Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Stochastic homogenization on perforated domains I: Extension operators

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submitted: June 21, 2021

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No. 2849
Berlin 2021

2020 *Mathematics Subject Classification.* 80M40, 60D05.

Key words and phrases. Stochastic homogenization, stochastic geometry, p-Laplace, nonlinear elasticity.

M. Heida is financed by Deutsche Forschungsgemeinschaft (DFG) through SPP2256.

Edited by
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Abstract

This preprint is part of a major rewriting and substantial improvement of WIAS Preprint 2742. In this first part of a series of 3 papers, we set up a framework to study the existence of uniformly bounded extension and trace operators for $W^{1,p}$-functions on randomly perforated domains, where the geometry is assumed to be stationary ergodic. We drop the classical assumption of minimaly smoothness and study stationary geometries which have no global John regularity. For such geometries, uniform extension operators can be defined only from $W^{1,p}$ to $W^{1,r}$ with the strict inequality $r < p$. In particular, we estimate the L^r-norm of the extended gradient in terms of the L^p-norm of the original gradient. Similar relations hold for the symmetric gradients (for $\mathbb{R}^d$-valued functions) and for traces on the boundary. As a byproduct we obtain some Poincaré and Korn inequalities of the same spirit.

Such extension and trace operators are important for compactness in stochastic homogenization. In contrast to former approaches and results, we use very weak assumptions: local (δ, M)-regularity to quantify statistically the local Lipschitz regularity and isotropic cone mixing to quantify the density of the geometry and the mesoscopic properties. These two properties are sufficient to reduce the problem of extension operators to the connectivity of the geometry.

In contrast to former approaches we do not require a minimal distance between the inclusions and we allow for globally unbounded Lipschitz constants and percolating holes. We will illustrate our method by applying it to the Boolean model based on a Poisson point process and to a Delaunay pipe process, for which we can explicitly estimate the connectivity terms.

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DOI 10.20347/WIAS.PREPRINT.2849 Berlin 2021

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1 Introduction

In 1979 Papanicolaou and Varadhan [22] and Kozlov [15] for the first time independently introduced concepts for the averaging of random elliptic operators. At that time, the periodic homogenization theory had already advanced to some extend (as can be seen in the book [23] that had appeared one year before) dealing also with non-uniformly elliptic operators [17] and domains with periodic holes [3]. The most recent and most complete work for extension operators on periodically perforated domains is [11].

In contrast, the homogenization on randomly perforated domains is still open to a large extend. Recent results focus on minimally smooth domains [9, 24] or on decreasing size of the perforations when the smallness parameter tends to zero [8] (and references therein). The main issue in homogenization on perforated domains compared to classical homogenization problems is compactness. For elasticity, this is completely open.

The results presented below are meant for application in quenched convergence. The estimates for the extension and trace operators which are derived strongly depends on the realization of the geometry - thus on ω. Nevertheless, if the geometry is stationary, a corresponding estimate can be achieved for almost every ω.

The Problem

In order to illustrate the issues in stochastic homogenization on perforated domains, we introduce the following example.

Let $\mathbf{P}(\omega) \subset \mathbb{R}^d$ be a stationary random open set and let $\varepsilon > 0$ be the smallness parameter and let $\tilde{\mathbf{P}}(\omega)$ be an infinitely connected component (i.e. an unbounded connected domain) of $\mathbf{P}(\omega)$. For a bounded open domain $\mathbf{Q}$, we consider $\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega) := \mathbf{Q} \cap \varepsilon\tilde{\mathbf{P}}(\omega)$ and $\Gamma^\varepsilon(\omega) := \mathbf{Q} \cap \varepsilon\partial\tilde{\mathbf{P}}(\omega)$ with outer normal $\nu_{\Gamma^\varepsilon(\omega)}$. For a sufficiently regular and $\mathbb{R}^d$-valued function u we denote $\nabla^s u := \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right)$ the symmetric part of ∇u. A typical homogenization problem then is the following::

$$\begin{aligned} -\operatorname{div}\left(|\nabla^s u^\varepsilon|^{p-2}\,\nabla^s u^\varepsilon\right) &= g(u^\varepsilon) && \text{on } \mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\,, \\ u &= 0 && \text{on } \partial\mathbf{Q} \cap (\varepsilon\mathbf{P})\,, \\ |\nabla^s u^\varepsilon|^{p-2}\,\nabla u^\varepsilon \cdot \nu_{\Gamma^\varepsilon(\omega)} &= f(u^\varepsilon) && \text{on } \Gamma^\varepsilon(\omega)\,. \end{aligned} \tag{1.1}$$

Note that for simplicity of illustration, the only randomness that we consider in this problem is due to $\mathbf{P}(\omega)$.

One way to prove homogenization of (1.1) is to prove Γ-convergence of

$$\mathcal{E}_{\varepsilon,\omega}(u) = \int_{\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)} \left(\frac{1}{p}\,|\nabla^s u|^p - G(u)\right) + \int_{\Gamma^\varepsilon(\omega)} F(u)\,,$$

in a suitably chosen space where $G' = g$ and $F' = f$. Conceptually, this implies convergence of the minimizers u^ε to a minimizer of a limit functional but if G or F are non-monotone, we need compactness. However, the minimizers are elements of $\mathbf{W}^{1,p}(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}) := W^{1,p}(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}};\mathbb{R}^d)$ and since this space changes with ε, there is apriori no compactness of u^ε, even though we have uniform apriori estimates on the gradients.

The *canonical* path to circumvent this issue in *periodic* homogenization is via uniformly bounded extension operators $\mathcal{U}_\varepsilon : W^{1,p}(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}) \to W^{1,p}(\mathbf{Q})$ that share the property that for some $C > 0$ independent from ε it holds for all $u \in W^{1,p}(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}})$ with $u|_{\mathbb{R}^d\setminus\mathbf{Q}} \equiv 0$

$$\|\nabla\mathcal{U}_\varepsilon u\|_{L^p(\mathbf{Q})} \leq C\,\|\nabla u\|_{L^p(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}})}\,, \qquad \|\mathcal{U}_\varepsilon u\|_{L^p(\mathbf{Q})} \leq C\,\|u\|_{L^p(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}})}\,, \tag{1.2}$$

see [11, 12], combined with uniformly bounded trace operators, see [7, 9]. Such operators have also been provided for elasticity problems [11, 21, 30, 31], i.e.

$$\|\nabla^s\mathcal{U}_\varepsilon u\|_{L^p(\mathbf{Q})} \leq C\,\|\nabla^s u\|_{L^p(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}})}\,.$$

The last estimate then allows to use Korn's inequality combined with Sobolev's embedding theorem to find $\mathcal{U}_\varepsilon u^\varepsilon \rightharpoonup u_0$ weakly in $\mathbf{W}^{1,p}(\mathbf{Q})$.

What is the classical strategy? The existing results on extension and trace operators for random domains are focused on a.s. minimally smooth domains. A connected domain $\mathbf{P} \subset \mathbb{R}^d$ is minimally smooth [26] if there exist (δ, M) such that for every $x \in \partial\mathbf{P}$ the set $\partial\mathbf{P} \cap \mathbb{B}_\delta(x)$ is the graph of a Lipschitz continuous function with Lipschitz constant less than M. It is further assumed that the complement $\mathbb{R}^d\backslash\mathbf{P}$ consists of uniformly bounded sets. This concept leads to almost sure construction of uniformly bounded extension operators $W^{1,p}_{\mathrm{loc}}(\mathbf{P}) \to W^{1,p}(\mathbb{R}^d)$ [9] in the sense that for every bounded $\mathbf{Q}$ and every $u \in W^{1,p}(\mathbf{Q}\cap\mathbf{P})$ with $u|_{\mathbb{R}^d\backslash\mathbf{Q}} \equiv 0$ holds

$$\|\nabla\mathcal{U}u\|_{L^p(\mathbf{Q})} \leq C\,\|\nabla u\|_{L^p(\mathbf{Q}\cap\mathbf{P})}\,, \qquad \|\mathcal{U}u\|_{L^p(\mathbf{Q})} \leq C\,\|u\|_{L^p(\mathbf{Q}\cap\mathbf{P})}\,, \tag{1.3}$$

with C independent from $\mathbf{Q}$. Similarly, one obtains for the trace $\mathcal{T}$ that [24]

$$\|\mathcal{T}u\|_{L^p(\mathbf{Q}\cap\partial\mathbf{P})} \leq C\left(\|u\|_{L^p(\mathbf{Q}\cap\mathbf{P})} + \|\nabla u\|_{L^p(\mathbf{Q}\cap\mathbf{P})}\right).$$

Using a scaling argument to obtain e.g. (1.2), such extension and trace operators are typically used in order to treat nonlinearities in homogenization problems.

Why does this work? The theory cited above is directly connected to the theory of Jones [13] and Duran and Muschietti [5] on so-called John domains. These are precisely the bounded domains $\mathbf{P}$ that admit extension operators $W^{1,p}(\mathbf{P}) \to W^{1,p}(\mathbb{R}^d)$ satisfying

$$\|\mathcal{U}u\|_{W^{1,p}(\mathbb{R}^d)} \leq C\,\|u\|_{W^{1,p}(\mathbf{Q}\cap\mathbf{P})}\,.$$

Definition (John domains)**.** A bounded domain $\mathbf{P} \subset \mathbb{R}^d$ is a John domain (a.k.a (ε,δ)-domain) if there exists $\varepsilon, \delta > 0$ such that for every $x, y \in \mathbf{P}$ with $|x-y| < \delta$ there exists a rectifiable path $\gamma:\,[0,1] \to \mathbf{P}$ from x to y such that

$$\begin{aligned} \mathrm{length}\gamma &\leq \frac{1}{\varepsilon}\,|x-y| \qquad \text{and} \\ \forall t \in (0,1):\quad \inf_{z\in\mathbb{R}^d\backslash\mathbf{P}} |\gamma(t)-z| &\geq \frac{\varepsilon\,|x-\gamma(t)|\,|\gamma(t)-y|}{|x-y|}\,. \end{aligned}$$

Because of the locality implied by δ, it is possible to glue together local extension operators on John domains such as done in [11] for periodic or [9] for minimally smooth domains. In the stochastic case one benefits a lot from the uniform boundedness of the components of $\mathbb{R}^d\backslash\mathbf{P}$, which allows to split the extension problem into independent extension problems on uniformly John-regular domains.

Why this is not enough for general random domains! As one could guess from the emphasis that is put on the above explanations, random geometries are merely minimally smooth. On an unbounded random domain $\mathbf{P}$, the constant M can locally become very large in points $x \in \partial\mathbf{P}$, while simultaneously, δ can become very small in the very same x. In fact, they are not even “uniformly John” as the following, yet deterministic example illustrates.

Example 1.1. Considering

$$\mathbf{P} := \left\{(x_1,x_2) \in \mathbb{R}^2:\ \exists n \in \mathbb{N}:\ x_1 - (2n+1) \in (-1,1],\ x_2 < \max\left\{1,\ n\,|x_1-(2n+1)|\right\}\right\}$$

the Lipschitz constant on $(2n, 2n+2)$ is n and it is easy to figure out that this non-uniformly Lipschitz domain violates the John condition due to the cups. Hence, a uniform estimate of the form (1.3) cannot exist.

Therefor, an alternative concept to measure the large scale regularity of a random geometry is needed. Since the classical results do not excluded the existence of an estimate

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\nabla\mathcal{U}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbb{B}_{\mathfrak{r}}(\mathbf{P})}|\nabla u|^p\right)^{\frac{r}{p}},\qquad \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\mathcal{U}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbb{B}_{\mathfrak{r}}(\mathbf{P})}|u|^p\right)^{\frac{r}{p}}, \tag{1.4}$$

or

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\nabla^s\mathcal{U}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbb{B}_{\mathfrak{r}}(\mathbf{P})}|\nabla^s u|^p\right)^{\frac{r}{p}},\qquad \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\mathcal{U}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbb{B}_{\mathfrak{r}}(\mathbf{P})}|u|^p\right)^{\frac{r}{p}}, \tag{1.5}$$

where $1\leq r<p$ and C is independent from $\mathbf{Q}$, such inequalities will be our goal.

Our results in a nutshell We will provide inequalities of the form (1.4)–(1.5) for a Voronoi-pipe model and for a Boolean model. On the way, we will provide several concepts and intermediate results that can be reused in further examples and general considerations such as planed in part III of this series. Scaled versions (replacing $\varepsilon=m^{-1}$ in Theorems 1.16 and 1.18) of (1.4)–(1.5) can be formulated for functions

$$u\in W^{1,p}_{0,\partial\mathbf{Q}}(\varepsilon\mathbf{P}\cap\mathbf{Q}):=\left\{u\in W^{1,p}(\mathbf{Q}\cap\varepsilon\mathbf{P}):\,u|_{(\varepsilon\mathbf{P})\cap\partial\mathbf{Q}}\equiv 0\right\},$$

and will be of the form

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbb{R}^d}|\nabla\mathcal{U}_\varepsilon u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\varepsilon\mathbf{P}}|\nabla u|^p\right)^{\frac{r}{p}},\qquad \frac{1}{|\mathbf{Q}|}\int_{\mathbb{R}^d}|\mathcal{U}_\varepsilon u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\varepsilon\mathbf{P}}|u|^p\right)^{\frac{r}{p}},$$

resp.

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbb{R}^d}|\nabla^s\mathcal{U}_\varepsilon u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\varepsilon\mathbf{P}}|\nabla^s u|^p\right)^{\frac{r}{p}},\qquad \frac{1}{|\mathbf{Q}|}\int_{\mathbb{R}^d}|\mathcal{U}_\varepsilon u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\varepsilon\mathbf{P}}|u|^p\right)^{\frac{r}{p}},$$

where the support of $\mathcal{U}_\varepsilon u$ lies within $\mathbb{B}_{\varepsilon^\beta}(\mathbf{Q})$ for ε small enough and some arbitrarily chosen but fixed $\beta\in(0,1)$.

Quantifying properties of random geometries

As a replacement for periodicity, we introduce the concept of mesoscopic regularity of a stationary random open set:

Definition 1.2 (Mesoscopic regularity)**.** Let $\mathbf{P}$ be a stationary ergodic random open set, let $\tilde{f}$ be a positive, monotonically decreasing function $\tilde{f}$ with $\tilde{f}(R)\to 0$ as $R\to\infty$ and let $\mathfrak{r}>0$ s.t.

$$\mathbb{P}\big(\exists x\in\mathbb{B}_R(0):\,\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x)\subset\mathbb{B}_R(0)\cap\mathbf{P}\big)\geq 1-\tilde{f}(R)\,. \tag{1.6}$$

Then $\mathbf{P}$ is called $(\mathfrak{r},\tilde{f})$-mesoscopic regular. $\mathbf{P}$ is called polynomially (exponentially) regular if $1/\tilde{f}$ grows polynomially (exponentially).

As a consequence of Lemmas 3.14, 3.16 and 3.17 we obtain the following.

Corollary 1.3 (All stationary ergodic random open sets are mesoscopic regular)**.** *Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set. Then there exists $\mathfrak{r}>0$ and a monotonically decreasing function with $\tilde{f}(R)\to 0$ as $R\to\infty$ such that $\mathbf{P}$ is $(\mathfrak{r},\tilde{f})$-mesoscopic regular. Furthermore, there exists a* *jointly stationary random point process* $\mathbb{X}_{\mathfrak{r}}(\omega) = (x_a)_{a\in\mathbb{N}}$ *and for every* $a \in \mathbb{N}$ *it holds* $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_a) \subset \mathbf{P}$ *and for all* $a, b \in \mathbb{N}$, $a \neq b$, *it holds* $|x_a - x_b| > 2\mathfrak{r}$. *Construct from* $\mathbb{X}_{\mathfrak{r}}$ *a Voronoi tessellation of cells* G_a *with diameter* $d_a = d(x_a)$. *Then for some constant* $C > 0$ *and some monotone decreasing* $f : (0,\infty) \to \mathbb{R}$ *and* $C > 0$ *with* $f(R) \leq C\tilde{f}(C^{-1}R)$ *it holds*

$$\mathbb{P}(d(x_a) > D) < f(D)\,.$$

$\mathfrak{r}$, $\mathbb{X}_{\mathfrak{r}}$ and f from Corollary 1.3 will play a central role in the analysis. We summarize some of these properties in the following.

Assumption 1.4. *Let* $\mathbf{P}$ *be a Lipschitz domain and assume there exists* $\mathbb{X}_{\mathfrak{r}} = (x_a)_{a\in\mathbb{N}}$ *be a set of points having mutual distance* $|x_a - x_b| > 2\mathfrak{r}$ *if* $a \neq b$ *and with* $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_a) \subset \mathbf{P}$ *for every* $a \in \mathbb{N}$ *(e.g.* $\mathbb{X}_{\mathfrak{r}}(\mathbf{P})$*, see (2.51)).*

The second important concept to quantify in a stochastic manner is that of local Lipschitz regularity.

Definition 1.5 (Local (δ, M)-Regularity)**.** Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set. $\mathbf{P}$ is called (δ, M)*-regular* in $p_0 \in \partial\mathbf{P}$ if there exists an open set $U \subset \mathbb{R}^{d-1}$ and a Lipschitz continuous function $\phi : U \to \mathbb{R}$ with Lipschitz constant greater or equal to M such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(p_0)$ is subset of the graph of the function $\varphi : U \to \mathbb{R}^d\,,\ \tilde{x} \mapsto (\tilde{x}, \phi(\tilde{x}))$ in some suitable coordinate system.

Every Lipschitz domain $\mathbf{P}$ is *locally* (δ, M)*-regular* in every $p_0 \in \partial\mathbf{P}$. In what follows, we bound δ from above by $\mathfrak{r}$ only for practical reasons in the proofs. The following quantities can be derived from local (δ, M)-regularity.

Definition 1.6. For a Lipschitz domain $\mathbf{P} \subset \mathbb{R}^d$ and for every $p \in \partial\mathbf{P}$ and $n \in \mathbb{N} \cup \{0\}$

$$\Delta(p) := \sup_{\delta<\mathfrak{r}} \{\exists M > 0 :\ \mathbf{P} \text{ is } (\delta, M)\text{-regular in } p\}\,, \quad \delta_\Delta(p) := \frac{\Delta(p)}{2}\,, \tag{1.7}$$

$$M_r(p) := \inf_{\eta>r} \inf \{M :\ \mathbf{P} \text{ is } (\eta, M)\text{-regular in } p\}\,, \tag{1.8}$$

$$\rho_n(p) := \sup_{r<\delta(p)} r\left(4M_r(p)^2 + 2\right)^{-\frac{n}{2}}\,, \tag{1.9}$$

If no confusion occurs, we write $\delta = \delta_\Delta$. Furthermore, for $c \in (0,1]$ let $\eta(p) = c\delta_\Delta(p)$ or $\eta(p) = c\rho_n(p)$, $n \in \mathbb{N}$ and $r \in C^{0,1}(\partial\mathbf{P})$ and define

$$\eta_{[r],\mathbb{R}^d}(x) := \inf\left\{\eta(\tilde{x})\ :\ \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \mathbb{B}_{r(\tilde{x})}(\tilde{x})\right\}\,, \tag{1.10}$$

$$M_{[r,\eta],\mathbb{R}^d}(x) := \sup\left\{M_{r(\tilde{x})}(\tilde{x})\ :\ \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \overline{\mathbb{B}_{\eta(\tilde{x})}(\tilde{x})}\right\}\,, \tag{1.11}$$

where $\inf\emptyset = \sup\emptyset := 0$ for notational convenience. We also write $M_{[\eta],\mathbb{R}^d}(x) := M_{[\eta,\eta],\mathbb{R}^d}(x)$ and $\eta_{\mathbb{R}^d}(x) := \eta_{[\eta],\mathbb{R}^d}(x)$. Of course, we can also consider $M_{[r],\partial\mathbf{P}} :\ p \mapsto M_{r(p)}(p)$ as a function on $\partial\mathbf{P}$, and we will do this once in Lemma 3.8.

When it comes to application of the abstract results found below, it is important to have in mind that η and M_r are quantities on $\partial\mathbf{P}$, while $\eta_{[r],\mathbb{R}^d}$ and $M_{[r,\eta],\mathbb{R}^d}$ are quantities on $\mathbb{R}^d$. Hence, while trivially

$$\mathbb{P}(\eta_{[r],\mathbb{R}^d} \in (\eta_1, \eta_2)) = \lim_{n\to\infty} n^{-d}\,|\mathbf{Q}|^{-1}\left|\left\{x \in n\mathbf{Q} :\ \eta_{[r],\mathbb{R}^d} \in (\eta_1, \eta_2)\right\}\right|$$

(and similarly for $M_{[r,\eta],\mathbb{R}^d}$) for every convex bounded open $\mathbf{Q}$, we have in mind

$$\mathbb{P}(\eta \in (\eta_1, \eta_2)) = \left(\lim_{n\to\infty} \mathcal{H}^{d-1}(\partial\mathbf{P} \cap n\mathbf{Q})\right)^{-1} \mathcal{H}^{d-1}(\{x \in (n\mathbf{Q}) \cap \partial\mathbf{P} :\ \eta \in (\eta_1, \eta_2)\})\,.$$

We will prove measurability of $\eta_{[r],\mathbb{R}^d}$ and $M_{[r,\eta],\mathbb{R}^d}$ in Lemma 3.11 and see how the weighted expectations of $\eta_{[r],\mathbb{R}^d}$ and $M_{[r,\eta],\mathbb{R}^d}$ can be estimated by weighted expectations of M and η in Lemma 3.12.

Traces

The first important result is the boundedness of the trace operator.

Theorem 1.7. *Let $\mathbf{P}\subset\mathbb{R}^d$ be a Lipschitz domain, $\frac{1}{8}>\mathfrak{r}>0$ and let $\mathbf{Q}\subset\mathbb{R}^d$ be a bounded open set and let $1\leq r<p_0<p$. Then the trace operator $\mathcal{T}$ satisfies for every $u\in W^{1,p}_{\mathrm{loc}}(\mathbf{P})$*

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}|\mathcal{T}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{1}{4}}(\mathbf{Q})\cap\mathbf{P}}|u|^p+|\nabla u|^p\right)^{\frac{r}{p}}$$

where for some constant C_0 depending only on p_0, p and r and d and for $\tilde\rho=2^{-5}\rho_1$ one has

$$C=C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{1}{4}\mathfrak{r}}(\mathbf{Q})\cap\partial\mathbf{P}}\tilde\rho_{\mathbb{R}^d}^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{1}{4}\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}}\left(1+\tilde M_{[\frac{1}{32}\delta],\mathbb{R}^d}\right)^{\left(\frac{1}{p_0}+1+\hat d\right)\frac{p}{p-p_0}}\right)^{\frac{p-p_0}{p_0p}},\tag{1.12}$$

$$C=C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{1}{4}\mathfrak{r}}(\mathbf{Q})\cap\partial\mathbf{P}}\left(\tilde\rho_{\mathbb{R}^d}\left(1+\tilde M_{[\frac{1}{32}\delta],\mathbb{R}^d}\right)\right)^{-\frac{1}{p-r}}\right)^{\frac{p-r}{p}}.\tag{1.13}$$

Proof. This is proved in Section 4.6. □

Local Covering of $\partial\mathbf{P}$

In view of Corollary 3.7, for every $n=1$ or $n=2$ there exist a complete covering of $\partial\mathbf{P}$ by balls $\mathbb{B}_{\tilde\rho_n\left(p_i^n\right)}(p_i^n)$, $(p_i^n)_{i\in\mathbb{N}}$, where $\tilde\rho_n(p):=2^{-5}\rho_n(p)$. We write $\tilde\rho_{n,i}:=\tilde\rho_n(p_i^n)$.

Definition 1.8 (Microscopic regularity and extension order)**.** The inner microscopic regularity α is

$$\alpha:=\inf\left\{\tilde\alpha\geq 0:\ \forall p\in\partial\mathbf{P}\exists y\in\mathbf{P}:\ \mathbb{B}_{\tilde\rho(p)/32(1+M_{\tilde\rho(p)}(p)^{\tilde\alpha})}(p)\subset\mathbb{B}_{\tilde\rho(p)/8}(p)\right\}.$$

In Lemma 3.1 we will see that indeed $\alpha\leq 1$.

