Dynamics of Spiral Waves on Unbounded Domains Using Center-Manifold Reductions

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Abstract

An equivariant center-manifold reduction near relative equilibria of G-equivariant semiflows on Banach spaces is presented. In contrast to previous results, the Lie group G is possibly non-compact. Moreover, it is not required that G induces a strongly continuous group action on the underlying function space. In fact, G may act discontinuously. The results are applied to bifurcations of stable patterns arising in reaction-diffusion systems on the plane or in three-space modeling chemical systems such as catalysis on platinum surfaces and Belousov-Zhabotinsky reactions. These systems are equivariant under the Euclidean symmetry group. Hopf bifurcations from rigidly-rotating spiral waves to meandering or drifting waves, and from twisted scroll rings are investigated.

1 Introduction

Spiral waves arise as stable spatio-temporal patterns in various chemical and physical systems. They have been observed experimentally, for instance, in catalysis on platinum surfaces [14], Belousov-Zhabotinsky reactions [10, 20], and the Rayleigh-Benard convection [17]. The dynamics of the first two systems is modeled by reaction-diffusion equations

$$u_t = D\Delta u + f(u,\mu), \qquad x \in \mathbb{R}^N,\ N = 2,3 \tag{1.1}$$

on the plane or in three-space. Here, D is a diagonal matrix with non-negative entries, and f is a smooth nonlinearity. The function $u : \mathbb{R}^N \to \mathbb{R}^M$ can be interpreted as a vector of spatially dependent concentrations of chemical species. Equation (1.1) is well-posed on the space $C^0_{\text{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ of uniformly continuous, bounded functions or, under certain additional growth conditions on f in case the diffusion matrix D is singular, on the space $L^2(\mathbb{R}^N, \mathbb{R}^M)$. On both spaces, it then generates a smooth local semiflow denoted by $\Phi_t(u,\mu)$, see [8].

The Euclidean group $SE(N)$ is the semi-direct product $SO(N)\dot{+}\mathbb{R}^N$ of the orthogonal group $SO(N)$ and the group of translations $\mathbb{R}^N$ with composition

$$(R,S)(\tilde{R},\tilde{S}) = (R\tilde{R}, S + R\tilde{S}) \tag{1.2}$$

on the product $SO(N) \times \mathbb{R}^N$. The Lie algebra $\mathbf{se}(N)$ of $SE(N)$ can be represented as the product $\mathbf{so}(N) \times \mathbb{R}^N$ of the Lie algebra $\mathbf{so}(N)$ of $SO(N)$ consisting of anti-symmetric matrices and $\mathbb{R}^N$, see [5]. The commutator and the exponential map on $\mathbf{so}(N) \times \mathbb{R}^N$ are given by

$$\begin{aligned} [(r,s),(\tilde{r},\tilde{s})] &= (r\tilde{r} - \tilde{r}r, r\tilde{s} - \tilde{r}s) \\ \exp((r,s)t) &= (\exp(r\,t), r^{-1}(\exp(r\,t) - \mathrm{id})s). \end{aligned} \tag{1.3}$$

The group $SE(N)$ acts on functions on $\mathbb{R}^N$ by

$$((R,S)u)(x) := u(R^{-1}(x-S)).$$

Equation (1.1) is equivariant with respect to this $SE(N)$-action, that is, $\Phi_t(u, \mu)$ is a solution whenever $(R, S)\Phi_t(u, \mu)$ is.

We consider bifurcations from relative equilibria of (1.1). *Relative equilibria* are solutions satisfying

$$\Phi_t(u_*, \mu_*) = (R(t), S(t))u_*,$$

with $(R(t), S(t)) = \exp((r_*, s_*)t)$ for suitable elements $(r_*, s_*) \in \mathbf{se}(N)$. In other words, u_* is a relative equilibrium if its time orbit is contained in its group orbit $SE(N)u_*$. *Rigidly-rotating spiral waves* u_* are rotating waves obeying

$$\Phi_t(u_*, \mu_*) = (R(t), 0)u_*,$$

where $R(t) = \exp(r_* t)$ is the one-parameter family of rotations generated by some fixed element $r_* \in \mathbf{so}(N)$. Thus, spiral waves are equilibria in a rotating frame $(x, t) \mapsto (\exp(-r_* t)x, t)$.

We shall further distinguish two kinds of modulated waves; these solutions are not relative equilibria. *Meandering spiral waves* are modulated rotating waves, that is, quasiperiodic solutions which are periodic in a rotating frame. In contrast, *drifting spiral waves* are modulated travelling waves, that is, periodic in a moving frame $(x, t) \mapsto (x - s_* t, t)$ generated by some element $s_* \in \mathbb{R}^N$.

Meandering spiral waves emanate from rigidly-rotating spiral waves by a Hopf bifurcation in the rotating frame. This has been verified numerically by Barkley [1]. Furthermore, in simulations of a two-parameter system, he observed a curve of drifting spiral waves emerging from the rotating wave if the rotation frequency of the rotating wave is a multiple of the eigenvalue leading to the Hopf bifurcation, see [2]. Barkley proposed a five-dimensional system of ordinary differential equations modeling the qualitative behavior of reaction-diffusion systems near Hopf bifurcations from rotating waves. However, a rigorous relation between the two systems has not been established yet. We remark that the system studied by Barkley has a singular diffusion matrix D, which seems to model the chemical situation more accurately. For that reason, we allow for degenerate diffusion matrices.

In three dimensions, Hopf instabilities of *twisted scroll rings* have been observed numerically in [15]. Mathematically, scroll rings are rotating waves which, at the same time, drift along the axis of rotation. Thus, they are relative equilibria with respect to the one-parameter family $(R(t), S(t)) = (\exp(r_* t), s_* t)$ for elements $(r_*, s_*) \in \mathbf{so}(3) \times \mathbb{R}^3 = \mathbf{se}(3)$ with $r_* s_* = 0$.

In this article, we will explain the phenomena mentioned above using an equivariant center-manifold reduction of the reaction-diffusion system (1.1). Standard results for center manifolds are not applicable since the group action of $SE(N)$ is *not* norm-continuous on either $C^0_{\text{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ or $L^2(\mathbb{R}^N, \mathbb{R}^M)$, see [22]. In fact, on $C^0_{\text{unif}}(\mathbb{R}^N, \mathbb{R}^M)$, rotations act not even as a strongly continuous semigroup: a counterexample is provided by the function

$u(x_1, x_2) = \cos x_1$. In addition, the group $SE(N)$ is not compact. Therefore, it is not clear how to obtain a smooth and equivariant center manifold. We emphasize that the spiral waves found analytically, for instance in [7], are Archimedean or logarithmic spirals, which are contained in C^0_{unif} but not in L^2. It seems then inevitable to consider discontinuous $SE(N)$-actions.

To circumvent these difficulties, we make the following hypotheses. Consider a smooth group orbit associated with a relative equilibrium. Assume that the center-unstable eigenspace of the linearization at the wave has a finite-dimensional generalized eigenspace. Note that the group action always enforces spectrum on the imaginary axis. Next, we assume that the group acts smoothly on elements in the center-unstable eigenspace, whence the center-unstable bundle along the group orbit will itself be smooth. Under these assumptions, we will prove the existence of a smooth center manifold M^{cu}_* tangent to the center bundle. The group will act smoothly on M^{cu}_*. Note that the group $SE(N)$ is not assumed to act smoothly on the whole function space. We shall emphasize that the result is optimal in the sense that whenever an invariant manifold M^{cu}_* with the above properties exists, the group will already act smoothly on the center bundle. In particular, the group orbit of u_* must be smooth.

We should comment on the satisfaction of these assumptions for the reaction-diffusion system (1.1). It turns out that $SE(2)$ acts smoothly on relative equilibria in either $C^0_{\text{unif}}(\mathbb{R}^2, \mathbb{R}^M)$ or $L^2(\mathbb{R}^2, \mathbb{R}^M)$. In addition, $SE(N)$ acts smoothly on vectors in the finite-dimensional eigenspace provided it acts smoothly on the underlying relative equilibrium. Therefore, the only hypothesis which is not automatically satisfied is that the eigenspace is indeed of finite dimension. This last assumption, however, has been verified numerically at Hopf-bifurcation points of spiral waves, see Barkley [1].

Therefore, at the outcome, we have reduced the infinite-dimensional dynamical system to ordinary differential equations on the center manifold. The structure of these equations has been clarified and analyzed in detail in the related paper [5]. In particular, drifting along the group orbit as well as bifurcations in the normal direction can be analyzed separately. We will apply these results to the phenomena mentioned above, that is, to Hopf bifurcations from spiral waves and twisted scroll rings, see Theorems 4 and 6 in section 5 and 6, respectively.

Similar results hold for relative periodic solutions of (1.1). They can be used to study secondary bifurcations of meandering or drifting waves to higher-dimensional tori, or to investigate the influence of periodic forcing. This is work in progress and will appear elsewhere.

Finally, we mention related results. Wulff [22] investigated Hopf bifurcations from rotating to meandering and drifting one-armed planar spiral waves using Lyapunov-Schmidt reduction in the largest subspace of $C^0_{\rm unif}$ on which the rotations act as a strongly continuous semigroup. This was the first rigorous result on bifurcations of spiral waves involving non-compact groups. Some of the results of this paper have been announced in [18]. Based on results by Krupa [12], Golubitsky et al. [6] used a formal center-bundle construction to derive ODEs describing bifurcations near ℓ-armed planar spiral waves. They exploited the structure of these ODEs using ideas from [5], and derived new conditions for drifting. Fiedler et al. [5] clarified the structure of the ODEs associated with relative equilibria with compact isotropy for general non-compact groups and gave conditions for drifting. In the present paper, these ODEs are derived rigorously using center-manifold reductions.

