Weak solution to some Penrose-Fife phase-field systems with temperature-dependent memory

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Abstract. In this paper a phase-field model of Penrose-Fife type is considered for a diffusive phase transition in a material in which the heat flux is a superposition of two different contributions: one part is proportional to the spatial gradient of the inverse temperature, while the other is of the form of the Gurtin-Pipkin law introduced in the theory of materials with thermal memory. It is shown that an initial-boundary value problem for the resulting state equations has a unique solution, thereby generalizing a number of recent results.

1 Introduction

This paper is devoted to the study of certain initial-boundary value problems for the phase-field model proposed by Penrose and Fife [27, 28]. We deal with the non-conserved case for the order parameter χ, which may represent the (local) liquid fraction in the solid-liquid phase transition. Moreover, our setting includes the possibility that the heat flux $\mathbf{q}$ also depends in a suitable way on the past history of the gradient of the absolute temperature θ.

The system of partial differential equations derived in [27] complies with the second principle of thermodynamics. In a quite general version allowing for non-differentiable free energies, it has the form

$$\partial_t(\theta + \lambda(\chi)) + \operatorname{div} \mathbf{q} = g \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

$$\mu\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\frac{\lambda'(\chi)}{\theta} \quad \text{in } Q, \tag{1.2}$$

where the nonlinearity β is an arbitrary maximal monotone graph in $\mathbb{R}^2$. The presence of the singular factor $1/\theta$ in the right-hand side of (1.2) and of a nonlinear function $\lambda(\chi)$ in (1.1) distinguishes the above system from the well-known Caginalp model [4], which can be viewed as a linearization of (1.1–2) around some equilibrium temperature.

The notation in (1.1–2) is standard, i. e., ∂_t is the time derivative, div the divergence, and ∇ and Δ denote the spatial gradient and Laplacian operators, respectively. The solid-liquid material is supposed to occupy the bounded domain $\Omega \subset \mathbb{R}^3$ during the fixed time interval $[0,T]$. We let Γ denote the smooth boundary of Ω and set $\Sigma := \Gamma \times (0,T)$. Concerning the data in (1.1–2), we notice that $g : Q \to \mathbb{R}$ stands for the heat supply, μ and ν are small positive coefficients, β is possibly multivalued and coincides with the subdifferential of a convex function $\widehat{\beta}$. A typical and significant example for $\widehat{\beta}$ is given by the indicator function (taking only the values 0 and $+\infty$) of the interval $[0,1]$; in this case, χ is forced to stay between 0 and 1. Finally, λ, $\sigma : \mathbb{R} \to \mathbb{R}$ are regular functions originating from the smooth part of the free energy, and λ is required to be convex. Both these functions are assumed Lipschitz continuous along with their derivatives on the domain of β.

Although the Penrose-Fife model is known only from the beginning of this decade (see [27]), it has nevertheless received a great deal of attention in the past few years. Of course, the energy balance equation (1.1) has to be supplied with a constitutive law for the heat flux $\mathbf{q}$, and initial and boundary conditions for (1.1–2) have to be prescribed. Many papers have been devoted to the mathematical analysis of the corresponding problems; see, quoting in a chronological order, [31, 30, 21, 23, 17, 24, 16, 20, 13]. There, questions like existence, uniqueness, regularity, large time behaviour, have been examined. All these results have been obtained assuming the relationship

$$\mathbf{q} = -\nabla\left(-\frac{\delta}{\theta}\right), \tag{1.3}$$

where $\delta > 0$ ([24] also addresses some extension of (1.3)). Moreover, a simplified version of (1.1–2) has been included in the family of problems considered in [2]. Furthermore, asymptotic analyses have been carried out with respect to the kinetic parameters μ and ν of (1.2), letting one of them or both tend to 0, in [11, 29, 12, 22].

The choice (1.3) turns out to have some advantages. Firstly, the forbidden value $\theta = 0$ (recall that θ is the absolute temperature) is penalized in (1.3) since $1/\theta$ blows up as θ approaches 0. A second remark concerns the mathematical treatment. Indeed, the sum of (1.1), tested by $-1/\theta$, and of (1.2), tested by $\delta\chi_t$, gives rise to a nice cancellation of terms in the derivation of *a priori* estimates. Therefore, despite of the fact that (1.3) is quite unusual in heat conduction equations, it opened the way to show rather interesting results; in fact, if one chooses the classical Fourier law in (1.1) instead, the investigation becomes much more difficult for evident reasons, and, to our knowledge, until now existence has only been proved in [25] for the particular case $\beta(\chi) = \chi^3$.

On the other hand, while it might look acceptable to postulate constitutive relations like (1.3) for low and intermediate temperatures (identifying appropriate constants δ), the behaviour of (1.3) for high temperatures is not satisfactory, since it does not furnish any sort of coerciveness as θ becomes larger and larger. To overcome this failure, some work has been done on (1.1–2) by replacing (1.3) by

$$\mathbf{q} = -\nabla \left(-\frac{\delta}{\theta} + \varepsilon\, \theta \right), \tag{1.4}$$

for some $\varepsilon > 0$, or generalizations thereof. The corresponding results are reported in [9, 10]. In our paper, in the same perspective, but moving from a different position, we add to $-\delta/\theta$ a contribution of the form

$$\int_{-\infty}^{t} k(t-s)\theta(x,s)ds, \qquad (x,t) \in \Omega \times (0,T), \tag{1.5}$$

where $k : [0,+\infty[\to \mathbb{R}$ is known and allows to account for memory effects in the phase transition. Note that if k was the Dirac mass multiplied by ε, then (1.5) would coincide with (1.4). Instead, we keep k as a smooth function, with the only natural restriction that $k(0) > 0$. Thus, we follow in parts a school of thinking which took its main motivation from trying to explain the occurrence of heat waves and to predict the finite speed of propagation for thermal disturbances. To give an idea of the interest on the subject and of the number of involved material scientists, it suffices to look over the review papers [18, 19]. In particular, for (1.5) we refer to Gurtin and Pipkin [15].

Now, let the history of θ be known up to $t = 0$, and introduce the notation

$$(a * b)(t) := \int_0^t a(s)b(t-s)\, ds, \qquad t \in [0,T],$$

for the convolution product with respect to time (where a and b may also depend on the space variables). Then, recalling (1.3) and (1.5), we can assume that

$$\mathbf{q} = -\nabla \left(-\frac{\delta}{\theta} + k * \theta \right), \tag{1.6}$$

provided we slightly modify the right-hand side g in the consequent equation (1.1). Also, as in [21] and [9], we supply (1.1) with a boundary condition that is linear with respect to the argument of the gradient in (1.6), namely,

$$\mathbf{q} \cdot \mathbf{n} = \gamma \left(-\frac{\delta}{\theta} + k * \theta - h \right) \quad \text{on } \Sigma. \tag{1.7}$$

Here, $\mathbf{n}$ indicates the outward normal vector, γ is a proportionality constant, and the datum $h : \Sigma \to \mathbb{R}$ depends on the outside temperature on the boundary and, possibly, on the values of the inside temperature for $t < 0$. Regarding the phase variable χ, we choose the usual no-flux condition

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \Sigma, \tag{1.8}$$

where $\partial_{\mathbf{n}}$ obviously denotes the outer normal derivative. Finally, the initial conditions

$$\theta(x,0) = \theta_0, \quad \chi(x,0) = \chi_0, \qquad x \in \Omega, \tag{1.9}$$

complete the formulation of the problem under study.

