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Abstract

A nonlinear extension of the Caginalp phase field system is considered that takes thermal memory into account. The resulting model, which is a first-order approximation of a thermodynamically consistent system, is inspired by the theories developed by Green and Naghdi. Two equations, resulting from phase dynamics and the universal balance law for internal energy, are written in terms of the phase variable (representing a non-conserved order parameter) and the so-called thermal displacement, i.e., a primitive with respect to time of temperature. Existence and continuous dependence results are shown for weak and strong solutions to the corresponding initial-boundary value problem. Then, an optimal control problem is investigated for a suitable cost functional, in which two data act as controls, namely, the distributed heat source and the initial temperature. Fréchet differentiability between suitable Banach spaces is shown for the control-to-state operator, and meaningful first-order necessary optimality conditions are derived in terms of variational inequalities involving the adjoint variables. Eventually, characterizations of the optimal controls are given.

1 Introduction

This paper is concerned with a phase field model for a non-isothermal phase transition with non-conserved order parameter describing the evolution in a container in terms of two physical variables. Well-posedness issues for weak and strong solutions and optimal control problems are investigated in detail. At first, we introduce the system of partial differential equations and related conditions.

1.1 The initial and boundary value problem

We assume that the phase transformation takes place in a fixed container $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, which is an open and bounded domain with smooth boundary $\Gamma := \partial\Omega$. For a positive fixed final time horizon T, we set,

$$Q_t := \Omega \times (0,t),\ 0 < t \le T, \quad Q := Q_T, \quad \Sigma := \Gamma \times (0,T).$$

Then the model under study reads as

$$\partial_t\varphi - \Delta\varphi + \gamma(\varphi) + \tfrac{2}{\theta_c}\pi(\varphi) - \tfrac{1}{\theta_c^2}\partial_t w\, \pi(\varphi) \ni 0 \qquad \text{in } Q, \tag{1.1}$$

$$\partial_{tt} w - \alpha\Delta(\partial_t w) - \beta\Delta w + \pi(\varphi)\partial_t\varphi = u \qquad \text{in } Q, \tag{1.2}$$

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}(\alpha\partial_t w + \beta w) = 0 \qquad \text{on } \Sigma, \tag{1.3}$$

$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = v_0 \qquad \text{in } \Omega. \tag{1.4}$$

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The primary variables of the system are φ, the order parameter of the phase transition, and w, the so-called *thermal displacement* or *freezing index*. The latter is directly connected to the absolute temperature θ of the system through the relation

$$w(\cdot,t) = w_0 + \int_0^t \theta(\cdot,s)\,\mathrm{d}s, \quad t\in[0,T]. \tag{1.5}$$

Moreover, α and β stand for prescribed positive coefficients that are related to the heat flux, θ_c for a (positive) critical temperature, and u for a distributed heat source. Besides, the nonlinearities $\gamma:\mathbb{R}\to 2^{\mathbb{R}}$ and $\pi:\mathbb{R}\to\mathbb{R}$ indicate, in this order, a maximal monotone graph and a Lipschitz continuous function. Finally, the symbol $\partial_{\mathbf{n}}$ represents the outward normal derivative on Γ, whereas $\varphi_0, w_0,$ and v_0 stand for some prescribed initial values.

Notice that the inclusion in (1.1) is of Allen–Cahn type and is suited for the case of non-conserved order parameters (while the case of a conserved order parameter would require a Cahn–Hilliard structure). The inclusion originates from the possibly multivalued nature of the graph γ. Typically, the maximal monotone graph γ is obtained as the subdifferential of a convex and lower semicontinuous function $\widehat{\gamma}:\mathbb{R}\to[0,+\infty]$, and well-known examples are given by the regular, logarithmic, and double obstacle potentials, defined, in the order, by

$$\widehat{\gamma}_{reg}(r) = \frac{r^4}{4}, \quad r\in\mathbb{R}, \tag{1.6}$$

$$\widehat{\gamma}_{log}(r) = \begin{cases} \frac{\kappa}{2}[(1+r)\ln(1+r)+(1-r)\ln(1-r)], & \text{if } r\in(-1,1),\\ \kappa\ln(2), & \text{if } r\in\{-1,1\},\\ +\infty, & \text{otherwise,}\end{cases} \tag{1.7}$$

$$\widehat{\gamma}_{dob}(r) = I_{[-1,1]}(r), \tag{1.8}$$

with a positive constant κ, where, for every subset $A\subset\mathbb{R}$, $I_A(\cdot)$ stands for the indicator function of A and is specified by

$$I_A(r) := \begin{cases} 0 & \text{if } r\in A,\\ +\infty & \text{otherwise.}\end{cases}$$

Let us point out that the inclusion (1.1) simply reduces to an equality in the case of (1.6) and of (1.7) for $-1<\varphi<1$, since the regularity of $\widehat{\gamma}$ ensures γ to be single valued.

Next, we present a possible physical derivation of the system in (1.1)–(1.4), trying to meet the requirement of thermodynamic consistency as much as possible. On the other hand, different approaches may be appealed and, in particular, we quote [2, 3, 4, 10, 19, 23, 24] as related references.

1.2 Thermodynamic derivation and modeling considerations

We start from the local specific Helmholtz free energy, acting on the absolute temperature $\theta>0$ and the dimensionless order parameter φ. With physical constants β_1,β_2,β_3, the specific local free energy F is assumed in the form

$$F(\theta,\varphi) = c_V\theta(1-\ln(\theta/\theta_1)) + \beta_1\widehat{\pi}(\varphi) + \beta_2\theta\,\widehat{\gamma}(\varphi) + \frac{\beta_3}{2}\,\theta\,|\nabla\varphi|^2, \tag{1.9}$$

where $c_V > 0$ denotes the specific heat (assumed constant), $\theta_1 > 0$ is some fixed reference temperature, $\widehat{\gamma}(\varphi)$ has been introduced above, and the real-valued function $\widehat{\pi}$ stands for a primitive of π. The last summand in (1.9) is a contribution that accounts for nearest-neighbor interactions.

By virtue of the general relations between the thermodynamic potentials, the expressions for local specific entropy S and local specific internal energy E are then given by

$$S(\theta,\varphi) = -\partial_\theta F(\theta,\varphi) = c_V \ln(\theta/\theta_1) - \beta_2\,\widehat{\gamma}(\varphi) - \frac{\beta_3}{2}|\nabla\varphi|^2, \tag{1.10}$$

$$E(\theta,\varphi) = F(\theta,\varphi) + \theta S(\theta,\varphi) = c_V\theta + \beta_1\widehat{\pi}(\varphi). \tag{1.11}$$

Now, we come to the evolution laws. As always, the universal balance law of internal energy must be obeyed. Under the assumption that velocity effects may be discarded, it has the general form

$$\rho\,\partial_t E(\theta,\varphi) + \operatorname{div}\mathbf{q} = \rho u, \tag{1.12}$$

where $\mathbf{q}$ denotes the heat flux, ρ is the mass density and ρu stands for the possible presence of distributed heat sources/sinks. Here, we consider the case when ρ varies only little during the phase transition and can be assumed constant.

Usually the Fourier law is assumed for $\mathbf{q}$, i.e.,

$$\mathbf{q} = -\kappa_V\nabla\theta, \tag{1.13}$$

where κ_V is the (positive) heat conductivity coefficient, together with the no-flux condition $\mathbf{q}\cdot\mathbf{n} = 0$ on the boundary.

In the present paper, we adopt a different approach for $\mathbf{q}$, the Fourier law (1.13) being generalized in the light of the works by Green and Naghdi [15, 16, 17] and Podio-Guidugli [24]. Indeed, these authors introduced a different approach for the study of heat conduction theory that leads to the notion of *thermal displacement*. We recall (1.5) and note that there w_0 represents a given datum at the (initial) reference time. This datum accounts for a possible previous thermal history of the phenomenon. Making use of this new variable w, Green and Naghdi proposed three theories for heat transmission labeled as type I–III. Let us now employ the symbols α and β for the coefficients which are assumed constant and positive. Type I theory, after suitable linearization, brings us back to the standard Fourier law

$$\mathbf{q} = -\alpha\nabla(\partial_t w) \qquad \text{(type I)}, \tag{1.14}$$

while linearized versions of type II and III yield the following heat-conduction laws:

$$\mathbf{q} = -\beta\nabla w \qquad \text{(type II)}, \tag{1.15}$$

$$\mathbf{q} = -\alpha\nabla(\partial_t w) - \beta\nabla w \qquad \text{(type III)}. \tag{1.16}$$

We point out that the thermal displacement w is useful to describe type II and III laws, whereas the type I law can be stated in terms of the temperature $\theta = \partial_t w$ alone.

This paper is concerned with the general type III theory. In fact, in view of (1.12) and (1.16), we infer that

$$\rho\left(c_V w_{tt} + \beta_1\pi(\varphi)\partial_t\varphi\right) - \alpha\Delta(\partial_t w) - \beta\Delta w = \rho u. \tag{1.17}$$

Observe that the no-flux condition $\mathbf{q}\cdot\mathbf{n} = 0$ then gives rise to the second boundary condition in (1.3).

It remains to derive the equation governing the evolution of the order parameter. To this end, we introduce the total entropy functional, which at any fixed time instant $t \in [0,T]$ is given by the expression

$$\mathcal{S}[\theta(t),\varphi(t)] = \int_\Omega \rho\, S(\theta(t),\varphi(t))\,,$$

with the usual notation $\theta(t) = \theta(\cdot,t), \varphi(t) = \varphi(\cdot,t)$.

For the dynamics of the order parameter, we postulate that it runs at each time instant $t \in (0,T]$ in a direction as to maximize total entropy subject to the constraint that the balance law (1.12) of internal energy be satisfied. To this end, observe that integration of (1.12) over $\Omega \times [0,t]$, using (1.11) and the no-flux boundary condition for $\mathbf{q}$, yields the identity

$$0 \,=\, \int_\Omega \rho\big(c_V\theta(t) - c_V\theta_0 + \beta_1\widehat{\pi}(\varphi(t)) - \beta_1\widehat{\pi}(\varphi_0) - R(t)\big),$$

where we again use the notation $R(t) = R(\cdot,t)$, and $R(x,t){:}= \int_0^t u(x,s)\,\mathrm{d}s$, $x \in \Omega$. We now consider the augmented entropy functional

$$\begin{aligned}
\mathcal{S}_\lambda[\theta(t),\varphi(t)] :=& \,\mathcal{S}[\theta(t),\varphi(t)] + \rho\int_\Omega \lambda(\cdot,t)\,\big(c_V\theta(t) - c_V\theta_0 + \beta_1\widehat{\pi}(\varphi(t)) - \beta_1\widehat{\pi}(\varphi_0) - R(t)\big)\\
=& \,\rho\int_\Omega \big[c_V\ln(\theta(t)/\theta_1) - \beta_2\,\widehat{\gamma}(\varphi(t)) - \frac{\beta_3}{2}\,|\nabla\varphi(t)|^2\\
&\quad + \lambda(\cdot,t)\,\big(c_V\theta(t) - c_V\theta_0 + \beta_1\widehat{\pi}(\varphi(t)) - \beta_1\widehat{\pi}(\varphi_0) - R(t)\big)\big],
\end{aligned}$$

where $\lambda(t) = \lambda(x,t)$, $x \in \Omega$, plays the role of a Lagrange multiplier. The search for critical points leads to the Euler–Lagrange equations obtained by taking the variational derivatives of $\mathcal{S}_\lambda$ with respect to φ and θ, namely,

$$\begin{aligned}
\delta_\varphi\mathcal{S}_\lambda[\theta(t),\varphi(t)] &= \rho\big[-\beta_2\gamma(\varphi(t)) + \beta_3\Delta\varphi(t) + \lambda(t)\,\beta_1\pi(\varphi(t))\big] \ni 0,\\
\delta_\theta\mathcal{S}_\lambda[\theta(t),\varphi(t)] &= \rho\big[c_V/\theta(t) + \lambda(t)\,c_V\big] = 0.
\end{aligned}$$

Then, from the second relation we can identify λ as $-1/\theta$, while we postulate that the evolution of φ runs in the direction of $\delta_\varphi\mathcal{S}_\lambda$ at a rate which is proportional to it. More precisely, we assume that the evolution of φ is governed by the equation

$$a_V(\theta,\varphi)\partial_t\varphi \,=\, \delta_\varphi\mathcal{S}_\lambda(\theta,\varphi),$$

that corresponds to

$$a_V(\theta,\varphi)\partial_t\varphi \,=\, \rho\big[-(\beta_1/\theta)\pi(\varphi) - \beta_2\,\gamma(\varphi) + \beta_3\,\Delta\varphi\big], \tag{1.18}$$

where a_V is a positive coefficient (assumed constant).

At this point, we simplify the exposition by generally assuming in the following that the numerical values of all of the physical constants $c_V, \rho, \beta_1, \beta_2, \beta_3, a_V$ equal unity, while their physical dimensions will be kept active so that they still match. This will have no bearing on the subsequent mathematical analysis and should not lead to any confusion. However, in a practical application of the model with real physical data, this would have to be accounted for. Under these premises, the balance of internal energy (1.17) takes the form (1.2), and (1.18) becomes

$$\partial_t\varphi - \Delta\varphi + \gamma(\varphi) + \frac{1}{\theta}\pi(\varphi) \ni 0. \tag{1.19}$$

From (1.19) we arrive at (1.1) with the help of (1.5) and of the first-order approximation

$$\frac{1}{\theta} \approx \frac{1}{\theta_c} - \frac{1}{\theta_c^2}(\theta - \theta_c)$$

about the critical temperature θ_c.

Initial conditions for $\varphi,\ w,\ \partial_t w$ are prescribed in (1.4) to complete the initial boundary value problem.

1.3 Comments and results

The full set of equations (1.1)–(1.4) turns out to be a variation of the Caginalp phase field model [4]. Some mathematical discussion of a simpler problem for (1.1)–(1.2) has already been given in [23]. The papers [5, 6] dealt with well-posedness issues and asymptotic analyses with respect to the positive coefficients $\alpha,\ \beta$ as one of them approaches zero. Other concerned results for this class of systems may be found in [13, 14]. Finally, let us notice that sliding mode control problems were investigated in [10].

The existence of a weak solution for (1.1)–(1.4) and its continuous dependence with respect to data are for the first time examined in the present paper, under very general assumptions on the convex function $\widehat{\gamma}$. Then, the regularity issue for obtaining strong solutions of the system is analyzed and an improved continuous dependence estimate is proved in a restricted framework for $\widehat{\gamma}$ that still allows for the cases (1.6) and (1.7) of regular and logarithmic potentials. However, the point of emphasis for this paper is the study of the optimal control problem, whose precise formulations is given at the beginning of Section 3 (cf. (3.1)–(3.2)). A tracking-type functional has to be minimized with respect to the variation of the distributed heat source u in (1.2) and of the initial value v_0 for the temperature $\partial_t w$. Indeed, both these data are taken as controls, and the existence of optimal controls is investigated along with first-order necessary optimality conditions. More specifically, the linearized problem is introduced, and it is shown that the control-to-state mapping is Fréchet differentiable between suitable spaces. The optimal controls are eventually characterized in terms of variational inequalities for the associated adjoint variables.

