\cap W^{1,2}(\Omega)$.

Remark 7.2. *The space* $\mathbb{Y}$ *equipped with the norm* $\|w\|_{\mathbb{Y}} := \|w\|_{L^2(V_1)} + \|\partial_t w\|_{L^2(V_2)}$ *is a Banach space.*

We now state the generalized integration by parts rule for the space $\mathbb{Y}$.

Lemma 7.3. *The space* $\mathbb{Y}$*, cf. Definition 7.1, is continuously embedded into* $C([0,T];L^2(\Omega))$ *and for arbitrary* $w,v\in\mathbb{Y}$*, we have*

$$(w(t),v(t))_{L^2(\Omega)} - (w(s),v(s))_{L^2(\Omega)} = \int_s^t \langle\partial_t w(\tau),v(\tau)\rangle + \langle\partial_t v(\tau),w(\tau)\rangle\,\mathrm{d}\tau \tag{66}$$

for all $0\le s\le t\le T$.

Before we turn to the proof, we note the following easy consequence of this lemma.

Corollary 7.4. *For* $w\in\mathbb{Y}$ *we have* $w^+ := \max\{0,w\}\in L^2(V_1)$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left\|w^+(t)\right\|^2_{L^2(\Omega)} = \langle\partial_t w(t),w^+(t)\rangle \tag{67}$$

holds for almost all $t\in[0,T]$.

Proof. This can be shown by first taking $w\in C^1([0,T];V_1)$. We then have $w^+\in\mathbb{Y}$ and we can apply the integration by parts rule from Lemma 7.3 to obtain (67). For arbitrary $w\in\mathbb{Y}$ we find an approximating sequence in $C^1([0,T];V_1)$ by Lemma 7.5, see below, and deduce that (67) remains true in the limit. □

In order to proof Lemma 7.3 we follow the standard procedure by moving to a dense subset of smooth functions.

Lemma 7.5. *The space $C^1([0,T];V_1)$ is dense in $\mathbb{Y}$.*

Proof. This can be shown by the use of mollifiers as it is done in [36, Lem. 7.2]. □

The following proof is conducted along the lines of [36, Lem. 7.3]. One cannot apply this Lemma directly, since $V_1 \hookrightarrow L^2(\Omega) \hookrightarrow V_2$ does not define an evolution triple, since $V_1^* \neq V_2$.

Proof (of Lemma 7.3). We start by proving the embedding into $C([0,T];L^2(\Omega))$, using the density from Lemma 7.5 to show that the embedding via the identity $i : C^1([0,T];V_1) \subseteq \mathbb{Y} \to C([0,T];L^2(\Omega))$, can be extended to the whole space $\mathbb{Y}$. First, we note that for arbitrary $w, v \in C^1([0,T];V_1)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t),v(t))_{L^2(\Omega)} = \langle \partial_t w(t), v(t)\rangle + \langle \partial_t v(t), w(t)\rangle \tag{68}$$

for all $t \in (0,T)$. Using the mean value theorem, we find $t_1 \in [0,T]$ such that $w(t_1) = \frac{1}{T}\int_0^T w(s)\,\mathrm{d}s$ holds. Using this identity and Young's inequality we get

$$\begin{aligned}
\|w(t)\|^2_{L^2(\Omega)} = \|w(t_1)\|^2_{L^2(\Omega)} + \Big(\|w(t)\|^2_{L^2(\Omega)} - \|w(t_1)\|^2_{L^2(\Omega)}\Big) \\
\leq \frac{1}{T}\int_0^T \|w(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + 2\int_{t_1}^t \langle \partial_t w(s), w(s)\rangle\,\mathrm{d}s \\
= \frac{1}{T}\int_0^T \|w(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + 2\int_{t_1}^t \langle \partial_t w(s)^1, w(s)\rangle + \langle \partial_t w(s)^2, w(s)\rangle\,\mathrm{d}s \\
\leq C\left(\|w\|_{L^2(V_1)} + \Big\|\big\|\partial_t w^1\big\|_{L^1(\Omega)} + \big\|\partial_t w^2\big\|_{W^{1,2}(\Omega)^*}\Big\|_{L^2(0,T)}\right)^2
\end{aligned}$$

for some $C > 0$ and all $t \in [0,T]$. Since this inequality holds for all decompositions of $\partial_t w(s)$ into its $L^1(\Omega)$ part $\partial_t w(s)^1$ and its $W^{1,2}(\Omega)^*$ part $\partial_t w(s)^2$, the inequality also holds for the infimum over all these decompositions and we obtain

$$\|w\|_{C([0,T];L^2(\Omega))} \leq C\,\|w\|_{\mathbb{Y}}\,.$$

We can extend this densely defined linear operator i to the whole space $\mathbb{Y}$ by the extension principle, see [43, Prop.18.29].