Definition 1.9 (Extension order)**.** The geometry is *of extension order* $n\in\mathbb{N}\cup\{0\}$ if there exists $C>0$ such that for almost every $p\in\partial\mathbf{P}$ there exists a local extension operator

$$\begin{aligned}&\mathcal{U}:\ W^{1,p}(\mathbb{B}_{\frac{1}{8}\delta(p)}(p)\cap\mathbf{P})\to W^{1,p}(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p)),\\&\qquad\|\nabla\mathcal{U}u\|_{L^p(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p))}\leq C\left(1+M_{\frac{1}{8}\delta(p)}(p)\right)\|\nabla u\|_{L^p(\mathbb{B}_{\frac{1}{8}\delta(p)}(p))}.\end{aligned}\tag{1.14}$$

The geometry is *of symmetric extension order* $n\in\mathbb{N}\cup\{0\}$ if there exists $C>0$ such that for almost every $p\in\partial\mathbf{P}$ there exists a local extension operator

$$\begin{aligned}&\mathcal{U}:\ \mathbf{W}^{1,p}(\mathbb{B}_{\frac{1}{8}\delta(p)}(p)\cap\mathbf{P})\to\mathbf{W}^{1,p}(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p)),\\&\qquad\|\nabla^s\mathcal{U}u\|_{L^p(\mathbb{B}_{\frac{1}{8}\rho_n(p)}(p))}\leq C\left(1+M_{\frac{1}{8}\delta(p)}(p)\right)^2\|\nabla^s u\|_{L^p(\mathbb{B}_{\frac{1}{8}\delta(p)}(p))}.\end{aligned}\tag{1.15}$$

Corollary 3.6 shows that every locally Lipschitz geometry is of extension order $n=1$ and every locally Lipschitz geometry is of symmetric extension order $n=2$. However, better results for n are possible, as we will see below.

Global Tessellation of $\mathbf{P}$

Let $\mathbb{X}=(x_a)_{a\in\mathbb{N}}$ be a jointly stationary point process with $\mathbf{P}$ such that $\mathbb{B}_{\mathfrak{r}}(\mathbb{X})\subset\mathbf{P}$. In this work, we will often assume that $|x_a-x_b|>2\mathfrak{r}$ for all $a\neq b$ for simplicity in Sections 5 and 6. The existence of such a process is always guarantied by Lemmas 3.14 and 3.16. Its choice in a concrete example is, however, delicate. Worth mentioning, for most of the theory developed until the end of Section 4 (Except for Lemmas 3.17 and 3.18 which are not used before Section 5), is completely independent from this mutual minimal distance assumption.

From $\mathbb{X}$ we construct a Voronoi tessellation with cells $(G_a)_{a\in\mathbb{N}}$ and we chose for each x_a a radius $\mathfrak{r}_a\leq\mathfrak{r}$ with $\mathbb{B}_{\mathfrak{r}_a}(x_a)\subset G_a\cap\mathbf{P}$. Again, using Corollary 1.3, we assume that $\mathfrak{r}_a=\mathfrak{r}$ is constant for simplicity.

Extensions I: Gradients

Notation 1.10. Given $n\in\{0,1\}$ and $\alpha\in[0,1]$ we chose

$$\mathfrak{r}_{n,\alpha,i}:=\tilde\rho_{n,i}/32(1+M_{\tilde\rho_{n,i}}(p_{n,i})^{\alpha}) \tag{1.16}$$

and some $y_{n,\alpha,i}$ such that

$$B_{n,\alpha,i}:=\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})\subset\mathbf{P}\cap\mathbb{B}_{\frac18\tilde\rho_{n,i}}(p_{n,i})\,. \tag{1.17}$$

and for every i and a, we define

$$\tau_{n,\alpha,i}u:=\fint_{B_{n,\alpha,i}}u\,,\qquad \mathcal{M}_a u:=\fint_{\mathbb{B}_{\frac{\mathfrak{r}_a}{16}}(x_a)}u\,,$$

local averages close to $\partial\mathbf{P}$ and in x_a. We say that $x_a\sim\!\sim x_b$ if $G_a\cap\mathbb{B}_{\mathfrak{r}}(G_b)\neq\emptyset$ and we say $x_a\in\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})$ if $\mathbb{B}_{\mathfrak{r}}(G_a)\cap\mathbf{Q}\neq\emptyset$. Based on (4.14) we obtain the following extension result.

Theorem 1.11. *Let $\mathfrak{r}>0$ and let $\mathbf{P}\subset\mathbb{R}^d$ be a stationary ergodic random Lipschitz domain such that Assumption 1.4 holds for $\mathbb{X}=(x_a)_{a\in\mathbb{N}}$ and $\mathbf{P}$ has microscopic regularity α with extension order n. Let $\mathbf{Q}\subset\mathbb{R}^d$ be a bounded open set with $\mathbb{B}_{\frac14}(0)\subset\mathbf{Q}$ and let $1\leq r<p$. Furthermore, let*

$$\mathbb{E}\left(\left(1+M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{nd}\left(1+M_{[\frac18\delta],\mathbb{R}^d}\right)^{r}\left(1+M_{[\tilde\rho_n],\mathbb{R}^d}\right)^{\alpha(d-1)}\right)<\infty$$

then there exist $C>0$ depending only on d, r and p such that for a.e. ω there exists an extension operator $\mathcal{U}_\omega:W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and $C_\omega>0$ such that for every $m\geq1$ and every $u\in W^{1,p}(\mathbf{P}(\omega))$ with $u|_{\mathbf{P}(\omega)\setminus m\mathbf{Q}}\equiv0$ it holds

$$\begin{aligned}\frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}}|\nabla(\mathcal{U}_\omega u)|^r\leq C_\omega\left(\frac{1}{m^d}\int_{\mathbf{P}\cap\mathbb{B}_{\mathfrak{r}}(m\mathbf{Q})}|\nabla u|^p\right)^{\frac{r}{p}}&\\ +C\frac{1}{m^d}\int_{\mathbf{P}\cap\mathbb{B}_{\mathfrak{r}}(m\mathbf{Q})}\sum_{i\neq0}\sum_a\tilde\rho_{\mathbf{P}}^{-r}\chi_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_a)}\chi_{\mathbb{B}_{\tilde\rho_{n,i}}(p_{n,i})}\left|\tau_{n,\alpha,i}u-\mathcal{M}_a u\right|^r&\\ +C\left|\frac{1}{m^d}\int_{\mathbf{P}\cap m\mathbf{Q}}\sum_a\sum_{a\sim\sim b}\chi_{\mathbb{B}_{\mathfrak{r}}(G_a)}\left|\mathcal{M}_a u-\mathcal{M}_b u\right|\right|^r&\,,\\ \frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}}|\mathcal{U}_\omega u|^r\leq C_\omega\left(\frac{1}{m^d}\int_{\mathbf{P}\cap\mathbb{B}_{\mathfrak{r}}(m\mathbf{Q})}|u|^p\right)^{\frac{r}{p}}&\,.\end{aligned}$$

Proof. This is a consequence of Lemma 4.7. □

In case one is interested in a weaker estimate on the extension operator, we propose the following:

Theorem 1.12. *Under the assumptions of Theorem 1.11 let additionally*

$$\mathbb{E}\left(\tilde{\rho}_{\mathbf{P}}^{-\frac{rp}{p-r}}\right) < \infty$$

then there exists an extension operator $\mathcal{U}_\omega : W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *such that for every* $m \geq 1$ *and every* $u \in W^{1,p}(\mathbf{P}(\omega))$ *with* $u|_{\mathbf{P}(\omega)\setminus m\mathbf{Q}} \equiv 0$ *it holds*

$$\frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}} \left(|\nabla\left(\mathcal{U}_\omega u\right)|^r + |\mathcal{U}_\omega u|^r\right) \leq C_\omega \left(\frac{1}{m^d}\int_{\mathbf{P}\cap\mathbb{B}_{\mathfrak{r}}(m\mathbf{Q})} \left(|\nabla u|^p + |u|^p\right)\right)^{\frac{r}{p}} .$$

Proof. This is a consequence of the proof of Lemma 4.7, replacing $M_a u$ in the definition of $\mathcal{U}u$ by 0. □

Percolation and Connectivity

The terms depending on $|\tau_{n,\alpha,i}u - \mathcal{M}_a u|$ or $|\mathcal{M}_a u - \mathcal{M}_a u|$ appearing on the right hand side in Theorem 1.11 need to be replaced by an integral over $|\nabla u|^p$. Here, the pathwise topology of the geometry comes into play. By this we mean that we have to integrate the gradient of u over a path connecting e.g. p_i and x_a. Here, the mesoscopic properties of the geometry will play a role. In particular, we need pathwise connectedness of the random domain, a phenomenon which is known as percolation in the theory of random sets. We will discuss two different examples to see that these terms can indeed be handled in application, but shift a general discussion of arbitrary geometries to a later publication.

Extensions II: Symmetric gradients

We now turn to the situation that u is a $\mathbb{R}^d$-valued function and that the given PDE system yields only estimates for $\nabla^s u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$. We introduce the following quantities:

Definition 1.13. Given $n \in \{0,1,2\}$ and $\alpha \in [0,1]$ such that such that (1.17) holds for $\mathfrak{r}_i = \mathfrak{r}_{n,\alpha,i}$ for every i let for i, a

$$\bar{\nabla}^{\perp}_{n,\alpha,i}u := \fint_{\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})} \left(\nabla u - \nabla^s u\right) , \quad \left[\tau^{\mathfrak{s}}_{n,\alpha,i}u\right](x) := \bar{\nabla}^{\perp}_{n,\alpha,i}u\,\left(x - y_{2,i}\right) + \fint_{\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})} u \,,$$

$$\bar{\nabla}^{\perp}_{a}u := \fint_{\mathbb{B}_{\frac{\mathfrak{r}_a}{16}}(x_a)} \left(\nabla u - \nabla^s u\right) , \qquad \left[\mathcal{M}^{\mathfrak{s}}_{a}u\right](x) := \bar{\nabla}^{\perp}_{a}u\,\left(x - x_a\right) + \fint_{\mathbb{B}_{\frac{\mathfrak{r}_a}{16}}(x_a)} u \,.$$

Using above introduced notation and $\mathbf{W}$ do denote $\mathbb{R}^d$-valued Sobolev spaces, we find the following.

Theorem 1.14. *Let* $\mathfrak{r} > 0$ *and let* $\mathbf{P} \subset \mathbb{R}^d$ *be a stationary ergodic random Lipschitz domain such that Assumption 1.4 holds for* $\mathbb{X} = (x_a)_{a\in\mathbb{N}}$ *and* $\mathbf{P}$ *has microscopic regularity* α *with symmetric extension order* $n \leq 2$. *Let* $\mathbf{Q} \subset \mathbb{R}^d$ *be a bounded open set with* $\mathbb{B}_{\frac{1}{4}}(0) \subset \mathbf{Q}$ *and let* $1 \leq r < p_0 < p$. *Furthermore, let*

$$\mathbb{E}\left(\left(1 + M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{nd}\left(1 + M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^{2r}\left(1 + M_{[\tilde{\rho}_n],\mathbb{R}^d}\right)^{\alpha(d-1)}\right) < \infty$$

then hen there exist $C > 0$ depending only on d, r, s and p such that for a.e. ω there exists an extension operator $\mathcal{U}_\omega : \mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to \mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and $C_\omega > 0$ such that for every $m \geq 1$ and every $u \in \mathbf{W}^{1,p}(\mathbf{P}(\omega))$ with $u|_{\mathbf{P}(\omega)\setminus\mathbf{Q}} \equiv 0$ it holds

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} |\nabla^s(\mathcal{U}_\omega u)|^r \leq C_\omega &\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}} |\nabla^s u|^p\right)^{\frac{r}{p}} \\ &+ C\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}} \sum_a \sum_{i\neq 0} \rho_{1,i}^{-r}\chi_{A_{1,i}}\chi_{\mathfrak{A}_{1,a}} \left|\tau^{\mathfrak{s}}_{n,\alpha,i}u - \mathcal{M}^{\mathfrak{s}}_a u\right|^r \\ &+ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left|\sum_{l=1}^d \sum_{a:\,\partial_l\Phi_a>0}\sum_{b:\,\partial_l\Phi_b<0} \frac{\partial_l\Phi_a\,|\partial_l\Phi_b|}{D^{\Phi}_{l+}}\left(\mathcal{M}^{\mathfrak{s}}_a u - \mathcal{M}^{\mathfrak{s}}_b u\right)\right|^r \\ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} |\mathcal{U}_\omega u|^r \leq C_\omega &\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}} |u|^p\right)^{\frac{r}{p}},\end{aligned}$$

Proof. This is a consequence of Lemma 4.9. □

Theorem 1.15. *Under the assumptions of Theorem 1.14 let additionally*

$$\mathbb{E}\left(\tilde{\rho}_{\mathbf{P}}^{-\frac{rp}{p-r}}\right) < \infty$$

then there exists an extension operator $\mathcal{U}_\omega : W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ such that for every $m \geq 1$ and every $u \in W^{1,p}(\mathbf{P}(\omega))$ with $u|_{\mathbf{P}(\omega)\setminus m\mathbf{Q}} \equiv 0$ it holds

$$\frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}} \left(|\nabla(\mathcal{U}_\omega u)|^r + |\mathcal{U}_\omega u|^r\right) \leq C_\omega \left(\frac{1}{m^d}\int_{\mathbf{P}\cap\mathbb{B}_{\mathfrak{r}}(m\mathbf{Q})} \left(|\nabla u|^p + |u|^p\right)\right)^{\frac{r}{p}}.$$

Proof. This is a consequence of the proof of Lemma 4.7, replacing $M^{\mathfrak{s}}_a u$ in the definition of $\mathcal{U}u$ by 0. □

Discussion: Random Geometries and Applicability of the Method

In Section 6 we discuss two standard models from the theory of stochastic geometries. The first one is a system of random pipes: Starting from a Poisson point process and deleting all points with nearest neighbor closer than $2\mathfrak{r}$ and introducing the Delaunay neighboring condition on the points, every two neighbors are connect through a pipe of random thickness 2δ, where δ is distributed i.i.d among the pipes and we complete the geometry by adding a ball of radius $\frac{\mathfrak{r}}{2}$ around each point. Defining for bounded open domains $\mathbf{Q} \subset \mathbb{R}^d$ and $n \in \mathbb{N}$

$$u \in W^{1,p}_{0,\partial(n\mathbf{Q})}(\mathbf{P}\cap n\mathbf{Q}) := \left\{u \in W^{1,p}(\mathbf{P}\cap n\mathbf{Q}) :\ u|_{\partial(n\mathbf{Q})} \equiv 0\right\},$$

and using $\mathbf{W}$ instead of W for $\mathbb{R}^d$-valued functions, we find our first result:

Theorem 1.16. *In the pipe model of Section 6.1 let $\mathbb{P}(\delta(x,y) < \delta_0) \leq C_\delta \delta_0^\beta$ and let $1 \leq r < s < p$ be such that $\max\left\{\frac{p(s+d)}{p-s}, \frac{p(2d-s-1)}{p-s}\right\} \leq \beta$ and $\frac{sr}{s-r} \leq \beta + d - 1$. Then $\alpha = n = 0$ both for extension and symmetric extension order and there almost surely exists an extension operator $\mathcal{U} : W^{1,p}_{\mathrm{loc}}(\mathbf{P}) \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and constants $C, R > 1$ such that for all $m \in \mathbb{N}$ and every $u \in W^{1,p}_{0,\partial(m\mathbf{Q})}(\mathbf{P}\cap m\mathbf{Q})$ it holds*

$$\frac{1}{|m\mathbf{Q}|}\int_{\mathbb{R}^d} |\nabla(\mathcal{U}u)|^r \leq C\left(\frac{1}{m^d}\int_{\mathbf{P}\cap m\mathbf{Q}} |\nabla u|^p\right)^{\frac{r}{p}}.$$

Furthermore there almost surely exists an extension operator $\mathcal{U}\,:\,\mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbf{P})\to\mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *and a constant* $C>0$ *such that for all* $m\in\mathbb{N}$ *and every* $u\in\mathbf{W}^{1,p}_{0,\partial(m\mathbf{Q})}(\mathbf{P}\cap m\mathbf{Q})$

$$\frac{1}{|m\mathbf{Q}|}\int_{\mathbb{R}^d}|\nabla^s(\mathcal{U}u)|^r\leq C\left(\frac{1}{m^d}\int_{\mathbf{P}\cap m\mathbf{Q}}|\nabla^s u|^p\right)^{\frac{r}{p}}.$$

In both cases for every $\beta\in(0,1)$ *the following holds: for some* $m_0>1$ *depending on* ω *and every* $m>m_0$ *the support of* $\mathcal{U}u$ *lies within* $\mathbb{B}_{m^{1-\beta}}(m\mathbf{Q})$.

Proof. The proof is given at the very end of Section 6.1. □

Corollary 1.17. *If* $\mathbb{P}(\delta(x,y)<\delta_0)\leq C_\delta e^{-\gamma\delta_0^{-1}}$ *then the last theorem holds for every* $1\leq r<p$.

In Section 6.2 we study the Boolean model based on a Poisson point process in the percolation case. Introduced in Example 2.48 we will consider a Poisson point process $\mathbb{X}_{\mathrm{pois}}(\omega)=(x_i(\omega))_{i\in\mathbb{N}}$ with intensity λ (recall Example 2.48). To each point x_i a random ball $B_i=\mathbb{B}_1(x_i)$ is assigned and the family $\mathbb{B}:=(B_i)_{i\in\mathbb{N}}$ is called the Poisson ball process. We say that $x_i\sim x_j$ if $|x_i-x_j|<2$. In case $\lambda>\lambda_c$ the union of these balls has a unique infinite connected component (that means we have percolation) and we denote $\mathbb{X}_{\mathrm{pois},\infty}$ the sellection of all points that contribute to the infinite component and $\mathbf{P}_\infty(\omega):=\bigcup_{i\in\mathbb{X}_{\mathrm{pois},\infty}}B_i$ this infinite open set and seek for a corresponding uniform extension operator. The connectedness of $\mathbf{P}_\infty$ is hereby essential. We use results from percolation theory that otherwise would not hold.

Here we can show that the micro- and mesoscopic assumptions are fulfilled, at least in case $\mathbf{P}$ is given as the union of balls. If we choose $\mathbf{P}$ as the complement of the balls, the situation becomes more involved. On one hand, Theorem 6.8 shows that α and n change in an unfortunate way. Furthermore, the connectivity estimate remains open. However, some of these problems might be overcome using a Matern modification of the Poisson process. For the moment, we state the following.

Theorem 1.18. *In the boolean model of Section 6.2 it holds* $\alpha=0$ *in case* $\mathbf{P}=\mathbf{P}_\infty$ *and both the extension order and the symmetric extension order are* $n=0$. *If* $d<p$ *and*

$$\frac{pr}{p-r}<2,\quad r<d+2$$

Then there almost surely exists an extension operator $\mathcal{U}\,:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P})\to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *and a constant* $C>0$ *such that for all* $m\in\mathbb{N}$ *and every* $u\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{P}\cap m\mathbf{Q})$

$$\frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}}|\nabla(\mathcal{U}u)|^r\leq C\left(\frac{1}{m^d}\int_{\mathbf{P}\cap m\mathbf{Q}}|\nabla u|^p\right)^{\frac{r}{p}}.$$

If furthermore

$$r<\frac{d+2}{2}$$

then there almost surele exists an extension operator $\mathcal{U}\,:\,\mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbf{P})\to\mathbf{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *and a constant* $C>0$ *such that for all* $m\in\mathbb{N}$ *and every* $u\in\mathbf{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{P}\cap m\mathbf{Q})$

$$\frac{1}{|m\mathbf{Q}|}\int_{m\mathbf{Q}}|\nabla^s(\mathcal{U}u)|^r\leq C\left(\frac{1}{m^d}\int_{\mathbf{P}\cap m\mathbf{Q}}|\nabla^s u|^p\right)^{\frac{r}{p}}.$$

In both cases for every $\beta\in(0,1)$ *the following holds: for some* $m_0>1$ *depending on* ω *and every* $m>m_0$ *the support of* $\mathcal{U}u$ *lies within* $\mathbb{B}_{m^{1-\beta}}(m\mathbf{Q})$.

Proof. The proof is given at the very end of Section 6.2. □

Notes

Structure of the article

We close the introduction by providing an overview over the article and its main contributions. In Section 2 we collect some basic concepts and inequalities from the theory of Sobolev spaces, random geometries and discrete and continuous ergodic theory. We furthermore establish local regularity properties for what we call η-regular sets, as well as a related covering theorem in Section 2.8. In Section 2.13 we will demonstrate that stationary ergodic random open sets induce stationary processes on $\mathbb{Z}^d$, a fact which is used later in the construction of the mesoscopic Voronoi tessellation in Section 3.2.

In Section 3 we introduce the regularity concepts of this work. More precisely, in Section 3.1 we introduce the concept of local (δ, M)-regularity and use the theory of Section 2.8 in order to establish a local covering result for $\partial\mathbf{P}$, which will allow us to infer most of our extension and trace results. In Section 3.2 we show how isotropic cone mixing geometries allow us to construct a stationary Voronoi tessellation of $\mathbb{R}^d$ such that all related quantities like "diameter" of the cells are stationary variables whose expectation can be expressed in terms of the isotropic cone mixing function f. Moreover we prove the important integration Lemma 3.18.

In Sections 4–5 we finally provide the aforementioned extension operators and prove estimates for these extension operators and for the trace operator. In Section 6 we study the sample geometries.

A Remark on Notation

This article uses concepts from partial differential equations, measure theory, probability theory and random geometry. Additionally, we introduce concepts which we believe have not been introduced before. This makes it difficult to introduce readable self contained notation (the most important aspect being symbols used with different meaning) and enforces the use of various different mathematical fonts. Therefore, we provide an index of notation at the end of this work. As a rough orientation, the reader may keep the following in mind:

We use the standard notation $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$ for natural (> 0), rational, real and integer numbers. $\mathbb{P}$ denotes a probability measure, $\mathbb{E}$ the expectation. Furthermore, we use special notation for some geometrical objects, i.e. $\mathbb{T}^d = [0,1)^d$ for the torus ($\mathbb{T}$ equipped with the topology of the torus), $\mathbb{I}^d = (0,1)^d$ the open interval as a subset of $\mathbb{R}^d$ (we often omit the index d), $\mathbb{B}$ a ball, $\mathbb{C}$ a cone and $\mathbb{X}$ a set of points. In the context of finite sets A, we write $\#A$ for the number of elements.

Bold large symbols ($\mathbf{U}$, $\mathbf{Q}$, $\mathbf{P}$,…) refer to open subsets of $\mathbb{R}^d$ or to closed subsets with $\partial\mathbf{P} = \partial\mathring{\mathbf{P}}$. The Greek letter Γ refers to a $d-1$ dimensional manifold (aside from the notion of Γ-convergence).

Calligraphic symbols ($\mathcal{A}$, $\mathcal{U}$, …) usually refer to operators and large Gothic symbols ($\mathfrak{B}, \mathfrak{C}, \dots$) indicate topological spaces, except for $\mathfrak{A}$.

Outlook

This work is the first part of a triology. In part II, we will see how to apply the extension and trace operators introduced above.

In part III we will discuss general quantifyable properties of the geometry that are eventually accessible also to computer algorithms that will allow to predict homogenization behavior of random geometries.

2 Preliminaries

We first collect some notation and mathematical concepts which will be frequently used throughout this paper. We first start with the standard geometric objects, which will be labeled by bold letters.

2.1 Fundamental Notation and Geometric Objects

Throughout this work, we use $(\mathbf{e}_i)_{i=1,\dots d}$ for the Euclidean basis of $\mathbb{R}^d$. By $C > 0$ we denote any constant that depends on p and d but no further dependencies unless explicitly mentioned. Such mentioning may expressed in some cases through the notation $C(a, b, \dots)$. Furthermore, we use the following notation.

Unit cube The torus $\mathbb{T} = [0, 1)^d$ is quipped with the topology of the metric

$$d(x, y) = \min_{z\in\mathbb{Z}^d} |x - y + z|$$

. In contrast, the open interval $\mathbb{I}^d := (0, 1)^d$ is considered as a subset of $\mathbb{R}^d$. We often omit the index d if this does not provoke confusion.