About optimal control problems for phase field systems, in particular of Caginalp type, we can quote the pioneering work [18]; one may also see the specific sections in the monograph [26]. For other contributions, we mention the article [21], dedicated to a thermodynamically consistent version of the phase field system described above, and the more recent papers[8] and [9], where the interested reader can find a list of related references.

1.4 Preliminaries

Let us set the notation we are going to employ throughout the paper. Given a Banach space X, we denote by $\|\cdot\|_X$ the corresponding norm, by X^* its topological dual space, and by $\langle\cdot,\cdot\rangle_X$ the related duality pairing between X^* and X. The standard Lebesgue and Sobolev spaces defined on Ω, for every $1 \le p \le \infty$ and $k \ge 0$, are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, and the associated norms by $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. For the special case $p = 2$, these become Hilbert spaces, and we denote by $\|\cdot\| = \|\cdot\|_2$ the norm of $L^2(\Omega)$ and employ the usual notation $H^k(\Omega) := W^{k,2}(\Omega)$.

For convenience, we also introduce the notation

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega)\,, \quad W := \{v \in H^2(\Omega) :\ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}. \tag{1.20}$$

Besides, for Banach spaces X and Y, we introduce the linear space $X \cap Y$, which becomes a Banach space when equipped with its natural norm $\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y$, for $v \in X \cap Y$. To conclude, for normed spaces X and $v \in L^1(0,T;X)$, we set

$$(1 * v)(t) := \int_0^t v(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{1.21}$$

and also introduce the notation

$$(1 \circledast v)(t) := \int_t^T v(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{1.22}$$

Throughout the paper, we employ the following convention: the capital-case symbol C is used to denote every constant that depends only on the structural data of the problem such as T, Ω, α, β, θ_c, the shape of the nonlinearities, and the norms of the involved functions. For this reason, its meaning may vary from line to line and even within formulas. Moreover, when a positive constant δ enters the computation, the related symbol C_δ denotes constants that depend on δ in addition.

1.5 Plan of the paper

The rest of the work is organized in the following way. Section 2 is devoted to the mathematical analysis of system (1.1)–(1.4). We prove the existence and uniqueness of a weak solution in a very general framework that includes singular and nonregular potentials like the double obstacle one. We then show that in the case of regular and logarithmic potentials, under natural assumptions for the initial data, the system admits a unique strong solution and that the phase variable enjoys the so-called separation property. This latter is of major importance for the mathematical analysis of phase field models involving singular potentials as it guarantees that the singularity of the potential γ is no longer an obstacle for the mathematical analysis. In fact, it ensures the phase field variable φ to range in some interval in which the potential is smooth. Next, in Section 3, by the results shown in Section 2, we discuss a nontrivial application to optimal control, where we seek optimal controls in the form of a distributed heat source and an initial temperature. The existence of an optimal strategy as well as first-order necessary optimality conditions are addressed.

2 Analysis of the system

The following assumptions will be in order throughout this paper.

A1 α, β, and θ_c are positive constants.

A2 $\widehat{\gamma} : \mathbb{R} \to [0,+\infty]$ is convex and lower semicontinuous with $\widehat{\gamma}(0) = 0$, so that $\gamma := \partial\widehat{\gamma}$ is a maximal monotone graph with $\gamma(0) \ni 0$. Moreover, we denote the effective domain of γ by $\mathrm{dom}(\gamma)$.

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A3 $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Let $\widehat{\pi} \in C^1(\mathbb{R})$ denote a primitive of π, i.e., $\pi(r) = \widehat{\pi}'(r)$ for every $r \in \mathbb{R}$.

The first result concerns the existence of weak solutions.

Theorem 2.1. *Assume that* **A1**–**A3** *hold. Moreover, let the initial data fulfill*

$$\varphi_0 \in V, \quad \widehat{\gamma}(\varphi_0) \in L^1(\Omega), \quad w_0 \in V, \quad v_0 \in H, \tag{2.1}$$

and, for the heat source, suppose that

$$u \in L^2(0,T;H). \tag{2.2}$$

Then there exists a weak solution (φ, w, ξ) to the system (1.1)–(1.4) *in the sense that*

$$\begin{aligned}
&\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\xi \in L^2(0,T;H), \quad \varphi \in \mathrm{dom}(\gamma) \quad \textit{and} \quad \xi \in \gamma(\varphi) \ a.e. \textit{ in } Q,\\
&w \in H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap H^1(0,T;V),
\end{aligned}$$

and that the variational equalities

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega \xi v + \frac{2}{\theta_c}\int_\Omega \pi(\varphi)v - \frac{1}{\theta_c^2}\int_\Omega \partial_t w\, \pi(\varphi) v = 0\,, \tag{2.3}$$

$$\langle \partial_{tt} w, v\rangle_V + \alpha\int_\Omega \nabla(\partial_t w)\cdot\nabla v + \beta\int_\Omega \nabla w\cdot\nabla v + \int_\Omega \pi(\varphi)\partial_t\varphi\, v = \int_\Omega uv\,, \tag{2.4}$$

are satisfied for every test function $v \in V$ and almost everywhere in $(0,T)$. Moreover, it holds that

$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = v_0.$$

Furthermore, there exists a constant $K_1 > 0$, which depends only on $\Omega, T, \alpha, \beta, \theta_c$ and the data of the system, such that

$$\begin{aligned}
&\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \|\widehat{\gamma}(\varphi)\|^{1/2}_{L^\infty(0,T;L^1(\Omega))}\\
&\quad + \|w\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le K_1.
\end{aligned} \tag{2.5}$$

Let us emphasize that the above result is very general and includes all of the choices for the potentials introduced in (1.6)–(1.8). Besides, notice that the second condition in (2.1) follows from the first one in the case of (1.6). In fact, we have that $\widehat{\gamma}_{\mathrm{reg}}(r) = \mathcal{O}(r^4)$ as $|r| \to \infty$, and in the three-dimensional case it turns out that $\varphi_0 \in V \subset L^6(\Omega)$. In view of the regularity of the solution, note that the initial conditions make sense at least in H, since, in particular, $\varphi \in C^0([0,T];V)$ and $w \in C^1([0,T];H)$ by interpolation properties. Moreover, terms like the last integrals on the left-hand sides of (2.3) and (2.4) are well defined thanks to Hölder's inequality, since $\partial_t w \in L^2(0,T;V)$, $\pi(\varphi) \in L^\infty(0,T;V)$, $\partial_t\varphi \in L^2(0,T;H)$, and $V \subset L^p(\Omega)$ for $1 \le p \le 6$.

Proof of Theorem 2.1. We proceed by formal estimates, referring, e.g., to the papers [7, 9] for the details on a regularization and Faedo–Galerkin approximation of a similar but abstract system.

First estimate: Note that (1.1) or, more precisely,

$$\partial_t\varphi - \Delta\varphi + \xi + \frac{2}{\theta_c}\pi(\varphi) - \frac{1}{\theta_c^2}\partial_t w\, \pi(\varphi) = 0 \quad \text{in } Q, \tag{2.6}$$

with $\xi \in \gamma(\varphi)$ almost everywhere in Q, and (1.2) are the equations related to the variational equalities (2.3) and (2.4), respectively. We test (2.6) by $\theta_c^2 \partial_t \varphi$ and (1.2) by $\partial_t w$. Then we add the resulting equalities and to both sides the term $\frac{\theta_c^2}{2}\big(\|\varphi(t)\|^2 - \|\varphi_0\|^2\big) = \theta_c^2 \int_{Q_t} \varphi\, \partial_t\varphi$. Note that there is a cancellation of two terms. Integrating by parts, we obtain that

$$
\begin{aligned}
&\theta_c^2 \int_{Q_t} |\partial_t\varphi|^2 + \frac{\theta_c^2}{2}\|\varphi(t)\|_V^2 + \theta_c^2 \int_\Omega \widehat{\gamma}(\varphi(t)) \\
&\quad + \frac{1}{2}\|\partial_t w(t)\|^2 + \alpha \int_{Q_t} |\nabla(\partial_t w)|^2 + \frac{\beta}{2}\|\nabla w(t)\|^2 \\
&\le \frac{\theta_c^2}{2}\|\varphi_0\|_V^2 + \theta_c^2 \int_\Omega \widehat{\gamma}(\varphi_0) + \frac{1}{2}\|v_0\|^2 + \frac{\beta}{2}\|\nabla w_0\|^2 \\
&\quad - 2\theta_c \int_{Q_t} \pi(\varphi)\partial_t\varphi + \int_{Q_t} u\, \partial_t w + \theta_c^2 \int_{Q_t} \varphi\, \partial_t \varphi.
\end{aligned}
$$

The first four terms on the right-hand side are easily bounded due to the assumption (2.1) on the initial data. As for the other three terms, we have, using (2.2), Young's inequality and the Lipschitz continuity of π, that

$$
\begin{aligned}
-2\theta_c \int_{Q_t} \pi(\varphi)\partial_t\varphi + \theta_c^2 \int_\Omega \varphi\, \partial_t\varphi &\le \frac{\theta_c^2}{2}\int_{Q_t} |\partial_t\varphi|^2 + C\int_{Q_t} (|\varphi|^2 + 1), \\
\int_{Q_t} u\, \partial_t w &\le \frac{1}{2}\int_{Q_t} |\partial_t w|^2 + C.
\end{aligned}
$$

Now, we can apply Gronwall's lemma, which finally entails that

$$
\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\widehat{\gamma}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}^{1/2} + \|w\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C. \tag{2.7}
$$

Second estimate: Next, we take an arbitrary function $v \in L^2(0,T;V)$ in (2.4), then use the linear growth of π, Hölder's inequality, and the continuous inclusion $V \subset L^6(\Omega)$, to infer that

$$
\begin{aligned}
&\Big| \int_0^T \langle \partial_{tt} w, v\rangle_V \,\mathrm{d}t \Big| \\
&\le C \int_0^T \Big(\|\nabla(\partial_t w)\|\|\nabla v\| + \|\nabla w\|\|\nabla v\| + \|u\|\|v\| \Big)\,\mathrm{d}t + C\int_Q (|\varphi| + 1)|\partial_t\varphi||v| \\
&\le C \int_0^T \Big(\|\nabla(\partial_t w)\| + \|\nabla w\| + \|u\| + (\|\varphi\|_3 + 1)\|\partial_t\varphi\| \Big)\|v\|_V \,\mathrm{d}t \\
&\le C\|v\|_{L^2(0,T;V)}.
\end{aligned}
$$

Thus, it is a standard matter to conclude that

$$
\|\partial_{tt} w\|_{L^2(0,T;V^*)} \le C. \tag{2.8}
$$

Third estimate: Next, we notice that (2.6) can be rewritten as the elliptic equation

$$
-\Delta\varphi + \xi = g, \quad \text{with} \quad g := -\,\partial_t\varphi - \frac{2}{\theta_c}\pi(\varphi) + \frac{1}{\theta_c^2}\partial_t w\, \pi(\varphi)
$$

and $\xi \in \gamma(\varphi)$ almost everywhere in Q. Due to the estimate (2.7), g is bounded in $L^2(0,T;H)$: indeed, it turns out that $\partial_t w \in L^2(0,T;L^4(\Omega))$ and $\pi(\varphi) \in L^\infty(0,T;L^4(\Omega))$. Thus, formally testing by $-\Delta\varphi$ and using monotonicity to infer that $\int_Q \xi(-\Delta\varphi) \geq 0$, we find that

$$\|\Delta\varphi\|_{L^2(0,T;H)} + \|\xi\|_{L^2(0,T;H)} \leq C.$$

Then, from (2.6), the smooth boundary condition (1.3) for φ, and well-known elliptic regularity results (see, e.g., [1]), it follows that

$$\|\varphi\|_{L^2(0,T;H^2(\Omega))} \leq C. \tag{2.9}$$

This ends the proof of the estimate (2.5), whence Theorem 2.1 is completely proved. □

Theorem 2.2. *Suppose that* **A1**–**A3** *hold. Then there exists a unique weak solution* (φ, w, ξ) *to the system* (1.1)–(1.4) *in the sense of Theorem 2.1. Moreover, let us denote by* $\{(\varphi_i, w_i, \xi_i)\}_{i=1,2}$ *a pair of weak solutions obtained by Theorem 2.1 and related to the initial data* $\{\varphi_{0,i}, w_{0,i}, v_{0,i}\}_{i=1,2}$ *and heat sources* $\{u_i\}_{i=1,2}$ *fulfilling* (2.1) *and* (2.2)*, respectively. Then it holds that*

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|w_1 - w_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\
&\quad \leq K_2\big(\|\varphi_{0,1} - \varphi_{0,2}\| + \|w_{0,1} - w_{0,2}\|_V + \|v_{0,1} - v_{0,2}\|\big) \\
&\qquad + K_2\|1 * (u_1 - u_2)\|_{L^2(0,T;H)}
\end{aligned} \tag{2.10}$$

with a positive constant K_2 *that depends only on* $\Omega, T, \alpha, \beta, \theta_c$ *and the data of the system.*

Proof of Theorem 2.2. We aim to prove the stability estimate (2.10). This will in turn guarantee the uniqueness of weak solutions. For convenience, let us set

$$\varphi := \varphi_1 - \varphi_2, \quad w := w_1 - w_2, \quad \xi := \xi_1 - \xi_2, \tag{2.11}$$

$$\rho_i := \pi(\varphi_i) \quad \text{for } i = 1, 2, \quad \rho := \rho_1 - \rho_2, \tag{2.12}$$

$$\varphi_0 := \varphi_{0,1} - \varphi_{0,2}, \quad w_0 := w_{0,1} - w_{0,2}, \quad v_0 := v_{0,1} - v_{0,2}, \quad u := u_1 - u_2. \tag{2.13}$$

Using this notation, we take the difference of the weak formulation (2.3)–(2.4) written for $\{(\varphi_i, w_i, \xi_i)\}_{i=1,2}$ and $\{\varphi_{0,i}, w_{0,i}, v_{0,i}, u_i\}_{i=1,2}$, obtaining that the differences fulfill

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega \xi v + \frac{2}{\theta_c}\int_\Omega \rho\, v - \frac{1}{\theta_c^2}\int_\Omega \partial_t w\, \rho_1 v - \frac{1}{\theta_c^2}\int_\Omega \partial_t w_2\, \rho\, v = 0, \tag{2.14}$$

$$\langle \partial_{tt} w, v\rangle_V + \alpha\int_\Omega \nabla(\partial_t w)\cdot\nabla v + \beta\int_\Omega \nabla w\cdot\nabla v + \int_\Omega \partial_t(\widehat{\pi}(\varphi_1) - \widehat{\pi}(\varphi_2))v = \int_\Omega uv, \tag{2.15}$$

for all $v \in V$ and almost everywhere in $(0,T)$. Note that, thanks to **A3**, we could write the terms $\rho_i\,\partial_t\varphi_i$ appearing in (1.2) as $\partial_t\widehat{\pi}(\varphi_i)$, $i = 1, 2$. Of course, also the initial conditions

