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Variational approach to fluid-structure interaction via GENERIC

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Abstract

We present a framework to systematically derive variational formulations for fluid-structure interaction problems based on thermodynamical driving functionals and geometric structures in different coordinate systems by suitable transformations within this formulation. Our approach provides a promising basis to construct structure-preserving discretization strategies.

1 Introduction

Fluid structure interaction (FSI) problems are solved with a large variety of methods, and offer countless possibilities to devise a zoo of discretization strategies, both in space and in time, allowing one to pick from any combination of finite volumes, finite elements, finite differences, and staggered, segregated, or monolithic approaches. The strong formulation of FSI problems is well understood, and results in a fully coupled nonlinear problem, where generally nonlinear equations of motion are formulated on time dependent domains, whose geometrical configuration depends on the solution itself. Such intricate structure makes the practical numerical implementation of FSI problems the subject of countless attempts to improve one aspect or the other of the numerical methods. The aim of this contribution is to shed some light on the GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) structure of FSI problems in variational formulation. The GENERIC framework $\partial_t \boldsymbol{q}(t) = \mathbb{J}(\boldsymbol{q})\mathrm{D}\mathscr{E}(\boldsymbol{q}) + \mathbb{K}(\boldsymbol{q})\mathrm{D}\mathscr{S}(\boldsymbol{q})$ with total energy $\mathscr{E}$ and entropy $\mathscr{S}$ has been used in the strong formulation in various applications for thermodynamically closed systems [7, 9, 16, 18, 19]. Here we systematically derive variational formulations of FSI problems in different coordinate systems, which could be used to construct structure-preserving discretization strategies.

In this work we consider the evolution of states $\boldsymbol{q} : [0, \mathrm{T}] \to \mathcal{Q}$ in a subset $\mathcal{Q}$ of a Banach space, $\mathcal{Q}$ representing a manifold. Its damped Hamiltonian evolution is driven by the free energy functional $\mathscr{F} : \mathcal{Q} \to \mathbb{R}$. The evolution equation is generated by a linear, skew-symmetric Poisson operator $\mathbb{J}(\boldsymbol{q}) : \mathcal{V}^* \to \mathcal{V}$ for reversible dynamics ($\mathbb{J}(\boldsymbol{q}) = -\mathbb{J}(\boldsymbol{q})^*$) and by a positively semi-definite, symmetric Onsager operator $\mathbb{K}(\boldsymbol{q}) : \mathcal{V}^* \to \mathcal{V}$ for dissipative dynamics ($\mathbb{K}(\boldsymbol{q}) = \mathbb{K}(\boldsymbol{q})^* \geq 0$). Here, $\mathcal{V}$ is assumed to be a Banach space with dual $\mathcal{V}^*$, and dual pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. For fixed $\boldsymbol{q}$ the space $\mathcal{V} = T_{\boldsymbol{q}}\mathcal{Q}$ can be seen as the (linear) tangent space of $\mathcal{Q}$ at $\boldsymbol{q}$. With these ingredients, the damped Hamiltonian evolution then reads

$$\partial_t \boldsymbol{q}(t) = [\mathbb{J}(\boldsymbol{q}) - \mathbb{K}(\boldsymbol{q})]\, \mathrm{D}\mathscr{F}(\boldsymbol{q}) \tag{1}$$

for $\partial_t \boldsymbol{q}(t) \in \mathcal{V}$ and $\mathrm{D}\mathscr{F}(\boldsymbol{q}) \in \mathcal{V}^*$. In [10] it is demonstrated for models describing fracture processes in dissipative solids that the GENERIC framework for systems bulk-interface interaction naturally leads to a weak formulation that encodes the bulk equations and the coupling conditions along the interface. While GENERIC is a 2-generator formalism where the operators $\mathbb{J}$ and $\mathbb{K}$ satisfy degeneracy conditions, here we outline the general approach of transforming weak formulations between coordinate frames for the 1-generator damped-Hamiltonian system, but this strategy also applies to GENERIC.

In this work we consider FSI problems for compressible fluids and solids. In particular, we are interested in deriving a weak formulation of (1) that is convenient for numerics and that maintains the damped Hamiltonian structure under changes of variables or coordinates. In an abstract setting, this can be done

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by introducing an auxiliary space $\mathcal{W}$ (space of generalized forces) and a corresponding transformation operator $\mathbb{M}^* : \mathcal{W} \to \mathcal{V}^*$, at this point assumed to be invertible. Then, with $\mathbb{M}^*\boldsymbol{\eta} = \mathrm{D}\mathscr{F}(\boldsymbol{q})$ and its adjoint operator $\mathbb{M} : \mathcal{V} \to \mathcal{W}^*$, we can rewrite (1) into the saddle-point structure

$$\begin{pmatrix} \mathbb{M}\left[\mathbb{J}(\boldsymbol{q}) - \mathbb{K}(\boldsymbol{q})\right]\mathbb{M}^* & -\mathbb{M} \\ \mathbb{M}^* & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta} \\ \partial_t \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{D}\mathscr{F}(\boldsymbol{q}) \end{pmatrix} \qquad \text{in } \mathcal{W}^* \times \mathcal{V}^* \,. \tag{2}$$

This provides a weak formulation of (1), where we seek $\boldsymbol{\eta} \in \mathcal{W}$ and $\partial_t \boldsymbol{q} \in \mathcal{V}$. With the operators from (1), we can introduce also suitable $\boldsymbol{q}$-dependent bilinear forms $\mathrm{a} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ and $\mathrm{b} : \mathcal{V} \times \mathcal{W} \to \mathbb{R}$, where $\mathrm{a}(\boldsymbol{w}, \boldsymbol{\eta}) := \mathrm{j}(\boldsymbol{w}, \boldsymbol{\eta}) - \mathrm{k}(\boldsymbol{w}, \boldsymbol{\eta})$ is the difference of a skew-symmetric part $\mathrm{j}(\boldsymbol{w}, \boldsymbol{\eta}) = \langle \mathbb{M}^*\boldsymbol{w}, \mathbb{J}(\boldsymbol{q})\mathbb{M}^*\boldsymbol{\eta}\rangle_{\mathcal{V}}$ and a positive semi-definite, symmetric part $\mathrm{k}(\boldsymbol{w}, \boldsymbol{\eta}) = \langle \mathbb{M}^*\boldsymbol{w}, \mathbb{K}(\boldsymbol{q})\mathbb{M}^*\boldsymbol{\eta}\rangle_{\mathcal{V}}$, and with the bilinear form b given by $\mathrm{b}(\boldsymbol{v}, \boldsymbol{\eta}) = \langle \mathbb{M}^*\boldsymbol{\eta}, \boldsymbol{v}\rangle_{\mathcal{V}}$. Accordingly, the weak formulation (2) can be equivalently reformulated as: For $t \in [0, \mathrm{T}]$ fixed, find $\boldsymbol{\eta} \in \mathcal{W}$ and $\partial_t \boldsymbol{q} \in \mathcal{V}$, such that for all $\boldsymbol{w} \in \mathcal{W}$ and $\boldsymbol{v} \in \mathcal{V}$ there holds

$$\mathrm{a}(\boldsymbol{w}, \boldsymbol{\eta}) - \mathrm{b}(\partial_t \boldsymbol{q}, \boldsymbol{w}) = 0 \,, \tag{3a}$$

$$\mathrm{b}(\boldsymbol{v}, \boldsymbol{\eta}) = \langle \mathrm{D}\mathscr{F}(\boldsymbol{q}), \boldsymbol{v}\rangle_{\mathcal{V}} \,. \tag{3b}$$

The weak formulations (2) and (3) are both saddle-point structures and formally equivalent to (1). In the GENERIC setting, the conservation of total energy is related to the skew symmetry of j and the production of entropy is related to k being symmetric, positive semidefinite. While its geometrical origin is not always emphasized, the benefits of skew symmetric or symmetric operators in the discretization of turbulent flows are well-studied, e.g. cf. [23]. Similar structure-preserving approaches based on variational principles are beneficial also for FSI, e.g. for the construction of time-discrete schemes by minimizing movements [3] or to preserve energy conservation and entropy production for GENERIC thermoviscoelasticity [4].

In this work, we apply this abstract framework to generate a weak formulation in saddle-point form for FSI problems. We demonstrate that changes of variables and coordinates can be incorporated in a particularly natural way by sufficiently smooth transformations

$$\mathbb{T} : \bar{\mathcal{Q}} \to \mathcal{Q}, \quad \boldsymbol{q} = \mathbb{T}(\bar{\boldsymbol{q}}) \,, \tag{4}$$

between state spaces that preserve the variational damped Hamiltonian structure of the problem in each step on the way: Consider two damped Hamiltonian systems with evolution laws $\partial_t \bar{\boldsymbol{q}}(t) = (\bar{\mathbb{J}}(\bar{\boldsymbol{q}}) - \bar{\mathbb{K}}(\bar{\boldsymbol{q}}))\mathrm{D}\bar{\mathscr{F}}(\bar{\boldsymbol{q}})$ and $\partial_t \boldsymbol{q}(t) = (\mathbb{J}(\boldsymbol{q}) - \mathbb{K}(\boldsymbol{q}))\mathrm{D}\mathscr{F}(\boldsymbol{q})$. Assume that both systems admit a weak formulation (3) and that their solutions $\bar{\boldsymbol{q}}(t)$ and $\boldsymbol{q}(t)$ satisfy the transformation relation $\boldsymbol{q}(t) = \mathbb{T}(\bar{\boldsymbol{q}}(t))$. We study the relation between the corresponding weak formulations (3). The linearization of $\mathbb{T}(\bar{\boldsymbol{q}})$, i.e., its (Fréchet-) derivative $\mathbb{L}(\bar{\boldsymbol{q}}) := \mathrm{D}_{\bar{\boldsymbol{q}}}\mathbb{T}(\bar{\boldsymbol{q}}) : \bar{\mathcal{V}} \to \mathcal{V}$, $\boldsymbol{v} = \mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}}$ for all $\bar{\boldsymbol{v}} \in \bar{\mathcal{V}}$, then can be used to carry out the transformation in the weak formulation (2). More precisely, if $\mathscr{F}(\boldsymbol{q}) = \bar{\mathscr{F}}(\bar{\boldsymbol{q}})$ and given the validity of the chain rule $\langle \mathrm{D}\mathscr{F}(\boldsymbol{q}), \mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}}\rangle_{\mathcal{V}} = \langle \mathrm{D}\bar{\mathscr{F}}(\bar{\boldsymbol{q}}), \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}}$ for all $\bar{\boldsymbol{v}} \in \bar{\mathcal{V}}$, the use of the map $\mathbb{M}$ from (2) implies that the following transformation relation between the test functions $\boldsymbol{w} \in \mathcal{W}$ and $\bar{\boldsymbol{w}} \in \bar{\mathcal{W}}$ from (3) holds true

$$\mathbb{L}^*(\bar{\boldsymbol{q}})\mathbb{M}^*\boldsymbol{w} = \bar{\mathbb{M}}^*\bar{\boldsymbol{w}} \,, \quad \text{i.e.,} \quad \langle \mathbb{M}^*\boldsymbol{w}, \mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}}\rangle_{\mathcal{V}} = \langle \bar{\mathbb{M}}^*\bar{\boldsymbol{w}}, \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}} \ \text{ for all } \bar{\boldsymbol{v}} \in \bar{\mathcal{V}} \,. \tag{5}$$

