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Abstract

A novel generalization of Γ-convergence applicable to a class of equilibrium problems is studied. After the introduction of the latter, a variety of its applications is discussed. The existence of equilibria with emphasis on Nash equilibrium problems is investigated. Subsequently, our Γ-convergence notion for equilibrium problems, generalizing the existing one from optimization, is introduced and discussed. The work ends with its application to a class of penalized generalized Nash equilibrium problems and quasi-variational inequalities.

1 Introduction

Equilibrium problems form an umbrella of various problems relevant in applied mathematics (see [KR18, Chapter 2] for an overview). Among them, especially optimization problems are a powerful tool for various applications. Thus, it is not surprising, that equilibrium problems such as Nash equilibrium problems and (quasi-)variational inequalities (abbr.: (Q)VIs) enjoyed growing interest in the recent literature (cf. [FK07], [PF05] for an overview, as well as the recent articles [HS13], [HSK15], [KKSW19], [AHR19]). Many of these problems contain constraints or suffer from a lack of smoothness and are often addressed via a sequence of more regular problems. Such a sequential approximation aims at approaching the original problem in the limit via a sequence of more tractable approximating problems. A powerful concept in this context utilized in optimization is Γ-*convergence* (cf. [Bra02], [DM12]).
Let a reflexive Banach space U be given. Consider a subset $U_{\mathrm{ad}} \subseteq U$ and functionals $(\mathcal{E}_n)_{n\in\mathbb{N}}, \mathcal{E} : U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$. The sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is called Γ-*convergent* to $\mathcal{E}$, denoted by $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$, if the following two conditions are fulfilled:

(i) For all sequences $u^n \to u$ holds $\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u^n)$.

(ii) For all $v \in U_{\mathrm{ad}}$ exists a sequence $v^n \to v$ with $\limsup_{n\to\infty} \mathcal{E}(v^n) \le \mathcal{E}(v)$,

where the sequences are chosen in U_{ad}.
The above displayed notion of Γ-convergence is tailored to optimization problems and in principle not immediately applicable to broader problem classes like general (quasi-)variational inequalities or Nash games. The main difficulty therein is the dependence of the constraint set on the solution, respectively of the objective on the stategies of the other players. This article provides a generalization of Γ-convergence to equilibrium problems covering the aforementioned applications.
The rest of this work is organized as follows: In Section 2, we introduce and discuss the equilibrium concept addressed in this work and provide applications embedding into our concept. In Section 3, we derive an abstract existence result and apply it to Nash games. Eventually, in Section 4, we derive our generalized Γ-convergence notion and apply it to a penalization technique addressing the applications discussed in Section 2.

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2 Equilibrium Problems

First, we introduce the type of equilibrium problem under investigation in this article. For this sake we denote for a given functional $F : U \to \mathbb{R} \cup \{+\infty\}$ the *domain* by $\mathrm{dom}\,(F) := \{u \in U : f(u) < \infty\}$.

Definition 1. Consider a reflexive Banach space U and a subset $U_{\mathrm{ad}} \subseteq U$ as well as a functional $\mathcal{E} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \overline{\mathbb{R}}$ with $\mathrm{dom}\,(\mathcal{E}(\,\cdot\,, u)) \neq \emptyset$ for all $u \in U_{\mathrm{ad}}$. A point $u \in U_{\mathrm{ad}}$ is called an *equilibrium (of $\mathcal{E}$)*, if

$$\mathcal{E}(u, u) \leq \mathcal{E}(v, u) \text{ holds for all } v \in U_{\mathrm{ad}}.$$

Occasionally, the first component in $\mathcal{E}$ is referred to as *control component* and the second component as *feedback component*. Evidently, optimization problems are a special instance of equilibrium problems without feedback component. In [BO94] the term *equilibrium problem* has been introduced as a problem of the following type:
Let a set $C \subseteq U$ and a bifunction $\Psi : C \times C \to \mathbb{R}$ be given. Seek $u \in C$, such that

$$\Psi(u, v) \leq 0 \text{ for all } v \in C.$$

There is a generalization of this problem called *quasi-equilibrium problem* (cf. [NO94]) incorporating a dependence of the feasible set on one of the variables. In that setting, a set-valued operator $C : U \rightrightarrows U$ is considered, leading to the problem of finding $u \in C(u)$, such that

$$\Psi(u, v) \leq 0 \text{ for all } v \in C(u).$$

These problems have been extensively discussed in the literature with special emphasis on providing existence results. Here, we refer again to [KR18] and [ACI17]. Using the difference in Definition 1 we can introduce the *Nikaido–Isoda functional* (compare to [NI55]) reading for $u, v \in U_{\mathrm{ad}}$ with $v \in \mathrm{dom}\,(\mathcal{E}(\,\cdot\,, u))$ as

$$\Psi(u, v) = \mathcal{E}(u, u) - \mathcal{E}(v, u). \tag{1}$$

Comparing to the concept given [BO94] the Nikaido–Isoda functional provides the link between (quasi)-equilibrium problems and the one given in Definition 1. Our approach addresses both problem classes simultaneously, as the set-valued mapping can be hidden in the feedback dependent domain. The Nikaido–Isoda functional allows another characterization related to optimization, which is given in the next theorem.

Theorem 2 (compare to [NI55, Lemma 3.1])**.** *Let a functional $\mathcal{E}$ as in Definition 1 be given. Then, u is an equilibrium, if and only if $V(u) = 0$, where $V : U_{\mathrm{ad}} \to [0, +\infty]$ denotes the value function defined by*

$$V(u) = \sup_{v \in \mathrm{dom}(\mathcal{E}(\cdot, u))} \Psi(u, v)$$

for given $u \in U_{\mathrm{ad}}$ with Ψ being the Nikaido–Isoda functional *defined in* (1).

Proof. Let $u \in U_{\mathrm{ad}}$ be an equilibrium of $\mathcal{E}$. By Definition 1 holds therefore $\mathcal{E}(u, u) \leq \mathcal{E}(v, u)$ for all $v \in U_{\mathrm{ad}}$. Thus, as $\mathrm{dom}\,(\mathcal{E}(\cdot, u)) \neq \emptyset$ also $\mathcal{E}(u, u) < +\infty$ respectively $u \in \mathrm{dom}\,(\mathcal{E}(\cdot, u))$ holds true. Hence

$$0 = \mathcal{E}(u, u) - \mathcal{E}(u, u) = \Psi(u, u) \leq V(u) = \sup_{v \in \mathrm{dom}(\mathcal{E}(\cdot, u))} \Psi(u, v) \leq 0,$$

which implies $V(u) = 0$.
For the other direction, assume $V(u) = 0$. Choosing an arbitrary $v \in \mathrm{dom}\,(\mathcal{E}(\cdot, u))$ yields

$$\mathcal{E}(u, u) - \mathcal{E}(v, u) \leq V(u) = 0,$$

implying u being an equilibrium. □

We note, that separating the equilibrium problem according to Definition 1 from the other presented ones helps to emphasize the optimization related structure: For every $u \in U_{\rm ad}$ we are interested in the optimization problem $\min \mathcal{E}(\,\cdot\,, u)$ having a solution. This motivates the formulation of the following set-valued operator.