Balls Given a metric space (M, d) we denote $\mathbb{B}_r(x)$ the open ball around $x \in M$ with radius $r > 0$. The surface of the unit ball in $\mathbb{R}^d$ is $\mathbb{S}^{d-1}$. Furthermore, we denote for every $A \subset \mathbb{R}^d$ by $\mathbb{B}_r(A) := \bigcup_{x\in A} \mathbb{B}_r(x)$.

Points A sequence of points will be labeled by $\mathbb{X} := (x_i)_{i\in\mathbb{N}}$.

A cone in $\mathbb{R}^d$ is usually labeled by $\mathbb{C}$. In particular, we define for a vector ν of unit length, $0 < \alpha < \frac{\pi}{2}$ and $R > 0$ the cone

$$\mathbb{C}_{\nu,\alpha,R}(x) := \{z \in \mathbb{B}_R(x) \;:\; z\cdot\nu > |z|\cos\alpha\} \quad \text{and} \quad \mathbb{C}_{\nu,\alpha}(x) := \mathbb{C}_{\nu,\alpha,\infty}(x)\,.$$

Inner and outer hull We use balls of radius $r > 0$ to define for a closed set $\mathbf{P} \subset \mathbb{R}^d$ the sets

$$\begin{aligned} \mathbf{P}_r &:= \overline{\mathbb{B}_r(\mathbf{P})} := \left\{x \in \mathbb{R}^d \;:\; \operatorname{dist}(x, \mathbf{P}) \le r\right\}\,, \\ \mathbf{P}_{-r} &:= \mathbb{R}^d \backslash \left[\mathbb{B}_r\big(\mathbb{R}^d \setminus \mathbf{P}\big)\right] := \left\{x \in \mathbb{R}^d \;:\; \operatorname{dist}\big(x, \mathbb{R}^d \setminus \mathbf{P}\big) \ge r\right\}\,. \end{aligned} \tag{2.1}$$

One can consider these sets as inner and outer hulls of $\mathbf{P}$. The last definition resembles a concept of “negative distance” of $x \in \mathbf{P}$ to $\partial\mathbf{P}$ and “positive distance” of $x \notin \mathbf{P}$ to $\partial\mathbf{P}$. For $A \subset \mathbb{R}^d$ we denote $\operatorname{conv}(A)$ the closed convex hull of A.

The natural geometric measures we use in this work are the Lebesgue measure on $\mathbb{R}^d$, written $|A|$ for $A \subset \mathbb{R}^d$, and the k-dimensional Hausdorff measure, denoted by $\mathcal{H}^k$ on k-dimensional submanifolds of $\mathbb{R}^d$ (for $k \le d$).

2.2 Simple Local Extensions and Traces

In the following, we formulate some extension and trace results. Although it is well known how such results are proved and the proofs are standard, we include them for completeness since we are particularly interested in the dependence of the operator norm on the local Lipschitz regularity of the boundary.

The following is well known:

Lemma 2.1. *For every $1 \leq p \leq \infty$ there exists $C_p > 0$ such that for every $R > 0$ there exists an extension operator $\mathcal{U} : W^{1,p}(\mathbb{B}_R(0)) \to W^{1,p}(\mathbb{B}_{2R}(0))$ such that*

$$\|\nabla \mathcal{U} u\|_{L^p(\mathbb{B}_{2R}(0))} \leq C_p \|\nabla u\|_{L^p(\mathbb{B}_R(0))} .$$

Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set and let $p \in \partial\mathbf{P}$ and $\delta > 0$ be a constant such that $\mathbb{B}_\delta(p) \cap \partial\mathbf{P}$ is graph of a Lipschitz function. We denote

$$M(p,\delta) := \inf \big\{ M : \ \exists \phi : U \subset \mathbb{R}^{d-1} \to \mathbb{R} \\ \phi \text{ Lipschitz, with constant } M \text{ s.t. } \mathbb{B}_\delta(p) \cap \partial\mathbf{P} \text{ is graph of } \phi \big\} . \tag{2.2}$$

Remark 2.2. For every p, the function $M(p,\cdot)$ is monotone increasing in δ.

Lemma 2.3 (Uniform Extension for Balls)**.** *Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set, $0 \in \partial\mathbf{P}$ and assume there exists $\delta > 0$, $M > 0$ and an open domain $U \subset \mathbb{B}_\delta(0) \subset \mathbb{R}^{d-1}$ such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(0)$ is graph of a Lipschitz function $\varphi : U \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ of the form $\varphi(\tilde{x}) = (\tilde{x}, \phi(\tilde{x}))$ in $\mathbb{B}_\delta(0)$ with Lipschitz constant M and $\varphi(0) = 0$. Writing $x = (\tilde{x}, x_d)$ and defining $\rho = \delta\sqrt{4M^2+2}^{-1}$ there exist an extension operator*

$$(\mathcal{U}u)(x) = \begin{cases} u(x) & \text{if } x_d < \phi(\tilde{x}) \\ u(\tilde{x}, -x_d + 2\phi(\tilde{x})) & \text{if } x_d > \phi(\tilde{x}) \end{cases} , \tag{2.3}$$

such that for

$$\mathcal{A}(0,\mathbf{P},\rho) := \{(\tilde{x}, -x_d + 2\phi(\tilde{x})) : (\tilde{x}, x_d) \in \mathbb{B}_\rho(0)\backslash\mathbf{P}\} \subset \mathbb{B}_\delta(0) , \tag{2.4}$$

and for every $p \in [1,\infty]$ the operator

$$\mathcal{U} : W^{1,p}(\mathcal{A}(0,\mathbf{P},\rho)) \to W^{1,p}(\mathbb{B}_\rho(0)) ,$$

is continuous with

$$\|\mathcal{U}u\|_{L^p(\mathbb{B}_\rho(0)\backslash\mathbf{P})} \leq \|u\|_{L^p(\mathcal{A}(0,\mathbf{P},\rho))} , \qquad \|\nabla\mathcal{U}u\|_{L^p(\mathbb{B}_\rho(0)\backslash\mathbf{P})} \leq 2M \|\nabla u\|_{L^p(\mathcal{A}(0,\mathbf{P},\rho))} . \tag{2.5}$$

Remark 2.4. In case $\phi(\tilde{x}) \geq 0$ we find $\mathcal{A}(0,\mathbf{P},\rho) \subset \mathbb{B}_\rho(0)$.

Proof of Lemma 2.3. In case $\phi(\tilde{x}) \equiv 0$ we consider the extension operator $\mathcal{U}_+ : W^{1,p}(\mathbb{R}^{d-1} \times (-\infty,0)) \to W^{1,p}(\mathbb{R}^d)$ having the form (compare also [6, chapter 5], [1])

$$(\mathcal{U}_+ u)(x) = \begin{cases} u(x) & \text{if } x_d < 0 \\ u(\tilde{x}, -x_d) & \text{if } x_d > 0 \end{cases} .$$

The general case follows from transformation. □

Lemma 2.5. *Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set, $0 \in \partial\mathbf{P}$ and assume there exists $\delta > 0$, $M > 0$ and an open domain $U \subset \mathbb{B}_\delta(0) \subset \mathbb{R}^{d-1}$ such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(0)$ is graph of a Lipschitz function $\varphi : U \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ of the form $\varphi(\tilde{x}) = (\tilde{x}, \phi(\tilde{x}))$ in $\mathbb{B}_\delta(0)$ with Lipschitz constant M and $\varphi(0) = 0$ and define $\rho = \delta\sqrt{4M^2+2}^{-1}$. Writing $x = (\tilde{x}, x_d)$ we consider the trace operator $\mathcal{T} : C^1(\mathbf{P} \cap \mathbb{B}_\delta(0)) \to C(\partial\mathbf{P} \cap \mathbb{B}_\rho(0))$. For every $p \in [1,\infty]$ and every $r < \frac{p(1-d)}{(p-d)}$ the operator $\mathcal{T}$ can be continuously extended to*

$$\mathcal{T} : W^{1,p}(\mathbf{P} \cap \mathbb{B}_\delta(0)) \to L^r(\partial\mathbf{P} \cap \mathbb{B}_\rho(0)) ,$$

such that

$$\|\mathcal{T}u\|_{L^r(\partial\mathbf{P}\cap\mathbb{B}_\rho(0))} \leq C_{r,p}\rho^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}+1} \|u\|_{W^{1,p}(\mathbf{P}\cap\mathbb{B}_\delta(0))} . \tag{2.6}$$

Proof. We proceed similar to the proof of Lemma 2.3.

Step 1: Writing $B_r = \mathbb{B}_r(0)$ together with $B_r^- = \{x \in B_r : x_d < 0\}$ and $\Sigma_r := \{x \in B_r : x_d = 0\}$ we recall the standard estimate

$$\left(\int_{\Sigma_1} |u|^r\right)^{\frac{1}{r}} \leq C_{r,p}\left(\left(\left(\int_{B_1^-} |\nabla u|^p\right)^{\frac{1}{p}} + \left(\int_{B_1^-} |u|^p\right)^{\frac{1}{p}}\right)\right) ,$$

which leads to

$$\left(\int_{\Sigma_\rho} |u|^r\right)^{\frac{1}{r}} \leq C_{r,p}\rho^{\frac{d(p-r)}{rp}-\frac{1}{r}}\left(\rho\left(\int_{B_\rho^-} |\nabla u|^p\right)^{\frac{1}{p}} + \left(\int_{B_\rho^-} |u|^p\right)^{\frac{1}{p}}\right) .$$

Step 2: Using the transformation rule and the fact that $1 \leq |\det D\varphi| \leq \sqrt{4M^2+2}$ we infer (2.6) similar to Step 2 in the proof of Lemma 2.3.

$$\begin{aligned}
\left(\int_{\partial\mathbf{P}\cap\mathbb{B}_\rho(0)} |u|^r\right)^{\frac{1}{r}} &\leq \sqrt{4M^2+2}^{\frac{1}{r}}\left(\int_{\Sigma_\rho} |u\circ\varphi|^r\right)^{\frac{1}{r}} \\
&\leq C_{r,p}\rho^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}}\left(\rho\left(\int_{B_\rho^-} |\nabla(u\circ\varphi)|^p\right)^{\frac{1}{p}} + \left(\int_{B_\rho^-} |u\circ\varphi|^p\right)^{\frac{1}{p}}\right) \\
&\leq C_{r,p}\rho^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}+1}\cdot \\
&\quad\cdot\left(\rho\left(\int_{B_\rho^-} |(\nabla u)\circ\varphi|^p \det D\varphi\right)^{\frac{1}{p}} + \left(\int_{B_\rho^-} |u\circ\varphi|^p \det D\varphi\right)^{\frac{1}{p}}\right)
\end{aligned}$$

and from this we conclude the Lemma with $\varphi^{-1}(B_\rho^-) \subset \mathbb{B}_\delta(0)$. □

2.3 Local Nitsche-Extensions

In this work, we will use bold letters for $\mathbb{R}^d$-valued function spaces. In particular, we introduce for $1 \leq p \leq \infty$

$$\begin{aligned}
\mathbf{L}^p(\mathbf{Q}) &:= L^p(u;\mathbb{R}^d) , \\
\mathbf{W}^{1,p}(\mathbf{Q}) &:= \left\{u \in \mathbf{L}^p(\mathbf{Q}) \,:\, \nabla u \in L^p\big(\mathbf{Q};\mathbb{R}^{d\times d}\big)\right\} .
\end{aligned}$$

From [5] we know that on general Lipschitz domains an estimate like the following holds:

Lemma 2.6. *For every $1 \leq p \leq \infty$ there exists a constant $C > 0$ depending only on the dimension $d \geq 2$ such that the following holds: For every radius $R > 0$ there exists an extension operator $\mathcal{U}_R : W^{1,p}(\mathbb{B}_R(0)) \to W^{1,p}(\mathbb{B}_{2R}(0))$ such that*

$$\|\nabla^s(\mathcal{U}_R u)\|_{W^{1,p}(\mathbb{B}_{2R}(0))} \leq C\,\|\nabla^s u\|_{W^{1,p}(\mathbb{B}_R(0))} .$$

Again, we will need a refined estimate on extensions on Lipschitz domains which explicitly accounts for the local Lipschitz constant.

Lemma 2.7 (Uniform Nitsche-Extension for Balls)**.** *For every $d \geq 2$ there exists a constant $C_{\mathcal{N}}$ depending only on the dimension d such that the following holds: Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set, $0 \in \partial\mathbf{P}$ and assume there exists $\delta > 0$, $M > 0$ and an open domain $U \subset \mathbb{B}_\delta(0) \subset \mathbb{R}^{d-1}$ such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(0)$ is graph of a Lipschitz function $\varphi : U \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ of the form $\varphi(\tilde{x}) = (\tilde{x}, \phi(\tilde{x}))$ in $\mathbb{B}_\delta(0)$ with Lipschitz constant M and $\varphi(0) = 0$. Writing $x = (\tilde{x}, x_d)$ and defining $\rho = \delta\sqrt{4M^2+2}^{-1}$ and*

$$\mathcal{A}(0,\mathbf{P},\rho) := \left\{(\tilde{x},x_d) \in \mathbf{P} \,:\, |\tilde{x}| < \rho,\, x_d \leq C_{\mathcal{N}}(1+M^2)\right\}\,, \tag{2.7}$$

and for every $p \in [1,\infty]$ there exists a continuous operator

$$\mathcal{U} : W^{1,p}(\mathcal{A}(0,\mathbf{P},\rho)) \to W^{1,p}(\mathbb{B}_\rho(0))\,,$$

such that for some constant C independent from (δ, M) and $\mathbf{P}$ it holds

$$\|\nabla^s \mathcal{U} u\|_{L^p(\mathbb{B}_\rho(0)\backslash\mathbf{P})} \leq C\,(1+M)^2\,\|\nabla^s u\|_{L^p(\mathcal{A}(0,\mathbf{P},\rho))}\,. \tag{2.8}$$

Remark 2.8. In case $\phi(\tilde{x}) \geq 0$ the proof reveals $\mathcal{A}(0,\mathbf{P},\rho) \subset \mathbb{B}_{c\rho}(0)$ for some c depending only on the dimension d.

In order to prove such a result we need the following lemma.

Lemma 2.9 ([26] Chapter 6 Section 1 Theorem 2)**.** *There exist constants $c_1, c_2, c_3 > 0$ such that for every open set $\mathbf{P} \subset \mathbb{R}^d$ with local Lipschitz boundary there exists a function $d_{\mathbf{P}} : \mathbb{R}^d\backslash\overline{\mathbf{P}^\complement} \to \mathbb{R}$ with*

$$\begin{aligned}
c_1 d_{\mathbf{P}}(x) &\leq \operatorname{dist}(x,\mathbf{P}) \leq c_2 d_{\mathbf{P}}(x)\,,\\
\forall i \in \{1,\dots,d\} : \quad & |\partial_i d_{\mathbf{P}}(x)| \leq c_3\,,\\
\forall i,k \in \{1,\dots,d\} : \quad & |\partial_i\partial_k d_{\mathbf{P}}(x)| \leq c_3\,|d_{\mathbf{P}}(x)|^{-1}\,.
\end{aligned}$$

From the theory presented by Stein [26] we will not get an explicit form of $C_{\mathcal{N}}$ but only an upper bound that grows exponentially with dimension d.

Proof of Lemma 2.7. We use an idea by Nitsche [20], which we transfer from $p = 2$ to the general case, thereby explicitly quantifying the influence of M. For simplicity we write $\mathbf{P}_\delta := \mathbf{P} \cap \mathbb{B}_\delta(0)$ and $\mathbf{P}_\delta^\complement := \mathbb{B}_\delta(0)\backslash\mathbf{P}$ and assume that $x \in \mathbf{P}_\delta$ iff $x \in \mathbb{B}_\delta(0)$ and $x_d < \phi(\tilde{x})$.

As observed by Nitsche, it holds

$$\forall x \in \mathbf{P}_\delta^\complement : \quad 0 < \left(1+M^2\right)^{-\frac{1}{2}}(x_d - \phi(\tilde{x})) \leq \operatorname{dist}(x,\partial\mathbf{P}) \leq x_d - \phi(\tilde{x})\,,$$

and together with Lemma 2.9, we can define $d_{\mathbf{P},M}(x) := 2c_2\,(1+M^2)^{\frac{1}{2}}\,d_{\mathbf{P}}(x)$ and find for $c > \max\left\{\frac{2c_2}{c_1}, 4c_2c_3\right\}$ that

$$\begin{aligned}
2\,(x_d - \phi(\tilde{x})) \leq \mathrm{d}_{\mathbf{P},M}(x) &\leq c\left(1+M^2\right)^{\frac{1}{2}}(x_d-\phi(\tilde{x}))\,,\\
\forall i \in \{1,\dots,d\} : \quad |\partial_i d_{\mathbf{P},M}(x)| &\leq c\left(1+M^2\right)^{\frac{1}{2}}\,,\\
\forall i,k \in \{1,\dots,d\} : \quad |\partial_i\partial_k d_{\mathbf{P},M}(x)| &\leq c\left(1+M^2\right)|d_{\mathbf{P},M}(x)|^{-1}\,.
\end{aligned}$$

If $\psi \in C([1,2])$ satisfies

$$\int_1^2 \psi(t)\,\mathrm{d}t = 1\,, \qquad \int_1^2 t\,\psi(t)\,\mathrm{d}t = 0. \tag{2.9}$$

Nitsche introduced $x_\lambda := (\tilde{x}, x_d - \lambda d_{\mathbf{P},M}(x))$ and proposed the following extension on $x \in \mathbf{P}_\delta^\complement$:

$$u_i(x) := \int_1^2 \psi(\lambda)\left(u_i(x_\lambda) + \lambda u_d(x_\lambda)\,\partial_i d_{\mathbf{P},M}(x)\right)\mathrm{d}\lambda\,.$$

One can quickly verify that this maps $C(\overline{\mathbf{P}_\delta})$ onto $C\left(\overline{\mathbb{B}_\rho(0)}\right)$. In what follows, we write $\varepsilon[u](x) := \nabla^s u(x)$ and particularly $\varepsilon_{ij}[u](x) := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$ as well as $\varepsilon_{ij}^\lambda[u](x) = \varepsilon_{ij}[u](x_\lambda)$ for $x \in \mathbf{P}_\delta^\complement$. Then for $x \in \mathbf{P}_\delta^\complement \cap \mathbb{B}_\rho(0)$

$$\varepsilon_{ij}[u](x) = \int_1^2 \psi(\lambda)\left(\varepsilon_{ij}^\lambda(x) + \lambda\partial_i d_{\mathbf{P},M}(x)\,\varepsilon_{jd}^\lambda(x) + \lambda\partial_j d_{\mathbf{P},M}(x)\,\varepsilon_{id}^\lambda(x)\right. \tag{2.10}$$

$$\left. + \lambda^2\partial_i d_{\mathbf{P},M}(x)\,\partial_j d_{\mathbf{P},M}(x)\,\varepsilon_{dd}^\lambda(x) + \lambda\partial_i\partial_j d_{\mathbf{P},M}(x)\,u_d(x_\lambda)\right) \tag{2.11}$$

From the fundamental theorem of calculus we find

$$u_d(x_\lambda) = u_d(x_1) + \delta(\tilde{x})\int_1^\lambda \partial_d u_d(x_t)\,\mathrm{d}t\,,$$

which leads by (2.9) to

$$\int_1^2 \psi(\lambda)\lambda\partial_i\partial_j d_{\mathbf{P},M}(x)\,u_d(x_\lambda)\,\mathrm{d}\lambda = \partial_i\partial_j d_{\mathbf{P},M}(x)\,d_{\mathbf{P},M}(x)\int_1^2 \varepsilon_{dd}[u](x_t)\,\mathrm{d}t\int_\mu^2 \psi(\lambda)\lambda\,\mathrm{d}\lambda\,.$$

We may now apply $|\cdot|^p$ on both sides of (2.10), integrate over $\mathbf{P}_\delta^\complement \cap \mathbb{B}_\rho(0)$ and use the integral transformation theorem for each λ to find

$$\|\varepsilon[u]\|_{L^p\left(\mathbf{P}_\delta^\complement\cap\mathbb{B}_\rho(0)\right)} \leq C\left(1+M^2\right)\|\varepsilon[u]\|_{L^p(\mathbf{P}_\delta)}\,.$$

□

2.4 Poincaré Inequalities

We denote for bounded open $A \subset \mathbb{R}^d$

$$W^{1,p}_{(0),r}(A) := \left\{u \in W^{1,p}(A)\;:\;\exists x:\;B_r(x)\subset A\;\vee\;\fint_{B_r(x)} u = 0\right\}.$$

Note that this is not a linear vector space.

Lemma 2.10. *For every $p \in [1,\infty)$ there exists $C_p > 0$ such that the following holds: Let $0 < r < R$ and $x \in \mathbb{B}_R(0)$ such that $\mathbb{B}_r(x) \subset \mathbb{B}_R(0)$ then for every $u \in W^{1,p}(\mathbb{B}_R(0))$*

$$\|u\|^p_{L^p(\mathbb{B}_R(0))} \leq C_p\left(R^p\frac{R^{d-1}}{r^{d-1}}\|\nabla u\|^p_{L^p(\mathbb{B}_R(0))} + \frac{R^d}{r^d}\|u\|^p_{L^p(\mathbb{B}_r(x))}\right), \tag{2.12}$$

and for every $u \in W^{1,p}_{(0),r}((\mathbb{B}_R(0))$ it holds

$$\|u\|^p_{L^p(\mathbb{B}_R(0))} \leq C_p R^p\left(\frac{r}{R}\right)^{1-d}\left(1+\left(\frac{r}{R}\right)^{p-1}\right)\|\nabla u\|^p_{L^p(\mathbb{B}_R(0))}\,. \tag{2.13}$$

Remark. In case $p \geq d$ we find that (2.13) holds iff $u(x) = 0$ for some $x \in \mathbb{B}_1(0)$.

Proof. In a first step, we assume $x = 0$ and $R = 1$. The underlying idea of the proof is to compare every $u(y)$, $y \in \mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ with $u(rx)$. In particular, we obtain for $y \in \mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ that

$$u(y) = u(ry) + \int_0^1 \nabla u(ry + t(1-r)y) \cdot (1-r)y \, \mathrm{d}t$$

and hence by Jensen's inequality

$$|u(y)|^p \leq C \left(\int_0^1 |\nabla u(ry + t(1-r)y)|^p \, (1-r)^p \, |y|^p \, \mathrm{d}t + |u(ry)|^p \right) .$$

We integrate the last expression over $\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ and find

$$\begin{aligned}
\int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(y)|^p \, \mathrm{d}y &\leq \int_{S^{d-1}} \int_r^1 C \left(\int_0^1 |\nabla u(rs\nu + t(1-r)s\nu)|^p \, (1-r)^p s^p \, \mathrm{d}t \right) s^{d-1} \mathrm{d}s \mathrm{d}\nu \\
&\quad + \int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(ry)|^p \, \mathrm{d}y \\
&\leq \int_{S^{d-1}} \int_r^1 C \left(\int_{rs}^s |\nabla u(t\nu)|^p \, (1-r)^{p-1} s^{p-1} \, \mathrm{d}t \right) s^{d-1} \mathrm{d}s \\
&\quad + \int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(ry)|^p \, \mathrm{d}y \\
&\leq C \int_r^1 \mathrm{d}s \; s^{d-1} \frac{1}{(rs)^{d-1}} \int_{rs}^s \mathrm{d}t \; t^{d-1} \int_{S^{d-1}} |\nabla u(t\nu)|^p \, (1-r)^{p-1} s^{p-1} \\
&\quad + \int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(ry)|^p \, \mathrm{d}y \\
&\leq C \frac{1}{r^{d-1}} \left\| \nabla u \right\|^p_{L^p(\mathbb{B}_1(0))} + \frac{1}{r^d} \left\| u \right\|^p_{L^p(\mathbb{B}_r(0))} .
\end{aligned}$$

For general $x \in \mathbb{B}_1(0)$, use the extension operator $\mathcal{U} : W^{1,p}(\mathbb{B}_1(0)) \to W^{1,p}(B_4(0))$ such that $\left\| \mathcal{U}u \right\|_{W^{1,p}(B_4(0))} \leq C \left\| u \right\|_{W^{1,p}(\mathbb{B}_1(0))}$ and $\left\| \nabla \mathcal{U}u \right\|_{W^{1,p}(B_4(0))} \leq C \left\| \nabla u \right\|_{W^{1,p}(\mathbb{B}_1(0))}$. Since $\mathbb{B}_1(0) \subset B_2(x) \subset B_4(0)$ we infer

$$\left\| u \right\|^p_{L^p(\mathbb{B}_1(0))} \leq \left\| \mathcal{U}u \right\|^p_{L^p(B_2(x))} \leq C \left(\frac{1}{r^{d-1}} \left\| \nabla \mathcal{U}u \right\|^p_{L^p(B_2(x))} + \frac{1}{r^d} \left\| \mathcal{U}u \right\|^p_{L^p(B_r(x))} \right) .$$

and hence (2.12). Furthermore, since there holds $\left\| u \right\|^p_{L^p(\mathbb{B}_1(0))} \leq C \left\| \nabla u \right\|^p_{L^p(\mathbb{B}_1(0))}$ for every $u \in W^{1,p}_{(0)}(\mathbb{B}_1(0))$, a scaling argument shows $\left\| u \right\|^p_{L^p(\mathbb{B}_r(0))} \leq C r^p \left\| \nabla u \right\|^p_{L^p(\mathbb{B}_r(0))}$ for every $u \in W^{1,p}_{(0),r}(\mathbb{B}_1(0))$ and hence (2.13). For general $R > 0$ use a scaling argument. □

A similar argument leads to the following, where we remark that the difference in the appearing of $\frac{1}{r}$ is due to the fact, that integrating the cylinder needs no surface element r^{d-1}.

