Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Controlled polyhedral sweeping processes: Existence, stability, and optimality conditions

René Henrion[1], Abderrahim Jourani[2], Boris S. Mordukhovich[3]

submitted: November 25, 2021

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: rene.henrion@wias-berlin.de

[2] Institut de Mathématiques de Bourgogne
Université de Bourgogne
Dijon
France
E-Mail: jourani@u-bourgogne.fr

[3] Department of Mathematics
Wayne State University
Detroit
Michigan 48202
USA
E-Mail: aa1086@wayne.edu

No. 2892
Berlin 2021

2020 *Mathematics Subject Classification.* 49J52, 49J53, 49K24, 49M25.

Key words and phrases. Sweeping process, moving polyhedra, existence of feasible solutions, qualitative stability, optimal control, discrete approximations, necessary optimality and suboptimality conditions.

The first author acknowledges support by the FMJH Program Gaspard Monge in optimization and operations research including support to this program by EDF as well as by the Deutsche Forschungsgemeinschaft for their support of project B04 within the CRC/Transregio 154. The work of the second author was partly supported by the EIPHI Graduate School (contract ANR-17-EURE-0002). Research of the third author was partly supported by the US National Science Foundation under grants DMS-1512846 and DMS-1808978, by the US Air Force Office of Scientific Research grant #15RT04, and by the Australian Research Council under grant DP-190100555.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Controlled polyhedral sweeping processes: Existence, stability, and optimality conditions

René Henrion, Abderrahim Jourani, Boris S. Mordukhovich

Abstract

This paper is mainly devoted to the study of controlled sweeping processes with polyhedral moving sets in Hilbert spaces. Based on a detailed analysis of truncated Hausdorff distances between moving polyhedra, we derive new existence and uniqueness theorems for sweeping trajectories corresponding to various classes of control functions acting in moving sets. Then we establish quantitative stability results, which provide efficient estimates on the sweeping trajectory dependence on controls and initial values. Our final topic, accomplished in finite-dimensional state spaces, is deriving new necessary optimality and suboptimality conditions for sweeping control systems with endpoint constrains by using constructive discrete approximations.

1 Introduction and Problem Formulation

In this paper we consider a family of sweeping processes with controlled polyhedral moving sets defined on a Hilbert space $\mathcal{H}$. To describe this family, fix some $x_0 \in \mathcal{H}$ and, for arbitrary control functions $(u,b) : [0,T] \to \mathcal{H}^m \times \mathbb{R}^m$ satisfying $x_0 \in C_{(u,b)}(0)$, define the *moving polyhedral set*

$$C_{(u,b)}(t) := \{x \in \mathcal{H} | \, \langle u_i(t), x\rangle \le b_i(t) \quad (i = 1, \dots, m)\} \quad (t \in [0,T]) . \tag{1.1}$$

This induces the *controlled sweeping process* $\left(\mathcal{S}_{(u,b)}\right)$ given by

$$-\dot{x}(t) \in N_{C_{(u,b)}(t)}\left(x(t)\right) \text{ a.e. } t \in [0,T], \; x(0) = x_0 \in C_{(u,b)}(0), \tag{1.2}$$

where $N_C(x)$ stands for the classical normal cone of convex analysis defined as

$$N_C(x) := \{v \in \mathcal{H} \bigm| \langle v, y-x\rangle \le 0\} \text{ if } x \in C \text{ and } N_C(x) := \emptyset \text{ else.} \tag{1.3}$$

We emphasize that the differential inclusion in (1.2) comes along with the hidden pointwise *state constraints* $x(t) \in C_{(u,b)}(t)$ for all $t \in [0,T]$, because otherwise the normal cone is empty by definition.

Uncontrolled sweeping processes were introduced and initially studied by Moreau [25, 26, 27] and then were extensively developed in the literature, where the main attention was paid to the existence and uniqueness of solutions and various applications; see, e.g., [1, 6, 7, 18, 20, 16] with their references.

Existence and uniqueness of *class-preserving* solutions $x_{(u,b)}$ to the sweeping dynamics (1.2) generated by *control* functions (u,b) in (1.1) from various classes in Hilbert spaces is the *first topic* of our paper. Note that the standard approach to this issue (see, e.g., [20]) consists of checking the Hausdorff Lipschitz continuity of the moving set (1.1). However, this does not make much sense when the moving set is an unbounded polyhedron. The $W^{1,2}$-preserving existence and uniqueness results for moving polyhedra were obtained by Tolstonogov [31, 32, 33] and more recently in [9] under certain

DOI 10.20347/WIAS.PREPRINT.2892 Berlin 2021

qualification conditions in Hilbert and finite-dimensional settings; see more discussions in Section 3. Here we develop a novel approach involving the *truncation* of polyhedra and deriving refined *error bounds*. This allows us obtain new class-preserving results, which shows that Lipschitz continuous (resp. absolutely continuous) controls in (1.1) uniquely generate Lipschitz continuous (resp. absolutely continuous) trajectories of (1.2) under an explicit and easily formulated *uniform Slater condition* for moving control polyhedra in separable Hilbert spaces.

The *second topic* of our study addresses *quantitative stability* issues on the *Hölderian* dependence of solutions to (1.2) on the corresponding perturbations of controls (u, b) in moving sets as well as the initial value x_0 in separable Hilbert spaces. To the best of our knowledge, such questions have never been posted for the sweeping processes formulated in (1.1) and (1.2). Based on the aforementioned truncation techniques and error bounds, we establish efficient results in this direction in the $W^{1,1}$ control-trajectory framework.

The *third topic* we investigate here concerns an *optimal control* problem for the sweeping process in (1.1) and (1.2) under the additional pointwise equality constraint on the u*-component of controls* and *geometric endpoint constraint* $x_{(u,b)} \in \Omega$ on trajectories. Optimal control theory for sweeping processes, with addressing the main issue of deriving necessary optimality conditions, has been started rather recently in [11] and then has been extensively developed in subsequent publications (see, e.g., [2, 5, 8, 9, 10, 12, 13, 14, 15, 35] and the references therein), which did not concern however systems with endpoint constraints. Problems of sweeping optimal control, that are governed by discontinuous differential inclusions with intrinsic pointwise and irregular state constraints, constitute one of the most challenging class in modern control theory. We develop here the *method of discrete approximation*, which allows us to constructively approximate the constrained control sweeping process under consideration by discrete-time sweeping systems with perturbed endpoint constraints so that feasible and optimal solutions to discrete approximations *strongly converge* to the designated feasible and locally optimal solutions of the original problem under the *uniform Slater condition* introduced above. Employing then advanced tools of first-order and second-order variational analysis and generalized differentiation, we derive new *necessary optimality conditions* for discrete approximations that gives us efficient *suboptimality conditions* for a general class of local minimizers in the original problem of sweeping optimal control.

The rest of the paper is organized as follow. Section 2 presents major technical developments on the truncation and error bounds, which are of their own interest while being widely used in deriving the main results of the paper. Section 3 is devoted to establishing the class-preserving existence and uniqueness theorems for the controlled sweeping process. Section 4 addresses stability issues for sweeping trajectories under control and initial value perturbations. In Section 5 we formulate an optimal control problems for the sweeping process ($\mathcal{S}_{(u,b)}$) with endpoint constraint and construct its well-posed discrete approximations with establishing the $W^{1,2}$-strong convergence of feasible and optimal solutions. The final Section 6 provides necessary optimality and suboptimality conditions for such control problems via advanced tools of generalized differentiation.

2 Error bounds and truncation of moving sets

This section plays a crucial role in describing and justifying our strategy to derive existence and stability results for sweeping processes with controlled polyhedra in both finite-dimensional and infinite-dimensional settings. The conventional by now theory of sweeping processes establishes the existence of Lipschitz continuous solutions of the sweeping dynamics via the Hausdorff Lipschitz continuity of moving sets; see, e.g., Theorem 2 in [20] and its proof. Unfortunately, this approach does not work for the case of unbounded moving polyhedra. For instance, in the case in moving *halfspaces*, i.e., for $m = 1$ in (1.1), the Hausdorff distance is either zero (if the two halfspaces coincide), or infinity otherwise. Hence the only “moving"halfspaces satisfying Hausdorff Lipschitz continuity are constant in time, which clearly does not offer any freedom for controlling the process. However, when *truncating* the moving polyhedron with a ball, the Hausdorff Lipschitz continuity may well be achieved. This suggests the following *strategy*, which will be implemented in the paper. *First* we intend to show that Lipschitzian controls lead us to *bounded* continuous solutions of the sweeping process and that the moving polyhedron *truncated* with a ball sufficiently large to contain this solution is Hausdorff Lipschitz, which hence verifies the actual Lipschitz continuity of the solution. The *second step* of our approach is to establish an appropriate *error bound* for the truncation moving polyhedra.

For the reader's convenience, we split this section into several subsections and present numerical examples providing the driving forces for our approach.

2.1 Hausdorff Lipschitz continuity of truncated moving polyhedra

As discussed above, it is generally hopeless to ensure a Hausdorff Lipschitz estimate for moving polyhedra (1.1) in the form

$$d_H\left(C_{(u,b)}(s), C_{(u,b)}(t)\right) \le \widehat{L}\,|s-t| \quad \forall s,t \in [0,T]\,. \tag{2.4}$$

Our efforts are now paid to establish a *truncated estimate* of type

$$d_H\left(C^r_{(u,b)}(s), C^r_{(u,b)}(t)\right) \le \widehat{L}\,|s-t| \quad \forall s,t \in [0,T]\,, \tag{2.5}$$

where $r \ge 0$ is appropriately given, and where $C^r := C \cap \mathbb{B}\,(0,r)$. To accomplish this, we proceed in following two steps. Our *first step* is to derive the *weakened Hausdorff estimate* given by

$$d\left(x, C_{(u,b)}(t)\right) \le L\left(\|x\|\right)|s-t| \quad \forall s,t \in [0,T] \;\; \forall x \in C_{(u,b)}(s) \tag{2.6}$$

with some monotonically increasing function $L(\cdot)$. Estimate (2.6) clearly yields

$$d\left(x, C_{(u,b)}(t)\right) \le \widehat{L}\,|s-t| \quad \forall s,t \in [0,T] \;\; \forall x \in C^r_{(u,b)}(s) \tag{2.7}$$

with $\widehat{L} := L\,(r)$. In the *second step* we prove the general estimate

$$d\left(x, C^r_{(u,b)}(t)\right) \le 3d\left(x, C_{(u,b)}(t)\right) \quad \forall t \in [0,T] \;\; \forall x \in \mathbb{B}\,(0,r) \tag{2.8}$$

for all r sufficiently large. Combining the latter with (2.7) will ensure the desired truncated estimate (2.5). Details follow.

2.1.1 Limitations of Hoffman's error bound

The first idea, which comes to our mind for proving (2.6), is the use of the classical *Hoffman's error bound*; see, e.g., [4, Theorem 2.200]. It guarantees in our setting that, for each $t \in [0,T]$, there exists some $\widetilde{L}\,(t) := L(t, u(t), b(t))$ ensuring the distance estimate

$$d\left(x, C_{(u,b)}(t)\right) \le \widetilde{L}\,(t) \max_{i=1,\dots,m} \left[\langle u_i(t), x\rangle - b_i(t)\right]_+ \quad \forall x \in \mathcal{H} \tag{2.9}$$

provided that $C_{(u,b)}(t) \neq \emptyset$. In particular, for $x \in C_{(u,b)}(s)$ it follows from $\langle u_i(s), x\rangle \leq b_i(s)$ for $i = 1, \ldots, m$, that

$$\begin{aligned} & [\langle u_i(t), x\rangle - b_i(t)]_+ \\ = \; & [\langle u_i(t), x\rangle - \langle u_i(s), x\rangle + \langle u_i(s), x\rangle - b_i(s) + b_i(s) - b_i(t)]_+ \\ \leq \; & [\langle u_i(t), x\rangle - \langle u_i(s), x\rangle + b_i(s) - b_i(t)]_+ \\ \leq \; & \|u_i(t) - u_i(s)\| \, \|x\| + |b_i(s) - b_i(t)| \quad \forall i = 1, \ldots, m. \end{aligned} \tag{2.10}$$

When (u, b) is Lipschitz continuous, this combines with the previous estimate to give us (with $\|\cdot\|_\infty$ referring to the maximum norm) the inequalities

$$\begin{aligned} d\left(x, C_{(u,b)}(t)\right) & \leq \widetilde{L}\,(t)\left(\|u(t) - u(s)\|_\infty \|x\| + \|b(s) - b(t)\|_\infty\right) \\ & \leq \widetilde{L}\,(t)\,(\|x\| + 1)\, K\, |s - t| \quad \forall x \in C_{(u,b)}(s), \end{aligned}$$

where K is a Lipschitz constant of (u, b). Therefore, if the function $\widetilde{L}\,(t)$ is bounded from above on $[0, T]$, say by L^*, then the desired estimate (2.6) would follow with the function $L\,(\tau) := (\tau + 1)\, L^*$, which is clearly monotonically increasing. Unfortunately, even for Lipschitzian controls (u, b), the function $\widetilde{L}\,(t)$ may be *unbounded from above* as can be seen from the following example.

Example 2.1 In (1.1) put $m := 2$, $\mathcal{H} := \mathbb{R}^2$, $T := 1$ and define the smooth (hence Lipschitz continuous) control pair

$$u_1\,(t) := (0, 1)\,;\; b_1\,(t) := 1;\; u_2\,(t) := (t, -1)\,;\; b_2\,(t) := 0.$$

For $t \in (0, 1]$, take $x\,(t) := (t^{-3}, 1)$ and observe that

$$d\left(x\,(t)\,, C_{(u,b)}(t)\right) = t^{-3} - t^{-1} \;\text{ and }\; \max_{i=1,\ldots,m} [\langle u_i(t), x\,(t)\rangle - b_i(t)]_+ = t^{-2} - 1.$$

It thus follows from (2.9) that $\widetilde{L}\,(t) \geq t^{-1}$ for all $t \in (0, 1]$. Therefore, the function $\widetilde{L}\,(t)$ is unbounded on $[0, T]$.

Remark 2.1 There are certain special cases in which Hoffman's error bound leads us to a *bounded* function $\widetilde{L}\,(t)$ in (2.6) on the interval $[0, T]$, even for non-Lipschitzian controls (u, b). We mention the following:

1 In the case of a *moving halfspace* (i.e., $m = 1$ and $u(t) \neq 0$ for all $t \in [0, 1]$) with a continuous control $u : [0, T] \to \mathcal{H}$ and an arbitrary control $b : [0, T] \to \mathbb{R}$, we have that

$$d\left(x, C_{(u,b)}(t)\right) = \|u(t)\|^{-1} [\langle u(t), x\rangle - b(t)]_+ \leq L^{-1} [\langle u(t), x\rangle - b(t)]_+$$

for all $t \in [0, 1]$ and all $x \in \mathcal{H}$, where $L := \inf_{t\in[0,1]} \|u(t)\| > 0$.

2 In the case where variable control functions are situated only on the *right-hand side* of (1.1), i.e, when $u\,(t) \equiv u \neq 0$) while $b : [0, T] \to \mathbb{R}$ is arbitrary, it follows from [19, Proposition 4.6] that

$$d\left(x, C_{(u,b)}(t)\right) \leq L \max_{i=1,\ldots,m} [\langle u_i(t), x\rangle - b_i(t)]_+ \quad \forall t \in [0, T] \;\; \forall x \in \mathcal{H}$$

whenever $C_{(u,b)}(t) \neq \emptyset$ for all $t \in [0, T]$.

Example 2.1 illustrates the drastic impact of fully controlled polyhedral moving sets on Hoffman's error bound starting from dimension two, even for smooth controls. Fortunately, it turns out that—despite the fact that the approach using Hoffman's error bound sketched above is not viable for our purposes—we may find an *alternative path* based on (2.6), in order to reach the desired goal. To support this idea, let us revisit Example 2.1 and observe that the sweeping process generated by the Lipschitzian control in this example does admit a unique Lipschitzian solution for an arbitrary initial point $x_0 \in C_{(u,b)}(0)$.