The main aim of this paper is to prove existence and uniqueness of a weak solution to (1.1–2), (1.6–9). The way leading to this result is not straightforward. In fact, from (1.1–2) and (1.6) one cannot extract any spatial regularity for θ that might help in the treatment of the *perturbation* due to $k * \theta$. Therefore, we first consider more or less the same problem, where (1.6) is replaced by (1.4), and generalize the previous approaches of [9, 10] in the sense of weak solutions. Then, including $\varepsilon\theta$ $(\varepsilon > 0)$ in (1.7), we employ a fixed-point technique to show that such approximating problems admit a unique solution. Finally, we take the limit as $\varepsilon \searrow 0$ to recover a pair of functions θ and χ solving (1.1–2), (1.6–9). The uniqueness is a consequence of a contracting estimate, which is also useful for the existence proof and is essentially based on a convexity argument devised by Kenmochi in [20].

We conclude the introduction by noticing that memory terms within the heat flux (and also the internal energy) have already been considered in the study of phase transition or phase field problems. For instance, in [7, 8] another combination (something like $\mathbf{q} = -\nabla(\varepsilon\theta + k * \theta)$) has been discussed, and in [1, 5, 6] the Caginalp model is investigated for the mere Gurtin-Pipkin law ((1.6) with $\delta = 0$). However, probably owing to the difficulty of the related problems, to our knowledge the present paper yields the first attempt to couple memory effects with the Penrose-Fife model.

2 Statement of the Problem

Consider the initial-boundary value problem (1.1–2), (1.6–9). We make the following general assumptions on the data of the system.

(A1) β is the subdifferential of a non-negative, proper, convex, and lower semicontinuous function $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = 0$. We denote by K the closure of the domain $D(\beta)$ of definition of β in $\mathbb{R}$ and point out that $0 \in \beta(0)$.

(A2) $\alpha(r) = -\,\delta/r$ for all $r \in (0, +\infty)$ and some fixed constant $\delta > 0$.

(A3) $\lambda \in C^{1,1}(K)$, $\lambda'' \geq 0$ a.e. in K, as well as $\sigma \in C^1(K)$ and $\sigma' \in C^{0,1}(K)$.

(A4) $k \in W^{2,1}(0,T)$, with $k(0) > 0$.

(A5) $g \in L^2(Q)$, $h \in L^2(\Sigma)$, with $h \leq 0$ a.e. in Σ.

(A6) $\theta_0 \in L^2(Q)$, $\theta_0 > 0$ a.e. in Ω, $\ln(\theta_0) \in L^1(\Omega)$.

(A7) $\chi_0 \in H^1(\Omega)$, $\widehat{\beta}(\chi_0) \in L^1(\Omega)$.

We now give a variational formulation of (1.1–2), (1.6–9). To this end, we denote by $(\cdot,\cdot)$ both the scalar product in $H := L^2(\Omega)$ and the dual pairing between V' and $V := H^1(\Omega)$. We also denote by

$$((v_1, v_2)) := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, + \, \gamma \int_\Gamma v_1\, v_2\,, \qquad v_1, v_2 \in V\,, \tag{2.1}$$

the scalar product in V. We define the Riesz isomorphism $J : V \to V'$ and the scalar product in V', respectively, by

$$(Jv_1, v_2) := ((v_1, v_2)), \quad v_1, v_2 \in V, \tag{2.2}$$

$$((w_1, w_2))_* := (w_1, J^{-1}w_2), \quad w_1, w_2 \in V'. \tag{2.3}$$

Then our problem can be stated as follows.

Problem $(\mathbf{P}_0)$. Find a quadruple (θ, u, χ, ξ) of functions such that the following conditions are fulfilled.

$$\theta > 0, \quad u = -\frac{1}{\theta}, \quad \chi \in D(\beta), \quad \xi \in \beta(\chi), \quad \text{a.e. in } Q, \tag{2.4}$$

$$\theta \in H^1(0,T;V') \cap L^\infty(0,T;H), \quad u, \alpha(\theta) \in L^2(0,T;V), \tag{2.5}$$

$$\chi \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \quad \xi \in L^2(Q), \tag{2.6}$$

$$k * \theta \in L^\infty(0,T;V), \tag{2.7}$$

$$\big(\theta + \lambda(\chi)\big)_t + J\alpha(\theta) + J(k * \theta) = f, \quad \text{in } V', \quad \text{a.e. in } (0,T), \tag{2.8}$$

where

$$(f, v) := \int_\Omega g\, v + \gamma \int_\Gamma h\, v, \quad \forall v \in V, \tag{2.9}$$

$$\mu\, (\chi_t(\cdot,t), v) + \nu \int_\Omega \nabla\chi(\cdot,t) \cdot \nabla v + ((\xi + \sigma'(\chi) - \lambda'(\chi)u)(\cdot,t), v) = 0, \quad \forall v \in V, \quad \text{for a.a. } t \in (0,T), \tag{2.10}$$

$$\theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0, \quad \text{a.e. in } \Omega. \tag{2.11}$$

The main result of this paper is the following.

Theorem 2.1 *Suppose that* **(A1)** *to* **(A7)** *hold. Then* $(\mathbf{P}_0)$ *has a unique solution.*

The proof of Theorem 2.1 will be achieved by passage to the limit as $\varepsilon \searrow 0$ using the following family of problems (which contains $(\mathbf{P}_0)$ as special case for $\varepsilon = 0$).

Problem $(\mathbf{P}_\varepsilon)$. Find (θ, u, χ, ξ) satisfying the conditions of $(\mathbf{P}_0)$, where $\alpha(\theta)$ is replaced by

$$\alpha_\varepsilon(\theta) := \alpha(\theta) + \varepsilon\,\theta, \quad \text{for fixed } \varepsilon \geq 0,$$

substitutes $\alpha(\theta) = \alpha_0(\theta)$ and, clearly, $\theta \in L^2(0,T;V)$ whenever $\varepsilon > 0$.

We have the following existence result for $\varepsilon > 0$.

Theorem 2.2 *Suppose that* **(A1)** *to* **(A7)** *are satisfied. Then for any* $\varepsilon > 0$ $(\mathbf{P}_\varepsilon)$ *has a unique solution.*

Remark 2.3 In the case $\varepsilon > 0$ the smoothness condition for k can be relaxed. Indeed, it then suffices that $k \in L^2(0,T)$, as pointed out in Remark 4.2 and Lemmas 4.3 to 4.5 below. Moreover, referring to the same lemmas, one easily verifies that the above statement holds true for an arbitrary $f \in L^2(0,T;V')$ and assumption **(A5)** can be omitted.

It turns out to be convenient to study a further family of problems, corresponding to the case $k = 0$, that deserve some attention by themselves.

Problem $(\mathbf{P}'_\varepsilon)$. Let $\varepsilon \geq 0$. Find (θ, u, χ, ξ) fulfilling the conditions of problem $(\mathbf{P}_\varepsilon)$ with the one exception that (2.8) is replaced by the equation

$$(\theta + \lambda(\chi))_t + J\alpha_\varepsilon(\theta) = F, \quad \text{in } V', \quad \text{a.e. in } (0,T), \tag{2.12}$$

where F only belongs to $L^2(0,T;V')$.

For this family of problems the following result will be proved.