Corollary 2.11. *For every $p \in [1, \infty)$ and $r > 0$ there exists $C_p > 0$ such that the following holds: Let $r < L$, $P_{L,r} := \mathbb{B}^{d-1}_r(0) \times (0, L)$ and $x \in P_{L,r}$ such that $\mathbb{B}_r(x) \subset P_{L,r}$ then for every $u \in W^{1,p}(P_{L,r})$*

$$\left\| u \right\|^p_{L^p(P_{L,r})} \leq C_p \left(L^p \left\| \nabla u \right\|^p_{L^p(P_{L,r})} + \frac{L}{r} \left\| u \right\|^p_{L^p(\mathbb{B}_r(x))} \right) , \tag{2.14}$$

and if additionally $\fint_{\mathbb{B}_r(x)} u = 0$ *then*

$$\|u\|^p_{L^p(P_{L,r})} \leq C_p \left(L^p \|\nabla u\|^p_{L^p(P_{L,r})} + Lr^{p-1} \|\nabla u\|^p_{L^p(\mathbb{B}_r(x))} \right) , \tag{2.15}$$

Let $y \in P_{L,r}$ *such that* $\mathbb{B}_r(y) \subset P_{L,r}$ *then for every* $u \in W^{1,p}(P_{L,r})$

$$\left| \fint_{\mathbb{B}_r(y)} u - \fint_{\mathbb{B}_r(x)} u \right|^p \leq C_p \left(L^{p-1} r^{1-d} \|\nabla u\|^p_{L^p(P_{L,r})} \right) . \tag{2.16}$$

2.5 Korn Inequalities

We introduce on open sets $A \subset \mathbb{R}^d$ the Sobolev space

$$\mathbf{W}^{1,p}_{\nabla^\perp(0)}(A) := \left\{ u \in \mathbf{W}^{1,p}(A) : \ \forall i,j : \int_A \partial_i u_j - \partial_j u_i = 0 \right\} .$$

To the authors best knowledge, the following is the most general Korn inequality in literature.

Theorem 2.12 ([5] Theorem 2.7 and Corollary 2.8)**.** *Let* $1 \leq p \leq \infty$ *and* $\varepsilon \in (0,1)$ *and* $\tilde{\delta} > 0$*. Then there exists a constant* $C_p > 0$ *depending only on* d*,* p*,* ε *and* $\tilde{\delta}$ *such that for every bounded open set* $A \in \mathbb{R}^d$ *with* $\delta > 0$ *such that* $\delta / \mathrm{diam} A \geq \tilde{\delta}$ *and with the property*

$$\left. \begin{aligned} &\forall x, y \in A, \ |x-y| < \delta : \quad \exists \gamma \in C^1([0,1];A), \ \gamma(0) = x, \ \gamma(1) = y \text{ such that:} \\ &l(\gamma) \leq \frac{1}{\varepsilon} |x-y| \quad \text{and} \quad \forall t \in (0,1) : \ \mathrm{dist}(\gamma(t), \partial A) \geq \frac{\varepsilon \, |x-\gamma(t)| \, |y-\gamma(t)|}{x-y} \end{aligned} \right\} \tag{2.17}$$

it holds

$$\forall u \in \mathbf{W}^{1,p}_{\nabla^\perp(0)}(A) : \ \|\nabla u\|_{L^p(A)} \leq C_p \|\nabla^s u\|_{L^p(A)} . \tag{2.18}$$

Remark 2.13. In the original work the claimed dependence of C_p was on d, p, ε, δ and A with the observation that (2.18) is invariant under scaling of A. However, this scale invariance results in the dependence on d, p, ε and $\delta / \mathrm{diam} A$ since ε, p and d are not sensitive to scaling of A.

Definition 2.14. Domains $A \subset \mathbb{R}^d$ satisfying (2.17) for some $\varepsilon \in (0,1)$ and $\delta > 0$ are called (ε, δ)-John domains or simply John domains.

Corollary 2.15. *For every* $1 \leq p \leq \infty$ *there exists* C_p *depending only on* d *and* p *such that for every bounded open convex set* $A \subset \mathbb{R}^d$ *the estimate (2.18) holds.*

We furthermore introduce the set

$$\mathbf{W}^{1,p}_{\nabla^\perp(0),r}(A) := \left\{ u \in \mathbf{W}^{1,p}(A) \ : \ \exists x : \mathbb{B}_r(x) \subset A \ \vee \ \forall i,j : \int_{\mathbb{B}_r(x)} \partial_i u_j - \partial_j u_i = 0 \right\}$$

which is **not** a vector space.

Lemma 2.16 (Mixed Korn inequality)**.** *Let* $1 \leq p \leq \infty$ *and* $\varepsilon, \delta \in (0,1)$*. Then there exists a constant* $\tilde{C}_p > 0$ *depending only on* d*,* p*,* ε *and* δ *such that for every* (ε, δ)*-John domain* $A \subset \mathbb{B}_1(0)$ *and for every* $r \in (0,1)$ *and every* $x \in A$ *with* $\mathbb{B}_r(x) \subset A$ *it holds*

$$\forall u \in \mathbf{W}^{1,p}(A) : \ \|\nabla u\|_{L^p(A)} \leq \tilde{C}_p \left(\frac{|A|}{r^d} \right)^{\frac{1}{p}} \left(\|\nabla^s u\|_{L^p(A)} + \|\nabla u\|_{L^p(\mathbb{B}_r(x))} \right) . \tag{2.19}$$

Furthermore,

$$\forall u \in \mathbf{W}^{1,p}_{\nabla^\perp(0),r}(A) : \ \|\nabla u\|_{L^p(A)} \leq \tilde{C}_p \left(\frac{|A|}{|\mathbb{S}^{d-1}| \, r^d} \right)^{\frac{1}{p}} \left(\|\nabla^s u\|_{L^p(A)} \right) . \tag{2.20}$$

Unfortunately, we do not have a reference for a comparable Lemma in the literature except for [25] in case $p = 2$. The author strongly supposes a proof must exist somewhere, however, we provide it for completeness.

Proof. Let C_p be the constant from Theorem 2.12 for domains with a diameter less than 2 and suppose (2.19) was wrong. Then there exists a sequence of (ε,δ)-John domains $A_n \subset \mathbb{B}_1(0)$ with $x_n \in A_n$, $r_n \in (0,1)$ with $\mathbb{B}_{r_n}(x_n) \subset A_n$ and functions $u_n \in \mathbf{W}^{1,p}(A_n)$ such that

$$1 = \|\nabla u_n\|_{L^p(A_n)} \geq C_p \left(\frac{|A_n|}{|\mathbb{S}^{d-1}|\, r_n^d}\right)^{\frac{1}{p}} n \left(\|\nabla^s u_n\|_{L^p(A_n)} + \|\nabla u_n\|_{L^p(\mathbb{B}_{r_n}(x_n))}\right) .$$

We define $\overline{\nabla_n^\perp}(u_n) := \fint_{A_n} (\nabla u_n - \nabla^s u_n)$ and $u_{n,\perp}(x) := u_n(x) - \overline{\nabla_n^\perp}(u_n)\, x$ with $\nabla^s u_{n,\perp} = \nabla^s u_n$. Hence by (2.18)

$$\left\|\nabla u_n - \overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} \leq C_p \|\nabla^s u_{n,\perp}\|_{L^p(A_n)} = C_p \|\nabla^s u_n\|_{L^p(A_n)} .$$

We directly infer with $C_n := \frac{|A_n|}{|\mathbb{S}^{d-1}| r_n^d}$

$$C_p C_n^{\frac{1}{p}} \left(\|\nabla^s u_n\|_{L^p(A_n)} + \|\nabla u_n\|_{L^p(\mathbb{B}_{r_n}(x_n))}\right) \to 0 , \qquad C_n^{\frac{1}{p}} \left\|\nabla u_n - \overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} \to 0 . \tag{2.21}$$

Furthermore, we find

$$\begin{aligned} 1 = \|\nabla u_n\|_{L^p(A_n)} &\geq \left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} - \left\|\nabla u_n - \overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} , \\ \left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} &\geq \|\nabla u_n\|_{L^p(A_n)} - \left\|\nabla u_n - \overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} , \end{aligned}$$

and hence $\left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} \to 1$ due to (2.21). Since $\overline{\nabla_n^\perp}(u_n)$ are constant, it holds

$$C_n \left\|\overline{\nabla_n^\perp}(u_n)\right\|^p_{L^p(\mathbb{B}_{r_n}(x_n))} = \left\|\overline{\nabla_n^\perp}(u_n)\right\|^p_{L^p(A_n)}$$

and we infer from a similar calculation

$$\begin{aligned} C_n^{\frac{1}{p}} \left(\|\nabla u_n\|_{L^p(\mathbb{B}_{r_n}(x_n))} + \left\|\nabla u_n - \overline{\nabla_n^\perp}(u_n)\right\|_{L^p(\mathbb{B}_{r_n}(x_n))}\right) &\geq C_n^{\frac{1}{p}} \left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(\mathbb{B}_{r_n}(x_n))} \\ &\geq \left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} . \end{aligned}$$

This implies $\left\|\overline{\nabla_n^\perp}(u_n)\right\|_{L^p(A_n)} \to 0$ by (2.21), a contradiction. Hence, (2.19) holds with $\tilde{C}_p = nC_p$ for some $n \in \mathbb{N}$.

Estimate (2.20) now follows from (2.19) and (2.18) and the definition of $\mathbf{W}^{1,p}_{\nabla^\perp(0),r}(\mathbb{B}_R(0))$. □

2.6 Korn-Poincaré Inequalities

Generalizing the above Korn inequality to a Korn-Poincaré inequality, we define

$$\mathbf{W}^{1,p}_{(0),\nabla^\perp(0),r}(\mathbb{B}_R(0)) := \Big\{u \in \mathbf{W}^{1,p}(\mathbb{B}_r(0)) \,:\, \exists x :\, B_r(x) \subset \mathbb{B}_R(0) \vee \int_{\mathbb{B}_r(x)} u_i = 0 \,\vee\, \forall i,j :\, \int_{\mathbb{B}_r(x)} \partial_i u_j - \partial_j u_i = 0 \Big\} .$$

Lemma 2.17 (Mixed Korn-Poincaré inequality on balls)**.** *For every $p \in [1, \infty)$ there exists $C_p > 0$ such that for every $R > 0$, $r \in (0, R)$ and every $x \in \mathbb{B}_R(0)$ with $\mathbb{B}_r(x) \subset \mathbb{B}_R(0)$ it holds*

$$\forall u \in \mathbf{W}^{1,p}_{(0),\nabla^\perp(0),r}(\mathbb{B}_R(0)) : \quad \|\nabla u\|^p_{L^p(\mathbb{B}_R(0))} \leq C_p \left(\frac{R}{r}\right)^d \|\nabla^s u\|^p_{L^p(\mathbb{B}_R(0))} , \tag{2.22}$$

$$\|u\|^p_{L^p(\mathbb{B}_R(0))} \leq C_p \left(\frac{R}{r}\right)^{2d-1} \left(1 + \left(\frac{R}{r}\right)^{1-p}\right) R^p \|\nabla^s u\|^p_{L^p(\mathbb{B}_R(0))} . \tag{2.23}$$

Proof. Apply Lemma 2.16 for $R = 1$ and use a simple scaling argument to obtain

$$\|\nabla u\|_{L^p(\mathbb{B}_R(0))} \leq C_p \left(\frac{R}{r}\right)^d \left(\|\nabla^s u\|_{L^p(\mathbb{B}_r(0))}\right) .$$

Afterwards apply Lemma 2.10. □

Lemma 2.18 (Mixed Korn-Poincaré inequality on cylinders)**.** *For every $p \in [1, \infty)$ and $r > 0$ there exists $C_p > 0$ such that the following holds: Let $r < L$, $P_{L,r} := (0, L) \times \mathbb{B}^{d-1}_r(0)$ and $x \in P_{L,r}$ such that $\mathbb{B}_r(x) \subset P_{L,r}$ then for every $u \in \mathbf{W}^{1,p}(P_{L,r})$*

$$\|\nabla u\|^p_{L^p(P_{L,r})} \leq C_p \left(\left(\frac{L}{r}\right)^p \|\nabla^s u\|^p_{L^p(P_{L,r})} + \frac{L}{r} \|\nabla u\|^p_{L^p(\mathbb{B}_r(x))}\right) . \tag{2.24}$$

Furthermore,

$$\|u\|^p_{L^p(P_{L,r})} \leq C_p \left(\frac{L^{2p}}{r^p} \|\nabla^s u\|^p_{L^p(P_{L,r})} + \frac{L^{p+1}}{r} \|\nabla u\|^p_{L^p(\mathbb{B}_r(x))} + \frac{L}{r} \|u\|^p_{L^p(\mathbb{B}_r(x))}\right) , \tag{2.25}$$

and if additionally $u \in \mathbf{W}^{1,p}_{(0),\nabla^\perp(0),r}(P_{L,r})$ then

$$\|\nabla u\|^p_{L^p(P_{L,r})} \leq C_p \frac{L^p}{r^p} \|\nabla^s u\|^p_{L^p(P_{L,r})} , \quad \|u\|^p_{L^p(P_{L,r})} \leq C_p \frac{L^{2p}}{r^p} \|\nabla^s u\|^p_{L^p(P_{L,r})} , \tag{2.26}$$

Defining $\overline{\nabla^\perp_{a,\delta}} u := ⨍_{\mathbb{B}_\delta(a)} (\nabla u - \nabla^s u)$ and

$$\left[\mathcal{M}^{\mathfrak{s},\delta}_a u\right](x) := \overline{\nabla^\perp_{a,\delta}} u(x - a) + ⨍_{\mathbb{B}_\delta(a)} u \tag{2.27}$$

we find for a, b with $\mathbb{B}_\delta(a), \mathbb{B}_\delta(b) \subset P_{L,r}$ for every $u \in \mathbf{W}^{1,p}(P_{L,r})$ that

$$\left|\left[\mathcal{M}^{\mathfrak{s},\delta}_a u\right](x) - \left[\mathcal{M}^{\mathfrak{s},\delta}_b u\right](x)\right|^p \leq C |x - a|^p \frac{|a - b|^{2p}}{\delta^{p+d}} \left(\int_{\mathrm{conv}(\mathbb{B}_\delta(a) \cup \mathbb{B}_\delta(b))} |\nabla^s u|^p\right) . \tag{2.28}$$

Furthermore, for every $\delta < r$ we find

$$\begin{aligned} &\left|\left[\mathcal{M}^{\mathfrak{s},r}_a u\right](x) - \left[\mathcal{M}^{\mathfrak{s},\delta}_a u\right](x)\right|^p \\ &\quad \leq C \left(\left(\frac{\delta}{r}\right)^{-d} |x - a|^p + \left(\frac{\delta}{r}\right)^{1-d} (1 + \left(\frac{\delta}{r}\right)^{p-d})\right) r^{p-d} \|\nabla^s u\|^p_{L^p(\mathbb{B}_r(a))} . \end{aligned} \tag{2.29}$$

Proof. Step1: W.l.o.g we assume $L\in\mathbb{N}$, $a=\frac{1}{2}\mathbf{e}_1$, $b=(L-\frac{1}{2})\mathbf{e}_1$, $r=\frac{1}{2}$ and define

$$\mathbf{P}_k:=\left(k\mathbf{e}_1+[0,1)\times\mathbb{B}^{d-1}_{\frac{1}{2}}(0)\right)\,,\quad \mathbf{B}_k:=k\mathbf{e}_1+\mathbb{B}_{\frac{1}{2}}\left(\frac{1}{2}\mathbf{e}_1\right)$$
$$\tau_k^{\mathfrak{s}}u(x):=\left[\mathcal{M}^{\mathfrak{s},\frac{1}{2}}_{\left(k+\frac{1}{2}\right)\mathbf{e}_1}u\right](x)=\left[\fint_{\mathbf{B}_k}(\nabla u-\nabla^s u)\right]x+\fint_{\mathbf{B}_k}u\,.$$

Then we find by Lemma 2.16

$$\begin{aligned}\|\nabla u\|^p_{L^p(\mathbf{P}_K)}&\leq C\left(\|\nabla\left(u-\tau_K^{\mathfrak{s}}u\right)\|^p_{L^p(\mathbf{P}_K)}+\|\nabla\tau_K^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_K)}\right)\\&\leq C\left(\|\nabla^s u\|^p_{L^p(\mathbf{P}_K)}+\|\nabla\tau_K^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_K)}\right)\,.\end{aligned}$$

Since $\nabla\tau_k^{\mathfrak{s}}u$ is constant, we find

$$\|\nabla\tau_K^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_K)}\leq C\,\|\nabla\tau_0^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_0)}+C\left(\sum_{k=0}^{K-1}\left\|\nabla\left(\tau_{k+1}^{\mathfrak{s}}u-\tau_k^{\mathfrak{s}}u\right)\right\|_{L^1(\mathbf{P}_{k+1})}\right)^p\,.$$

Furthermore, we find

$$\begin{aligned}\tau_k^{\mathfrak{s}}\big(u-\tau_{k+1}^{\mathfrak{s}}u\big)&=\fint_{\mathbf{B}_k}\left(\nabla u-\fint_{\mathbf{B}_{k+1}}(\nabla u-\nabla^s u)-\nabla^s u\right)x+\fint_{\mathbf{B}_k}\left(u-\fint_{\mathbf{B}_{k+1}}u\right)\\&=\tau_k^{\mathfrak{s}}u-\tau_{k+1}^{\mathfrak{s}}u=\tau_{k+1}^{\mathfrak{s}}(u-\tau_k^{\mathfrak{s}}u)\,.\end{aligned}$$

This implies by $\nabla\tau_{k+1}^{\mathfrak{s}}\left(u-\tau_k^{\mathfrak{s}}u\right)=\fint_{\mathbf{B}_{k+1}}\left(\nabla-\nabla^s\right)\left(u-\tau_k^{\mathfrak{s}}u\right)$ and Lemma 2.16 and Theorem 2.12

$$\begin{aligned}\left\|\nabla\left(\tau_{k+1}^{\mathfrak{s}}u-\tau_k^{\mathfrak{s}}u\right)\right\|^p_{L^p(\mathbf{P}_{k+1})}&\leq C\left\|\nabla\tau_{k+1}^{\mathfrak{s}}\left(u-\tau_k^{\mathfrak{s}}u\right)\right\|^p_{L^p(\mathbf{B}_{k+1})}\\&\leq C\left\|\nabla\left(u-\tau_k^{\mathfrak{s}}u\right)\right\|^p_{L^p(\mathbf{B}_{k+1})}\\&\overset{2.16}{\leq}\leq C\left(\|\nabla^s\left(u-\tau_k^{\mathfrak{s}}u\right)\|^p_{L^p(\mathbf{P}_{k+1}\cup\mathbf{P}_k)}+\|\nabla\left(u-\tau_k^{\mathfrak{s}}u\right)\|^p_{L^p(\mathbf{B}_k)}\right)\\&\overset{2.12}{\leq}C\,\|\nabla^s u\|^p_{L^p(\mathbf{P}_{k+1}\cup\mathbf{P}_k)}\,.\end{aligned}$$

Since the last inequality implies

$$\left(\sum_{k=0}^{K-1}\left\|\nabla\left(\tau_{k+1}^{\mathfrak{s}}u-\tau_k^{\mathfrak{s}}u\right)\right\|_{L^1(\mathbf{P}_{k+1})}\right)^p\leq K^{p-1}C\,\|\nabla^s u\|^p_{L^p\left((0,K)\times\mathbb{B}_1^{d-1}(0)\right)}$$

and $\|\nabla\tau_0^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_0)}\leq C\left(\|\nabla^s u\|^p_{L^p(\mathbf{P}_0)}+\|\nabla u\|^p_{L^p(\mathbf{B}_0)}\right)$ by Lemma 2.16 we find in total

$$\|\nabla\tau_K^{\mathfrak{s}}u\|^p_{L^p(\mathbf{P}_K)}\leq C\,\|\nabla u\|^p_{L^p(\mathbf{B}_0)}+CK^{p-1}\,\|\nabla^s u\|^p_{L^p\left((0,K)\times\mathbb{B}_1^{d-1}(0)\right)}\,.$$

Adding the last inequality from $K=0$ to $K=L$ implies (2.24) through scaling. Applying Corollary 2.11 we infer that (2.25) and (2.26).

Step 2: We observe that Step 1 also holds for $P_{L,r}$ being replaced by $\operatorname{conv}(\mathbb{B}_\delta(a)\cup\mathbb{B}_\delta(b))$. Writing $u_b:=u-\mathcal{M}_b^{\mathfrak{s},\delta}u$ we find from the above calculations

$$\begin{aligned}\left|\mathcal{M}_a^{\mathfrak{s},\delta}u-\mathcal{M}_b^{\mathfrak{s},\delta}u\right|^p(x)&=\left|\mathcal{M}_a^{\mathfrak{s},\delta}\left(u-\mathcal{M}_b^{\mathfrak{s},\delta}u\right)\right|^p(x)\\&\leq C\frac{1}{\delta^d}\left(|x-a|^p\int_{\mathbb{B}_\delta(a)}|\nabla u_b-\nabla^s u_b|^p+\int_{\mathbb{B}_\delta(a)}|u_b|^p\right)\,.\end{aligned}$$

Using that $u_b \in \mathbf{W}^{1,p}_{(0),\nabla^\perp(0),r}(\mathrm{conv}(\mathbb{B}_\delta(a)\cup\mathbb{B}_\delta(b)))$, we find (2.28) with help of (2.26) and Lemma 2.17.

Step 3: W.l.o.g. $a=0$. Writing $\bar u(y) := u(y) - \left(\overline{\nabla^\perp_{a,\delta}} u\right) y$ with $⨍_{\mathbb{B}_r(0)} u = ⨍_{\mathbb{B}_r(0)} \overline{u}$ we infer (2.29) from Lemmas 2.16 and 2.10 via

$$\begin{aligned}\left|\left[\mathcal{M}_0^{\mathfrak{s},1}u\right](x) - \left[\mathcal{M}_0^{\mathfrak{s},\delta}u\right](x)\right|^p &\le C\left|\int_{\mathbb{B}_1(0)} \nabla u - \nabla^s u - \overline{\nabla^\perp_{a,\delta}} u\right|^p |x|^p + \left|⨍_{\mathbb{B}_1(0)} \overline{u} - ⨍_{\mathbb{B}_\delta(0)} \overline{u}\right|^p \\ &\le C\int_{\mathbb{B}_1(0)} \left(\left|\nabla u - \overline{\nabla^\perp_{a,\delta}} u\right|^p + |\nabla^s u|^p\right)|x|^p + ⨍_{\mathbb{B}_1(0)}\left|\overline{u} - ⨍_{\mathbb{B}_\delta(0)} \overline{u}\right|^p \\ &\le C\left(\delta^{-d}|x|^p \|\nabla^s u\|^p_{L^p(\mathbb{B}_1(0))} + \delta^{1-d}\left(1+\delta^{p-d}\right)\|\nabla\overline{u}\|^p_{L^p(\mathbb{B}_1(0))}\right).\end{aligned}$$

□

2.7 Voronoi Tessellations and Delaunay Triangulation

Definition 2.19 (Voronoi Tessellation)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $x_i \neq x_k$ if $i \neq k$. For each $x \in \mathbb{X}$ let

$$G(x) := \left\{y \in \mathbb{R}^d \,:\, \forall \tilde x \in \mathbb{X}\setminus\{x\} : \, |x-y| < |\tilde x - y|\right\} .$$

Then $(G(x_i))_{i\in\mathbb{N}}$ is called the *Voronoi tessellation* of $\mathbb{R}^d$ w.r.t. $\mathbb{X}$. For each $x\in\mathbb{X}$ we define $d(x) := \mathrm{diam} G(x)$.