By this transformation relation, the bilinear forms $\bar{\mathrm{a}} : \bar{\mathcal{W}} \times \bar{\mathcal{W}} \to \mathbb{R}$ and $\bar{\mathrm{b}} : \bar{\mathcal{V}} \times \bar{\mathcal{W}} \to \mathbb{R}$ transform into corresponding bilinear forms $\mathrm{a} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ and $\mathrm{b} : \mathcal{V} \times \mathcal{W} \to \mathbb{R}$ accordingly. It is the aim of this work to demonstrate the versatility of the above described transformation framework and the weak formulation in terms of a saddle point problem for FSI problems. As indicated in Fig. 1, for the formulation of this coupled problem it will be important to switch with the aid of suitable transformations between the Lagrangian and the Eulerian frame of coordinates. We introduce the general notation as well as the weak formulation of the FSI problem as a saddle point problem in Lagrangian and Eulerian coordinates in Sec. 2. In particular, in Sec. 2.2 the above described change of coordinates is performed from the Lagrangian to the Eulerian frame under preservation of the saddle point structure. We present some conclusions and perspectives in Section 3.

2 Variational formulations in different coordinate systems

Lagrangian and Eulerian domains. Let $\bar{\Omega} \subset \mathbb{R}^d$ be an open domain with sufficiently smooth boundary $\partial\bar{\Omega}$, representing the configuration of the fluid and solid domains at time $t = 0$. Let $\mathrm{cl}\,\bar{\Omega}$ be the closure of $\bar{\Omega}$, and $\bar{\Gamma} \subset \bar{\Omega}$ a closed, sufficiently smooth $(d-1)$-dimensional subset, dividing $\bar{\Omega}$ into its solid and fluid domains. We call $\bar{\Omega}$ the *Lagrangian* or *reference domain* and the points $\bar{\boldsymbol{x}} \in \mathrm{cl}\,\bar{\Omega}$ the *Lagrangian coordinates* or *material coordinates*, and we indicate with $\bar{\nabla}$ the gradient w.r.t. $\bar{\boldsymbol{x}}$. Let $\bar{\chi} : [0,\mathrm{T}] \times \bar{\Omega} \to \mathbb{R}^d$ be a continuously differentiable map with continuously differentiable inverse and with the property that $\chi(0,\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}$ for all $\bar{\boldsymbol{x}} \in \bar{\Omega}$, describing the trajectory of each material point $\bar{\boldsymbol{x}}$ at time t in the observer frame of reference. We call $\bar{\chi}$ a motion – resp. – a flow map, and we denote by $\bar{\boldsymbol{F}} := \bar{\nabla}\bar{\chi}$ its Jacobian, by $\bar{V} := \det(\bar{\boldsymbol{F}})$ the determinant of its Jacobian, and by $\boldsymbol{\alpha}$ its inverse map at time t, i.e. the map for which $\boldsymbol{\alpha}(t,\bar{\chi}(t,\bar{\boldsymbol{x}})) := \bar{\boldsymbol{x}}$ for all $(t,\bar{\boldsymbol{x}}) \in [0,\mathrm{T}] \times \bar{\Omega}$.

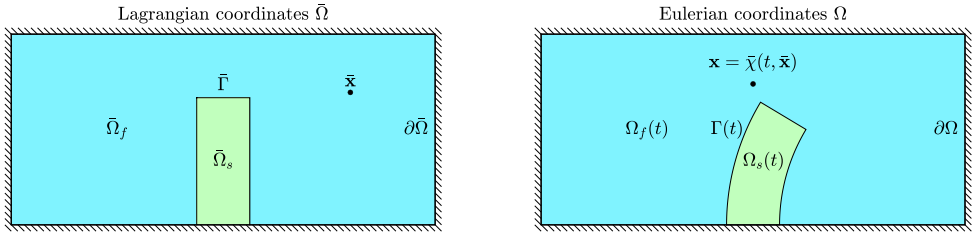

Figure 1: Sketch of typical FSI problem with Lagrangian and Eulerian coordinates.

We indicate with $\boldsymbol{x}$ the *Eulerian coordinates*, i.e., the coordinates in the observer frame of reference, and with ∇ the gradient w.r.t. $\boldsymbol{x}$. The *current* or *Eulerian domain* is then given by $\Omega = \Omega(t) = \bar{\chi}(t,\bar{\Omega}) := \{\bar{\chi}(t,\bar{\boldsymbol{x}}),\ \bar{\boldsymbol{x}} \in \bar{\Omega}\}$, and it represents the current geometrical configuration of the domain $\bar{\Omega}$ in the observer frame of reference. Its boundary $\partial\Omega = \bar{\chi}(t,\partial\bar{\Omega})$ and the Eulerian interface $\Gamma = \bar{\chi}(t,\bar{\Gamma})$ are defined accordingly. We point out that the current domain and its boundary parts may be time-dependent subsets of $\mathbb{R}^d$.

We use the bar notation $\bar{\cdot}$ to indicate fields defined on $\bar{\Omega}$, say $\bar{a}(t,\bar{\boldsymbol{x}})$. Fields without bars indicate an Eulerian field, or the representation of a Lagrangian field according to $a(t,\boldsymbol{x}) := \bar{a}(t,\bar{\chi}^{-1}(t,\boldsymbol{x}))$. We notice that the chain rule applies, and allows one to express the gradient of a Lagrangian vector field $\bar{\nabla}\bar{a}$ in terms of the gradient of Eulerian fields and $\bar{\boldsymbol{F}}$, i.e., $\bar{\nabla}\bar{a}(t,\bar{\boldsymbol{x}}) = \nabla a(t,\bar{\chi}(t,\bar{\boldsymbol{x}}))\bar{\boldsymbol{F}}(t,\bar{\boldsymbol{x}})$, and vice-versa, i.e., $\nabla a(t,\boldsymbol{x}) = (\bar{\nabla}\bar{a})(t,\bar{\chi}^{-1}(t,\boldsymbol{x}))\boldsymbol{F}^{-1}(t,\boldsymbol{x})$.

Lagrangian and Eulerian states. Based on the flow map $\bar{\chi}$, the *Lagrangian velocity* is defined by the map $\bar{\boldsymbol{u}}(t,\boldsymbol{x}) = \partial_t\bar{\chi}(t,\bar{\boldsymbol{x}})$, which is the time derivative of $\bar{\chi}$ for all $(t,\bar{\boldsymbol{x}}) \in [0,\mathrm{T}] \times \bar{\Omega}$, while we denote by $\bar{\varrho}_0 : \bar{\Omega} \to \mathbb{R}$ the mass density of a material point in Lagrangian coordinates at time $t = 0$, and with $\bar{\varrho}(t,\bar{\boldsymbol{x}}) = \bar{\varrho}_0(\bar{\boldsymbol{x}})/\bar{V}(t,\bar{\boldsymbol{x}})$ the mass density expressed in Lagrangian coordinates. Similarly, we indicate with $\bar{\boldsymbol{p}}_0(t,\bar{\boldsymbol{x}}) = \bar{\varrho}_0(\bar{\boldsymbol{x}})\bar{\boldsymbol{u}}(t,\bar{\boldsymbol{x}})$ for all $(t,\bar{\boldsymbol{x}}) \in [0,\mathrm{T}] \times \bar{\Omega}$ the momentum density w.r.t. the initial configuration, and with $\bar{\boldsymbol{p}}(t,\bar{\boldsymbol{x}}) = \bar{\varrho}(t,\bar{\boldsymbol{x}})\bar{\boldsymbol{u}}(t,\bar{\boldsymbol{x}})$ for all $(t,\bar{\boldsymbol{x}}) \in [0,\mathrm{T}] \times \bar{\Omega}$ the current momentum density, expressed in Lagrangian coordinates. The corresponding Eulerian fields $\boldsymbol{u}, \boldsymbol{F}, \varrho_0, \varrho, \boldsymbol{p}_0$, and $\boldsymbol{p}$ are all obtained by composition with $\boldsymbol{\alpha}(t,\boldsymbol{x})$. We note that the Eulerian counterpart of $\bar{\boldsymbol{F}}$ can equivalently be expressed as $\boldsymbol{F} = (\nabla\boldsymbol{\alpha})^{-1}$. In accordance with our choice later on for the fluid-structure interaction problem we here introduce the Lagrangian state vector $\bar{\boldsymbol{q}}$ and the Eulerian state vector $\boldsymbol{q}$ to be given by the following variables

$$\bar{\boldsymbol{q}} := (\bar{\chi}, \bar{\boldsymbol{p}}_0) \quad \text{and} \quad \boldsymbol{q} := (\boldsymbol{q}_\mathrm{f}, \boldsymbol{q}_\mathrm{s}), \ \text{where } \boldsymbol{q}_\mathrm{f} := (\varrho_\mathrm{f}, \boldsymbol{p}_\mathrm{f}) \text{ for fluid and } \boldsymbol{q}_\mathrm{s} := (\boldsymbol{\alpha}, \boldsymbol{p}_\mathrm{s}) \text{ for solid}, \quad (6)$$

and we refer to Sections 2.1 and 2.2 for more details about this choice. In particular, (6) also indicates the transformation $\mathbb{T}$ mentioned in (4) and used for the Lagrangian-Eulerian change of coordinates subsequently in Sec. 2.2. Function spaces defined with respect to the Lagrangian coordinate frame are denoted by $\bar{\mathcal{Q}}$, $\bar{\mathcal{V}}$, $\bar{\mathcal{W}}$ and their Eulerian counterparts by $\mathcal{Q}, \mathcal{V}, \mathcal{W}$, respectively.