Definition 3 (Best response operator)**.** Consider a functional $\mathcal{E} : U_{\rm ad} \times U_{\rm ad} \to \mathbb{R} \cup \{+\infty\}$ as in Definition 1. The *best response operator* is the set-valued mapping $\mathcal{B} : U_{\rm ad} \rightrightarrows U_{\rm ad}$ defined by

$$\mathcal{B}(u) := \operatorname{argmin}_{v \in U_{\rm ad}} \mathcal{E}(v, u).$$

In principle, this operator is allowed to have empty values. Within the scope of this work however we are interested in functionals, where the associated minimization problem yields a solution. By Definition 1, a point $u \in U_{\rm ad}$ is an equilibrium of $\mathcal{E}$, if and only if

$$u \in \mathcal{B}(u)$$

holds true. Thus, the equilibrium problem proposed in Definition 1 can equivalently be interpreted as the minimization of a merit functional as proposed in Theorem 2 or a fixed point problem of a set-valued operator. Regarding existence especially the latter will be of importance for us. But before addressing this issue, we draw our attention to a few examples to demonstrate the applicability of the introduced concept.

2.1 Application to Partial Differential Equations

Consider for instance the following partial differential equation (abbr.: PDE) for a constant $\alpha > 0$ related to a simplified Ginzburg–Landau model for superconductivity in absence of a magnetic field, see [Tin04, Chapter 1,4]:
Seek $u \in H_0^1(\Omega)$, such that

$$\begin{aligned} -\Delta u + u^3 - \alpha u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{PDE}$$

where $\Omega \subseteq \mathbb{R}^d$ denotes an open, bounded domain with $\partial\Omega$ its boundary. Besides its relation to physics this example is of mathematical interest, since it contains a non-monotone operator. One can indeed interpret it as the first order system of an optimization problem. However, (PDE) does not need to be a sufficient optimality condition. Moreover, one can show (e.g. using the techniques in [BS10, Section 2.3.2]), that for sufficiently large α this system has besides its trivial solution as well another non-trivial one. As the operator is odd, (PDE) has at least three solutions. The trivial solution might not be local minimizer and then, this equation cannot be approached using optimization techniques only to obtain all solutions. Therefore, we seek to embed this equation into the setting of Definition 1:
For this sake, consider for an arbitrary $u \in H_0^1(\Omega)$ the following equation:
Seek $v \in H_0^1(\Omega)$, such that

$$\begin{aligned} -\Delta v + v^3 &= \alpha u \quad \text{in } \Omega, \\ v &= 0 \quad\;\; \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Here, $H_0^1(\Omega)$ denotes the classical Sobolev space as defined, e.g., in [AF03, Definition 3.2]. Using standard arguments of calculus of variations and convex analysis it is straightforward to see, that (2) can be interpreted as a first order system of a convex optimization problem in v using $U = U_{\rm ad} = H_0^1(\Omega)$ and the functional $\mathcal{E} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}(v, u) := \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{1}{4}\|v\|^4_{L^4(\Omega)} - \alpha(v, u)_{L^2(\Omega)}.$$

Returning to (PDE) and using the best response operator $\mathcal{B}$ of the above defined $\mathcal{E}$ (cf. Definition 3) we can rewrite (2) as $\mathcal{B}(u) = v$. Then, it is evident, that (PDE) is equivalent to $u = \mathcal{B}(u)$. This leads to the assertion, that u is an equilibrium of $\mathcal{E}$, if and only if u solves (PDE).
Alternatively, one could have associated the equation to the following one:
Seek $v \in H_0^1(\Omega)$, such that

$$\begin{aligned} -\Delta v &= \alpha u - u^3 && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then, one can relate the following functional

$$\mathcal{E}(v,u) := \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)} + (u^3 - \alpha u, v)_{L^2(\Omega)}$$

instead. Returning again to Definition 1, we associate (PDE) to the equilibrium of the second functional. Hence, we see that a given system can be put into our equilibrium framework in different ways.

2.2 Nash Equilibrium Problems

Let a family of Banach spaces U_i for $i = 1,\dots,N$ for $N \geq 1$ be given. Define the space $U := U_1 \times \dots \times U_N$ as well as the *strategy sets* $U^i_{\mathrm{ad}} \subseteq U_i$ and the joint strategy set $U_{\mathrm{ad}} := \prod_{i=1}^N U^i_{\mathrm{ad}}$ together with a family of real-valued functionals $\mathcal{J}_i : U_{\mathrm{ad}} \to \mathbb{R}$ for all $i = 1,\dots,N$. The idea is, that every index is associated with a player seeking to minimize his objective choosing an argument called *strategy* from a feasible set. With the index $-i$ we denote strategies, where the i-th component has been omitted. A joint strategy $(u_1,\dots,u_{i-1},v_i,u_{i+1},\dots,u_N) \in U$ is written as (v_i, u_{-i}) – with no change of the ordering. Consider the *Nash equilibrium problem* (abbr.: NEP) (cf. [Nas90], [Nas50]) reading as follows:
Seek $u \in U_{\mathrm{ad}}$, such that for all indices $i = 1,\dots,N$ the relation

$$\mathcal{J}_i(u_i, u_{-i}) \leq \mathcal{J}_i(v_i, u_{-i}) \text{ for all } v_i \in U^i_{\mathrm{ad}} \tag{NEP}$$

holds true. Now, we are dealing with a system of N coupled optimization problems instead of only one. Note, however, that they are formulated in separate components, i.e.: only u_i is used to minimize $\mathcal{J}_i(\cdot, u_{-i})$. In the sense of Definition 1 we formulate the following functional

$$\mathcal{E}^{\mathrm{NEP}}(v,u) := \sum_{i=1}^N \mathcal{J}_i(v_i, u_{-i}).$$

By the product structure, it is straightforward to show that the equilibrium problem induced by $\mathcal{E}^{\mathrm{NEP}}$ is equivalent to (NEP).
In a similar fashion we can as well treat *generalized* NEPs (abbr.: GNEPs), see [FK07]. Here, an additional restriction is imposed via the strategy mapping of the i-th player $C_i : U^{-i}_{\mathrm{ad}} \rightrightarrows U^i_{\mathrm{ad}}$. Combining all the C_i's leads to the (joint) strategy mapping $C : U_{\mathrm{ad}} \rightrightarrows U_{\mathrm{ad}}$ defined by $C(u) := C_1(u_{-1}) \times \dots \times C_N(u_{-N})$ and to the problem of finding $u \in U_{\mathrm{ad}}$ with $u \in C(u)$, such that for all $i = 1,\dots,N$ the relation

$$\mathcal{J}_i(u_i, u_{-i}) \leq \mathcal{J}_i(v_i, u_{-i}) \text{ for all } v_i \in C_i(u_{-i}) \tag{GNEP}$$

holds true. Then, the functional $\mathcal{E}^{\mathrm{NEP}}$ can be modified by adding the indicator realizing the additional constraint yielding

$$\mathcal{E}^{\mathrm{GNEP}}(v,u) := \sum_{i=1}^N \left(\mathcal{J}_i(v_i, u_{-i}) + I_{C_i(u_{-i})}(v_i)\right).$$