Example 2.2 Consider the control pair (u, b) defined in Example 2.1 and fix an arbitrary initial point $x_0 \in C_{(u,b)}(0)$. We subdivide the initial polyhedron as $C_{(u,b)}(0) = \Omega_1 \cup \Omega_2$ with the sets

$$\Omega_1 := \left\{x \in C_{(u,b)}(0) \;\middle|\; x_2 < x_1\right\} \text{ and } \Omega_2 := \left\{x \in C_{(u,b)}(0) \middle|\, x_2 \geq x_1\right\}.$$

If $x_0 \in \Omega_2$, then for an arbitrary time $t \in (0,1)$ the boundaries of the two controlled halfspaces have no contact with x_0. Consequently, $\dot{x}(t) = 0$ for all $t \in (0,1)$, and hence $x\,(t) = x_0$ for all $t \in [0,1]$. In contrast, for $x_0 \in \Omega_1$ we get

$$x\,(t) = \begin{cases} x_0 & t \in [0,t_1] \\ y\,(t) & t \in (t_1,t_2) \\ (1/t,1) & t \in [t_2,1] \end{cases}, \; t_1 = \frac{x_{0,2}}{x_{0,1}}, \; t_2 = \begin{cases} \frac{1}{\sqrt{\|x_0\|^2-1}} & \text{if } \|x_0\| \geq \sqrt{2} \\ \infty & \text{else} \end{cases},$$

$$y_1\,(t) = \frac{\|x_0\|}{\sqrt{1+t^2}}, \text{ and } y_2\,(t) = \frac{\|x_0\|}{\sqrt{1+t^2}}t.$$

Here t_1 denotes the time when the second halfspace (the moving one) becomes binding for x_0 for the first time, i.e., when $tx_{0,1} = x_{0,2}$. This gives us the indicated formula for t_1. For $t < t_1$ both halfspaces are nonbinding for x_0; so $\dot{x}(t) = 0$, and hence $x\,(t) = x_0$ for all $t \in [0,t_1]$. For $t \geq t_1$ the second halfspace is binding. The first halfspace also becomes binding at a certain time $t_2 > t_1$; so we have $x_2\,(t) = 1$ for all $t \in [t_2,1]$. Since the second halfspace keeps binding, it follows that $tx_1\,(t) = x_2\,(t) = 1$ from where we conclude that $x_1\,(t) = 1/t$ during this period of time. It remains to determine the trajectory $x\,(t)$ for $t \in (t_1,t_2)$, as well as the switching time t_2. Since in this interval only the second halfspace is binding, we derive the following relations from the sweeping dynamics:

$$-\dot{x}(t) \in N_{C_{(u,b)}(t)}\,(x(t)) = \mathbb{R}_+\,(t,-1) \quad \forall t \in (t_1,t_2)\,.$$

Consequently, there exists a function $\lambda\,(t) \leq 0$ such that

$$\dot{x}_1(t) = t\lambda\,(t)\,; \quad \dot{x}_2(t) = -\lambda\,(t) \quad \forall t \in (t_1,t_2)\,.$$

On the other hand, with the second halfspace being binding, we also have that $tx_1\,(t) = x_2\,(t)$ for all $t \in [t_1,t_2)$. This tells us therefore that

$$\dot{x}_1(t) = -t\dot{x}_2(t) = -\frac{x_2\,(t)}{x_1\,(t)}\dot{x}_2(t) \Longleftrightarrow \dot{x}_1(t)x_1\,(t) + \dot{x}_2(t)x_2\,(t) = 0 \quad \forall t \in (t_1,t_2)\,.$$

The solution to the latter differential equation is given by $x_1^2\,(t) + x_2^2\,(t) = C$, where the constant C can be identified from the fact that $x\,(t_1) = x_0$, which yields $C = \|x_0\|^2$. Along with the equality $tx_1\,(t) = x_2\,(t)$, we identify the function $y(t)$ indicated in the formula above. Finally, the switching time t_2 is determined from the relation $y_2\,(t_2) = 1$. Observe that for $\|x_0\| < \sqrt{2}$ the first halfspace is never binding in the given time interval $[0,1]$. It is easy to check that the determined solution $x(t)$ is Lipschitz continuous on the entire interval $[0,1]$, and as such it has to be unique due [20, Theorem 3].

2.1.2 Uniform Slater condition and weakened Hausdorff estimate

As shown in our subsequent analysis, the reason why the announced result—that Lipschitzian controls yield Lipschitzian solutions of the sweeping process—can be maintained in Example 2.1 despite the fact that an argumentation via Hoffman's error bound does not apply, consists in the fulfillment of an appropriate *constraint qualification*. Now we introduce this qualification condition, which plays a crucial role not only in establishing existence and stability results presented in what follows, but also in the two last sections of the paper dealing with the verification of the strong convergence of discrete approximations and the derivation of necessary optimality conditions for sweeping optimal control.

Here is this easy formulated and natural qualification condition.

Definition 2.1 *We say that the moving polyhedron in* (1.1) *generated by the given control pair* (u, b) *satisfies the* UNIFORM SLATER CONDITION *if*

$$\forall t \in [0, T] \;\; \exists x \in \mathcal{H} \;\; \text{such that} \;\; \langle u_i\,(t)\,, x\rangle < b_i\,(t) \quad \forall i = 1, \ldots, m. \tag{2.11}$$

We emphasize that, unlike the boundedness of $\widetilde{L}(t)$ in Hoffman's error bound estimate (2.9), this constraint qualification is *essential* for our desired result. Indeed, a simple two-dimensional example taken from [13, Example 2.3] shows that, even for smooth control functions, the sweeping process (1.2) may not admit a solution when (2.11) is violated. On the other hand, we see below that (2.11) yields the weakened Hausdorff estimate (2.6), which is the first step mentioned in the introduction to this section.

Before deriving (2.6) via (2.11), we show that the following seemingly stronger version of (2.11) has been used in the earlier work on the existence of solutions to sweeping processes defined by moving polyhedra [9, Assumption (H4)]:

$$\exists \varepsilon > 0 \; \forall t \in [0, T] \;\; \exists x \in \mathcal{H} \; \text{with} \;\; \langle u_i\,(t)\,, x\rangle \le b_i\,(t) - \varepsilon \quad \forall i = 1, \ldots, m \tag{2.12}$$

It turns out, however, that this “strong uniform Slater conditionïs *equivalent* to the uniform Slater condition formulated in (2.11).

Proposition 2.1 *Assume that the control* (u, b) *in* (1.1) *is continuous. Then conditions* (2.11) *and* (2.12) *are equivalent.*

Proof. Since (2.12) obviously yields (2.11), it remains to verify the opposite implication. Assume that (2.12) fails, which tells us that

$$\forall n \in \mathbb{N} \; \exists t_n \in [0, T] \;\; \forall x \in \mathcal{H} \; \exists i \in \{1, \ldots, m\} \;\; \text{with} \;\; \langle u_i\,(t_n)\,, x\rangle > b_i\,(t_n) - \frac{1}{n}.$$

For some subsequence $t_{n_k} \in [0, T]$, there exists $\bar{t} \in [0, T]$ such that $t_{n_k} \to_k \bar{t}$. Fix an arbitrary vector $x \in \mathcal{H}$ and then get

$$\forall k \in \mathbb{N} \; \exists i_k \in \{1, \ldots, m\} \;\; \text{with} \;\; \langle u_{i_k}\,(t_{n_k})\,, x\rangle > b_{i_k}\,(t_{n_k}) - \frac{1}{n_k}.$$

Selecting another subsequence, find $i^* \in \{1, \ldots, m\}$ such that $i_{k_l} \equiv i^*$. Therefore, we have the inequalities

$$\left\langle u_{i^*}\left(t_{n_{k_l}}\right), x\right\rangle > b_{i^*}\left(t_{n_{k_l}}\right) - \frac{1}{n_{k_l}} \;\; \text{for all} \;\; l \in \mathbb{N}.$$

Passing there to the limit as $l \to \infty$ gives us $\langle u_{i^*}(\bar{t}), x\rangle \geq b_{i^*}(\bar{t})$. Since $x \in \mathcal{H}$ was chosen arbitrarily, we arrive at

$$\exists \bar{t} \in [0,T] \;\; \forall x \in \mathcal{H} \; \exists i^* \in \{1,\ldots,m\} \text{ with } \langle u_{i^*}(\bar{t}), x\rangle \geq b_{i^*}(\bar{t}),$$

which contradicts (2.11) and thus completes the proof of the proposition. □

Now we turn to the announced proof of the weakened Hausdorff estimate (2.6). Given $\delta > 0$, define the $\delta-$*moving polyhedron* by

$$C^{(\delta)}_{(u,b)}(t) := \left\{x \in \mathcal{H} \;\middle|\; \langle u_i(t), x\rangle \leq b_i(t) - \delta \;\; (i = 1,\ldots,m)\right\} \;\; (t \in [0,T]). \tag{2.13}$$

To proceed, we first present the following crucial technical lemma involving continuous controls $(u,b) \in \mathcal{C}([0,T],\mathcal{H}^m) \times \mathcal{C}([0,T],\mathbb{R}^m)$ in the moving polyhedron (1.1) endowed with the maximum norm

$$\|(u,b)\|_\infty := \max_{t\in[0,T],i=1,\ldots,m} \|u_i(t)\| + \max_{t\in[0,T],i=1,\ldots,m} |b_i(t)|.$$

The associated closed ball in this space centered at (u,b) with radius $r > 0$ is denoted by $\mathbb{B}_\infty((u,b),r)$.

Lemma 2.1 *Fix continuous control* $(\bar{u},\bar{b}) \in \mathcal{C}([0,T],\mathcal{H}^m) \times \mathcal{C}([0,T],\mathbb{R}^m)$ *satisfying the uniform Slater condition* (2.11)*. Then there exists* $\varepsilon > 0$ *such that whenever* $\gamma \in (0,\varepsilon)$ *we can find a continuous function* $\widehat{x} \in \mathcal{C}([0,T],\mathcal{H})$ *for which*

$$\widehat{x}(t) \in C^{(\gamma)}_{(u,b)}(t) \;\forall t \in [0,T] \;\; \forall (u,b) \in \mathcal{B} := \mathbb{B}_\infty\left((\bar{u},\bar{b}), \frac{\varepsilon - \gamma}{3\,(1 + \|\widehat{x}\|_\infty)}\right). \tag{2.14}$$

Furthermore, we have the estimate

$$d(x, C_{(u,b)}(t)) \leq \frac{f_{(u,b)}(t,x)}{f_{(u,b)}(t,x) - f_{(u,b)}(t,\widehat{x}(t))} \|x - \widehat{x}(t))\| \;\forall t \in [0,T] \tag{2.15}$$

for all $t \in [0,T]$*, all* $x \in \mathcal{H}\backslash C_{(u,b)}(t)$*, and all* $(u,b) \in \mathcal{B}$*, where* $f_{(u,b)}(t,x) := \max_{i=1,\cdots,m}\langle u_i(t), x\rangle - b_i(t)$*. Finally,*

$$\begin{aligned} &d(x, C_{(u',b')}(t)) \leq \\ &\|x - \widehat{x}(t)\| \min\left\{1, \gamma^{-1} \max_{i=1,\cdots,m} [\langle u'_i(t) - u_i(s), x\rangle + b_i(s) - b'_i(t)]_+\right\} \end{aligned} \tag{2.16}$$

for all $(u,b),(u',b') \in \mathcal{B}$*, all* $s,t \in [0,T]$*, and all* $x \in C_{(u,b)}(s)$*.*

Proof. As shown in Proposition 2.1, the imposed uniform Slater condition (2.11) is equivalent to (2.12) for $(u,b) := (\bar{u},\bar{b})$. Using the latter and choosing $\varepsilon > 0$ therein, pick an arbitrary number $\gamma \in (0,\varepsilon)$ and define

$$\delta := \frac{2\varepsilon + \gamma}{3} \in (0,\varepsilon).$$

Then condition (2.12) tells us that

$$\forall t \in [0,T] \;\; \exists x \in \mathcal{H} \text{ with } \langle \bar{u}_i(t), x\rangle \leq \bar{b}_i(t) - \varepsilon < \bar{b}_i(t) - \delta \quad \forall i = 1,\ldots,m.$$

In other words, for each $t \in [0,T]$ the convex set $C^{(\delta)}_{(\bar{u},\bar{b})}(t)$ admits a Slater point. This ensures the inclusion

$$C^{(\delta)}_{(\bar{u},\bar{b})}(t) \subseteq \mathrm{cl}\left\{x \in \mathcal{H} \;\middle|\; \langle \bar{u}_i(t), x\rangle < \bar{b}_i(t) - \delta\right\} \quad \forall t \in [0,T]$$

which in turn allows to conclude (by invoking, e.g., [3, Theorem 3.1.5]) that $C^{(\delta)}_{(u,b)} : [0,T] \rightrightarrows \mathcal{H}$ is a lower semicontinuous multifunction. Since the images $C^{(\delta)}_{(\bar{u},\bar{b})}(t)$ are closed and convex for all $t \in [0,T]$, the classical Michael selection theorem ensures the existence of a continuous function $\widehat{x} \in \mathcal{C}([0,T],\mathcal{H})$ with

$$\widehat{x}(t) \in C^{(\delta)}_{(\bar{u},\bar{b})}(t) \quad \forall t \in [0,T].$$

Next we fix an arbitrary continuous control $(u,b) \in \mathcal{B}$ and get by the definition of δ the following inequalities:

$$\begin{aligned} \langle u_i(t), \widehat{x}(t)\rangle - b_i(t) &\le \langle \bar{u}_i(t), \widehat{x}(t)\rangle + \|u_i(t) - \bar{u}_i(t)\| \cdot \|\widehat{x}(t)\| - b_i(t) \\ &\le \bar{b}_i(t) - \delta + \|u_i(t) - \bar{u}_i(t)\| \cdot \|\widehat{x}(t)\| - b_i(t) \\ &\le \frac{2}{3}(\varepsilon - \gamma) - \delta \le -\gamma \quad \forall t \in [0,T] \ \forall i = 1,\ldots,m. \end{aligned}$$

Thus $\widehat{x} \in \mathcal{C}([0,T],\mathcal{H})$ and $\widehat{x}(t) \in C^{(\gamma)}_{(u,b)}(t)$ for all $t \in [0,T]$, which verify (2.14).

Addressing the second assertion of the lemma, fix arbitrary elements $t \in [0,T]$, $(u,b) \in \mathcal{B}$, and $x \in \mathcal{H}\backslash C_{(u,b)}(t)$. Remembering the construction of $f_{(u,b)}$, we have that $f_{(u,b)}(t,x) > 0$ by $x \in \mathcal{H}\backslash C_{(u,b)}(t)$ and $f_{(u,b)}(t,\widehat{x}(t)) \le -\gamma < 0$ by the already proved relation (2.14), define

$$\lambda := \frac{f_{(u,b)}(t,x)}{f_{(u,b)}(t,x) - f_{(u,b)}(t,\widehat{x}(t))} \in (0,1).$$

It follows from the convexity of $f_{(u,b)}(t,\cdot)$ that

$$f_{(u,b)}(t,(1-\lambda)x + \lambda\widehat{x}(t)) \le (1-\lambda) f_{(u,b)}(t,x) + \lambda f_{(u,b)}(t,\widehat{x}(t)) = 0,$$

and so $(1-\lambda)x + \lambda\widehat{x}(t) \in C_{(u,b)}(t)$. This verifies (2.15), which can be written as

$$d(x, C_{(u,b)}(t)) \le \|x - ((1-\lambda)x + \lambda\widehat{x}(t))\| = \lambda\|x - \widehat{x}(t)\|.$$

It remains to justify the final assertion of the lemma. To proceed, fix arbitrary elements $s,t \in [0,T]$, $(u,b),(u',b') \in \mathcal{B}$, and $x \in C_{(u,b)}(s)$. If $x \in C_{(u',b')}(t)$, then (2.16) holds trivially. Supposing now that $x \notin C_{(u',b')}(t)$ gives us $f_{(u',b')}(t,x) > 0$ and $f_{(u,b)}(t,\widehat{x}(t)) \le -\gamma$ by (2.14). Therefore, (2.15) yields

$$\begin{aligned} & d(x, C_{(u',b')}(t)) \\ & \le \frac{f_{(u',b')}(t,x)}{f_{(u',b')}(t,x) - f_{(u',b')}(t,\widehat{x}(t))}\|x - \widehat{x}(t))\| \le \gamma^{-1} f_{(u',b')}(t,x)\|x - \widehat{x}(t))\| \\ & \le \gamma^{-1}\left(f_{(u',b')}(t,x) - f_{(u,b)}(s,x)\right)\|x - \widehat{x}(t))\| \quad \big(\text{because of } x \in C_{(u,b)}(s)\big) \\ & \le \gamma^{-1}\|x - \widehat{x}(t)\| \max_{i=1,\cdots,m} \left[\langle u'_i(t) - u_i(s), x\rangle + b_i(s) - b'_i(t)\right]_+. \end{aligned}$$

Since $\widehat{x}(t) \in C^{(\gamma)}_{(u',b')}(t) \subseteq C_{(u',b')}(t)$ by (2.14), we also have that $d(x, C_{(u',b')}(t)) \le \|x - \widehat{x}(t))\|$. Combining the above verifies (2.16) and completes the proof. □

We are now in a position to derive the weakened Hausdorff estimate (2.6).