The paper is organized as follows. In section 2, an abstract result for the existence of center manifolds is given. It is proved in section 3. In section 4, we verify the smoothness hypothesis for the Euclidean group $SE(N)$. We apply the results to Hopf bifurcations of spiral waves and twisted scroll rings in section 5 and 6, respectively.

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2 Center-manifold reduction near relative equilibria

Consider a semilinear differential equation

$$u_t = -Au + F(u), \tag{2.1}$$

on some Banach space X. We assume that A is sectorial and F is a C^{k+2}-function from $Y = X^\alpha$ to X for some $k \geq 1$ and $\alpha \in [0,1)$, see Henry [8] for the notation. The norms for vectors and operators on Y are denoted $|\cdot|$ and $\|\cdot\|$, respectively. The local semiflow on Y associated with (2.1) is denoted by $\Phi_t(u)$. Let G be a finite-dimensional but possibly non-compact Lie group, and $\rho : G \to GL(Y)$, $g \mapsto \rho_g$ be a representation of G in the space of bounded invertible operators. We assume that there exists a constant K such that $\|\rho_g\| \leq K$ for all $g \in G$. After introducing an equivalent norm on Y, we may assume that $\|\rho_g\| = 1$ for all g, see Lemma 3.1. We suppose that $\Phi_t(u)$ is G-equivariant, that is, $\Phi_t(\rho_g u) = \rho_g \Phi_t(u)$ for $t \geq 0$, $g \in G$, and $u \in Y$.

Throughout, we fix a point u_* and denote its group orbit and the isotropy group by Gu_* and H, respectively, that is, we set $Gu_* = \{\rho_g u_*;\ g \in G\}$ and $H = \{g \in G;\ \rho_g u_* = u_*\}$. Suppose that the element u_* chosen is a relative equilibrium of (2.1):

Hypothesis 1 *Let $u_* \in Y$ and assume that there exists an element $\xi_* \in \mathbf{alg}(G)$ in the Lie algebra of G such that*

$$\Phi_t(u_*) = \rho_{g_*(t)} u_*,$$

where $g_*(t) = \exp(\xi_* t) \in G$ *is the one-parameter family generated by* ξ_*.

Next, we consider the linearization of the flow evaluated at u_*.

Hypothesis 2 *Assume that* $\{\lambda \in \mathbb{C};\ |\lambda| \geq 1\}$ *is a spectral set for the linearization*

$$\rho_{\exp(-\xi_*)} D\Phi_1(u_*) \in L(Y)$$

with associated projection $P_* \in L(Y)$ *such that the generalized eigenspace* $E_*^{cu} = R(P_*)$ *is finite-dimensional.*

Note that the isotropy H acts on E_*^{cu}. Hence, whenever H is non-compact and does not possess any finite-dimensional representation on the space Y, the spectral hypothesis 2 must be violated.

Finally, as announced in the introduction, we impose smoothness conditions.

Hypothesis 3 *(i)* $\rho_g u_*$ *is* C^{k+2} *in* $g \in G$.

(ii) For any $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $|\rho_g u_* - u_*| \geq \delta$ *for all* $g \in G$ *satisfying* $\mathrm{dist}(g, H) \geq \epsilon$.

(iii) $\rho_g v$ *is* C^{k+1} *in* $g \in G$ *for any point* v *in* E_*^{cu}.

(iv) The projections $\rho_g P_* \rho_{g^{-1}}$ *are* C^{k+1} *in* $g \in G$ *in the operator norm.*

It follows from Hypotheses 3(i) and (ii) that the group orbit Gu_* is an embedded C^{k+2}-manifold. In many applications, Hypothesis 3 follows from Hypothesis 2, see section 4. We remark that, if the group G were compact and the G-action on Y smooth, Hypothesis 3 would always be satisfied.

We have then the following theorem, which is proved in section 3.

Theorem 1 *Assume that Hypotheses 1 – 3 are obeyed. Under these conditions, there exists a* G*-invariant manifold* $M_*^{cu} \subset Y$ *which is locally invariant under* Φ_t *for any* $t \geq 0$. *The manifold* M_*^{cu} *and the action of* G *on* M_*^{cu} *are of class* C^{k+1}. *Furthermore,* M_*^{cu} *is locally exponentially attracting and contains all solutions which stay close to the group orbit of* u_* *for all backward times.*

Similar results are valid for the equation

$$u_t = -Au + F(u) + \mu G(u, \mu), \qquad (u, \mu) \in Y \times \mathbb{R}^p, \tag{2.2}$$

with $|\mu| < \delta$ for some small $\delta > 0$ whenever the nonlinearity $G : Y \times \mathbb{R}^p \to X$ is C^{k+2}. The resulting manifold is C^{k+1} in μ.

We shall investigate the structure of the vector field on the center manifold. For that purpose, we need to introduce more notation. The adjoint representation of G on $\mathbf{alg}(G)$ is defined by

$$\mathrm{Ad}_g\,\xi = g\,\xi\,g^{-1} = \frac{d}{dt}\Big(g\,\exp(\xi t)\,g^{-1}\Big)\Big|_{t=0}, \qquad g \in G,\, \xi \in \mathbf{alg}(G).$$

The isotropy group H acts naturally on the eigenspace E^{cu}_* and the tangent space $T_{u_*}(Gu_*) \subset E^{cu}_*$ of the group orbit, and both spaces are invariant under the H-action. Actually, the representation of H is via the image of ρ, that is, $\rho(H) \subset GL(E^{cu}_*)$ acts on E^{cu}_*. Since the latter space is finite-dimensional and group elements are isometries, we see that $\mathrm{clos}\,\rho(H) \subset GL(E^{cu}_*)$ is compact. Using the Haar measure associated with $\mathrm{clos}\,\rho(H)$, we can construct an H-equivariant projection $Q_* : E^{cu}_* \to E^{cu}_*$ with kernel $N(Q_*) = T_{u_*}(Gu_*)$. Its range $V_* := R(Q_*)$ is an H-invariant complement of $T_{u_*}(Gu_*)$. We then consider the manifold $G \times V_*$ with an H-action defined by $(g,v) \to (gh^{-1}, \rho_h v)$ for $(g,v) \in G \times V_*$ and $h \in H$.

Theorem 2 *Suppose that the assumptions of Theorem 1 are met, and that the isotropy group H is compact. The manifold M^{cu}_* is then diffeomorphic to $(G \times V_*)/\sim$ where the equivalence relation on $G \times V_*$ is defined by identifying orbits under the above H-action, that is, $(g,v) \sim (gh^{-1}, \rho_h v)$ for $(g,v) \in G \times V_*$ and $h \in H$. Furthermore, there exist C^k-functions $f_G : V_* \to \mathbf{alg}(G)$ and $f_N : V_* \to V_*$ such that any solution of*

$$\begin{pmatrix} \dot g \\ \dot v \end{pmatrix} = \begin{pmatrix} g f_G(v) \\ f_N(v) \end{pmatrix} \tag{2.3}$$

on $G \times V_*$ corresponds to a solution of the vector field on M^{cu}_* under the identification. The vector field (2.3) is H-equivariant: $f_G(\rho_h v) = \mathrm{Ad}_h\, f_G(v) = h f_G(v) h^{-1}$ and $f_N(\rho_h v) = \rho_h f_N(v)$ for all $h \in H$ and $v \in V_*$. Finally, $f_G(0) = \xi_*$ and $f_N(0) = 0$.

We say that the vector field (2.3) is the pull-back of the vector field on M^{cu}_* to $G \times V_*$. Note that it is of skew-product form. We refer to [5] for more properties of the pull-back.

Proof. The statement follows from [5, Theorem 1.1] provided the Lie group G induces a proper action on M^{cu}_*. We prove that this is indeed the case. The action being proper means that if $y_n \in M^{cu}_*$ and $g_n \in G$ are sequences such that $y_n \to y$ and $\rho_{g_n} y_n \to \tilde y$, then $\{g_n\}$ has a convergent subsequence. The action restricted to the group orbit satisfies this condition, and thus is proper, since Gu_* is embedded on account of Hypothesis 3(ii). We show that the above condition is an open property using that each ρ_g is an isometry.

Due to Hypothesis 3(ii), compactness of the isotropy group H, and local compactness of G, there exist $\delta > 0$ and a neighborhood U of H in G such that U is precompact and

$$|\rho_g u_* - u_*| \geq \delta > 0 \tag{2.4}$$

for all $g \notin U$. Note that the same estimate is valid with u_* and U replaced by $\tilde{g}u_*$ and $\tilde{g}\,U\,\tilde{g}^{-1}$, respectively, for any $\tilde{g} \in G$ since $\|\rho_g\| = 1$ for all g.

Suppose now that $y_n \to y$ and $\rho_{g_n} y_n \to \tilde{y}$ in M_*^{cu} as $n \to \infty$. Since ρ_g is linear and of norm one, $|\rho_{g_n} y_n - \rho_{g_n} y| \le |y_n - y|$. Therefore, $\rho_{g_n} y \to \tilde{y}$ in M_*^{cu}. We have to show that $\{g_n\}$ has a convergent subsequence.