We will need the following result on Voronoi tessellation of a minimal diameter.

Lemma 2.20. *Let $\mathfrak{r} > 0$ and let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $|x_i - x_k| > 2\mathfrak{r}$ if $i \neq k$. For $x \in \mathbb{X}$ let $\mathcal{I}(x) := \{y \in \mathbb{X} : G(y)\cap\mathbb{B}_\mathfrak{r}(G(x)) \neq \emptyset\}$. Then $y \in \mathcal{I}(x)$ implies $|x-y| \le 4d(x)$ and*

$$\#\mathcal{I}(x) \le \left(\frac{4d(x)}{\mathfrak{r}}\right)^d . \tag{2.30}$$

Proof. Let $\mathbb{X}_k = \left\{x_j \in \mathbb{X} \,:\, \mathcal{H}^{d-1}(\partial G_k \cap \partial G_j) \ge 0\right\}$ the neighbors of x_k and $d_k := d(x_k)$. Then all $x_j \in \mathbb{X}$ satisfy $|x_k - x_j| \le 2d_k$. Moreover, every $\tilde x \in \mathbb{X}$ with $|\tilde x - x_k| > 4d_k$ has the property that $\mathrm{dist}(\,\partial G(\tilde x)\,,\, x_k\,) > 2d_k > d_k + \mathfrak{r}$ and $\tilde x \notin \mathcal{I}_k$. Since every Voronoi cell contains a ball of radius $\mathfrak{r}$, this implies that $\#\mathcal{I}_k \le |\mathbb{B}_{4d_k}(x_k)| \,/\, |\mathbb{B}_\mathfrak{r}(0)| = \left(\frac{4d_k}{\mathfrak{r}}\right)^d$. □

Definition 2.21 (Delaunay Triangulation)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $x_i \neq x_k$ if $i\neq k$. The Delaunay triangulation is the dual unoriented graph (see Def. **??** below) of the Voronoi tessellation, i.e. we say $\mathbb{D}(\mathbb{X}) := \left\{(x,y) :\, \mathcal{H}^{d-1}(\partial G(x)\cap\partial G(y)) \neq 0\right\}$.

2.8 Local η-Regularity

Definition 2.22 (η- regularity)**.** For a function $\eta : \partial\mathbf{P} \to (0,r]$ we call $\mathbf{P}$ η-regular if

$$\forall p \in \partial\mathbf{P},\, \varepsilon \in \left(0,\frac{1}{2}\right),\, \tilde p \in \mathbb{B}_{\varepsilon\eta(p)}(p)\cap\partial\mathbf{P} :\, \eta(\tilde p) > (1-\varepsilon)\eta(p) . \tag{2.31}$$

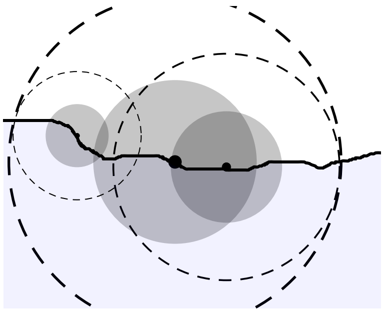

Figure 1: An illustration of η-regularity. In Theorem 2.25 we will rely on a "gray" region like in this picture.

Remark 2.23. This concept and its consequences from Lemma 2.24 and Theorem 2.25 will be extensively used later to cover $\partial\mathbf{P}$ by a suitable family of open balls.

Lemma 2.24. *Let $\mathbf{P}$ be a locally η-regular set for $\eta:\partial\mathbf{P}\to(0,\mathfrak{r})$. Then $\eta:\mathbf{P}\to\mathbb{R}$ is locally Lipschitz continuous with Lipschitz constant 1 and for every $\varepsilon\in\left(0,\frac{1}{2}\right)$ and $\tilde{p}\in\mathbb{B}_{\varepsilon\eta}(p)\cap\mathbf{P}$ it holds*

$$\frac{1-\varepsilon}{1-2\varepsilon}\eta(p)>\eta(\tilde{p})>\eta(p)-|p-\tilde{p}|>(1-\varepsilon)\,\eta(p)\,. \tag{2.32}$$

Furthermore,

$$|p-\tilde{p}|\leq\varepsilon\max\left\{\eta(p),\eta(\tilde{p})\right\}\quad\Rightarrow\quad|p-\tilde{p}|\leq\frac{\varepsilon}{1-\varepsilon}\min\left\{\eta(p),\eta(\tilde{p})\right\} \tag{2.33}$$

Proof. Let $p,\tilde{p}$ such that $|\tilde{p}-p|<\frac{1}{2}\eta(p)$ with $\varepsilon_{p,\tilde{p}}:=\inf\left\{\varepsilon:\,|\tilde{p}-p|<\varepsilon\eta(p)\right\}$. This means $\varepsilon\in[\varepsilon_{p,\tilde{p}},\frac{1}{2})$ iff $\eta(\tilde{p})\geq(1-\varepsilon)\,\eta(p)$ and we find

$$\eta(\tilde{p})\geq\eta(p)-|p-\tilde{p}|=\eta(p)-\varepsilon_{p,\tilde{p}}\eta(p)>(1-\varepsilon)\,\eta(p)$$

which implies $|\tilde{p}-p|<\frac{\varepsilon}{1-\varepsilon}\eta(\tilde{p})$ and the local Lipschitz continuity by a symmetry argument in $p,\tilde{p}$. This in turn leads to $\eta(p)>\left(1-\frac{\varepsilon}{1-\varepsilon}\right)\eta(\tilde{p})$ or

$$\eta(p)=\frac{1-\varepsilon}{1-\varepsilon}\eta(p)<\frac{1}{1-\varepsilon}\left(\eta(p)-|p-\tilde{p}|\right)<\frac{1}{1-\varepsilon}\eta(\tilde{p})\leq\frac{1}{1-2\varepsilon}\eta(p)\,,$$

implying (2.32) and continuity of η.

In order to prove (2.33), w.l.o.g. let $\eta(\tilde{p})\leq\eta(p)$. Then

$$|p-\tilde{p}|\leq\varepsilon\eta(p)\leq\frac{\varepsilon}{1-\varepsilon}\eta(\tilde{p})\,.$$

□

Theorem 2.25. *Let $\Gamma\subset\mathbb{R}^d$ be a closed set and let $\eta(\cdot)\in C(\Gamma)$ be bounded and satisfy for every $\varepsilon\in\left(0,\frac{1}{2}\right)$ and for $|p-\tilde{p}|<\varepsilon\eta(p)$*

$$\frac{1-\varepsilon}{1-2\varepsilon}\,\eta(p)>\eta(\tilde{p})>\eta(p)-|p-\tilde{p}|>(1-\varepsilon)\,\eta(p)\,. \tag{2.34}$$

and define $\tilde{\eta}(p)=2^{-K}\eta(p)$, $K\geq 2$. Then for every $C\in(0,1)$ there exists a locally finite covering of Γ with balls $\mathbb{B}_{\tilde{\eta}(p_k)}(p_k)$ for a countable number of points $(p_k)_{k\in\mathbb{N}}\subset\Gamma$ such that for every $i\neq k$ with $\mathbb{B}_{\tilde{\eta}(p_i)}(p_i)\cap\mathbb{B}_{\tilde{\eta}(p_k)}(p_k)\neq\emptyset$ it holds

$$\begin{aligned}&\frac{2^{K-1}-1}{2^{K-1}}\tilde{\eta}(p_i)\leq\tilde{\eta}(p_k)\leq\frac{2^{K-1}}{2^{K-1}-1}\tilde{\eta}(p_i)\\ \text{and}\quad&\frac{2^K-1}{2^{K-1}-1}\min\left\{\tilde{\eta}(p_i),\tilde{\eta}(p_k)\right\}\geq|p_i-p_k|\geq C\max\left\{\tilde{\eta}(p_i),\tilde{\eta}(p_k)\right\}\end{aligned} \tag{2.35}$$

Proof. We chose $\delta > 0$, $n \in \mathbb{N}$ such that $\left(1-\frac{1}{n}\right)(1-\delta) > C$. W.o.l.g. assume $\tilde{\eta} < (1-\delta)$. Consider $\tilde{Q} := \left[0, \frac{1}{n}\right]^d$, let $q_{1,\dots,n^d}$ denote the n^d elements of $[0,1)^d \cap \frac{\mathbb{Q}^d}{n}$ and let $\tilde{Q}_{z,i} = \tilde{Q} + z + q_i$, $z \in \mathbb{Z}^d$. We set $B_{(0)} := \emptyset$, $\Gamma_1 = \Gamma$, $\eta_k := (1-\delta)^k$ and for $k \geq 1$ we construct the covering using inductively defined open sets $B_{(k)}$ and closed set Γ_k as follows:

1 Define $\Gamma_{k,1} = \Gamma_k$. For $i = 1, \dots, n^d$ do the following:

1.1 For every $z \in \mathbb{Z}^d$ do

if $\exists p \in \left(\eta_k \tilde{Q}_{z,i}\right) \cap \Gamma_{k,i},\ \tilde{\eta}(p) \in (\eta_k, \eta_{k-1}]$ then set $b_{z,i} = \mathbb{B}_{\tilde{\eta}(p)}(p)\,,\ \mathbb{X}_{z,i} = \{p\}$

otherwise set $b_{z,i} = \emptyset\,,\ \mathbb{X}_{z,i} = \emptyset\,.$

1.2 Define $B_{(k),i} := \bigcup_{z \in \mathbb{Z}^d} b_{z,i}$ and $\Gamma_{k,i+1} = \Gamma_{k,i} \backslash B_{(k),i}$ and $\mathbb{X}_{(k),i} := \bigcup_{z \in \mathbb{Z}^d} \mathbb{X}_{z,i}$. Observe: $p_1, p_2 \in \mathbb{X}_{(k),i}$ implies $|p_1 - p_2| > \left(1-\frac{1}{n}\right)\eta_k$ and $p_3 \in \mathbb{X}_{(k),j}$, $j < i$ implies $p_1 \notin \mathbb{B}_{\eta_k}(p_3)$ and hence $|p_1 - p_3| > \eta_k$. Similar, $p_3 \in \mathbb{X}_l$, $l < k$, implies $|p_1 - p_3| > \eta_l > \eta_k$.

2 Define $\Gamma_{k+1} := \Gamma_{k,n^d+1}$, $\mathbb{X}_k := \bigcup_i \mathbb{X}_{(k),i}$.

The above covering of Γ is complete in the sense that every $x \in \Gamma$ lies in one of the balls (by contradiction). We denote $\mathbb{X} := \bigcup_k \mathbb{X}_k = (p_i)_{i \in \mathbb{N}}$ the family of centers of the above constructed covering of Γ and find the following properties: Let $p_1, p_2 \in \mathbb{X}$ be such that $\mathbb{B}_{\tilde{\eta}(p_1)}(p_1) \cap \mathbb{B}_{\tilde{\eta}(p_2)}(p_2) \neq \emptyset$. W.l.o.g. let $\tilde{\eta}(p_1) \geq \tilde{\eta}(p_2)$. Then the following two properties are satisfied due to (2.34)

1 It holds $|p_1 - p_2| \leq 2\tilde{\eta}(p_1) \leq \frac{1}{2^{K-1}}\eta(p_1)$ and hence $\mathbb{B}_{\tilde{\eta}(p_2)}(p_2) \subset \mathbb{B}_{2^{2-K}\eta(p_1)}(p_1)$ and $\eta(p_2) \geq \frac{2^{K-1}-1}{2^{K-1}}\eta(p_1)$. Furthermore $\tilde{\eta}(p_1) \geq \tilde{\eta}(p_2) \geq \frac{2^{K-1}-1}{2^{K-1}}\tilde{\eta}(p_1)$.

2 Let k such that $\tilde{\eta}(p_1) \in (\eta_k, \eta_{k+1}]$. If also $\tilde{\eta}(p_2) \in (\eta_k, \eta_{k+1}]$ then the observation in Step 1.(b) implies $|p_1 - p_2| \geq \left(1-\frac{1}{n}\right)\eta_k \geq \left(1-\frac{1}{n}\right)(1-\delta)\,\tilde{\eta}(p_1)$. If $\tilde{\eta}(p_2) \notin [\eta_k, \eta_{k+1})$ then $\tilde{\eta}(p_2) < \eta_k$ and hence $p_2 \notin \mathbb{B}_{\tilde{\eta}(p_1)}(p_1)$, implying $|p_1 - p_2| > \tilde{\eta}(p_1)$.

Due to our choice of n and δ, this concludes the proof. □

2.9 Dynamical Systems

Assumption 2.26. *Throughout this work we assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space with countably generated σ-algebra $\mathscr{F}$.*

Due to the insight in [10], shortly sketched in the next two subsections, after a measurable transformation the probability space Ω can be assumed to be metric and separable, which always ensures Assumption 2.26.

Definition 2.27 (Dynamical system)**.** A dynamical system on Ω is a family $(\tau_x)_{x \in \mathbb{R}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$ satisfying (i)-(iii):

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ (Group property)

(ii) $\mathbb{P}(\tau_{-x}B) = \mathbb{P}(B) \quad \forall x \in \mathbb{R}^d,\ B \in \mathscr{F}$ (Measure preserving)

(iii) $A: \mathbb{R}^d \times \Omega \to \Omega \qquad (x,\omega) \mapsto \tau_x\omega$ is measurable (Measurability of evaluation)

A set $A \subset \Omega$ is almost invariant if $\mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0$. The family

$$\mathscr{I} = \left\{ A \in \mathscr{F} \,:\, \forall x \in \mathbb{R}^d \, \mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0 \right\} \tag{2.36}$$

of almost invariant sets is σ-algebra and

$$\mathbb{E}\left(f|\mathscr{I}\right) \text{ denotes the expectation of } f: \Omega \to \mathbb{R} \text{ w.r.t. } \mathscr{I}\,. \tag{2.37}$$

A concept linked to dynamical systems is the concept of stationarity.

Definition 2.28 (Stationary)**.** Let X be a measurable space and let $f : \Omega \times \mathbb{R}^d \to X$. Then f is called (weakly) stationary if $f(\omega, x) = f(\tau_x\omega, 0)$ for (almost) every x.

Definition 2.29. A family $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is called convex averaging sequence if

(i) each A_n is convex

(ii) for every $n \in \mathbb{N}$ holds $A_n \subset A_{n+1}$

(iii) there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$ such that $B_{r_n}(0) \subseteq A_n$.

We sometimes may take the following stronger assumption.

Definition 2.30. A convex averaging sequence A_n is called regular if

$$|A_n|^{-1} \# \left\{ z \in \mathbb{Z}^d \,:\, (z + \mathbb{T}) \cap \partial A_n \neq \emptyset \right\} \to 0\,.$$

The latter condition is evidently fulfilled for sequences of cones or balls. Convex averaging sequences are important in the context of ergodic theorems.

Theorem 2.31 (Ergodic Theorem [4] Theorems 10.2.II and also [27])**.** *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f : \Omega \to \mathbb{R}$ be measurable with $|\mathbb{E}(f)| < \infty$. Then for almost all $\omega \in \Omega$*

$$|A_n|^{-1} \int_{A_n} f(\tau_x\omega)\,\mathrm{d}x \to \mathbb{E}(f|\mathscr{I})\,. \tag{2.38}$$

We observe that $\mathbb{E}\left(f|\mathscr{I}\right)$ is of particular importance. For the calculations in this work, we will particularly focus on the case of trivial $\mathscr{I}$. This is called ergodicity, as we will explain in the following.

Definition 2.32 (Ergodicity and mixing)**.** A dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *mixing* if for every measurable $A, B \subset \Omega$ it holds

$$\lim_{\|x\|\to\infty} \mathbb{P}(A \cap \tau_x B) = \mathbb{P}(A)\,\mathbb{P}(B)\,. \tag{2.39}$$

A dynamical system is called *ergodic* if

$$\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \mathbb{P}(A \cap \tau_x B)\mathrm{d}x = \mathbb{P}(A)\,\mathbb{P}(B)\,. \tag{2.40}$$

Remark 2.33. a) Let $\Omega = \{\omega_0 = 0\}$ with the trivial σ-algebra and $\tau_x \omega_0 = \omega_0$. Then τ is evidently mixing. However, the realizations are constant functions $f_\omega(x) = c$ on $\mathbb{R}^d$ for some constant c.

b) A typical ergodic system is given by $\Omega = \mathbb{T}$ with the Lebesgue σ-algebra and $\mathbb{P} = \mathcal{L}$ the Lebesgue measure. The dynamical system is given by $\tau_x y := (x + y) \mod \mathbb{T}$.

c) It is known that $(\tau_x)_{x\in\mathbb{R}^d}$ is ergodic if and only if every almost invariant set $A \in \mathscr{I}$ has probability $\mathbb{P}(A) \in \{0, 1\}$ (see [4] Proposition 10.3.III) i.e.

$$[\ \forall x\, \mathbb{P}((\tau_x A \cup A) \setminus (\tau_x A \cap A)) = 0\ ] \ \Rightarrow\ \mathbb{P}(A) \in \{0, 1\}\ . \tag{2.41}$$

d) It is sufficient to show (2.39) or (2.40) for A and B in a ring that generates the σ-algebra $\mathscr{F}$. We refer to [4], Section 10.2, for the later results.

A further useful property of ergodic dynamical systems, which we will use below, is the following:

Lemma 2.34 (Ergodic times mixing is ergodic)**.** *Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ be probability spaces with dynamical systems $(\tilde{\tau}_x)_{x\in\mathbb{R}^d}$ and $(\hat{\tau}_x)_{x\in\mathbb{R}^d}$ respectively. Let $\Omega := \tilde{\Omega} \times \hat{\Omega}$ be the usual product measure space with the notation $\omega = (\tilde{\omega}, \hat{\omega}) \in \Omega$ for $\tilde{\omega} \in \tilde{\Omega}$ and $\hat{\omega} \in \hat{\Omega}$. If $\tilde{\tau}$ is ergodic and $\hat{\tau}$ is mixing, then $\tau_x(\tilde{\omega}, \hat{\omega}) := (\tilde{\tau}_x \tilde{\omega}, \hat{\tau}_x \hat{\omega})$ is ergodic.*

Proof. Relying on Remark 2.33.c) we verify (2.40) by proving it for sets $A = \tilde{A} \times \hat{A}$ and $B = \tilde{B} \times \hat{B}$ which generate $\mathscr{F} := \tilde{\mathscr{F}} \otimes \hat{\mathscr{F}}$. We make use of $A \cap B = \left(\tilde{A} \cap \tilde{B}\right) \times \left(\hat{A} \cap \hat{B}\right)$ and observe that

$$\begin{aligned} \mathbb{P}(A \cap \tau_x B) &= \mathbb{P}\Big(\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right) \times \left(\hat{A} \cap \hat{\tau}_x \hat{B}\right)\Big) = \hat{\mathbb{P}}\left(\hat{A} \cap \hat{\tau}_x \hat{B}\right) \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right) \\ &= \hat{\mathbb{P}}\left(\hat{A} \cap \hat{B}\right) \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right) + \left[\hat{\mathbb{P}}\left(\hat{A} \cap \hat{\tau}_x \hat{B}\right) - \hat{\mathbb{P}}\left(\hat{A} \cap \hat{B}\right)\right] \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right). \end{aligned}$$

Using ergodicity, we find that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \hat{\mathbb{P}}\left(\hat{A} \cap \hat{B}\right) \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right) \mathrm{d}x &= \hat{\mathbb{P}}\Big(\left(\hat{A} \cap \hat{B}\right)\Big) \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{B}\right) \\ &= \mathbb{P}(A \cap B)\,. \end{aligned} \tag{2.42}$$

Since $\hat{\tau}$ is mixing, we find for every $\varepsilon > 0$ some $R > 0$ such that $\|x\| > R$ implies

$$\left|\hat{\mathbb{P}}\left(\hat{A} \cap \hat{\tau}_x \hat{B}\right) - \hat{\mathbb{P}}\left(\hat{A} \cap \hat{B}\right)\right| < \varepsilon.$$

For $n > R$ we find

$$\begin{aligned} &\frac{1}{(2n)^d} \int_{[-n,n]^d} \left|\hat{\mathbb{P}}\left(\hat{A} \cap \hat{\tau}_x \hat{B}\right) - \hat{\mathbb{P}}\left(\hat{A} \cap \hat{B}\right)\right| \tilde{\mathbb{P}}\left(\tilde{A} \cap \tilde{\tau}_x \tilde{B}\right) \\ &\qquad \leq \frac{1}{(2n)^d} \int_{[-n,n]^d} \varepsilon + \frac{1}{(2n)^d} \int_{[-R,R]^d} 2 \to \varepsilon \quad \text{as } n \to \infty\,. \end{aligned} \tag{2.43}$$

The last two limits (2.42) and (2.43) imply (2.40). □

Remark 2.35. The above proof heavily relies on the mixing property of $\hat{\tau}$. Note that for $\hat{\tau}$ being only ergodic, the statement is wrong, as can be seen from the product of two periodic processes in $\mathbb{T} \times \mathbb{T}$ (see Remark 2.33). Here, the invariant sets are given by $I_A := \{((y + x) \mod \mathbb{T}\,,\, x) \,:\, y \in A\}$ for arbitrary measurable $A \subset \mathbb{T}$.

2.10 Random Measures and Palm Theory

We recall some facts from random measure theory (see [4]) which will be needed for homogenization. Let $\mathfrak{M}(\mathbb{R}^d)$ denote the space of locally bounded Borel measures on $\mathbb{R}^d$ (i.e. bounded on every bounded Borel-measurable set) equipped with the Vague topology, which is generated by the sets

$$\left\{\mu\,:\,\int f\,\mathrm{d}\mu\in A\right\}\text{ for every open }A\subset\mathbb{R}^d\text{ and }f\in C_c\big(\mathbb{R}^d\big)\,.$$

This topology is metrizable, complete and countably generated. A random measure is a measurable mapping

$$\mu_\bullet\,:\,\Omega\to\mathfrak{M}(\mathbb{R}^d)\,,\qquad\omega\mapsto\mu_\omega$$

which is equivalent to both of the following conditions

1 For every bounded Borel set $A\subset\mathbb{R}^d$ the map $\omega\mapsto\mu_\omega(A)$ is measurable

2 For every $f\in C_c(\mathbb{R}^d)$ the map $\omega\mapsto\int f\,\mathrm{d}\mu_\omega$ is measurable.

A random measure is stationary if the distribution of $\mu_\omega(A)$ is invariant under translations of A that is $\mu_\omega(A)$ and $\mu_\omega(A+x)$ share the same distribution. From stationarity of μ_ω one concludes the existence ([10, 22] and references therein) of a dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on Ω such that $\mu_\omega\left(A+x\right)=\mu_{\tau_x\omega}\left(A\right)$. By a deep theorem due to Mecke (see [19, 4]) the measure

$$\mu_{\mathcal{P}}(A)=\int_\Omega\int_{\mathbb{R}^d}g(s)\,\chi_A(\tau_s\omega)\,\mathrm{d}\mu_\omega(s)\,\mathrm{d}\mathbb{P}(\omega)$$

can be defined on Ω for every positive $g\in L^1(\mathbb{R}^d)$ with compact support. $\mu_{\mathcal{P}}$ is independent from g and in case $\mu_\omega=\mathcal{L}$ we find $\mu_{\mathcal{P}}=\mathbb{P}$. Furthermore, for every $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\Omega)$-measurable non negative or $\mu_{\mathcal{P}}\times\mathcal{L}$- integrable functions f the Campbell formula

$$\int_\Omega\int_{\mathbb{R}^d}f(x,\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)=\int_{\mathbb{R}^d}\int_\Omega f(x,\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\omega)\,\mathrm{d}x$$

holds. The measure μ_ω has finite intensity if $\mu_{\mathcal{P}}(\Omega)<+\infty$.