Theorem 2.1 *Let* (u,b) *be a Lipschitz continuous control along which the moving polyhedron* (1.1) *satisfies the uniform Slater condition* (2.11)*. Then there exist constants* $K_1, K_2 \ge 0$ *such that the weakened Hausdorff estimate* (2.6) *holds with the monotonically increasing function* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$L(r) := K_1(r+1)(r+K_2) \quad (r \ge 0). \tag{2.17}$$

Proof. We again employ the uniform Slater condition (2.11) in the equivalent form (2.12) by Proposition 2.1. Then we get from (2.16) in Lemma 2.1 that

$$d(x, C_{(u,b)}(t)) \leq \frac{2}{\varepsilon}\|x - \widehat{x}(t)\| \max_{i=1,\cdots,m} \left[\langle u_i(t) - u_i(s), x\rangle + b_i(s) - b_i(t)\right]_+$$

along a continuous function $\widehat{x}(\cdot)$ for all $s, t \in [0, T]$ and all $x \in C_{(u,b)}(s)$. Define $\varkappa := \max_{t\in[0,T]} \|\widehat{x}\,(t)\| \geq 0$ and denote by $K \geq 0$ a Lipschitz constant of the control pair (u, b). Then we have the estimate

$$d(x, C_{(u,b)}(t)) \leq \frac{2K}{\varepsilon}\|x - \widehat{x}(t)\| \left(\|x\| + 1\right)|s - t| \leq \frac{2K}{\varepsilon}\left(\|x\| + \varkappa\right)\left(\|x\| + 1\right)|s - t|$$

for all $s, t \in [0, T]$ and all $x \in C_{(u,b)}(s)$. This is exactly (2.6) with the monotonically increasing function $L\,(r) := \delta^{-1}K\,(r + \varkappa)\,(r + 1)$. □

Remark 2.2 The moving polyhedron $C_{(u,b)}$ defined in Example 2.1 does satisfy the uniform Slater condition. To see this, select the constant solution $x\,(t) \equiv (0, 0.5)$ in (2.11). Thus the estimate (2.6) can be verified in this example via Theorem 2.1, while the usage of Hoffman's error bound does not lead us to the desired result. The reason is that Hoffman's error bound—if applicable as in the special cases mentioned in Remark 2.1—would necessarily bring us to an *affine function* L in (2.6); see the discussion above in Example 2.1. Yet, a closer inspection of the example shows that such an affine function L cannot work in this example. Indeed, consider the sequences

$$x^{(n)} := (2n, 0) \in C_{(u,b)}\,(0)\,; \quad t_n := n^{-1} \quad (n \in \mathbb{N})\,.$$

Assuming that estimate (2.6) holds with an affine function $L\,(r) := ar + b$ and choosing $s := 0$, we arrive at the following contradiction

$$\begin{array}{rcl} n & \leq & \sqrt{1+n^2} = d\left(x^{(n)}, C_{(u,b)}(t_n)\right) \leq \left(a\left\|x^{(n)}\right\| + b\right)t_n \\ & = & (2an + b)\,n^{-1} \leq 2a + |b| \quad \forall n \in \mathbb{N}. \end{array}$$

On the other hand, the choice of the *quadratic function* (2.17) by Theorem 2.1 allows us to derive the weakened Hausdorff estimate (2.6) in this example.

2.1.3 General truncation lemma

The last subsection of this section accomplishes the *second step* of our approach outlined in the introduction to this section. The following general truncation result clearly implies the desired estimates (2.8) for truncating polyhedra.

Lemma 2.2 *Let $(X, \|\cdot\|)$ be a normed space, and let C be a nonempty, closed, and convex subset of X. Define the truncating set $C^r := C \cap \mathbb{B}\,(0, r)$ for $r > 0$. Then we have the estimate*

$$d(x, C^r) \leq \frac{2r}{r - d(0, C)} d(x, C) \quad \forall x \in \mathbb{B}\,(0, r)\ \ \forall r > d\,(0, C)\,. \tag{2.18}$$

Consequently, it follows that

$$d(x, C^r) \leq 3d(x, C) \quad \forall x \in \mathbb{B}\,(0, r)\ \ \forall r > 3d\,(0, C)\,. \tag{2.19}$$

Proof. Pick arbitrary elements $r > d(0,C)$, $x \in \mathbb{B}(0,r)$, and ε with $0 < \varepsilon < r - d(0,C)$. If $x \in C$, then $x \in C^r$ and (2.18) holds trivially. Assume now that $x \notin C$, and so $d(x,C) > 0$. Choose $x_0, y \in C$ such that

$$\|x_0\| \le \beta := d(0,C) + \varepsilon,\ \|x - y\| \le d(x,C) + \min\{\varepsilon, d(x,C)\}. \tag{2.20}$$

If $\|y\| \le r$, then $y \in C^r$, and (2.18) follows from the inequality in (2.20). Therefore, it remains to examine the case where $\|y\| > r$. The equality in (2.20) combined with $\varepsilon < r - d(0,C)$ gives us the estimate $\|x_0\| \le \beta < r$. Therefore, there exists $\gamma \in (0,1)$ such that $\|z\| = r$ for $z := (1-\gamma)y + \gamma x_0$. The convexity of C readily ensures that $z \in C^r$. Then we have

$$r \le (1-\gamma)\|y\| + \gamma\|x_0\| \quad \text{or, equivalently,} \quad \gamma(\|y\| - \|x_0\|) \le \|y\| - r.$$

Due to $\|y\| > r > \beta \ge \|x_0\|$, the latter implies that

$$\|z - y\| = \gamma\|y - x_0\| \le \frac{\|y\| - r}{\|y\| - \beta}(\|y\| + \beta).$$

Taking into account that $\|x\| \le r$ brings us to

$$\|y\| \le \|y - x\| + \|x\| \le d(x,C) + \varepsilon + r,$$

and therefore we arrive at the estimate

$$\|z - y\| \le \frac{\|y\| + \beta}{\|y\| - \beta}(d(x,C) + \varepsilon).$$

Combining all the above leads us to the relationships

$$\|z - x\| \le \|z - y\| + \|y - x\| \le (1 + \frac{\|y\| + \beta}{\|y\| - \beta})(d(x,C) + \varepsilon) \le \left(2 + \frac{2\beta}{r - \beta}\right)(d(x,C) + \varepsilon).$$

Since $z \in C^r$ and ε was chosen arbitrarily with $0 < \varepsilon < r - d(0,C)$, we get

$$d(x, C_r) \le \left(2 + \frac{2d(0,C)}{r - d(0,C)}\right) d(x,C),$$

which verifies (2.18) and thus completes the proof of the truncation lemma. □

3 Existence and uniqueness of sweeping solutions

The main goal of this section is establishing two class-preservation *existence and uniqueness* theorems for polyhedral controlled sweeping processes defined in (1.1) and (1.2) under the uniform Slater condition (2.11) in the setting of *separable Hilbert spaces*. Namely, we aim at proving that *Lipschitz continuous* controls (u,b) uniquely generate Lipschitz continuous trajectories of $\mathcal{S}_{(u,b)}$ and that *absolutely continuous* (of class $W^{1,1}$) controls uniquely generate sweeping trajectories of the same class. Note that results of this type in the $W^{1,2}$ control-trajectory framework we obtained in [31, 32, 33] for various types of sweeping processes under appropriate assumptions in separable Hilbert spaces. Similar preservation results of class $W^{1,2}$ were established in [9] in finite dimensions under the strong uniform Slater condition (2.12) reducing to (2.11) as we now know. Observe also that results of this type in class of $W^{1,1}$ were derived in [13, 12] for polyhedral sweeping processes in finite-dimensional spaces under essentially stronger qualification conditions than (2.11) used in what follows. Our approach below is strongly based on the truncation procedure and error bound estimates developed in the previous section.

Here is the first theorem dealing with Lipschitzian controls.

Theorem 3.1 *Let $\mathcal{H}$ be a separable Hilbert space. Assume that (u,b) is Lipschitz continuous control and that the moving polyhedron $C_{(u,b)}$ in* (1.1) *satisfies the uniform Slater condition* (2.11) *along this control pair. Then the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ admits a unique Lipschitz continuous solution.*

Proof. Theorem 2.1 ensures the existence of a monotonically increasing function $L:\mathbb{R}_+\to\mathbb{R}_+$ satisfying the weakened Hausdorff estimate (2.6). This gives us for each $r>0$ a constant $\widehat{L}_r := L(r)$ such that (2.7) holds. Thus for all $r>0$, all $s,t\in[0,T]$, and all $x\in C_{(u,b)}(s)$ with $\|x\|\le r$ there is $y\in C_{(u,b)}(t)$ satisfying

$$\|x-y\|\le\left(\widehat{L}_r+1\right)|s-t|\,.$$

Indeed, the latter is obvious with the choice of $y:=x$ in the case where $s=t$, and this follows from (2.7) and from $d\left(x,C_{(u,b)}(t)\right)<\left(\widehat{L}_r+1\right)|s-t|$ in the case where $s\neq t$. Since the linear function $s\longmapsto\left(\widehat{L}_r+1\right)s$ trivially belongs to $W^{1,2}[0,T]$, it is r-*weakly uniformly lower semicontinuous from the right* for $p=2$ in the sense of Tolstonogov [31, eq. (2.2)]. Therefore, we deduce from [31, Lemma 2.1 and Lemma 3.1] that the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ has a unique solution $x^*\in W^{1,2}\left([0,T],\mathcal{H}\right)$. In particular, the trajectory $x^*(t)$ is absolutely continuous on $[0,T]$. It remains to show that $x^*(t)$ is Lipschitz continuous on this interval. To proceed, define

$$\rho:=\max_{t\in[0,T]}\|x^*(t)\|\,;\quad r:=3\rho+1 \tag{3.21}$$

and then fix arbitrary $s,t\in[0,T]$ and

$$x\in C^r_{(u,b)}(s):=C_{(u,b)}(s)\cap\mathbb{B}\left(0,r\right).$$

As a solution to $\left(\mathcal{S}_{(u,b)}\right)$, the function $x^*(t)$ satisfies the hidden state constraint $x^*(t)\in C_{(u,b)}(t)$. Therefore, we obtain

$$r=3\rho+1\ge3\|x^*(t)\|+1>3d\left(0,C_{(u,b)}(t)\right).$$

This allows us to invoke the truncation result from Lemma 2.2 to get

$$d\left(x,C^r_{(u,b)}(t)\right)\le3d\left(x,C_{(u,b)}(t)\right). \tag{3.22}$$

On the other hand, Theorem 2.1 yields (2.6) and hence gives us a constant $\widehat{L}$ such that (2.7) holds for our selected $s,t\in[0,T]$. Combining this with (3.22), and recalling that s,t,x were chosen arbitrarily, we arrive at the estimate

$$d\left(x,C^r_{(u,b)}(t)\right)\le3\widehat{L}\,|s-t|\quad\forall s,t\in[0,T]\ \forall x\in C^r_{(u,b)}(s).$$

Interchanging the roles of s and t readily yields the desired Lipschitz Hausdorff estimate (2.5) of the truncated moving polyhedron with modulus $3\widehat{L}$. Employing the standard existence result from [20, Theorem 2]) leads us to deducing from the obtained estimate that the truncated sweeping process $\left(\widetilde{\mathcal{S}}_{(u,b)}\right)$ defined as

$$-\dot{x}(t)\in N_{C^r_{(u,b)}(t)}\left(x(t)\right)\ \text{a.e.}\ t\in[0,T],\quad x(0)=x_0\in C^r_{(u,b)}(0) \tag{3.23}$$

admits a Lipschitz continuous solution $\widetilde{x}(\cdot)$. It follows from the definitions in (3.21) that for all $r>\rho$ we have the inclusions

$$x^*(t)\in C_{(u,b)}(t)\cap\mathbb{B}\left(0,\rho\right)\subseteq C_{(u,b)}(t)\cap\operatorname{int}\mathbb{B}\left(0,r\right)\subset C^r_{(u,b)}(t)\quad\forall t\in[0,T]\,.$$

On the one hand, the resulting inclusion justifies the feasibility of the initial point in $\left(\widetilde{\mathcal{S}}_{(u,b)}\right)$ due to $x_0 = x^*(0)$. On the other hand, it tells us that

$$N_{C^{\tau}_{(u,b)}(t)}\left(x^*(t)\right) = N_{C_{(u,b)}(t)}\left(x^*(t)\right) \quad \forall\, t \in [0,T]\,.$$

Therefore, $x^*(\cdot)$ being a solution to $\left(\mathcal{S}_{(u,b)}\right)$ is also a solution to $\left(\widetilde{\mathcal{S}}_{(u,b)}\right)$. Since $x^*(t)$ is absolutely continuous on $[0,T]$ as an element of $W^{1,2}\left([0,T]\,,\mathcal{H}\right)$, and since $\left(\widetilde{\mathcal{S}}_{(u,b)}\right)$ can have at most one absolutely continuous solution by [20, Theorem 3], we conclude that $x^*(\cdot) = \widetilde{x}(\cdot)$. This ensures that $x^*(t)$ is Lipschitz continuous on $[0,T]$, since $\widetilde{x}(t)$ is so. Thus we complete the proof. □

Our next goal in this section is establish the existence of a unique *absolutely continuous* solution of the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ generated by any absolutely control (u,b) in the moving polyhedron (1.1) under the same uniform Slater condition. Recall that the norms on the spaces of absolutely continuous functions $W^{1,1}([0,T],\mathcal{H}^m)$ and $W^{1,1}([0,T],\mathbb{R}^m)$ are defined, respectively, by

$$\|u\|_{1,1} := \sum_{i=1}^{m}\|u_i(0)\| + \sum_{i=1}^{m}\int_0^T \|\dot{u}_i(t)\|dt,\ \|b\|_{1,1} := \sum_{i=1}^{m}|b_i(0)| + \sum_{i=1}^{m}\int_0^T |\dot{b}_i(t)|dt.$$

The norm on the product space $W^{1,1}([0,T],\mathcal{H}^m)\times W^{1,1}([0,T],\mathbb{R}^m))$ is $\|(u,b)\|_{1,1} := \|u\|_{1,1} + \|b\|_{1,1}$, and the induced ball around (u,b) with radius r is $\mathbb{B}_{1,1}\left((u,b)\,,r\right)$.

The proof of the following theorem elaborates a reduction idea from [30] that allows us to deal with non-Lipschitzian controls of the sweeping dynamics.

Theorem 3.2 *Let $\mathcal{H}$ be a separable Hilbert space. Take $\left(\bar{u},\bar{b}\right) \in W^{1,1}([0,T],\mathcal{H}^m)\times W^{1,1}([0,T],\mathbb{R}^m)$ and suppose that the moving polyhedron $C_{(u,b)}$ in* (1.1) *satisfies the uniform Slater condition* (2.11)*. Then the control pair (u,b) generates a unique solution $x \in W^{1,1}\left([0,T]\,,\mathcal{H}\right)$ of the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ in* (1.2)*.*

Proof. It follows from the Newton-Leibniz formula that

$$\|f\left(t\right) - f\left(s\right)\| \le \int_s^t \left\|\dot{f}\left(r\right)\right\| dr \quad \forall f \in W^{1,1}\left([0,T]\,,\mathcal{H}\right)$$

whenever $s,t \in [0,T]$ with $s \le t$. Therefore, for all such s,t we have

$$\begin{aligned}\sum_{i=1}^{m}\|u_i(t)-u_i(s)\| + |b_i(t)-b_i(s)| &\le \left|\int_s^t \sum_{i=1}^{m}\|\dot{u}_i(r)\| + |\dot{b}_i(r)|dr + t - s\right| \\ &= |\gamma(t)-\gamma(s)| \end{aligned} \tag{3.24}$$

with the strongly increasing and absolutely continuous function

$$\gamma(t) := t + \int_0^t \sum_{i=1}^{m}\|\dot{u}_i(r)\| + |\dot{b}_i(r)|dr \tag{3.25}$$

For each index $i = 1,\ldots,m$, introduce the pair $(u_i',b_i') : [0,\gamma(T)] \to H \times \mathbb{R}$ by

$$\left(u_i',b_i'\right)(\tau) := \left(u_i,b_i\right)\left(\gamma^{-1}(\tau)\right), \quad \tau \in [0,\gamma(T)].$$

Then we readily have the relationship

$$C_{(u',b')}(\tau) = C_{(u,b)}(\gamma^{-1}(\tau)), \quad \tau \in [0,\gamma(T)]. \tag{3.26}$$

Since $\gamma^{-1}(0) = 0$, it follows from (3.26) that $x_0 \in C_{(u,b)}(0) = C_{(u',b')}(0)$. Therefore, the sweeping process

$$\left(\mathcal{S}'_{(u',b')}\right): \qquad -\dot{x}(\tau) \in N_{C_{(u',b')}(\tau)}\left(x(\tau)\right) \quad \text{a.e. } \tau \in [0, \gamma(T)], \quad x(0) = x_0$$

is exactly of type $\left(\mathcal{S}_{(u,b)}\right)$ as in (1.2). Furthermore, (3.24) yields

$$\|u'_i(\tau_1) - u'_i(\tau_2)\| + |b'_i(\tau_1) - b'_i(\tau_2)| \le |\tau_1 - \tau_2| \quad \forall \tau_1, \tau_2 \in [0, \gamma(T)] \;\; \forall i = 1, \ldots, m,$$

which tells us that the control (u', b') is Lipschitz continuous on the interval $[0, \gamma(T)]$. Observe also that $C_{(u',b')}$ satisfies the uniform Slater condition (2.11) on this interval since $C_{(u,b)}$ does so on the original interval $[0, T]$). This allows us to invoke Theorem 3.1, applied now to the control (u', b'), and conclude that the modified sweeping process $\left(\mathcal{S}'_{(u',b')}\right)$ admits a unique Lipschitzian solution $y(\cdot)$ with some modulus K. For all $t \in [0, T]$, set $z\,(t) := y\left(\gamma(t)\right)$, which implies that $\dot{z}\,(t) := \dot{y}\left(\gamma(t)\right)\dot{\gamma}(t)$ for a.e. $t \in [0, T]$. Hence

$$\|\dot{z}\,(t)\| \le \|\dot{y}\left(\gamma(t)\right)\|\,\dot{\gamma}(t) \le K\dot{\gamma}(t) \quad \text{a.e. } \; t \in [0, T].$$

Since $y(\cdot)$ is a solution to $\left(\mathcal{S}'_{(u',b')}\right)$ while $\dot{\gamma}(t) > 0$ for a.e. $t \in [0, T]$, we get by using (3.26) that

$$\begin{aligned} -\dot{z}\,(t) = \dot{y}\left(\gamma(t)\right)\dot{\gamma}(t) &\in \dot{\gamma}(t) N_{C_{(u',b')}(\gamma(t))}\left(y(\gamma(t))\right) = N_{C_{(u',b')}(\gamma(t))}\left(z(t)\right) \\ &= N_{C_{(u,b)}(t)}\left(z(t)\right) \text{ a.e. } t \in [0.T]. \end{aligned}$$

It follows from (3.25) that $\gamma \in W^{1,1}\left([0, T], \mathbb{R}\right)$, and so $z \in W^{1,1}([0, T], H)$ as well. Furthermore, we have that $z\,(0) = y\left(\gamma(0)\right) = y(0) = x_0$ because $y(\cdot)$ is a solution of $\left(\mathcal{S}'_{(u',b')}\right)$. This allows us to conclude that $z(\cdot)$ is a solution of the original sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ and—being absolutely continuous on $[0, T]$—it is unique by [20, Theorem 3]. $\square$

Finally in this section, we present a consequence of Theorem 3.2 ensuring the result of this type for the $\delta-$ moving polyhedron (2.13). This result is important to our applications to stability in the next section.