Due to the proof of Theorem 1 in section 3, any point on M_*^{cu} is of the form $\rho_g(u_* + v_* + \sigma_\#(u_* + v_*))$ with $v_* \in V_*$ where $\sigma_\#$ is a smooth and G-equivariant map satisfying $\|D\sigma_\#\| \le 1$ and $\sigma_\#(u_*) = 0$. Hence, without loss of generality, we may assume that $y = (\mathrm{id} + \sigma_\#)(u_* + v_*)$, and

$$(2.5) \qquad \rho_{g_n}(\mathrm{id} + \sigma_\#)(u_* + v_*) \to (\mathrm{id} + \sigma_\#)(u_* + \tilde{v}_*)$$

for some $\tilde{v}_* \in E_*^{cu}$. Indeed, $\tilde{y} = \rho_{\tilde{g}}(\mathrm{id} + \sigma_\#)(u_* + \tilde{v}_*)$ for some $\tilde{g} \in G$ and $\tilde{v}_* \in E_*^{cu}$, and we may replace the sequence $\{g_n\}$ by $\{\tilde{g}^{-1} g_n\}$.

We will argue by contradiction. Assume that the sequence $\{g_n\}$ has no convergent subsequence. We may then assume that $g_n \notin U$ for all n since the neighborhood U of H is precompact. Therefore, for the sequence appearing in (2.5), we obtain

$$\begin{aligned} &|\rho_{g_n}(\mathrm{id} + \sigma_\#)(u_* + v_*) - (\mathrm{id} + \sigma_\#)(u_* + \tilde{v}_*)| \\ &\quad \ge \; |\rho_{g_n} u_* - u_*| - |\rho_{g_n} v_* - \tilde{v}_*| - |\sigma_\#(u_* + v_*)| - |\sigma_\#(u_* + \tilde{v}_*)| \ge \delta - 2(|v_*| + |\tilde{v}_*|), \end{aligned}$$

using (2.4) and the properties of the map $\sigma_\#$ mentioned above. For $|v_*|, |\tilde{v}_*| \le \delta/8$, this contradicts convergence of the sequence. Therefore, G acts properly on a $\delta/8$-neighborhood of Gu_* in M_*^{cu} and the theorem is proved. ■

We shall comment on the relation between the spectral assumption 2 and the spectrum of the reduced vector field (2.3).

Lemma 2.1 *Suppose that assumptions 1 – 3 are obeyed, and that H is compact. Under these conditions, there exists a matrix $B_* \in L(E_*^{cu})$ such that*

$$(2.6) \qquad e^{B_* t} v := \rho_{g_*^{-1}(t)} D\Phi_t(u_*) v$$

for any $v \in E_*^{cu}$ and $t \ge 0$, and

$$(2.7) \qquad B_* = \begin{pmatrix} -[\xi_*, \cdot] & Df_G(0) \\ 0 & Df_N(0) \end{pmatrix},$$

using $E_*^{cu} = T_{u_*}(Gu_*) \times V_*$.

Proof. Notice that the matrix B_* is well-defined. Indeed, $\rho_{g_*^{-1}(t)} D\Phi_t(u_*)$ maps the space E_*^{cu} into itself and, by equivariance, meets the semiflow properties, whence [16, Corollary

1.4] applies. It remains to show that B_* satisfies (2.7). The linearization of (2.3) at the relative equilibrium $\Phi_t(u_*) = \rho_{\exp(\xi_* t)} u_*$ is given by

$$\begin{pmatrix} \dot\xi \\ \dot v \end{pmatrix} = \begin{pmatrix} \xi\xi_* + \exp(\xi_* t) Df_G(0)v \\ Df_N(0)v \end{pmatrix},$$

using $f_G(0) = \xi_*$. Solving the second component, we may write its solution as $(\xi(t), e^{Df_N(0)t}v_0)$ with $v(0) = v_0$. Using the variation-of-constant formula and multiplying by $\exp(-\xi_* t)$, we obtain the expression

$$\begin{aligned} \exp(-\xi_* t)\,\xi(t) &= \exp(-\xi_* t)\,\xi_0\,\exp(\xi_* t) + \\ &\quad \textstyle\int_0^t \exp(-\xi_*(t-\tau)) Df_G(0)(e^{Df_N(0)\tau}v_0)\exp(\xi_*(t-\tau))\,d\tau \end{aligned}$$

for the first component with $\xi(0) = \xi_0$. Comparing its derivative with respect to t with the first component of $B_*(\xi_0, v_0)$ proves (2.7). ■

3 Graph transform near group orbits

In this section, the center-manifold theorem will be proved using the graph transform. We will show how the set-up of the previous section fits into the standard framework. For the remaining part of the proof, we then refer to [4, 9, 19, 21], where the reader may also find background in graph transform. The graph transform requires a first approximation of the desired manifold, normal hyperbolicity, and a property called overflowing. We outline their verification. The first approximation is constructed using the group orbit Gu_* with the spaces $\rho_g V_*$ attached to it. Normal hyperbolicity means that the linearization of the flow near the group orbit contracts vectors in the center direction with a smaller rate than in the direction normal to it. This property will follow from the spectral hypothesis 2. Finally, for the overflowing property, we show that solutions starting at the boundary of the first approximation leave a fixed neighborhood of it immediately. This will be achieved by modifying the vector field in a G-equivariant fashion. Complications arise due to the presence of Jordan blocks and since the cut-off function used for this purpose has to be G-invariant and smooth.

As claimed in the previous section, an equivalent norm may be chosen such that group elements act as isometries on the underlying Banach space.

Lemma 3.1 *There exists a norm $\|\cdot\|$ on Y such that $\|\rho_g\| = 1$ for all $g \in G$. Moreover, the old and new norm are equivalent.*

Proof. Define $\|y\| := \sup_{g\in G} |\rho_g y|$. It is straightforward to verify that this norm satisfies the properties claimed in the lemma. ■

From now on, we assume that the above norm replaces the original norm on Y.

3.1 Jordan blocks in $\mathbb{R}^l$

To outline the basic idea of the cut-off mechanism, consider

$$\dot{v} = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} v, \qquad v \in \mathbb{R}^2, \tag{3.1}$$

for $K \neq 0$. We seek a small neighborhood $\hat{U}$ of zero such that any solution starting on the boundary $\partial\hat{U}$ will leave $\hat{U}$ immediately. Such neighborhoods are called *overflowing*. Apparently, for (3.1), overflowing neighborhoods do not exist. Therefore, we add an outward-directed vector field of norm $\delta > 0$,

$$\dot{v} = \begin{pmatrix} \delta & K \\ 0 & \delta \end{pmatrix} v. \tag{3.2}$$

For small $\epsilon > 0$, we may then choose $\hat{U} = \{v;\ |v_1| < \epsilon, |v_2| < \epsilon\frac{\delta}{2|K|}\}$. Indeed, for the first component, and with $v_1 > 0$, say, we obtain $\dot{v_1} = \delta v_1 + K v_2 > 0$ whenever $v \in \partial\hat{U}$. However, we should not change the vector field near zero. Thus, we consider

$$\dot{v} = \begin{pmatrix} \delta\chi(\frac{v_1}{\epsilon}) & K \\ 0 & \delta\chi(\frac{2Kv_2}{\delta\epsilon}) \end{pmatrix} v. \tag{3.3}$$

Here, $\chi(\cdot)$ is the standard cutoff-function on $[0,1]$, that is, $\chi(\tau) \in [0,1]$, $\chi(\tau) = 0$ and $\chi(\tau) = 1$ for τ close to zero and one, respectively. Then, (3.3) coincides with (3.2) on the boundary $\partial\hat{U}$, while it coincides with (3.1) near zero. Moreover, the derivative

$$\left\| \begin{pmatrix} \delta\chi(\frac{v_1}{\epsilon}) + \delta\frac{v_1}{\epsilon}D\chi(\frac{v_1}{\epsilon}) & 0 \\ 0 & \delta\chi(\frac{2Kv_2}{\delta\epsilon}) + \delta\frac{2Kv_2}{\delta\epsilon}D\chi(\frac{2Kv_2}{\delta\epsilon}) \end{pmatrix} \right\| \leq \delta\,(1 + \|D\chi\|)$$

of the perturbation is small since $v \in \hat{U}$. Note that we have to choose a vector-valued cut-off function for obtaining the above bound.

We consider now the set-up of section 2. Recall that the space $E_*^{cu} = T_{u_*}(Gu_*) \oplus V_*$ can be decomposed into two H-invariant subspaces. The projection onto V_* along the tangent space $T_{u_*}(Gu_*)$ is denoted Q_*. Moreover, by Lemma 2.1, there exists a matrix $B_* \in L(E_*^{cu})$ with

$$e^{B_* t} v = \rho_{g_*^{-1}(t)} D\Phi_t(u_*) v,$$

for all $v \in E_*^{cu}$. Let $A_* := Q_* B_*|_{V_*}$ in $L(V_*)$. We will define an H-invariant neighborhood $\hat{U}$ of zero in V_*, which depends on small parameters δ and ϵ, such that any solution of

$$\dot{v} = (A_* + \delta\,\mathrm{id})v, \qquad v(0) \in \partial\hat{U},$$

will leave $\hat{U}$ immediately.