Theorem 2.4 *Suppose that the conditions* **(A1)** *to* **(A3)**, **(A6)**, *and* **(A7)**, *are satisfied, and assume that* $F \in L^2(0,T;V')$. *Then* $(\mathbf{P}'_\varepsilon)$ *admits a unique solution for any* $\varepsilon > 0$. *If, in addition,*

$$F = f, \quad \textit{with } f \textit{ specified by (2.9) and } \mathbf{(A5)}, \tag{2.13}$$

then also $(\mathbf{P}'_0)$ *has a unique solution.*

Remark 2.5 In the special case when $\alpha_\varepsilon(\theta) = \alpha(\theta) + \varepsilon\theta$ for some $\varepsilon > 0$, Theorem 2.4 generalizes Theorem 2.3 in [9] to right-hand sides $F \in L^2(0,T;V')$. It also provides a different approach, in the sense of weak solutions, to the main result in [10].

Remark 2.6 In the case $\varepsilon = 0$, Theorem 2.4 constitutes a generalization of the existence result proved in [29]. Also, it can be compared with a very recent result by Damlamian and Kenmochi [13] where the system (2.4–10) is formulated as Cauchy problem for an evolution equation generated by subdifferential operators.

3 Analysis of Problem $(\mathbf{P}'_\varepsilon)$

We first consider the problems $(\mathbf{P}'_\varepsilon)$ for $\varepsilon \geq 0$. We begin with a continuous dependence property.

Lemma 3.1 *Let* $\varepsilon \geq 0$, *and suppose that* $(\theta_i, u_i, \chi_i, \xi_i)$ *denote solutions to* $(\mathbf{P}'_\varepsilon)$ *corresponding to the data* $(F_i, \theta_{0i}, \chi_{0i})$, $i = 1,2$. *Let*

$$e_i = \theta_i + \lambda(\chi_i), \quad e_{0i} = \theta_{0i} + \lambda(\chi_{0i}), \quad i = 1,2, \tag{3.1}$$

as well as

$$\begin{aligned} &\chi = \chi_1 - \chi_2, \quad e = e_1 - e_2, \quad \theta = \theta_1 - \theta_2, \quad u = u_1 - u_2, \\ &e_0 = e_{01} - e_{02}, \quad \chi_0 = \chi_{01} - \chi_{02}, \quad F = F_1 - F_2. \end{aligned} \tag{3.2}$$

Then there is some constant $C > 0$, depending only on the data, such that

$$\|e(\cdot,t)\|_{V'}^2 + \varepsilon\,\|\theta\|_{L^2(0,t;H)}^2 + \mu\,\delta\,\|\chi(\cdot,t)\|_H^2 + 2\,\nu\,\delta \int_0^t\!\!\int_\Omega |\nabla\chi|^2$$
$$\leq \|e_0\|_{V'}^2 + \mu\,\delta\,\|\chi_0\|_H^2 + C\int_0^t \|\chi(\cdot,s)\|_H^2\,ds$$
$$+ 2\int_0^t (F(\cdot,s), J^{-1}e(\cdot,s))\,ds\,, \quad \forall\, t\in[0,T]\,. \tag{3.3}$$

In particular, $(\mathbf{P}'_\varepsilon)$ *admits at most one solution.*

Proof. At first, we subtract the respective equations (2.12) for $(\theta_i, u_i, \chi_i, \xi_i)$, $i = 1,2$, from each other, apply the result to $J^{-1}e$, and integrate over $[0,t]$. Next, we choose $v = \delta\,\chi$ in (2.10) for $(\theta_i, u_i, \chi_i, \xi_i)$, $i = 1,2$, take the difference, and integrate over $[0,t]$. We then find that

$$\frac{1}{2}\|e(\cdot,t)\|_{V'}^2 + \delta\int_0^t (Ju(\cdot,s), J^{-1}e(\cdot,s))\,ds + \varepsilon\int_0^t (J\theta(\cdot,s), J^{-1}e(\cdot,s))\,ds$$
$$+ \frac{\mu\,\delta}{2}\|\chi(\cdot,t)\|_H^2 + \nu\,\delta\int_0^t\!\!\int_\Omega |\nabla\chi|^2 + \delta\int_0^t\!\!\int_\Omega \xi\,\chi$$
$$- \delta\int_0^t ((\lambda'(\chi_1)\,u_1 - \lambda'(\chi_2)\,u_2)(\cdot,s), \chi(\cdot,s))\,ds$$
$$= \frac{1}{2}\|e_0\|_{V'}^2 + \frac{\mu\,\delta}{2}\|\chi_0\|_H^2 - \delta\int_0^t\!\!\int_\Omega (\sigma'(\chi_1) - \sigma'(\chi_2))\,\chi + \int_0^t (F(\cdot,s), J^{-1}e(\cdot,s))\,ds\,. \tag{3.4}$$

Now, note that if $v\in V$ and $w\in H$ then

$$(Jv, J^{-1}w) = ((Jv, w))_* = ((w, Jv))_* = (w, J^{-1}(Jv)) = (w, v)\,. \tag{3.5}$$

Hence, using the convexity of λ and the fact that $u_i \leq 0$, $i = 1,2$, we can employ the same argument as in the proof of Lemma 3.1 in Kenmochi [20] to conclude that

$$(Ju(\cdot,s), J^{-1}e(\cdot,s)) \;\geq\; ((\lambda'(\chi_1)\,u_1 \;-\; \lambda'(\chi_2)\,u_2)(\cdot,s), \chi(\cdot,s))\,, \tag{3.6}$$

for a. a. $s\in(0,t)$. In addition, we have that

$$\varepsilon\,(J\theta(\cdot,s), J^{-1}e(\cdot,s)) = \varepsilon\,\|\theta(\cdot,s)\|_H^2 + \varepsilon\,(\theta(\cdot,s), ((\lambda(\chi_1) - \lambda(\chi_2))(\cdot,s))$$

and consequently, by **(A3)** and Young's inequality,

$$\varepsilon\,(J\theta(\cdot,s), J^{-1}e(\cdot,s)) \geq \frac{\varepsilon}{2}\|\theta(\cdot,s)\|_H^2 - \frac{\varepsilon}{2}\|\lambda'\|_{L^\infty(K)}^2\,\|\chi(\cdot,s)\|_H^2\,. \tag{3.7}$$

for a. a. $s\in(0,t)$. Moreover, the monotonicity of β entails

$$\xi\,\chi \geq 0\,, \quad \text{a.e. in } Q\,, \tag{3.8}$$

and, since $\sigma'\in C^{0,1}(K)$, it turns out that

$$|(\sigma'(\chi_1) - \sigma'(\chi_2))\,\chi| \leq \|\sigma''\|_{L^\infty(K)}\,|\chi|^2\,, \quad \text{a.e. in } Q\,. \tag{3.9}$$

Combining (3.4) to (3.9), we obtain (3.3), whence the uniqueness result easily follows using Gronwall's lemma. Let us point out that the thesis actually holds also for the case $\varepsilon = 0$. □

We now derive further estimates for $(\mathbf{P}'_\varepsilon)$. In the sequel, we denote by $\overline{C}_i$, C_i, $i\in\mathbb{N}$, any constant that may depend on the data of the system but neither on ε nor on $t\in[0,T]$. In addition, the dependence on $\|F\|_{L^2(0,T;V')}$ will always be specified explicitly.