Here, as well as in the remainder of this article, the indicator function of a set $M \subseteq U$ is defined by

$$I_M(u) = \begin{cases} 0 & \text{if } x \in M, \\ \infty & \text{else.} \end{cases}$$

A frequently encountered special case is the one of *shared* constraints or *joint constraints*, where the set-valued mapping C can be characterized via a single set $\mathcal{F} \subseteq U_{\text{ad}}$ by the relation $v_i \in C_i(u_{-i})$, if and only if $(v_i, u_{-i}) \in \mathcal{F}$.
This motivates the formulation of a modified solution concept called *variational equilibrium* (also known as *normalized equilibrium*) that has been introduced in [Ros65]. Here, the values of the strategy mapping are replaced by the set $\mathcal{F}$. This leads to the problem of finding $u \in \mathcal{F}$, such that

$$\sum_{i=1}^{N} \mathcal{J}_i(u_i, u_{-i}) \leq \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) \text{ for all } v \in \mathcal{F}$$

and to the functional

$$\mathcal{E}^{\text{VEP}}(v, u) = \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) + I_{\mathcal{F}}(v).$$

In conclusion, Nash equilibrium problems are covered by the framework discussed in this paper.

2.3 Quasi-Variational Inequalities

Consider the following type of quasi-variational inequality:
Seek $u \in C(u)$, such that

$$f(u) \in Au + N_{C(u)}(u) \tag{QVI}$$

holds. Here, $C : U \rightrightarrows U$ is a set-valued mapping with non-empty, closed, convex values and a single-valued operator $f : U \to U^*$ where U denotes a Banach space and U^* its topological dual. Further, $A \in \mathcal{L}(U, U^*)$ is a bounded, linear operator and assumed to be coercive. Also, $N_{C(u)}(\cdot)$ denotes the normal cone mapping associated with $C(u)$; see e.g. [AF90, Definition 4.4.2]. We rewrite (QVI) as follows: First, we decompose $A = A_{\text{sym}} + A_{\text{anti}}$ into its symmetric and anti-symmetric part and formulate the corresponding variational inequality for given $u \in U$ as:
Seek $v \in U$, such that

$$f(u) - A_{\text{anti}}u \in A_{\text{sym}}v + N_{C(u)}(v) \text{ in } U^*.$$

This can be interpreted as v being the best response of the functional

$$\mathcal{E}^{\text{QVI}}(v, u) := \frac{1}{2}\langle A_{\text{sym}}v, v\rangle_{U^*,U} + \langle A_{\text{anti}}u - f(u), v\rangle_{U^*,U} + I_{C(u)}(v).$$

In many cases, this can be transferred to non-linear, cyclically monotone operators A (cf. [BC17, Theorem 22.18]). The above shows, that QVIs can formally be brought into the framework of Definition 1.

2.4 Eigenvalue Problems

Consider a (real) Hilbert space H with inner product $(\cdot,\cdot)_H$ and the dual space H^* identified with H and a linear, bounded operator $A \in \mathcal{L}(H, H)$. The associated *eigenvalue problem* (EVP) reads as:
Seek $\lambda \in \mathbb{R}$ and $u \in H$, such that

$$Au = \lambda u. \tag{EVP}$$

First, note that any eigenvalue of A fulfils the well-known inequality $|\lambda| \leq \|A\|_{\mathcal{L}(H,H)}$. Hence, we can take $\alpha = \|A\|_{\mathcal{L}(H,H)} + 1$ and rewrite (EVP) equivalently as

$$(A + \alpha \cdot \mathrm{id})v = (\lambda + \alpha)v.$$

Thus, we obtain again an eigenvalue problem with a coercive operator with shifted eigenvalues. Hence, we can without loss of generality assume A to be coercive with modulus ≥ 1. Then, we formulate the following QVI:
Seek $u \in C(u)$, such that

$$0 \in Au + N_{C(u)}(u), \tag{3}$$

with $C(u) := \{v \in H : (v,u)_H = 1\}$. To show the equivalence of (EVP) and (3) we calculate the normal cone of $C(u)$ in $u \in C(u)$:
For this sake, take $v \in C(u)$ and decompose it as $v = \mu u + u^\perp$ for $\mu \in \mathbb{R}$ with $(u^\perp, u)_H = 0$. To determine μ we calculate

$$1 = (v,u)_H = \mu \|u\|_H^2 = \mu$$

as $v, u \in C(u)$. To obtain the other direction, it can easily be seen, that every $v = u + u^\perp$ belongs to $C(u)$. Thus, for $u^* \in N_{C(u)}(u)$ it holds that

$$0 \geq (u^*, v - u)_H = (u^*, u^\perp)_H,$$

and thus $(u^*, u^\perp)_H = 0$ for all $u^\perp$ with $(u^\perp, u)_H = 0$. Hence, $u^* \in \mathbb{R}u$.
The other direction $\mathbb{R}u \subseteq N_{C(u)}(u)$ holds by the same calculation as well and yields $N_{C(u)}(u) = \mathbb{R}u$. Thus, the QVI in (3) is equivalent to (EVP). The reformulation into the form in Definition 1 follows the discussion in Subsection 2.3.

2.5 Existence

For the sake of self-containment, we want to draw the attention shortly to the existence of equilibria. There are several approaches for equilibria based on bifunctions (cf. [KR18], [Yua99]). However, we seek to apply fixed point results involving the best response operator. One of the classical results is the Kakutani fixed point theorem (cf. [Kak41]). As we aim at the infinite-dimensional situation, we cite here the Glicksberg fixed point theorem serving as the corresponding generalization of Kakutani's result.

Theorem 4 (cf. [Gli52])**.** *Given a closed point-to-(non-void)-convex-set mapping $\Phi : Q \rightrightarrows Q$ of a convex compact subset Q of a convex Hausdorff linear topological space into itself, then there exists a fixed point $x \in \Phi(x)$.*

Next, we apply Theorem 4 to derive an existence result for an equilibrium problem of the type presented in Definition 1. The existence discussion follows in close proximity the arguments used for the existence of Nash equilibria. We just refer to [Dut13] for finite-dimensional Nash games. The following result generalizes the aforementioned existence.