As remarked in the previous section, without loss of generality, we may assume that H is compact since its action on V_* is induced by the bounded subgroup $\rho(H) \subset GL(V_*)$. Furthermore, we may choose an H-invariant scalar product using the Haar measure associated with $\rho(H) \subset GL(V_*)$. Thus, by an H-invariant change of coordinates, we can transform A_* into complex Jordan normal form. Let $K > 0$ be a bound for the off-diagonal elements of the matrix A_* written in normal form. Without loss of generality, we consider the case that $\mathrm{spec}(A_*) = \{\lambda\}$ for some eigenvalue λ on the imaginary axis. Otherwise, apply the results below for each eigenvalue, which is possible since generalized eigenspaces are H-invariant. It follows that there exists an H-invariant decomposition of $V_* = \bigoplus_{i=1}^{l} V_*^i$ such that

$$N(A_* - \lambda\,\mathrm{id})^j = \bigoplus_{i=1}^{j} V_*^i$$

for any $j \leq l$, and A_* maps $\bigoplus_{i=1}^{j} V_*^i$ into itself. We write any vector $v \in V_*$ as $v = (v_1, ..., v_l)$ with $v_i \in V_*^i$. In these coordinates, the matrix A_* acts according to

$$A_* v = (\lambda v_1 + A_2 v_2, \lambda v_2 + A_3 v_3, ..., \lambda v_l),$$

where the matrices A_i have norm less than K. We define the H-invariant neighborhood $\hat{U}$ by

$$\hat{U} = \Big\{ v \in V_* = \bigoplus_{i=1}^{l} V_*^i;\ |v_i| < \epsilon \left(\frac{\delta}{2K}\right)^{i-1},\ i = 1, ..., l \Big\}, \tag{3.4}$$

for any $\epsilon > 0$ small.

Finally, define the function

$$\hat{F}(v) := \delta\Big(\chi\Big(\frac{|v_1|}{\epsilon}\Big) v_1, \chi\Big(\frac{2K|v_2|}{\delta\epsilon}\Big) v_2, ..., \chi\Big(\frac{(2K)^{l-1}|v_l|}{\delta^{l-1}\epsilon}\Big) v_l\Big)^*, \tag{3.5}$$

where the cut-off function χ has been defined above. Notice that $\hat{F}$ is H-equivariant and smooth since the norm induced by the H-invariant scalar product is smooth. Moreover, as before,

$$\|D\hat{F}(v)\| \leq \delta\,(1 + \|D\chi\|), \qquad v \in \hat{U}, \tag{3.6}$$

uniformly in (δ, ϵ). It is straightforward to verify that any solution $v(t)$ of

$$\dot{v} = A_* v + \hat{F}(v),$$

with $v(0) \in \partial\hat{U}$ leaves $\hat{U}$ immediately. Indeed, $\hat{F}(v) = \delta v$ for any $v \in \partial\hat{U}$ by construction, and the eigenvalues of A_* have non-negative real part. Therefore, $(A_* + \delta\,\mathrm{id})v$ points outwards of $\partial\hat{U}$ for $v \in \partial\hat{U}$.

3.2 Normal hyperbolicity

In this paragraph, we define a global parametrization of a neighborhood of the group orbit Gu_* which is adapted to the spectral decomposition assumed in Hypothesis 2.

Lemma 3.2 *The complementary projections*

$$Q_G(\rho_g u_*) := \rho_g(\mathrm{id} - Q_*)P_*\rho_{g^{-1}}, \;\; Q_V(\rho_g u_*) := \rho_g Q_* P_* \rho_{g^{-1}}, \;\; Q_S(\rho_g u_*) := \rho_g(\mathrm{id} - P_*)\rho_{g^{-1}}$$

are C^{k+1} in $g \in G$ and depend only on $u = \rho_g u_* \in Gu_*$. They satisfy

$$R(Q_G(\rho_g u_*)) = T_{gu_*}(Gu_*), \;\; R(Q_V(\rho_g u_*)) = \rho_g V_*, \;\; R(Q_S(\rho_g u_*)) = \rho_g W_* := \rho_g N(P_*).$$

In particular, the sets $\{\rho_g u_* + w;\ w \in \rho_g W_*\}$ and $\{\rho_g(u_* + v);\ v \in V_*\}$ are C^{k+1}-bundles over Gu_*, to which we refer as the stable and center bundle.

Proof. The assertions are consequences of Hypothesis 3(iv). ∎

We obtain the following parametrization of a neighborhood of the group orbit Gu_*. There exists an $\eta > 0$ such that, if $|y - Gu_*| < \eta$, then $y = u(y) + v(y) + w(y)$. Here, $u(y) = \rho_{g(y)}u_* \in Gu_*$, $v(y) \in \rho_{g(y)}V_*$, and $w(y) \in \rho_{g(y)}W_*$ are C^{k+1} in y. Since G/H is diffeomorphic to Gu_*, we may choose $g(y)$ locally as a C^{k+1}-function. Indeed, since H is a submanifold of G, we find a submanifold Σ of G transverse to H at $g = \mathrm{id}$ such that the map $\Sigma \to Gu_*$, $g \mapsto gu_*$ is a diffeomorphism locally near $g = \mathrm{id}$. Thus, there exist smooth local charts near any point $u \in Gu_*$. These charts may not fit together globally, though they do if the isotropy group H is compact, see [5].

Using the set $\hat{U}$, see (3.4), we define the G-invariant set

$$N^{cu} := \{\rho_g(u_* + v);\ g \in G,\ v \in \hat{U} \subset V_*\}, \tag{3.7}$$

for any $\delta, \epsilon \in (0, \eta)$. Note that N^{cu} is well-defined since $\hat{U}$ is H-invariant. Thus, for fixed u in Gu_*, it is not important which $g \in G$ with $gu_* = u$ we choose. It is a consequence of Lemma 3.2 and the discussion above that N^{cu} is a C^{k+1}-manifold. Finally, let

$$\tilde{U} := \{\rho_g(u_* + v) + w;\ \rho_g(u_* + v) \in N^{cu},\ w \in \rho_g W_*,\ |w| < \epsilon\} \tag{3.8}$$

be an adapted neighborhood of N^{cu}.

On account of the spectral hypothesis 2 and G-equivariance, there exist constants $C > 0$, $l \in \mathbb{N}$, and $\gamma^s > 0$ such that

$$\|D\Phi_t(\rho_g u_*)|_{\rho_g W_*}\| < Ce^{-\gamma^s t}, \qquad \|D\Phi_{-t}(\rho_g u_*)|_{T_{gu_*}(Gu_*)\oplus\rho_g V_*}\| < C(1 + t^l), \tag{3.9}$$

for $t > 0$ uniformly in $g \in G$. Indeed, Hypothesis 2 and equation (2.6) show that we have

$$D\Phi_t(\rho_g u_*)|_{T_{gu_*}(Gu_*)\oplus\rho_g V_*} = \rho_g \rho_{g_*(t)} e^{B_* t}\rho_{g^{-1}}$$

for some matrix B_* with $\mathrm{Re}\,\mathrm{spec}(B_*) \geq 0$, and, by Lemma 3.1, $\|\rho_g\| = 1$ for all g. Thus, normal hyperbolicity is established.

3.3 Overflowing of N^{cu}

In this paragraph, we extend the nonlinear perturbation $\hat{F}$ as defined in (3.5) to the manifold N^{cu}, and show that N^{cu} is overflowing.

Lemma 3.3 *For $y \in \tilde{U}$, let*

$$\tilde{F}(y) := \rho_{g(y)} \hat{F}(\rho_{g(y)^{-1}} v(y)), \tag{3.10}$$

then $\tilde{F}$ is well-defined, smooth, and G-equivariant.

Proof. We start by verifying that $\tilde{F}$ is well-defined. Without loss of generality, we may again restrict to the case of a single Jordan block since, using the notation of section 3.1, the isotropy group H and the function $\hat{F}$ defined in (3.5) map each subspace V_*^i into itself. For the proof that $\tilde{F}$ is well-defined, assume that

$$\tilde{F}(\rho_{g_j}(u_* + v_j) + w_*) = \rho_{g_j} \chi\Big(\frac{|v_j|}{\epsilon}\Big) v_j$$

for $j = 1, 2$ such that $\rho_{g_1^{-1}} \rho_{g_2} \in H$ and $\rho_{g_1} v_1 = \rho_{g_2} v_2$. Using that χ is a scalar function and the norm is H-invariant, it is straightforward to show that $\tilde{F}$ does not depend on the choice of g_1 and g_2. Equivariance follows in a similar fashion. It is also clear that $\tilde{F}$ is smooth since the charts $g(y)$ are. ■

By (3.6), we have

$$\|D\tilde{F}(y)\| \le C\delta, \qquad y \in \tilde{U}, \tag{3.11}$$

for some constant $C > 0$ uniformly in ϵ. Moreover, by definition of $\hat{F}$,

$$\tilde{F}(y) = \delta v(y), \tag{3.12}$$

for any $y = \rho_{g(y)} u_* + v(y) + w(y)$ with $\rho_{g(y)} u_* + v(y) \in \partial N^{cu}$.

Finally, we modify the vector field in $\tilde{U}$ to achieve overflowing of the boundary of N^{cu}. Consider the equation

$$y_t = -Ay + F(y) + \tilde{F}(y), \qquad y \in \tilde{U}. \tag{3.13}$$

Solving this equation with $y_0 = y(0) \in \tilde{U}$, yields a G-equivariant semiflow denoted by $\tilde{\Phi}_t(y)$.