Corollary 3.1 *Let $\mathcal{H}$ be a separable Hilbert space, and let the uniform Slater condition* (2.11) *be satisfied along a given control $\left(\bar{u}, \bar{b}\right) \in W^{1,1}([0, T], \mathcal{H}^m) \times W^{1,1}([0, T], \mathbb{R}^m)$. Then there exists $\varepsilon > 0$ such that for all numbers $\delta \in [0, \varepsilon)$ the perturbed sweeping process*

$$-\dot{x} \in N(C^{(\delta)}_{\left(\bar{u},\bar{b}\right)}(t), x(t)) \quad \text{a.e. } t \in [0, T], \quad x(0) = \widehat{x}\,(0) \in C^{(\delta)}_{\left(\bar{u},\bar{b}\right)}(0) \tag{3.27}$$

admits a unique absolutely continuous solution. Here $C^{(\delta)}_{\left(\bar{u},\bar{b}\right)}$ is defined in (2.13) *and $\widehat{x}(\cdot)$ is the continuous selection $\widehat{x}(t) \in C^{(\delta)}_{\left(\bar{u},\bar{b}\right)}(t)$ taken from* (2.14).

Proof. As in the proof of Lemma 2.1, choose $\varepsilon > 0$ from (2.12) and pick $\delta \in [0, \varepsilon)$. Then $C_{\left(\bar{u},\widetilde{b}\right)} = C^{(\delta)}_{\left(\bar{u},\bar{b}\right)}$, with $\widetilde{b}$ defined by $\widetilde{b}_i := b_i - \delta$ as $i = 1, \ldots, m$, also satisfies the uniform Slater condition. The result now follows from Theorem 3.2. $\square$

4 Quantitative stability of the perturbed sweeping dynamics

In this section, we investigate the stability of solutions to controlled polyhedral sweeping processes with respect to perturbations of controls and initial values of the sweeping dynamics. Theorem 3.2 allows us to associate with each absolutely continuous control (u,b) satisfying (2.11) and with the initial value $x(0)=x_0\in C_{(u,b)}(0)$ the unique absolutely continuous solution $x_{(u,b)}$ of the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$. In contrast with the previous analysis, where the initial point x_0 was fixed, we now compare solutions of $\left(\mathcal{S}_{(u,b)}\right)$ corresponding not only to different controls but also to different initial points. To emphasize this dependence, let us write $\left(\mathcal{S}_{(u,b,x_0)}\right)$ for the sweeping process $\left(\mathcal{S}_{(u,b)}\right)$ corresponding to the initial condition $x(0)=x_0\in C_{(u,b)}(0)$ and denote its unique solution by $x_{(u,b,x_0)}$. We begin with the following estimate, which is based on Lemma 2.1 and uses the arguments from the proof of Proposition 3 in [17].

Lemma 4.1 *Assume that $\mathcal{H}$ is a separable Hilbert space, and that the uniform Slater condition* (2.11) *holds for some given control $(\bar u,\bar b)\in W^{1,1}([0,T],\mathcal{H}^m)\times W^{1,1}([0,T],\mathbb{R}^m)$. Then there exists $\varepsilon>0$ such that for all $\delta\in(0,\varepsilon)$, for all controls $(u,b)\in\mathbb{B}_{1,1}\left((\bar u,\bar b),\frac{\delta}{1+\|y_\delta\|_\infty}\right)$, and for all corresponding solutions $x(\cdot)$ to the sweeping processes $\left(\mathcal{S}_{(u,b,x_0)}\right)$ we have the estimate*

$$\|\dot x(t)\|\le\frac{1}{\delta}\left(\|\widehat{x}\|_\infty+\|y_\delta\|_\infty+\alpha_\delta\right)\left(1+\|y_\delta\|_\infty+\alpha_\delta\right)\sum_{i=1}^m\left(\|\dot u_i(t)\|+|\dot b_i(t)|\right)$$
$$\text{a.e. } t\in[0,T]. \tag{4.28}$$

Here $\widehat{x}(\cdot)$ stands for the continuous selection $\widehat{x}(t)\in C^{(\delta)}_{(\bar u,\bar b)}(t)$ taken from (2.14)*, $y_\delta(\cdot)$ refers to the associate unique solution of the perturbed sweeping process* (3.27) *guaranteed by Corollary* 3.1*, and the constant α_δ is defined by*

$$\alpha_\delta:=\int_0^T\|\dot y_\delta(t)\|dt+\sqrt{\left(\int_0^T\|\dot y_\delta(t)\|dt\right)^2+\|x(0)-\widehat{x}\,(0)\,\|^2}. \tag{4.29}$$

Proof. As in previous proofs, we choose $\varepsilon>0$ from perturbed uniform Slater condition (2.12) equivalent to the the assumed one (2.11) by Proposition 2.1. Fix an arbitrary $\delta\in(0,\varepsilon)$, then fix an arbitrary control pair

$$(u,b)\in\mathbb{B}_{1,1}\left((\bar u,\bar b),\frac{\delta}{1+\|y_\delta\|_\infty}\right), \tag{4.30}$$

and denote by $x(\cdot)$ the corresponding unique solution of the sweeping process $\left(\mathcal{S}_{(u,b,x_0)}\right)$ due to Theorem 3.2. By the absolute continuity of the triple (u,b,x), the derivatives $\dot x(t)$, $\dot u_i(t)$ and $\dot b_i(t)$ exist for almost all $t\in[0,1]$. Fixing now any such time t and then get

$$\begin{aligned}&x(t-s)=x(t)-s(\dot x(t)+\alpha_x(s)),\quad u_i(t-s)=u_i(t)-s(\dot u_i(t)+\alpha_{u,i}(s))\\&b_i(t-s)=b_i(t)-s(\dot b_i(t)+\alpha_{b,i}(s)),\end{aligned}$$

where $\lim_{s\to0}\alpha_x(s)=0,\lim_{s\to0}\alpha_{u,i}(s)=0$ and $\lim_{s\to0}\alpha_{b,i}(s)=0$. Since $x(t-s)\in C_{(u,b)}(t-s)$ for all s, we deduce from (2.16) that

$$\begin{aligned}&x(t-s)\in C_{(u,b)}(t)+\\&\frac{1}{\delta}\|x(t-s)-\widehat{x}(t)\|\sum_{i=1}^m\left(\|u_i(t-s)-u_i(t)\|\cdot\|x(t-s)\|+|b_i(t-s)-b_i(t))\right|\mathbb{B},\end{aligned}$$

where $\mathbb{B}$ refers as usual to the unit ball in $\mathcal{H}$. Using the convexity of the $C_{(u,b)}(t)$ and passing to the limit $s\downarrow0$, gives us the inclusion

$$-\dot x(t)\in T(C_{(u,b)}(t),x(t))+\frac{1}{\delta}\|x(t)-\widehat{x}(t)\|\sum_{i=1}^m\left(\|\dot u_i(t)\|\cdot\|x(t)\|+|\dot b_i(t)|\right)\mathbb{B},$$

where $T(S,u)$ stands for the tangent cone to a convex set S at u in the sense of convex analysis. As $-\dot{x}(t)\in N(C_{(u,b)}(t),x(t))$, we arrive at

$$\|\dot{x}(t)\|^2\le\|\dot{x}(t)\|\cdot\frac{1}{\delta}\|x(t)-\widehat{x}(t)\|\sum_{i=1}^{m}\left(\|\dot{u}_i(t)\|\cdot\|x(t)\|+|\dot{b}_i(t)|\right),$$

which in turn implies, since t was arbitrarily chosen from a subset of full measure on $[0,T]$, the derivative norm estimate

$$\|\dot{x}(t)\|\le\frac{1}{\delta}\|x(t)-\widehat{x}\,(t)\,\|\sum_{i=1}^{m}\left(\|\dot{u}_i(t)\|\cdot\|x(t)\|+|\dot{b}_i(t)|\right)\quad\text{a.e.}\,t\in[0,T]. \tag{4.31}$$

To proceed further, let $y_\delta(\cdot)$ be the unique absolutely continuous solution to the sweeping process (3.27) according to Corollary 3.1. Since $\langle\bar{u}_i\,(t)\,,y_\delta(t)\rangle\le\bar{b}_i\,(t)-\delta$ for all $t\in[0,T]$ and all $i=1,\ldots,m$, we deduce from (4.30) that

$$\begin{array}{rcl}\langle u_i(t),y_\delta(t)\rangle-b_i(t)&\le&\langle u_i(t)-\bar{u}_i(t),y_\delta(t)\rangle+\bar{b}_i(t)-b_i(t)-\delta\\&\le&\|u-\bar{u}\|_\infty\|y_\delta\|_\infty+\|b-\bar{b}\|_\infty-\delta\\&\le&\|u-\bar{u}\|_{1,1}\|y_\delta\|_\infty+\|b-\bar{b}\|_{1,1}-\delta\\&\le&\|(u,b)-(\bar{u},\bar{b})\|_{1,1}(1+\|y_\delta\|_\infty)-\delta\le0\quad\forall t\in[0,T].\end{array}$$

Therefore, $y_\delta(t)\in C_{(u,b)}(t)$ for all $t\in[0,T]$. Remembering that $x(\cdot)$ solves the original sweeping process $\left(\mathcal{S}_{(u,b,x_0)}\right)$, it follows that $-\dot{x}(t)\in N_{C_{(u,b)}(t)}\,(x(t))$ for a.e. $t\in[0,T]$, and hence we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|x(t)-y_\delta(t)\|^2&=\langle\dot{x}(t)-\dot{y}_\delta(t),x(t)-y_\delta(t)\rangle\\&=\langle\dot{x}(t),x(t)-y_\delta(t)\rangle+\langle-\dot{y}_\delta(t),x(t)-y_\delta(t)\rangle\\&\le\langle-\dot{y}_\delta(t),x(t)-y_\delta(t)\rangle\le\|\dot{y}_\delta(t)\|\cdot\|x(t)-y_\delta(t)\|_\infty.\end{aligned}$$

This brings us to the estimate

$$\frac{\|x(t)-y_\delta(t)\|^2}{2}-\frac{\|x(0)-\widehat{x}\,(0)\,\|^2}{2}\le\|x-y_\delta\|_\infty\cdot\int_0^T\|\dot{y}_\delta(t)\|dt\quad\forall t\in[0,T],$$

which implies on turn that

$$\frac{\|x-y_\delta\|_\infty^2}{2}-\frac{\|x(0)-\widehat{x}\,(0)\,\|^2}{2}\le\|x-y_\delta\|_\infty\cdot\int_0^T\|\dot{y}_\delta(t)\|dt.$$

Consequently, we arrive at the inequality

$$\|x-y_\delta\|_\infty^2-2\left(\int_0^T\|\dot{y}_\delta(t)\|dt\right)\|x-y_\delta\|_\infty-\|x(0)-\widehat{x}\,(0)\,\|^2\le0.$$

Invoking the definition of α_δ in (4.29) gives us the estimate

$$\|x-y_\delta\|_\infty\le\alpha_\delta, \tag{4.32}$$

which being combined with (4.31) verifies the claimed inequality (4.28) and thus completes the proof of the lemma. □

Now we are ready to establish the main stability result.

Theorem 4.1 *Let $\mathcal{H}$ be a separable Hilbert space, and let the uniform Slater condition* (2.11) *hold for a given control pair $(\bar{u},\bar{b}) \in W^{1,1}([0,T],\mathcal{H}^m) \times W^{1,1}([0,T],\mathbb{R}^m)$. Then there exist a number $\rho > 0$ and a continuous function $K : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that for all control pairs*

$$(u,b),(u',b') \in \left[W^{1,1}([0,T],\mathcal{H}^m) \times W^{1,1}([0,T],\mathbb{R}^m)\right] \cap \mathbb{B}_{1,1}\left((\bar{u},\bar{b}),\rho\right), \tag{4.33}$$

for all initial values $x_0 \in C_{(u,b)}(0)$, $x_0' \in C_{(u',b')}(0)$, and the associated solutions x, x' to the sweeping processes $\left(\mathcal{S}_{(u,b,x_0)}\right)$ and $\left(\mathcal{S}_{(u',b',x_0')}\right)$, respectively, we have

$$\|x(t) - x'(t)\|^2 \le \|x_0 - x_0'\|^2 + K(x_0,x_0')\|(u-u',b-b')\|_\infty \quad \forall t \in [0,T]. \tag{4.34}$$

Proof. As above, we employ the equivalent description (2.12) of the uniform Slater condition (2.11) and take $\varepsilon > 0$ from Proposition 2.1. Fixing an arbitrary number $\delta \in (0,\varepsilon)$, define the quantity

$$\rho := \min\left\{\frac{\delta}{1+\|y_\delta\|_\infty}, \frac{\varepsilon-\delta}{3\left(1+\|\widehat{x}\|_\infty\right)}\right\}, \tag{4.35}$$

where $\widehat{x}(\cdot)$ is the continuous selection $\widehat{x}(t) \in C^{(\delta)}_{(\bar{u},\bar{b})}(t)$ satisfying (2.14), and where $y_\delta(\cdot)$ is the unique absolutely continuous solution to the perturbed sweeping process (3.27) taken from Corollary 3.1. Select arbitrary controls $(u,b),(u',b')$ from (4.33), arbitrary initial values $x_0 \in C_{(u,b)}(0)$, $x_0' \in C_{(u',b')}(0)$, and the associated solutions x, x' to the sweeping processes $\left(\mathcal{S}_{(u,b,x_0)}\right)$ and $\left(\mathcal{S}_{(u',b',x_0')}\right)$, respectively. Then it follows from (4.32) that

$$\|x - y_\delta\|_\infty \le \alpha_\delta \text{ and } \|x' - y_\delta\|_\infty \le \alpha_\delta' \tag{4.36}$$

for α_δ defined in (4.29) and α_δ' defined by the same formula with the initial value $x(0) = x_0$ replaced by the initial value $x'(0) = x_0'$. Lemma 4.1 gives us estimate (4.28) for the control (u,b) as well as the corresponding estimate

$$\|\dot{x}'(t)\| \le \delta^{-1}\left(\|\widehat{x}\|_\infty + \|y_\delta\|_\infty + \alpha_\delta'\right)\left(1 + \|y_\delta\|_\infty + \alpha_\delta'\right)\sum_{i=1}^m \left(\|\dot{u}_i'(t)\| + |\dot{b}_i'(t)|\right)$$
$$\text{a.e. } t \in [0,T] \tag{4.37}$$

for the control (u',b'). Denoting now

$$\begin{aligned} C &:= \left(\alpha_\delta + \|y_\delta\|_\infty + \|\widehat{x}\|_\infty\right)\left(1 + \alpha_\delta + \|y_\delta\|_\infty\right), \\ C' &:= \left(\alpha_\delta' + \|y_\delta\|_\infty + \|\widehat{x}\|_\infty\right)\left(1 + \alpha_\delta' + \|y_\delta\|_\infty\right) \end{aligned} \tag{4.38}$$

and integrating (4.28) ensure that

$$\int_0^t \|\dot{x}(s)\|ds \le \delta^{-1}C\sum_{i=1}^m \int_0^t \left(\|\dot{u}_i(s)\| + |\dot{b}_i(s)|\right)ds \quad \forall\, t \in [0,T].$$

Therefore, recalling that $(u,b) \in \mathbb{B}_{1,1}\left((\bar{u},\bar{b}),\rho\right)$ yields

$$\int_0^t \|\dot{x}(s)\|ds \le \delta^{-1}C\,\|(u,b)\|_{1,1} \le \delta^{-1}C\left(\rho + \left\|(\bar{u},\bar{b})\right\|_{1,1}\right). \tag{4.39}$$

Similarly, the integration of (4.37) gives us

$$\int_0^t \|\dot{x}'(s)\|ds \le \delta^{-1}C'\left(\rho + \left\|(\bar{u},\bar{b})\right\|_{1,1}\right). \tag{4.40}$$

Let now $t \in [0,T]$ be from a subset of full measure such that $\dot{x}(t)$ and $\dot{x}'(t)$ exist. We clearly have $x(t) \in C_{(u,b)}(t)$ and $x'(t) \in C_{(u',b')}(t)$. Since a ball in the $\|\cdot\|_{1,1}$-norm is contained in a ball of the same radius in the $\|\cdot\|_\infty$-norm, the construction of ρ in (4.35) allows us to employ the error bound (2.16) from Lemma 2.1. This ensures the existence of $x_1 \in C_{(u',b')}(t)$ and $x_1' \in C_{(u,b)}(t)$ with

$$\begin{aligned}
&\|x(t) - x_1\| \le \delta^{-1}\|x(t) - \widehat{x}(t)\| \max_{i=1,\ldots,m} [\langle u_i'(t) - u_i(t), x\rangle + b_i(t) - b_i'(t)]_+ \\
&\le \delta^{-1}\|x(t) - \widehat{x}(t)\| \left(\|u(t) - u'(t)\| \, \|x(t)\| + \|b(t) - b'(t)\|\right) \\
&\le \delta^{-1}\|x(t) - \widehat{x}(t)\|(1 + \|x(t)\|)(\|(u(t) - u'(t), b(t) - b'(t))\|.
\end{aligned}$$