$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = v_0, \quad \text{hold a.e. in } \Omega. \tag{2.16}$$

First, we add the term $\int_\Omega \varphi\, v$ to both sides of (2.14), then take $v = \varphi$ and integrate with respect to time. We deduce that

$$\begin{aligned}
&\frac{1}{2}\|\varphi(t)\|^2 + \int_0^t \|\varphi(s)\|_V^2\,\mathrm{d}s + \int_{Q_t} \xi\,\varphi \\
&\quad = \frac{1}{2}\|\varphi_0\|^2 + \int_{Q_t}\Big(\varphi - \frac{2}{\theta_c}\rho\Big)\varphi + \frac{1}{\theta_c^2}\int_{Q_t}\partial_t w\,\rho_1\,\varphi + \frac{1}{\theta_c^2}\int_{Q_t}\partial_t w_2\,\rho\,\varphi
\end{aligned} \tag{2.17}$$

for all $t \in [0,T]$. Due to the monotonicity of γ, we immediately conclude that the third term on the left-hand side is nonnegative. Using the Lipschitz continuity of π along with the regularities $\partial_t w_i \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\varphi_i \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $i = 1,2$, we infer from Theorem 2.1 that

$$\int_{Q_t} \Big(\varphi - \frac{2}{\theta_c}\rho\Big)\varphi \le C \int_{Q_t} |\varphi|^2 ,$$

and, with the help of Hölder's inequality and of the continuous embedding $V \subset L^4(\Omega)$,

$$\begin{aligned}\frac{1}{\theta_c^2}\int_{Q_t} \partial_t w\, \rho_1 \varphi &\le C \int_0^t \|\partial_t w\| \Big(\|\varphi_1\|_4 + 1\Big) \|\varphi\|_4 \,\mathrm{d}s \\ &\le C\Big(\|\varphi_1\|_{L^\infty(0,T;V)} + 1\Big) \int_0^t \|\partial_t w\|\, \|\varphi\|_V \,\mathrm{d}s \le \frac{1}{4}\int_0^t \|\varphi\|_V^2 \,\mathrm{d}s + D_1 \int_{Q_t} |\partial_t w|^2,\end{aligned}$$

where D_1 is a computable and by now fixed constant. Moreover, we have that

$$\begin{aligned}\frac{1}{\theta_c^2}\int_{Q_t} \partial_t w_2\, \rho\, \varphi &\le C \int_0^t \|\partial_t w_2\|_4\, \|\varphi\| \|\varphi\|_4 \,\mathrm{d}s \\ &\le C\int_0^t \|\partial_t w_2\|_V\, \|\varphi\|\, \|\varphi\|_V \,\mathrm{d}s \le \frac{1}{4}\int_0^t \|\varphi\|_V^2 \,\mathrm{d}s + C\int_0^t \|\partial_t w_2\|_V^2\, \|\varphi\|^2 \,\mathrm{d}s,\end{aligned}$$

where the function $t \mapsto \|\partial_t w_2(t)\|_V^2$ belongs to $L^1(0,T)$ due to Theorem 2.1. Therefore, collecting the above estimates, it follows from (2.17) that

$$\begin{aligned}&\frac{1}{2}\|\varphi(t)\|^2 + \frac{1}{2}\int_0^t \|\varphi(s)\|_V^2 \,\mathrm{d}s \\ &\quad\le \frac{1}{2}\|\varphi_0\|^2 + C\int_0^t \Big(1 + \|\partial_t w_2\|_V^2\Big)\, \|\varphi\|^2 \,\mathrm{d}s + D_1 \int_{Q_t} |\partial_t w|^2. \end{aligned} \tag{2.18}$$

Next, we integrate (2.15) with respect to time using (2.16), then take $v = \partial_t w$, and integrate once more over $(0,t)$, for an arbitrary $t \in [0,T]$. Addition of the terms $\frac{\alpha}{2}\big(\|w(t)\|^2 - \|w_0\|^2\big) = \alpha \int_{Q_t} w\, \partial_t w$ to both sides leads to

$$\begin{aligned}\int_{Q_t} |\partial_t w|^2 + \frac{\alpha}{2}\|w(t)\|_V^2 &= \int_{Q_t} v_0\, \partial_t w + \int_{Q_t} (\widehat{\pi}(\varphi_{0,1}) - \widehat{\pi}(\varphi_{0,2}))\partial_t w \\ &\quad + \alpha \int_{Q_t} \nabla w_0 \cdot \nabla(\partial_t w) + \frac{\alpha}{2}\|w_0\| - \beta\int_{Q_t} (1 * \nabla w)\cdot \nabla(\partial_t w) \\ &\quad - \int_{Q_t} (\widehat{\pi}(\varphi_1) - \widehat{\pi}(\varphi_2))\partial_t w + \int_{Q_t} (1 * u)\partial_t w + \alpha \int_{Q_t} w\, \partial_t w. \end{aligned} \tag{2.19}$$

We estimate each term on the right-hand side individually. Let us recall that the mean value theorem and the Lipschitz continuity of π yield the existence of a positive constant C such that

$$|\widehat{\pi}(r) - \widehat{\pi}(s)| \le C(|r| + |s| + 1)|r - s| \quad \text{for all } r,\ s \in \mathbb{R}. \tag{2.20}$$

By Young's inequality, we easily have

$$\int_{Q_t} v_0\, \partial_t w \le \frac{1}{8}\int_{Q_t} |\partial_t w|^2 + C\|v_0\|^2.$$

Using integration over time, Hölder's inequality, (2.20), and the continuous embedding $V \subset L^4(\Omega)$, we find that

$$\begin{aligned}\int_{Q_t} (\widehat{\pi}(\varphi_{0,1}) - \widehat{\pi}(\varphi_{0,2}))\partial_t w &= \int_\Omega (\widehat{\pi}(\varphi_{0,1}) - \widehat{\pi}(\varphi_{0,2}))(w(t) - w_0)\\ &\le C\big\||\varphi_{0,1}| + |\varphi_{0,2}| + 1\big\|_4 \|\varphi_{0,1} - \varphi_{0,2}\| \big(\|w(t)\|_4 + \|w_0\|_4\big)\\ &\le C\big(\|\varphi_{0,1}\|_V + \|\varphi_{0,2}\|_V + 1\big)\|\varphi_0\|\big(\|w(t)\|_V + \|w_0\|_V\big)\\ &\le \frac{\alpha}{8}\big(\|w(t)\|_V^2 + \|w_0\|_V^2\big) + C\big(\|\varphi_{0,1}\|_V^2 + \|\varphi_{0,2}\|_V^2 + 1\big)\|\varphi_0\|^2.\end{aligned}$$

Next, the third term on the right-hand side of (2.19) can be bounded as

$$\alpha \int_{Q_t} \nabla w_0 \cdot \nabla(\partial_t w) = \alpha \int_\Omega \nabla w_0 \cdot (\nabla w(t) - \nabla w_0) \le \frac{\alpha}{8}\|\nabla w(t)\|^2 + C\|\nabla w_0\|^2.$$

Then, by using the identity

$$\int_{Q_t} (1 * \nabla w) \cdot \nabla(\partial_t w) = \int_\Omega (1 * \nabla w(t)) \cdot \nabla w(t) - \int_{Q_t} |\nabla w|^2,$$

the fact that $\|1 * \nabla w(t)\|^2 \le \Big(\int_0^t \|\nabla w\|\Big)^2 \le T \int_{Q_t} |\nabla w|^2$, and Young's inequality, we infer that

$$-\beta \int_{Q_t} (1 * \nabla w) \cdot \nabla(\partial_t w) \le \frac{\alpha}{8}\|\nabla w(t)\|^2 + C \int_{Q_t} |\nabla w|^2.$$

To handle the sixth term on the right-hand side of (2.19), we owe once more to (2.20) and the continuous and compact embedding $V \subset L^p(\Omega)$, $1 \le p < 6$. By the Hölder and Young inequalities, and thanks to (2.5) and the Ehrling lemma (see, e.g., [22, Lemme 5.1, p. 58]), we can deduce that

$$\begin{aligned}-\int_{Q_t} (\widehat{\pi}(\varphi_1) - \widehat{\pi}(\varphi_2))\partial_t w &\le C \int_0^t \big\||\varphi_1| + |\varphi_2| + 1\big\|_4 \|\varphi_1 - \varphi_2\|_4 \|\partial_t w\| \,\mathrm{d}s\\ &\le \frac{1}{8}\int_{Q_t} |\partial_t w|^2 + C(\|\varphi_1\|^2_{L^\infty(0,T;V)} + \|\varphi_2\|^2_{L^\infty(0,T;V)} + 1)\int_0^t \|\varphi\|_4^2 \,\mathrm{d}s\\ &\le \frac{1}{8}\int_{Q_t} |\partial_t w|^2 + \delta \int_0^t \|\varphi\|_V^2 \,\mathrm{d}s + C_\delta \int_{Q_t} |\varphi|^2,\end{aligned}$$

for any positive coefficient δ (yet to be chosen). Lastly, Young's inequality easily produces

$$\int_{Q_t} (1 * u)\partial_t w + \alpha \int_{Q_t} w\,\partial_t w \le \frac{1}{4}\int_{Q_t} |\partial_t w|^2 + C\int_{Q_t} |1 * u|^2 + C\int_{Q_t} |w|^2.$$

Thus, in view of (2.19), upon collecting the above computations, we realize that

$$\begin{aligned}&\frac{1}{2}\int_{Q_t} |\partial_t w|^2 + \frac{\alpha}{8}\|w(t)\|_V^2\\ &\quad\le C\|v_0\|^2 + C\big(\|\varphi_{0,1}\|_V^2 + \|\varphi_{0,2}\|_V^2 + 1\big)\|\varphi_0\|^2 + C\|w_0\|_V^2\\ &\qquad + \delta \int_0^t \|\varphi\|_V^2 \,\mathrm{d}s + C_\delta \int_{Q_t} |\varphi|^2 + C\int_{Q_t} |1 * u|^2 + C\int_0^t \|w\|_V^2 \,\mathrm{d}s. \qquad (2.21)\end{aligned}$$

At this point, we multiply (2.21) by $4D_1$ and add it to (2.18); then, fixing $\delta > 0$ such that $4D_1\delta < 1/2$, and applying the Gronwall lemma, we obtain the estimate

$$\begin{aligned}&\|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|w\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\ &\quad \le C(\|\varphi_0\| + \|w_0\|_V + \|v_0\| + \|1 * u\|_{L^2(0,T;H)}),\end{aligned}$$

where C depends also on $\|\varphi_{0,i}\|_V$, $i = 1, 2$. Due to our notation in (2.11)–(2.13), this is actually (2.10), and the proof of Theorem 2.2 is complete. □

To improve the regularity results of Theorem 2.1, as well as the stability estimate (2.10), we are forced to require more regularity on structural elements, in particular, for the nonlinearity $\widehat{\gamma}$. In the following lines, we state general conditions under which we are able to extend the existence and uniqueness results to a stronger framework.

B1 There exists an interval (r_-, r_+) with $-\infty \le r_- < 0 < r_+ \le +\infty$ such that the restriction of $\widehat{\gamma}$ to (r_-, r_+) belongs to $C^2(r_-, r_+)$. Thus, γ coincides with the derivative of $\widehat{\gamma}$ in (r_-, r_+).

B2 It holds that $\lim_{r\searrow r_-} \gamma(r) = -\infty$ and $\lim_{r\nearrow r_+} \gamma(r) = +\infty$.

B3 $\gamma \in C^2(r_-, r_+)$ and $\pi \in C^2(\mathbb{R})$.

Notice that **B1**–**B3** are fulfilled by the regular and the logarithmic potentials (1.6) and (1.7), whereas the double obstacle nonlinearity (1.8) is no longer allowed. Again, we remark that, due to **B1**, we no longer need to consider any selection $\xi \in \partial\gamma(\varphi)$ as $\gamma = \widehat{\gamma}\,'$ in (r_-, r_+) . This also entails that (1.1) becomes an equality.

The next result dealing with regularity of the solution does not need the condition **B3**.

Theorem 2.3. *Assume that* **A1**–**A3** *and* **B1**–**B2** *are fulfilled. Furthermore, let the heat source u fulfill* (2.2)*, and let the initial data, in addition to* (2.1)*, satisfy*

$$\varphi_0 \in W, \quad v_0 \in V, \quad \varphi_0' := \Delta\varphi_0 - \gamma(\varphi_0) - \tfrac{2}{\theta_c}\pi(\varphi_0) + \tfrac{1}{\theta_c^2} v_0 \pi(\varphi_0) \in H. \tag{2.22}$$

Then there exists a strong solution (φ, w) to system (1.1)–(1.4) *in the sense that*

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.23}$$
$$w \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W), \tag{2.24}$$

and that the equations (1.1)–(1.4) *are fulfilled almost everywhere in Q, on Σ, or in Ω, respectively. In addition, assume that the heat source u fulfills*

$$u \in L^\infty(0,T;H) \tag{2.25}$$

and that

$$w_0,\, v_0 \in L^\infty(\Omega), \quad r_- < \min_{x\in\overline{\Omega}} \varphi_0(x) \le \max_{x\in\overline{\Omega}} \varphi_0(x) < r_+. \tag{2.26}$$

Then it holds that

$$\partial_t w \in L^\infty(Q)\,,$$

and the phase variable φ enjoys the so-called separation property, which means that there exist two values r_*, r^*, depending only on $\Omega, T, \alpha, \beta, \theta_c$ and the data of the system, such that

$$r_- < r_* \le \varphi \le r^* < r_+ \quad \text{a.e. in } Q. \tag{2.27}$$

Furthermore, there exists a constant $K_3 > 0$ such that

$$\begin{aligned}&\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))}\\ &\quad + \|w\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + \|\partial_t w\|_{L^\infty(Q)} \le K_3.\end{aligned} \tag{2.28}$$

Here, we point out that the regularities in (2.24) imply $w \in C^0(\overline{Q})$ thanks to the Sobolev embedding results. Moreover, since the embedding $W \subset C^0(\overline{\Omega})$ is compact, it follows from [25, Sect. 8, Cor. 4] that also $\varphi \in C^0(\overline{Q})$. In particular, the separation property (2.27) is valid even pointwise in $\overline{Q}$.

Proof of Theorem 2.3. In what follows, we perform the estimate directly on the system (1.1)–(1.4) underlying that now also equation (1.1) turns to an equality as $\gamma(\cdot) = \widehat{\gamma}'(\cdot)$ is single valued. A rigorous proof would need some approximation, but please take into account that we already have proved the existence and uniqueness of the weak solution.