Functionals and geometric structures in Lagrangian and Eulerian coordinates. In the Lagrangian frame we consider integral functionals $\bar{\mathscr{G}} : \bar{\mathcal{V}} \to \mathbb{R}$ with densities $\bar{G} : (\bar{\Omega}\backslash\bar{\Gamma}) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ which are functions of the state vector $\bar{\boldsymbol{q}} \in \mathbb{R}^d \times \mathbb{R}^d$ from (6) and of the deformation gradient $\bar{\boldsymbol{F}}$. With respect to Eulerian coordinates the functional is denoted by $\mathscr{G} : \mathcal{V} \to \mathbb{R}$ and $G : (\Omega\backslash\Gamma) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ its density, which depends on the Eulerian state vector $\boldsymbol{q}$ from (6) and on the Eulerian deformation gradient $\boldsymbol{F}$. Due to the independence of the quantity with respect to the choice of the coordinate frame there holds

$$\bar{\mathscr{G}}(\bar{\boldsymbol{q}}) = \mathscr{G}(\boldsymbol{q}), \text{ and hence } \quad \bar{\mathscr{G}}(\bar{\boldsymbol{q}}) = \int_{\bar{\Omega}\backslash\bar{\Gamma}} \bar{G}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{F}}) \,\mathrm{d}\bar{\boldsymbol{x}} = \int_{\Omega\backslash\Gamma} G(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{F}) \,\mathrm{d}\boldsymbol{x} = \mathscr{G}(\boldsymbol{q}) \,,$$

Note that these functionals explicitly exclude integration over the interface $\bar{\Gamma}$, resp. Γ. Similarly, geometric structures in the Lagrangian and Eulerian frame are denoted by $\bar{\mathbb{J}}, \bar{\mathbb{K}}(\bar{\boldsymbol{q}}) : \bar{\mathcal{V}}^* \to \bar{\mathcal{V}}$ and $\mathbb{J}, \mathbb{K}(\boldsymbol{q}) : \mathcal{V}^* \to \mathcal{V}$.

Derivatives of functionals in the Lagrangian frame. The first variation of $\bar{\mathscr{G}}$ with respect to $\bar{\boldsymbol{v}} = (\bar{\boldsymbol{v}}_{\bar{\chi}}, \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}}) \in \bar{\mathcal{V}}$ is defined by

$$\langle \mathrm{D}\bar{\mathscr{G}}(\bar{\boldsymbol{q}}), \bar{\boldsymbol{v}} \rangle_{\bar{\mathcal{V}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (\bar{\mathscr{G}}(\bar{\boldsymbol{q}} + \varepsilon\bar{\boldsymbol{v}}) - \bar{\mathscr{G}}(\bar{\boldsymbol{q}})) = \int_{\bar{\Omega}\backslash\bar{\Gamma}} \tfrac{\partial \bar{G}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{F}})}{\partial \bar{\boldsymbol{q}}} \cdot \bar{\boldsymbol{v}} + \tfrac{\partial \bar{G}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{F}})}{\partial \bar{\boldsymbol{F}}} : \bar{\nabla} \bar{\boldsymbol{v}}_{\bar{\chi}} \,\mathrm{d}\bar{\boldsymbol{x}} \,.$$

We assume that all quantities involved are sufficiently smooth, that $\bar{\boldsymbol{v}}_{\bar{\chi}}$ satisfies homogeneous boundary conditions along the outer boundary $\partial\bar{\Omega}$, and that it is continuous across $\bar{\Gamma}$. Integration by parts gives

$$\langle \mathrm{D}\bar{\mathscr{G}}(\bar{\boldsymbol{q}}), \bar{\boldsymbol{v}} \rangle_{\bar{\mathcal{V}}} = \int_{\bar{\Omega}\backslash\bar{\Gamma}} \Big(\tfrac{\partial \bar{G}}{\partial \bar{\chi}} - \bar{\nabla} \cdot \tfrac{\partial \bar{G}}{\partial \bar{\boldsymbol{F}}} \Big) \cdot \bar{\boldsymbol{v}}_{\bar{\chi}} + \tfrac{\partial \bar{G}}{\partial \bar{\boldsymbol{p}}_0} \cdot \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}} \,\mathrm{d}\bar{\boldsymbol{x}} + \int_{\bar{\Gamma}} \Big(\gamma_{\mathrm{s}}\Big(\tfrac{\partial \bar{G}}{\partial \bar{\boldsymbol{F}}}\Big) - \gamma_{\mathrm{f}}\Big(\tfrac{\partial \bar{G}}{\partial \bar{\boldsymbol{F}}}\Big) \Big) \bar{\boldsymbol{\nu}} \cdot \gamma_{\mathrm{s}}(\bar{\boldsymbol{v}}_{\bar{\chi}}) \,\mathrm{d}\bar{s} \tag{7}$$

with $\bar{G} = \bar{G}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{F}})$, with γ_{s} and γ_{f} trace operators from the subdomains $\bar{\Omega}_{\mathrm{s}}$, $\bar{\Omega}_{\mathrm{f}}$ onto the interface $\bar{\Gamma}$, with $\boldsymbol{\nu}$ the outer unit normal vector of $\bar{\Omega}_{\mathrm{s}}$ along $\bar{\Gamma}$, and $\bar{s}$ representing a suitable surface measure.

2.1 Lagrangian fluid-structure interaction problem

In the Lagrangian setup of Fig. 1 (left) the Lagrangian domain $\bar{\Omega} = \bar{\Omega}_{\mathrm{s}} \cup \bar{\Omega}_{\mathrm{f}} \cup \bar{\Gamma}$ is given by the disjoint union of the Lagrangian solid and fluid domains, $\bar{\Omega}_{\mathrm{s}}$ and $\bar{\Omega}_{\mathrm{f}}$, and their joint interface $\bar{\Gamma}$. For simplicity, we define here the unit normal vector $\bar{\boldsymbol{\nu}}$ along $\bar{\Gamma}$ to be $\bar{\boldsymbol{\nu}} := \bar{\boldsymbol{\nu}}_{\mathrm{s}} = -\bar{\boldsymbol{\nu}}_{\mathrm{f}}$ given by the outer unit normal along of $\bar{\Omega}_{\mathrm{s}}$ along $\bar{\Gamma}$. With $i \in \{\mathrm{s}, \mathrm{f}\}$, for any function $\bar{a} : \bar{\Omega} \to \mathbb{R}^m$, $m \in \{1, d\}$, we denote by $\bar{a}_i = \bar{a}|_{\bar{\Omega}_i}$ its restriction to the subdomain $\bar{\Omega}_i$ and $\gamma_i(\bar{a})$ denotes the trace of $\bar{a}_i : \bar{\Omega}_i \to \mathbb{R}^m$ on $\bar{\Gamma}$. For sufficiently smooth functions $\bar{a}$ we have for $\bar{\boldsymbol{x}} \in \bar{\Gamma}$ that $\gamma_i(\bar{a})(\bar{\boldsymbol{x}}) = \bar{a}_i(\bar{\boldsymbol{x}})$ and if misinterpretation is excluded we also write $\bar{a}_i$ for the trace of $\bar{a}_i$ on $\bar{\Gamma}$. Above notation is applied in particular to the Lagrangian states $\bar{\boldsymbol{q}} = (\bar{\chi}, \bar{\boldsymbol{p}}_0)$, the deformation gradient $\bar{\boldsymbol{F}}$, and the Lagrangian mass density $\bar{\varrho}_0$. We point out that $\bar{\varrho}_0 : \bar{\Omega} \to \mathbb{R}$ is a piecewise constant function in Lagrangian coordinates, i.e., $\bar{\varrho}_{0i} = \mathrm{const.}$ on each of the subdomains $\bar{\Omega}_i$. Moreover, $\bar{\chi} : \bar{\Omega} \to \mathbb{R}^d$ has to satisfy boundary condition along the outer boundary $\partial\bar{\Omega}$. In this work we consider no-slip conditions $\bar{\chi}(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}$ for all $\bar{\boldsymbol{x}} \in \partial\bar{\Omega}$, which implies $\boldsymbol{u}(t, \boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \partial\Omega$ for the Eulerian velocity. For the joint interface to be mapped to a joint interface, it is required that the images of $\bar{\Gamma}$ under $\bar{\chi}_i$ coincide, i.e., $\Gamma = \bar{\chi}_{\mathrm{s}}(\bar{\Gamma}) = \bar{\chi}_{\mathrm{f}}(\bar{\Gamma})$. This property holds true if the Eulerian velocities satisfy $(\boldsymbol{u}_{\mathrm{s}}(t, \boldsymbol{x}) - \boldsymbol{u}_{\mathrm{f}}(t, \boldsymbol{x})) \cdot \boldsymbol{\nu} = 0$ for any $\boldsymbol{x} \in \Gamma$. This condition is also called *impermeability* and is valid for moving interfaces $\Gamma(t)$ in the absence of mass transfer from the solid to the fluid phase. Alternatively, the joint interface can be mapped appropriately by directly requiring a no-slip condition $\bar{\chi}_{\mathrm{s}}(\bar{\boldsymbol{x}}) = \bar{\chi}_{\mathrm{f}}(\bar{\boldsymbol{x}})$ for all $\bar{\boldsymbol{x}} \in \bar{\Gamma}$, which implies $\boldsymbol{u}_{\mathrm{s}}(t, \boldsymbol{x}) = \boldsymbol{u}_{\mathrm{f}}(t, \boldsymbol{x})$ for all $\boldsymbol{x} \in \Gamma(t)$ for the Eulerian velocities. The free energy $\bar{\mathscr{F}} : \bar{\mathcal{Q}} \to \mathbb{R}$ of the fluid-structure interaction system