Similar considerations bring us to the estimate

$$\|x'(t) - x_1'\| \le \delta^{-1}\|x'(t) - \widehat{x}(t)\|(1 + \|x'(t)\|)(\|(u(t) - u'(t), b(t) - b'(t))\|.$$

Since $x(\cdot)$ and $x'(\cdot)$ are absolutely continuous solutions to $\left(\mathcal{S}_{(u,b,x_0)}\right)$ and $\left(\mathcal{S}_{\left(u',b',x_0'\right)}\right)$, respectively, we deduce from $-\dot{x}(t) \in N_{C_{(u,b)}(t)}\left(x(t)\right)$, $-\dot{x}'(t) \in N_{C_{(u',b')}(t)}\left(x'(t)\right)$, and the obtained estimates of $\|x(t) - x_1\|$ and $\|x'(t) - x_1'\|$ that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left\|x(t) - x'(t)\right\|^2 \\
= \;& \langle \dot{x}(t) - \dot{x}'(t), x(t) - x'(t)\rangle = \langle \dot{x}(t), x(t) - x'(t)\rangle - \langle \dot{x}'(t), x(t) - x'(t)\rangle \\
= \;& \langle \dot{x}(t), x(t) - x_1'\rangle + \langle \dot{x}(t), x_1' - x'(t)\rangle + \langle \dot{x}'(t), x'(t) - x_1\rangle + \langle \dot{x}(t), x_1 - x(t)\rangle \\
\le \;& \langle \dot{x}(t), x_1' - x'(t)\rangle + \langle \dot{x}(t), x_1 - x(t)\rangle \\
\le \;& \|\dot{x}(t)\|\|x_1' - x'(t)\| + \|\dot{x}'(t)\|\|x_1 - x(t)\| \\
\le \;& \delta^{-1}\left(\|\dot{x}(t)\|\|x'(t) - \widehat{x}(t)\|(1 + \|x'(t)\|) + \|\dot{x}'(t)\|\|x(t) - \widehat{x}(t)\|(1 + \|x(t)\|)\right) \\
& \cdot\|(u(t) - u'(t), b(t) - b'(t))\|.
\end{aligned}$$

For all $t \in [0,T]$ define the function

$$\chi\left(t\right) := \delta^{-1}\left(\|x'(t) - \widehat{x}(t)\|(1 + \|x'(t)\|) + \|x(t) - \widehat{x}(t)\|(1 + \|x(t)\|)\right).$$

Then the latter estimate can be rewritten as

$$\frac{d}{dt}\frac{1}{2}\left\|x(t) - x'(t)\right\|^2 \le \chi\left(t\right)\left(\|\dot{x}(t)\| + \|\dot{x}'(t)\|\right)\left\|(u(t) - u'(t), b(t) - b'(t))\right\|. \tag{4.41}$$

It follows from (4.36) and (4.38) that $\chi\left(t\right) \le \delta^{-1}\left(C + C'\right)$. As t was arbitrarily chosen from a subset of full measure of $[0,T]$, we integrate (4.41) and then employ (4.39) and (4.40) to get

$$\begin{aligned}
& \left\|x(t) - x'(t)\right\|^2 - \left\|x(0) - x'(0)\right\|^2 \\
\le \;& \delta^{-1}\left(C + C'\right)\int_0^t \left(\|\dot{x}(s)\| + \|\dot{x}'(s)\|\right)\left\|(u(s) - u'(s), b(s) - b'(s))\right\| ds \\
\le \;& \delta^{-1}\left(C + C'\right)\left\|(u - u', b - b')\right\|_\infty \int_0^t \left(\|\dot{x}(s)\| + \|\dot{x}'(s)\|\right) ds \\
\le \;& \underbrace{\delta^{-2}\left(C + C'\right)^2\left(\rho + \left\|\left(\bar{u}, \bar{b}\right)\right\|_{1,1}\right)}_{K(x_0, x_0')}\left\|(u - u', b - b')\right\|_\infty
\end{aligned}$$

for all $t \in [0,T]$. By (4.38), C and C' depend continuously on α_δ and α_δ', respectively, which in turn depend continuously on x_0 and x_0' by (4.29)). Thus we verify that the obtained continuous function $K(x_0, x\prime_0)$ ensures the claimed estimate (4.34), and we are done with the proof of the theorem. $\square$

To conclude this section, we present a direct consequence of Theorem 4.1 for the case where the initial value x_0 in (1.2) is fixed. In this case the function $K(\cdot)$ in the estimate (4.34) is constant.

Corollary 4.1 *Let $\mathcal{H}$ be a separable Hilbert space, let the uniform Slater condition* (2.11) *hold for a given control $(\bar{u},\bar{b}) \in W^{1,1}([0,T],\mathcal{H}^m) \times W^{1,1}([0,T],\mathbb{R}^m)$, and let $x_0 \in C_{(\bar{u},\bar{b})}$ be an arbitrarily given initial value in* (1.2)*. Then there exist positive numbers ρ and K such that for all controls $(u,b),(u',b')$ satisfying* (4.33) *and the corresponding solutions $x(\cdot)$ and $x'(\cdot)$ of the sweeping processes $\big(\mathcal{S}_{(u,b,x_0)}\big)$ and $\big(\mathcal{S}_{(u',b',x_0)}\big)$ with $x_0 \in C_{(u,b)}(0) \cap C_{(u',b')}(0)$, respectively, we have*

$$\|x(t)-x'(t)\|^2 \le K\|(u-u',b-b')\|_\infty \quad \forall t \in [0,T].$$

5 Discrete approximations of controlled sweeping processes

The last two sections of the paper are devoted to the study of the following *optimal control problem* for the sweeping process (1.2) with controls in polyhedral moving sets (1.1) and additional *endpoint constraints* as well as the *pointwise equality constraints* on the u*-control* functions:

$$\min\big\{f(u,b)|(u,b) \in W^{1,2}([0,T],\mathbb{R}^{nm}\times\mathbb{R}^m),\ \|u_i(t)\|=1\ (i=1,\ldots,m) \\ x_{(u,b)}(T) \in \Omega\big\}, \tag{P}$$

where $\Omega \subseteq \mathbb{R}^n$ is a closed subset, f is a cost function (specified later on), and $x_{(u,b)}$ is the unique trajectory of the polyhedral sweeping process $\big(\mathcal{S}_{(u,b)}\big)$ from (1.2) generated by a control pair $(u,b) = (u(\cdot),b(\cdot))$ of the above class on $[0,T]$. Such a control pair (u,b) is called a *feasible solution* to (P) if $\|u(t)\|$=1 for all $t \in [0,T]$ and $x_{(u,b)}(T) \in \Omega$ for the corresponding trajectory of (1.2). Note that our focus in what follows is on *Lipschitzian* controls in (P), which uniquely generate by Theorem 3.1 Lipschitzian sweeping trajectory under the imposed *uniform Slater condition* (2.11). At the current stage of our developments for (P), we have to restrict ourselves to the case of finite-dimensional state spaces.

Our main goal here is to develop the *method of discrete approximations* to investigate the sweeping control problem (P) and its discrete counterparts from both viewpoints of *stability* and deriving *necessary suboptimality* and *optimality conditions*. Stability issues address the construction of finite-difference approximations of sweeping differential inclusions such that their feasible solutions strongly approximate a broad class of *canonical controls* in the original sweeping process; this notion is introduced in the paper for the *first time*. Furthermore, we construct a sequence of discrete-time optimal control problems (P_k) always admitting optimal solutions, which $W^{1,2}$*-strongly converge* to a prescribed *local minimizer* of the *intermediate* class (between weak and strong, including the latter) of the original sweeping control problem (P). This opens the door to derive *necessary optimality conditions* for such minimizers of (P) by using advanced tools of variational analysis and (first-order and second-order) generalized differentiation. Furnishing this approach, we concentrate here on deriving necessary optimality conditions for problems (P_k) with the approximation number $k \in I\!N$ being sufficiently large. The obtained necessary optimality conditions for (P_k) serve as constructive *suboptimality* conditions for intermediate local minimizers of (P) that are convenient for numerical implementations. This is a *clear advantage* of the method of discrete approximations in comparison with other methods of deriving necessary optimality conditions for continuous-time variational and control problems. In our separate publication, we are going to realize the involved limiting procedure of passing to the limit from the obtained necessary optimality conditions for (P_k) (i.e., suboptimality conditions for (P)) to derive *exact* necessary optimality conditions for intermediate local minimizers of continuous-time sweeping control problems of type (P).

The method of discrete approximations was developed in [21, 22] to establish necessary suboptimality and optimality conditions for *Lipschitzian* differential inclusions. Sweeping differential inclusions are highly *discontinuous*, and the machinery of Lipschitzian variational analysis is not applicable in the sweeping framework. Further developments of this method in various sweeping control settings can be found in [2, 9, 8, 10, 13, 14] and the references therein. However, neither these publications, nor those of [5, 15, 35] exploring other approaches to deriving optimality conditions in different models of sweeping optimal control address additional endpoint constraints $x(T) \in \Omega$ on sweeping trajectories.

In this section we focus on the construction of discrete approximations for the constrained sweeping dynamics and local minimizers of (P) with obtaining stability/convergence results, while the next section is devoted to reviewing the required tools of generalized differentiation and their applications to necessary optimality conditions for discrete approximation problems (P_k) giving us suboptimality conditions for intermediate local minimizers of (P).

Let us start with introducing a new notion of *canonical controls* for problem (P) that plays a crucial role in our developments.

Definition 5.1 *We say that a control pair $(u,b) \in W^{1,2}([0,T],\mathbb{R}^{nm} \times \mathbb{R})$ is* CANONICAL *for problem (P) if the following conditions hold:*
- *The functions $u(\cdot)$ and $b(\cdot))$ are Lipschitz continuous on $[0,T]$.*
- *The uniform Slater condition* (2.11) *is satisfied along (u,b).*
- *We have the constraints*

$$\|u_i(t)\| = 1 \ \text{ for all } \ t \in [0,T] \ \text{ and } \ i = 1,\ldots,m.$$

- *The derivatives $\dot{u}(\cdot)$ and $\dot{b}(\cdot)$ are of bounded variation (BV) on $[0,T]$ together with the derivative of the unique Lipschitz continuous trajectory $x(\cdot)$ of* (1.2) *generated by the control pair (u,b).*

Observe that the corresponding trajectory to (1.2) generated by a canonical control pair *may not* satisfy the endpoint constraint $x_{(u,b)}(T) \in \Omega$, i.e., not any canonical pair is feasible for (P).

To proceed with our approach, we construct a sequence of discrete approximations of the sweeping process $(\mathcal{S}_{(u,b)})$ from (1.2) over the controlled polyhedron (1.1) *without any appeal to optimization* as in (P). For each $k \in \mathbb{N}$ define the discrete mesh on $[0,T]$ by

$$\Delta_k := \big\{0 = t_0^k < t_1^k < \ldots < t_{\nu(k)-1}^k < t_{\nu(k)}^k = T\big\} \tag{5.1}$$

with $h_j^k := t_{j+1}^k - t_j^k \downarrow 0$, $j = 0,\ldots,\nu(k)-1$, as $k \to \infty$. Denote

$$F(u,b,x) := N_{C(u,b)},\ C(u,b) := \big\{x \in \mathbb{R}^n \ \big|\ \langle u_i, x\rangle \le b_i \ \ (i=1,\ldots,m)\big\}. \tag{5.2}$$

The following theorem tells us that *any canonical* control pair (u,b) and the corresponding sweeping trajectory $x(\cdot)$ can be $W^{1,2}$*-strongly* approximated by feasible solutions to discrete sweeping processes defined on the partition Δ_k from (5.1) and appropriately extended to the continuous-time interval $[0,T]$.

Theorem 5.1 *Let $\big(\bar{u}(\cdot),\bar{b}(\cdot)\big)$ be a canonical control pair for (P), and let $\bar{x}(\cdot)$ be the corresponding unique solution of the Cauchy problem in* (1.2)*. Then there exist a mesh Δ_k in* (5.1)*, a sequence of piecewise linear functions $(\widetilde{u}^k(\cdot),\widetilde{b}^k(\cdot),\widetilde{x}^k(\cdot))$ on $[0,T]$, and a sequence of positive numbers $\delta_k \downarrow 0$ as $k \to \infty$ such that $(\widetilde{u}^k(0),\widetilde{b}^k(0),\widetilde{x}^k(0)) = (\bar{u}(0),\bar{b}(0),x_0)$,*

$$1 - \delta_k \le \big\|\widetilde{u}_i^k(t_j^k)\big\| \le 1 + \delta_k \ \text{ for all } \ t_j^k \in \Delta_k, \quad i = 1,\ldots,m, \tag{5.3}$$

$$\widetilde{x}^k(t) = \widetilde{x}^k(t_j^k) + (t - t_j^k)\widetilde{v}_j^k, \ \ t_j^k \le t \le t_{j+1}^k \ \text{ with } \ -\widetilde{v}_j^k \in F\big(\widetilde{u}^k(t_j^k),\widetilde{b}^k(t_j^k),\widetilde{x}^k(t_j^k)\big)$$

for $j = 0,\ldots,\nu(k)-1$, $k \in \mathbb{N}$, and the sequence $\{(\widetilde{u}^k(\cdot),\widetilde{b}^k(\cdot),\widetilde{x}^k(\cdot))\}$ converges to $(\bar{u}(\cdot),\bar{b}(\cdot),\bar{x}(\cdot))$ as $k \to \infty$ in the $W^{1,2}$-norm topology on $[0,T]$.

Proof. As mentioned, the existence of the unique Lipschitz continuous trajectory $\bar{x}(\cdot)$ of the Cauchy problem for the polyhedral sweeping process in (1.2) generated by the given canonical control pair $(\bar{u}(\cdot), \bar{b}(\cdot))$ follows from Theorem 3.1. Now we are in a position to deduce the claimed assertions from [9, Theorem 4.1] under the BV assumption on $\dot{\bar{u}}(\cdot)$, $\dot{\bar{b}}(\cdot)$, and $\dot{\bar{x}}(\cdot)$. Indeed, the qualification condition (H4) from [9, Theorem 4.1] is equivalent to the uniform Slater condition (2.11) by our new result obtained in Proposition 2.1. Thus the application of [9, Theorem 4.1] gives us all the assertions claimed in this theorem. $\square$

From now on, we consider for simplicity problem (P), where the cost function is defined in the *Mayer form* via a given terminal state function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(u,b) := \varphi\big(x_{u,b}(T)\big).$$

If the function φ is lower semicontinuous, then problem (P) admits a (global) *optimal solution* in $W^{1,2}([0,T], \mathbb{R}^{nm} \times \mathbb{R}^m)$ provided that there is a bounded minimizing sequence of feasible solutions; see [9, Theorem 3.1] and its proof. Since our main attention is paid to deriving necessary (sub)optimality conditions in (P), it is natural to define an appropriate notion of *local minimizers*.

The notion of local minimizers of our study in this paper occupies an *intermediate* position between the classical notions of *weak* and *strong* minimizers in variational and control problems, while encompassing the latter. Following [21], where this notion was initiated for Lipschitzian differential inclusions (see also [22] for more details), we keep the name “intermediate"for the version of this notion in the setting of our sweeping control problem (P).

Definition 5.2 *We say that a feasible control pair $(\bar{u}, \bar{b})$ for (P) is an* INTERMEDIATE LOCAL MINIMIZER *in this problem if there exists $\varepsilon > 0$ such that*

$$\varphi\big(x_{\bar{u},\bar{b}}(T)\big) \le \varphi\big(x_{u,b}(T)\big)$$

for any feasible solution to (P) satisfying the condition

$$\|(u,b) - (\bar{u}, \bar{b})\|_{W^{1,2}} + \|x_{u,b} - x_{\bar{u},\bar{b}}\|_{W^{1,2}} \le \varepsilon. \tag{5.4}$$

The notion of *strong local minimizer* for (P) is a particular case of Definition 5.2, where the norm $\|x_{u,b} - x_{\bar{u},\bar{b}}\|_{W^{1,2}}$ in (5.4) is replaced with the norm $\|x_{u,b} - x_{\bar{u},\bar{b}}\|_{\mathcal{C}}$ in the space of continuous functions $\mathcal{C}([0,T], \mathbb{R}^n)$. It is not hard to construct examples showing that there exist intermediate local minimizers to (P) that fail to be strong ones; see [21, 22, 34] even for simpler problems.