First estimate: To begin with, we formally differentiate (1.1) with respect to time and multiply the resulting identity by $\theta_c^2 \partial_t \varphi$; then we add (1.2) tested by $\partial_{tt} w$, and integrate over Q_t. Note that a cancellation occurs and that, after some rearrangements, one obtains

$$\begin{aligned}&\frac{\theta_c^2}{2}\|\partial_t\varphi(t)\|^2 + \theta_c^2\int_{Q_t}|\nabla(\partial_t\varphi)|^2 + \theta_c^2\int_{Q_t}\gamma'(\varphi)|\partial_t\varphi|^2 + \int_{Q_t}|\partial_{tt}w|^2 + \frac{\alpha}{2}\|\nabla(\partial_t w)(t)\|^2\\ &\quad \le \frac{\theta_c^2}{2}\|\varphi_0'\|^2 + \frac{\alpha}{2}\|\nabla v_0\|^2 - 2\theta_c\int_{Q_t}\pi'(\varphi)|\partial_t\varphi|^2 + \int_{Q_t}\partial_t w\,\pi'(\varphi)|\partial_t\varphi|^2\\ &\qquad - \beta\int_{Q_t}\nabla w\cdot\nabla(\partial_{tt}w) + \int_{Q_t}u\,\partial_{tt}w.\end{aligned}$$

Owing to the monotonicity of γ, we infer that the third term on the left-hand side is nonnegative. The first two terms on the right-hand side are controlled due to the conditions (2.22) on the initial data. As for the third term on the right-hand side, we note that $\partial_t\pi(\varphi)$ makes sense as $\pi'(\varphi)\partial_t\varphi$, in view of the global Lipschitz continuity of π. Now, we use the boundedness of π' and estimate (2.5), obtaining that

$$-2\theta_c\int_{Q_t}\pi'(\varphi)|\partial_t\varphi|^2 \le C\int_{Q_t}|\partial_t\varphi|^2 \le C.$$

Next, as $V \subset L^4(\Omega)$ with compact embedding, we employ Hölder's inequality, (2.5), and Ehrling's lemma, to deduce that

$$\begin{aligned}\int_{Q_t}\partial_t w\,\pi'(\varphi)|\partial_t\varphi|^2 &\le C\int_0^t\|\partial_t w\|\|\partial_t\varphi\|_4^2\,\mathrm{d}s\\ &\le C\int_0^t\|\partial_t\varphi\|_4^2\,\mathrm{d}s \le \frac{\theta_c^2}{2}\int_{Q_t}|\nabla(\partial_t\varphi)|^2 + C\int_{Q_t}|\partial_t\varphi|^2.\end{aligned}$$

The fifth term on the right-hand side can be controlled by integrating by parts and using the above estimate along with Young's inequality and assumptions (2.22), so that

$$
\begin{aligned}
&- \beta \int_{Q_t} \nabla w \cdot \nabla(\partial_{tt} w) \\
&= \beta \int_{Q_t} |\nabla(\partial_t w)|^2 - \beta \int_\Omega \nabla w(t) \cdot \nabla(\partial_t w(t)) + \beta \int_\Omega \nabla w_0 \cdot \nabla v_0 \\
&\le \beta \int_{Q_t} |\nabla(\partial_t w)|^2 + \frac{\alpha}{4} \|\nabla(\partial_t w)(t)\|^2 + C\|w\|^2_{L^\infty(0,T;V)} + C(\|w_0\|_V^2 + \|v_0\|_V^2).
\end{aligned}
$$

Finally, the last term can be easily handled by Young's inequality, namely,

$$
\int_{Q_t} u\, \partial_{tt} w \le \frac{1}{2} \int_{Q_t} |\partial_{tt} w|^2 + \frac{1}{2} \|u\|^2_{L^2(0,T;H)}.
$$

Hence, upon collecting the above computations, the Gronwall lemma yields that

$$
\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|w\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)} \le C. \tag{2.29}
$$

Second estimate: By comparison in equation (1.1), we deduce that

$$
\|-\Delta\varphi + \gamma(\varphi)\|_{L^\infty(0,T;H)} \le C.
$$

Then, arguing as in the proof of Theorem 2.1 (cf. the Third estimate there), and using the elliptic regularity theory, we infer that

$$
\|\varphi\|_{L^\infty(0,T;H^2(\Omega))} + \|\gamma(\varphi)\|_{L^\infty(0,T;H)} \le C. \tag{2.30}
$$

Third estimate: We then rewrite (1.2) as a parabolic equation in the new variable $y := \alpha\partial_t w + \beta w$. Thanks to equations (1.3)–(1.4), we have that

$$
\begin{cases}
\frac{1}{\alpha}\partial_t y - \Delta y = g := u - \pi(\varphi)\partial_t\varphi + \frac{\beta}{\alpha}\partial_t w & \text{in } Q, \\
\partial_{\mathbf{n}} y = 0 & \text{on } \Sigma, \\
y(0) = y_0 := \alpha v_0 + \beta w_0 & \text{in } \Omega.
\end{cases} \tag{2.31}
$$

By analyzing system (2.31), we realize that $g \in L^2(0,T;H)$ and $y_0 \in V$, so that the parabolic regularity theory entails that

$$
\|y\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \le C. \tag{2.32}
$$

In fact, since the ODE relation $\alpha\partial_t w + \beta w = y$ holds true in Q, then

$$
w(t) = e^{-\beta t/\alpha} w_0 + \frac{1}{\alpha} \int_0^t e^{-\beta(t-s)/\alpha} y(s)\mathrm{d}s, \quad t \in [0,T]. \tag{2.33}
$$

Thus, w and its derivative $\partial_t w$ possess the same regularity as y and satisfy estimates like (2.32), where the constant on the right-hand side has the same dependencies. Therefore, we eventually conclude that

$$
\|w\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} \le C. \tag{2.34}
$$

Fourth estimate: Let us consider again system (2.31). Due to the above estimates and to (2.25), we have that g is bounded in $L^\infty(0,T;H)$. Thanks to (2.1), (2.22), and the first condition in (2.26), it turns out that the initial datum y_0 is bounded in $V \cap L^\infty(\Omega)$. Hence, an application of [20, Thm. 7.1, p. 181] yields that

$$\|y\|_{L^\infty(Q)} = \|\alpha\partial_t w + \beta w\|_{L^\infty(Q)} \le C.$$

Moreover, arguing as above, this in particular leads to

$$\|w\|_{L^\infty(Q)} + \|\partial_t w\|_{L^\infty(Q)} \le C. \tag{2.35}$$

As a consequence, by virtue of (2.29), (2.30), and (2.34), the estimate (2.28) eventually follows.

Separation property: Now, with the help of the regularity result proved above, we are in a position to prove the separation property for the phase variable φ. This can be shown by following the same lines of argumentation as in [11, Proof of Theorem 2.2] (see also [12]). Observe that φ is bounded in $L^\infty(Q)$ due to (2.30) and the Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$ (as noted above, we even have $\varphi \in C^0(\overline{Q})$). Hence, if we rewrite (1.1) as

$$\partial_t\varphi - \Delta\varphi + \gamma(\varphi) = g, \quad \text{where now} \quad g{:} = -\tfrac{2}{\theta_c}\pi(\varphi) + \tfrac{1}{\theta_c^2}\partial_t w\, \pi(\varphi), \tag{2.36}$$

then it turns out that g is bounded in $L^\infty(Q)$, due to **A3** and (2.35). This entails the existence of a positive constant g^* for which $\|g\|_{L^\infty(Q)} \le g^*$. Furthermore, the growth assumptions **B1**–**B2** ensure the existence of some constants r_* and r^* such that $r_- < r_* \le r^* < r^+$ and

$$r_* \le \min_{x\in\overline{\Omega}} \varphi_0(x), \quad r^* \ge \max_{x\in\overline{\Omega}} \varphi_0(x), \tag{2.37}$$

$$\gamma(r) + g^* \le 0 \quad \forall r \in (r_-, r_*), \quad \gamma(r) - g^* \ge 0 \quad \forall r \in (r^*, r_+). \tag{2.38}$$

Then, if we set $\lambda = (\varphi - r^*)^+$, where $(\cdot)^+ := \max\{\cdot, 0\}$ denotes the positive part function, and multiply equation (2.36) by λ, then integration over Q_t and by parts leads to

$$\frac{1}{2}\|\lambda(t)\|^2 + \int_{Q_t} |\nabla\lambda|^2 + \int_{Q_t} (\gamma(\varphi) - g)\lambda = 0,$$

for all $t \in [0,T]$, where we also applied (2.37) to conclude that $\lambda(0) = 0$. Moreover, (2.38) yields that the last term on the left-hand side of the above identity is nonnegative, so that it follows $\lambda = (\varphi - r^*)^+ = 0$, which means that $\varphi \le r^*$ almost everywhere in Q. The same argument can be applied with the choice $\lambda = -(\varphi - r_*)^-$, with $(\cdot)^- := -\min\{0, \cdot\}$, to derive the other bound $\varphi \ge r_*$ almost everywhere in Q. Thus, we end up with the property (2.27) and conclude the proof. □

Finally, in the more regular framework we can provide a refined continuous dependence result that complements Theorem 2.2.

Theorem 2.4. *Suppose that* **A1**–**A3** *and* **B1**–**B3** *hold. Denote by* $\{(\varphi_i, w_i)\}_{i=1,2}$ *two pairs of strong solutions obtained by Theorem 2.3 in correspondence with the initial data* $\{\varphi_{0,i}, w_{0,i}, v_{0,i}\}_{i=1,2}$ *fulfilling* (2.1)*,* (2.22)*,* (2.26)*, and heat sources* $\{u_i\}_{i=1,2}$ *as in* (2.25)*. Then it holds that*

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|w_1 - w_2\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \\
&\quad \le K_4\big(\|\varphi_{0,1} - \varphi_{0,2}\|_W + \|w_{0,1} - w_{0,2}\|_V + \|v_{0,1} - v_{0,2}\|_V\big) \\
&\qquad + K_4\|u_1 - u_2\|_{L^2(0,T;H)},
\end{aligned} \tag{2.39}$$

with a positive constant K_4 *that depends only on* $\Omega, T, \alpha, \beta, \theta_c$ *and the data of the system.*

Proof of Theorem 2.4. First, let us recall the notation introduced in (2.11)–(2.13) and again consider the variational system (2.14)–(2.15). Now, owing to the regularity assumption **B1**, we have $\xi_i = \gamma(\varphi_i)$ for $i = 1, 2$. Moreover, the separation property (2.27) enjoyed by both φ_i, $i = 1, 2$, combined with **B1**–**B2**, yields that γ is Lipschitz continuous when restricted to $[r_*, r^*]$. Besides, due to the improved regularity at disposal, we may now express the difference $\partial_t(\widehat{\pi}(\varphi_1) - \widehat{\pi}(\varphi_2))$ in (2.15) as $\rho_1\partial_t\varphi_1 - \rho_2\partial_t\varphi_2 = \rho\partial_t\varphi_1 + \rho_2\partial_t\varphi$.

Let us now move on checking the estimate (2.39).

First estimate: We test (2.14) by $\partial_t\varphi$, (2.15) by $\partial_t w$, add the resulting identities, and integrate over $(0, t)$ to infer that

$$\begin{aligned}
&\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\|\nabla\varphi(t)\|^2 + \frac{1}{2}\|\partial_t w(t)\|^2 + \alpha\int_{Q_t} |\nabla(\partial_t w)|^2 + \frac{\beta}{2}\|\nabla w(t)\|^2 \\
&\quad = \frac{1}{2}\|\nabla\varphi_0\|^2 + \frac{1}{2}\|v_0\|^2 + \frac{\beta}{2}\|\nabla w_0\|^2 - \int_{Q_t} \big(\gamma(\varphi_1) - \gamma(\varphi_2)\big)\partial_t\varphi \\
&\qquad - \frac{2}{\theta_c}\int_{Q_t} \rho\,\partial_t\varphi + \frac{1}{\theta_c^2}\int_{Q_t} \partial_t w\,\rho_1\,\partial_t\varphi + \frac{1}{\theta_c^2}\int_{Q_t} \partial_t w_2\,\rho\,\partial_t\varphi \\
&\qquad - \int_{Q_t} \rho\,\partial_t\varphi_1\,\partial_t w - \int_{Q_t} \rho_2\,\partial_t\varphi\,\partial_t w + \int_{Q_t} u\,\partial_t w.
\end{aligned} \tag{2.40}$$

The fourth, fifth, and last terms on the right-hand side can be easily handled using Young's inequality and the Lipschitz continuity of π and γ, namely,

$$\begin{aligned}
&- \int_{Q_t} \big(\gamma(\varphi_1) - \gamma(\varphi_2)\big)\partial_t\varphi - \frac{2}{\theta_c}\int_{Q_t} \rho\,\partial_t\varphi + \int_{Q_t} u\,\partial_t w \\
&\quad \le \frac{1}{4}\int_{Q_t} |\partial_t\varphi|^2 + C\int_{Q_t} (|\varphi|^2 + |u|^2 + |\partial_t w|^2).
\end{aligned}$$

Due to Theorem 2.3, we have that φ_i, and consequently ρ_i, are uniformly bounded in $L^\infty(Q)$ for $i = 1, 2$, so that also the sixth and ninth terms can be easily controlled in a similar fashion as

$$\begin{aligned}
&\frac{1}{\theta_c^2}\int_{Q_t} \partial_t w\,\rho_1\,\partial_t\varphi - \int_{Q_t} \rho_2\,\partial_t\varphi\,\partial_t w \\
&\quad \le \frac{1}{4}\int_{Q_t} |\partial_t\varphi|^2 + C\big(\|\rho_1\|^2_{L^\infty(Q)} + \|\rho_2\|^2_{L^\infty(Q)}\big)\int_{Q_t} |\partial_t w|^2.
\end{aligned}$$

As for the remaining two terms, we recall that $\|\partial_t\varphi_i\|_{L^\infty(0,T;H)}$ and $\|\partial_t w_i\|_{L^\infty(0,T;V)}$ are bounded for $i = 1, 2$, so that the Hölder and Young inequalities and the continuous embedding $V \subset L^4(\Omega)$ imply that

$$\begin{aligned}
&\frac{1}{\theta_c^2}\int_{Q_t} \partial_t w_2\,\rho\,\partial_t\varphi - \int_{Q_t} \rho\,\partial_t\varphi_1\,\partial_t w \\
&\quad \le C\int_0^t \|\partial_t w_2\|_4\,\|\varphi\|_4\,\|\partial_t\varphi\|\,\mathrm{d}s + C\int_0^t \|\varphi\|_4\,\|\partial_t\varphi_1\|\,\|\partial_t w\|_4\,\mathrm{d}s \\
&\quad \le \frac{1}{4}\int_{Q_t} |\partial_t\varphi|^2 + C\,\|\partial_t w_2\|^2_{L^\infty(0,T;V)}\int_0^t \|\varphi\|_V^2\,\mathrm{d}s \\
&\qquad + \frac{\alpha}{2}\int_{Q_t} \big(|\partial_t w|^2 + |\nabla(\partial_t w)|^2\big) + C\,\|\partial_t\varphi_1\|^2_{L^\infty(0,T;H)}\int_0^t \|\varphi\|_V^2\,\mathrm{d}s.
\end{aligned}$$

At this point, we can collect the above estimates and combine them with (2.40). Then we either apply the Gronwall lemma or take advantage of the already shown inequality (2.10) to bound the right-hand side. Thus, we arrive at

$$\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|w\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \\ \leq C\big(\|\varphi_0\|_V + \|w_0\|_V + \|v_0\| + \|u\|_{L^2(0,T;H)}\big). \tag{2.41}$$