Lemma 3.4 *Take any point $y = \rho_g u_* + v + w \in \operatorname{clos} \tilde{U}$ with $\rho_g u_* + v \in \partial N^{cu}$, then $\tilde{\Phi}_t(y) \notin \operatorname{clos} \tilde{U}$ for any $t > 0$ small.*

Proof. Without loss of generality, by equivariance, we may consider $y_0 = u_* + v_* + w_*$ with $v_* \in V_*$ and $w_* \in W_*$. Denote the corresponding solution of (3.13) by $y(t) = \tilde{\Phi}_t(y_0)$, and let $u(t) = \Phi_t(u_*)$ be the solution of the original equation (2.1)

$$y_t = -Ay + F(y)$$

with $u(0) = u_*$. Let $\Psi(t, \tau)$ denote the evolution of the linearized equation

$$y_t = -Ay + DF(u(t))y.$$

It is useful to introduce the difference

$$x(t) = y(t) - u(t) = \tilde{\Phi}_t(y_0) - \Phi_t(u_*),$$

then $x(t)$ satisfies the integral equation

$$x(t) = \Psi(t, 0)x_0 + \int_0^t \Psi(t, \tau)\Big(G(\tau, x(\tau)) + \tilde{F}(u(\tau) + x(\tau))\Big)d\tau$$

with $x_0 = v_* + w_*$ and

$$G(t, x) := F(u(t) + x) - F(u(t)) - DF(u(t))x = \mathrm{O}(|x|^2).$$

Since t is small and $|x(0)| \leq \epsilon$ by assumption, we may write

$$\begin{aligned} x(t) &= \Psi(t, 0)(v_* + w_*) + \textstyle\int_0^t \Psi(t, \tau)\tilde{F}(u(\tau) + x(\tau))\, d\tau + \mathrm{O}(\epsilon^2) \\ &= \Psi(t, 0)(v_* + w_*) + \mathrm{O}(t) + \mathrm{O}(\epsilon^2) \end{aligned} \tag{3.14}$$

uniformly for $t \in [0, \eta]$ for some fixed $\eta > 0$. Indeed, the values of the nonlinearity $\tilde{F}$ are in $D(A)$. We will compare the solution $x(t)$ with the function

$$z(t) = \Psi(t, 0)(v_* + w_*) + \int_0^t \Psi(t, \tau)\delta\Psi(\tau, 0)v_*\, d\tau = \Psi(t, 0)((1 + \delta t)v_* + w_*).$$

Substituting the expansion (3.14) of $x(t)$ into $\tilde{F}(u(t)+x(t))$ and using the definition (3.10) of $\tilde{F}$, it is straightforward to calculate that $|x(t) - z(t)| = \mathrm{O}(\epsilon^2 + t^2)$. Therefore,

$$\begin{aligned} x(t) &= \Psi(t, 0)((1 + \delta t)v_* + w_*) + \mathrm{O}(\epsilon^2 + t^2) \\ &= \rho_{\exp(\xi_* t)} e^{B_* t}(1 + \delta t)v_* + \Psi(t, 0)w_* + \mathrm{O}(\epsilon^2 + t^2) \end{aligned}$$

and the claim follows from section 3.1 and the definition (3.8) of $\tilde{U}$. ∎

Summarizing, the modified vector field (3.13) has been constructed such that N^{cu} is overflowing. In addition, the estimates

$$\|\tilde{\Phi}_T(y) - \Phi_T(y)\| \leq C_T\delta\epsilon, \qquad \|D\tilde{\Phi}_T(y) - D\Phi_T(y)\| \leq C_T\delta, \tag{3.15}$$

are true for all $T > 0$. Indeed, the derivative of the term $\tilde{F}(y)$ is of order δ, see (3.11) and an application of the Gronwall lemma proves (3.15).

3.4 The graph transform

The graph transform works as follows. We consider the closed metric space $\Sigma_\#$ of Lipschitz continuous sections of the stable bundle defined by

$$\Sigma_\# := \{\sigma \in C^{0,1}(N^{cu}, Y);\ \sigma(u+v) \in \rho_{g(u)} W_*,\ |\sigma(u+v)| < \epsilon,\ \mathrm{Lip}(\sigma) \le 1\},$$

equipped with the metric $|\sigma - \hat\sigma| := \sup_{y \in N^{cu}} |\sigma(y) - \hat\sigma(y)|$. The time-$T$ map $\tilde\Phi_T$ will induce a contraction $\Phi_\#$ on $\Sigma_\#$ for any sufficiently large T by mapping σ to $\tilde\sigma$ where the latter is defined by

$$y + \tilde\sigma(y) \in \{\tilde\Phi_T(x + \sigma(x));\ x \in N^{cu}\} \tag{3.16}$$

for all $y \in N^{cu}$.

Normal hyperbolicity and overflowing of $\tilde\Phi_T$ have been obtained in equations (3.9), (3.15) and in Lemma 3.4, respectively. Therefore, we may apply the general results described, for instance, in [4, 9, 19, 21] to conclude that $\Phi_\#$ is well-defined and a contraction on $\Sigma_\#$. We can also infer the existence of a unique C^{k+1}-manifold M_*^{cu} which is locally invariant and attracting under $\tilde\Phi_T$, and tangent to N^{cu} at the group orbit Gu_*, see the articles listed above for the details.

It remains to prove that M_*^{cu} is G-invariant and invariant under $\tilde\Phi_t$ for any $t \ge 0$. The first claim follows since $\rho_g M_*^{cu}$ is also invariant under $\tilde\Phi_T$. Indeed, by construction, $\tilde\Phi_T$ is G-equivariant. By uniqueness of M_*^{cu}, we have $\rho_g M_*^{cu} = M_*^{cu}$. By a similar token, we obtain $M_*^{cu} \subset \tilde\Phi_t M_*^{cu}$ for any $t \ge 0$. Since Φ_t and $\tilde\Phi_t$ coincide in a small neighborhood of Gu_*, we see that M_*^{cu} is actually locally invariant under Φ_t. Finally, we prove that the G-action restricted to M_*^{cu} is C^{k+1}. Any point in M_*^{cu} is given by $u + v + \sigma_\#(u+v)$ with $u = \rho_g u_*$ and $v \in \rho_g V_*$. Here, $\sigma_\#$ denotes the fixed point of $\Phi_\#$. Since, by the above discussion, $\sigma_\#$ is G-equivariant and the group acts smoothly on the center bundle, the claim follows immediately.

This completes the proof of Theorem 1.

4 $SE(N)$-equivariant reaction-diffusion equations

Isotropic and excitable media are described by reaction-diffusion systems (1.1)

$$u_t = D\Delta u + f(u,\mu), \qquad x \in \mathbb{R}^N,\ N = 2,3 \tag{4.1}$$

where $D = \mathrm{diag}(d_j)$ is diagonal with non-negative entries $d_j \ge 0$, $u \in \mathbb{R}^M$, and $f : \mathbb{R}^M \times \mathbb{R}^p \to \mathbb{R}^M$ is a C^{k+2}-function for some $k \ge 1$, see section 1. We consider (4.1) on the space $Y = C^0_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ or $Y = L^2(\mathbb{R}^N, \mathbb{R}^M)$. Recall that (4.1) generates a smooth semiflow $\Phi_t(u,\mu)$ on both spaces. More precisely, we require growth conditions on the nonlinearity if the diffusion matrix D is singular and $Y = L^2$, see [8]. Equation (4.1) is equivariant with respect to the action of $SE(N)$ stated in the introduction.

4.1 Isotropy subgroups of relative equilibria

The next lemma classifies the possible isotropy subgroups of relative equilibria u_* and shows that group orbits are embedded provided $SE(N)$ acts smoothly on u_*.

Lemma 4.1 *Suppose that u_* satisfies Hypothesis 3(i), that is, $(R,S)u_*$ is C^{k+2} in $(R,S) \in SE(N)$. Under this condition, Hypothesis 3(ii) is met. In particular, the group orbit of u_* is embedded. In addition, for $N = 2$, the isotropy subgroup H of u_* is $SE(2)$, S^1, or $\mathbb{Z}_\ell$. Similarly, for $N = 3$, the isotropy of u_* is either $SE(3)$ or a compact subgroup of $SO(3)$.*

Proof. We prove the lemma for $N = 2$ and $Y = C^0_{\rm unif}$ since the proofs for $N = 3$ or $Y = L^2$ are similar. We start with the first assertion and argue by contradiction. Throughout, we use the notation $(\varphi, a) \in S^1 \dot{+} \mathbb{R}^2 = SO(2) \dot{+} \mathbb{R}^2 = SE(2)$. The action of (φ, a) on u is then denoted $\rho_{(\varphi,a)}u$. The generator of the rotations is $\frac{\partial}{\partial\varphi}$ with functions written in polar coordinates. Observe that $u_* \in D(\frac{\partial}{\partial\varphi})$ by assumption.