$$\bar{\mathscr{F}}(\bar{\boldsymbol{q}}) = \int_{\bar{\Omega}\backslash\bar{\Gamma}} \bar{F}(\bar{\boldsymbol{q}}) \,\mathrm{d}\bar{\boldsymbol{x}} = \int_{\bar{\Omega}\backslash\bar{\Gamma}} \tfrac{1}{2\bar{\varrho}_0} |\bar{\boldsymbol{p}}_0|^2 + \bar{W}(\bar{\boldsymbol{x}}, \bar{\chi}, \bar{\boldsymbol{F}}) \,\mathrm{d}\bar{\boldsymbol{x}} \tag{8}$$

consists of the kinetic energy and the internal energy with density $\bar{W}$, here mainly accounting for mechanical effects. In particular, the main difference of $\bar{W}$ for the fluid and solid phase can be encoded in space-dependent coefficients of $\bar{W} : \bar{\Omega} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$, which make sure that $\bar{V}_{\rm f}\bar{W}_{\rm f} := \bar{W}|_{\bar{\Omega}_{\rm f}}$ only depends on $\bar{\boldsymbol{F}}_{\rm f}$ through its Jacobi determinant, whereas the stored elastic energy density of the solid $\bar{W}_{\rm s} := \bar{W}|_{\bar{\Omega}_{\rm s}}$ may depend on all minors of $\bar{\boldsymbol{F}}_{\rm s}$, see Example 1 below. Hence, we set

$$\bar{W}(\bar{\boldsymbol{x}}, \bar{\chi}, \bar{\boldsymbol{F}}) = \bar{V}_{\rm f}\bar{W}_{\rm f}(\bar{\chi}, \bar{\varrho}_{\rm f}/\bar{V}_{\rm f}) \text{ for } \bar{\boldsymbol{x}} \in \bar{\Omega}_{\rm f} \quad \text{and} \quad \bar{W}(\bar{\boldsymbol{x}}, \bar{\chi}, \bar{\boldsymbol{F}}) = \bar{W}_{\rm s}(\bar{\boldsymbol{F}}_{\rm s}) \text{ for } \bar{\boldsymbol{x}} \in \bar{\Omega}_{\rm s}. \tag{9}$$

Example 1. *Gravity and compressibility of the fluid is taken into account by setting* $\bar{V}_{\rm f}\bar{W}_{\rm f}(\bar{\chi}, \varrho_{\rm f}) = \bar{\chi} \cdot \boldsymbol{g}\varrho_{\rm f} + \frac{\kappa}{2}(\varrho_{\rm f} - \varrho_{eq})^2$*. Moreover, stored elastic energy densities in finite strain elasticity depend on the minors of* $\bar{\boldsymbol{F}}_{\rm s}$*, hence nonlinearly on* $\bar{\boldsymbol{F}}_{\rm s}$*, e.g.,*

$$\bar{W}_{\rm s}(\bar{\boldsymbol{F}}_{\rm s}) = \sum_{i=1}^{M} \alpha_i {\rm tr}(\bar{\boldsymbol{F}}_{\rm s}^{\top}\bar{\boldsymbol{F}}_{\rm s})^{\delta_i/2} + \sum_{j=1}^{N} \beta_j {\rm tr}({\rm Cof}(\bar{\boldsymbol{F}}_{\rm s}^{\top}\bar{\boldsymbol{F}}_{\rm s}))(\bar{\boldsymbol{F}}_{\rm s}^{\top}\bar{\boldsymbol{F}}_{\rm s})^{\hat{\delta}_j/2} + \kappa({\rm det}\bar{\boldsymbol{F}}_{\rm s}),$$

for an Ogden material or $\bar{W}_{\rm s}(\bar{\boldsymbol{F}}_{\rm s}) = \alpha|\bar{\boldsymbol{F}}_{\rm s}|^2 + \beta|{\rm Cof}\bar{\boldsymbol{F}}_{\rm s}|^2 + \kappa({\rm det}\bar{\boldsymbol{F}}_{\rm s})$ *for Mooney-Rivlin materials for* $d = 3$*, where* $\kappa({\rm det}\bar{\boldsymbol{F}}_{\rm s}) \to \infty$ *as* ${\rm det}\bar{\boldsymbol{F}}_{\rm s} \to 0$*, see [6, p. 155]. Hence, the bilinear form* $\bar{\rm a}|_{\bar{\Omega}_{\rm s}}$ *is rather defined on* $\bar{\mathcal{W}}^* \times \bar{\mathcal{W}}$*, with the component* $\mathcal{W}_{\bar{\chi}_{\rm s}} = \Pi_{i=1}^{d}\bar{\mathcal{W}}_{{\rm M}_i(\bar{\boldsymbol{F}})}$ *being a product of suitable* L^p*-spaces providing coercivity for the minors* ${\rm M}_i(\bar{\boldsymbol{F}})$ *and* $\mathcal{W}^*$ *its dual. For* $p = 2$ *for all minors we indeed have* $\mathcal{W}^*_{\bar{\chi}_{\rm s}} \equiv \mathcal{W}_{\bar{\chi}_{\rm s}}$ *and hence the mapping property stated in the introduction. The abovementioned no-slip condition along* $\partial\bar{\Omega}$ *and the interfacial coupling conditions can be incorporated by the choice of the state space* $\bar{\mathcal{Q}}$*. Additional jump conditions for* $\bar{\chi}$ *across* $\bar{\Gamma}$ *can be incorporated by specific interfacial energy and dissipation terms. Incompressibility* $\bar{V}_i \equiv {\rm det}\,\bar{\boldsymbol{F}}_i = 1$ *of the fluid or solid phase is incorporated by a further restriction of* $\bar{\mathcal{Q}}$ *and typically leads to a problem formulation with Lagrange multipliers, which we do not discuss here.* ⋆

Qualitative characterization of the ingredients to (3)**:** The auxiliary space $\bar{\mathcal{W}}$ and the operator $\bar{\mathbb{M}}^* : \bar{\mathcal{W}} \to \bar{\mathcal{V}}^*$ are introduced in (3) to give a suitable representation of elements ${\rm D}\bar{\mathscr{F}}(\bar{\boldsymbol{q}}) \in \bar{\mathcal{V}}^*$. As discussed in Ex. 1 the space $\bar{\mathcal{W}}$ is given as the product of suitable $L^p(\bar{\Omega})$-spaces, while the state space $\bar{\mathcal{Q}}$ and the velocity space $\bar{\mathcal{V}}$ are (subsets of) suitable Sobolev spaces. In this representation in particular the coupling conditions for the states and the stresses of the fluid-structure interaction along $\bar{\Gamma}$ should be captured. To this aim we define the operator $\bar{\mathbb{M}}^* : \bar{\mathcal{W}} \to \bar{\mathcal{V}}^*$ and the bilinear form $\bar{\rm b} : \bar{\mathcal{V}} \times \bar{\mathcal{W}} \to \mathbb{R}$ in (3b) by

$$\bar{\rm b}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}) = \langle\bar{\mathbb{M}}^*\bar{\boldsymbol{w}}, \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}} := \int_{\bar{\Omega}\setminus\bar{\Gamma}} (\bar{\boldsymbol{w}}_{\bar{\chi}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}} + \bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}} \cdot \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}})\,{\rm d}\bar{x} + \int_{\bar{\Gamma}} \bar{\boldsymbol{w}}_{\bar{\Gamma}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}}\,{\rm d}\bar{s} \tag{10}$$

for any $\bar{\boldsymbol{v}} = (\bar{\boldsymbol{v}}_{\bar{\chi}}, \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}}) \in \bar{\mathcal{V}}$ and $\bar{\boldsymbol{w}} = (\bar{\boldsymbol{w}}_{\bar{\chi}}, \bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}}, \bar{\boldsymbol{w}}_{\Gamma}) \in \bar{\mathcal{W}}$, where $\bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}_{\rm f}} = \bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}_{\rm s}} = \bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}}$ on $\bar{\Gamma}$. Based on this, in (3b), the representation $\bar{\boldsymbol{\eta}} \in \bar{\mathcal{W}}$ of ${\rm D}\bar{\mathscr{F}}(\bar{\boldsymbol{q}}) \in \bar{\mathcal{V}}^*$ for $\bar{\mathscr{F}}$ from (8) is introduced as

$$\bar{\rm b}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\eta}}) := \langle\bar{\mathbb{M}}^*\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}} = \langle{\rm D}\bar{\mathscr{F}}(\bar{\boldsymbol{q}}), \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}} = \int_{\bar{\Omega}\setminus\bar{\Gamma}} \left(\frac{\bar{\boldsymbol{p}}_0}{\bar{\varrho}_0} \cdot \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}} + \partial_{\bar{\chi}}\bar{W} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}} + \partial_{\bar{\boldsymbol{F}}}\bar{W} : \bar{\nabla}\bar{\boldsymbol{v}}_{\bar{\chi}}\right) {\rm d}\bar{\boldsymbol{x}}.$$

Assuming sufficient regularity of all quantities involved, an integration by parts, as in (7), identifies

$$\bar{\boldsymbol{\eta}}_{\bar{\chi}} = \frac{\partial\bar{W}}{\partial\bar{\chi}} - \bar{\nabla}\cdot\frac{\partial\bar{W}}{\partial\bar{\boldsymbol{F}}} \quad \text{and} \quad \bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}_0} = \frac{\bar{\boldsymbol{p}}_0}{\bar{\varrho}_0} \text{ in } \bar{\Omega}\setminus\bar{\Gamma} \quad \text{and} \quad \bar{\boldsymbol{\eta}}_{\bar{\Gamma}} = \Big(\frac{\partial\bar{W}_{\rm s}}{\partial\bar{\boldsymbol{F}}_{\rm s}} - \frac{\partial(\bar{V}_{\rm f}\bar{W}_{\rm f})}{\partial\bar{\boldsymbol{F}}_{\rm f}}\Big) \cdot \bar{\boldsymbol{\nu}} \text{ on } \bar{\Gamma}\,. \tag{11}$$

Geometric structures and bilinear forms: Next, we characterize the Poisson and Onsager operators $\bar{\mathbb{J}}$ and $\bar{\mathbb{K}}$ as well as the corresponding bilinear forms $\bar{\rm j}$ and $\bar{\rm k}$. Thus, in what follows, we shall consider functionals $\bar{\mathscr{G}}, \bar{\mathscr{H}} : \bar{\mathcal{Q}} \to \mathbb{R}$ sufficiently smooth and the representations $\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\boldsymbol{w}}^{\bar{H}} \in \bar{\mathcal{W}}$ of their functional derivatives by (10), i.e., $\bar{\mathbb{M}}^*\bar{\boldsymbol{w}}^{\bar{G}} = {\rm D}\bar{\mathscr{G}}(\bar{\boldsymbol{q}})$ and $\bar{\mathbb{M}}^*\bar{\boldsymbol{w}}^{\bar{H}} = {\rm D}\bar{\mathscr{H}}(\bar{\boldsymbol{q}})$ in $\bar{\mathcal{V}}^*$.