Lemma 3.2 *Let (θ, u, χ, ξ) solve $(\mathbf{P}'_\varepsilon)$ for some $\varepsilon > 0$. Then there is some $\overline{C}_1 > 0$ such that, for all $t \in [0,T]$,*

$$
\begin{aligned}
&\frac{1}{4}\,\|\theta(\cdot\,,t)\|_H^2 \,+\, \frac{1}{4}\,\|\ln(\theta(\cdot\,,t))\|_{L^1(\Omega)} \,+\, \frac{\delta}{2}\,\|u\|^2_{L^2(0,t;V)} \,+\, \varepsilon\,\|\theta\|^2_{L^2(0,t;V)} \\
&\quad +\, \delta \int_0^t\!\!\int_\Omega |\nabla(\ln(\theta))|^2 \,+\, \frac{\mu}{4}\,\|\chi_t\|^2_{L^2(0,t;H)} \,+\, \frac{\nu}{2}\int_\Omega |\nabla\chi(\cdot\,,t)|^2 \\
&\le\; \overline{C}_1\left(1 \,+\, \varepsilon \,+\, \|F\|^2_{L^2(0,t;V')} \,+\, \int_0^t \Big(\|\theta(\cdot\,,s)\|_H^2 \,+\, \|\chi_t\|^2_{L^2(0,s;H)}\Big)\,ds\right) \\
&\quad +\, \int_0^t (F(\cdot\,,s),\theta(\cdot\,,s))\,ds\,,
\end{aligned}
\tag{3.10}
$$

where the norm in V is the one induced by the scalar product defined in (2.1).

Proof. Since $u,\theta \in L^2(0,T;V)$, we may apply both sides of (2.12) to $v = u+\theta$ to obtain, for any $t \in [0,T]$,

$$
\begin{aligned}
&\int_\Omega \left(\frac{\theta^2}{2} \,-\, \ln(\theta)\right)(\cdot\,,t) \,-\, \int_\Omega \left(\frac{\theta_0^2}{2} \,-\, \ln(\theta_0)\right) \,+\, \int_0^t\!\!\int_\Omega \lambda'(\chi)\,\chi_t\,(u+\theta) \\
&\quad +\, \int_0^t\!\!\int_\Omega \nabla\alpha_\varepsilon(\theta)\cdot\nabla(u+\theta) \,+\, \gamma\int_0^t\!\int_\Gamma \alpha_\varepsilon(\theta)\,(u+\theta) \\
&=\; \int_0^t (F(\cdot\,,s),(u+\theta)(\cdot\,,s))\,ds\,.
\end{aligned}
\tag{3.11}
$$

Next, we refer to Lemma 3.3, in particular formula (3.21), of [9] for the estimate (this can be verified formally multiplying (1.2) by χ_t and then integrating over space and time)

$$
\begin{aligned}
&\frac{\mu}{2}\int_0^t\!\!\int_\Omega \chi_t^2 \,+\, \frac{\nu}{2}\int_\Omega |\nabla\chi(\cdot\,,t)|^2 \,+\, \int_\Omega \widehat{\beta}(\chi(\cdot\,,t)) \\
&\le\; \int_0^t\!\!\int_\Omega \lambda'(\chi)\,\chi_t\,u \,+\, \frac{\nu}{2}\int_\Omega |\nabla\chi_0|^2 \,+\, \int_\Omega \widehat{\beta}(\chi_0) \\
&\quad +\, C_0\left(1 \,+\, \|\chi_0\|_H^2\right) \,+\, C_0\int_0^t \|\chi_t\|^2_{L^2(0,s;H)}\,ds\,,
\end{aligned}
\tag{3.12}
$$

where C_0 depends only on μ, $|\sigma'(0)|$, $\|\sigma''\|_{L^\infty(K)}$, $|\Omega|$, and T. Next, adding (3.11) and (3.12), we obtain from **(A3)**, **(A6)**, and **(A7)**, that

$$
\begin{aligned}
&\int_\Omega \left(\frac{\theta^2}{2} \,-\, \ln(\theta) \,+\, \widehat{\beta}(\chi)\right)(\cdot\,,t) \,+\, \int_0^t\!\!\int_\Omega \nabla\alpha_\varepsilon(\theta)\cdot\nabla(u+\theta) \\
&\quad +\, \gamma\int_0^t\!\int_\Gamma \alpha_\varepsilon(\theta)\,(u+\theta) \,+\, \frac{\mu}{2}\int_0^t\!\!\int_\Omega \chi_t^2 \,+\, \frac{\nu}{2}\int_\Omega |\nabla\chi(\cdot\,,t)|^2 \\
&\le\; C_1 \,-\, \int_0^t\!\!\int_\Omega \lambda'(\chi)\,\chi_t\,\theta \,+\, C_0\int_0^t \|\chi_t\|^2_{L^2(0,s;H)}\,ds \\
&\quad +\, \int_0^t (F(\cdot\,,s),(u+\theta)(\cdot\,,s))\,ds\,.
\end{aligned}
\tag{3.13}
$$

We have, thanks to **(A3)** and to Young's inequality,

$$
\Big|\int_0^t\!\!\int_\Omega \lambda'(\chi)\,\chi_t\,\theta\Big| \le \frac{\mu}{4}\int_0^t\!\!\int_\Omega \chi_t^2 \,+\, \frac{1}{\mu}\,\|\lambda'\|^2_{L^\infty(K)}\int_0^t\!\!\int_\Omega \theta^2\,.
\tag{3.14}
$$

Moreover, (2.4) and the definition of α_ε imply

$$
\int_0^t\!\!\int_\Omega \nabla\alpha_\varepsilon(\theta)\cdot\nabla(u+\theta) = \delta\int_0^t\!\!\int_\Omega |\nabla u|^2 \,+\, \varepsilon\int_0^t\!\!\int_\Omega |\nabla\theta|^2 \,+\, (\delta+\varepsilon)\int_0^t\!\!\int_\Omega \frac{|\nabla\theta|^2}{\theta^2}\,,
\tag{3.15}
$$

as well as

$$
\begin{aligned}
\gamma \int_0^t\!\!\int_\Gamma \alpha_\varepsilon(\theta)\,(u+\theta) &= \gamma \int_0^t\!\!\int_\Gamma (\delta\, u + \varepsilon\,\theta)\,(u+\theta) \\
&= \gamma \int_0^t\!\!\int_\Gamma (\delta\, u^2 + \varepsilon\,\theta^2) \;-\; \int_0^t\!\!\int_\Gamma (\delta + \varepsilon)\,.
\end{aligned} \tag{3.16}
$$

Finally, one easily sees that

$$
\begin{aligned}
\int_0^t (F(\cdot\,,s), u(\cdot\,,s))\,ds &\le \int_0^t \|F(\cdot\,,s)\|_{V'}\;\|u(\cdot\,,s)\|_V\,ds \\
&\le \frac{\delta}{2}\,\|u\|^2_{L^2(0,t;V)} \;+\; \frac{1}{2\,\delta}\,\|F\|^2_{L^2(0,t;V')}\,.
\end{aligned} \tag{3.17}
$$

Now recall that for any $r>0$ there holds

$$
\frac{r^2}{2} \;-\; \ln(r) \ge \frac{1}{4}\,(r^2 \;+\; |\ln(r)|)\,.
$$

Thus, combining the inequalities (3.13) to (3.17), and invoking that $\widehat{\beta}$ is non-negative, we obtain (3.10), which concludes the proof of the lemma. □

We deduce further estimates.