Theorem 5 (Existence)**.** *Consider a bounded, closed, convex, non-empty subset $U_{\mathrm{ad}} \subseteq U$ of a reflexive Banach space U and a functional $\mathcal{E} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$. There exists an equilibrium of $\mathcal{E}$, if the following assumptions are fulfilled:*

(i) For all $u \in U_{\mathrm{ad}}$ the functional $\mathcal{E}(\,\cdot\,, u)$ is quasi-convex and bounded from below.

(ii) The functional $\mathcal{E}$ is weakly lower semi-continuous and the following recovery condition is fulfilled: For all sequences $u^n \rightharpoonup u$ and $v \in U_{\mathrm{ad}}$ there exists $v^n \rightharpoonup v$ such that

$$\limsup_{n\to\infty} \mathcal{E}(v^n, u^n) \leq \mathcal{E}(v, u).$$

(iii) *The set-valued operator $u \mapsto \operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$ has as effective domain the set U_{ad} and has a weakly closed graph.*

Proof. To prove the existence of an equilibrium, we prove the existence of a fixed point of $\mathcal{B}: U_{\mathrm{ad}} \rightrightarrows U_{\mathrm{ad}}$ (cf. Definition 3). For this sake we use Theorem 4 and check the assumptions therein. As a set we use U_{ad} and equip it with the weak topology. Due to its closedness and convexity it is weakly closed. Since U_{ad} is as well bounded and U is reflexive, it is weakly compact as well.
As for given $u \in U_{\mathrm{ad}}$ the functional $\mathcal{E}(\,\cdot\,,u)$ is bounded from below we choose an infimizing sequence $(v^n)_{n\in\mathbb{N}}$. Since $\operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$ is non-empty and $\mathcal{E}(\,\cdot\,,u)$ is quasi-convex, all sublevel sets are convex and thus the domain is convex as well. As the graph of $u \mapsto \operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$ is closed, the domain is closed as well. Since U_{ad} is weakly compact, we can extract a weakly convergent subsequence with limit $v \in U_{\mathrm{ad}}$. Using the convexity of the domain, the Lemma of Mazur yields $v \in \operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$ and the assumed lower semi-continuity implies

$$\inf \mathcal{E}(\,\cdot\,,u) \le \mathcal{E}(v,u) \le \liminf_{n\to\infty} \mathcal{E}(v^n,u) = \inf \mathcal{E}(\,\cdot\,,u).$$

Thus, $\mathcal{B}(u)$ is non-empty. By the quasi-convexity and the lower semi-continuity we obtain, that $\mathcal{B}(u)$ is non-empty, closed and convex.
It is left to show the closedness of $\operatorname{gph}(\mathcal{B})$: Therefore, take a sequence $(v^n,u^n) \subseteq \operatorname{gph}(\mathcal{B})$ with $(v^n,u^n) \rightharpoonup (v,u)$. Moreover, take without loss of generality an arbitrary $w \in \operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$. By Assumption (ii) we can find for every sequence $u^n \rightharpoonup u$ a recovery sequence $w^n \rightharpoonup w$ with $\limsup_{n\to\infty} \mathcal{E}(w^n,u^n) \le \mathcal{E}(w,u)$ and using v^n being minimizers of $\mathcal{E}(\,\cdot\,,u^n)$ we obtain

$$\mathcal{E}(v,u) \le \liminf_{n\to\infty} \mathcal{E}(v^n,u^n) \le \liminf_{n\to\infty} \mathcal{E}(w^n,u^n) \le \mathcal{E}(w,u).$$

Thus, $v \in \mathcal{B}(u)$ and $\mathcal{B}$ has a closed graph. Subsequently, we can use Theorem 4 and obtain the existence of a fixed point of $\mathcal{B}$, which is equivalent to the existence of an equilibrium of $\mathcal{E}$. □

It is worth noting, that the assumption on the existence of a recovery sequence implies the domain to be a lower semi-continuous set-valued operator, i.e. for all $v \in \operatorname{dom}(\mathcal{E}(\,\cdot\,,u))$ and all sequences $u^n \rightharpoonup u$ there exists a sequence $v^n \rightharpoonup v$ with $v^n \in \operatorname{dom}(\mathcal{E}(\,\cdot\,,u^n))$.
Next, we seek to utilize Theorem 5 and the arguments in its proof to derive the existence of an equilibrium for (generalized) Nash equilibrium problems.

Theorem 6 (Existence for GNEPs)**.** *Let $U^i_{\mathrm{ad}} \subseteq U_i$ be a family of non-empty, convex, closed and bounded sets and consider the generalized Nash equilibrium problem:*
Seek $u \in U_{\mathrm{ad}}$ with $u \in C(u)$, such that

$$\mathcal{J}_i(u_i,u_{-i}) \le \mathcal{J}_i(v_i,u_{-i}) \text{ for all } v_i \in C_i(u_{-i}).$$

There exists a Nash equilibrium, if the following assumptions are fulfilled:

(i) *The objectives $v_i \mapsto \mathcal{J}_i(v_i,u_{-i})$ are quasi-convex and bounded from below for all $i = 1,\dots,N$ and $u_{-i} \in U^{-i}_{\mathrm{ad}}$.*

(ii) *The set-valued operator $C : U_{\mathrm{ad}} \rightrightarrows U_{\mathrm{ad}}$ has non-empty, bounded, closed and convex values and has a weakly closed graph and its effective domain is the set U_{ad}.*

(iii) *Let C be a* completely *lower semi-continuous mapping, i.e., for all sequences $u^n \rightharpoonup u$ and all $v \in C(u)$ there exists a sequence $v^n \in C(u^n)$, such that $v^n \to v$.*

(iv) *The functionals $u \mapsto \mathcal{J}_i(u)$ are weakly lower semi-continuous on $\operatorname{gph}(C)$ and moreover upper semi-continuous on $\operatorname{gph}(C)$ with respect to the strong topology on U_i and the weak topology on U_{-i}, i.e.: for all $i = 1,\dots,N$ and all sequences $u^n_i \to u_i$ in U_i and $u^n_{-i} \rightharpoonup u_{-i}$ in U_{-i} it holds that $\mathcal{J}_i(u) \ge \limsup_{n\to\infty} \mathcal{J}_i(u^n)$ (cf. [AF90, Definition 1.4.2]).*