Using compactness of S^{N-1} and the $SO(N)$-component of $SE(N)$, it suffices to consider the following: suppose that there exists a sequence $a_n \in \mathbb{R}$ with $a_n \to \infty$ and some $\epsilon > 0$ such that $\mathrm{dist}((0,(a_n,0)), H) \geq \epsilon$ and $\rho_{(0,(a_n,0))}u_* \to u_*$ as $n \to \infty$. In other words, $u_*(x_1 - a_n, x_2) \to u_*(x_1, x_2)$ uniformly in $(x_1, x_2) \in \mathbb{R}^2$. We will infer a contradiction to $u_* \in D(\frac{\partial}{\partial\varphi})$. Note that either there exist numbers y_1, y_2 and $\tilde{y}_2$ such that $u_*(y_1, y_2) \neq u_*(y_1, \tilde{y}_2)$, or else the function $u_*(x_1, x_2)$ is independent of x_2.

Suppose the former is true, that is, $u_*(y_1, y_2) \neq u_*(y_1, \tilde{y}_2)$ for some y_1, y_2 and $\tilde{y}_2$. Using $\rho_{(0,(a_n,0))}u_* \to u_*$, there exist $\delta > 0$ and numbers $y_2^{(n)} \in [y_2, \tilde{y}_2]$ such that

$$\left|\Big(\frac{\partial}{\partial x_2}u_*\Big)(y_1 - a_n, y_2^{(n)})\right| \geq \delta > 0$$

for any $n \in \mathbb{N}$. The derivative of u_* with respect to φ evaluated at $(y_1, y_2^{(n)})$ is given by

$$\Big(\frac{\partial}{\partial\varphi}u_*\Big)(y_1 - a_n, y_2^{(n)}) = (y_1 - a_n)\Big(\frac{\partial}{\partial x_2}u_*\Big)(y_1 - a_n, y_2^{(n)}) - y_2^{(n)}\Big(\frac{\partial}{\partial x_1}u_*\Big)(y_1 - a_n, y_2^{(n)}).$$

Since $a_n \to \infty$, we obtain a contradiction to boundedness of $\frac{\partial}{\partial\varphi}u_*$ as $\frac{\partial}{\partial x_1}u_*(x)$ is bounded uniformly in $x \in \mathbb{R}^2$.

Next, suppose that the function $u_*(x_1, x_2) = u_*(x_2)$ is independent of x_2. Using the above arguments in the x_1-direction for $x_2 \to \infty$, we conclude that u_* is in fact a constant function reaching a contradiction to $\mathrm{dist}((0,(a_n,0)), H) \geq \epsilon$. Thus the first assertion of the lemma is proved.

If the isotropy subgroup were to contain a translation, we could apply the above results. They show that u_* is in fact a constant function. Otherwise we would reach a contradiction to $u_* \in D(\frac{\partial}{\partial\varphi})$. ∎

Remark 4.2 In passing, we note that, since $SE(N)$, $N = 2, 3$, has no finite-dimensional representations on C^0_{unif}, the isotropy subgroup H of u_* must be compact once the spectral hypothesis 2 is satisfied. Unless, of course, u_* is a constant function and $E^{cu}_* = \{0\}$ is trivial.

4.2 Satisfaction of Hypothesis 3

In this section, we show that Hypotheses 3(iii) and (iv) are satisfied provided the relative equilibrium meets Hypothesis 2, and $SE(N)$ acts smoothly on u_*.

Theorem 3 *Assume that u_* is a relative equilibrium of (4.1) for $N = 2, 3$ on C^0_{unif} or L^2, and satisfies Hypotheses 2 and 3(i). If some of the entries of the diffusion matrix D vanish, assume in addition that u_* is a rotating wave, that is, the generator $(r_*, s_*) = (r_*, 0)$ is a pure rotation. Under these conditions, Hypotheses 3(iii) and (iv) are also satisfied.*

Thus, we have to prove that $(R, S)v$ is C^{k+1} in $(R, S) \in SE(N)$ for any $v \in E^{cu}_*$, and that the spectral projections are C^{k+1}. We start with the latter.

Lemma 4.3 *Under the assumptions of Theorem 3, Hypothesis 3(iv) is obeyed.*

Proof. Since u_* is a relative equilibrium, it satisfies $\Phi_t(u_*, \mu_*) = \exp((r_*, s_*)t)u_*$ for some element $(r_*, s_*) \in \mathbf{so}(N) \times \mathbb{R}^N$. Without loss of generality, we may therefore assume that $\Phi_1(u_*, \mu_*) = (\text{id}, S_*)u_*$, see (1.3). Note that it is here where we use that $N = 2, 3$, since the subgroup $SO(N)$ contains non-trivial tori for $N > 3$. Hence, by (1.2),

$$(R, S)\,(\text{id}, S_*)\,(R, S)^{-1} = (\text{id}, RS_*) \tag{4.2}$$

is a pure translation which depends smoothly on the rotational component R. We claim that the operator

$$L_{(R,S)} := (R, S)\,(\text{id}, S_*)\,(R, S)^{-1}\,D\Phi_1((R, S)u_*, \mu_*) \tag{4.3}$$

depends smoothly on $(R, S) \in SE(N)$ as a map from C^0_{unif} or L^2 into itself. Assume for the moment that the claim is true. Using Dunford-Taylor calculus, we see that the spectral projections associated with $L_{(R,S)}$ are smooth in (R, S). Moreover, by equivariance, they coincide with the projections $(R, S)\,P_*\,(R, S)^{-1}$ appearing in Hypothesis 3(iv). Therefore, it suffices to prove the above claim in order to verify Hypothesis 3(iv).

First, we consider the case that the diffusion matrix D is singular. Then, by assumption, $S_* = 0$ and therefore $L_{(R,S)} = D\Phi_1((R,S)u_*, \mu_*)$, see (4.2) and (4.3). In particular, $L_{(R,S)}$ is smooth in (R,S) and the arguments given above go through.

Next, consider non-singular diffusion matrices D. We argue for the space C^0_{unif}. As explained above, the operator $L_{(R,S)}$ is the composition of a translation and the operator $D\Phi_1((R,S)u_*, \mu_*)$. Since the diffusion matrix is non-singular, $D\Phi_1((R,S)u_*, \mu_*)$ depends smoothly on (R,S) as a map from $C^0_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ into $C^{k+2}_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$, see [8]. Finally, the translations $(\mathrm{id}, R^{-1}S_*)$ are C^{k+1} in R considered as maps from C^{k+2}_{unif} into C^0_{unif}. Therefore, $L_{(R,S)} \in L(C^0_{\mathrm{unif}})$ is C^{k+1} in (R,S). This proves the claim for the space C^0_{unif}. Since the proof for L^2 is similar, we will omit it. ∎

It remains to prove that $(R,S)v$ is C^{k+1} in $(R,S) \in SE(N)$ for any $v \in E^{cu}_*$.

Lemma 4.4 *Under the assumptions of Theorem 3, Hypothesis 3(iii) is obeyed.*

Proof. Throughout the proof, the action of $SE(N)$ on functions u is denoted by either (R,S) or ρ_g with $g = (R,S) \in SE(N)$. Note that Hypothesis 3(iv) is met by the previous lemma. Therefore, by Hypotheses 3(i) and (iv), the set

$$N^{cu}_* := \{\rho_g u_* + \rho_g\, P_* \rho_{g^{-1}}\, v;\ g \in SE(N),\, v \in V_*\}$$

is a C^{k+1}-manifold locally near u_*. Here, V_* has been defined in section 2 as a complement of the tangent space $T_{u_*}SE(N)u_*$ in the eigenspace $E^{cu}_* = R(P_*)$, where P_* is the spectral projection appearing in Hypothesis 2. We claim that $SE(N)$ acts continuously on N^{cu}_*. Suppose the claim is true. Since $SE(N)$ operates continuously on the finite-dimensional smooth manifold N^{cu}_*, the action is in fact smooth, see, for instance, [13, Theorem 5.3], and the assertion of the Lemma follows.

Thus, it remains to prove the claim. Since $SE(N)$ acts smoothly on the group orbit of u_*, it suffices to show that $\rho_g v$ is continuous in $g \in SE(N)$ for any $v \in V_*$.

For $v \in E_{cu} = R(P_*)$ and $g \in SE(N)$,

$$|(1-P_*)(\rho_g v - v)| = |(1-P_*)\rho_g v| \le |(1-P_*)\rho_g P_* \rho_{g^{-1}}|\, |\rho_g v|,$$

because $(\rho_g P_* \rho_{g^{-1}})\rho_g v = \rho_g v$. Since $\|(1-P_*)(\rho_g P_* \rho_{g^{-1}})\| \to 0$ as $g \to \mathrm{id}$, we infer that $(1-P_*)(\rho_g v - v)$ is continuous at $g = \mathrm{id}$.

It remains to show that $P_* \rho_{g_n} v$ converges to $P_* v$ for any sequence $g_n \to \mathrm{id}$ in $SE(N)$ as $n \to \infty$. We argue by contradiction: suppose that there is some $\epsilon > 0$ such that $|P_* \rho_{g_n} v - v| \ge \epsilon$ for all n. Since $P_* \rho_{g_n} v$ is bounded and E^{cu}_* is finite-dimensional, there exists a convergent subsequence, which we again denote by g_n, such that $P_* \rho_{g_n} v \to \tilde{v}$ for some $\tilde{v} \in E^{cu}_*$. This, however, implies $v = \tilde{v}$ and a contradiction is obtained. Indeed, for the representation of $SE(N)$ on C^0_{unif} or L^2, if $(R_n, S_n) \to (\mathrm{id}, 0)$ and $(R_n, S_n)v \to \tilde{v}$ as $n \to \infty$ then $v = \tilde{v}$. ∎

Remark 4.5 Note that no use has been made in the proof of Lemma 4.4 of particular features of equation (4.1) or the function spaces involved except for the property: if $g_n \to \mathrm{id}$ and $\rho_{g_n} v \to \tilde{v}$ as $n \to \infty$, then $v = \tilde{v}$.