Lemma 3.3 *Let* (θ, u, χ, ξ) *solve* $(\mathbf{P}'_\varepsilon)$ *for some* $\varepsilon \ge 0$. *Then there is some* $\overline{C}_2 > 0$ *such that, for all* $t \in [0,T]$,

$$
\|\chi\|^2_{L^2(0,t;H^2(\Omega))} \;+\; \|\xi\|^2_{L^2(0,t;H)} \le \overline{C}_2 \left(1 \;+\; \|\chi\|^2_{L^2(0,t;H)} \;+\; \|u\|^2_{L^2(0,t;H)}\right), \tag{3.18}
$$

$$
\|\theta_t\|_{L^2(0,t;V')} \le \overline{C}_2 \left(\|\chi_t\|_{L^2(0,t;H)} \;+\; \delta\,\|u\|_{L^2(0,t;V)} \;+\; \varepsilon\,\|\theta\|_{L^2(0,t;V)} \;+\; \|F\|_{L^2(0,t;V')}\right). \tag{3.19}
$$

Proof. Consider the initial-boundary value problem

$$
\mu\,\partial_t\chi_m \;-\; \nu\,\Delta\chi_m \;+\; \beta_m(\chi_m) = G \quad \text{a.e. in } Q\,, \tag{3.20}
$$

$$
\partial_{\mathbf{n}}\chi_m = 0 \quad \text{a.e. in } \Sigma\,, \tag{3.21}
$$

$$
\chi_m(\cdot\,,0) = \chi_0 \quad \text{a.e. in } \Omega\,, \tag{3.22}
$$

where $\beta_m := m\,(I \;-\; (I \;+\; \frac{1}{m}\,\beta)^{-1})$ denotes the (monotone and Lipschitz continuous) Yosida approximation to β for $m \in \mathbb{N}$, and where $G := -\sigma'(\chi) \;+\; \lambda'(\chi)\,u$ belongs to $L^2(Q)$. It is not difficult to check (see, e. g., [9, Lemma 3.3]) that the unique solution

$$
\chi_m \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega))
$$

to (3.20–22) fulfils

$$
\|\partial_t\chi_m\|^2_{L^2(0,t;H)} \;+\; \|\chi_m(\cdot\,,t)\|^2_V \;+\; \int_\Omega \widehat{\beta}_m(\chi_m(\cdot\,,t)) \le C_2\,\left(1 \;+\; \|G\|^2_{L^2(0,t;H)}\right), \tag{3.23}
$$

for any $t \in [0,T]$, letting $\widehat{\beta}_m$ specify the antiderivative of β_m such that $\widehat{\beta}_m(0) = 0$. In addition, testing (3.20) by $\beta_m(\chi_m)$ and using the monotonicity of β_m (cf. **(A1)** and **(A7)** as well), we easily find that

$$
\|\beta_m(\chi_m)\|^2_{L^2(0,t;H)} \;\le\; C_3\,\left(1 \;+\; \|G\|^2_{L^2(0,t;H)}\right). \tag{3.24}
$$

Moreover, by comparison in (3.20), in view of (3.21), (3.23), and (3.24) we also have

$$\|\chi_m\|^2_{L^2(0,t;H^2(\Omega))} \le C_4 \left(1 + \|G\|^2_{L^2(0,t;H)}\right) . \tag{3.25}$$

Hence, χ_m and $\xi_m := \beta_m(\chi_m)$ satisfy (3.18). Besides, thanks to (3.23–25) there are functions $\overline{\chi}$, $\overline{\xi}$ such that, possibly for a subsequence of $m \nearrow \infty$,

$$\chi_m \to \overline{\chi} \qquad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \tag{3.26}$$

$$\xi_m \to \overline{\xi} \qquad \text{weakly in } L^2(Q). \tag{3.27}$$

From this point, we can argue as in the passage-to-the-limit procedure of the proof of Lemma 3.1 in [11], for instance, in order to conclude that $\overline{\xi} \in \beta(\overline{\chi})$. Thus, we easily deduce that $\overline{\chi}$ is a solution to the problem

$$\mu\,\overline{\chi}_t - \nu\,\Delta\overline{\chi} + \beta(\overline{\chi}) \ni -\sigma'(\chi) + \lambda'(\chi)u \quad \text{a.e. in } Q, \tag{3.28}$$

$$\partial_{\mathbf{n}}\overline{\chi} = 0 \quad \text{a.e. in } \Sigma, \quad \overline{\chi}(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.29}$$

The unique solvability of (3.28–29) implies $\overline{\chi} = \chi$ as well as $\overline{\xi} = \xi$. Using the lower semicontinuity of norms, we realize that (3.18) holds. Finally, we obtain (3.19) directly from (2.12) and **(A3)**. The assertion of the lemma is proved. □

We draw a straightforward consequence from the previous Lemmas 3.2 and 3.3 for the case when $\varepsilon > 0$.

Corollary 3.4 *Let (θ, u, χ, ξ) solve $(\mathbf{P}'_\varepsilon)$ for some $\varepsilon > 0$. Then there is some $\overline{C}_3 > 0$ such that, for all $t \in [0,T]$,*

$$\begin{aligned} &\|\theta\|^2_{H^1(0,t;V')\cap C^0([0,t];H)} + \varepsilon\,\|\theta\|^2_{L^2(0,t;V)} + \|u\|^2_{L^2(0,t;V)} \\ &+ \|\chi\|^2_{H^1(0,t;H)\cap C^0([0,t];V)\cap L^2(0,t;H^2(\Omega))} + \|\xi\|^2_{L^2(0,t;H)} \\ \le\ &\overline{C}_3 \left(1 + \varepsilon + \left(1 + \varepsilon^{-1}\right) \|F\|^2_{L^2(0,t;V')}\right) . \end{aligned} \tag{3.30}$$

Proof. Recalling (3.10) and observing that

$$\int_0^t (F(\cdot,s), \theta(\cdot,s))\, ds \le \frac{\varepsilon}{2}\|\theta\|^2_{L^2(0,t;V)} + \frac{1}{2\,\varepsilon}\|F\|^2_{L^2(0,t;V')} , \tag{3.31}$$

it follows plainly from Gronwall's lemma that there exists some constant $C_5 > 0$ satisfying

$$\begin{aligned} &\frac{1}{4}\|\theta(\cdot,t)\|^2_H + \frac{\delta}{2}\|u(\cdot,t)\|^2_{L^2(0,t;V)} + \frac{\varepsilon}{2}\|\theta\|^2_{L^2(0,t;V)} \\ &+ \frac{\mu}{4}\|\chi_t\|^2_{L^2(0,t;H)} + \frac{\nu}{2}\int_\Omega |\nabla\chi(\cdot,t)|^2 \\ \le\ &C_5 \left(1 + \varepsilon + \left(1 + \varepsilon^{-1}\right) \|F\|^2_{L^2(0,t;V')}\right) , \end{aligned} \tag{3.32}$$

whence (3.30) can be easily derived with the help of (3.18–19). □

We are now in the position to prove Theorem 2.4.

Proof of Theorem 2.4. At first, let $\varepsilon > 0$ be fixed. We argue by approximation. To this end, we pick any sequence $\{F_n\} \subset L^2(Q)$ satisfying

$$F_n \to F \quad \text{in } L^2(0,T;V'), \quad \text{as } n \nearrow \infty, \tag{3.33}$$

and we put $\theta_{0n} := \max\{\theta_0, \frac{1}{n}\}$, for $n \in \mathbb{N}$. Clearly, $\{\theta_{0n}\} \subset L^2(\Omega)$ is a pointwise a. e. decreasing sequence, and Beppo Levi's theorem implies that

$$\theta_{0n} \to \theta_0 \quad \text{in } L^2(\Omega), \quad \ln(\theta_{0n}) \to \ln(\theta_0) \quad \text{in } L^1(\Omega). \tag{3.34}$$

In particular, there is some $C_6 > 0$ such that

$$\int_\Omega \left(\frac{\theta_{0n}^2}{2} - \ln(\theta_{0n}) \right) \le C_6 . \tag{3.35}$$