Proof. Unfortunately, we cannot use the existence result derived in Theorem 5 directly, as the quasi-convexity of each $v_i \mapsto \mathcal{J}_i(v_i, u_{-i})$ does not imply the quasi-convexity of $\mathcal{E}(\,\cdot\,, u)$. However, we can still guarantee the best response operator to have non-empty, closed, convex values: For that sake, we exploit the product structure of the strategy mapping in the underlying minimization problem and rewrite

$$\begin{aligned}\mathcal{B}(u) &= \operatorname{argmin}_{v\in C(u)} \left(\sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i})\right)\\ &= \prod_{i=1}^{N} \operatorname{argmin}_{v_i\in C_i(u_{-i})} \mathcal{J}_i(v_i, u_{-i}) = \prod_{i=1}^{N} \mathcal{B}_i(u_{-i}),\end{aligned}$$

where the product is taken in the canonical ordering $1, \dots, N$. However, the arguments used in Theorem 5 can be used as well to prove non-emptyness, closedness and convexity of $\mathcal{B}_i(u_{-i})$ and thus also of $\mathcal{B}(u)$.
The remaining assumptions translate using the example given in Subsection 2.2. As $\mathcal{J}_i$, $i = 1, \dots, N$, are defined on U_{ad} we obtain $\operatorname{dom}(\mathcal{E}(\,\cdot\,, u)) = C(u)$. Thus, the remaining requirements on C are translated accordingly. The continuity requirements on $\mathcal{E}$ are translated via the continuity requirements on the functionals $\mathcal{J}_i$. To check the continuity condition on $(v, u) \mapsto I_{C(u)}(v)$ take first a sequence $(v^n, u^n) \rightharpoonup (v, u)$. There are two cases:
(i): There are infinitely many indices with $v^n \in C(u^n)$. Then, along this subsequence we obtain $\liminf_{n\to\infty} I_{C(u^n)}(v^n) = 0$ and by the assumed weak closedness of the graph $v \in C(u)$. Thus, $I_{C(u)}(v) = 0$ and the desired weak lower semi-continuity is proven.
(ii): Otherwise, if for almost all indices $v^n \notin C(u^n)$ holds true, then
$\lim_{n\to\infty} I_{C(u^n)}(v^n) = \infty$ and the desired lower semi-continuity is proven as well.
For the recovery condition, take without loss of generality, $w \in C(u)$ and a sequence $u^n \rightharpoonup u$. Then, there exists a sequence $w^n \to w$ with $w^n \in C(u^n)$. By the upper semi-continuity condition on $\mathcal{J}_i$ for $i = 1, \dots, N$ we obtain

$$\limsup_{n\to\infty} \mathcal{E}(w^n, u^n) = \limsup_{n\to\infty} \sum_{i=1}^{N} \mathcal{J}_i(w_i^n, u_{-i}^n) \le \sum_{i=1}^{N} \mathcal{J}_i(w_i, u_{-i}) = \mathcal{E}(w, u).$$

Thus, by the remaining arguments in Theorem 5, we obtain the existence of a Nash equilibrium. □

In comparison to Theorem 5 we demanded a strongly convergent recovery sequence, but only combined it with an upper semi-continuity condition, that used strong continuity in the control component. Alternatively, one could have used weak convergence for both instead. This choice however might depend on the application in mind.
Analogously, we proceed with the existence of variational equilibria.

Theorem 7 (Existence for VEPs)**.** *Let $U_{\text{ad}}^i \subseteq U_i$ be a family of non-empty, convex, closed and bounded sets and consider the following variational equilibrium problem: Seek $u \in \mathcal{F}$, such that*

$$\sum_{i=1}^{N} \mathcal{J}_i(u_i, u_{-i}) \le \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) \text{ for all } v \in \mathcal{F}.$$

There exists a variational equilibrium, if the following assumptions are fulfilled:

(i) The objective $v \mapsto \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i})$ is quasi-convex and bounded from below for all $u \in U_{\text{ad}}$.

(ii) The set of shared constraints $\mathcal{F} \subseteq U_{\text{ad}}$ is non-empty, closed and convex.

(iii) *The functional* $(v,u) \mapsto \sum_{i=1}^N \mathcal{J}_i(v_i, u_{-i})$ *is weakly lower semi-continuous on* $\mathcal{F}$*. Moreover, assume, that for every weakly convergent sequence* $u^n \rightharpoonup u$ *in* $\mathcal{F}$ *and* $v \in \mathcal{F}$ *there exists a sequence* $(v^n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ *with* $v^n \to v$*, such that*

$$\limsup_{n\to\infty} \sum_{i=1}^N \mathcal{J}_i(v_i^n, u_{-i}^n) \leq \sum_{i=1}^N \mathcal{J}_i(v_i, u_{-i}).$$

Proof. The proof uses Theorem 5 and the arguments in the proof of Theorem 6 and is omitted for brevity. □

Here, we decided again to use a strongly convergent recovery sequence.
With these results at hand, we close our existence discussion. It is worth noting, that the aforementioned approach via a bifunction can alternatively be used. Moreover, especially in the context of QVIs other fixed point results can be used (cf. [AHR19], [Bir73]).

3 Γ-Convergence of Equilibrium Problems

Having a constraint in the optimization problem, which induces the equilibrium problem, leads to analytical and as a consequence to numerical difficulties. To address these difficulties one group of techniques in use is formed by penalization and regularization schemes, see e.g.: [HK06], [HR15], [AHS18]. Therein, the addressed functional is substituted by a sequence of more regular objects, that are easier to handle. Then, in the limit one hopes to recover the originally formulated problem. A successful concept to provide such a convergence statement for optimization problems is Γ*-convergence*.
For our purpose, we want to generalize that concept to equilibrium problems as introduced in Definition 1.

Definition 8 (Γ-convergence)**.** Let U_{ad} be a subset of a reflexive Banach space U. A sequence of functionals $(\mathcal{E}_n)_{n\in\mathbb{N}} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$ is called (weakly) Γ*-convergent* to a functional $\mathcal{E} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$, denoted by $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ (resp. $\mathcal{E}_n \xrightharpoonup{\Gamma} \mathcal{E}$), if the following two conditions hold:

(i) For all sequences $u^n \to u$ ($u^n \rightharpoonup u$) it holds that

$$\mathcal{E}(u,u) \leq \liminf_{n\to\infty} \mathcal{E}_n(u^n, u^n).$$

(ii) For all $v \in \mathcal{U}$ and all sequences $u^n \to u$ ($u^n \rightharpoonup u$) there exists a sequence $v^n \to v$ ($v^n \rightharpoonup v$), such that

$$\mathcal{E}(v,u) \geq \limsup_{n\to\infty} \mathcal{E}_n(v^n, u^n).$$

However, a similar concept for Nash equilibrium problems has been proposed in [GP09, First Definition on p.226] called *multi epi-convergence*, which in fact can be interpreted as a special case of Definition 8.
A strengthened concept addressing the differences between strong and weak convergence in the infinite dimensional case is *Mosco*-convergence, which we generalize next.