Theorem 3 is a consequence of Lemmata 4.3 and 4.4.

5 Spiral waves in two-dimensional excitable media

Consider the set-up of section 4 with $N = 2$. We will use a slightly different notation for the group action, namely

$$(\rho_{(\varphi,a)}u)(x) := u(R_{-\varphi}(x-a)),$$

where $(\varphi, a) \in S^1 \dot{+} \mathbb{R}^2 = SO(2) \dot{+} \mathbb{R}^2 = SE(2)$. The matrix R_φ denotes the rotation by the angle φ around zero in $\mathbb{R}^2$.

5.1 Center manifolds near spiral waves

For the sake of clarity, we formulate the results for the space C^0_{unif} though they are also true for L^2, then with H^k replacing C^k.

We assume that $u_* \in C^0_{\mathrm{unif}}$ is a rotating wave of (4.1) for $\mu = \mu_*$ satisfying Hypothesis 1, that is,

$$\Phi_t(u_*, \mu_*) = \rho_{(\omega_* t, 0)} u_*$$

for some ω_*. First, it is shown that Hypothesis 3(i) is satisfied.

Lemma 5.1 *Assume that u_* is a rotating wave. If the diffusion matrix D is singular, assume in addition that $u_* \in C^{k+2}_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$. Under these conditions, Hypothesis 3(i) is satisfied.*

Proof. If D is positive, we observe that u_* is of class C^{k+2} by regularity properties of (4.1), see [8]. Therefore, the translations $\rho_{(0,a)} : u_*(\cdot) \mapsto u_*(\cdot - a)$ act smoothly on u_*. The one-parameter family of rotations $\rho_{(\varphi,0)}$ act smoothly on u_* since, by definition, the action coincides with the time evolution of the rotating wave u_* provided $\omega_* \neq 0$. If $\omega_* = 0$, the steady state is a rotating wave for any frequency $\omega_* > 0$, and thus also smooth. ∎

We have then the following application of Theorem 1.

Theorem 4 *Let u_* be a rotating wave of (4.1). Suppose that the spectral hypothesis 2 is met. If the diffusion matrix D is singular, assume in addition that $u_* \in C^{k+2}_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$.*

Then, for any μ with $|\mu - \mu_*|$ sufficiently small, there exists an $SE(2)$-invariant, locally flow-invariant manifold M^{cu}_μ contained in C^0_{unif}. The manifold M^{cu}_μ and the action of $SE(2)$ on M^{cu}_μ are of class C^{k+1} and depend C^{k+1}-smoothly on the parameter μ. Furthermore, M^{cu}_μ contains all solutions which stay close to the group orbit of u_* for all negative times. Finally, M^{cu}_μ is locally exponentially attracting.

Proof. We have to show that the assumptions of Theorem 1 are obeyed. Hypothesis 3(i) is met by Lemma 5.1. Therefore, we may apply Lemma 4.1 and Theorem 3 to conclude that Hypotheses 3(ii), (iii) and (iv) are satisfied. This completes the proof. ∎

Remark 5.2 Assume that the diffusion matrix D is singular. Using the results of [22], it is possible to prove that $\rho_{(\varphi,a)}u_*$ is C^{k+2} in $\rho_{(\varphi,a)} \in SE(2)$ whenever the group acts continuously on u_*. Therefore, the assumption $u_* \in C^{k+2}_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$ appearing in Theorem 4 can be replaced by the following weaker one: $SE(2)$ acts continuously on u_*.

We also remark that Theorem 4 remains true for more general relative equilibria provided Hypothesis 3(i) is met.

Under the assumptions of Theorem 4, the isotropy H of u_* is either $\mathbb{Z}_\ell$ or S^1, see Lemma 4.1 and Remark 4.2. Thus, we can apply the results of [5], see Theorem 2, and obtain the following theorem. As in section 2, we choose an H-invariant complement V_* of $T_{u_*}(SE(2)u_*)$ in the generalized eigenspace E^{cu}_*.

Theorem 5 *Suppose that the assumptions of Theorem 4 are met. The isotropy subgroup H of u_* is then either $\mathbb{Z}_\ell$ or S^1. The manifold M^{cu}_μ is diffeomorphic to $(SE(2) \times V_*)/\sim$, where the equivalence relation on $SE(2) \times V_* = S^1 \times \mathbb{C} \times V_*$ is defined by $(\varphi, a, v) \sim (\varphi + \hat\varphi, a, \rho_{(-\hat\varphi,0)}v)$ for any $(\hat\varphi, 0)$ in the isotropy H of u_*. Furthermore, the pull-back of the vector field on M^{cu}_μ to $SE(2) \times V_*$ as defined in Theorem 2 is of skew-product form*

$$\begin{aligned} \dot\varphi &= f_1(v,\mu) \\ \dot a &= e^{i\varphi} f_2(v,\mu) \\ \dot v &= f_N(v,\mu), \end{aligned} \tag{5.1}$$

and H-equivariant:

$$(f_1, f_2, f_N)(\rho_{(\hat\varphi,0)}v, \mu) = (f_1, e^{i\hat\varphi} f_2, \rho_{(\hat\varphi,0)} f_N)(v, \mu).$$

Finally, $(f_1, f_2, f_N)(0, \mu_*) = (\omega_*, 0, 0)$.

Proof. The theorem follows from Theorem 2 and 4 once the adjoint representation has been computed. Identifying $SE(2)$ with $S^1 \dot{+} \mathbb{C}$, and its Lie algebra $\mathbf{se}(2)$ with $\mathbb{R} \times \mathbb{C}$, the group structure on $SE(2)$ is given by

$$(\tilde{\varphi}, \tilde{a})(\varphi, a) = (\varphi + \tilde{\varphi}, e^{\mathrm{i}\tilde{\varphi}} a + \tilde{a}).$$

In particular, the inverse of (φ, a) is

$$(\varphi, a)^{-1} = (-\varphi, -e^{-\mathrm{i}\varphi} a).$$

Thus, the adjoint action $\mathrm{Ad}_{(\varphi,a)}$ of $SE(2)$ on the Lie algebra $\mathbf{se}(2)$ is given by

$$\mathrm{Ad}_{(\varphi,a)}(r, s) = (\varphi, a)\,(r, s)\,(\varphi, a)^{-1} = (r, e^{\mathrm{i}\varphi} s - \mathrm{i} r a)$$

and, in particular,

$$\mathrm{Ad}_{(\tilde{\varphi},0)}(r, s) = (r, e^{\mathrm{i}\tilde{\varphi}} s). \tag{5.2}$$

Any element in the isotropy group H is of the form $(\tilde{\varphi}, 0)$. Thus the theorem is proved. ■

5.2 The spectral hypothesis 2

We remark that Lemma 2.1 relates the spectral assumption 2 to the spectrum of the linearization of (5.1) at the rotating wave u_*. It is possible to make this relation more explicit. For that purpose, we have to work in either $L^2(\mathbb{R}^2, \mathbb{R}^M)$ or else the subspace $C^0_{\mathrm{eucl}}(\mathbb{R}^2, \mathbb{R}^M)$ of $C^0_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$ which is defined as the closure of $D(\frac{\partial}{\partial\varphi})$ in C^0_{unif}, see [22]. On L^2 and C^0_{eucl}, the one-parameter family of rotations acts as a strongly continuous semigroup. It is then possible to write Hypothesis 2 in terms of the spectrum of the operator

$$L := D\Delta - \omega_* \frac{\partial}{\partial\varphi} + D_u f(u_*, \mu_*), \tag{5.3}$$

that is, the linearization of the spiral wave in a rotating frame. Note that L generates a C^0-semigroup on either $C^0_{\mathrm{eucl}}(\mathbb{R}^2, \mathbb{R}^M)$ or $L^2(\mathbb{R}^2, \mathbb{R}^M)$, see [22], but not necessarily on $C^0_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$.

Lemma 5.3 *Consider equation (4.1) on either $C^0_{\mathrm{eucl}}(\mathbb{R}^2, \mathbb{R}^M)$ or $L^2(\mathbb{R}^2, \mathbb{R}^M)$. Furthermore, assume that u_* is a rotating wave solution. Suppose that* $\mathrm{spec}(L) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq 0\}$ *is a spectral set with spectral projection P_*. If* $\dim P_*(E^{cu}_*) < \infty$ *and the semigroup e^{Lt} satisfies*

$$\|e^{Lt}|_{(1-P_*)E^{cu}_*}\| \leq Ce^{-\beta t}$$

for some $\beta > 0$, then Hypothesis 2 is true. In that case, we have $\mathrm{spec}(Df_N(0, \mu_*)) = \mathrm{spec}(Q_* L|_{V_*})$, *where $V_* = R(Q_*)$ is an H-invariant complement of $T_{u_*}(SE(2)u_*) = N(Q_*)$ in E^{cu}_* with associated projection Q_*.*

Proof. Since the operator L generates a C^0-semigroup on either space, we have $D\Phi_t(u_*, \mu_*) = \rho_{(\omega_* t, 0)} e^{Lt}$, see [22, Lemma 3.7]. In particular, $D\Phi_{2\pi/\omega_*}(u_*, \mu_*) = e^{2\pi/\omega_* L}$. The remaining assertions follow from Lemma 2.1. ∎

Finally, consider the operator L on L^2.