We then consider the initial-boundary value problem

$$\partial_t(\theta_n + \lambda(\chi_n)) - \Delta\alpha_\varepsilon(\theta_n) = F_n, \quad \text{in } \mathcal{D}'(Q), \tag{3.36}$$

$$\mu\,\partial_t\chi_n - \nu\,\Delta\chi_n + \beta(\chi_n) + \sigma'(\chi_n) \ni \lambda'(\chi_n)\,u_n, \quad \text{a.e. in } Q, \tag{3.37}$$

$$\partial_{\mathbf{n}}\,\alpha_\varepsilon(\theta_n) + \gamma\,\alpha_\varepsilon(\theta_n) = 0, \quad \text{in } H^{-1/2}(\Gamma), \quad \text{a.e. in } (0,T), \tag{3.38}$$

$$\partial_{\mathbf{n}}\chi_n = 0, \quad \text{a.e. in } \Sigma, \tag{3.39}$$

$$\theta_n(\cdot,0) = \theta_{0n}, \quad \chi_n(\cdot,0) = \chi_0, \quad \text{a.e. in } \Omega. \tag{3.40}$$

Note that (3.36–40) represents just the problem $(\mathbf{P}'_\varepsilon)$, where $F \in L^2(0,T;V')$ is replaced by $F_n \in L^2(Q)$, and θ_0 by θ_{0n}, respectively. Therefore, we may combine Theorem 2.3 in [9] with Corollary 3.4 to conclude that (3.36–40) admits a solution $(\theta_n, u_n, \chi_n, \xi_n)$ fulfilling

$$\theta_n \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{3.41}$$

$$\varepsilon\,\theta_n,\ u_n \in L^2(0,T;V), \tag{3.42}$$

$$\chi_n \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)), \tag{3.43}$$

$$\xi_n \in L^2(Q), \tag{3.44}$$

$$\theta_n > 0, \quad u_n = -1/\theta_n, \quad \chi_n \in D(\beta), \quad \xi_n \in \beta(\chi_n), \quad \text{a.e. in } Q, \tag{3.45}$$

this solution being uniquely determined because of Lemma 3.1. In addition, using (3.35) (see also (3.11)) and Corollary 3.4, the estimate (3.30) holds if (θ, u, χ, ξ) is replaced by $(\theta_n, u_n, \chi_n, \xi_n)$, with a constant $\overline{C}_3 > 0$ that is independent of $n \in \mathbb{N}$. Consequently, there are functions θ, u, χ, ξ such that (at first only for a subsequence, but by the uniqueness of the limit eventually for the entire sequence)

$$\theta_n \to \theta \quad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;H), \tag{3.46}$$

$$\varepsilon\,\theta_n \to \varepsilon\,\theta \quad \text{weakly in } L^2(0,T;V), \tag{3.47}$$

$$u_n \to u \quad \text{weakly in } L^2(0,T;V), \tag{3.48}$$

$$\chi_n \to \chi \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \text{ and strongly in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.49}$$

$$\xi_n \to \xi \quad \text{weakly in } L^2(Q). \tag{3.50}$$

Here we have applied some compactness results and, in particular, the well-known Ascoli theorem and the Aubin lemma (cf., e. g., [26, p. 58]). Next, using (3.49) and the Lipschitz continuity of λ, λ', and σ', we see that

$$\lambda(\chi_n) \to \lambda(\chi)\,, \quad \lambda'(\chi_n) \to \lambda'(\chi)\,, \quad \sigma'(\chi_n) \to \sigma'(\chi) \quad \text{strongly in } C^0([0,T];H)\,, \tag{3.51}$$

whence

$$\lambda'(\chi_n)\,\partial_t\chi_n \to \lambda'(\chi)\,\chi_t\,, \quad \lambda'(\chi_n)\,u_n \to \lambda'(\chi)\,u\,, \quad \text{both weakly in } L^1(Q)\,. \tag{3.52}$$

But, in view of the boundednes of λ', the limit identification yields

$$(\lambda(\chi_n))_t = \lambda'(\chi_n)\,\partial_t\chi_n \to \lambda'(\chi)\,\chi_t \quad \text{weakly in } L^2(0,T;H)\,, \tag{3.53}$$

$$\lambda'(\chi_n)\,u_n \to \lambda'(\chi)\,u \quad \text{weakly in } L^2(0,T;H)\,. \tag{3.54}$$

Owing to (3.46), we also have

$$\theta_n \to \theta \quad \text{strongly in } C^0([0,T];V')\,. \tag{3.55}$$

This helps us to show that $\theta > 0$ and $\theta = -1/u$ a. e. in Q. Indeed, denoting by ρ the maximal monotone graph defined by $D(\rho) = (-\infty, 0)$ and $\rho(r) = -1/r$ for $r < 0$, it turns out that $\theta_n \in \rho(u_n)$ a. e. in Q, and (3.48) and (3.55) entail

$$\int_0^T\!\!\int_\Omega \theta_n\,u_n = \int_0^T (\theta_n(\cdot,t), u_n(\cdot,t))\,dt \to \int_0^T (\theta(\cdot,t), u(\cdot,t))\,dt = \int_0^T\!\!\int_\Omega \theta\,u\,, \tag{3.56}$$

whence $\theta \in \rho(u)$ a. e. in Q follows. The proof that $\xi \in \beta(\chi)$ is essentially the same, exploiting (3.49) and (3.50). Since it is a standard matter to recover (2.10–12) from (3.36–40) and from the listed convergences, we conclude that (θ, u, χ, ξ) is a solution to $(\mathbf{P}'_\varepsilon)$. By Lemma 3.1, the solution is unique. This ends the proof for the case $\varepsilon > 0$. □

It remains to consider the case $\varepsilon = 0$. To this end, let $F = f$ be as in (2.9), and let $(\theta_\varepsilon, u_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ be the solution to $(\mathbf{P}'_\varepsilon)$ for $\varepsilon > 0$. We aim to get the unique (owing to Lemma 3.1) solution to $(\mathbf{P}'_0)$ by passage to the limit in $(\mathbf{P}'_\varepsilon)$ as $\varepsilon \searrow 0$. To obtain uniform bounds, we recall (3.10), estimating the last term of the right-hand side in the form

$$\int_0^t (f(\cdot,s), \theta_\varepsilon(\cdot,s))\,ds \le \int_0^t \|g(\cdot,s)\|_H\,\|\theta_\varepsilon(\cdot,s)\|_H\,ds + \gamma\int_0^t\!\!\int_\Gamma h\,\theta_\varepsilon\,. \tag{3.57}$$

By virtue of **(A5)**, the latter summand is non-positive. From this point, using Young's and Gronwall's inequalities, as well as Lemma 3.3, it is straightforward to recover an estimate like (3.30) with a constant of the form

$$\overline{C}_4\left(1 + \varepsilon + \|f\|^2_{L^2(0,t;V')}\right)$$

on the right-hand side (the term $1/\varepsilon$ disappears!). Then the passage to the limit as $\varepsilon \searrow 0$ can be performed exactly as the previous one for $n \nearrow \infty$; the only difference is that (3.47) becomes

$$\varepsilon\,\theta_\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2(0,T;V)\,. \tag{3.58}$$

In this connection, observe that (3.47) has not been used in the subsequent considerations. With this, the proof of Theorem 2.4 is complete. □

4 Existence for Problem $(\mathbf{P}_\varepsilon)$ in the case $\varepsilon > 0$

We now analyse problem $(\mathbf{P}_\varepsilon)$ for $\varepsilon > 0$. We begin with a uniqueness result that also holds for $(\mathbf{P}_0)$.