Second estimate: Arguing as in (2.31), we can rewrite (2.15) as a parabolic system in the variable $y = \alpha\partial_t w + \beta w$ with source term $g := u - \rho\partial_t\varphi_1 - \rho_2\partial_t\varphi + \frac{\beta}{\alpha}\partial_t w$. Since

$$\|\rho\,\partial_t\varphi_1\|^2_{L^2(0,T;H)} \leq C\int_0^T \|\varphi\|_4^2\,\|\partial_t\varphi_1\|_4^2\,\mathrm{d}s \leq C\|\varphi\|^2_{L^\infty(0,T;V)}\|\partial_t\varphi_1\|^2_{L^2(0,T;V)},$$

and as (2.28) holds, it turns out that

$$\|g\|_{L^2(0,T;H)} \leq C\big(\|\varphi_0\|_V + \|w_0\|_V + \|w_0'\| + \|u\|_{L^2(0,T;H)}\big).$$

Moreover, the initial value $y(0) = \alpha v_0 + \beta w_0$ lies in V. Therefore, using parabolic regularity and the representation given in (2.33) (which holds as well), we easily infer that

$$\|w\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \|\partial_t w\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \\ \leq C\big(\|\varphi_0\|_V + \|w_0\|_V + \|v_0\|_V + \|u\|_{L^2(0,T;H)}\big). \tag{2.42}$$

Third estimate: First, we observe that (2.14) can be rewritten as

$$\int_\Omega \partial_t\varphi\, v = -\int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega hv \quad \text{for every } v\in V, \text{ a.e. in } (0,T). \tag{2.43}$$

Here, recalling the notation in (2.11)–(2.13), h is specified by

$$h = -\gamma(\varphi_1) + \gamma(\varphi_2) - \frac{2}{\theta_c}(\pi(\varphi_1) - \pi(\varphi_2)) + \frac{1}{\theta_c^2}\big(\partial_t w\,\pi(\varphi_1) + \partial_t w_2(\pi(\varphi_1) - \pi(\varphi_2))\big).$$

Now, in view of the regularity properties in (2.27) and (2.28) that hold for both (φ_1, w_1) and (φ_2, w_2), we can check that every term of h belongs to $H^1(0,T;H)$ and that

$$\partial_t h = -(\gamma'(\varphi_1) - \gamma'(\varphi_2))\partial_t\varphi_1 - \gamma'(\varphi_2)\partial_t\varphi - \frac{2}{\theta_c}(\pi'(\varphi_1) - \pi'(\varphi_2))\partial_t\varphi_1 - \frac{2}{\theta_c}\pi'(\varphi_2)\partial_t\varphi \\ + \frac{1}{\theta_c^2}\big(\partial_{tt}w\,\pi(\varphi_1) + \partial_t w\,\pi'(\varphi_1)\,\partial_t\varphi_1 + \partial_{tt}w_2(\pi(\varphi_1) - \pi(\varphi_2))\big) \\ + \frac{1}{\theta_c^2}\big(\partial_t w_2(\pi'(\varphi_1) - \pi'(\varphi_2))\partial_t\varphi_1 + \partial_t w_2\,\pi'(\varphi_2)\partial_t\varphi\big)\,. \tag{2.44}$$

Moreover, from (2.43) we can recover the expression of $\partial_t\varphi(0)$, which is given by (cf. (2.22))

$$\partial_t\varphi(0) = \varphi_{0,1}' - \varphi_{0,2}' := \Delta\varphi_0 - (\gamma(\varphi_{0,1}) - \gamma(\varphi_{0,2})) - \frac{2}{\theta_c}(\pi(\varphi_{0,1}) - \pi(\varphi_{0,2})) \\ + \frac{1}{\theta_c^2}\big(v_0\,\pi(\varphi_{0,1}) + v_{0,2}(\pi(\varphi_{0,1}) - \pi(\varphi_{0,2}))\big) \tag{2.45}$$

and belongs to H, due to the assumptions on the initial data. Therefore, since we also have that $\varphi = \varphi_1 - \varphi_2$ is in $H^1(0,T;V)$, a comparison in (2.43) yields that $\partial_t \varphi \in H^1(0,T;V^*)$, and consequently we can differentiate (2.43) with respect to time and then test by $v = \partial_t \varphi \in L^2(0,T;V)$. A subsequent integration leads to

$$\frac{1}{2}\|\partial_t \varphi(t)\|^2 + \int_{Q_t} |\nabla(\partial_t \varphi)|^2 = \frac{1}{2}\|\partial_t \varphi(0)\|^2 + \int_{Q_t} \partial_t h \, \partial_t \varphi \,, \tag{2.46}$$

for all $t \in [0,T]$ (indeed, we also have $\partial_t \varphi \in C^0([0,T];H)$). Now, in view of (2.45) and (2.22), (2.26), it is straightforward to check that

$$\frac{1}{2}\|\partial_t \varphi(0)\|^2 \le C\bigl(\|\varphi_0\|_W^2 + \|v_0\|_H^2\bigr),$$

while, on account of the boundedness and Lipschitz continuity of γ and π in $[r_*, r^*]$ (cf. (2.27)), the Hölder and Young inequalities, and the continuous embedding $V \subset L^4(\Omega)$, we can infer from (2.44) that

$$\begin{aligned}
&\int_{Q_t} \partial_t h \, \partial_t \varphi \\
&\quad \le C \int_0^t \|\varphi\|_4 \|\partial_t \varphi_1\|_4 \|\partial_t \varphi\| \, \mathrm{d}s + C \int_{Q_t} |\partial_t \varphi|^2 + C \int_{Q_t} |\partial_{tt} w|^2 \\
&\qquad + C \int_0^t \|\partial_t w\|_4 \|\partial_t \varphi_1\|_4 \|\partial_t \varphi\| \, \mathrm{d}s + C \int_0^t \|\partial_{tt} w_2\| \|\varphi\|_4 \|\partial_t \varphi\|_4 \, \mathrm{d}s \\
&\quad \le C\bigl(\|\varphi\|_{L^\infty(0,T;V)} + \|\partial_t w\|_{L^\infty(0,T;V)}\bigr) \|\partial_t \varphi_1\|_{L^2(0,T;V)} \|\partial_t \varphi\|_{L^2(0,T;H)} + C\|\partial_t \varphi\|_{L^2(0,T;H)}^2 \\
&\qquad + C\|\partial_{tt} w\|_{L^2(0,T;H)}^2 \bigl(1 + \|\varphi\|_{L^\infty(0,T;V)}^2\bigr) + \frac{1}{2} \int_{Q_t} \bigl(|\partial_t \varphi(t)|^2 + |\nabla(\partial_t \varphi)|^2\bigr).
\end{aligned}$$

Then, by virtue of (2.41) and (2.42), combining the last two inequalities with (2.46) plainly leads to the estimate

$$\|\varphi\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V)} \le C\bigl(\|\varphi_0\|_W + \|w_0\|_V + \|v_0\|_V + \|u\|_{L^2(0,T;H)}\bigr). \tag{2.47}$$

Fourth estimate: Now, from (2.43), that reproduces (2.14), and the regularity of solutions we deduce that

$$-\Delta \varphi = h - \partial_t \varphi \quad \text{a.e. in } Q,$$

with the right-hand side that is under control in $L^\infty(0,T;H)$. Then, by elliptic regularity we easily derive the estimate

$$\|\varphi\|_{L^\infty(0,T;H^2(\Omega))} \le C\bigl(\|\varphi_0\|_W + \|w_0\|_V + \|v_0\|_V + \|u\|_{L^2(0,T;H)}\bigr). \tag{2.48}$$

Therefore, upon collecting (2.41), (2.42), (2.47), and (2.48), we obtain (2.39) and conclude the proof of Theorem 2.4. □

3 Optimal control theory

In this section, we aim at solving an optimal control problem whose governing state equation is given by the system (1.1)–(1.4) analyzed in the previous section. We seek optimal controls in the form of a distributed heat source, represented by u in (1.2), and an initial temperature, which corresponds to v_0 in (1.4). As we aim at covering the cases of polynomial and regular singular potentials, including, in particular, (1.6) and (1.7), we are from now on restricting ourselves to the framework of strong solutions (cf. Theorems 2.3 and 2.4).

The control problem under investigation reads as follows:

CP Minimize the cost functional

$$\begin{aligned}\mathcal{J}(u, v_0, \varphi, w) := & \frac{k_1}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)} + \frac{k_2}{2}\|\varphi(T) - \varphi_\Omega\|^2 + \frac{k_3}{2}\|w - w_Q\|^2_{L^2(Q)} \\ & + \frac{k_4}{2}\|w(T) - w_\Omega\|^2 + \frac{k_5}{2}\|\partial_t w - w'_Q\|^2_{L^2(Q)} + \frac{k_6}{2}\|\partial_t w(T) - w'_\Omega\|^2 \\ & + \frac{\nu_1}{2}\|u\|^2_{L^2(Q)} + \frac{\nu_2}{2}\|v_0\|^2_V \end{aligned} \tag{3.1}$$

subject to the state system (1.1)–(1.4) and to the control constraint

$$(u, v_0) \in \mathcal{U}_{\text{ad}},$$

where $\mathcal{U} := L^\infty(Q) \times (V \cap L^\infty(\Omega))$ and the set of *admissible controls* is

$$\begin{aligned}\mathcal{U}_{\text{ad}} := \big\{(u, v_0) \in \mathcal{U} : u_* \le u \le u^* \text{ a.e. in } Q, \\ v_* \le v_0 \le v^* \text{ a.e. in } \Omega, \quad \|v_0\|_V \le M\big\}.\end{aligned} \tag{3.2}$$

Above, the symbols $k_1, ..., k_6$ and ν_1, ν_2 denote some nonnegative constants which are not all zero, while $\varphi_Q, w_Q, w'_Q \in L^2(Q)$ and $\varphi_\Omega, w_\Omega, w'_\Omega \in L^2(\Omega)$ denote some prescribed targets. As for the set of admissible controls $\mathcal{U}_{\text{ad}}$, we assume that u_* and u^* are prescribed functions in $L^\infty(Q)$; moreover, v_* and v^* are given in $L^\infty(\Omega)$, and $M > 0$ is a fixed constant such that

$$\mathcal{U}_{\text{ad}} \text{ is a nonempty, closed and convex subset of the control space } \mathcal{U}.$$

Note that closedness and convexity can be easily verified from (3.2). Furthermore, we can select a value $R > 0$ big enough such that the open ball

$$\mathcal{U}_R := \big\{(u, v_0) \in \mathcal{U} : \|(u, v_0)\|_{\mathcal{U}} < R\big\} \text{ contains } \mathcal{U}_{\text{ad}}. \tag{3.3}$$

Let us remark that from a physical viewpoint it is more relevant investigating the evolution of $\partial_t w$ instead that of w, as the first one denotes the temperature of the system. This is the reason why the terms in (3.1) related to k_5 and k_6 are more significant than the ones associated with k_3 and k_4; nonetheless, we believe that those less physical terms are still worth considering from a mathematical viewpoint through the way in which they appear in the adjoint system (cf. system (3.35)–(3.38)). Also, note that the quantities v_* and v^* appearing in (3.2) represent threshold values for the initial temperature distribution v_0, while the condition $\|v_0\|_V \le M$ prevents extremely large variations for this distribution.

By virtue of Theorems 2.1–2.4, the *control-to-state operator*

$$\mathcal{S} : \mathcal{U}_R \subset \mathcal{U} \to \mathcal{Y}, \quad \mathcal{S} : (u, v_0) \mapsto (\varphi, w),$$

is well-defined as a mapping from $\mathcal{U}$ into the solution space $\mathcal{Y}$, with the latter being defined by (cf. Theorem 2.3)

$$\begin{aligned} \mathcal{Y} := & \big(W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\big) \\ & \times \big(H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W)\big). \end{aligned} \tag{3.4}$$

Moreover, we also set

$$\begin{aligned} \mathcal{X} := & \big(H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\big) \\ & \times \big(H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W)\big) \end{aligned}$$

and observe that $\mathcal{Y} \subset \mathcal{X}$ with continuous embedding. Then, the solution operator allows us to define the *reduced cost functional* as follows:

$$\mathcal{J}_{\rm red} : \mathcal{U} \to \mathbb{R}, \quad \mathcal{J}_{\rm red}(u, v_0) := \mathcal{J}\big(u, v_0, \mathcal{S}(u, v_0)\big). \tag{3.5}$$

Moreover, notice that Theorems 2.2 and 2.4 already ensure that the solution operator $\mathcal{S}$ is Lipschitz continuous in $\mathcal{U}_R$ when viewed as a mapping from $L^2(Q) \times V$ into the space $\mathcal{Y}$. Namely, for arbitrary controls $(u_i, v_{0,i}) \in \mathcal{U}_R$, $i = 1, 2$, the stability estimate (2.39) yields that

$$\|\mathcal{S}(u_1, v_{0,1}) - \mathcal{S}(u_2, v_{0,2})\|_{\mathcal{Y}} \le C\big(\|u_1 - u_2\|_{L^2(0,T;H)} + \|v_{0,1} - v_{0,2}\|_V\big).$$

For the control problem, some additional assumptions are in order:

C1 $\widehat{\gamma} \in C^3(r_-, r_+)$ and $\widehat{\pi} \in C^3(\mathbb{R})$.

C2 $k_1, k_2, k_3, k_4, k_5, k_6, \nu_1, \nu_2$ are nonnegative constants, not all zero.

C3 The target functions fulfill $\varphi_Q, w_Q, w'_Q \in L^2(Q)$, $\varphi_\Omega, w_\Omega \in H$, and $w'_\Omega \in V$.

C4 The functions u_*, u^* belong to $L^\infty(Q)$ with $u_* \le u^*$ a.e. in Q, and v_*, v^* are fixed in $L^\infty(\Omega)$ such that $v_* \le v^*$ a.e. in Ω. Moreover, $M > 0$, and the set $\mathcal{U}_{\rm ad}$ defined by (3.2) is nonempty.

The first result we address concerns the existence of an optimal strategy, that is of an optimal control pair.