Reversible contribution by $\bar{\mathbb{J}}$, resp. $\bar{\mathrm{j}}$ in (2) & (3): The Poisson operator $\bar{\mathbb{J}}$ and the corresponding bilinear form $\bar{\mathrm{j}}$ are related by $\langle \mathrm{D}\bar{\mathscr{G}}, \bar{\mathbb{J}}\mathrm{D}\bar{\mathscr{H}}\rangle_{\bar{\mathcal{V}}} = \langle \bar{\mathbb{M}}^*\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\mathbb{J}}\bar{\mathbb{M}}^*\bar{\boldsymbol{w}}^{\bar{H}}\rangle_{\bar{\mathcal{V}}} = \bar{\mathrm{j}}(\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\boldsymbol{w}}^{\bar{H}})$, where

$$\langle \mathrm{D}\bar{\mathscr{G}}, \bar{\mathbb{J}}\mathrm{D}\bar{\mathscr{H}}\rangle_{\bar{\mathcal{V}}} = \int_{\bar{\Omega}\setminus\bar{\Gamma}} \begin{pmatrix} \bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\chi}} \\ \bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\boldsymbol{p}}} \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbb{I}_d \\ -\mathbb{I}_d & 0 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\chi}} \\ \bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\boldsymbol{p}}} \end{pmatrix} \mathrm{d}\bar{\boldsymbol{x}} + \int_{\bar{\Gamma}} \left[\bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\boldsymbol{p}}} \cdot \bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\Gamma}} - \bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\boldsymbol{p}}} \cdot \bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\Gamma}} \right] \mathrm{d}\bar{s}, \tag{12}$$

which is adopted from [14] using the identity matrix $\mathbb{I}_d \in \mathbb{R}^{d\times d}$. Additionally we write $\bar{\mathrm{j}}(\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\boldsymbol{w}}^{\bar{H}}) := \bar{\mathrm{j}}_{\mathrm{f}}(\bar{\boldsymbol{w}}^{\bar{G}}_{\mathrm{f}}, \bar{\boldsymbol{w}}^{\bar{H}}_{\mathrm{f}}) + \bar{\mathrm{j}}_{\mathrm{s}}(\bar{\boldsymbol{w}}^{\bar{G}}_{\mathrm{s}}, \bar{\boldsymbol{w}}^{\bar{H}}_{\mathrm{s}}) + \bar{\mathrm{j}}_{\bar{\Gamma}}(\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\boldsymbol{w}}^{\bar{H}})$, where $\bar{\mathrm{j}}_i$ with $i \in \{\mathrm{s}, \mathrm{f}\}$ is the restriction of the first term of (12) to the subdomain $\bar{\Omega}_{\mathrm{i}}$ and $\bar{\mathrm{j}}_{\bar{\Gamma}}$ stands for the interface term. We point out that the coupling between the two phases is established by joint variables in $\mathrm{j}_{\bar{\Gamma}}$ or by continuity conditions in the state space.

Viscous dissipation generated by $\bar{\mathbb{K}}$, resp. $\bar{\mathrm{k}}$: For simplicity we use the standard Newtonian viscosity

$$\bar{\mathrm{k}}(\bar{\boldsymbol{w}}^{\bar{G}}, \bar{\boldsymbol{w}}^{\bar{H}}) = \int_{\bar{\Omega}\setminus\bar{\Gamma}} \bar{\mu}(\bar{\boldsymbol{x}})\ \mathrm{sym}\left(\nabla\bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\boldsymbol{p}}}\right) : \mathrm{sym}\left(\nabla\bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\boldsymbol{p}}}\right) \bar{V}\, \mathrm{d}\bar{\boldsymbol{x}} = \int_{\bar{\Omega}\setminus\bar{\Gamma}} \bar{\nabla}\bar{\boldsymbol{w}}^{\bar{G}}_{\bar{\boldsymbol{p}}} : \bar{M}\bar{\nabla}\bar{\boldsymbol{w}}^{\bar{H}}_{\bar{\boldsymbol{p}}}\, \mathrm{d}\bar{\boldsymbol{x}}\,, \tag{13}$$

with $\mathrm{sym}(A) = (A + A^\top)/2$ and the viscosity $\bar{\mu}$ piecewise constant on the subdomains $\bar{\Omega}_i$, i.e., $\bar{\mu}_i = \bar{\mu}|_{\Omega_i} = \mu_i > 0$ for $i \in \{\mathrm{s}, \mathrm{f}\}$. The tensor $\bar{M} = \bar{M}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{F}})$ depends on $\bar{\boldsymbol{F}}$ since the Eulerian gradient is $\nabla\boldsymbol{u}_i = \bar{\nabla}\bar{\boldsymbol{u}}_i\bar{\boldsymbol{F}}^{-1}$.

Weak formulation (3): With the bilinear form $\bar{\mathrm{a}}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{\eta}}) = \bar{\mathrm{j}}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{\eta}}) - \bar{\mathrm{k}}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{\eta}})$, the Lagrangian weak formulation of the damped Hamiltonian system for fluid-structure interaction in terms of (3) reads

$$\bar{\mathrm{a}}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{\eta}}) - \bar{\mathrm{b}}(\partial_t\bar{\boldsymbol{q}}, \bar{\boldsymbol{w}}) = \bar{\mathrm{a}}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{\eta}}) - \int_{\bar{\Omega}} \left(\bar{\boldsymbol{w}}_{\bar{\chi}} \cdot \partial_t\bar{\chi} + \bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}} \cdot \partial_t\bar{\boldsymbol{p}}_0\right) \mathrm{d}\bar{x} + \int_{\bar{\Gamma}} \bar{\boldsymbol{w}}_{\bar{\Gamma}} \cdot \partial_t\bar{\chi}\, \mathrm{d}\bar{s} = 0, \tag{14a}$$

$$\bar{\mathrm{b}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\eta}}) = \int_{\bar{\Omega}} \left(\bar{\boldsymbol{\eta}}_{\bar{\chi}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}} + \bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}} \cdot \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}}\right) \mathrm{d}\bar{x} + \int_{\bar{\Gamma}} \bar{\boldsymbol{\eta}}_{\bar{\Gamma}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}}\, \mathrm{d}\bar{s} = \langle \mathrm{D}\bar{\mathscr{F}}(\boldsymbol{q}), \bar{\boldsymbol{v}}\rangle_{\bar{\mathcal{V}}}, \tag{14b}$$

with $\bar{\boldsymbol{\eta}} = (\bar{\boldsymbol{\eta}}_{\bar{\chi}}, \bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}}, \bar{\boldsymbol{\eta}}_{\bar{\Gamma}}) \in \bar{\mathcal{W}}$ identified in (11). Then, for sufficiently smooth quantities, an integration by parts in (14) reveals the strong formulation (1) and the following coupling conditions along $\bar{\Gamma}$

$$\partial_t\bar{\chi} = \bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}} = \bar{\boldsymbol{p}}_0/\bar{\varrho}_0 \qquad \text{in } \bar{\Omega}\setminus\bar{\Gamma}\,, \tag{15a}$$

$$\partial_t\bar{\boldsymbol{p}}_0 = -\bar{\boldsymbol{\eta}}_{\bar{\chi}} + \bar{\nabla}\cdot(\bar{M}\bar{\nabla}\bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}}) = \bar{\nabla}\cdot\bar{\boldsymbol{\Sigma}} \qquad \text{in } \bar{\Omega}\setminus\bar{\Gamma}\,, \tag{15b}$$

$$0 = \bar{\boldsymbol{\eta}}_{\bar{\Gamma}} + \bar{M}_{\mathrm{s}}\bar{\nabla}\bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}_{\mathrm{s}}} \cdot \bar{\boldsymbol{\nu}}_{\mathrm{s}} + \bar{M}_{\mathrm{f}}\bar{\nabla}\bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}_{\mathrm{f}}} \cdot \bar{\boldsymbol{\nu}}_{\mathrm{f}} = (\bar{\boldsymbol{\Sigma}}_{\mathrm{s}} - \bar{\boldsymbol{\Sigma}}_{\mathrm{f}}) \cdot \bar{\boldsymbol{\nu}}_{\mathrm{s}} \qquad \text{on } \bar{\Gamma}\,, \tag{15c}$$

$$\partial_t\bar{\chi} = \bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}} = \tfrac{\bar{\boldsymbol{p}}_0}{\bar{\varrho}_0} \qquad \text{on } \bar{\Gamma}, \tag{15d}$$

with $\bar{\boldsymbol{\Sigma}}|_{\bar{\Omega}_i} = \bar{\boldsymbol{\Sigma}}_i$ the total stress given by the sum of elastic and viscous stresses $\bar{\boldsymbol{\Sigma}}_i = \partial_{\bar{\boldsymbol{F}}_i}\bar{W}_i + \bar{M}_i\bar{\nabla}\bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}_i}$. Note that (15a) is obtained by a test with $\bar{\boldsymbol{w}}_{\bar{\chi}}$, (15b) by testing with $\bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}}$ in $\bar{\Omega}\setminus\bar{\Gamma}$, (15c) by a test with $\bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}}$, and (15d) by $\bar{\boldsymbol{w}}_{\bar{\Gamma}}$ on $\bar{\Gamma}$. Finally, we mention that Betsch and Schiebl [4] use $\partial_{\bar{\boldsymbol{q}}}\bar{G}$ and $\partial_{\bar{\boldsymbol{F}}}\bar{G}$ to represent derivatives and the operators $\bar{\mathbb{J}}, \bar{\mathbb{K}}$ to formulate GENERIC for open systems, while Marsden et al. [14] used Dirac measures and defined $\langle \mathrm{D}\bar{\mathscr{F}}, \boldsymbol{v}\rangle = \int_\Omega \frac{\delta E}{\delta\bar{\boldsymbol{q}}} \cdot \boldsymbol{v}\, \mathrm{d}x$. In [20] extensions for boundary dissipation and inclusion of interfacial energies are discussed in the context of moving contact lines.