Definition 9 (Mosco-convergence)**.** Let U_{ad} be a subset of a reflexive Banach space U. A sequence of functionals $(\mathcal{E}_n)_{n\in\mathbb{N}} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$ is called *Mosco-convergent* to a functional $\mathcal{E} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$, denoted by $\mathcal{E}_n \xrightarrow{M} \mathcal{E}$, if the following two conditions hold:

(i) For all sequences $u^n \rightharpoonup u$ it holds that

$$\mathcal{E}(u,u) \leq \liminf_{n\to\infty} \mathcal{E}_n(u^n,u^n).$$

(ii) For all $v \in \mathcal{U}$ and all sequences $u^n \rightharpoonup u$ there exists a sequence $v^n \to v$, such that

$$\mathcal{E}(v,u) \geq \limsup_{n\to\infty} \mathcal{E}_n(v^n,u^n).$$

The direct comparison between Γ-convergence with respect to weak topology and Mosco-convergence yields a strong convergence for the recovery sequence with a weak convergent sequence in the feedback component. It is worth noting, that the above definitions do only require the first condition to hold true on the diagonal.
Alternatively, one can for a given sequence $u^n \rightharpoonup u$ consider the sequence $\mathcal{E}(\,\cdot\,,u^n)$ and use the usual Γ-, Mosco-convergence known from optimization, which we refer to as 'feedbackwise' convergence.

Definition 10 (Feedbackwise Γ- and Mosco-convergence)**.** Let U_{ad} be a subset of a reflexive Banach space U. A sequence of functionals $(\mathcal{E}_n)_{n\in\mathbb{N}} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$ is called *feedbackwise (weakly)* Γ*-convergent*, resp. *feedbackwise Mosco-convergent* to a functional $\mathcal{E} : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R} \cup \{+\infty\}$, if for all $u^n \rightharpoonup u$ the convergence $\mathcal{E}_n(\,\cdot\,,u^n) \xrightarrow{\Gamma} \mathcal{E}(\,\cdot\,,u)$ ($\mathcal{E}_n(\,\cdot\,,u^n) \overset{\Gamma}{\rightharpoonup} \mathcal{E}(\,\cdot\,,u)$), respectively $\mathcal{E}_n(\,\cdot\,,u^n) \xrightarrow{M} \mathcal{E}(\,\cdot\,,u)$ holds true.

It is straightforward to see, that feedbackwise convergence implies Γ- resp. Mosco-convergence, as the first requirement in Definition 8 resp. Definition 9 is placed on the whole product set and not only on the diagonal. In fact, considering the sequence $\mathcal{E}(v,u) = \mathcal{E}_n(v,u) := (v,u)_U$ with U being a real Hilbert space yields the Mosco-convergence, but not feedbackwise Mosco-convergence, as for two weakly convergent sequences $v^n \rightharpoonup v$ and $u^n \rightharpoonup u$ the relation $(v,u)_U \leq \liminf_{n\to\infty}(v^n,u^n)_U$ does not need to hold. Moreover, Mosco-convergence of convex sets (cf. [Mos69]) as used in the existence discussion of solutions of QVIs can be interpreted as a case of feedbackwise Mosco-convergence for the indicator sets $I_{C(u)}$ of a set-valued mapping $C : U \rightrightarrows U$. However, it is straightforward to show, that (quasi-)convexity in the control component is preserved by the stronger notion of feedbackwise convergence. This is not so clear to the authors for the convergence concept introduced in Definition 8 and Definition 9. As we are solving a sequence of equilibrium problems we wish them to cluster around an equilibrium of the original problem. Indeed, this holds true, as presented in the following result.

Theorem 11. *Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a (weakly) Γ-convergent sequence of functionals with limit $\mathcal{E}$ like in Definition 8. Then, every (weak) accumulation point of a sequence of corresponding equilibria $(u^n)_{n\in\mathbb{N}}$ is an equilibrium of the limit.*

Proof. Let u be a (weak) accumulation point of $(u^n)_{n\in\mathbb{N}}$ along a (not relabeled) subsequence. Let $v \in U_{\mathrm{ad}}$ be arbitrary. Then, there exists a recovery sequence $v^n \to v$ ($v^n \rightharpoonup v$) by the second property for u^n. We deduce, that

$$\mathcal{E}(u,u) \leq \liminf_{n\to\infty} \mathcal{E}_n(u^n,u^n) \leq \limsup_{n\to\infty} \mathcal{E}_n(v^n,u^n) \leq \mathcal{E}(v,u),$$

which proves the assertion. □

Next, we return to the aforementioned penalization technique and apply it to a selection of equilibrium problems for Nash games as well as quasi-variational inequalities.

3.1 Application to Penalized Nash Equilibrium Problems

For a given set-valued mapping $C : U_{\mathrm{ad}} \rightrightarrows U_{\mathrm{ad}}$ we associate to it a penalty functional $\pi_C : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to [0, \infty)$ with the property $\pi_C(v,u) = 0$, if and only if $v \in C(u)$. In the same fashion, we associate to a given set $\mathcal{F} \subseteq U_{\mathrm{ad}}$ a penalty functional $\pi_{\mathcal{F}} : U_{\mathrm{ad}} \to [0,\infty)$ with $\pi_{\mathcal{F}}(v) = 0$, if and only if $v \in \mathcal{F}$. A fairly general example of such penalty functionals is given by

$$\begin{aligned} \pi_C(v,u) &:= \operatorname{dist}(v, C(u)) \text{ and } \pi_{\mathcal{F}}(v) := \operatorname{dist}(v,\mathcal{F}) \text{ with} \\ \operatorname{dist}(v,M) &:= \inf\{\|v - v'\|_U : v' \in M\} \text{ for a subset } M \subseteq U, \end{aligned} \tag{4}$$

where we assume the sets $C(u)$ and $\mathcal{F}$ to be closed as the closedness of a set M guarantees $u \in M$ if and only if $\operatorname{dist}(u, M) = 0$. Next, we establish a Γ-convergence result for the penalized version of (GNEP).

Theorem 12 (Convergence of penalized GNEPs)**.** *Let a sequence of positive penalty parameters $\gamma_n \to \infty$ be given. Moreover, assume the following conditions to be fulfilled:*

(i) The set-valued operator $C : U_{\mathrm{ad}} \rightrightarrows U_{\mathrm{ad}}$ has the set U_{ad} as its domain and has a weakly closed graph.

(ii) Moreover, let C be a completely *lower semi-continuous mapping, i.e., for all sequences $u^n \rightharpoonup u$ and all $v \in C(u)$ there exists a sequence $v^n \in C(u^n)$, such that $v^n \to v$.*

(iii) The functionals $U_{\mathrm{ad}} \ni u \mapsto \mathcal{J}_i(u) \in \mathbb{R}$ are bounded from below and weakly lower semi-continuous on U_{ad} as well as upper semi-continuous on U_{ad} with respect to the strong topology on U_i and the weak topology on U_{-i}, i.e.: for all $i = 1, \dots, N$ and all sequences $u_i^n \to u_i$ in U_i and $u_{-i}^n \rightharpoonup u_{-i}$ in U_{-i} holds $\mathcal{J}_i(u) \geq \limsup_{n\to\infty} \mathcal{J}_i(u^n)$.