We denote by

$$\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I}):=\int_\Omega f\text{ the expectation of }f\text{ w.r.t. the }\sigma\text{-algebra }\mathscr{I}\text{ and }\mu_{\mathcal{P}}\,.\tag{2.44}$$

For random measures we find a more general version of Theorem 2.31.

Theorem 2.36 (Ergodic Theorem [4] 12.2.VIII). *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $(A_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f\,:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega|f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$*

$$|A_n|^{-1}\int_{A_n}f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\,.\tag{2.45}$$

Given a bounded open (and convex) set $\mathbf{Q}\subset\Omega$, it is not hard to see that the following generalization holds:

Theorem 2.37 (General Ergodic Theorem)**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $\mathbf{Q}\subset\mathbb{R}^d$ be a bounded open set with $0\in\mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega |f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$ it holds*

$$\forall\varphi\in C_0(\mathbf{Q}):\quad n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\int_{\mathbf{Q}}\varphi\,. \tag{2.46}$$

Sketch of proof. Chose a countable dense family of functions $\varphi\in C_0(\mathbf{Q})$ that spans $L^1(\mathbf{Q})$ and that have support on a ball. Use a Cantor argument and Theorem 2.36 to prove the statement for a countable dense family of $C_0(\mathbf{Q})$. From here, we conclude by density.

The last result can be used to prove the most general ergodic theorem which we will use in this work: □

Theorem 2.38 (General Ergodic Theorem for the Lebesgue measure)**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $\mathbf{Q}\subset\mathbb{R}^d$ be a bounded open set with $0\in\mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f\in L^p(\Omega;\mu_{\mathcal{P}})$ and $\varphi\in L^q(\mathbf{Q})$, where $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$ it holds*

$$n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}x\to\mathbb{E}(f)\int_{\mathbf{Q}}\varphi\,.$$

Proof. Let $\varphi_\delta\in C(\overline{\mathbf{Q}})$ with $\|\varphi-\varphi_\delta\|_{L^q(\mathbf{Q})}<\delta$. Then

$$\begin{aligned}&\left|n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}x-\mathbb{E}(f)\int_{\mathbf{Q}}\varphi\right|\\&\qquad\leq\|\varphi-\varphi_\delta\|_{L^q(\mathbf{Q})}\left(n^{-d}\int_{n\mathbf{Q}}|f(\tau_x\omega)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\&\qquad\quad+\left|n^{-d}\int_{n\mathbf{Q}}\varphi_\delta(x)f(\tau_x\omega)\,\mathrm{d}x-\mathbb{E}(f)\int_{\mathbf{Q}}\varphi_\delta\right|+\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\int_{\mathbf{Q}}|\varphi-\varphi_\delta|\,,\end{aligned}$$

which implies the claim. □

2.11 Random Sets

The theory of random measures and the theory of random geometry are closely related. In what follows, we recapitulate those results that are important in the context of the theory developed below and shed some light on the correlations between random sets and random measures.

Let $\mathfrak{F}(\mathbb{R}^d)$ denote the set of all closed sets in $\mathbb{R}^d$. We write

$$\mathfrak{F}_V:=\ \left\{F\in\mathfrak{F}(\mathbb{R}^d)\,:\,F\cap V\neq\emptyset\right\}\quad\text{if } V\subset\mathbb{R}^d\ \text{ is an open set}\,, \tag{2.47}$$

$$\mathfrak{F}^K:=\ \left\{F\in\mathfrak{F}(\mathbb{R}^d)\,:\,F\cap K=\emptyset\right\}\quad\text{if } K\subset\mathbb{R}^d\ \text{ is a compact set}\,. \tag{2.48}$$

The *Fell-topology* $\mathscr{T}_F$ is created by all sets $\mathfrak{F}_V$ and $\mathfrak{F}^K$ and the topological space $(\mathfrak{F}(\mathbb{R}^d),\mathscr{T}_F)$ is compact, Hausdorff and separable[18].

Remark 2.39. We find for closed sets F_n,F in $\mathbb{R}^d$ that $F_n\to F$ if and only if [18]

1 for every $x\in F$ there exists $x_n\in F_n$ such that $x=\lim_{n\to\infty}x_n$ and

2 if F_{n_k} is a subsequence, then every convergent sequence x_{n_k} with $x_{n_k} \in F_{n_k}$ satisfies $\lim_{k\to\infty} x_{n_k} \in F$.

If we restrict the Fell-topology to the compact sets $\mathfrak{K}(\mathbb{R}^d)$ it is equivalent with the Hausdorff topology given by the Hausdorff distance

$$\mathrm{d}(A,B) = \max\left\{\sup_{y\in B}\inf_{x\in A}|x-y| \;,\; \sup_{x\in A}\inf_{y\in B}|x-y|\right\} .$$

Remark 2.40. For $A \subset \mathbb{R}^d$ closed, the set

$$\mathfrak{F}(A) := \left\{F \in \mathfrak{F}(\mathbb{R}^d) \;:\; F \subset A\right\}$$

is a closed subspace of $\mathfrak{F}\big(\mathbb{R}^d\big)$. This holds since

$$\mathfrak{F}\big(\mathbb{R}^d\big)\backslash\mathfrak{F}(A) = \left\{B \in \mathfrak{F}\big(\mathbb{R}^d\big) \;:\; B \cap \big(\mathbb{R}^d\backslash A\big) \neq \emptyset\right\} = \mathfrak{F}_{\mathbb{R}^d\backslash A} \quad \text{is open.}$$

..

Lemma 2.41 (Continuity of geometric operations)**.** *The maps* $\tau_x : A \mapsto A+x$ *and* $b_\delta : A \mapsto \overline{\mathbb{B}_\delta(A)}$ *are continuous in* $\mathfrak{F}\left(\mathbb{R}^d\right)$*.*

Proof. We show that preimages of open sets are open. For open sets V we find

$$\tau_x^{-1}\left(\mathfrak{F}_V\right) = \left\{F \in \mathfrak{F}(\mathbb{R}^d) \;:\; \tau_x F \cap V \neq \emptyset\right\} = \left\{F \in \mathfrak{F}(\mathbb{R}^d) \;:\; F \cap \tau_{-x}V \neq \emptyset\right\} = \mathfrak{F}_{\tau_{-x}V} \,,$$
$$b_\delta^{-1}\left(\mathfrak{F}_V\right) = \left\{F \in \mathfrak{F}(\mathbb{R}^d) \;:\; \overline{\mathbb{B}_\delta(F)} \cap V \neq \emptyset\right\} = \left\{F \in \mathfrak{F}(\mathbb{R}^d) \;:\; F \cap \mathbb{B}_\delta(V) \neq \emptyset\right\} = \mathfrak{F}_{(b_\delta V)^\circ} \,.$$

The calculations for $\tau_x^{-1}\big(\mathfrak{F}^K\big) = \mathfrak{F}^{\tau_{-x}K}$ and $b_\delta^{-1}\big(\mathfrak{F}^K\big) = \mathfrak{F}^{b_\delta K}$ are analogue. □

Remark 2.42. The *Matheron-σ-field* $\sigma_{\mathfrak{F}}$ is the Borel-σ-algebra of the Fell-topology and is fully characterized either by the class $\mathfrak{F}_V$ of $\mathfrak{F}^K$.

Definition 2.43 (Random closed / open set according to Choquet (see [18] for more details))**.**

a) Let $(\Omega,\sigma,\mathbb{P})$ be a probability space. Then a *Random Closed Set (RACS)* is a measurable mapping
$$A : (\Omega,\sigma,\mathbb{P}) \longrightarrow (\mathfrak{F},\sigma_{\mathfrak{F}})$$

b) Let τ_x be a dynamical system on Ω. A random closed set is called stationary if its characteristic functions $\chi_{A(\omega)}$ are stationary, i.e. they satisfy $\chi_{A(\omega)}(x) = \chi_{A(\tau_x\omega)}(0)$ for almost every $\omega \in \Omega$ for almost all $x \in \mathbb{R}^d$. Two random sets are jointly stationary if they can be parameterized by the same probability space such that they are both stationary.

c) A random closed set $\Gamma : (\Omega,\sigma,P) \longrightarrow (\mathfrak{F},\sigma_{\mathfrak{F}}) \quad \omega \mapsto \Gamma(\omega)$ is called a *Random closed C^k-Manifold* if $\Gamma(\omega)$ is a piece-wise C^k-manifold for P almost every ω.

d) A measurable mapping
$$A : (\Omega,\sigma,\mathbb{P}) \longrightarrow (\mathfrak{F},\sigma_{\mathfrak{F}})$$
is called *Random Open Set (RAOS)* if $\omega \mapsto \mathbb{R}^d\backslash A(\omega)$ is a RACS.

The importance of the concept of random geometries for stochastic homogenization stems from the following Lemma by ZÃd'hle. It states that every random closed set induces a random measure. Thus, every stationary RACS induces a stationary random measure.

Lemma 2.44 ([32] Theorem 2.1.3 resp. Corollary 2.1.5)**.** *Let $\mathfrak{F}_m \subset \mathfrak{F}$ be the space of closed m-dimensional sub manifolds of $\mathbb{R}^d$ such that the corresponding Hausdorff measure is locally finite. Then, the σ-algebra $\sigma_{\mathfrak{F}} \cap \mathfrak{F}_m$ is the smallest such that*

$$M_B : \mathfrak{F}_m \to \mathbb{R} \quad M \mapsto \mathcal{H}^m(M \cap B)$$

is measurable for every measurable and bounded $B \subset \mathbb{R}^d$.

This means that

$$M_{\mathbb{R}^d} : \mathfrak{F}_m \to \mathfrak{M}(\mathbb{R}^d) \quad M \mapsto \mathcal{H}^m(M \cap \cdot)$$

is measurable with respect to the σ-algebra created by the Vague topology on $\mathfrak{M}(\mathbb{R}^d)$. Hence a random closed set always induces a random measure. Based on Lemma 2.44 and on Palm-theory, the following useful result was obtained in [10] (See Lemma 2.14 and Section 3.1 therein). We can thus assume w.l.o.g that Ω is a separable metric space.

Theorem 2.45. *Let (Ω, σ, P) be a probability space with an ergodic dynamical system τ. Let $A : (\Omega, \sigma, P) \longrightarrow (\mathfrak{F}, \sigma_{\mathfrak{F}})$ be a stationary random closed m-dimensional C^k-Manifold.*

There exists a separable metric space $\tilde{\Omega} \subset \mathfrak{M}(\mathbb{R}^d)$ with an ergodic dynamical system $\tilde{\tau}$ and a mapping $\tilde{A} : (\tilde{\Omega}, \mathcal{B}_{\tilde{\Omega}}, \mathbb{P}) \to (\mathfrak{F}, \sigma_{\mathfrak{F}})$ such that A and $\tilde{A}$ have the same law and such that $\tilde{A}$ still is stationary. Furthermore, $(x, \omega) \mapsto \tau_x\omega$ is continuous. We identify $\tilde{\Omega} = \Omega$, $\tilde{A} = A$ and $\tilde{\tau} = \tau$.

Also the following result will be useful below.

Lemma 2.46. *Let μ be a Radon measure on $\mathbb{R}^d$ and let $\mathbf{Q} \subset \mathbb{R}^d$ be a bounded open set. Let $\mathfrak{F}_0 \subset \mathfrak{F}\left(\overline{\mathbf{Q}}\right)$ be such that $\mathfrak{F}_0 \to \mathbb{R}$, $A \mapsto \mu(A)$ is continuous. Then*

$$m : \mathfrak{F} \times \mathfrak{F}_0 \to \mathfrak{M}\big(\mathbb{R}^d\big), \quad (P, B) \mapsto \begin{cases} A \mapsto \mu(A \cap B) & B \subset P \\ 0 & \textit{else} \end{cases}$$

is measurable.

Proof. For $f \in C_c(\mathbb{R}^d)$ we introduce m_f through

$$m_f : (P, B) \mapsto \begin{cases} \int_B f \, \mathrm{d}\mu & B \subset P \\ 0 & \text{else} \end{cases}$$

and observe that m is measurable if and only if for every $f \in C_c\big(\mathbb{R}^d\big)$ the map m_f is measurable (see Section 2.10). Hence, if we prove the latter property, the lemma is proved.

We assume $f \geq 0$ and we show that the mapping m_f is even upper continuous. In particular, let $(P_n, B_n) \to (P, B)$ in $\mathfrak{F} \times \mathfrak{F}_0$ and assume that $B_n \subset P_n$ for all $n > N_0$. Since $\overline{\mathbf{Q}}$ is compact, Remark 2.39. 2. implies that $B \subset P \cap \overline{\mathbf{Q}}$. Furthermore, since f has compact support, we find $\left|\int_{B_n} f \, \mathrm{d}\mu - \int_B f \, \mathrm{d}\mu\right| \leq \|f\|_\infty |\mu(B_n) - \mu(B)| \to 0$. On the other hand, if there exists a subsequence such that $B_n \not\subset P_n$ for all n, then either $B \not\subset P$ and $m_f(P_n, B_n) = 0 \to m_f(P, B) = 0$ or $B \subset P$ and $0 = \lim_{n\to\infty} m_f(P_n, B_n) \leq \int_B f \mathrm{d}\mu = m_f(P, B)$. For $f \leq 0$ we obtain lower semicontinuity and for general f the map m_f is the sum of an upper and a lower semicontinuous map, hence measurable. ☐

2.12 Point Processes

Definition 2.47 ((Simple) point processes)**.** A $\mathbb{Z}$-valued random measure μ_ω is called point process. In what follows, we consider the particular case that for almost every ω there exist points $(x_k(\omega))_{k\in\mathbb{N}}$ and values $(a_k(\omega))_{k\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$\mu_\omega = \sum_{k\in\mathbb{N}} a_k \delta_{x_k(\omega)} \,.$$

The point process μ_ω is called simple if almost surely for all $k\in\mathbb{N}$ it holds $a_k \in \{0,1\}$.

Example 2.48 (Poisson process)**.** A particular example for a stationary point process is the Poisson point process $\mu_\omega = \mathbb{X}_\omega$ with intensity λ. Here, the probability $\mathbb{P}(\mathbb{X}(A)=n)$ to find n points in a Borel-set A with finite measure is given by a Poisson distribution

$$\mathbb{P}(\mathbb{X}(A)=n) = e^{-\lambda|A|}\frac{\lambda^n\,|A|^n}{n!} \tag{2.49}$$

with expectation $\mathbb{E}(\mathbb{X}(A)) = \lambda\,|A|$. Shift-invariance of (2.49) implies that the Poisson point process is stationary.

We can use a given random point process to construct further processes.

Example 2.49 (Hard core Matern process)**.** The hard core Matern process is constructed from a given point process $\mathbb{X}_\omega$ by mutually erasing all points with the distance to the nearest neighbor smaller than a given constant r. If the original process $\mathbb{X}_\omega$ is stationary (ergodic), the resulting hard core process is stationary (ergodic) respectively.

Example 2.50 (Hard core Poisson–Matern process)**.** If a Matern process is constructed from a Poisson point process, we call it a Poisson–Matern point process.

Lemma 2.51. *Let μ_ω be a simple point process with $a_k = 1$ almost surely for all $k\in\mathbb{N}$. Then $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ is a random closed set of isolated points with no limit points. On the other hand, if $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ is a random closed set that almost surely has no limit points then μ_ω is a point process.*

Proof. Let μ_ω be a point process. For open $V\subset\mathbb{R}^d$ and compact $K\subset\mathbb{R}^d$ let

$$f_{V,R}(x) = \operatorname{dist}\big(\,x,\,\mathbb{R}^d\backslash\left(V\cap\mathbb{B}_R(0)\right)\big)\,,\qquad f_\delta^K(x) = \max\left\{\,1-\frac{1}{\delta}\operatorname{dist}(x,K)\,,\,0\,\right\}.$$

Then $f_{V,R}$ is Lipschitz with constant 1 and f_δ^K is Lipschitz with constant $\frac{1}{\delta}$ and support in $\mathbb{B}_\delta(K)$. Moreover, since μ_ω is locally bounded, the number of points x_k that lie within $\mathbb{B}_1(K)$ is bounded. In particular, we obtain

$$\mathbb{X}^{-1}(\mathfrak{F}_V) = \bigcup_{R>0}\left\{\omega\,:\,\int_{\mathbb{R}^d} f_{V,R}\,\mathrm{d}\mu_\omega > 0\right\}\,,$$
$$\mathbb{X}^{-1}\big(\mathfrak{F}^K\big) = \bigcap_{\delta>0}\left\{\omega\,:\,\int_{\mathbb{R}^d} f_\delta^K\,\mathrm{d}\mu_\omega > 0\right\}\,,$$

are measurable. Since $\mathfrak{F}_V$ and $\mathfrak{F}^K$ generate the σ-algebra on $\mathfrak{F}\left(\mathbb{R}^d\right)$, it follows that $\omega\to\mathbb{X}_\omega$ is measurable.

In order to prove the opposite direction, let $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ be a random closed set of points. Since $\mathbb{X}_\omega$ has almost surely no limit points the measure μ_ω is locally bounded almost surely. We prove that μ_ω is a random measure by showing that

$$\forall f \in C_c\big(\mathbb{R}^d\big)\ : \qquad F\,:\,\omega \mapsto \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_\omega \text{ is measurable.}$$

For $\delta > 0$ let $\mu_\omega^\delta(A) := \left(\left|\mathbb{S}^{d-1}\right|\delta^d\right)^{-1}\mathcal{L}(A\cap\mathbb{B}_\delta(\mathbb{X}_\omega))$. By Lemmas 2.41 and 2.46 we obtain that $F_\delta\,:\,\omega\mapsto\int_{\mathbb{R}^d} f\,\mathrm{d}\mu_\omega^\delta$ are measurable. Moreover, for almost every ω we find $F_\delta(\omega)\to F(\omega)$ uniformly and hence F is measurable. □

Corollary 2.52. *A random simple point process μ_ω is stationary iff $\mathbb{X}_\omega$ is stationary.*

Hence we can provide the following definition based on Definition 2.43.

Definition 2.53. A point process μ_ω and a random set $\mathbf{P}$ are jointly stationary if $\mathbf{P}$ and $\mathbb{X}$ are jointly stationary.

Lemma 2.54. *Let $\mathbb{X}_\omega = (x_i)_{i\in\mathbb{N}}$ be a Matern point process from Example 2.49 with distance r and let for $\delta < \frac{r}{2}$ be $\mathbf{B}(\omega) := \bigcup_i \overline{B_\delta(x_i)}$. Then $\mathbf{B}(\omega)$ is a random closed set.*

Proof. This follows from Lemma 2.41: $\mathbb{X}_\omega$ is measurable and $\mathbb{X}\mapsto\overline{B_\delta(\mathbb{X})}$ is continuous. Hence $\mathbf{B}(\omega)$ is measurable. □

2.13 Dynamical Systems on $\mathbb{Z}^d$

Definition 2.55. Let $\left(\hat\Omega,\hat{\mathscr{F}},\hat{\mathbb{P}}\right)$ be a probability space. A discrete dynamical system on $\hat\Omega$ is a family $(\hat\tau_z)_{z\in r\mathbb{Z}^d}$ of measurable bijective mappings $\hat\tau_z : \hat\Omega\mapsto\hat\Omega$ satisfying (i)-(iii) of Definition 2.27 with $\mathbb{R}^d$ replaced by $\mathbb{Z}^d$. A set $A\subset\hat\Omega$ is almost invariant if for every $z\in r\mathbb{Z}^d$ it holds $\mathbb{P}\left((A\cup\hat\tau_z A)\setminus(A\cap\hat\tau_z A)\right) = 0$ and $\hat\tau$ is called ergodic w.r.t. $r\mathbb{Z}^d$ if every almost invariant set has measure 0 or 1.

Similar to the continuous dynamical systems, also in this discrete setting an ergodic theorem can be proved.

Theorem 2.56 (See Krengel and Tempel'man [16, 27])**.** *Let $(A_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a convex averaging sequence, let $(\hat\tau_z)_{z\in r\mathbb{Z}^d}$ be a dynamical system on $\hat\Omega$ with invariant σ-algebra $\mathscr{I}$ and let $f\,:\,\hat\Omega\to\mathbb{R}$ be measurable with $|\mathbb{E}(f)|<\infty$. Then for almost all $\hat\omega\in\hat\Omega$*

$$|A_n|^{-1}\sum_{z\in A_n\cap r\mathbb{Z}^d} f(\hat\tau_z\hat\omega)\to r^{-d}\mathbb{E}(f|\mathscr{I})\,. \tag{2.50}$$

In the following, we restrict to $r=1$ for simplicity of notation.

Let $\Omega_0\subset\mathbb{R}^d$. We consider an enumeration $(\xi_i)_{i\in\mathbb{N}}$ of $\mathbb{Z}^d$ such that $\hat\Omega := \Omega_0^{\mathbb{Z}^d} = \Omega_0^{\mathbb{N}}$ and write $\hat\omega = (\hat\omega_{\xi_1},\hat\omega_{\xi_2},\dots) = (\hat\omega_1,\hat\omega_2,\dots)$ for all $\hat\omega\in\hat\Omega$. We define a metric on $\hat\Omega$ through

$$d(\hat\omega_1,\hat\omega_2) = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|\hat\omega_{1,\xi_k}-\hat\omega_{2,\xi_k}|}{1+|\hat\omega_{1,\xi_k}-\hat\omega_{2,\xi_k}|}\,.$$

We write $\Omega_n := \Omega_0^n$ and $\mathbb{N}_n := \{k\in\mathbb{N}\,:\,k\geq n+1\}$. The topology of $\hat\Omega$ is generated by the open sets $A\times\Omega_0^{\mathbb{N}_n}$, where for some $n>0$, $A\subset\Omega_n$ is an open set. In case Ω_0 is compact, the space $\hat\Omega$ is compact. Further, $\hat\Omega$ is separable in any case since Ω_0 is separable (see [14]).

Lemma 2.57. *Suppose for every $n \in \mathbb{N}$ there exists a probability measure $\mathbb{P}_n$ on Ω_n such that for every measurable $A_n \subset \Omega_n$ it holds $\mathbb{P}_{n+k}\big(A_n \times \Omega^k\big) = \mathbb{P}_n(A_n)$. Then $\mathbb{P}$ defined as follows defines a probability measure on Ω:*

$$\mathbb{P}\big(A_n \times \Omega_0^{\mathbb{N}_n}\big) := \mathbb{P}_n(A_n)\,.$$

Proof. We consider the ring

$$\mathcal{R} = \bigcup_{n\in\mathbb{N}} \left\{A \times \Omega_0^{\mathbb{N}_n} \;:\; A \subset \Omega_n \text{ is measurable}\right\}$$

and make the observation that $\mathbb{P}$ is additive and positive on $\mathcal{R}$ and $\mathbb{P}(\emptyset) = 0$. Next, let $(A_j)_{j\in\mathbb{N}}$ be an increasing sequence of sets in $\mathcal{R}$ such that $A := \bigcup_j A_j \in \mathcal{R}$. Then, there exists $\tilde{A}_1 \subset \Omega_0^n$ such that $A_1 = \tilde{A}_1 \times \Omega_0^{\mathbb{N}_n}$ and since $A_1 \subset A_2 \subset \dots \subset A$, for every $j > 1$, we conclude $A_j = \tilde{A}_j \times \Omega_0^{\mathbb{N}_n}$ for some $\tilde{A}_j \subset \Omega_n$. Therefore, $\mathbb{P}(A_j) = \mathbb{P}_n(\tilde{A}_j) \to \mathbb{P}_n(\tilde{A}) = \mathbb{P}(A)$ where $A = \tilde{A} \times \Omega_0^{\mathbb{N}_n}$. We have thus proved that $\mathbb{P} : \mathcal{R} \to [0,1]$ can be extended to a measure on the Borel-σ-Algebra on Ω (See [2, Theorem 6-2]). □

We define for $z \in \mathbb{Z}^d$ the mapping

$$\hat{\tau}_z : \hat{\Omega} \to \hat{\Omega}\,, \qquad \hat{\omega} \mapsto \hat{\tau}_z\hat{\omega}\,, \quad \text{where } \left(\hat{\tau}_z\hat{\omega}\right)_{\xi_i} = \hat{\omega}_{\xi_i+z} \text{ component wise}\,.$$

Remark 2.58. In this paper, we consider particularly $\Omega_0 = \{0,1\}$. Then $\hat{\Omega} := \Omega_0^{\mathbb{Z}^d}$ is equivalent to the power set of $\mathbb{Z}^d$ and every $\hat{\omega} \in \hat{\Omega}$ is a sequence of 0 and 1 corresponding to a subset of $\mathbb{Z}^d$. Shifting the set $\hat{\omega} \subset \mathbb{Z}^d$ by $z \in \mathbb{Z}^d$ corresponds to an application of $\hat{\tau}_z$ to $\hat{\omega} \in \hat{\Omega}$.