2.2 Eulerian fluid-structure interaction problem

We now carry out the transformation of the weak Lagrangian FSI problem (14) to the Eulerian frame by applying the abstract framework described in (4)–(6) to the fluid and the solid component. This procedure is well known and was first introduced by Arnold [1]. For this, we shall use the variables introduced in (6) and the decomposition of bilinear forms by restriction of the domain of integration and denoted by the indices f, s, and Γ the fluid, solid, and interface terms. In particular, we have $\mathrm{a}(\boldsymbol{w}, \boldsymbol{\eta}) = \mathrm{a}_{\mathrm{s}}(\boldsymbol{w}_{\mathrm{s}}, \boldsymbol{\eta}_{\mathrm{s}}) + \mathrm{a}_{\mathrm{f}}(\boldsymbol{w}_{\mathrm{f}}, \boldsymbol{\eta}_{\mathrm{f}}) + \mathrm{a}_\Gamma(\boldsymbol{w}, \boldsymbol{\eta})$. This notation underlines that the coupling between solid

and fluid is communicated by the interface bilinear form a_Γ. The auxiliary space $\mathcal{W}$ consists of elements $\boldsymbol{w} = (w_{\varrho_\mathrm{f}}, \boldsymbol{w}_\Gamma, \boldsymbol{w}_{\boldsymbol{\alpha}}, \boldsymbol{w}_{\boldsymbol{p}})$, while elements of the velocity space $\mathcal{V}$ are denoted by $\boldsymbol{q} = (\varrho_\mathrm{f}, \boldsymbol{p}_\mathrm{f}, \boldsymbol{\alpha}, \boldsymbol{p}_\mathrm{s})$. To apply (4)–(6) we introduce the Eulerian state variables $\boldsymbol{q} \in \mathcal{Q}$, the transformation $\mathbb{T} : \bar{\mathcal{Q}} \to \mathcal{Q}$ and its Fréchet derivative $\mathbb{L} : \bar{\mathcal{Q}} \to \mathcal{Q}$ as follows

$$\mathbb{T}(\bar{\boldsymbol{q}}) := \boldsymbol{q} = \begin{pmatrix} \varrho_\mathrm{f} : \Omega_\mathrm{f} \to \mathbb{R} \\ \boldsymbol{p}_\mathrm{f} : \Omega_\mathrm{f} \to \mathbb{R}^d \\ \boldsymbol{\alpha} : \Omega_\mathrm{s} \cup \Gamma \to \bar{\Omega}_\mathrm{s} \cup \bar{\Gamma} \subset \mathbb{R}^d \\ \boldsymbol{p}_\mathrm{s} : \Omega_\mathrm{s} \to \mathbb{R}^d \end{pmatrix} \in \mathcal{Q}, \qquad \mathbb{L}(\bar{\boldsymbol{q}}) := \begin{pmatrix} -\nabla \cdot (\bar{\varrho}_\mathrm{f} \square_\mathrm{f}) & 0 \\ -\nabla \cdot (\bar{\boldsymbol{p}}_\mathrm{f} \otimes \square_\mathrm{f}) & \frac{\square_\mathrm{f}}{\bar{V}_\mathrm{f}} \\ -(\nabla \boldsymbol{\alpha}) \square_\mathrm{s} & 0 \\ -\nabla \cdot (\bar{\boldsymbol{p}}_\mathrm{s} \otimes \square_\mathrm{s}) & \frac{\square_\mathrm{s}}{\bar{V}_\mathrm{s}} \end{pmatrix}, \tag{16}$$

using that $\bar{\varrho} = \bar{\varrho}_0/\bar{V}$ and $\bar{\boldsymbol{p}} = \bar{\boldsymbol{p}}_0/\bar{V}$ and $\mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}} = (v_{\varrho_\mathrm{f}}, \boldsymbol{v}_{\boldsymbol{p}_\mathrm{f}}, \boldsymbol{v}_{\boldsymbol{\alpha}}, \boldsymbol{v}_{\boldsymbol{p}_\mathrm{s}})^\top$. We refer to [25] for more details on such transformations in the context of reactive fluid flows. Moreover, we define $\mathbb{M}^* : \mathcal{W} \to \mathcal{V}^*$ as follows

$$\langle \mathbb{M}^* \boldsymbol{w}, \boldsymbol{v} \rangle_{\mathcal{V}} = \int_{\Omega_\mathrm{f}} w_{\varrho_\mathrm{f}} v_{\varrho_\mathrm{f}} + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \cdot \boldsymbol{v}_{\boldsymbol{p}_\mathrm{f}} \mathrm{d}\boldsymbol{x} + \int_{\Omega_\mathrm{s}} \boldsymbol{w}_{\boldsymbol{\alpha}} \cdot \boldsymbol{v}_{\boldsymbol{\alpha}} + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \cdot \boldsymbol{v}_{\boldsymbol{p}_\mathrm{s}} \mathrm{d}\boldsymbol{x} + \int_\Gamma \boldsymbol{w}_\Gamma \cdot \boldsymbol{v}_{\boldsymbol{\alpha}} \mathrm{d}s =: \mathrm{b}(\boldsymbol{v}, \boldsymbol{w}) \,.$$

From this, we compute the weak form of the transformation via (5), i.e., $\langle \mathbb{M}^* \boldsymbol{w}, \mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}} \rangle_{\mathcal{V}} = \langle \bar{\mathbb{M}}^* \bar{\boldsymbol{w}}, \bar{\boldsymbol{v}} \rangle_{\bar{\mathcal{V}}}$ via

$$\begin{aligned} \langle \mathbb{M}^* \boldsymbol{w}, \mathbb{L}(\bar{\boldsymbol{q}})\bar{\boldsymbol{v}} \rangle_{\mathcal{V}} = & \int_{\Omega_\mathrm{f}} -w_{\varrho_\mathrm{f}} \nabla \cdot (\bar{\varrho}_\mathrm{f} \bar{\boldsymbol{v}}_{\bar{\chi}}) + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \cdot (\tfrac{\bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}_\mathrm{f}}}{\bar{V}_\mathrm{f}} - \nabla \cdot (\bar{\boldsymbol{p}}_\mathrm{f} \otimes \bar{\boldsymbol{v}}_{\bar{\chi}})) \, \mathrm{d}\boldsymbol{x} \\ & + \int_{\Omega_\mathrm{s}} \boldsymbol{w}_{\boldsymbol{\alpha}} \cdot (-(\nabla \boldsymbol{\alpha}) \bar{\boldsymbol{v}}_{\bar{\chi}}) + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \cdot (\tfrac{\bar{\boldsymbol{v}}_{\boldsymbol{p}_\mathrm{s}}}{\bar{V}_\mathrm{s}} - \nabla \cdot (\bar{\boldsymbol{p}}_\mathrm{s} \otimes \bar{\boldsymbol{v}}_{\bar{\chi}})) \, \mathrm{d}\boldsymbol{x} \\ & + \int_\Gamma \boldsymbol{w}_\Gamma \cdot (-(\nabla \boldsymbol{\alpha}) \bar{\boldsymbol{v}}_{\bar{\chi}}) \, \mathrm{d}s \\ = & \int_{\bar{\Omega}_\mathrm{f}} \underbrace{(\bar{\varrho}_\mathrm{f} \nabla w_{\varrho_\mathrm{f}} + (\nabla \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}})^\top \bar{\boldsymbol{p}}_\mathrm{f}) \bar{V}_\mathrm{f}}_{\bar{\boldsymbol{w}}_{\bar{\chi}_\mathrm{f}}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}_\mathrm{f}} + \underbrace{\boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}}}_{\bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}_\mathrm{f}}} \cdot \bar{\boldsymbol{v}}_{\bar{\boldsymbol{p}}_\mathrm{f}} \, \mathrm{d}\bar{\boldsymbol{x}} \\ & + \int_{\bar{\Omega}_\mathrm{s}} \underbrace{-((\nabla \boldsymbol{\alpha})^\top \boldsymbol{w}_{\boldsymbol{\alpha}} + (\nabla \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}})^\top \boldsymbol{p}_\mathrm{s}) \bar{V}_\mathrm{s}}_{\bar{\boldsymbol{w}}_{\bar{\chi}_\mathrm{s}}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}_\mathrm{s}} + \underbrace{\boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}}}_{\bar{\boldsymbol{w}}_{\bar{\boldsymbol{p}}_\mathrm{s}}} \cdot \bar{\boldsymbol{v}}_{\boldsymbol{p}_\mathrm{s}} \, \mathrm{d}\bar{\boldsymbol{x}} \\ & + \int_{\bar{\Gamma}} \underbrace{\Big[-\nabla \boldsymbol{\alpha}^\top \boldsymbol{w}_\Gamma + \Big((w_{\varrho_\mathrm{f}} \bar{\varrho}_\mathrm{f} + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \cdot \bar{\boldsymbol{p}}_\mathrm{f}) |\mathrm{Cof} \bar{\boldsymbol{F}} \cdot \bar{\boldsymbol{\nu}}_\mathrm{f}| - \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \cdot \bar{\boldsymbol{p}}_\mathrm{s} |\mathrm{Cof} \bar{\boldsymbol{F}} \cdot \bar{\boldsymbol{\nu}}_\mathrm{s}| \Big) \cdot \bar{\boldsymbol{\nu}}_\mathrm{s} \Big]}_{\bar{\boldsymbol{w}}_{\bar{\Gamma}}} \cdot \bar{\boldsymbol{v}}_{\bar{\chi}} \, \mathrm{d}\bar{s} \end{aligned} \tag{17}$$