(iv) Let the penalty functional $\pi_C : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to [0, \infty)$ be weakly lower semi-continuous.

Then, the sequence of functionals $\mathcal{E}_\gamma : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R}$ defined by

$$\mathcal{E}_\gamma(v,u) := \sum_{i=1}^N \mathcal{J}_i(v_i, u_{-i}) + \gamma \pi_C(v,u) \tag{5}$$

is Mosco-convergent to

$$\mathcal{E}(v,u) := \sum_{i=1}^N \left(\mathcal{J}_i(v_i, u_{-i}) + I_{C_i(u_{-i})}(v_i) \right).$$

Proof. Take a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n > 0$, $\gamma_n \to \infty$. To check Condition (i) in Definition 8 take an arbitrary sequence $(u^n)_{n\in\mathbb{N}} \subseteq U_{\mathrm{ad}}$ with $u^n \rightharpoonup u$ in U. Since U_{ad} is assumed to be a non-empty, closed, convex set we have $u \in U_{\mathrm{ad}}$. First, consider the case $u \in C(u)$. By the lower semi-continuity of the functionals $\mathcal{J}_i : U \to \mathbb{R}$ one obtains

$$\begin{aligned} \mathcal{E}(u,u) = \sum_{i=1}^N \mathcal{J}_i(u_i, u_{-i}) &\leq \liminf_{n\to\infty} \left(\sum_{i=1}^N \mathcal{J}_i(u_i^n, u_{-i}^n) \right) \\ &\leq \liminf_{n\to\infty} \left(\sum_{i=1}^N \mathcal{J}_i(u_i^n, u_{-i}^n) + \gamma_n \pi_C(u^n, u^n) \right) = \liminf_{n\to\infty} \mathcal{E}_{\gamma_n}(u^n, u^n). \end{aligned}$$

In the case of $u \notin C(u)$ it holds that $\pi_C(u,u) > 0$. Using the assumed weak lower semi-continuity of π_C yields

$$0 < \pi_C(u,u) \leq \liminf_{n\to\infty} \pi_C(u^n, u^n).$$

Hence $\pi_C(u^n, u^n) \geq \frac{1}{2}\pi_C(u, u)$ holds for almost all indices n and therefore

$$\lim_{n\to\infty} \gamma_n \pi_C(u^n, u^n) = \infty.$$

In combination with the boundedness of $\mathcal{J}_i$ from below we obtain

$$\lim_{n\to\infty} \left(\sum_{i=1}^{N} \mathcal{J}_i(u_i^n, u_{-i}^n) + \gamma_n \pi_C(u^n, u^n) \right) = \infty = \mathcal{E}(u, u)$$

and hence Condition (i) in Definition 8.
Checking Condition (ii) in Definition 8, choose again an arbitrary sequence $u^n \rightharpoonup u$ in U with $u^n \in U_{\mathrm{ad}}$. Moreover, take an arbitrary $v \in C(u)$, then $\pi_C(v, u) = 0$ holds. Taking by assumption a sequence $(v^n)_{n\in\mathbb{N}} \subset U$ with $v^n \in C(u^n)$ and $v^n \to v$ yields

$$\begin{aligned} \mathcal{E}(v, u) &= \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) = \lim_{n\to\infty} \sum_{i=1}^{N} \mathcal{J}_i(v_i^n, u_{-i}^n) = \lim_{n\to\infty} \sum_{i=1}^{N} \mathcal{J}_i(v_i^n, u_{-i}^n) \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{N} \mathcal{J}_i(v_i^n, u_{-i}^n) + \gamma_n \pi_C(v^n, u^n) \right) = \lim_{n\to\infty} \mathcal{E}_n(v^n, u^n). \end{aligned}$$

□

As in the existence results Theorem 6 and Theorem 7 the treatment of the weak convergence in infinite dimensions was of significant importance. Analogously, the corresponding result for variational equilibrium is derived in the following theorem.

Theorem 13 (Convergence of penalized VEPs)**.** *Let a sequence of penalty parameters $\gamma_n \to \infty$ be given and let the following assumptions be fulfilled:*

(i) The set of shared constraints $\mathcal{F} \subseteq U_{\mathrm{ad}}$ is non-empty, bounded, closed and convex.

(ii) The functional $(v, u) \mapsto \sum_{i=1} \mathcal{J}_i(v_i, u_{-i})$ is weakly lower semi-continuous on $\mathcal{F}$. Moreover, assume, that for every weakly convergent sequence $u^n \rightharpoonup u$ and $v \in \mathcal{F}$ there exists a sequence $(v^n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ with $v^n \to v$, such that

$$\limsup_{n\to\infty} \sum_{i=1}^{N} \mathcal{J}_i(v_i^n, u_{-i}^n) \leq \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}).$$

(iii) The penalty functional $\pi_{\mathcal{F}} : U_{\mathrm{ad}} \to [0, \infty)$ is weakly lower semi-continuous.

Then, the sequence of functionals $\mathcal{E}_\gamma : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to \mathbb{R}$ defined by

$$\mathcal{E}_\gamma(v, u) := \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) + \gamma \pi_{\mathcal{F}}(v) \tag{6}$$

Mosco-convergent to

$$\mathcal{E}(v, u) := \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) + I_{\mathcal{F}}(v).$$

Proof. The proof is analogous to the one of Theorem 12. □

Returning to the penalty functionals in (4) we would like to discuss the interplay between the conditions listed in Theorem 12 and Theorem 13. Let us first consider the latter setting. If $\mathcal{F}$ is convex, closed and non-empty, also $v \mapsto \mathrm{dist}(v, \mathcal{F})$ is continuous and convex and thus weakly lower semi-continuous. Thus, the condition on the feasible set $\mathcal{F}$ induces the requested properties

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of the penalty functional.
The reasoning for π_C requires more effort: Take weakly convergent sequences $u^n \rightharpoonup u$ and $v^n \rightharpoonup v$. Then, take the subsequence (not relabeled) realizing $\liminf_{n\to\infty} \operatorname{dist}(v^n, C(u^n))$. Then, for an arbitrary sequence $\varepsilon^n \searrow 0$, exists a sequence $(w^n)_{n\in\mathbb{N}}, w^n \in C(u^n)$ such that $\|v^n - w^n\|_U \leq \operatorname{dist}(v^n, C(u^n)) + \varepsilon^n$. For showing its boundedness we take an arbitrary $w \in C(u)$. By the complete lower semi-continuity of C there exists a sequence $\tilde{w}^n \to w$ with $\tilde{w}^n \in C(u^n)$. Then, a direct estimate yields

$$\begin{aligned}\|w^n\|_U &\leq \|v^n - w^n\|_U + \|v^n\|_U \leq \operatorname{dist}(v^n, C(u^n)) + \|v^n\|_U + \varepsilon^n \\ &\leq \|v^n - \tilde{w}^n\|_U + \|v^n\|_U + \varepsilon^n \leq 2\|v^n\|_U + \|\tilde{w}^n\|_U + \varepsilon^n\end{aligned}$$