Having $F(u,b,x)$ from (5.2), fix a Lipschitz continuous intermediate local minimizer $(\bar{u}, \bar{b})$ for (P) with the corresponding sweeping trajectory $\bar{x}(\cdot) := x_{\bar{u},\bar{b}}$ and assume that the uniform Slater condition (2.11) holds along $(\bar{u}, \bar{b})$. Take the mesh Δ_k from (5.1) and identify the points t_j^k with the index j if no confusion arises. Consider now discrete triples (u^k, b^k, x^k) with the components

$$(u^k, b^k, x^k) := (u_0^k, u_1^k, \ldots, u_{\nu(k)}^k, b_0^k, b_1^k, \ldots, b_{\nu(k)}^k, x_0^k, x_1^k, \ldots, x_{\nu(k)}^k)$$

and form the sequence of *discrete approximation* problems (P_k) by:

$$\text{minimize} \quad \varphi\big(x_{\nu(k)}^k\big) + \tag{5.5}$$
$$\frac{1}{2} \sum_{j=0}^{\nu(k)-1} \int_{t_j^k}^{t_{j+1}^k} \left\| \left(\frac{u_{j+1}^k - u_j^k}{h_j^k}, \frac{b_{j+1}^k - b_j^k}{h_j^k}, \frac{x_{j+1}^k - x_j^k}{h_j^k} \right) - \big(\dot{\bar{u}}(t), \dot{\bar{b}}(t), \dot{\bar{x}}(t)\big) \right\|^2 dt$$

DOI 10.20347/WIAS.PREPRINT.2892

over the triples (u^k, b^k, x^k) subject to the following constraints:

$$x^k_{j+1} \in x^k_j - h^k_j F(u^k_j, b^k_j, x^k_j),\ j = 0, \ldots, \nu(k) - 1, \tag{5.6}$$

$$x^k_0 = x_0 \in C_{\bar{u},\bar{b}}(0),\ (u^k_0, b^k_0) = \big(\bar{u}(0), \bar{b}(0)\big),\ x^k_{\nu(k)} \in \Omega + \xi_k I\!B, \tag{5.7}$$

$$1 - \delta_k \le \big\|u^k_i(t^k_j)\big\| \le 1 + \delta_k \ \text{ for all } \ t^k_j \in \Delta_k, \quad i = 1, \ldots, m, \tag{5.8}$$

$$\sum_{j=0}^{\nu(k)-1} \int_{t^k_j}^{t^k_{j+1}} \Big\|\big(u^k_j, b^k_j, x^k_j\big) - \big(\bar{u}(t), \bar{b}(t), \bar{x}(t)\big)\Big\|^2 dt \le \frac{\varepsilon}{2}, \tag{5.9}$$

$$\sum_{j=0}^{\nu(k)-1} \int_{t^k_j}^{t^k_{j+1}} \Bigg\|\bigg(\frac{u^k_{j+1} - u^k_j}{h^k_j}, \frac{b^k_{j+1} - b^k_j}{h^k_j}, \frac{x^k_{j+1} - x^k_j}{h^k_j}\bigg) - \big(\dot{\bar{x}}(t), \dot{\bar{a}}(t), \dot{\bar{b}}(t)\big)\Bigg\|^2 dt \le \frac{\varepsilon}{2}, \tag{5.10}$$

where $\{\delta_k\}$ in (5.8) is taken from Theorem 5.1 applied to $(\bar{u}, \bar{b})$ and can be chosen such that both inequalities in (5.8) are strict, where $\varepsilon > 0$ in (5.9) and (5.10) is taken from Definition 5.2 of the intermediate local minimizer $(\bar{u}, \bar{b})$ for (P), and where the sequence $\{\xi_k\}$ of the endpoint perturbations in (5.7) is defined by

$$\xi_k := \|\widetilde{x}^k(T) - \bar{x}(T)\| \to 0 \ \text{ as } \ k \in \mathbb{N} \tag{5.11}$$

via the sequence $\{\widetilde{x}^k(\cdot)\}$ approximating $\bar{x}(\cdot)$ in Theorem 5.1.

The next theorem establishes the existence of optimal solutions to problems (P_k) for all $k \in \mathbb{N}$ and then shows that any sequence of optimal controls $\{(\bar{u}^k, \bar{b}^k)\}$ to (P_k) constructed for the given *canonical intermediate local minimizer* $(\bar{u}, \bar{b})$ of (P), together with the corresponding sequence of discrete trajectories $\{\bar{x}^k\}$ piecewise linearly extended to the whole interval $[0, T]$, *strongly* $W^{1,2}$-*converge* as $k \to \infty$ to the prescribed local optimal triple $(\bar{u}, \bar{b}, \bar{x})$ for (P).

Theorem 5.2 *Let $(\bar{u}, \bar{b})$ be a canonical intermediate local minimizer for (P) with the corresponding sweeping trajectory $\bar{x}(\cdot)$. The following assertions hold:*

(i) *If the cost function φ is lower semicontinuous around $\bar{x}(T)$, then each problem (P_k) admits an optimal solution whenever $k \in \mathbb{N}$ is sufficiently large.*

(ii) *If in addition φ is continuous around $\bar{x}(T)$, then every sequence of optimal solutions $\{(\bar{u}^k, \bar{b}^k)\}$ to (P_k) and the corresponding sequence of discrete trajectories $\{\bar{x}^k\}$, being piecewise linearly extended to $[0, T]$, converge to $(\bar{u}, \bar{b}, \bar{x})$ as $k \to \infty$ in the norm topology of $W^{1,2}([0, T], \mathbb{R}^{mn} \times \mathbb{R}^m \times \mathbb{R}^n)$.*

Proof. To verify (i), observe first that the set of feasible solutions to problem (P_k) is *nonempty* for all large $k \in \mathbb{N}$. Namely, we show that the approximating sequence $\{(\widetilde{u}^k, \widetilde{b}^k, \widetilde{x}^k)\}$ from Theorem 5.1, being applied to the given *canonical* intermediate local minimizer $(\bar{u}, \bar{b})$ of the original problem (P), consists of feasible solutions to (P_k) when k is sufficiently large. Indeed, the discrete sweeping inclusions (5.6) with the initial data in (5.7) are clearly satisfied for $\{(\widetilde{u}^k, \widetilde{b}^k, \widetilde{x}^k)\}$ together with the control constraints (5.8), the conditions in (5.9) and (5.10) also hold for large k by the $W^{1,2}$-convergence of the extended triples $\{(\widetilde{u}^k(t), \widetilde{b}^k(t), \widetilde{x}^k(t))\}$ to $(\bar{u}(t), \bar{b}(t), \bar{x}(t))$ on $[0, T]$ as $k \to \infty$, and the fulfillment of the endpoint constraint in (5.7) for the approximating trajectories $\widetilde{x}^k(\cdot)$ follows from $\bar{x}(T) \in \Omega$ and the definition of ξ_k in (5.11) by Theorem 5.1 applied to the canonical intermediate local minimizer $(\bar{u}, \bar{b})$. It follows from the construction of (P_k) and the structure of F in (5.2) that the set of feasible solutions to (P_k) is closed. Furthermore, the constraints in (5.8)–(5.10) ensure the boundedness of the latter set. Since φ is assumed to be lower semicontinuous around $\bar{x}(T)$, the existence of optimal solutions to such (P_k) follows from the classical Weierstrass existence theorem in finite dimensions.

Now we verify assertion (ii) of the theorem. Consider an arbitrary sequence $\{(\bar{u}^k(\cdot),\bar{b}^k(\cdot),\bar{x}^k(\cdot))\}$ of optimal controls to (P_k) and the associated trajectories of (5.6) that are piecewise linearly extended to $[0,T]$. We aim at proving

$$\lim_{k\to\infty}\int_0^T \big\|\big(\dot{\bar{u}}^k(t),\dot{\bar{b}}^k(t),\dot{\bar{x}}^k(t)\big)-\big(\dot{\bar{u}}(t),\dot{\bar{b}}(t),\dot{\bar{x}}(t)\big)\big\|^2 dt=0, \tag{5.12}$$

which readily yields the claimed convergence in (ii). Supposing on the contrary that (5.12) fails gives us a subsequence of $k\to\infty$ (no relabeling) along which the limit in (5.12) equals to some $\sigma>0$. Due to (5.10), the sequence $\{(\dot{\bar{u}}^k(t),\dot{\bar{b}}^k(t),\dot{\bar{x}}^k(t))\}$ is weakly compact in $L^2([0,T],\mathbb{R}^{mn}\times\mathbb{R}^m\times\mathbb{R}^n)$, and hence it contains a subsequence that converges to some triple $(\vartheta^u(\cdot),\vartheta^b(\cdot),\vartheta^x(\cdot))\in L^2([0,T],\mathbb{R}^{mn}\times\mathbb{R}^m\times\mathbb{R}^n)$ weakly in this space. Employing Mazur's weak closure theorem tells us that there is a sequence of convex combinations of $(\dot{\bar{u}}^k(\cdot),\dot{\bar{b}}^k(\cdot),\dot{\bar{x}}^k(\cdot))$, which converges to $(\vartheta^u(\cdot),\vartheta^b(\cdot),\vartheta^x(\cdot))$ strongly in $L^2([0,T],\mathbb{R}^{mn}\times\mathbb{R}^m\times\mathbb{R}^n)$, and hence almost everywhere on $[0,T]$ along a subsequence. Define

$$\big(\widehat{u}(t),\widehat{b}(t),\widehat{x}(t)\big):=(\bar{u}(0),\bar{b}(0),x_0)+\int_0^t\big(\vartheta^u(\tau),\vartheta^b(\tau),\vartheta^x(\tau)\big)d\tau\ \text{ for all }\ t\in[0,T]$$

and get that $(\dot{\widehat{u}}(t),\dot{\widehat{b}}(t),\dot{\widehat{x}}(t))=(\vartheta^u(t),\vartheta^b(t),\vartheta^x(t))$ for a.e. $t\in[0,T]$. It follows from the construction of $(\widehat{u}(t),\widehat{b}(t),\widehat{x}(t))$ and the passage to the limit as $k\to\infty$ in (5.7)–(5.10) that $\|\widehat{u}(t)\|=1$ on $[0,T]$, that $\widehat{x}(T)\in\Omega$, and that $(\widehat{u}(t),\widehat{b}(t),\widehat{x}(t))$ belongs to the ε-neighborhood of $(\bar{u}(\cdot),\bar{b}(\cdot),\bar{x}(\cdot))$ in the norm topology of $W^{1,2}([0,T],\mathbb{R}^{mn}\times\mathbb{R}^m\times\mathbb{R}^n)$. Let us now check that the limiting triple $(\widehat{u}(\cdot),\widehat{b}(\cdot),\widehat{x}(\cdot))$ satisfies the sweeping inclusion

$$-\dot{x}(t)\in N_{C_{(u(t),b(t))}}\big(x(t)\big)\ \text{ for a.e. }\ t\in[0,T] \tag{5.13}$$

over the controlled polyhedron. It follows from (5.6) due to (1.1) and (5.2) that

$$\big\langle\bar{u}_i^k(t_j),\bar{x}^k(t_j)\big\rangle\le\bar{b}_i^k(t_j)\ \text{ for all }\ i=1,\ldots,m,\ \text{ all }\ j=0,\ \nu(k)-1,\ \text{ and }\ k\in\mathbb{N}.$$

Passing there to the limit as $k\to\infty$ ensures the conditions

$$\langle\widehat{u}_i(t),\widehat{x}(t)\rangle\le\widehat{b}_i(t)\ \text{ for all }\ i=1,\ldots,m\ \text{ and }\ t\in[0,T], \tag{5.14}$$

i.e., $\widehat{x}(t)\in C_{(\widehat{u}(t),\widehat{b}(t))}$ on $[0,T]$. To proceed further, we use the construction of F in (5.2) and rewrite (5.6) along the optimal triple $(\bar{u}^k,\bar{b}^k,\bar{x}^k)$ for (P_k) as

$$-\frac{\bar{x}^k(t_{j+1})-\bar{x}^k(t_j)}{h_{k_j}}\in N_{C_{(\bar{u}^k(t_j),\bar{b}^k(t_j))}}\big(\bar{x}^k(t_j)\big)\quad(j=0,\ldots,\nu(k)-1,\ k\in\mathbb{N}). \tag{5.15}$$

Recalling the piecewise linear extensions $(\bar{u}^k(t),\bar{b}^k(t),\bar{x}^k(t))$ of the discrete triples $(\bar{u}^k,\bar{b}^k,\bar{x}^k)$ and their strong $W^{1,2}$-convergence to the triple $(\widehat{u}(t),\widehat{b}(t),\widehat{x}(t))$ satisfying (5.14) tells us by passing to the limit in (5.15) as $k\to\infty$ that the sweeping inclusion (5.13) holds for $(\widehat{u}(t),\widehat{b}(t),\widehat{x}(t))$. The verification of the latter involves the usage of the aforementioned Mazur theorem and the outer semicontinuity (closed-graph) property of the convex normal cone (1.3) with respect to pointwise perturbations of the moving polyhedron $C_{(u,b)}$ in (5.13).

All the above shows that the limiting triple $(\widehat{u},\widehat{b},\widehat{x})$ is a feasible solution to problem (P) while satisfying the ε-localization condition (5.4). Passing finally to the limit in (P_k) with taking into account the assumed continuity of φ and remembering the value $\sigma>0$ of the chosen limiting point of the sequence in (5.12), we get that $\varphi(\widehat{x}(T))<\varphi(\bar{x}(T))$. This contradicts the imposed local optimality of $(\bar{u},\bar{b})$ in (P) and hence completes the proof of theorem. □

6 Optimality conditions via discrete approximations

The results of the previous section show that optimal solutions to the finite-dimensional discrete-time problem (P_k) are approximating *suboptimal* solutions to the original sweeping control problem (P) of infinite-dimensional dynamic optimization. Therefore, necessary optimality conditions for solutions to problems (P_k), when $k \in I\!N$ is sufficiently large, can be viewed as (necessary) *suboptimality conditions* for the prescribed intermediate local minimizers of (P). This observation allows us to justify solving the original sweeping control problem by applying appropriate numerical techniques based on necessary optimality conditions for the discrete approximations.

Each discrete-time problem (P_k) can be reduced to a nondynamic problem of *mathematical programming* in finite-dimensional spaces. As we see, problems (P_k) contain constraints of special types, the most challenging of which are given by *increasingly many inclusions* in (5.6) that come from the sweeping dynamics. The latter constraints of the *graphical type* require appropriate tools of generalized differentiation to deal with. In particular, Clarke's nonsmooth analysis cannot be apply here, since his normal cone is usually too large for graphical sets associated with velocity mappings in (1.2) and (5.6). In fact, the only (known to us) machinery of generalized differentiation suitable for these purposes is the one introduced by the third author and then developed by many researchers; see, e.g., the books [22, 23, 29] and the references therein. We now briefly review what is needed in this paper.

Given a set $\Theta \subset \mathbb{R}^n$ locally closed around $\bar{z} \in \Theta$, the (Mordukhovich basic/limiting) *normal cone* to Θ at $\bar{z}$ is defined by

$$
\begin{aligned}
&N(\bar{z};\Theta) = N_\Theta(\bar{z}) := \\
&\big\{v \in \mathbb{R}^n \;\big|\; \exists\, z_k \to \bar{z},\ w_k \in \Pi(z_k;\Omega),\ \alpha_k \geq 0 \ \text{ with } \ \alpha_k(z_k - w_k) \to v\big\},
\end{aligned}
\tag{6.1}
$$

where $\Pi(z;\Theta) := \{w \in \Theta \mid \|z-w\| = d(z,\Theta)\}$ is the Euclidean projector of $z \in \mathbb{R}^n$ onto Θ. While for convex sets Θ the normal cone (6.1) agrees with the classical one (1.3), in general the set of normals (6.1) may be nonconvex even for simple sets as, e.g., the graph of the absolute value function $|\cdot|$ at $\bar{z} = (0,0) \in \mathbb{R}^2$. Nevertheless, the normal cone (6.1) for sets, as well as the coderivatives of set-valued mappings and (first-order and second-order) subdifferentials of extended-real-valued functions generated by (6.1), enjoy *comprehensive calculus rules* that are based on *variational and extremal principles* of variational analysis.

Given further a set-valued mapping $\mathcal{F}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the graph $\operatorname{gph}\mathcal{F} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \mathcal{F}(x)\}$ locally closed around $(\bar{x},\bar{y}) \in \operatorname{gph}\mathcal{F}$, the *coderivative* of $\mathcal{F}$ at $(\bar{x},\bar{y})$ is defined by

$$
D^*\mathcal{F}(\bar{x},\bar{y})(u) := \big\{v \in \mathbb{R}^n \;\big|\; (v,-u) \in N\big((\bar{x},\bar{y});\operatorname{gph}\mathcal{F}\big)\big\}, \quad u \in \mathbb{R}^m. \tag{6.2}
$$

Given finally an extended-real-valued function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty,\infty]$ lower semicontinuous around $\bar{x}$ with $f(\bar{x}) < \infty$ and the epigraph $\operatorname{epi} f := \{(x,\alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq f(x)\}$, the (first-order) *subdifferential* of f at $\bar{x}$ can be defined geometrically via the normal cone (6.1) as

$$
\partial f(\bar{x}) := \big\{v \in \mathbb{R}^n \big|\; (v,-1) \in N\big((\bar{x},f(\bar{x}));\operatorname{epi} f\big)\big\}, \tag{6.3}
$$

while it admits various analytic descriptions that can be found in the aforementioned books. Observe that the normal cone (6.1) is the subdifferential (6.3) of the indicator function $\delta_\Theta(x)$ of Θ, which equals 0 for $x \in \Theta$ and ∞ otherwise. The *second-order subdifferential* of f at $\bar{x}$ relative to $\bar{x} \in \partial f(\bar{x})$ is defined as the coderivative of the first-order subdifferential mapping by

$$
\partial^2 f(\bar{x},\bar{v})(d) := \big(D^*\partial f\big)(\bar{x},\bar{v})(d), \quad d \in \mathbb{R}^n. \tag{6.4}
$$

This construction naturally arises in optimal control of sweeping processes of type (1.2), where the right-hand side is described by the normal cone mapping. We look for second-order evaluations of the coderivative in (6.4) applied to the normal cone mapping F in (5.2) generated by the control-dependent convex polyhedron $C(u,b)$ in the sweeping process (1.2). The result needed in this paper follows from [13, Theorem 4.3], where it was derived by using calculations in [24] and Robinson's theorem of the calmness property of polyhedral multifunctions [28]. To formulate the required result, consider the matrix

$$A := [u_{ij}]\ (i=1,\ldots,m; j=1,\ldots,n)$$

with the vector columns u_i as well as the transpose matrix A^T. As usual, the symbol $^\perp$ indicates the orthogonal complement of a vector in the corresponding space. Having the controlled polyhedron $C(u,b)$ in (5.2), take its *active indices* at (u,b,x) with $x \in C(u,b)$ denoted by

$$I(u,b,x) := \big\{i \in \{1,\ldots,m\}\ \big|\ \langle u_i, x\rangle = b_i\big\}.$$

The *positive linear independence constraint qualification* (PLICQ) at (u,b,x) is

$$\Big[\sum_{i\in I(x,u,b)} \alpha_i u_i = 0,\ \alpha_i \ge 0\Big] \Longrightarrow \big[\alpha_i = 0\ \text{ for all }\ i \in I(x,u,b)\big]. \tag{6.5}$$

This condition is significantly weaker than the classical *linear independence constraint qualification* (LICQ), which corresponds to (6.5) with $\alpha_i \in \mathbb{R}$ while not being used in this paper. Considering the moving polyhedron as in (1.1), it is not hard to check that our basic uniform Slater condition from (2.11) is equivalent to the fulfillment of PLICQ along the feasible triple $(x(t),u(t),b(t))$ for all $t \in [0,T]$; see [9] for more discussions on this topic.