Lemma 4.1 *For any $\varepsilon \geq 0$ the problem $(\mathbf{P}_\varepsilon)$ admits at most one solution.*

Proof. Let $\varepsilon \geq 0$ be fixed, and suppose that $(\theta_i, u_i, \chi_i, \xi_i)$, $i = 1, 2$, fulfil the conditions of $(\mathbf{P}_\varepsilon)$. We then put $F_i := f - J(k * \theta_i)$, $i = 1, 2$, and $F := F_1 - F_2$. Since $F_i \in L^2(0,T;V')$ (cf. (2.7)), it follows that $(\theta_i, u_i, \chi_i, \xi_i)$ solves $(\mathbf{P}'_\varepsilon)$ for the right-hand side F_i, $i = 1, 2$. Hence, using the notations of Lemma 3.1, for all $t \in [0,T]$ we find that

$$\begin{aligned} &\|e(\cdot,t)\|_{V'}^2 + \varepsilon\,\|\theta\|_{L^2(0,t;H)}^2 + \mu\,\delta\,\|\chi(\cdot,t)\|_H^2 + 2\,\nu\,\delta \int_0^t\!\!\int_\Omega |\nabla\chi|^2 \\ \leq\ & C\|\chi\|_{L^2(0,t;H)}^2 - 2\int_0^t (J(k*\theta)(\cdot,s), J^{-1}e\,(\cdot,s))\,ds\,. \end{aligned} \tag{4.1}$$

Using (3.5) and (3.1), we have

$$\begin{aligned} I(t) \ &:= \ -2\int_0^t (J(k*\theta)(\cdot,s), J^{-1}e\,(\cdot,s))\,ds \\ &\leq \ -2\int_0^t ((k*\theta)(\cdot,s), \theta(\cdot,s))\,ds \\ &\quad\ + 2\|k*\theta\|_{L^2(0,t;H)}\,\|\lambda(\chi_1) - \lambda(\chi_2)\|_{L^2(0,t;H)}\,. \end{aligned} \tag{4.2}$$

Now, recall the well-known identities

$$a * b \ = \ a(0)\,(1*b) + a' * (1*b)\,, \tag{4.3}$$

$$(a*b)_t \ = \ a(0)\,b + a' * b\,, \tag{4.4}$$

holding whenever they make sense, as well as Young's theorem

$$\begin{aligned} &\|a*b\|_{L^r(0,T;X)} \ \leq \ \|a\|_{L^p(0,T)}\,\|b\|_{L^q(0,T;X)}\,, \\ &\qquad \text{for} \quad 1 \leq p\,,q\,,r \leq \infty \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1\,, \end{aligned} \tag{4.5}$$

where X denotes a normed space. Hence, from integration by parts in (4.2) and Young's inequality we conclude that

$$\begin{aligned} I(t) \ \leq\ & -k(0)\|(1*\theta)(\cdot,t)\|_H^2 - 2\,((k'*1*\theta)(\cdot,t), (1*\theta)(\cdot,t)) \\ & + 2\int_0^t ((k'(0)(1*\theta) + k''*1*\theta)(\cdot,s), (1*\theta)(\cdot,s))\,ds \\ & + \left(|k(0)|^2 + \|k'\|_{L^1(0,T)}^2\right)\|1*\theta\|_{L^2(0,t;H)}^2 + 2\,\|\lambda'\|_{L^\infty(K)}^2\,\|\chi\|_{L^2(0,t;H)}^2\,. \end{aligned} \tag{4.6}$$

Combining (4.1) and (4.6), we bring the positive term (cf. **(A4)**) $k(0)\,\|(1*\theta)\,(\cdot,t)\|_H^2$ to the left and estimate the remaining terms on the right. With the help of (4.5) we infer that

$$\begin{aligned} & 2\,|((k'*1*\theta)(\cdot,t), (1*\theta)(\cdot,t))| \\ \leq\ & 2\,\|k'*1*\theta\|_{C^0([0,t];H)}\,\|(1*\theta)(\cdot,t)\|_H \\ \leq\ & \frac{2}{k(0)}\,\|k'\|_{L^2(0,T)}^2\,\|1*\theta\|_{L^2(0,t;H)}^2 + \frac{k(0)}{2}\,\|(1*\theta)(\cdot,t)\|_H^2\,. \end{aligned} \tag{4.7}$$

Moreover, since $k'' \in L^1(0,T)$, Hölder's inequality and (4.5) lead us to

$$\begin{aligned} & 2\left|\int_0^t ((k'(0)(1*\theta) + k''*1*\theta)(\cdot,s),(1*\theta)(\cdot,s))\,ds\right| \\ \leq\ & 2\left(|k'(0)| + \|k''\|_{L^1(0,T)}\right)\|1*\theta\|^2_{L^2(0,t;H)}\,. \end{aligned} \tag{4.8}$$

Finally, collecting (4.7–8) it is straightforward to determine a constant C_7 such that

$$\begin{aligned} & \|e(\cdot,t)\|^2_{V'} + \varepsilon\,\|\theta\|^2_{L^2(0,t;H)} + \|(1*\theta)(\cdot,t)\|^2_H + \|\chi(\cdot,t)\|^2_H + \|\chi\|^2_{L^2(0,t;V)} \\ \leq\ & C_7\int_0^t \left(\|\chi(\cdot,s)\|^2_H + \|(1*\theta)(\cdot,s)\|^2_H\right)ds\,. \end{aligned} \tag{4.9}$$

for any $t\in[0,T]$. Therefore, in order to conclude the proof it suffices to apply the Gronwall's lemma in (4.9). □

Remark 4.2 If $\varepsilon>0$, one can get the same uniqueness result assuming only $k\in L^2(0,T)$ in place of **(A4)**. In fact, one then uses the contribution $\varepsilon\|\theta\|_{L^2(0,t;H)}$ on the left-hand side of (4.1) and estimates the last integral in (4.2) this way (by means of (4.5) for $r=\infty$, $p=q=2$ and Young's inequality),

$$-\,2\int_0^t ((k*\theta)(\cdot,s),\theta(\cdot,s))\,ds \leq \frac{2}{\varepsilon}\int_0^t \|k\|^2_{L^2(0,T)}\|\theta\|^2_{L^2(0,s;H)}\,ds + \frac{\varepsilon}{2}\,\|\theta\|^2_{L^2(0,s;H)}\,. \tag{4.10}$$

Then, observing that $\|1*\theta\|^2_{L^2(0,t;H)} \leq T\int_0^t \|\theta\|^2_{L^2(0,s;H)}\,ds$, the assertion still follows from Gronwall's lemma.

Lemma 4.3 *Let $\varepsilon>0$ and $k\in L^1(0,T)$. Let A_ε be the operator assigning to each function $\Theta\in C^0([0,T];H)\cap L^2(0,T;V)$ the solution component θ of problem $(\mathbf{P}'_\varepsilon)$ where F is replaced by $f-J(k*\Theta)$. Then for any $t\in[0,T]$ there holds*

$$\|\theta\|^2_{H^1(0,t;V')\cap C^0([0,t];H)} + \varepsilon\,\|\theta\|^2_{L^2(0,t;V)} \leq R_1(\varepsilon) + R_2(\varepsilon)\,\|k\|^2_{L^1(0,t)}\,\|\Theta\|^2_{L^2(0,t;V)}\,, \tag{4.11}$$

with

$$R_1(\varepsilon) := \overline{C}_3\left(1+\varepsilon+2\left(1+\varepsilon^{-1}\right)\|f\|^2_{L^2(0,T;V')}\right), \qquad R_2(\varepsilon) := 2\,\overline{C}_3\left(1+\varepsilon^{-1}\right),$$

and $\overline{C}_3$ is the same constant as in (3.30). Moreover, setting $\theta_i=A_\varepsilon(\Theta_i)$, $i=1,2$, there is a constant $\overline{C}_4$ such that, for all $t\in[0,T]$,

$$\varepsilon\,\|\theta_1-\theta_2\|^2_{L^2(0,t;H)} \leq \overline{C}_4\left(1+\varepsilon^{-1}\right)\|k\|^2_{L^1(0,t)}\,\|\Theta_1-\Theta_2\|^2_{L^2(0,t;H)}\,. \tag{4.12}$$