implying the boundedness of $(w^n)_{n\in\mathbb{N}}$ by the boundedness of $(v^n)_{n\in\mathbb{N}}$ and $(\tilde{w}^n)_{n\in\mathbb{N}}$. As U is reflexive we can extract a weakly convergent subsequence (not relabeled) converging towards $w^* \in U$. By the assumed weak closedness of the graph of C we deduce using $\tilde{w}^n \in C(u^n)$ that $w^* \in C(u)$. Then, we obtain using the weak convergence along the previously constructed subsequence the estimate

$$\begin{aligned}\operatorname{dist}(v, C(u)) &\leq \|v - w^*\|_U \leq \liminf_{n\to\infty} \|v^n - w^n\|_U \\ &\leq \lim_{n\to\infty} (\operatorname{dist}(v^n, C(u^n)) + \varepsilon^n) = \liminf_{n\to\infty} \operatorname{dist}(v^n, C(u^n)),\end{aligned}$$

or in other words the weak lower semi-continuity of π_C on $U_{\mathrm{ad}} \times U_{\mathrm{ad}}$.

3.2 Application to Penalized Quasi-Variational Inequalities

In principle, the arguments in the proofs of Theorem 12 and Theorem 13 can be used to derive an analogous result for penalized quasi-variational inequalities, which we provide in the next theorem.

Theorem 14 (Convergence of penalized QVIs)**.** *Let a sequence of penalty parameters $\gamma_n \to \infty$ be given and let the following assumptions be fulfilled:*

(i) The set-valued operator $C : U \rightrightarrows U$ has the set U as its domain and has a weakly closed graph.

(ii) Moreover, let C be a completely *lower semi-continuous mapping, i.e., for all sequences $u^n \rightharpoonup u$ and all $v \in C(u)$ there exists a sequence $v^n \in C(u^n)$, such that $v^n \to v$.*

(iii) The operator $f : U \to U$ is weakly continuous, i.e., $u^n \rightharpoonup u$ implies $f(u^n) \to f(u)$, and for every $u^n \rightharpoonup u$ it holds that

$$\limsup_{n\to\infty} \langle f(u^n), u^n\rangle \leq \langle f(u), u\rangle.$$

(iv) Let the penalty functional $\pi_C : U_{\mathrm{ad}} \times U_{\mathrm{ad}} \to [0, \infty)$ be weakly lower semi-continuous.

Then, the functionals $\mathcal{E}_\gamma : U \times U \to \mathbb{R}$, $\gamma > 0$, defined by

$$\mathcal{E}_\gamma(v, u) := \frac{1}{2}\langle A_{\mathrm{sym}} v, v\rangle + \langle A_{\mathrm{anti}} u, v\rangle - \langle f(u), v\rangle + \gamma\pi_C(v, u) \tag{7}$$

are Mosco-convergent to

$$\mathcal{E}(v, u) := \frac{1}{2}\langle A_{\mathrm{sym}} v, v\rangle + \langle A_{\mathrm{anti}} u, v\rangle - \langle f(u), v\rangle + I_{C(u)}(v)$$

as $\gamma \to \infty$.

Proof. Take a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n > 0$, $\gamma_n \to \infty$ and another sequence $u^n \rightharpoonup u$ and consider the case $u \in C(u)$. Then we obtain by the anti-symmetry $\langle A_{\text{anti}} u^n, u^n\rangle = \langle A_{\text{anti}} u, u\rangle = 0$. By the coercivity of A_{sym} the functional $v \mapsto \frac{1}{2}\langle A_{\text{sym}} v, v\rangle$ is continuous and convex and thus weakly lower semi-continuous. Thus, we obtain

$$\begin{aligned}\mathcal{E}(u,u) &= \frac{1}{2}\langle A_{\text{sym}} u, u\rangle - \langle f(u), u\rangle \leq \liminf_{n\to\infty}\left(\frac{1}{2}\langle A_{\text{sym}} u^n, u^n\rangle - \langle f(u^n), u^n\rangle\right)\\ &\leq \mathcal{E}_n(u^n, u^n).\end{aligned}$$

In the case of $u \notin C(u)$ it holds that $\pi_C(u,u) > 0$. Using the assumed weak lower semi-continuity of π_C yields

$$0 < \pi_C(u,u) \leq \liminf_{n\to\infty} \pi_C(u^n, u^n).$$

Hence $\pi_C(u^n,u^n) \geq \frac{1}{2}\pi_C(u,u)$ holds for almost all indices n and therefore

$$\lim_{n\to\infty} \gamma_n \pi_C(u^n, u^n) = \infty.$$

The rest of the functional is bounded, as $\frac{1}{2}\langle A_{\text{sym}} u^n, u^n\rangle \geq 0$ by coercivity and by the weak continuity of f we obtain the boundedness of $\|f(u^n)\|_{U^*}$ leading to

$$\mathcal{E}_n(u^n,u^n) \geq -\|f(u^n)\|_{U^*}\cdot\|u^n\|_U + \gamma_n\pi_C(u^n,u^n) \to \infty \text{ as } n\to\infty,$$

and thus $\mathcal{E}(u,u) \leq \liminf_{n\to\infty}\mathcal{E}_n(u^n,u^n)$.
Let now again a sequence $u^n \rightharpoonup u$ and $v \in U$ be given and assume without loss of generality $v \in C(u)$. Choose as recovery sequence the one for the operator C, such that $v^n \to v$ and $v^n \in C(u^n)$. Then, we obtain by the weak continuity of f and the strong convergence of $(v^n)_{n\in\mathbb{N}}$ the convergence

$$\begin{aligned}\lim_{n\to\infty}\mathcal{E}_n(v^n,u^n) &= \lim_{n\to\infty}\left(\frac{1}{2}\langle A_{\text{sym}} v^n, v^n\rangle + \langle A_{\text{anti}} v^n, u^n\rangle - \langle f(u^n), v^n\rangle\right)\\ &= \frac{1}{2}\langle A_{\text{sym}} v, v\rangle + \langle A_{\text{anti}} v, u\rangle - \langle f(u), v\rangle = \mathcal{E}(v,u).\end{aligned}$$

Thus, we obtain the requested Mosco-convergence. □

4 Conclusion

Within the scope of this text, we discussed a type of equilibrium problem, formulated equivalent characterizations of equilibria and derived existence results in the abstract case as well as for Nash-type equilibrium problems. The generalized Γ-convergence concept has been analyzed and applied to a penalization technique for Nash games and QVIs. We expect, that the presented results are as well suitable for other approximation techniques in the context of equilibrium problems and serve as a strong theoretical foundation of a convergence analysis and its numerical realization.

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