Given $x \in C(u,b)$ and $v \in N(x;C(u,b))$, define the sets

$$Q(p) := \begin{cases} q_i = 0\ \text{ for all }\ i\ \text{ with either }\ \langle u_i,x\rangle < b_i\ \text{ or }\ p_i = 0,\ \text{ or } \langle u_i,y\rangle < 0, \\ q_i \ge 0\ \text{ for all }\ i\ \text{ such that }\ \langle u_i,x\rangle = b_i,\ p_i = 0,\ \text{ and }\ \langle u_i,y\rangle > 0, \end{cases}$$

$$P(y) := \big\{p \in N_{\mathbb{R}^m_-}(Ax-b)\ \big|\ A^T p = v\big\}\ \text{ for }\ y \in \bigcap_{\{i\,|\,p_i>0\}} u_i^\perp,$$

where the normal cone to the nonpositive orthant $\mathbb{R}^m_-$ is easy to compute.

Now we are ready to present the required evaluation of the coderivative of the normal cone mapping $F(x,u,b)$ generated by the controlled polyhedron in (5.2). The following lemma is a slight modification of [13, Theorem 4.3].

Lemma 6.1 *Taking F and $C(u,b)$ from* (5.2)*, suppose that the active vector columns $\{u_i \mid i \in I(u,b,x)\}$ are positively linearly independent for any (u,b,x) with $x \in C(u,b)$. Then for all such (u,b,x), all $v \in N(x;C(u,b))$, and all $y \in \cap_{\{i\,|\,p_i>0\}} u_i^\perp$ we have the coderivative upper estimate*

$$D^*F(u,b,x,v)(y) \subset \bigcup_{\substack{p\in P(y)\\ q\in Q(p)}} \left\{\begin{pmatrix} A^T q \\ \hline p_1 y + q_1 x \\ \vdots \\ p_m y + q_m x \\ \hline -q \end{pmatrix}\right\}. \tag{6.6}$$

Note that imposing the LICQ condition instead of PLICQ ensures that the set $P(y)$ is a singleton and that the inclusion in (6.6) holds as equality; see [13, Theorem 4.3]. However, for the purpose of this paper it is sufficient to have the inclusion in (6.6) under the less restrictive PLICQ.

To proceed further, we need one more auxiliary result giving us necessary optimality conditions for a finite-dimensional nondynamic problem of *mathematical programming* with finitely many equality, inequality and inclusion (geometric) constraints. The next lemma is obtained by combining the necessary optimality conditions from [23, Theorem 6.5] for mathematical programs containing one geometric constraint and the intersection rule for limiting normals taken from [23, Corollary 2.17]. Arguing in this way, we can derive necessary optimality conditions for mathematical programs described by lower semicontinuous cost and inequality constraint functions as well as continuous functions describing equality constraints. However, we confine ourselves to considering problems with just locally Lipschitzian functions for cost and inequality constraints and smooth functions for equality constraints, since only such functional constraints appear in mathematical programs to which we reduce the discrete-time sweeping control problems (P_k).

Lemma 6.2 *Consider the following problem of mathematical programming:*

$$\begin{cases} \text{minimize } \ f_0(z) \ \text{ as } \ z \in \mathbb{R}^d \ \text{ subject to} \\ f_i(z) \le 0 \ \text{ for } \ i = 1, \ldots, s, \\ g_j(z) = 0 \ \text{ for } \ j = 0, \ldots, r, \\ z \in \Theta_j \ \text{ for } \ j = 0, \ldots, l, \end{cases} \tag{MP}$$

where all the functions f_i and g_j are real-valued. Given a local minimizer $\bar{z}$ to (MP), assume that the functions f_i are locally Lipschitzian around $\bar{z}$ for $i = 0, \ldots, s$, the functions g_j are continuously differentiable around this point for $j = 0, \ldots, r$, and the sets Θ_j are locally closed around $\bar{z}$ for all $j = 0, \ldots, l$. Then there exist nonnegative numbers $\lambda_0, \ldots, \lambda_s$, real numbers $\mu_0, \ldots, \mu_r$, and vectors $z_j^* \in \mathbb{R}^d$ for $j = 0, \ldots, l$, not equal to zero simultaneously, such that

$$\lambda_i f_i(\bar{z}) = 0 \ \text{ for } \ i = 1, \ldots, s,$$

$$z_j^* \in N(\bar{z}; \Theta_j) \ \text{ for } \ j = 0, \ldots, l,$$

$$-\sum_{j=0}^{l} z_j^* \in \sum_{i=0}^{s} \lambda_i \partial f_i(\bar{z}) + \sum_{j=0}^{r} \mu_i \nabla g_j(\bar{z}),$$

Having Lemma 6.1 and Lemma 6.2 in hand, we are now in a position to establish necessary conditions for optimal solutions to problems (P_k) from (5.5)–(5.10) whenever the approximation number $k \in I\!N$ is sufficiently large. The obtained relationships involve the given intermediate local minimizer for the sweeping optimal control problem (P) and thus present necessary suboptimality conditions for the original continuous-time problem due to Theorem 5.2. For any $x \in \mathbb{R}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^{nm}$ with $y_i \in \mathbb{R}^n$ $(i = 1, \ldots, m)$, and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ we use the symbols

$$\text{rep}_m(x) := (x, \ldots, x) \in \mathbb{R}^{nm} \ \text{ and } \ [\alpha, y] := (\alpha_1 y_1, \ldots, \alpha_m y_m) \in \mathbb{R}^{nm}.$$

Theorem 6.1 *Let $(\bar{u}, \bar{b})$ be a canonical intermediate local minimizer of (P) generated the trajectory $\bar{x} = \bar{x}(\cdot)$ of the controlled polyhedral sweeping process* (1.2) *such that the cost function φ is locally Lipschitzian around $\bar{x}(T)$. Fix an optimal triple $(\bar{u}^k, \bar{b}^k, \bar{x}^k)$ in problem (P_k) with the components*

$$(\bar{u}^k, \bar{b}^k, \bar{x}^k) := (\bar{u}_0^k, \bar{u}_1^k, \ldots, \bar{u}_{\nu(k)}^k, \bar{b}_0^k, \bar{b}_1^k, \ldots, \bar{b}_{\nu(k)}^k, \bar{x}_0^k, \bar{x}_1^k, \ldots, \bar{x}_{\nu(k)}^k)$$

and choose $k \in I\!N$ to be sufficiently large. Denote the quantities

$$\theta_j^{uk} := \int_{t_j^k}^{t_{j+1}^k} \left(\frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{h_j^k} - \dot{\bar{u}}(t) \right) dt, \quad \theta_j^{bk} := \int_{t_j^k}^{t_{j+1}^k} \left(\frac{\bar{b}_{j+1}^k - \bar{b}_j^k}{h_j^k} - \dot{\bar{b}}(t) \right) dt,$$

$$\theta_j^{xk} := \int_{t_j^k}^{t_{j+1}^k} \left(\frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_j^k} - \dot{\bar{x}}(t) \right) dt$$

and define the set $\Omega_k := \Omega + \xi_k I\!B$, *where* ξ_k *is taken from the construction of problem* (P_k). *Then there exist a multiplier* $\lambda^k \geq 0$, *an adjoint triple* $p_j^k = (p_j^{xk}, p_j^{ak}, p_j^{bk}) \in \mathbb{R}^{n+mn+m}$ $(j = 0, \ldots, \nu(k))$, *as well as vectors* $\eta^k = (\eta_0^k, \ldots, \eta_{\nu(k)}^k) \in \mathbb{R}_+^{m(\nu(k)+1)}$, $\alpha^{1k} = \left(\alpha_0^{1k}, \ldots, \alpha_{\nu(k)}^{1k}\right) \in \mathbb{R}_+^{m(\nu(k)+1)}$, $\alpha^{2k} = (\alpha_0^{2k}, \ldots, \alpha_{\nu(k)}^{2k}) \in \mathbb{R}_+^{m(\nu(k)+1)}$, *and* $\gamma^k = (\gamma_0^k, \ldots, \gamma_{\nu(k)-1}^k) \in \mathbb{R}^{m\nu(k)}$ *such that*

$$\lambda^k + \left\|\alpha^{1k} - \alpha^{2k}\right\| + \left\|\eta_{\nu(k)}^k\right\| + \sum_{j=0}^{\nu(k)-1} \left\|p_j^{xk}\right\| + \left\|p_0^{ak}\right\| + \left\|p_0^{bk}\right\| \neq 0, \tag{6.7}$$

$$\lambda^k + \left\|\alpha^{1k} - \alpha^{2k}\right\| + \left\|\gamma^k\right\| + \left\|p_{\nu(k)}^{ak}\right\| + \left\|p_{\nu(k)}^{bk}\right\| \neq 0, \tag{6.8}$$

and we have the following conditions:

- DYNAMIC RELATIONSHIPS, *which are satisfied for all indices* $j = 0, \ldots, \nu(k)-1$ *and* $i = 1, \ldots, m$:

$$-\frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_j^k} = \sum_{i=1}^m \eta_{ij}^k \bar{u}_{ij}^k, \tag{6.9}$$

$$\begin{aligned} &\frac{p_{j+1}^{uk} - p_j^{uk}}{h_j^k} - \frac{2}{h_j^k}\left[\alpha_j^{1k} - \alpha_j^{2k}, \bar{u}_j^k\right] \\ &= \left[\gamma_j^k, \operatorname{rep}_m(\bar{x}_j^k)\right] + \left[\eta_j^k, \operatorname{rep}_m\left(-\frac{1}{h_j^k}\lambda^k\theta_j^{xk} - \lambda^k + p_{j+1}^{xk}\right)\right], \end{aligned} \tag{6.10}$$

$$\frac{p_{j+1}^{bk} - p_j^{bk}}{h_j^k} = -\gamma_j^k, \tag{6.11}$$

$$\frac{p_{j+1}^{xk} - p_j^{xk}}{h_j^k} = \sum_{i=1}^m \gamma_{ij}^k \bar{u}_{ij}^k, \tag{6.12}$$

where the components of the vectors γ_j^k *are such that*

$$\begin{cases} \gamma_{ij}^k = 0 \ \textit{ if } \ \langle \bar{u}_{ij}^k, \bar{x}_j^k\rangle < \bar{b}_{ij}^k \ \textit{ or } \ \eta_{ij}^k = 0, \ \left\langle \bar{u}_{ij}^k, -\frac{1}{h_j^k}\lambda^k\theta_j^{xk} + p_{j+1}^{xk}\right\rangle < 0, \\ \gamma_{ij}^k \geq 0 \ \textit{ if } \ \langle \bar{u}_{ij}^k, \bar{x}_j^k\rangle = \bar{b}_{ij}^k, \ \eta_{ij}^k = 0, \ \left\langle \bar{u}_{ij}^k, -\frac{1}{h_j^k}\lambda^k\theta_j^{xk} + p_{j+1}^{xk}\right\rangle > 0, \\ \gamma_{ij}^k \in \mathbb{R} \ \textit{ if } \ \eta_{ij}^k > 0, \ \left\langle \bar{u}_{ij}^k, -\frac{1}{h_j^k}\lambda^k\theta_j^{xk} + p_{j+1}^{xk}\right\rangle = 0. \end{cases} \tag{6.13}$$

- COMPLEMENTARY SLACKNESS CONDITIONS*:*

$$\alpha_{ij}^{1k}\left(\left\|u_{ij}^k\right\| - (1+\delta_k)\right) = 0 \quad (i = 1, \ldots, m, \ \ j = 0, \ldots, \nu(k)), \tag{6.14}$$

$$\alpha_{ij}^{2k}\left(\left\|u_{ij}^k\right\| - (1-\delta_k)\right) = 0 \quad (i = 1, \ldots, m, \ \ j = 0, \ldots, \nu(k)), \tag{6.15}$$

$$\left[\langle u_{ij}^k, \bar{x}_j^k\rangle < \bar{b}_{ij}^k\right] \Longrightarrow \eta_{ij}^k = 0 \quad (i = 1, \ldots, m, \ \ j = 0, \ldots, \nu(k)-1), \tag{6.16}$$

$$\left[\langle \bar{u}_{i\nu(k)}^k, \bar{x}_{\nu(k)}^k\rangle < \bar{b}_{i\nu(k)}^k\right] \Longrightarrow \eta_{i\nu(k)}^k = 0 \quad (i = 1, \ldots, m, \ \ j = 0, \ldots, \nu(k)-1), \tag{6.17}$$

$$\eta_{ij}^k > 0 \Longrightarrow \left[\left\langle \bar{u}_{ij}^k, -\frac{1}{h_j^k}\lambda^k\theta_j^{xk} + p_{j+1}^{xk}\right\rangle = 0\right] \ (i = 1,\ldots,m, \ \ j = 0,\ldots,\nu(k)-1). \quad (6.18)$$

- TRANSVERSALITY RELATIONSHIPS *at the right end of the trajectory:*

$$-p_{\nu(k)}^{xk} \in \lambda^k \partial\varphi(\bar{x}_{\nu(k)}^k) + N\big(\bar{x}_{\nu(k)}^k;\Omega_k\big) + \sum_{i=1}^m \eta_{i\nu(k)}^k \bar{u}_{i\nu(k)}^k, \quad (6.19)$$

$$p_{\nu(k)}^{uk} = -2\left[\alpha_{\nu(k)}^{1k} - \alpha_{\nu(k)}^{2k}, \bar{u}_{\nu(k)}^k\right] - \left[\eta_{\nu(k)}^k, \operatorname{rep}_m(\bar{x}_{\nu(k)}^k)\right], \quad (6.20)$$

$$p_{i\nu(k)}^{bk} = \eta_{i\nu(k)}^k \ge 0, \ \langle \bar{u}_{i\nu(k)}^k, \bar{x}_{\nu(k)}^k\rangle < \bar{b}_{i\nu(k)}^k \Longrightarrow p_{i\nu(k)}^{bk} = 0 \quad (i = 1,\ldots,m). \quad (6.21)$$

Proof. To reduce problem (P_k) from (5.5)–(5.10) for each fixed $k \in I\!N$ to a mathematical program of type (MP) formulated in Lemma 6.2, we form the multidimensional vector

$$z := \big(u_0^k,\ldots,u_{\nu(k)}^k, b_0^k,\ldots,b_{\nu(k)}^k, x_0^k,\ldots,x_{\nu(k)}^k, v_0^k,\ldots,v_{\nu(k)-1}^k,$$
$$w_0^k,\ldots,w_{\nu(k)-1}^k, y_0^k,\ldots,y_{\nu(k)-1}^k\big)$$

and consider the problem of minimizing the cost function

$$f_0(z) := \varphi(x_{\nu(k)}^k) + \frac{1}{2}\sum_{j=0}^{\nu(k)-1}\int_{t_j^k}^{t_{j+1}^k} \big\|\big(v_j^k - \dot{\bar{u}}(t), w_j^k - \dot{\bar{b}}(t), y_j^k - \dot{\bar{x}}(t)\big)\big\|^2 dt$$

subject to the five groups of inequality constraints

$$f_1(z) := \sum_{j=0}^{\nu(k)-1}\int_{t_j^k}^{t_{j+1}^k} \big\|\big(u_j^k, b_j^k, x_j^k\big) - \big(\bar{u}(t), \bar{b}(t), \bar{x}(t)\big)\big\|^2 dt - \frac{\varepsilon}{2} \le 0,$$

$$f_2(z) := \sum_{j=0}^{\nu(k)-1}\int_{t_j^k}^{t_{j+1}^k} \Big\|\big(v_j^k, w_j^k, y_j^k\big) - \big(\dot{\bar{u}}(t), \dot{\bar{b}}(t), \dot{\bar{x}}(t)\big)\Big\|^2 dt - \frac{\varepsilon}{2} \le 0,$$

$$f_{ij}(z) := \|u_{ij}^k\|^2 - (1+\delta_k)^2 \le 0 \ \text{ for } \ i = 1,\ldots,m, \ \ j = 0,\ldots,\nu(k),$$

$$\widetilde{f}_{ij}(z) := (1-\delta_k)^2 - \|u_{ij}^k\|^2 \le 0, \ \text{ for } \ i = 1,\ldots,m, \ \ j = 0,\ldots,\nu(k),$$

$$\widehat{f}_i(z) := \big\langle u_{i\nu(k)}^k, x_{\nu(k)}^k\big\rangle - b_{i\nu(k)}^k \le 0 \ \text{ for } \ i = 1,\ldots,m,$$

the three groups of equality constraints

$$g_j^u(z) := u_{j+1}^k - u_j^k - h_j^k v_j^k = 0 \ \text{ for } \ j = 0,\ldots,\nu(k)-1,$$

$$g_j^b(z) := b_{j+1}^k - b_j^k - h_j^k w_j^k = 0 \ \text{ for } \ j = 0,\ldots,\nu(k)-1,$$

$$g_j^x(z) := x_{j+1}^k - x_j^k - h_j^k y_j^k = 0, \ \text{ for } \ j = 0,\ldots,\nu(k)-1,$$

and the two groups of inclusion constraints

$$z \in \Theta_j := \big\{z \;\big|\; -y_j^k \in F(u_j^k, b_j^k, x_j^k)\big\} \ \text{ for } \ j = 0,\ldots,\nu(k)-1,$$

$$z \in \Theta_{\nu(k)} := \big\{ z \;\big|\; (u_0^k, b_0^k, x_0^k) \text{ are fixed, } x_{\nu(k)}^k \in \Omega_k \big\},$$

where those for $j = 0, \ldots, \nu(k) - 1$ incorporate the constraints $x_j^k \in C(u_j^k, b_j^k)$ for such j due to the construction of F in (5.2).