The integration by parts formula for arbitrary $w, v \in \mathbb{Y}$ follows by a density argument. We take approximating sequences $(v_n)_n, (w_n)_n \subseteq C^1([0,T];V_1)$ such that $v_n \to v$ and $w_n \to w$ in $\mathbb{Y}$ holds. We then have that the identity (68) holds for w and v replaced by w_n and v_n respectively. Integrating this identity over (s,t) we find that (66) holds, again with w and v replaced by w_n and v_n. Finally we can conclude that the equality remains true in the limit. □

Next we show a generalized product rule.

Lemma 7.6 (Product rule)**.** *For $w \in \mathbb{Y}$ and $v \in W^{1,2}(0,T;V_1)$ the product rule $\partial_t(wv) = v\partial_t w + w\partial_t v$ holds, where*

$$W^{1,2}(0,T;V_1) := \big\{v \in L^2(0,T;V_1) \mid \partial_t v \in L^2(0,T;V_1)\big\}\,.$$

Proof. The proof is analogous to the proof of the product rule for weak derivatives. First we show that it holds for $w \in C^1([0,T];V_1)$ and $v \in C_0^\infty([0,T]) \otimes V_1$. In this case v has enough regularity and can be split up to join the test functions φ and ϕ from the definition of the generalized time derivative, cf. Definition 7.1. By Lemma 5.12 in [38, p. 70], we have the density of $C_0^\infty([0,T]) \otimes V_1$ in $W^{1,2}(0,T;V_1)$. With this density we can deduce the product rule for all $u \in C^1([0,T];V_1)$ and $v \in W^{1,2}(0,T;V_1)$, where we use the continuous embedding of $W^{1,2}(0,T;V_1)$ into $C([0,T];V_1)$ to pass to the limit. With the density of $C^1([0,T];V)$ in $\mathbb{Y}$, cf. Lemma 7.5, we finally get the product rule as stated in the lemma. □

7.2 Auxiliary results

We reprove a nice pointwise inequality resulting from the Fenchel–Young inequality. The proof is taken from Lemma 26 in [19].

Lemma 7.7 (An application of the Fenchel–Young inequality)**.** *Let $u, \tilde{u}, w, \tilde{w} \in \mathbb{R}$ be non-negative, $r > 0$ such that $w, \tilde{w} \in B_r(0)$ and $\tilde{u}, u > 0$ holds. We then have*

$$(w - \tilde{w})(u - \tilde{u}) \leq \max\left\{\frac{1}{4r}, 4r\right\}\left(\tilde{u}\,|w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})\right). \tag{69}$$

Proof. First, we define the proper convex function $g : \mathbb{R} \to \overline{\mathbb{R}}$ via

$$g(x) := \begin{cases} x - \ln(x+1) & \text{for } x \in (-1,\infty), \\ +\infty & \text{otherwise.} \end{cases}$$

The convex conjugate, see Section 2.1.4 in [3, p. 75], is easily computed and given by $g^* : \mathbb{R} \to \overline{\mathbb{R}}$ with

$$g^*(y) = \begin{cases} -\ln(1-y) - y & \text{for } y \in (-\infty, 1), \\ +\infty & \text{otherwise.} \end{cases}$$

Choosing $y = \frac{w - \tilde{w}}{4r}$ such that $|y| < \frac{1}{2}$ and $x = (\frac{u}{\tilde{u}} - 1)$, we find that

$$g(x) = \frac{u}{\tilde{u}} - 1 - \ln\left(\frac{u}{\tilde{u}}\right) = \frac{u}{\tilde{u}} - 1 - (\ln u - \ln \tilde{u}).$$

We estimate $g^*(y)$ by writing it via the Taylor expansion using the integral form of the remainder term,

$$\begin{aligned} g^*(y) = g^*(0) + (g^*)'(0)y + \int_0^1 (1-s)(g^*)''(sy)\,\mathrm{d}s\,y^2 &= \int_0^1 \frac{1-s}{(1-sy)^2}\,\mathrm{d}s\,y^2 \\ &\leq \int_0^1 \frac{1-s}{(1-\frac{s}{2})^2}\,\mathrm{d}s\,y^2 \leq 2\int_0^1 (1-s)\,\mathrm{d}s\,y^2 = y^2, \end{aligned}$$

since $|y| \leq \frac{1}{2}$ by our choice of y. An application of the Fenchel–Young inequality, again see Section 2.1.4 in [3], yields

$$\begin{aligned} (w - \tilde{w})(u - \tilde{u}) = (4r\tilde{u})\frac{w - \tilde{w}}{4r}\left(\frac{u}{\tilde{u}} - 1\right) &= (4r\tilde{u})xy \leq 4r\tilde{u}\left(g^*(y) + g(x)\right) \\ &\leq 4r\tilde{u}\left(\frac{|w - \tilde{w}|^2}{16r^2} + \frac{u}{\tilde{u}} - 1 - (\ln u - \ln \tilde{u})\right) \\ &\leq \max\left\{\frac{1}{4r}, 4r\right\}\left(\tilde{u}|w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}\,(\ln u - \ln \tilde{u})\right). \end{aligned}$$

This concludes our proof of the inequality (69). □

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