for any $\boldsymbol{w} = (w_{\varrho_\mathrm{f}}, \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}}, \boldsymbol{w}_{\boldsymbol{\alpha}}, \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}}) \in \mathcal{W}$, and $\bar{\boldsymbol{v}} \in \bar{\mathcal{V}}$. We are now ready to compute the transformed bilinear forms (12) and (13). Firstly, we focus on terms with $\boldsymbol{w}_\mathrm{f} = (w_{\varrho_\mathrm{f}}, \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}})$ and use $\mathrm{j}_\mathrm{f}(\boldsymbol{w}_\mathrm{f}, \boldsymbol{\eta}_\mathrm{f}) = \bar{\mathrm{j}}_\mathrm{f}(\bar{\boldsymbol{w}}_\mathrm{f}, \bar{\boldsymbol{\eta}}_\mathrm{f})$ to get

$$\begin{aligned} \mathrm{j}_\mathrm{f}(\boldsymbol{w}_\mathrm{f}, \boldsymbol{\eta}_\mathrm{f}) = & \int_{\Omega_\mathrm{f}} \Big(\varrho_\mathrm{f} \nabla w_{\varrho_\mathrm{f}} + (\nabla \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}})^\top \boldsymbol{p}_\mathrm{f} \Big) \cdot \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} - \Big(\varrho_\mathrm{f} \nabla \eta_{\varrho_\mathrm{f}} + (\nabla \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}})^\top \boldsymbol{p}_\mathrm{f} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \, \mathrm{d}\boldsymbol{x} \\ = & - \int_{\Omega_\mathrm{f}} \Big(\varrho_\mathrm{f} \nabla \eta_{\varrho_\mathrm{f}} + (\nabla \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}})^\top \boldsymbol{p}_\mathrm{f} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} + w_{\varrho_\mathrm{f}} \nabla \cdot \Big(\varrho_\mathrm{f} \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} \Big) + \nabla \cdot \Big(\boldsymbol{p}_\mathrm{f} \otimes \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \, \mathrm{d}\boldsymbol{x} \\ & + \int_\Gamma \Big[w_{\varrho_\mathrm{f}} \varrho_\mathrm{f} \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} + (\boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \cdot \boldsymbol{p}_\mathrm{f}) \, \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} \Big] \cdot \boldsymbol{\nu}_\mathrm{f} \, \mathrm{d}s \,. \end{aligned} \tag{18a}$$

Note that the bulk term of the fluid agrees with the bracket for ideal fluids in the work by Morrison [17]. Analogously by setting $\mathrm{j}_\mathrm{s}(\boldsymbol{w}_\mathrm{s}, \boldsymbol{\eta}_\mathrm{s}) = \bar{\mathrm{j}}_\mathrm{s}(\bar{\boldsymbol{w}}_\mathrm{s}, \bar{\boldsymbol{\eta}}_\mathrm{s})$, one gets the reversible bilinear form for the solid subdomain

$$\begin{aligned} \mathrm{j}_\mathrm{s}(\boldsymbol{w}_\mathrm{s}, \boldsymbol{\eta}_\mathrm{s}) = & \int_{\Omega_\mathrm{s}} \Big((-\nabla \boldsymbol{\alpha})^\top \boldsymbol{w}_{\boldsymbol{\alpha}} + (\nabla \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}})^\top \boldsymbol{p}_\mathrm{s} \Big) \cdot \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} + \Big((\nabla \boldsymbol{\alpha})^\top \boldsymbol{\eta}_{\boldsymbol{\alpha}} - (\nabla \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}})^\top \boldsymbol{p}_\mathrm{s} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \, \mathrm{d}\boldsymbol{x} \\ = & \int_{\Omega_\mathrm{s}} \Big((\nabla \boldsymbol{\alpha})^\top \boldsymbol{\eta}_{\boldsymbol{\alpha}} - (\nabla \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}})^\top \boldsymbol{p}_\mathrm{s} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} - (\nabla \boldsymbol{\alpha})^\top \boldsymbol{w}_{\boldsymbol{\alpha}} \cdot \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} + \nabla \cdot \Big(\boldsymbol{p}_\mathrm{s} \otimes \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} \Big) \cdot \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \, \mathrm{d}\boldsymbol{x} \\ & + \int_\Gamma (\boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \cdot \boldsymbol{p}_\mathrm{s}) \, \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} \cdot \boldsymbol{\nu}_\mathrm{s} \, \mathrm{d}s \,. \end{aligned} \tag{18b}$$

Finally, the skew-symmetric boundary bilinear form is

$$\begin{aligned}\mathrm{j}_\Gamma(\boldsymbol{w},\boldsymbol{\eta}) = \bar{\mathrm{j}}_{\bar\Gamma}(\bar{\boldsymbol{w}},\bar{\boldsymbol{\eta}}) = &\int_\Gamma \left[-\nabla\boldsymbol{\alpha}^\top \boldsymbol{w}_\Gamma + \left(w_{\varrho_\mathrm{f}}\varrho_\mathrm{f} + \boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}}\cdot\boldsymbol{p}_\mathrm{f} - \boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}}\cdot\boldsymbol{p}_\mathrm{s}\right)\boldsymbol{\nu}_\mathrm{s}\right]\cdot\boldsymbol{\eta}_{\boldsymbol{p}}\,\mathrm{d}s \\ &- \int_\Gamma \left[-\nabla\boldsymbol{\alpha}^\top \boldsymbol{\eta}_\Gamma + \left(\eta_{\varrho_\mathrm{f}}\varrho_\mathrm{f} + \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}\cdot\boldsymbol{p}_\mathrm{f} - \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}\cdot\boldsymbol{p}_\mathrm{s}\right)\boldsymbol{\nu}_\mathrm{s}\right]\cdot\boldsymbol{w}_{\boldsymbol{p}}\,\mathrm{d}s\,. \end{aligned} \tag{18c}$$

The transformation of the dissipative bilinear form $\bar{\mathrm{k}}$ is straight-forward, since $\bar{\boldsymbol{\eta}}_{\bar{\boldsymbol{p}}} \equiv \boldsymbol{\eta}_{\boldsymbol{p}}$ and thus

$$\begin{aligned}\mathrm{k}(\boldsymbol{w},\boldsymbol{\eta}) = \bar{\mathrm{k}}(\bar{\boldsymbol{w}},\bar{\boldsymbol{\eta}}) &= \int_{\Omega\setminus\Gamma} \mu(\boldsymbol{x})\mathrm{sym}(\nabla\boldsymbol{\eta}_{\boldsymbol{p}}) : \mathrm{sym}(\nabla\boldsymbol{w}_{\boldsymbol{p}})\,\mathrm{d}\boldsymbol{x} \\ &= \sum_{i\in\{\mathrm{s},\mathrm{f}\}} -\int_{\Omega_i} \nabla\cdot\left[\mu_i\,\mathrm{sym}\left(\nabla\boldsymbol{\eta}_{\boldsymbol{p}_i}\right)\right]\cdot\boldsymbol{w}_{\boldsymbol{p}_i}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \left(\mu_\mathrm{s}\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}^\top\boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} - \mu_\mathrm{f}\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}^\top\boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}}\right)\cdot\boldsymbol{\nu}_\mathrm{s}\,\mathrm{d}s. \end{aligned} \tag{19}$$

The Eulerian free energy $\mathscr{F} = \mathscr{F}_\mathrm{s} + \mathscr{F}_\mathrm{f}$ is

$$\mathscr{F}_\mathrm{f}(\boldsymbol{q}) = \int_{\Omega_\mathrm{f}} \frac{|\boldsymbol{p}_\mathrm{f}|^2}{2\varrho_\mathrm{f}} + \bar{W}_\mathrm{f}(\boldsymbol{x},\varrho_\mathrm{f})\,\mathrm{d}\boldsymbol{x}, \qquad \mathscr{F}_\mathrm{s}(\boldsymbol{q}) = \int_{\Omega_\mathrm{s}} \frac{|\boldsymbol{p}_\mathrm{s}|^2}{2\varrho_\mathrm{s}} + V_\mathrm{s}^{-1}\bar{W}_\mathrm{s}(\boldsymbol{F}_\mathrm{s})\,\mathrm{d}\boldsymbol{x}, \tag{20}$$

defining $\varrho_\mathrm{s} = \varrho_{0\mathrm{s}}/V_\mathrm{s}$ and $V_\mathrm{s} = \det\boldsymbol{F}_\mathrm{s}$ and $\boldsymbol{F}_\mathrm{s} = (\nabla\boldsymbol{\alpha})^{-1}$. For the derivative of $\mathscr{F}$ we get

$$\begin{aligned}
&\eta_{\varrho_\mathrm{f}} = \partial_\varrho\bar{W}_\mathrm{f} - \tfrac{|\boldsymbol{p}_\mathrm{f}|^2}{2\varrho_\mathrm{f}^2}, \qquad \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}} = \tfrac{\boldsymbol{p}_\mathrm{f}}{\varrho_\mathrm{f}}, \qquad \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} = \tfrac{\boldsymbol{p}_\mathrm{s}}{\varrho_\mathrm{s}}\,, \\
&\boldsymbol{\eta}_\Gamma = -\nabla\boldsymbol{\alpha}^{-\top}\left[\left(\tfrac{|\boldsymbol{p}_\mathrm{s}|^2}{2\varrho_\mathrm{s}} + V_s^{-1}\bar{W}_\mathrm{s}\right) - \left(\tfrac{|\boldsymbol{p}_\mathrm{f}|^2}{2\varrho_\mathrm{f}} + \bar{W}_\mathrm{f}\right)\right]\boldsymbol{\nu}_\mathrm{s} - \left(\boldsymbol{F}_\mathrm{s}^\top\partial_{\boldsymbol{F}}\left(\tfrac{\bar{W}_\mathrm{s}}{V}\right)\boldsymbol{F}_\mathrm{s}^\top\right)\boldsymbol{\nu}_\mathrm{s} - \boldsymbol{F}_\mathrm{s}^\top\frac{|\boldsymbol{p}_\mathrm{s}|^2}{2\varrho_\mathrm{s}}\boldsymbol{\nu}_\mathrm{s}\,, \\
&\boldsymbol{\eta}_{\boldsymbol{\alpha}} = \nabla\cdot\left(\boldsymbol{F}_\mathrm{s}^\top\frac{|\boldsymbol{p}_\mathrm{s}|^2}{2\varrho_\mathrm{s}}\right) - \frac{|\boldsymbol{p}_\mathrm{s}|^2}{2\varrho_\mathrm{s}^2}\frac{\boldsymbol{F}_\mathrm{s}^\top}{V_\mathrm{s}}\nabla\left(\varrho_\mathrm{s}V_\mathrm{s}\right) + \nabla\cdot\left(\boldsymbol{F}_\mathrm{s}^\top\partial_{\boldsymbol{F}}\left(\tfrac{\bar{W}_\mathrm{s}}{V}\right)\boldsymbol{F}_\mathrm{s}^\top\right)\,.
\end{aligned}$$