Now, let $\mathbf{P}(\omega)$ be a stationary ergodic random open set and let $r > 0$. Recalling (2.1) the map $\omega \mapsto \mathbf{P}_{-r}(\omega)$ is measurable due to Lemma 2.41 and we can define $\mathbb{X}_r(\mathbf{P}(\omega)) := 2r\mathbb{Z}^d \cap \mathbf{P}_{-\frac{r}{2}}(\omega)$.

Lemma 2.59. *If $\mathbf{P}$ is a stationary ergodic random open set then the set*

$$\mathbb{X} = \mathbb{X}_r(\omega) := \mathbb{X}_r(\mathbf{P}(\omega)) := 2r\mathbb{Z}^d \cap \mathbf{P}_{-r}(\omega) \tag{2.51}$$

is a stationary random point process w.r.t. $2r\mathbb{Z}^d$.

Proof. By a simple scaling we can w.l.o.g. assume $2r = 1$ and write $\mathbb{X} = \mathbb{X}_r$. Evidently, $\mathbb{X}$ corresponds to a process on $\mathbb{Z}^d$ with values in $\Omega_0 = \{0,1\}$ writing $\mathbb{X}(z) = 1$ if $z \in \mathbb{X}$ and $\mathbb{X}(z) = 0$ if $z \notin \mathbb{X}$. In particular, we write $(\omega, z) \mapsto \mathbb{X}(\omega, z)$. This process is stationary as the shift invariance of $\mathbf{P}$ induces a shift-invariance of $\hat{\mathbb{P}}$ with respect to $\hat{\tau}_z$. It remains to observe that the probabilities $\mathbb{P}(\mathbb{X}(z) = 1)$ and $\mathbb{P}(\mathbb{X}(z) = 0)$ induce a random measure on $\hat{\Omega}$ in the way described in Remark 2.58. □

Remark 2.60. If $\mathbf{P}$ is mixing one can follow the lines of the proof of Lemma 2.34 to find that $\mathbb{X}_r(\mathbf{P}(\omega))$ is ergodic. However, in the general case $\mathbb{X}_r(\mathbf{P}(\omega))$ is not ergodic. This is due to the fact that by nature $(\tau_z)_{z\in\mathbb{Z}^d}$ on Ω has more invariant sets than$(\tau_x)_{x\in\mathbb{R}^d}$. For sufficiently complex geometries the map $\Omega \to \hat{\Omega}$ is onto.

Definition 2.61 (Jointly stationary)**.** We call a point process $\mathbb{X}$ with values in $2r\mathbb{Z}^d$ to be strongly jointly stationary with a random set $\mathbf{P}$ if the functions $\chi_{\mathbf{P}(\omega)}$, $\chi_{\mathbb{X}(\omega)}$ are jointly stationary w.r.t. the dynamical system $(\tau_{2rx})_{x\in\mathbb{Z}^d}$ on Ω.

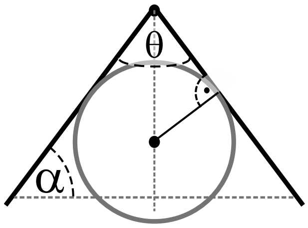

Figure 2: How to fit a ball into a cone.

3 Quantifying Nonlocal Regularity Properties of the Geometry

3.1 Microscopic Regularity

Lemma 3.1. *Let $\mathbf{P}$ be a Lipschitz domain. Then for every $p_0 \in \partial\mathbf{P}$ with $\delta(p_0) > 0$ the following holds: For every $\delta < \delta(p_0)$ and $M := M_\delta(p_0) > 0$ there exists $y \in \mathbf{P}$ with $|p_0 - y| = \frac{\delta}{4}$ such that with $r(p_0) := \frac{\delta}{4(1+M)}$ it holds $\mathbb{B}_{r(p_0)}(y) \subset \mathbb{B}_{\delta/2}(p_0)$.*

Proof. We can assume that $\partial\mathbf{P}$ is locally a cone as in Figure 2. With regard to Figure 2, for $p_0 \in \partial\mathbf{P}$ with δ and M as in the statement we can place a right circular cone with vertex (apex) p_0 and axis ν and an aperture $\theta = \pi - 2\arctan M$ inside $\mathbb{B}_\delta(p_0)$, where $\alpha = \arctan M$. In other words, it holds $\tan(\alpha) = \tan\left(\frac{\pi-\theta}{2}\right) = M$. Along the axis we may select y with $|p_0 - y| = \frac{\delta}{4}$. Then the distance R of y to the cone is given through

$$|y - p_0|^2 = R^2 + R^2\tan^2\left(\frac{\pi-\theta}{2}\right) \quad \Rightarrow \quad R = \frac{|y-p_0|}{\sqrt{1+M^2}}\,.$$

In particular $r(p_0)$ as defined above satisfies the claim. □

Continuity properties of δ, M and ϱ

Lemma 3.2. *Let $\mathfrak{r} > 0$, $\mathbf{P}$ be a Lipschitz domain and recall (1.7). Then $\partial\mathbf{P}$ is δ_Δ-regular in the sense of Definition 2.22. In particular, $\delta_\Delta : \partial\mathbf{P} \to \mathbb{R}$ is locally Lipschitz continuous with Lipschitz constant 4 and for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\tilde{p} \in \mathbb{B}_{\varepsilon\delta}(p) \cap \partial\mathbf{P}$ it holds*

$$\frac{1-\varepsilon}{1-2\varepsilon}\delta_\Delta(p) > \delta_\Delta(\tilde{p}) > \delta_\Delta(p) - |p - \tilde{p}| > (1-\varepsilon)\,\delta_\Delta(p)\,. \tag{3.1}$$

Remark 3.3. The latter lemma does ***not*** imply global Lipschitz regularity of δ_Δ. It could be that $2\delta_\Delta(p) < |p - \tilde{p}| < 3\delta_\Delta(p)$ and p and $\tilde{p}$ are connected by a path inside $\partial\mathbf{P}$ with the shortest path of length $10\delta_\Delta(p)$. Then Lemma 3.2 would have to be applied successively along this path yielding an estimate of $|\delta_\Delta(p) - \delta_\Delta(\tilde{p})| \leq 40\,|p - \tilde{p}|$.

Proof of Lemma 3.2. It is straight forward to verify that $|p - \tilde{p}| < \varepsilon\delta_\Delta(p)$ implies $\delta_\Delta(\tilde{p}) > (1-\varepsilon)\delta_\Delta(p)$ and we conclude with Lemma 2.24. □

With regard to Lemma 2.3, the local extension operator is related to $\delta(p)/\sqrt{4M(p)^2+2}$, where $M(p)$ is the related Lipschitz constant. While we can quantify $\delta(p)$ in terms of $\delta(\tilde{p})$ and $|p - \tilde{p}|$, this does not work for $M(p)$. Hence we cannot quantify $\delta(p)/\sqrt{4M(p)^2+2}$ in terms of its neighbors. This drawback is compensated by a variational trick in the following statement.

Lemma 3.4. *Let $\mathbf{P}$ be a Lipschitz domain and let $\delta \leq \delta_\Delta$ satisfy (3.1) such that $\partial\mathbf{P}$ is δ-regular. For $p \in \partial\mathbf{P}$ and let $M_r(p)$ be given in (1.8) and define for $n, K \in \mathbb{N}$*

$$\rho_n(p) := \sup_{r<\delta(p)} r\sqrt{4M_r(p)^2+2}^{\,-n}\,, \tag{3.2}$$

$$\hat\rho_{n,K}(p) := \inf\left\{\delta \leq \delta(p)\,:\, \sup_{r<2^{-K}\delta} r\sqrt{4M_r(p)^2+2}^{\,-n} \geq 2^{-K}\rho_n(p)\right\}. \tag{3.3}$$

Then for fixed $p \in \partial\mathbf{P}$ the functions $r \mapsto M_r(p)$ is right continuous and monotone increasing (i.e. u.s.c.). Furthermore, ρ_n is positive and locally Lipschitz continuous on $\partial\mathbf{P}$ with Lipschitz constant 4 and $\partial\mathbf{P}$ is ρ-regular in the sense of Definition 2.22. In particular, for $|p-\tilde p| < \varepsilon\rho_n(p)$ it holds

$$\frac{1-\varepsilon}{1-2\varepsilon}\rho_n(p) > \rho_n(\tilde p) > \rho_n(p) - |p-\tilde p| > (1-\varepsilon)\,\rho_n(p)\,. \tag{3.4}$$

Furthermore, $\hat\rho_{n,K} \leq \delta$ is well defined.

Remark 3.5. Like in Remark 3.3 this does ***not*** imply global Lipschitz regularity of ρ_n or $\hat\rho_n$.

Corollary 3.6. *Every Lipschitz domain $\mathbf{P}$ has extension order 1 and symmetric extension order 2.*

Proof. This follows from $\hat\rho_{n,3} \leq \delta$ and Lemmas 2.3 and 2.7 applied to $\mathbb{B}_{\frac18\hat\rho_{n,3}}(p_0)$ and $\mathbb{B}_{\frac18\rho_n}(p_0)$. □

Proof of Lemma 3.4. Right continuity of $r \mapsto M_r(p)$ follows because for every $0 < r < R$ because M being Lipschitz constant of $\partial\mathbf{P}$ in $\mathbb{B}_R(p)$ implies M being Lipschitz constant of $\partial\mathbf{P}$ in $\mathbb{B}_r(p)$.

Let $|p-\tilde p| < \varepsilon\rho(p) < \varepsilon\delta(p)$ implying $\delta(\tilde p) \geq (1-\varepsilon)\,\delta(p)$ by Lemma 3.2. For every $\eta > 0$ let $r_\eta \in (\rho(p), \delta(p))$ such that $\rho(p) \leq (1+\eta)\, r_\eta\sqrt{4M_{r_\eta}(p)^2+2}^{\,-n}$. Since $r_\eta > \rho(p)$ and $|p-\tilde p| < \varepsilon\rho(p)$ we find $\mathbb{B}_{r_\eta}(p) \supset \mathbb{B}_{(1-\varepsilon)r_\eta}(\tilde p)$ and hence $M_{(1-\varepsilon)r_\eta}(\tilde p) \leq M_{r_\eta}(p)$. This implies at the same time that $\partial\mathbf{P}$ is ρ-regular and that

$$\rho(\tilde p) \geq \frac{(1-\varepsilon)\, r_\eta}{\sqrt{4M_{(1-\varepsilon)r_\eta}(\tilde p)^2+2}^{\,n}} \geq \frac{(1-\varepsilon)\, r_\eta}{\sqrt{4M_{r_\eta}(p)^2+2}^{\,n}} \geq \frac{(1-\varepsilon)}{(1+\eta)}\rho(p)\,.$$

Since η was arbitrary, we conclude $\rho(\tilde p) \geq (1-\varepsilon)\,\rho(p)$. Moreover, we find $|p-\tilde p| < \frac{\varepsilon}{1-\varepsilon}\rho(\tilde p)$. And we conclude the first part with Lemma 2.24.

Second, it holds for every $r < \delta$ and $\varepsilon \in (0,1)$ that

$$\varepsilon r\sqrt{4M_r(p)^2+2}^{\,-n} \leq \varepsilon r\sqrt{4M_{\varepsilon r}(p)^2+2}^{\,-n}$$

and choosing $\varepsilon = 2^{-K}$ and taking the supremum on both sides, we infer $\hat\rho_{n,K} \leq \delta$. □

Corollary 3.7. *Let $\mathfrak{r} > 0$ and let $\mathbf{P} \subset \mathbb{R}^d$ be a locally (δ, M)-regular open set, where we restrict δ by $\delta(\cdot) \leq \frac{\mathfrak{r}}{4}$. Then there exists a countable number of points $(p_k)_{k\in\mathbb{N}} \subset \partial\mathbf{P}$ such that $\partial\mathbf{P}$ is completely covered by balls $\mathbb{B}_{\tilde\rho(p_k)}(p_k)$ where $\tilde\rho\,(p) := 2^{-5}\rho_n\,(p)$ for some $n \in \mathbb{N}$. Writing*

$$\tilde\rho_k := \tilde\rho(p_k)\,, \qquad \delta_k := \delta(p_k)\,.$$

For two such balls with $\mathbb{B}_{\tilde\rho_k}(p_k) \cap \mathbb{B}_{\tilde\rho_i}(p_i) \neq \emptyset$ it holds

$$\begin{aligned} &\frac{15}{16}\tilde\rho_i \leq \tilde\rho_k \leq \frac{16}{15}\tilde\rho_i \\ \text{and}\quad &\frac{31}{15}\min\{\tilde\rho_i, \tilde\rho_k\} \geq |p_i - p_k| \geq \frac12\max\{\tilde\rho_i, \tilde\rho_k\}\,. \end{aligned} \tag{3.5}$$

Furthermore, there exists $\mathfrak{r}_k \geq \frac{\tilde\rho_k}{32\left(1+M_{\tilde\rho(p_k)}(p_k)\right)}$ *and* y_k *such that* $\mathbb{B}_{\mathfrak{r}_k}(y_k) \subset \mathbb{B}_{\tilde\rho_k/8}(p_k)\cap\mathbf{P}$ *and* $\mathbb{B}_{2\mathfrak{r}_k}(y_k)\cap\mathbb{B}_{2\mathfrak{r}_j}(y_j)=\emptyset$ *for* $k\neq j$.

Proof. The existence of the points and Balls satisfying (3.5) follows from Theorem 2.25, in particular (2.35). It holds for $\mathbb{B}_{\tilde\rho_k}(p_k)\cap\mathbb{B}_{\tilde\rho_i}(p_i)\neq\emptyset$

$$|p_i-p_k|\leq\tilde\rho_i+\tilde\rho_k\leq\left(\frac{16}{15}+1\right)\tilde\rho_i\,.$$

Lemma 3.1 yields existence of y_k such that $\mathbb{B}_{\mathfrak{r}_k}(y_k)\subset\mathbb{B}_{\tilde\rho_k/8}(p_k)\cap\mathbf{P}$. The latter implies $\mathbb{B}_{\mathfrak{r}_k}(y_k)\cap\mathbb{B}_{\mathfrak{r}_j}(y_j)=\emptyset$ for $k\neq j$. □

Lemma 3.8. *Let* $\mathfrak{r}>0$, $\mathbf{P}\subset\mathbb{R}^d$ *be a locally* (δ,M)*-regular open set and let* $M_0\in(0,+\infty]$ *such that for every* $p\in\partial\mathbf{P}$ *there exists* $\delta>0$, $M<M_0$ *such that* $\partial\mathbf{P}$ *is* (δ,M)*-regular in* p. *For* $\alpha\in(0,1]$ *let* $\eta(p)=\alpha\delta_\Delta(p)$ *from Lemma 3.2 or* $\eta(p)=\alpha\rho_n(p)$ *from Lemma 3.4 and define*

$$\mathrm{M}_{[\eta]}(p):=\inf_{\delta>\eta(p)}\inf_{M}\left\{\,\mathbf{P}\text{ is }(\delta,M)\text{-regular in }p\right\}\,. \tag{3.6}$$

Then, for fixed ξ, $\mathrm{M}_{[\eta]}(\cdot):\ \partial\mathbf{P}\to\mathbb{R}$ *is upper semicontinuous and on each bounded measurable set* $A\subset\mathbb{R}^d$ *the quantity*

$$\mathrm{M}_{[\eta]}(A):=\sup_{p\in\overline{A}\cap\partial\mathbf{P}}M_{[\eta]}(p) \tag{3.7}$$

with $\mathrm{M}_{[\eta]}(A)=0$ *if* $\overline{A}\cap\partial\mathbf{P}=\emptyset$ *is well defined. The functions*

$$\mathrm{M}_{[\eta]}(A,\cdot):\ \mathbb{R}^d\to\mathbb{R}\,,\quad\mathrm{M}_{[\eta]}(A,x):=\mathrm{M}_{[\eta]}(A+x)\quad\text{with }\mathrm{M}_{[\eta]}(A,0)=\mathrm{M}_{[\eta]}(A)$$

are upper semicontinuous.

Remark 3.9. Note at this point that $M_{[\eta,r],\mathbb{R}^d}$ defined in (1.11) is a function on $\mathbb{R}^d$ and different from $M_{[\eta]}$.

Notation 3.10. The infimum in (3.6) is a $\liminf$ for $\delta\searrow\eta(p)$. We sometimes use the special notation

$$M_{[\eta],\mathfrak{r}}(x):=M_{[\eta],\mathbb{B}_{\mathfrak{r}}(0)}(x)\,.$$

Proof of Lemma 3.8. Let $p,\tilde p\in\partial\mathbf{P}$ with $|p-\tilde p|<\varepsilon\eta(p)$. Writing $\tilde\varepsilon:=\frac{\varepsilon}{1-\varepsilon}$ and $r\left(p,\varepsilon\right):=\left(\frac{1}{1-2\varepsilon}+\varepsilon\right)\eta(p)$ and

$$M(p,\varepsilon):=\inf_M\left\{\mathbb{B}_{r(p,\varepsilon)}(p)\cap\partial\mathbf{P}\text{ is }M\text{-Lipschitz graph}\right\}$$

as well as we observe from η-regularity that $\mathbb{B}_{\eta(\tilde p)}(\tilde p)\subset\mathbb{B}_{r(p,\varepsilon)}(p)$ and $\mathbb{B}_{\eta(p)}(p)\subset\mathbb{B}_{r(\tilde p,\tilde\varepsilon)}(\tilde p)$. Hence we find

$$\mathrm{M}_{[\eta]}(\tilde p)\leq M(p,\varepsilon)\,.$$

Observing that $M(p,\varepsilon)\searrow\mathrm{M}_{[\eta]}(p)$ as $\varepsilon\to0$ we find $\limsup_{\tilde p\to p}\mathrm{M}_{[\eta]}(\tilde p)\leq\mathrm{M}_{[\eta]}(p)$ and M is u.s.c.

Let $x\to0$. First observe that $\mathrm{M}_{[\eta]}(A)=\max_{y\in\overline{A}}\mathrm{M}_{[\eta]}(y)$. The set $\overline{A}$ is compact and hence $\overline{A}+x\to\overline{A}$ in the Hausdorff metric as $x\to0$. Let $y_x\in\overline{A}+x$ such that $\mathrm{M}_{[\eta]}(y_x)=\mathrm{M}_{[\eta]}\left(A,x\right)$. Since $\overline{A}+x\to\overline{A}$ w.l.o.g. we find $y_x\to y$ converges and $y\in\overline{A}$. Hence

$$\mathrm{M}_{[\eta]}(y)\geq\limsup_{x\to0}\mathrm{M}_{[\eta]}(y_x)=\limsup_{x\to0}\mathrm{M}_{[\eta]}(A,x)\,.$$

In particular, $M_{[\eta],A}(\cdot)$ is u.s.c. The u.s.c of $\mathfrak{m}_{[\eta]}(p,\xi)$ can be proved similarly. □

Measurability and Integrability of Extended Variables

Lemma 3.11. *Let $\mathfrak{r} > 0$, let $\mathbf{P} \subset \mathbb{R}^d$ be a Lipschitz domain and let $\eta, r : \partial\mathbf{P} \to \mathbb{R}$ be continuous such that $\eta \le \mathfrak{r}$ and $\mathbf{P}$ is η- and r-regular. For $\varepsilon \in (0,1]$ let $\eta(p) = \varepsilon\delta(p)$ from Lemma 3.2 or $\eta(p) = \varepsilon\rho_n(p)$, $n \in \mathbb{N}$, from Lemma 3.4. Then $\eta_{[r],\mathbb{R}^d}$ from (1.10) is measurable and $M_{[\eta,r],\mathbb{R}^d}$ from (1.11) is upper semicontinuous.*

In what follows, we write $A_{\eta,\mathfrak{r}} := F^{-1}\big((0, \frac{3}{2}\mathfrak{r})\big)$ for

$$F := \inf_{p\in\partial\mathbf{P}} f_p\,, \qquad f_p(x) := \begin{cases} \eta(p) & \text{if } x \in \mathbb{B}_{r(p)}(p) \\ 2\mathfrak{r} & \text{else} \end{cases} .$$

Proof. Step 1: We write $A = A_{\eta,\mathfrak{r}}$ for simplicity. Let $(p_i)_{i\in\mathbb{N}} \subset \partial\mathbf{P}$ be a dense subset. If $x \in \mathbb{B}_{r(p)}(p)$ for some $p \in \partial\mathbf{P}$ then also $x \in \mathbb{B}_{r(\tilde p)}(\tilde p)$ for $|p - \tilde p|$ sufficiently small, by continuity of η. Hence every f_p is upper semicontinuous and it holds $F = \inf_{i\in\mathbb{N}} f_{p_i}$. In particular, F is measurable and so is the set A. This implies $\eta_{[r],\mathbb{R}^d} = \chi_A F$ is measurable.

Step 2: We show that for every $a \in \mathbb{R}$ the preimage $M^{-1}_{[\eta,r],\mathbb{R}^d}([a,+\infty))$ is closed. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence with $M_{[\eta,r],\mathbb{R}^d}(x_k) \in [a,+\infty)$. Let $(p_k) \subset \partial\mathbf{P}$ be a sequence with $|x_k - p_k| \le r(p_k)$. W.l.o.g. assume $p_k \to p \in \partial\mathbf{P}$ and $x_k \to x \in \mathbb{R}^d$. Since r is continuous, it follows $|x - p| \le r(p)$. On the other hand $M_{[\eta]}(p) \ge \limsup_{k\to\infty} M_{[\eta]}(p_k)$ and thus $M_{[\eta,r],\mathbb{R}^d}(x) \ge M_{[\eta,r]}(p) \ge a$. □

Lemma 3.12. *Under the assumptions of Lemma 3.11 let $\tilde\eta := \eta_{[\frac{\eta}{8}],\mathbb{R}^d}$. Then there exists a constant $C > 0$ only depending on the dimension d such that for every bounded open domain $\mathbf{Q}$ and $k \in [0,4)$ it holds*

$$\int_{A_{\eta,\mathfrak{r}}\cap\mathbf{Q}} \chi_{\tilde\eta>0}\tilde\eta^{-\alpha} \le C\int_{\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}} \eta^{1-\alpha} M^{d-2}_{[\frac{\eta}{4}],\mathbb{R}^d}\,, \tag{3.8}$$

$$\int_{A_{\eta,\mathfrak{r}}\cap\mathbf{Q}} \tilde\eta^{-\alpha} M^r_{[k\frac{\eta}{8},\frac{\eta}{8}],\mathbb{R}^d} \le C\int_{\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}} \eta^{1-\alpha} M^r_{[k\frac{\eta}{8},\frac{\eta}{4}],\mathbb{R}^d} M^{d-2}_{[\frac{\eta}{4}],\mathbb{R}^d}\,. \tag{3.9}$$

Finally, it holds

$$x \in \mathbb{B}_{\frac{1}{8}\eta(p)}(p) \quad\Rightarrow\quad \eta(p) > \tilde\eta(x) > \frac{3}{4}\eta(p)\,. \tag{3.10}$$

Remark 3.13. Estimates (3.8)–(3.9) are only rough estimates and better results could be obtained via more sophisticated calculations that make use of particular features of given geometries.