Hypothesis 4 *Assume that the spectrum of the operator* $A_\infty := D\Delta + D_u f(0, \mu_*)$ *on* $L^2(\mathbb{R}^2, \mathbb{R}^M)$ *satisfies* $\operatorname{spec}(A_\infty) < -\beta < 0$.

Lemma 5.4 *Consider equation (4.1) on* $L^2(\mathbb{R}^2, \mathbb{R}^M)$. *Let the diffusion matrix* D *be non-singular. We assume that* $u_* \in L^2(\mathbb{R}^2, \mathbb{R}^M)$ *is a rotating wave such that* $u_*(x) \to 0$ *uniformly in* $|x| \to \infty$. *Suppose that Hypothesis 4 is met. Under these conditions, Hypothesis 3 is obeyed. In fact,*

$$\operatorname{spec}(e^L) \cap \{\lambda \in \mathbb{C};\ |\lambda| \geq 1\} = \exp\Big(\operatorname{spec}(L) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geq 0\}\Big)$$

is a spectral set and $\dim E^{cu}_* < \infty$ *is true for the associated generalized eigenspace. Moreover,* $\operatorname{spec}(Df_N(0, \mu_*))$ *and* $\operatorname{spec}(L)$ *are related as in Lemma 5.3.*

Proof. The proof is motivated by [3, Chapter 4]. Note that $-A_\infty$ is sectorial with domain $H^2(\mathbb{R}^2, \mathbb{R}^M)$ since the diffusion matrix D is positive. Therefore, $\operatorname{spec}(e^{A_\infty})$ lies inside the circle of radius $e^{-\beta}$, see [8], and

$$\|e^{A_\infty t}\| \leq C e^{-\beta t} \tag{5.4}$$

for some positive C and all $t > 0$. The operator

$$L_\infty = D\Delta - \omega_* \frac{\partial}{\partial\varphi} + D_u f(0, \mu_*)$$

generates a strongly semigroup given by $e^{L_\infty t} = \rho_{(-\omega_* t, 0)} e^{A_\infty t}$. Since the rotations $\rho_{(-\omega_* t, 0)}$ have norm one, $\operatorname{spec}(e^{L_\infty})$ is also contained inside the circle of radius $e^{-\beta}$. Indeed, use the estimate (5.4), and the relation between spectral radius and the norm of powers of the operator. We claim that $e^{Lt} - e^{L_\infty t}$ is compact for any $t > 0$. Suppose for the moment that the claim is true. Then, by [11, Theorem IV.5.35], the essential spectra

$$\operatorname{spec}_{\mathrm{ess}}(e^L) = \operatorname{spec}_{\mathrm{ess}}(e^{L_\infty}) \subset \operatorname{spec}(e^{A_\infty}) \subset \{\lambda \in \mathbb{C};\ |\lambda| < e^{-\beta}\}$$

coincide. Here, the essential spectrum $\operatorname{spec}_{\mathrm{ess}}$ denotes the complement (in the spectrum) of the set of isolated eigenvalues with finite multiplicity. Therefore, Hypothesis 3 is satisfied. Also, the relation between the point spectra of L and e^L outside the circle of radius $e^{-\beta}$ is a consequence of [16, Theorem 2.2.4, p. 46]. It remains to prove that

$$e^{Lt} - e^{L_\infty t} = \int_0^t e^{L(t-\tau)} K e^{L_\infty \tau}\, d\tau \tag{5.5}$$

is compact for positive t. Here, the bounded operator K is given by $K = D_u f(u_*, \mu_*) - D_u f(0, \mu_*)$. Note that K is compact from $H^2(R^2, \mathbb{R}^M)$ to L^2 since $u_*(x) \to 0$ uniformly as $|x| \to \infty$, see [3, pp. 27–28]. Therefore,

$$K e^{L_\infty \tau} = K \rho_{(-\omega_* \tau, 0)} e^{A_\infty \tau} \in L(L^2)$$

is compact for $\tau > 0$ since $e^{A_\infty \tau}$ maps L^2 into H^2 and $\rho_{(-\omega_* \tau, 0)} \in L(H^2)$. By the arguments given in [3, p. 28], the integrand appearing on the right hand side of (5.5) is norm-continuous in $\tau \in (0, t]$. Thus, $\int_\nu^t e^{L(t-\tau)} K e^{L_\infty \tau} \, d\tau$ is compact for any $\nu > 0$. Since the set of compact operators is closed in the norm-topology, and $\| \int_0^\nu e^{L(t-\tau)} K e^{L_\infty \tau} \, d\tau \| \leq C\nu$ in norm, the integral in (5.5) is compact. This proves the claim and thus the lemma. ∎

5.3 Bifurcations of spiral waves

Summarizing, a center-manifold reduction to a smooth and $SE(2)$-equivariant manifold near ℓ-armed spiral waves has been obtained. The skew-product structure of the vector field on the center manifold has been proved in [5]. Finally, at least on $C^0_{\text{eucl}}(\mathbb{R}^2, \mathbb{R}^M)$ and $L^2(\mathbb{R}^2, \mathbb{R}^M)$, the spectrum of the reduced vector field (5.1) has been explicitly related to the spectrum of the linearization of (4.1). Thus, we may investigate bifurcations of the H-equivariant normal component

$$\dot{v} = f_N(v, \mu)$$

of (5.1) and study the drift along the group orbit using the H-equivariant equation

$$\begin{pmatrix} \dot{\varphi} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} f_1(v, \mu) \\ e^{i\varphi} f_2(v, \mu) \end{pmatrix}.$$

For Hopf bifurcations from rigidly-rotating ℓ-armed spiral waves to meandering or drifting waves, this program has been carried out in [6] and [5] to which we refer for more details. In [6], the consequences of Takens-Bogdanov bifurcations for one-armed spirals have been discussed. By Theorem 5, similar statements hold for ℓ-armed waves. Note that the formal reduction given in [6] requires that the center bundle is trivial. Takens-Bogdanov bifurcations near ℓ-armed spiral waves may result in non-trivial bundles. However, the center-manifold theorem 4 and the associated reduction described in [5], see Theorem 5, do not suffer from this drawback. Therefore, our results cover Takens-Bogdanov bifurcations near ℓ-armed spiral waves.

6 Twisted scroll rings in $SE(3)$-equivariant systems

In numerical simulations of reaction-diffusion systems on $\mathbb{R}^3$, *twisted scroll rings* have been observed in [15]. These are relative equilibria with finite isotropy group $\mathbb{Z}_\ell$ which rotate around the x_3-axis, say, and additionally drift along the same axis with constant speed. We may think of a one-parameter family of ℓ-armed spirals with a core aligned along the unit circle parallel to the (x_1, x_2)-plane. The spiral patterns occur, locally, in the bundle of normal planes to the core circle. Such patterns are called *scroll waves*. Hopf bifurcations of scroll waves can be analyzed using Theorem 1.

Consider the reaction-diffusion system (4.1) with $N = 3$ and for positive diffusion matrix D. In mathematical terms, a twisted scroll ring u_* satisfies

$$\Phi_t(u_*) = \exp(\xi_* t)u_*,$$

where $\xi_* = (r_*, s_*) \in \mathbf{so}(3) \times \mathbb{R}^3 = \mathbf{se}(3)$ in the Lie algebra of $SE(3)$ has the special form

$$r_* = \begin{pmatrix} 0 & -\omega_* & 0 \\ \omega_* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; s_* = \begin{pmatrix} 0 \\ 0 \\ c_* \end{pmatrix}. \tag{6.1}$$

The temporal evolution of the twisted scroll ring is then given by

$$\Phi_t(u_*)(x) = u_*(R_*(-\omega_* t)x - s_* t),$$

where $R_*(\varphi)$ denotes the rotation by the angle φ around the x_3-axis in $\mathbb{R}^3$.

We assume that the group orbit $SE(3)u_*$ is smooth which is clearly satisfied for the scroll waves observed in numerical simulations.

Theorem 6 *Assume that the relative equilibrium u_* meets Hypotheses 2 and 3(i). The conclusions of Theorem 4 are then valid with $SE(3)$ replacing $SE(2)$.*

Proof. By assumption, Hypothesis 3(i) is met. Thus, Theorem 3 applies, and Hypotheses 3(iii) and (iv) are obeyed. Similarly, by Lemma 4.1, Hypothesis 3(ii) is satisfied. Finally, Theorem 1 proves the assertion of the theorem. ■

Remark 6.1 Using an extension of the results of [22], we can prove that Hypothesis 3(i) is satisfied whenever $SE(3)$ acts continuously on u_*. Therefore, the assertion of Theorem 6 is true provided $SE(3)$ acts continuously on the scroll wave u_* and Hypothesis 2 is met. The proof will appear elsewhere. We should also emphasize that the relation (6.1) on the generator ξ_* has not been assumed in Theorem 6.

Using the reduced differential equations (2.3), the dynamics near Hopf bifurcations from twisted scroll waves can be analyzed. It turns out that bifurcating solutions drift approximately in the x_3-direction. In a plane perpendicular to the vertical propagation direction, the bifurcating scroll rings perform a planar meandering or drifting motion. We refer to [5, Section 6] for the details.

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