Theorem 3.1. *Suppose that* **A1**–**A3***,* **B1**–**B3***,* **C2**–**C4** *hold in addition to the assumptions* (2.1)*,* (2.22)*,* (2.26) *on* φ_0, w_0*. Then the minimization problem* $\mathbf{CP}$ *admits a solution, that is, there exists at least one optimal pair* $(\overline{u}, \overline{v}_0) \in \mathcal{U}_{\rm ad}$ *such that*

$$\mathcal{J}_{\rm red}(\overline{u}, \overline{v}_0) \le \mathcal{J}_{\rm red}(u, v_0) \quad \forall (u, v_0) \in \mathcal{U}_{\rm ad}.$$

Proof of Theorem 3.1. The existence of a minimizer $(\overline{u}, \overline{v}_0)$ plainly follows from applying the direct method of the calculus of variations. In fact, we can pick a minimizing sequence $\{(u_n, v_{0,n})\}_n \subset \mathcal{U}_{\rm ad}$ for the functional $\mathcal{J}_{\rm red}$, and let, for every $n \in \mathbb{N}$, $(\varphi_n, w_n) = \mathcal{S}(u_n, v_{0,n})$ denote the corresponding strong solution to the system (1.1)–(1.4). Then, due to (3.2), by compactness it turns out that there exist a subsequence, still denoted by $\{(u_n, v_{0,n})\}_n$, and a pair $(\overline{u}, \overline{v}_0) \in \mathcal{U}_{\rm ad}$ such that

$$u_n \to \overline{u} \quad \text{weakly star in } L^\infty(Q),$$
$$v_{0,n} \to \overline{v}_0 \quad \text{weakly star in } V \cap L^\infty(\Omega),$$

as $n \nearrow \infty$. Correspondingly, in view of Theorem 2.3, and taking advantage of [25, Sect. 8, Cor. 4], it turns out that there is a pair $(\overline{\varphi}, \overline{w})$ satisfying

$$\varphi_n \to \overline{\varphi} \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \text{ and strongly in } C^0(\overline{Q}), \tag{3.6}$$
$$w_n \to \overline{w} \quad \text{weakly star in } H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W) \text{ and strongly in } C^1([0,T];H) \cap H^1(0,T;V), \tag{3.7}$$

in principle for another subsequence. Indeed, as for (3.6) note that $W \subset C^0(\overline{\Omega})$ with compact embedding. At this point, it is a standard matter to check that passage to the limit as $n \nearrow \infty$ in the system (1.1)–(1.4), written for $\{\varphi_n, w_n, u_n, v_{0,n}\}$, leads to the same system written for the limits $\{\overline{\varphi}, \overline{w}, \overline{u}, \overline{v}_0\}$. Then, taking into account Theorem 2.4 as well, we infer that $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$, (3.6) and (3.7) hold for the selected subsequence, and, by the lower semicontinuity of norms,

$$\mathcal{J}_{\rm red}(\overline{u}, \overline{v}_0) \leq \liminf_{n\to\infty} \mathcal{J}_{\rm red}(u_n, v_{0,n}).$$

Hence, $(\overline{u}, \overline{v}_0)$ is a global minimizer for $\mathcal{J}_{\rm red}$, as $\mathcal{J}_{\rm red}(u_n, v_{0,n})$ converges to the infimum of $\mathcal{J}_{\rm red}$. The assertion is thus proved. □

We are now interested in finding optimality conditions that every minimizer has to satisfy. To this end, recall the reduced form (3.5) and the fact that $\mathcal{U}_{\rm ad}$ is a nonempty, closed, and convex subset of the control space $\mathcal{U}$. Standard results of convex analysis (see, e.g., [26]) entail the first-order necessary condition for $\mathcal{J}_{\rm red}$ at every minimizer $(\overline{u}, \overline{v}_0)$ in terms of a suitable variational inequality of the form

$$D\mathcal{J}_{\rm red}(\overline{u}, \overline{v}_0)(u - \overline{u}, v_0 - \overline{v}_0) \geq 0 \quad \forall (u, v_0) \in \mathcal{U}_{\rm ad}, \tag{3.8}$$

where $D\mathcal{J}_{\rm red}$ stands for the derivative of the reduced cost functional in a proper mathematical sense (cf. Theorem 3.3). The quadratic structure of $\mathcal{J}$ directly yields its Fréchet differentiability, so that it suffices to show the differentiability of the solution operator $\mathcal{S}$ in order to derive the first-order necessary conditions from (3.8) by means of the chain rule.

For this purpose, we fix a control pair $(\overline{u}, \overline{v}_0) \in \mathcal{U}_R$ with corresponding state $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$. We introduce the linearized system to (1.1)–(1.4), which reads, for every $(h, h^0) \in L^2(Q) \times V$,

$$\partial_t \xi - \Delta \xi + \gamma'(\overline{\varphi})\xi + \tfrac{2}{\theta_c}\pi'(\overline{\varphi})\xi - \tfrac{1}{\theta_c^2}\partial_t \eta\, \pi(\overline{\varphi}) - \tfrac{1}{\theta_c^2}\partial_t \overline{w}\, \pi'(\overline{\varphi})\xi = 0 \quad \text{a.e. in } Q, \tag{3.9}$$
$$\partial_{tt}\eta - \alpha\Delta(\partial_t \eta) - \beta\Delta\eta + \pi'(\overline{\varphi})\xi\partial_t\overline{\varphi} + \pi(\overline{\varphi})\partial_t\xi = h \quad \text{a.e. in } Q, \tag{3.10}$$
$$\partial_{\mathbf{n}}\xi = \partial_{\mathbf{n}}(\alpha\partial_t\eta + \beta\eta) = 0 \quad \text{a.e. on } \Sigma, \tag{3.11}$$
$$\xi(0) = 0, \quad \eta(0) = 0, \quad \partial_t\eta(0) = h^0 \quad \text{a.e. in } \Omega. \tag{3.12}$$

Its well-posedness is stated in the following result.

Theorem 3.2. *Assume that* **A1**–**A3** *and* **B1**–**B3** *are fulfilled in addition to the assumptions* (2.1), (2.22), (2.26) *on* φ_0, w_0. *Let* $(\overline{u}, \overline{v}_0) \in \mathcal{U}_R$ *be given and* $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$. *Then the linearized system* (3.9)–(3.12) *has for every* $(h, h^0) \in L^2(Q) \times V$ *a unique solution* $(\xi, \eta) \in \mathcal{X}$.

Proof of Theorem 3.2. Since the problem is linear, we can prove existence and, at the same time, uniqueness, by performing suitable estimates on the solution $(\xi,\eta)\in\mathcal{X}$ in terms of the data $(h,h^0)\in L^2(Q)\times V$, with linear dependence. As in the case of the state problem, we here avoid to implement a Faedo–Galerkin scheme and argue directly on the linearized problem.

First estimate: We first add ξ to both sides of (3.9) and then test (3.9) by $\theta_c^2\partial_t\xi$ and (3.10) by $\partial_t\eta$. Next, we sum up the resulting equalities and integrate by parts to infer that a cancellation occurs, obtaining the identity

$$\begin{aligned}
&\theta_c^2\int_{Q_t}|\partial_t\xi|^2+\frac{\theta_c^2}{2}\|\xi(t)\|_V^2+\frac12\|\partial_t\eta(t)\|^2+\alpha\int_{Q_t}|\nabla(\partial_t\eta)|^2+\frac{\beta}{2}\|\nabla\eta(t)\|^2\\
&=\frac12\|h^0\|^2+\int_{Q_t}\big(\theta_c^2-\theta_c^2\gamma'(\overline{\varphi})-2\theta_c\pi'(\overline{\varphi})\big)\xi\,\partial_t\xi\\
&\quad+\int_{Q_t}\partial_t\overline{w}\,\pi'(\overline{\varphi})\xi\,\partial_t\xi-\int_{Q_t}\pi'(\overline{\varphi})\partial_t\overline{\varphi}\,\xi\,\partial_t\eta+\int_{Q_t}h\,\partial_t\eta=:\sum_{i=1}^{5}\mathbb{I}_i.
\end{aligned}$$

Since $(\overline{\varphi},\overline{w})$ is a strong solution to (1.1)–(1.4), we deduce from (2.27)–(2.28) that $\gamma'(\overline{\varphi})$, $\pi'(\overline{\varphi})$, $\partial_t\overline{w}\in L^\infty(Q)$ and $\partial_t\overline{\varphi}\in L^\infty(0,T;H)\cap L^2(0,T;V)$. We thus have that

$$\mathbb{I}_2+\mathbb{I}_3\le\frac{\theta_c^2}{2}\int_{Q_t}|\partial_t\xi|^2+C\int_{Q_t}|\xi|^2\,,$$

and, with the help of the continuous embedding $V\subset L^4(\Omega)$,

$$\mathbb{I}_4\le C\int_0^t\|\partial_t\overline{\varphi}\|_4\|\xi\|_4\|\partial_t\eta\|\,\mathrm{d}s\le C\int_0^t\|\partial_t\overline{\varphi}\|_V^2\|\xi\|_V^2\,\mathrm{d}s+C\int_{Q_t}|\partial_t\eta|^2\,,$$

where the function $t\mapsto\|\partial_t\overline{\varphi}(t)\|_V^2$ belongs to $L^1(0,T)$. As for the last term, we simply employ Young's inequality and obtain

$$\mathbb{I}_5\le C\int_{Q_t}(|h|^2+|\partial_t\eta|^2).$$

We collect the above estimates and apply Gronwall's lemma. Then, observing that (cf. (3.12)) $\|\eta(t)\|_V^2\le T\int_{Q_t}\|\partial_t\eta\|_V^2$ for $t\in[0,T]$, by the Hölder inequality, we can conclude that

$$\|\xi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}+\|\eta\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)}\le C(\|h\|_{L^2(Q)}+\|h^0\|). \tag{3.13}$$

Second estimate: Next, (3.13) (in particular, the boundedness of $\|\partial_t\xi\|_{L^2(0,T;H)}$) and a comparison of terms in (3.9) easily produce that

$$\|\Delta\xi\|_{L^2(0,T;H)}\le C(\|h\|_{L^2(Q)}+\|h^0\|),$$

so that elliptic regularity entails that

$$\|\xi\|_{L^2(0,T;W)}\le C(\|h\|_{L^2(Q)}+\|h^0\|). \tag{3.14}$$

Third estimate: As done in the third estimate of Theorem 2.3, we add to both sides of (3.10) the term $\frac{\beta}{\alpha}\partial_t\eta$ and rewrite it as a parabolic equation in terms of the new variable $y:=\alpha\partial_t\eta+\beta\eta$. Precisely, we deduce that

$$\begin{cases} \frac{1}{\alpha}\partial_t y - \Delta y = g & \text{in } Q, \\ \partial_{\mathbf{n}} y = 0 & \text{on } \Sigma, \\ y(0) = \alpha h^0 & \text{in } \Omega, \end{cases}$$

with $g := -\pi'(\overline{\varphi})\xi\partial_t\overline{\varphi} - \pi(\overline{\varphi})\partial_t\xi + h + \frac{\beta}{\alpha}\partial_t\eta$, here. Due to (3.13), we have that the norm of g in $L^2(0,T;H)$ is under control. Besides, we are assuming that $h^0 \in V$, so that parabolic regularity theory entails that

$$\|\alpha\partial_t\eta + \beta\eta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C(\|h\|_{L^2(Q)} + \|h^0\|_V). \tag{3.15}$$

Now, arguing as in (2.33), it follows that (3.15) implies the same estimate for η and $\partial_t\eta$, whence

$$\|\eta\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \le C(\|h\|_{L^2(Q)} + \|h^0\|_V). \tag{3.16}$$

Then, by collecting (3.13), (3.14), and (3.16), we end the proof. □

After proving Theorem 3.2, we are in a position to show that the control-to-state operator $\mathcal{S}$ is Fréchet differentiable as a mapping between suitable Banach spaces. Here is the related result.

Theorem 3.3. *Suppose that the conditions* **A1**–**A3**, **B1**–**B3**, *and* **C1** *are fulfilled. Moreover, let the initial data* φ_0 *and* w_0 *satisfy* (2.1), (2.22), (2.26), *and let* $(\overline{u},\overline{v}_0) \in \mathcal{U}_R$ *with* $(\overline{\varphi},\overline{w}) = \mathcal{S}(\overline{u},\overline{v}_0)$. *Then the solution operator* $\mathcal{S}$ *is Fréchet differentiable at* $(\overline{u},\overline{v}_0)$ *as a mapping from* $\mathcal{U}$ *into* $\mathcal{X}$. *Moreover, for every* $\mathbf{h} := (h,h^0) \in \mathcal{U}$, *the Fréchet derivative* $D\mathcal{S}(\overline{u},\overline{v}_0) \in \mathcal{L}(\mathcal{U},\mathcal{X})$ *is given by the identity* $D\mathcal{S}(\overline{u},\overline{v}_0)(\mathbf{h}) = (\xi,\eta)$, *where* (ξ,η) *is the unique solution to the linearized system* (3.9)–(3.9) *associated with* $\mathbf{h}$.

Proof of Theorem 3.3. Since $\mathcal{U}_R$ is open, provided that we consider small ε-perturbations in the $\mathcal{U}$-norm, we surely have that $(\overline{u}+h,\overline{v}_0+h^0) \in \mathcal{U}_R$ as well, that is, there exists some $\varepsilon > 0$ such that

$$(\overline{u}+h,\overline{v}_0+h^0) \in \mathcal{U}_R \quad \forall\, \mathbf{h} \in \mathcal{U} \quad \text{such that} \quad \|\mathbf{h}\|_{\mathcal{U}} \le \varepsilon.$$

For the rest of the proof, we agree that this condition is met by all of the appearing increments $\mathbf{h}$.

We claim that $D\mathcal{S}(\overline{u},\overline{v}_0)(\mathbf{h}) = (\xi,\eta)$, with (ξ,η) being the unique solution to the linearized system (3.9)–(3.12). We prove this claim directly by showing that

$$\mathcal{S}(\overline{u}+h,\overline{v}_0+h^0) = \mathcal{S}(\overline{u},\overline{v}_0) + (\xi,\eta) + o(\|\mathbf{h}\|_{\mathcal{U}}) \quad \text{in } \mathcal{X} \quad \text{as } \|\mathbf{h}\|_{\mathcal{U}} \to 0. \tag{3.17}$$

Upon setting

$$(\overline{\varphi}^{\mathbf{h}},\overline{w}^{\mathbf{h}}) = \mathcal{S}(\overline{u}+h,\overline{v}_0+h^0), \quad \psi := \overline{\varphi}^{\mathbf{h}} - \overline{\varphi} - \xi, \quad z := \overline{w}^{\mathbf{h}} - \overline{w} - \eta, \tag{3.18}$$

the condition (3.17) becomes

$$\|(\psi,z)\|_{\mathcal{X}} = o(\|\mathbf{h}\|_{\mathcal{U}}) \quad \text{as } \|\mathbf{h}\|_{\mathcal{U}} \to 0, \tag{3.19}$$

which is the identity we are going to prove. Accounting for the notation in (3.18), we infer that the variables ψ and z solve the initial-boundary value problem

$$\partial_t\psi - \Delta\psi + \Lambda_1 - \tfrac{1}{\theta_c^2}\pi(\overline{\varphi})\partial_t z = 0 \qquad \text{in } Q, \tag{3.20}$$

$$\partial_{tt} z - \alpha\Delta(\partial_t z) - \beta\Delta z + \Lambda_2 + \pi(\overline{\varphi})\partial_t\psi = 0 \qquad \text{in } Q, \tag{3.21}$$

$$\partial_{\mathbf{n}}\psi = \partial_{\mathbf{n}}(\alpha\partial_t z + \beta z) = 0 \qquad \text{on } \Sigma, \tag{3.22}$$

$$\psi(0) = 0, \quad z(0) = 0, \quad \partial_t z(0) = 0 \qquad \text{in } \Omega, \tag{3.23}$$

where the terms Λ_1 and Λ_2 are defined by

$$\begin{aligned}
\Lambda_1 &= [\gamma(\overline{\varphi}^{\mathbf{h}}) - \gamma(\overline{\varphi}) - \gamma'(\overline{\varphi})\xi] + \tfrac{2}{\theta_c}[\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}) - \pi'(\overline{\varphi})\xi] \\
&\quad - \tfrac{1}{\theta_c^2}\Big((\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}))(\partial_t\overline{w}^{\mathbf{h}} - \partial_t\overline{w}) + \partial_t\overline{w}\,[\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}) - \pi'(\overline{\varphi})\xi]\Big), \\
\Lambda_2 &= (\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}))(\partial_t\overline{\varphi}^{\mathbf{h}} - \partial_t\overline{\varphi}) + \partial_t\overline{\varphi}\,[\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}) - \pi'(\overline{\varphi})\xi].
\end{aligned}$$