As we see, the formulated nondynamic equivalent of problem (P_k) is written in the mathematical programming form (MP) as in Lemma 6.2 with the fulfillment all the assumptions imposed in the lemma. Thus we can readily apply the conclusions of the lemma by taking into account the particular structure of the functions and sets in the formulated equivalent of (P_k). Employing now the necessary optimality conditions of Lemma 6.2 to the optimal solution

$$\begin{aligned}\bar z := \bar z^k = \big(&\bar u_0^k, \ldots, \bar u_{\nu(k)}^k, \bar b_0^k, \ldots, \bar b_{\nu(k)}^k, \bar x_0^k, \ldots, \bar x_{\nu(k)}^k, \bar v_0^k, \ldots, \bar v_{\nu(k)-1}^k,\\ &\bar w_0^k, \ldots, \bar w_{\nu(k)-1}^k, \bar y_0^k, \ldots, \bar y_{\nu(k)-1}^k\big)\end{aligned}$$

of problem $(MP) \equiv (P_k)$, observe by Theorem 5.2 that the inequality constraints defined by the functions f_1 and f_2 above are *inactive* at $\bar z$ for sufficiently large k, and thus the corresponding multipliers will not appear in optimality conditions. Taking this into account, we find by Lemma 6.2 multipliers $\lambda^k \ge 0$, $(\beta_1^k, \ldots, \beta_m^k) \in \mathbb{R}_+^m$, $p_j^k = (p_j^{uk}, p_j^{bk}, p_j^{xk}) \in \mathbb{R}^{mn+n+m}$ for $j = 1, \ldots, \nu(k)$, as well as vectors

$$\begin{aligned}z_j^* := \big(&u_{0j}^*, \ldots, u_{\nu(k)j}^*, b_{0j}^*, \ldots, b_{\nu(k)j}^*, x_{0j}^*, \ldots, x_{\nu(k)j}^*, v_{0j}^*, \ldots, v_{(\nu(k)-1)j}^*,\\ &w_{0j}^*, \ldots, w_{(\nu(k)-1)j}^*, y_{0j}^*, \ldots, y_{(\nu(k)-1)j}^*\big)\end{aligned}$$

for $j = 0, \ldots, \nu(k)$, $\alpha^{1k} = (\alpha_0^{1k}, \ldots, \alpha_{\nu(k)}^{1k}) \in \mathbb{R}_+^{\nu(k)+1}$, $\alpha^{2k} = (\alpha_0^{2k}, \ldots, \alpha_{\nu(k)}^{2k}) \in \mathbb{R}_-^{\nu(k)+1}$ such that the complementary slackness conditions (6.14), (6.15), and

$$\beta_i^k \big(\langle \bar u_{i\nu(k)}^k, \bar x_{\nu(k)}^k \rangle - \bar b_{i\nu(k)}^k\big) = 0 \;\text{ for }\; i = 1, \ldots, m \tag{6.22}$$

hold together with the normal cone inclusions

$$z_j^* \in N(\bar z; \Theta_j) \;\text{ for }\; j = 0, \ldots, \nu(k) \tag{6.23}$$

and the generalized Lagrangian condition

$$\begin{aligned}-\sum_{j=0}^{\nu(k)} z_j^* \;&\in \lambda^k \partial f_0(\bar z) + \sum_{i=1}^m \beta_j^k \nabla \widehat{f}_i(\bar z) + \sum_{j=0}^{\nu(k)-1} \nabla g_j(\bar z)^T p_{j+1}^k\\ &+ \sum_{j=0}^{\nu(k)} \sum_{i=1}^m \Big[\alpha_{ij}^{1k} \nabla f_{ij}(\bar z) + \alpha_{ij}^{2k} \nabla \widetilde{f}_{ij}(\bar z)\Big],\end{aligned} \tag{6.24}$$

where $g_j = (g_j^u, g_j^b, g_j^x)$, and where the dual elements λ^k, β_i^k, p_j^k, z_j^*, α^{1k}, and α^{2k} are not all zero simultaneously.

Looking at the graphical structure of the geometric constraints $z \in \Theta_j$ for $j = 0, \ldots, \nu(k) - 1$, we readily deduce from (6.23) that

$$(u_{jj}^*, b_{jj}^*, x_{jj}^*, -y_{jj}^*) \in N\Big(\Big(\bar u_j^k, \bar b_j^k, \bar x_j^k, -\frac{\bar x_{j+1}^k - \bar x_j^k}{h_j^k}\Big); \operatorname{gph} F\Big) \;\; (j = 0, \ldots, \nu(k) - 1)$$

with all the other components of z_j^* equal to zero for these indices j. It follows from the coderivative definition (6.2) that the obtained normal cone inclusion can be equivalently written as

$$(u_{jj}^*, b_{jj}^*, x_{jj}^*) \in D^* F\Big(\bar u_j^k, \bar b_j^k, \bar x_j^k, -\frac{\bar x_{j+1}^k - \bar x_j^k}{h_j^k}\Big)(y_{jj}^*) \;\text{ for }\; j = 0, \ldots, \nu(k) - 1. \tag{6.25}$$

Since the mapping F is given in the particular form (5.2), we are able to use the coderivative evaluation in (6.25) provided the fulfillment of PLICQ (6.5) along the discrete optimal solutions for all k sufficiently large. As discussed above, the assumed uniform Slater condition (2.11) for the given canonical intermediate local minimizer $(\bar{u}, \bar{b})$ of (P) yields PLICQ at $(\bar{u}, \bar{b}, \bar{x})$. Since the latter condition is *robust* with respect to perturbations of the initial triple and since the discrete optimal solutions strongly converge to $(\bar{u}(\cdot), \bar{b}(\cdot), \bar{x}(\cdot))$ by Theorem 5.2, we are in a position to use Lemma 6.1 in the coderivative inclusion (6.25). Prior to this, let us calculate the other terms in the generalized Lagrangian condition (6.24).

First observe that the summation term in the cost function is smooth. Therefore, the usage of the subdifferential sum rule from [23, Proposition 1.30(ii)] gives the precise calculation

$$\partial f_0(\bar{z}) = \partial\varphi(\bar{x}^k_{\nu(k)}) + \sum_{j=0}^{\nu(k)-1} \left(0, \ldots, 0, \theta_j^{uk}, \theta_j^{bk}, \theta_j^{xk}\right)$$

where zeros stands for the all components of $\bar{z}$ till $\bar{v}^k_j$, and where $\theta_j^{uk}, \theta_j^{bk}, \theta_j^{xk}$ are defined in the formulation of the theorem. Further, with the usage of our notation presented before the formulation of this theorem, we easily get

$$\sum_{i=1}^m \beta_i^k \nabla \widehat{f}_i(\bar{z}) = \left(\sum_{i=1}^m \beta_i^k \bar{u}^k_{ik}, \left[\beta^k, \operatorname{rep}_m(\bar{x}^k_{\nu(k)})\right], -\beta^k \right),$$

$$\left(\sum_{j=0}^{\nu(k)-1} \nabla g_j(\bar{z})^T p^k_{j+1} \right)_{(u_j, b_j, x_j)} = \begin{cases} -p_1^k & \text{if} \quad j=0 \\ p_j^k - p^k_{j+1} & \text{if} \quad j = 1, \ldots, \nu(k)-1 \\ p^k_{\nu(k)} & \text{if} \quad j = \nu(k) \end{cases},$$

$$\left(\sum_{j=0}^{\nu(k)-1} \nabla g_j(\bar{z})^T p^k_{j+1} \right)_{(v_j, w_j, y_j)} = \left(-h_0^k p_1^{uk}, -h_1^k p_2^{uk}, \ldots, -h^k_{\nu(k)-1} p^{uk}_{\nu(k)}, \right.$$
$$\left. -h_0^k p_1^{bk}, -h_1^k p_2^{bk}, \ldots, -h^k_{\nu(k)-1} p^{bk}_{\nu(k)}, -h_0^k p_1^{xk}, -h_1^k p_2^{xk}, \ldots, -h^k_{\nu(k)-1} p^{xk}_{\nu(k)} \right),$$

$$\sum_{j=0}^{\nu(k)} \sum_{i=1}^m \alpha_{ij}^{1k} \nabla f_{ij}(\bar{z}) = 2\left[\alpha_j^{1k}, \bar{u}_j^k\right], \quad \sum_{j=0}^{\nu(k)} \sum_{i=1}^m \alpha_{ij}^{2k} \nabla \widetilde{f}_{ij}(\bar{z}) = -2\left[\alpha_j^{2k}, \bar{u}_j^k\right]$$
$$(j = 0, \ldots, \nu(k)).$$

To proceed with (6.24), it remains to express the dual element $z^*_{\nu(k)} \in N(\bar{z}; \Theta_{\nu(k)})$ in (6.23) corresponding the last geometric constraint $\bar{z}_{\nu(k)} \in \Theta_{\nu(k)}$ in terms of the data of (P_k). We directly conclude from the structure of $\Theta_{\nu(k)}$ that the components of $z^*_{\nu(k)}$ corresponding to (u_0^k, b_0^k, x_0^k) are free (i.e., just belong to $\mathbb{R}^{mn} \times \mathbb{R}^m \times \mathbb{R}^n$), that $x^*_{\nu(k)\nu(k)} \in N(\bar{x}^k_{\nu(k)}; \Omega_k)$, and that all the other components are equal to zero. The fulfillment of PLICQ along $(\bar{u}^k, \bar{b}^k, \bar{x}^k)\}$ for all k sufficiently large allows us to find unique vectors $\eta_j^k \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^m \eta^k_{ij} \bar{u}^k_{ij} = -\frac{\bar{x}^k_{j+1} - \bar{x}^k_j}{h^k_j} \quad \text{for} \quad j = 0, \ldots, \nu(k)-1.$$

For the last index $j = \nu(k)$, we put $\eta^k_{\nu(k)} := \beta^k \in \mathbb{R}^m_+$. Substituting all the above into the Lagrangian inclusion (6.24) with taking into account the coderivative upper estimate from Lemma 6.1 gives us the claimed necessary optimality conditions (6.9)–(6.21). Finally, the nontriviality conditions in (6.7) and (6.8) follows directly from (6.9)–(6.21) and the nontriviality of the dual elements in Lemma 6.2 for the mathematical program (MP) equivalent to (P_k). Therefore, we complete the proof of the theorem. $\square$

References

[1] S. Adly, *A Variational Approach to Nonsmooth Dynamics. Applications in Unilateral Mechanics and Electronics*, Springer, Cham. Switzerland, 2018.

[2] L. Adam and J. V. Outrata, On optimal control of a sweeping process coupled with an ordinary differential equation, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), 2709–2738.

[3] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Non-Linear Parametric Optimization*, Akademie-Verlag, Berlin, 1982.

[4] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[5] M. Brokate and P. Krejčí, Optimal control of ODE systems involving a rate independent variational inequality, *Discrete Contin. Dyn. Syst. Ser. B* **18** (2013), 331–348.

[6] B. Brogliato, *Nonsmooth Mechanics*, 3rd edition, Springer, Cham, Switzerland, 2016.

[7] B. Brogliato and A. Tanwani, Dynamical systems coupled with monotone set-valued operators: formalisms, applications, well-posedness, and stability, *SIAM Rev.* **62** (2020), 3–129.

[8] T. H. Cao and B. S. Mordukhovich, Optimal control of a nonconvex perturbed sweeping process, *J. Diff. Eqs.* **266** (2019), 1003–1050.

[9] T. H. Cao, G. Colombo, B. S. Mordukhovich and D. Nguyen, Optimization and discrete approximation of sweeping processes with controlled moving sets and perturbations, *J. Diff. Eqs.* **274** (2021), 461–509.

[10] G. Colombo and P. Gidoni, On the optimal control of rate-independent soft crawlers, *J. Mathém. Pures Appl.* **146** (2021), 127–157.

[11] G. Colombo, R. Henrion, N. D. Hoang and B. S. Mordukhovich, Optimal control of the sweeping process, *Dyn. Contin. Discrete Impuls. Syst. Ser. B* **19** (2012), pp. 117–159.

[12] G. Colombo, R. Henrion, N.D. Hoang and B.S. Mordukhovich, Discrete approximations of a controlled sweeping process, *Set-Valued Var. Analysis*, **23** (2015), 69–86.

[13] G. Colombo, R. Henrion, N. D. Hoang and B. S. Mordukhovich, Optimal control of the sweeping process over polyhedral controlled sets, *J. Diff. Eqs.* **260** (2016), 3397–3447.

[14] G. Colombo, B. S. Mordukhovich and D. Nguyen, Optimization of a perturbed sweeping process by discontinuous controls, *SIAM J. Control Optim.* **58** (2020), 2678–2709.

[15] M. d. R. de Pinho, M. M. A. Ferreira and G. V. Smirnov, Optimal control involving sweeping processes, *Set-Valued Var. Anal.* **27** (2019), 523–548.

[16] I. Gudoshnikov, M. Kamenskii, O. Makarenkov and N. Voskovskaia, One-period stability analysis of polygonal sweeping processes with application to an elastoplastic model, *Math. Model. Nat. Phenom.* **15** (2020, 1-18.

[17] T. Haddad, A. Jourani and L. Thibault, Reduction of sweeping process to unconstrained differential inclusion, *Pacific J. Optim.* **4** (2008), 493–512.

[18] A. Jourani and E. Vilches, A differential equation approach to implicit sweeping processes, *J. Diff. Eqs.* **266** (2019), 5168–5184.

[19] A. Jourani and D. Zagrodny, The positiveness of lower limits of the Hoffman constant in parametric polyhedral programs, *J. Global Optim.* **53** (2012), 641–661.

[20] M. Kunze and M. D. P. Monteiro Marques, An Introduction to Moreau's Sweeping Process. In: B. Brogliato et al.(eds.) *Impacts in Mechanical Systems*, Lecture Notes in Physics, vol. 551, pp. 1–60, 2000, Springer, Berlin, 2000.

[21] B. S. Mordukhovich, Discrete approximations and refined Euler-Lagrange conditions for differential inclusions, *SIAM J. Control Optim.* **33** (1995), 882–915.

[22] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications*, Springer, Berlin, 2006.

[23] B. S. Mordukhovich, *Variational Analysis and Applications*, Springer, Cham, Switzerland, 2018.

[24] B. S. Mordukhovich and J. V. Outrata, Coderivative analysis of quasi-variational inequalities with applications to stability and optimization, *SIAM J. Optim.* **18** (2007), 389–412.

[25] J. Moreau, Rafle par un convexe variable, I. In: *Travaux du Séminaire d'Analyse Convexe*, Vol. I, Exp. No. 15, pp. 1–43, U.É.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1971.

[26] J. Moreau, Rafle par un convexe variable, II. In: *Travaux du Séeminaire d'Analyse Convexe*, Vol. II, 245 Exp. No. 3, pp. 1–36, U.É.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1972.

[27] J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, *J. Diff. Eqs.* **26** (1977), 347–374.

[28] S. M. Robinson, Some continuity properties of polyhedral multifunctions, *Math. Program. Stud.* **19** (1981), 206–214.

[29] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[30] L. Thibault, Regularization of nonconvex sweeping process in Hilbert space, *Set-Valued Anal.* **16** (2008), 319–333.

[31] A. A. Tolstonogov, Sweeping process with unbounded nonconvex perturbation, *Nonlinear Anal.* **108** (2014), 291–301.

[32] A. A. Tolstonogov, Polyhedral sweeping processes with unbounded nonconvex-valued perturbation, *J. Diff. Eqs.* **63** (2017), 7965–7983.

[33] A. A. Tolstonogov, Polyhedral multivalued mappings: properties and applications, *Sib. Math. J.* **61** (2020), 338–358.

[34] R. B. Vinter, *Optimal Control*, Birkhäuser, Boston, MA, 2000.

[35] V. Zeidan, C. Nour and H. Saoud, A nonsmooth maximum principle for a controlled nonconvex sweeping process, *J. Diff. Eqs.* **269** (2020), 9531–9582.