Strong formulation. From (18)–(20) we obtain the following strong formulation by testing with

$$\begin{aligned}
\boldsymbol{w}_{\varrho_\mathrm{f}} \text{ in } \Omega_\mathrm{f}\text{: } \partial_t\varrho_\mathrm{f} &= -\nabla\cdot\left(\varrho_\mathrm{f}\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}\right), && \text{(22a)}\\
\boldsymbol{w}_{\boldsymbol{p}_\mathrm{f}} \text{ in } \Omega_\mathrm{f}\text{: } \partial_t\boldsymbol{p}_\mathrm{f} &= -\left[\nabla\cdot(\boldsymbol{p}_\mathrm{f}\otimes\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}) + \varrho_\mathrm{f}\nabla\eta_{\varrho_\mathrm{f}} + (\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}})\boldsymbol{p}_\mathrm{f}\right] + \nabla\cdot(\mu_\mathrm{f}\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}) \\
&= -\nabla\cdot(\boldsymbol{p}_\mathrm{f}\otimes\boldsymbol{u}_\mathrm{f}) + \nabla\cdot\boldsymbol{\sigma}_\mathrm{f}, && \text{(22b)}\\
\boldsymbol{w}_{\boldsymbol{\alpha}} \text{ in } \Omega_\mathrm{s}\text{: } \partial_t\boldsymbol{\alpha} &= -(\nabla\boldsymbol{\alpha})\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} = -(\nabla\boldsymbol{\alpha})\boldsymbol{u}_\mathrm{s}\,, && \text{(22c)}\\
\boldsymbol{w}_{\boldsymbol{p}_\mathrm{s}} \text{ in } \Omega_\mathrm{s}\text{: } \partial_t\boldsymbol{p}_\mathrm{s} &= -\left[\nabla\cdot(\boldsymbol{p}_\mathrm{s}\otimes\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}) - \nabla\boldsymbol{\alpha}^\top\boldsymbol{\eta}_{\boldsymbol{\alpha}} + \nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}^\top\boldsymbol{p}_\mathrm{s}\right] + \nabla\cdot(\mu_\mathrm{s}\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}) \\
&= -\nabla\cdot(\boldsymbol{p}_\mathrm{s}\otimes\boldsymbol{u}_\mathrm{s}) + \nabla\cdot\boldsymbol{\sigma}_\mathrm{s}, && \text{(22d)}\\
\boldsymbol{w}_{\boldsymbol{p}} \text{ on } \Gamma\text{: } \quad 0 &= \nabla\boldsymbol{\alpha}_s^\top\boldsymbol{\eta}_\Gamma + (\eta_{\varrho_\mathrm{f}}\varrho_\mathrm{f} + \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}}\cdot\boldsymbol{p}_\mathrm{f})\boldsymbol{\nu}_\mathrm{f} + \boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}\cdot\boldsymbol{p}_\mathrm{s}\boldsymbol{\nu}_\mathrm{s} \\
&\quad - \left(\mu_\mathrm{s}\mathrm{sym}(\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}}) - \mu_\mathrm{f}\mathrm{sym}(\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{f}})\right)\cdot\boldsymbol{\nu}_\mathrm{s} \\
&= (\boldsymbol{\sigma}_\mathrm{s} - \boldsymbol{\sigma}_\mathrm{f})\,\boldsymbol{\nu}_\mathrm{s}\,, && \text{(22e)}\\
\boldsymbol{w}_\Gamma \text{ on } \Gamma\text{: } \partial_t\boldsymbol{\alpha} &= -(\nabla\boldsymbol{\alpha})\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{s}} = -(\nabla\boldsymbol{\alpha})\boldsymbol{u}_\mathrm{s}\,, && \text{(22f)}
\end{aligned}$$

where $\boldsymbol{\sigma}_\mathrm{s} = \boldsymbol{\sigma}_\mathrm{s}^\mathrm{visc} + \boldsymbol{\sigma}_\mathrm{s}^\mathrm{el}$ and $\boldsymbol{\sigma}_\mathrm{f} = \boldsymbol{\sigma}_\mathrm{f}^\mathrm{visc} - \pi\mathbb{I}$ are the Cauchy stresses for solid and fluid. We denote $\boldsymbol{\sigma}_\mathrm{i}^\mathrm{visc} = \mu_\mathrm{i}\mathrm{sym}(\nabla\boldsymbol{\eta}_{\boldsymbol{p}_\mathrm{i}})$ with $i\in\{\mathrm{s},\mathrm{f}\}$ the viscous stresses, $\pi = \varrho_\mathrm{f}\partial_{\varrho_\mathrm{f}}\bar{W}_\mathrm{f} - \bar{W}_\mathrm{f}$ the fluid pressure and $\boldsymbol{\sigma}_\mathrm{s}^\mathrm{el} = \bar{V}_\mathrm{s}^{-1}\partial_{\boldsymbol{F}_s}\bar{W}_\mathrm{s}\boldsymbol{F}_s^\top$ the elastic solid stress, where in order to obtain $\nabla\cdot\boldsymbol{\sigma}_\mathrm{s}^\mathrm{el}$ we used Piola's identity $\nabla\cdot(V^{-1}\boldsymbol{F}^\top) = 0$ and the fact that $\nabla_i W = \boldsymbol{F}^{-1}_{mi}\boldsymbol{F}_{jl}(\partial_{\boldsymbol{F}}W)_{kl}\nabla_j\boldsymbol{F}_{km}$. Testing by $\boldsymbol{w}_{\varrho_\mathrm{f}}$ on Γ produces an equation that is always satisfied, in a similar way to the Lagrangian formulation when testing by $\bar{\boldsymbol{w}}_{\boldsymbol{\chi}}$ on $\bar\Gamma$.

3 Conclusions and outlook

In this paper we introduced a variational approach to FSI based on the framework of GENERIC in the special case of an isothermal damped Hamiltonian system for compressible materials, which naturally leads to a weak formulation of the coupled problem and allows it to transform between different sets of coordinates. This approach is convenient to include additional interfacial processes by suitably extending state space, driving functionals, and geometric structures. Perhaps one of the most elegant formulations of FSI problems is the Arbitrary Lagrangian Eulerian (ALE) formulation, dating back to the early seventies [13], in which conservation equations are formulated in an arbitrary frame of reference, typically chosen to be Lagrangian in the solid (i.e., identifying material particles), and Arbitrary-Lagrangian Eulerian in the fluid, mapping the fluid equations (defined on a moving domain) to a fixed reference (artificial) domain, not associated to material particles. Over the years, ALE frameworks have become a standard, and are used in many current applications with FSI, e.g., [2, 24, 26]. Related variational approaches to FSI are currently investigated using incremental minimization schemes [3], and similar FSI problem for compressible fluids have been analysed recently in [15, 22].

To fix the ideas, consider an invertible map $\widehat{\boldsymbol{A}} : [0,T] \times \widehat{\Omega} \mapsto \Omega(t)$, with Jacobian $\widehat{\boldsymbol{F}}_A$, and Jacobi determinant $\widehat{V_A}$. Indicating with $\widehat{\cdot}$ fields in $\widehat{\Omega}$, we can apply the framework presented in (5) to build a GENERIC formulation of ALE FSI problems by choosing as state variable $\widehat{\boldsymbol{q}} := (\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\alpha}}, \widehat{\varrho}_A := \widehat{\varrho}\widehat{V_A}, \widehat{\boldsymbol{p}}_A := \widehat{\boldsymbol{p}}\widehat{V_A})$. In the spirit of ALE FSI, setting $\widehat{\boldsymbol{A}}$ to be the identity we fall back to the Eulerian state $\boldsymbol{q}$, while setting $\widehat{\boldsymbol{A}} \equiv \bar{\chi}$, we recover $\bar{\boldsymbol{q}}$. The GENERIC framework guarantees a structure preserving construction also in all intermediate choices, including the classical choice in ALE FSI, where $\widehat{\boldsymbol{A}}_{\mathrm{s}} \equiv \bar{\chi}$ in the solid domain, and where $\widehat{\boldsymbol{A}}$ is a geometrically conforming extension operator of $\widehat{\boldsymbol{A}}_{\mathrm{s}}$ in the fluid domain (most commonly pseudo-elastic, harmonic, or bi-harmonic operators [21]). Alternative formulations of FSI problems relying on non-matching techniques, like Immersed Finite Element Methods (IFEM) [5, 8, 11, 12] can also be casted in this framework by choosing as state variables a mixture of Lagrangian and Eulerian variables $\tilde{\boldsymbol{q}} := (\bar{\chi}_{\mathrm{s}}, \boldsymbol{p}, \varrho)$, where the momentum and density fields are treated in the Eulerian framework, and the solid displacement is kept in the Lagrangian framework.

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