Before moving on, let us recall that the continuous dependence result in Theorem 2.4, applied to the solutions $(\overline{\varphi}^{\mathbf{h}}, \overline{w}^{\mathbf{h}})$ and $(\overline{\varphi}, \overline{w})$, yields that

$$\begin{aligned}
&\|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\overline{w}^{\mathbf{h}} - \overline{w}\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} \\
&\quad \le K_4\big(\|h\|_{L^2(Q)} + \|h^0\|_V\big).
\end{aligned} \tag{3.24}$$

Besides, $(\overline{\varphi}^{\mathbf{h}}, \overline{w}^{\mathbf{h}})$ and $(\overline{\varphi}, \overline{w})$, as strong solution to (1.1)–(1.4), satisfy (2.27) and (2.28). Moreover, we recall Taylor's formula with integral remainder: let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function with Lipschitz continuous derivative g'. Then, for $\overline{x} \in \mathbb{R}$ it holds that

$$g(x) = g(\overline{x}) + g'(\overline{x})(x - \overline{x}) + (x - \overline{x})^2 \int_0^1 g''(\overline{x} + s(x - \overline{x}))(1 - s)\,\mathrm{d}s, \quad x \in \mathbb{R}. \tag{3.25}$$

An application of (3.25) to π and γ yields that

$$\gamma(\overline{\varphi}^{\mathbf{h}}) - \gamma(\overline{\varphi}) - \gamma'(\overline{\varphi})\xi = \gamma'(\overline{\varphi})\psi + R_\gamma^{\mathbf{h}}\,(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2, \tag{3.26}$$

$$\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi}) - \pi'(\overline{\varphi})\xi = \pi'(\overline{\varphi})\psi + R_\pi^{\mathbf{h}}\,(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2, \tag{3.27}$$

with the remainders

$$R_\gamma^{\mathbf{h}} := \int_0^1 \gamma''(\overline{\varphi} + s(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}))(1 - s)\,\mathrm{d}s, \quad R_\pi^{\mathbf{h}} := \int_0^1 \pi''(\overline{\varphi} + s(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}))(1 - s)\,\mathrm{d}s.$$

Due to assumptions **C1**, it directly follows that

$$\|R_\gamma^{\mathbf{h}}\|_{L^\infty(Q)} + \|R_\pi^{\mathbf{h}}\|_{L^\infty(Q)} \le C. \tag{3.28}$$

We now prove some estimates that will imply (3.19).

First estimate: Add ψ to both sides of (3.20) and test it by $\theta_c^2\partial_t\psi$; then, test (3.21) by $\partial_t z$ and sum up the resulting equalities. After integration by parts, we obtain that a cancellation occurs and that

$$\begin{aligned}
&\theta_c^2\int_{Q_t}|\partial_t\psi|^2 + \frac{\theta_c^2}{2}\|\psi(t)\|_V^2 + \frac{1}{2}\|\partial_t z(t)\|^2 + \alpha\int_{Q_t}|\nabla(\partial_t z)|^2 + \frac{\beta}{2}\|\nabla z(t)\|^2 \\
&\quad = \theta_c^2\int_{Q_t}(\psi - \Lambda_1)\partial_t\psi - \int_{Q_t}\Lambda_2\,\partial_t z.
\end{aligned}$$

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The first term on the right-hand side can be controlled by employing Taylor's formulae (3.26)–(3.27), the uniform bounds (2.27)–(2.28) for $(\overline{\varphi}, \overline{w})$, the Young and Hölder inequalities, the stability estimates (3.24), (3.28), and the continuous embedding $V \subset L^4(\Omega)$. We infer that

$$
\begin{aligned}
&\theta_c^2 \int_{Q_t} (\psi - \Lambda_1)\partial_t \psi \\
&\le \theta_c^2 \int_{Q_t} \big|\psi - \gamma'(\overline{\varphi})\psi - R^{\mathbf{h}}_{\gamma}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2\big| |\partial_t\psi| + 2\theta_c \int_{Q_t} \big|\pi'(\overline{\varphi})\psi + R^{\mathbf{h}}_{\pi}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2\big| |\partial_t\psi| \\
&\quad + \int_{Q_t} |\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi})| |\partial_t \overline{w}^{\mathbf{h}} - \partial_t \overline{w}| |\partial_t\psi| + \int_{Q_t} |\partial_t \overline{w}| \big|\pi'(\overline{\varphi})\psi + R^{\mathbf{h}}_{\pi}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2\big| |\partial_t\psi| \\
&\le \delta \int_{Q_t} |\partial_t\psi|^2 + C_\delta \int_{Q_t} |\psi|^2 + C_\delta \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_4^4 \,\mathrm{d}s \\
&\quad + C_\delta \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_4^2 \|\partial_t \overline{w}^{\mathbf{h}} - \partial_t \overline{w}\|_4^2 \,\mathrm{d}s \\
&\le \delta \int_{Q_t} |\partial_t\psi|^2 + C_\delta \int_{Q_t} |\psi|^2 + C_\delta \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_V^2 \big(\|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_V^2 + \|\partial_t \overline{w}^{\mathbf{h}} - \partial_t \overline{w}\|_V^2\big) \,\mathrm{d}s \\
&\le \delta \int_{Q_t} |\partial_t\psi|^2 + C_\delta \int_{Q_t} |\psi|^2 + C_\delta\big(\|h\|^4_{L^2(Q)} + \|h^0\|_V^4\big),
\end{aligned}
\tag{3.29}
$$

for a positive δ yet to be chosen. Similar arguments allow us to bound the second term on the right-hand side, concluding that

$$
\begin{aligned}
&-\int_{Q_t} \Lambda_2\, \partial_t z \\
&\le \int_{Q_t} |\pi(\overline{\varphi}^{\mathbf{h}}) - \pi(\overline{\varphi})| |\partial_t \overline{\varphi}^{\mathbf{h}} - \partial_t \overline{\varphi}| |\partial_t z| + \int_{Q_t} |\partial_t \overline{\varphi}| \big|\pi'(\overline{\varphi})\psi + R^{\mathbf{h}}_{\pi}(\overline{\varphi}^{\mathbf{h}} - \overline{\varphi})^2\big| |\partial_t z| \\
&\le C \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_4 \|\partial_t \overline{\varphi}^{\mathbf{h}} - \partial_t \overline{\varphi}\|_4 \|\partial_t z\| \,\mathrm{d}s + C \int_0^t \|\partial_t \overline{\varphi}\|_6 \|\partial_t z\| \big(\|\psi\|_3 + \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_6^2\big) \,\mathrm{d}s \\
&\le C \int_0^t \big(1 + \|\partial_t \overline{\varphi}\|_V^2\big) \|\partial_t z\|^2 \,\mathrm{d}s + C \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|^2_{L^\infty(0,T;V)} \int_0^t \|\partial_t \overline{\varphi}^{\mathbf{h}} - \partial_t \overline{\varphi}\|_V^2 \,\mathrm{d}s \\
&\quad + C \int_0^t \|\psi\|_V^2 \,\mathrm{d}s + C \int_0^t \|\overline{\varphi}^{\mathbf{h}} - \overline{\varphi}\|_V^4 \,\mathrm{d}s \\
&\le C \int_0^t \big(1 + \|\partial_t \overline{\varphi}\|_V^2\big) \|\partial_t z\|^2 \,\mathrm{d}s + C \int_0^t \|\psi\|_V^2 \,\mathrm{d}s + C\big(\|h\|^4_{L^2(Q)} + \|h^0\|_V^4\big),
\end{aligned}
\tag{3.30}
$$

where we notice that the function $t \mapsto (1 + \|\partial_t \overline{\varphi}\|_V^2)$ is in $L^1(0,T)$, due to (2.28). Upon choosing $0 < \delta < \theta_c^2$, Gronwall's lemma yields that

$$
\|\psi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|z\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C\big(\|h\|^2_{L^2(Q)} + \|h^0\|_V^2\big).
\tag{3.31}
$$

Second estimate: A closer inspection of the estimate in (3.29), along with the bound (3.31), shows that $\|\Lambda_1\|_{L^2(0,T;H)}$ is bounded as well by an analogous term. Then, a comparison argument in (3.20) directly leads to

$$
\|\Delta\psi\|_{L^2(0,T;H)} \le C\big(\|h\|^2_{L^2(Q)} + \|h^0\|_V^2\big),
$$

so that (3.31) and elliptic regularity yield that

$$\|\psi\|_{L^2(0,T;W)} \le C\big(\|h\|^2_{L^2(Q)} + \|h^0\|^2_V\big). \tag{3.32}$$

Third estimate: Repeating the argument employed in the third estimate of the proof of Theorem 2.3 (cf., in particular, (2.31)), we can in view of (3.21)–(3.23) state a parabolic system in the variable $y = \alpha\partial_t z + \beta z$, with source term $\frac{\beta}{\alpha}\partial_t z - \Lambda_2 - \pi(\overline{\varphi})\partial_t\psi$ and null initial value. With the help of (3.31), it is not difficult to verify that

$$\big\|\tfrac{\beta}{\alpha}\partial_t z - \Lambda_2 - \pi(\overline{\varphi})\partial_t\psi\big\|_{L^2(0,T;H)} \le C\big(\|h\|^2_{L^2(Q)} + \|h^0\|^2_V\big).$$

Therefore, using parabolic regularity and the fact that

$$z(t) = \frac{1}{\alpha}\int_0^t e^{-\beta(t-s)/\alpha} y(s)\mathrm{d}s, \quad t \in [0,T],$$

we can deduce that

$$\|z\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \le \big(\|h\|^2_{L^2(Q)} + \|h^0\|^2_V\big). \tag{3.33}$$

A combination of the estimates (3.31)–(3.33) concludes the proof, since the continuous embedding of $\mathcal{U} \subset L^2(Q) \times V$, namely,

$$\|h\|_{L^2(0,T;H)} + \|h^0\|_V \le C\|\mathbf{h}\|_{\mathcal{U}} \quad \text{for every } \mathbf{h} = (h, h^0) \in \mathcal{U},$$

ensures that (3.19) is fulfilled. □

Remark 3.4. Let us point out that the Fréchet differentiability of $\mathcal{S}$ at the fixed control pair $(\overline{u}, \overline{v}_0)$ is defined from $\mathcal{U}_R$ into $\mathcal{X}$ and not from an open bounded subset of $L^2(Q) \times V$, as it may appear (incorrectly) from the estimates above. The reason is that for controls $(\overline{u}, \overline{v}_0)$ just in $L^2(Q) \times V$ we cannot guarantee the existence of a strong solution (cf. Theorem 2.3). Nevertheless, the above estimates show that, due to the density of the embedding of $\mathcal{U}$ in $L^2(Q) \times V$, the Fréchet derivative $D\mathcal{S}(\overline{u}, \overline{v}_0) \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ can be continuously extended to a linear and continuous operator from $L^2(Q) \times V$ into $\mathcal{X}$. In particular, denoting that extension with the same symbol $D\mathcal{S}(\overline{u}, \overline{v}_0)$, the identity $D\mathcal{S}(\overline{u}, \overline{v}_0)(\mathbf{h}) = (\xi, \eta)$ continues to hold also for $\mathbf{h} = (h, h^0) \in L^2(Q) \times V$.

It is now a standard matter to derive the first-order optimality conditions for $\mathbf{CP}$ by combining (3.8), Theorem 3.3, and the chain rule.

Theorem 3.5. *Suppose that* **A1**–**A3**, **B1**–**B3**, **C1**–**C4** *are satisfied. Moreover, let the initial data* φ_0 *and* w_0 *satisfy* (2.1), (2.22), (2.26), *and let* $(\overline{u}, \overline{v}_0)$ *be an optimal control with* $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$. *Then the optimal pair* $(\overline{u}, \overline{v}_0)$ *necessarily fulfills the variational inequality*

$$\begin{aligned} &k_1 \int_Q (\overline{\varphi} - \varphi_Q)\xi + k_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\xi(T) + k_3 \int_Q (\overline{w} - w_Q)\eta + k_4 \int_\Omega (\overline{w}(T) - w_\Omega)\eta(T) \\ &\quad + k_5 \int_Q (\partial_t\overline{w} - w_Q')\partial_t\eta + k_6 \int_\Omega (\partial_t\overline{w}(T) - w_\Omega')\partial_t\eta(T) + \nu_1 \int_Q \overline{u}(u - \overline{u}) \\ &\quad + \nu_2 \int_\Omega \Big(\overline{v}_0(v_0 - \overline{v}_0) + \nabla\overline{v}_0 \cdot \nabla(v_0 - \overline{v}_0)\Big) \ge 0 \quad \forall (u, v_0) \in \mathcal{U}_{\mathrm{ad}}, \end{aligned} \tag{3.34}$$

where (ξ, η) *denotes the unique solution of the linearized system* (3.9)–(3.12) *associated with the choice* $\mathbf{h} = (u - \overline{u}, v_0 - \overline{v}_0)$.

We now want to rewrite the optimality conditions in terms of the solution to the adjoint problem, in order to simplify the above variational inequality. The backward-in-time system characterizing the adjoint problem is given, in a strong form, by

$$
\begin{aligned}
& -\partial_t p - \pi(\overline{\varphi})\,\partial_t q - \Delta p + \gamma'(\overline{\varphi})\,p + \tfrac{2}{\theta_c}\pi'(\overline{\varphi})\,p - \tfrac{1}{\theta_c^2}\partial_t\overline{w}\,\pi'(\overline{\varphi})\,p \\
& \qquad = k_1(\overline{\varphi} - \varphi_Q) \qquad \text{in } Q, \qquad (3.35)\\
& -\partial_t q - \alpha\Delta q + \beta\Delta(1 \circledast q) - \tfrac{1}{\theta_c^2}\pi(\overline{\varphi})\,p \\
& \qquad = k_3(1 \circledast (\overline{w} - w_Q)) + k_5(\partial_t\overline{w} - w_Q') + k_4(\overline{w}(T) - w_\Omega) \qquad \text{in } Q, \qquad (3.36)\\
& \partial_{\mathbf{n}} p = \partial_{\mathbf{n}} q = 0 \qquad \text{on } \Sigma, \qquad (3.37)\\
& p(T) = k_2(\overline{\varphi}(T) - \varphi_\Omega) - k_6\pi(\overline{\varphi}(T))(\partial_t\overline{w}(T) - w_\Omega'),\\
& \quad q(T) = k_6(\partial_t\overline{w}(T) - w_\Omega') \qquad \text{in } \Omega, \qquad (3.38)
\end{aligned}
$$

where the product $\circledast$ is defined in (1.22). For convenience, let us denote by f_q the source term in (3.36), that is,

$$
f_q := k_3(1 \circledast (\overline{w} - w_Q)) + k_5(\partial_t\overline{w} - w_Q') + k_4(\overline{w}(T) - w_\Omega)
$$

and notice that the last part $k_4(\overline{w}(T) - w_\Omega)$ is constant in time. Moreover, due to **C3** and to the fact that $\overline{w}$ is a strong solution in the sense of Theorem 2.3, f_q satisfies

$$
\|f_q\|_{L^2(0,T;H)} \le C(\|\overline{w}\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + 1) \le C, \qquad (3.39)
$$

where the above constant certainly depends on T.