Proof. The first assertion follows easily from (3.30), (2.2), and (4.5), while (4.12) is a consequence of (3.3) with $e_0=\chi_0=0$ and $F=-J(k*(\Theta_1-\Theta_2))$, once one argues as in (4.2), notices that $(\Theta=\Theta_1-\Theta_2)$

$$\begin{aligned} 2\int_0^t (F(\cdot,s)\,,\,J^{-1}e(\cdot,s))\,ds \leq\ & \left(2\,\varepsilon^{-1}+1\right)\|k\|^2_{L^1(0,t)}\,\|\Theta\|^2_{L^2(0,t;H)} \\ & + \frac{\varepsilon}{2}\,\|\theta\|^2_{L^2(0,t;H)} + \|\lambda'\|^2_{L^\infty(K)}\int_0^t \|\chi(\cdot,s)\|^2_H\,ds\,, \end{aligned}$$

and applies Gronwall's lemma. □

Lemma 4.4 *Let $\varepsilon > 0$ and $k \in L^1(0,T)$. Then there exists some $T_0 \in (0,T]$ such that* $(\mathbf{P}'_\varepsilon)$ *has a unique solution on* $[0,T_0]$.

Proof. Choose $T_0 > 0$ small enough so that

$$R_2(\varepsilon)\,\|k\|^2_{L^1(0,T_0)}\,\frac{2R_1(\varepsilon)}{\varepsilon} \le R_1(\varepsilon) \tag{4.13}$$

and that

$$\overline{C}_4\left(1 + \varepsilon^{-1}\right)\,\|k\|^2_{L^1(0,T_0)} \le \frac{\varepsilon}{2}\,. \tag{4.14}$$

Then A_ε maps the set

$$Y_0 = \left\{ v \in L^2(0,T_0;V) \;\middle|\; \varepsilon\|v\|^2_{L^2(0,T_0;V_0)} \le 2R_1(\varepsilon) \right\} \tag{4.15}$$

into itself because of (4.11). Moreover, if we endow Y_0 with the distance

$$d\,(v_1,v_2) = \|v_1 - v_2\|_{L^2(0,T_0;H)}\,, \quad v_1, v_2 \in Y_0\,, \tag{4.16}$$

then A_ε is a contraction mapping. Note that Y_0 is a complete metric space owing to the weak lower semicontinuity of the norm $\|\cdot\|_{L^2(0,T_0;V)}$ and to the coincidence of strong and weak limits in $L^2(0,T_0;H)$. At this point, it only remains to invoke Banach's fixed point theorem. □

Lemma 4.5 *Let $\varepsilon > 0$ and $k \in L^2(0,T)$. Then Problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution on the whole time interval* $[0,T]$.

Proof. Thanks to Lemma 4.4, it suffices to prove an estimate independent of T_0. Note that the solution (θ,u,χ,ξ) of $(\mathbf{P}_\varepsilon)$ solves $(\mathbf{P}'_\varepsilon)$ for $F = f - J(k*\theta)$. Therefore, recalling the inequality (3.30) and observing that (see (4.5))

$$\begin{aligned}\|F\|^2_{L^2(0,t;V')} &\le 2\left(\|f\|^2_{L^2(0,T;V')} + \int_0^t \|(k*\theta)(\cdot,s)\|^2_V\,ds\right)\\ &\le 2\left(\|f\|^2_{L^2(0,T;V')} + \int_0^t \|k\|^2_{L^2(0,T)}\|\theta\|^2_{L^2(0,s;V)}\,ds\right),\end{aligned} \tag{4.17}$$

one can apply Gronwall's lemma and conclude the proof. □

5 Existence for Problem $(\mathbf{P}_0)$

Let $(\theta_\varepsilon, u_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ be the solution of $(\mathbf{P}_\varepsilon)$ for $\varepsilon > 0$. We make use of estimate (3.10), being

$$F = f - J(k*\theta_\varepsilon)\,.$$

Owing to (3.57) and **(A5)**, we have

$$\int_0^t (f(\cdot,s)\,,\,\theta_\varepsilon(\cdot,s))\,ds \le C_8\left(1 + \int_0^t \|\theta_\varepsilon(\cdot,s)\|^2_H\,ds\right). \tag{5.1}$$

For the other contribution, integrating by parts and using (4.3–4) we infer that

$$
\begin{aligned}
& - \int_0^t (J(k * \theta_\varepsilon)(\cdot\,,s)\,,\, \theta_\varepsilon(\cdot\,,s))\, ds \\
= \; & - \frac{k(0)}{2} \|(1 * \theta_\varepsilon)(\cdot\,,t)\|_V^2 - (\!((k' * 1 * \theta_\varepsilon)(\cdot\,,t)\,,\, (1 * \theta_\varepsilon)(\cdot\,,t))\!) \\
& + \int_0^t (\!((k'(0)(1 * \theta_\varepsilon) + k'' * 1 * \theta_\varepsilon)(\cdot\,,s)\,,\, (1 * \theta_\varepsilon)(\cdot\,,s))\!)\, ds\,, \qquad (5.2)
\end{aligned}
$$

so that we gain a further positive term on the left-hand side of (3.10). On the other hand, referring also to (3.19) we remark that

$$
\|F\|_{L^2(0,t;V')} \le \|f\|_{L^2(0,T;V')} + \Big(|k(0)| + \|k'\|_{L^1(0,T)}\Big) \int_0^t \|(1 * \theta_\varepsilon)(\cdot\,,s)\|_V\, ds\,. \qquad (5.3)
$$

Hence everything reduces to estimate the other terms in (5.2) suitably (cf. the computations performed in (4.7–8)), to apply Gronwall's lemma in the inequality resulting from (3.10), and then to take advantage of Lemma 3.3. In conclusion, we get the uniform bound

$$
\begin{aligned}
& \|\theta_\varepsilon\|^2_{H^1(0,t;V') \cap C^0([0,t];H)} + \varepsilon \|\theta_\varepsilon\|^2_{L^2(0,t;V)} + \|1 * \theta_\varepsilon\|^2_{C^0([0,t];V)} \\
& + \|\chi_\varepsilon\|^2_{H^1(0,t;H) \cap C^0([0,t];V) \cap L^2(0,t;H^2(\Omega))} + \|\xi_\varepsilon\|_{L^2(0,t;H)} \le C_9 \qquad \forall\, t \in [0,T]\,. \qquad (5.4)
\end{aligned}
$$

Then one can pass to the limit as $\varepsilon \searrow 0$ as in the analogous analysis for $(\mathbf{P}'_\varepsilon)$ and $(\mathbf{P}'_0)$, by just noting that here we have the additional convergence

$$
k * \theta_\varepsilon \to k * \theta \quad \text{weakly star in } L^\infty(0,T;V),
$$

due to (5.4), **(A4)**, and (4.3). □

Remark 5.1. We conjecture that all our results hold in any spatial dimension since we used the fact that $\Omega \subseteq \mathbb{R}^3$ just to dispose of approximating solutions found in other papers. But in our estimates we never exploit the dimension 3 of the space.

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