The above system reveals why we did also include the possibly redundant objective terms associated to k_3 and k_4 in (3.1). Indeed, the way they appear in the adjoint system above is nonstandard. Another remark concerns the fact that only first-order time derivatives appear in (3.35)–(3.36), while the corresponding state system, as well as the linearized one, contains an equation with a second-order time derivative as well. However, note that if (3.36) is interpreted as an equation in the time-integrated variable $1 \circledast q$, then it turns out that $-\partial_t q = \partial_{tt}(1 \circledast q)$, and the system (3.35)–(3.38) looks more natural.

The well-posedness result, as well as the notion of solution to the above system, is specified in the following theorem.

Theorem 3.6. *Assume that* **A1**–**A3**, **B1**–**B3**, **C1**–**C3** *hold true. Let the initial data φ_0 and w_0 satisfy* (2.1), (2.22), (2.26), *and let $(\overline{u}, \overline{v}_0) \in \mathcal{U}_{\rm ad}$ be an optimal control for $\mathbf{CP}$ with the associated state $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$. Then the adjoint system* (3.35)–(3.38) *admits a unique weak solution (p, q) with*

$$
p \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \qquad (3.40)
$$
$$
q \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \qquad (3.41)
$$

that satisfies the variational equalities

$$
\begin{aligned}
& -\langle \partial_t p, v\rangle_V - \int_\Omega \pi(\overline{\varphi})\,\partial_t q\,v + \int_\Omega \nabla p\cdot\nabla v + \int_Q \gamma'(\overline{\varphi})\,p\,v \\
& \quad + \frac{2}{\theta_c}\int_Q \pi'(\overline{\varphi})\,p\,v - \frac{1}{\theta_c^2}\int_\Omega \partial_t\overline{w}\,\pi'(\overline{\varphi})\,p\,v = \int_\Omega k_1(\overline{\varphi} - \varphi_Q)\,v, \qquad (3.42)
\end{aligned}
$$

$$
-\int_\Omega \partial_t q\,v + \alpha\int_\Omega \nabla q\cdot\nabla v - \beta\int_\Omega \nabla(1 \circledast q)\cdot\nabla v - \frac{1}{\theta_c^2}\int_\Omega \pi(\overline{\varphi})p\,v = \int_\Omega f_q\zeta, \qquad (3.43)
$$

for every $v \in V$, almost everywhere in $(0,T)$, and the final conditions

$$p(T) = k_2(\overline{\varphi}(T) - \varphi_\Omega) - k_6\pi(\overline{\varphi}(T))(\partial_t\overline{w}(T) - w'_\Omega) \qquad \text{a.e. in } \Omega, \tag{3.44}$$

$$q(T) = k_6(\partial_t\overline{w}(T) - w'_\Omega) \qquad \text{a.e. in } \Omega. \tag{3.45}$$

Proof of Theorem 3.6. We again proceed formally by pointing out the estimates that will imply the existence of a solution. These computations can however easily be reproduced in a rigorous framework. Moreover, before moving on, let us set $Q_t^T := \Omega \times (t,T)$.

First estimate: We take $v = p$ in (3.42), $v = -\theta_c^2\partial_t q$ in (3.36), add the resulting equalities and note that two terms cancel out. Then, integration over (t,T) and by parts yields

$$\begin{aligned} &\frac{1}{2}\|p(t)\|^2 + \int_{Q_t^T}|\nabla p|^2 + \int_{Q_t^T}\gamma'(\overline{\varphi})|p|^2 + \theta_c^2\int_{Q_t^T}|\partial_t q|^2 + \frac{\alpha\theta_c^2}{2}\|\nabla q(t)\|^2 \\ &\quad = \frac{1}{2}\|p(T)\|^2 + \frac{\alpha\theta_c^2}{2}\|\nabla q(T)\|^2 + k_1\int_{Q_t^T}(\overline{\varphi}-\varphi_Q)p - \frac{2}{\theta_c}\int_{Q_t^T}\pi'(\overline{\varphi})\,p^2 \\ &\qquad + \frac{1}{\theta_c^2}\int_{Q_t^T}\partial_t\overline{w}\,\pi'(\overline{\varphi})\,p^2 + \beta\theta_c^2\int_{Q_t^T}\nabla(1\circledast q)\cdot\nabla(\partial_t q) - \theta_c^2\int_{Q_t^T}f_q\,\partial_t q. \end{aligned} \tag{3.46}$$

Notice that the third term on the left-hand side is nonnegative due to the monotonicity of γ. As for the sixth term on the right-hand side, we note that $(1\circledast q)(T) = 0$ in Ω, thus the Young and Hölder inequalities allow us to deduce that

$$\begin{aligned} &\beta\theta_c^2\int_{Q_t^T}\nabla(1\circledast q)\cdot\nabla(\partial_t q) \\ &\quad = -\beta\theta_c^2\int_\Omega\nabla(1\circledast q)(t)\cdot\nabla q(t) + \beta\theta_c^2\int_{Q_t^T}|\nabla q|^2 \\ &\quad \le \frac{\alpha\theta_c^2}{4}\|\nabla q(t)\|^2 + C\int_{Q_t^T}|\nabla q|^2. \end{aligned}$$

Concerning the third and last terms on the right-hand side, we recall that $(\overline{\varphi},\overline{w})$ satisfies (2.27)–(2.28) and that **C3** and (3.39) hold as well. Hence, it follows from Young's inequality that

$$k_1\int_{Q_t^T}(\overline{\varphi}-\varphi_Q)p - \theta_c^2\int_{Q_t^T}f_q\,\partial_t q \le \frac{\theta_c^2}{2}\int_{Q_t^T}|\partial_t q|^2 + C\int_{Q_t^T}\big(|p|^2+1\big).$$

Still on the right-hand side, the first terms involving the terminal conditions are bounded by a constant due to (3.38) and **C3**, while for the remaining terms we owe to the fact that $\overline{\varphi}, \partial_t\overline{w} \in L^\infty(Q)$ (cf. Theorem 2.3). Hence, with the help of **C1**, we have that

$$-\frac{2}{\theta_c}\int_{Q_t^T}\pi'(\overline{\varphi})\,p^2 + \frac{1}{\theta_c^2}\int_{Q_t^T}\partial_t\overline{w}\,\pi'(\overline{\varphi})\,p^2 \le C\int_{Q_t^T}|p|^2.$$

Upon collecting the above computations, we can apply the Gronwall lemma and infer that

$$\|p\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|q\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C.$$

Second estimate: Next, we proceed with comparison in equation (3.36) to deduce that

$$\big\|\Delta\big(\alpha q + \beta(1\circledast q)\big)\big\|_{L^2(0,T;H)} \le C.$$

Then, setting $g = \alpha q + \beta(1 \circledast q)$, the elliptic regularity theory entails that $\|g\|_{L^2(0,T;W)} \leq C$. Hence, solving the equation $\alpha q + \beta(1 \circledast q) = g$ with respect to $1 \circledast q$ (which is equal to 0 at the time T), we eventually obtain that

$$\|1 \circledast q\|_{L^2(0,T;W)} + \|q\|_{L^2(0,T;W)} \leq C.$$

Third estimate: Finally, we take an arbitrary test function $v \in L^2(0,T;V)$ in (3.42) and compare the terms. Using the above estimates, it is then a standard matter to realize that

$$\|\partial_t p\|_{L^2(0,T;V^*)} \leq C.$$

This concludes the proof. In fact, let us recall that the above estimates also imply the uniqueness of the weak solution, as the system (3.42)–(3.45) is linear. □

By combining Theorem 3.5 with Theorem 3.6, we can obtain a more effective version of the variational inequality (3.34).

Theorem 3.7. *Suppose that* **A1**–**A3***,* **B1**–**B3***, and* **C1**–**C4** *are satisfied. Moreover, assume that the initial data* φ_0 *and* w_0 *satisfy* (2.1)*,* (2.22)*,* (2.26)*, and let* $(\overline{u}, \overline{v}_0) \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $\mathbf{CP}$ *with associated state* $(\overline{\varphi}, \overline{w}) = \mathcal{S}(\overline{u}, \overline{v}_0)$*. Finally, let* (p,q) *be the unique solution to the adjoint system* (3.35)–(3.38) *as given by Theorem 3.6. Then the optimal pair* $(\overline{u}, \overline{v}_0)$ *necessarily verifies*

$$\int_Q (q + \nu_1 \overline{u})(u - \overline{u}) + \int_\Omega (q(0) + \nu_2 \overline{v}_0)(v_0 - \overline{v}_0) \\ + \nu_2 \int_\Omega \nabla \overline{v}_0 \cdot \nabla (v_0 - \overline{v}_0) \geq 0 \quad \forall (u, v_0) \in \mathcal{U}_{\rm ad}. \tag{3.47}$$

Remark 3.8. Let us point out that the regularity in (3.41) entails that $q \in C^0([0,T];H)$, so that $q(0)$ makes sense in $L^2(\Omega)$.

Proof of Theorem 3.7. Starting from Theorem 3.5 and comparing (3.34) with (3.47), we realize that, in order to prove Theorem 3.7, it suffices to check that

$$\int_Q qh + \int_\Omega q(0)h^0 \geq k_1 \int_Q (\overline{\varphi} - \varphi_Q)\xi + k_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\xi(T) \\ + k_3 \int_Q (\overline{w} - w_Q)\eta + k_4 \int_\Omega (\overline{w}(T) - w_\Omega)\eta(T) \\ + k_5 \int_Q (\partial_t \overline{w} - w_Q')\partial_t \eta + k_6 \int_\Omega (\partial_t \overline{w}(T) - w_\Omega')\partial_t \eta(T), \tag{3.48}$$

with (ξ, η) denoting the unique solution to (3.9)–(3.12) associated with the increment $(h, h^0) = (u - \overline{u}, v_0 - \overline{v}_0)$. To this end, we test (3.9) by p, (3.10) by q, and integrate over time and by parts to infer that

$$\begin{aligned}
0 = & \int_Q \Big[\partial_t\xi - \Delta\xi + \gamma'(\overline{\varphi})\xi + \tfrac{2}{\theta_c}\pi'(\overline{\varphi})\xi - \tfrac{1}{\theta_c^2}\partial_t\eta\,\pi(\overline{\varphi}) - \tfrac{1}{\theta_c^2}\partial_t\overline{w}\,\pi'(\overline{\varphi})\xi\Big]p \\
& + \int_Q \Big[\partial_{tt}\eta - \alpha\Delta(\partial_t\eta) - \beta\Delta\eta + \pi'(\overline{\varphi})\xi\partial_t\overline{\varphi} + \pi(\overline{\varphi})\partial_t\xi\Big]q - \int_Q hq \\
= & - \int_0^T \langle\partial_t p, \xi\rangle_V \mathrm{d}t + \int_Q \nabla p\cdot\nabla\xi \\
& + \int_Q \Big[\gamma'(\overline{\varphi})p + \tfrac{2}{\theta_c}\pi'(\overline{\varphi})p - \tfrac{1}{\theta_c^2}\partial_t\overline{w}\,\pi'(\overline{\varphi})\,p - \partial_t q\,\pi(\overline{\varphi})\Big]\xi \\
& + \int_Q \Big[- \partial_t q\,\partial_t\eta + \alpha\nabla q\cdot\nabla(\partial_t\eta) + \beta\nabla(1\circledast q)\cdot\nabla(\partial_t\eta) - \tfrac{1}{\theta_c^2}\pi(\overline{\varphi})p\,\partial_t\eta\Big] \\
& + \int_\Omega \Big[p(T)\xi(T) + \partial_t\eta(T)q(T) + \pi(\overline{\varphi}(T))\xi(T)q(T)\Big] \\
& - \int_Q qh - \int_\Omega q(0)h^0.
\end{aligned}$$

By using (3.42)–(3.45), we simplify the above identity, obtaining that

$$\begin{aligned}
0 = & \, k_1 \int_Q (\overline{\varphi} - \varphi_Q)\xi + k_2 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\xi(T) \\
& + \int_Q \Big(k_3(1\circledast(\overline{w} - w_Q)) + k_4(\overline{w}(T) - w_\Omega)\Big)\partial_t\eta \\
& + k_5 \int_Q (\partial_t\overline{w} - w_Q')\partial_t\eta + k_6 \int_\Omega (\partial_t\overline{w}(T) - w_\Omega')\partial_t\eta(T) \\
& - \int_Q qh - \int_\Omega q(0)h^0.
\end{aligned}$$

Now, we integrate by parts the second line, using the initial condition $\eta(0) = 0$ and the fact that $1\circledast(\overline{w} - w_Q)(T) = 0$. Then, it is shown that (3.48) holds, and the proof is concluded. □

Finally, let us notice that from (3.47) we obtain the standard characterization for the minimizers $\overline{u}$ and $\overline{v}_0$ if ν_1 and ν_2 are positive. Prior to the statement, we recall the definition (3.2) of $\mathcal{U}_{\mathrm{ad}}$.

Corollary 3.9. *Suppose that the assumptions of Theorem 3.7 hold, and let $\nu_1 > 0$. Then, $\overline{u}$ is the $L^2(0,T;H)$-orthogonal projection of $-\nu_1^{-1}q$ onto the closed and convex subspace $\{u \in L^\infty(Q) : u_* \le u \le u^*$ a.e. in $Q\}$, and*

$$\overline{u}(x,t) = \max\big\{u_*(x,t), \min\{u^*(x,t), -\nu_1^{-1}q(x,t)\}\big\} \quad \text{for a.a. } (x,t)\in Q.$$

Likewise, if $\nu_2 > 0$, then we infer from Stampacchia's theorem (see, e.g., [1, Thm. 5.6, p. 138]) that $\overline{v}_0$ is characterized by

$$\frac{\nu_2}{2}\|\overline{v}_0\|_V^2 + \int_\Omega q(0)\overline{v}_0 = \min_{v_0\in\mathcal{C}} \Big\{\frac{\nu_2}{2}\|v_0\|_V^2 + \int_\Omega q(0)v_0\Big\},$$

where $\mathcal{C}$ denotes the nonempty, closed and convex subset

$$\{v_0 \in V :\ v_* \le u \le v^* \text{ a.e. in } \Omega,\ \|v_0\|_V \le M\}.$$

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