Proof. We write $A = A_{\eta,\mathfrak{r}}$ for simplicity. Step 1: Given $x \in \mathbb{R}^d$ with $\tilde\eta(x) > 0$ let

$$p_x \in \operatorname{argmin}\left\{\eta(\tilde x)\ :\ \tilde x \in \partial\mathbf{P} \text{ s.t. } x \in \overline{\mathbb{B}_{\frac{1}{8}\eta(\tilde x)}(\tilde x)}\right\}\,. \tag{3.11}$$

Such p_x exists because $\partial\mathbf{P}$ is locally compact. We observe with help of the definition of p_x, the triangle inequality and (2.34)

$$x \in \mathbb{B}_{\frac{1}{8}\eta(p)}(p) \quad\Rightarrow\quad \eta(p_x) \le \eta(p) \quad\Rightarrow\quad |p - p_x| < \frac{\eta(p)}{4} \quad\Rightarrow\quad \eta(p_x) > \frac{3}{4}\eta(p)\,.$$

The last line particularly implies (3.10) and

$$\forall p \in \partial\mathbf{P}\ \forall x \in \mathbb{B}_{\frac{\eta(p)}{8}}(p) : \quad \tilde\eta(x) > \frac{3\eta(p)}{4}\,.$$

Step 2: By Theorem 2.25 we can chose a countable number of points $(p_k)_{k\in\mathbb{N}} \subset \partial\mathbf{P}$ such that $\Gamma = \partial\mathbf{P}$ is completely covered by balls $B_k := \mathbb{B}_{\xi(p_k)}(p_k)$ where $\xi(p) := 2^{-4}\eta(p)$. For simplicity of notation we write $\eta_k := \eta(p_k)$ and $\xi_k := \xi(p_k)$. Assume $x \in A$ with $p_x \in \Gamma$ given by (3.11). Since the balls B_k cover Γ, there exists p_k with $|p_x - p_k| < \xi_k = 2^{-4}\eta_k$, implying $\eta(p_x) < \frac{2^4}{2^4-1}\eta_k$ and hence $|x - p_k| \leq \left(2^{-4} + \frac{2^{-3}2^4}{2^4-1}\right)\eta_k < \frac{3}{16}\eta_k$. Hence we find

$$\forall x \in A\ \exists p_k\ :\quad x \in \mathbb{B}_{\frac{3}{16}\eta_k}(p_k)\,.$$

Step 3: For $p \in \Gamma$ with $x \in \mathbb{B}_{\frac{1}{4}\eta(p)}(p) \cap \mathbb{B}_{\frac{1}{8}\eta(p_x)}(p_x)$ we can distinguish two cases:

1 $\eta(p) \geq \eta(p_x)$: Then $p_x \in \mathbb{B}_{\frac{3}{8}\eta(p)}(p)$ and hence $\eta(p_x) \geq \frac{5}{8}\eta(p)$ by (2.34).

2 $\eta(p) < \eta(p_x)$: Then $p \in \mathbb{B}_{\frac{3}{8}\eta(p_x)}(p_x)$ and hence$\eta(p_x) > \frac{1-\frac{3}{8}}{1-\frac{6}{8}}\eta(p) = \frac{5}{2}\eta(p)$ by (2.34).

and hence

$$x \in \mathbb{B}_{\frac{1}{4}\eta(p)}(p) \qquad \Rightarrow \qquad \tilde{\eta}(x) = \eta(p_x) > \frac{5}{8}\eta(p)\,.$$

Step 4: Let $k \in \mathbb{N}$ be fixed and define $B_k = \mathbb{B}_{\frac{1}{4}\eta_k}(p_k)$, $M_k := M_{\frac{1}{4}\eta_k}(p_k)$. By construction, every B_j with $B_j \cap B_k \neq \emptyset$ satisfies $\eta_j \geq \frac{1}{2}\eta_k$ and hence if $B_j \cap B_k \neq \emptyset$ and $B_i \cap B_j \neq \emptyset$ we find $|p_j - p_i| \geq \frac{1}{4}\eta_k$ and $|p_j - p_k| \leq 3\eta_k$. This implies that

$$\exists C > 0:\ \forall k \quad \#\left\{j\ :\ B_j \cap B_k \neq \emptyset\right\} \leq C\,.$$

We further observe that the minimal surface of $B_k \cap \partial\mathbf{P}$ is given in case when $B_k \cap \partial\mathbf{P}$ is a cone with opening angle $\frac{\pi}{2} - \arctan M(p_k)$. The surface area of $B_k \cap \partial\mathbf{P}$ in this case is bounded by $\frac{1}{d-1}\left|\mathbb{S}^{d-2}\right|\eta_k^{d-1}\left(M_k+1\right)^{2-d}$. This particularly implies up to a constant independent from k:

$$\begin{aligned}\int_{A\cap\mathbf{Q}\cap\mathbf{P}}\tilde{\eta}^{-\alpha} &\lesssim \sum_{k:\,B_k\cap\mathbf{Q}\neq\emptyset}\int_{A\cap B_k\cap\mathbf{P}}\eta_k^{-\alpha}\\ &\lesssim \sum_{k:\,B_k\cap\mathbf{Q}\neq\emptyset}\int_{A\cap B_k\cap\partial\mathbf{P}}\eta^{1-\alpha}M_{[\frac{\eta}{4}]}^{d-2}\\ &\lesssim \int_{A\cap\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}}\eta^{1-\alpha}M_{[\frac{\eta}{4}]}^{d-2}\,.\end{aligned}$$

The second integral formula follows in a similar way. □

3.2 Mesoscopic Regularity and Isotropic Cone Mixing

Lemma 3.14. *Let $\mathbf{P}(\omega)$ be a stationary and ergodic random open set such that*

$$\mathbb{P}(\mathbf{P}\cap\mathbb{I}=\emptyset) < 1\,.$$

Then there exists $\mathfrak{r} > 0$ and a positive, monotonically decreasing function $\tilde{f}$ such that almost surely $\mathbf{P}(\omega)$ is $(\mathfrak{r}, \tilde{f})$-mesoscopic regular.

Proof. Step 1: For some $\mathfrak{r} > 0$ and with positive probability $p_{\mathfrak{r}} > 0$ the set $(0,1)^d \cap \mathbf{P}$ contains a ball with radius $5\sqrt{d}\mathfrak{r}$. Otherwise, for every $r > 0$ the set $(0,1)^d \cap \mathbf{P}$ almost surely does not contain an open ball with radius r. In particular with probability 1 the set $(0,1)^d \cap \mathbf{P}$ does not contain any ball. Hence $(0,1)^d \cap \mathbf{P} = \emptyset$ almost surely, contradicting the assumptions.

Step 2: We define

$$\tilde{f}(R) := \mathbb{P}\big(\nexists x : \mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x) \subset \mathbb{B}_R(0) \cap \mathbf{P}(\omega)\big) .$$

The stationary ergodic random measure $\tilde{\mu}_\omega(\,\cdot\,) := \mathcal{L}\big(\,\cdot\, \cap \mathbf{P}_{-4\sqrt{d}\mathfrak{r}}(\omega)\big)$ has positive intensity $\tilde{\lambda}_0 > p_{\mathfrak{r}} \left|\mathbb{S}^{d-1}\left(\sqrt{d}\mathfrak{r}\right)^d\right|$ and it holds $\tilde{\mu}_\omega(A) \neq 0$ implies the existence of $\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x) \subset \mathbf{P} \cap \mathbb{B}_{4\sqrt{d}\mathfrak{r}}(A)$. Assuming that $\liminf_{R\to\infty} \tilde{f} > 0$ there exists for every $R > 0$ a set $\Omega_R \subset \Omega$ with $\tilde{\mu}_\omega(\mathbb{B}_R(0)) = 0$ for every $\omega \in \Omega_R$ with $\Omega_{R+1} \subset \Omega_R$ and

$$\Omega_\infty := \bigcap_{R>0} \Omega_R \quad \text{satisfies} \quad \mathbb{P}(\Omega_\infty) = \liminf_{R\to\infty} \tilde{f}(R) > 0 .$$

But for almost every $\omega \in \Omega_\infty$ it holds by the ergodic theorem

$$\lim_{R\to\infty} |\mathbb{B}_R(0)|^{-1} \tilde{\mu}_\omega(\mathbb{B}_R(0)) \geq \lambda_0 ,$$

which implies the existence of $\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x) \subset \mathbb{B}_R(0) \cap \mathbf{P}(\omega)$, a contradiction. □

Definition 3.15 (Isotropic cone mixing)**.** A random set $\mathbf{P}(\omega)$ is isotropic cone mixing if there exists a jointly stationary point process $\mathbb{X}$ in $\mathbb{R}^d$ or $2\mathfrak{r}\mathbb{Z}^d$, $\mathfrak{r} > 0$, such that almost surely two points $x, y \in \mathbb{X}$ have mutual minimal distance $2\mathfrak{r}$ and such that $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbb{X}(\omega)) \subset \mathbf{P}(\omega)$. Further there exists a function $f(R)$ with $f(R) \to 0$ as $\mathbb{R} \to \infty$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that with $\mathbf{E} := \{e_1, \dots e_d\} \cup \{-e_1, \dots - e_d\}$ ($\{e_1, \dots e_d\}$ being the canonical basis of $\mathbb{R}^d$)

$$\mathbb{P}(\forall e \in \mathbf{E} :\ \mathbb{X} \cap \mathbb{C}_{e,\alpha,R}(0) \neq \emptyset) \geq 1 - f(R) . \tag{3.12}$$

Lemma 3.16 (A simple sufficient criterion for (3.12))**.** *Let $\mathbf{P}$ be stationary ergodic and $(\mathfrak{r}, \tilde{f})$-regular. Then $\mathbf{P}$ is isotropic cone mixing with $f(R) = 2d\tilde{f}\Big(\left((\tan\alpha)^{-1} + 1\right)^{-1} R\Big)$ and with*

$$\mathbb{X}(\omega) := \mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega)) = 2\mathfrak{r}\mathbb{Z}^d \cap \mathbf{P}_{-\mathfrak{r}}(\omega) = \left\{x \in 2\mathfrak{r}\mathbb{Z}^d \ :\ \mathbb{B}_{\frac{\mathfrak{r}}{2}}(x) \subset \mathbf{P}\right\} \tag{3.13}$$

from Lemma 2.59. Vice versa, if $\mathbf{P}$ is isotropic cone mixing for f then $\mathbf{P}$ satisfies (1.6) with $\tilde{f} = f$.

Proof of Lemma 3.16. Because of $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ it holds for $a > 1$

$$\mathbb{P}\big(\exists e \in \mathbf{E} \ :\ \nexists x \in \mathbb{B}_R(aRe) : \mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x) \subset \mathbb{B}_R(aRe) \cap \mathbf{P}\big) \leq 2d\tilde{f}(R) .$$

The existence of $\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x) \subset \mathbb{B}_R(aRe) \cap \mathbf{P}(\omega)$ implies that there exists at least one $x \in \mathbb{X}_{\mathfrak{r}}\left(\mathbf{P}\left(\omega\right)\right)$ such that $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x) \subset \mathbb{B}_R(aRe) \cap \mathbf{P}(\omega)$ and we find

$$\mathbb{P}\Big(\exists e \in \mathbf{E} \ :\ \nexists x \in \mathbb{X}_{\mathfrak{r}}(\mathbf{P}) : \mathbb{B}_{\frac{\mathfrak{r}}{2}}(x) \subset \mathbb{B}_R(aRe) \cap \mathbf{P}\Big) \leq 2d\tilde{f}(R) .$$

In particular, for $\alpha = \arctan a^{-1}$ and R large enough we discover

$$\mathbb{P}\left(\exists e \in \mathbf{E} \ :\ \mathbb{X}_{\mathfrak{r}}(\mathbf{P}) \cap \mathbb{C}_{e,\alpha,(a+1)R}\left(0\right) = \emptyset\right) \leq 2d\tilde{f}(R) .$$

The relation (3.12) holds with $f(R) = 2d\tilde{f}\big((a+1)^{-1} R\big)$.

The other direction is evident. □

Properties of $\mathbb{X}$

The formulation of Definition 3.15 is particularly useful for the following statement.

Lemma 3.17 (Size distribution of cells)**.** *Let $\mathbf{P}(\omega)$ be a stationary and ergodic random open set that is isotropic cone mixing for $\mathbb{X}(\omega)$, $\mathfrak{r} > 0$, $f : (0,\infty) \to \mathbb{R}$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then $\mathbb{X}$ and its Voronoi tessellation have the following properties:*

1 *If $G(x)$ is the open Voronoi cell of $x \in \mathbb{X}(\omega)$ with diameter $d(x)$ then d is jointly stationary with $\mathbb{X}$ and for some constant $C_\alpha > 0$ depending only on α*

$$\mathbb{P}(d(x) > D) < f\left(C_\alpha^{-1}\frac{D}{2}\right). \tag{3.14}$$

2 *For $x \in \mathbb{X}(\omega)$ let $\mathcal{I}(x) := \{y \in \mathbb{X} \,:\, G(y) \cap \mathbb{B}_{\mathfrak{r}}(G(x)) \neq \emptyset\}$. Then*

$$\#\mathcal{I}(x) \leq \left(\frac{4d(x)}{\mathfrak{r}}\right)^d. \tag{3.15}$$

Proof. 1. W.l.o.g. let $x_k = 0$. The first part follows from the definition of isotropic cone mixing: We take arbitrary points $x_{\pm j} \in C_{\pm \mathrm{e}_j,\alpha,R}(0) \cap \mathbb{X}$. Then the planes given by the respective equations $\left(x - \frac{1}{2}x_{\pm j}\right) \cdot x_{\pm j} = 0$ define a bounded cell around 0, with a maximal diameter $D(\alpha, R) = 2C_\alpha R$ which is proportional to R. The constant C_α depends nonlinearly on α with $C_\alpha \to \infty$ as $\alpha \to \frac{\pi}{2}$. Estimate (3.14) can now be concluded from the relation between R and $D(\alpha, R)$ and from (3.12).

2. This follows from Lemma 2.30. □

Lemma 3.18. *Let $\mathbb{X}_{\mathfrak{r}}$ be a stationary and ergodic random point process with minimal mutual distance $2\mathfrak{r}$ for $\mathfrak{r} > 0$ and let $f : (0,\infty) \to \mathbb{R}$ be such that the Voronoi tessellation of $\mathbb{X}$ has the property*

$$\forall x \in \mathfrak{r}\mathbb{Z}^d : \quad \mathbb{P}(d(x) > D) = f(D).$$

Furthermore, let $n, s : \mathbb{X}_{\mathfrak{r}} \to [1,\infty)$ be measurable and i.i.d. among $\mathbb{X}_{\mathfrak{r}}$ and let n, s, d be independent from each other. Let either

$$G_{n(x)}(x) = \begin{cases} x + n(x)\,(G(x) - x) & \text{or} \\ \mathbb{B}_{n(x)d(x)}(x) \end{cases}$$

be the cell $G(x)$ enlarged by the factor $n(x)$ or a ball of radius $n(x)d(x)$ arround x, let $d(x) = \mathrm{diam} G(x)$ and let

$$\mathfrak{b}_n(y) := \sum_{x \in \mathbb{X}_{\mathfrak{r}}} \chi_{G_n(x)} d(x)^\eta s(x)^\xi n(x)^\zeta,$$

where $\eta, \xi, \zeta > 0$ are fixed a constant. Then $\mathfrak{b}_n$ is jointly stationary with $\mathbb{X}_{\mathfrak{r}}$ and for every $r > 1$ there exists $C \in (0, +\infty)$ such that

$$\mathbb{E}(\mathfrak{b}_n^p) \leq C\left(\sum_{k,N,S=1}^{\infty} (k+1)^{d(p+1)+\eta p + r(p-1)}\,(S+1)^{\xi p + r(p-1)}\,(N+1)^{d(p+1)+\zeta p + r(p-1)}\,\mathbb{P}_{d,k}\mathbb{P}_{n,N}\mathbb{P}_{s,S}\right). \tag{3.16}$$

where

$$\begin{aligned} \mathbb{P}_{d,k} &:= \mathbb{P}(d(x) \in [k, k+1)) = f(k) - f(k+1), \\ \mathbb{P}_{n,N} &:= \mathbb{P}(n(x) \in [N, N+1)), \\ \mathbb{P}_{s,S} &:= \mathbb{P}(s(x) \in [S, S+1)). \end{aligned}$$

Corollary 3.19. *Under the assumptions of Lemma 3.18 let additionally* $n = const$, $s = const$. *Then*

$$\mathbb{E}(\mathfrak{b}^p) \leq C \sum_{k,N=1}^{\infty} (k+1)^{d+(d+\eta+1)p} f(k) \,.$$

Proof of Lemma 3.18. We write $\mathbb{X}_{\mathfrak{r}} = (x_i)_{i\in\mathbb{N}}$, $d_i = d(x_i)$, $n_i = n(x_i)$, $s_i := s(x_i)$. Let

$$\begin{aligned} X_{k,N,S}(\omega) &:= \left\{x_i \in \mathbb{X}_{\mathfrak{r}} \,:\, d_i \in [k,k+1),\; n_i \in [N,N+1),\; s_i \in [S,S+1)\right\} \,, \\ A_{k,N,S} &:= \bigcup_{x\in X_{k,N,S}} G_{n(x)}(x) \,, \qquad A_{k,N} := \bigcup_{S\in\mathbb{N}} A_{k,N,S} \,, \qquad X_{k,N} := \bigcup_{S\in\mathbb{N}} X_{k,N,S} \,. \end{aligned}$$

We observe that the mutual minimal distance implies

$$\forall x \in \mathbb{R}^d : \quad \#\left\{x_i \in X_{k,N,S} : x \in G_{n(x_i)}(x_i)\right\} \leq \mathbb{S}^{d-1} (N+1)^d \, (k+1)^d \, \mathfrak{r}^{-d} \,, \tag{3.17}$$

which follows from the uniform boundedness of cells $G_{n(x)}(x)$, $x \in X_{k,N}$ and the minimal distance of $|x_i - x_j| > 2\mathfrak{r}$. Then, writing $B_R := \mathbb{B}_R(0)$ for every $y \in \mathbb{R}^d$ it holds by stationarity and the ergodic theorem

$$\begin{aligned} \mathbb{P}(y \in G_{n_i}(x_i) \,:\, x_i \in X_{k,N,S}) &= \lim_{R\to\infty} |B_R|^{-1} \left|A_{k,N} \cap B_R\right| \mathbb{P}_{s,S} \\ &\leq \lim_{R\to\infty} |B_R|^{-1} \left| B_R \cap \bigcup_{x_i \in X_{k,N}} G_{n_i}(x_i) \right| \mathbb{P}_{s,S} \\ &\leq \lim_{R\to\infty} |B_R|^{-1} \sum_{x_i \in X_{k,N}\cap B_R} \left|\mathbb{S}^{d-1}\right| (N+1)^d \, (k+1)^d \, \mathfrak{r}^{-d} \mathbb{P}_{s,S} \\ &\to \mathbb{P}_{d,k}\mathbb{P}_{n,N}\mathbb{P}_{s,S} \, (N+1)^d \left|\mathbb{S}^{d-1}\right| (k+1)^d \, \mathfrak{r}^{-d} \,. \end{aligned}$$

In the last inequality we made use of the fact that every cell $G_{n(x)}(x)$, $x \in X_{k,N}$, has volume smaller than $\mathbb{S}^{d-1} \, (N+1)^d \, (k+1)^d$. We note that for $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} &\int_{\mathbf{Q}} \left(\sum_{x\in\mathbb{X}_{\mathfrak{r}}} \chi_{G_{n(x)}} d(x)^\eta s(x)^\xi n(x)^\zeta \right)^p \\ &\quad \leq \int_{\mathbf{Q}} \left(\sum_{k=1}^{\infty}\sum_{N=1}^{\infty}\sum_{S=1}^{\infty} \left(\sum_{x\in X_{k,N,S}} \chi_{G_{n(x)}(x)} \, (k+1)^\eta \, (N+1)^\xi (S+1)^\zeta \right) \right)^p \\ &\quad \leq \int_{\mathbf{Q}} \left(\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^q \right)^{\frac{p}{q}} \left(\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^{-p} \left(\sum_{x\in X_{k,N,S}} \chi_{G_{n(x)}(x)} \, (k+1)^\eta \, (N+1)^\xi (S+1)^\zeta \right)^p \right) \,. \end{aligned}$$

Due to (3.17) we find

$$\sum_{x\in X_{k,N,S}} \chi_{G_{n(x)}(x)} \leq \chi_{A_{k,N,S}} \, (N+1)^d \, (k+1)^d \left|\mathbb{S}^{d-1}\right|$$

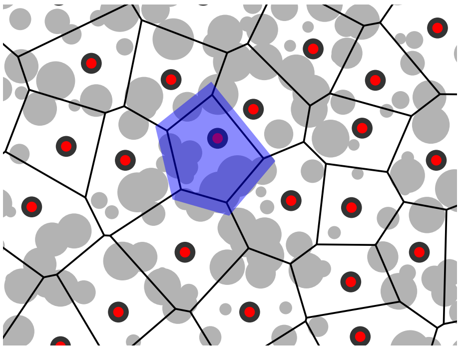

Figure 3: Gray: a Poisson ball process. Black balls: balls of radius $\mathfrak{r} > 0$. Red Balls: radius $\frac{\mathfrak{r}}{2}$. The Voronoi tessellation is generated from the centers of the red balls. The existence of such tessellations is discussed in Section 3.2. Blue region: $\mathfrak{A}_{1,k}$.

and obtain for $q = \frac{p}{p-1}$ and $C_q := \left(\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^q\right)^{\frac{p}{q}} \left|\mathbb{S}^{d-1}\right|^p$:

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R}\left(\sum_{x\in\mathbb{X}_\mathfrak{r}} \chi_{G_n(x)} d(x)^\eta s(x)^\xi n(x)^\zeta\right)^p \\ &\leq C_q \frac{1}{|B_R|}\int_{B_R}\left(\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^{-p}\chi_{A_{k,N,S}}\left(N+1\right)^{dp+\zeta p}\left(k+1\right)^{dp+\eta p}\left(S+1\right)^{\xi p}\right) \\ &\to C_q\left(\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^{-p}\left(k+1\right)^{d(p+1)+\eta p}\left(N+1\right)^{d(p+1)+\zeta p}\left(S+1\right)^{\xi p}\mathbb{P}_{s,S}\mathbb{P}_{d,k}\mathbb{P}_{n,N}\right)\end{aligned}$$

For the sum $\sum_{k,N,S=1}^{\infty} \alpha_{k,N,S}^q$ to converge, it is sufficient that $\alpha_{k,N,S}^q = (k+1)^{-r}(N+1)^{-r}(S+1)^{-r}$ for some $r > 1$. Hence, for such r it holds $\alpha_{k,N,S} = (k+1)^{-r/q}(N+1)^{-r/q}(S+1)^{-r/q}$ and thus (3.16). □

4 Extension and Trace Properties from (δ, M)-Regularity

4.1 Preliminaries

For this whole section, let $\mathbf{P}$ be a Lipschitz domain which furthermore satisfies the following assumption.

Remark 4.1. All calculations that follow in the present Section 4 equally work for arbitrarily distributed radii $\mathfrak{r}_a$ associated to x_a and replacing the constant $\mathfrak{r}$, e.g. with

$$\mathcal{M}_a u := \fint_{\mathbb{B}_{\frac{\mathfrak{r}_a}{16}}(x_a)} u\,, \qquad \overline{\nabla_{\mathcal{M},a}^{\perp}} u := \fint_{\mathbb{B}_{\frac{\mathfrak{r}_a}{16}}(x_a)} \left(\nabla - \nabla^s\right) u\,.$$

However, for simplicity of presentation, we chose to work with constant $\mathfrak{r}$ from the start.

Assumption 4.2. *Let $\mathbf{P}$ be an open (unbounded) set and let $\mathbb{X}_\mathfrak{r} = (x_a)_{a\in\mathbb{N}}$ be a set of points having mutual distance $|x_a - x_b| > 2\mathfrak{r}$ if $a \neq b$ and with $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_a) \subset \mathbf{P}$ for every $a \in \mathbb{N}$ (e.g. $\mathbb{X}_\mathfrak{r}(\mathbf{P})$, see (2.51)). We construct from $\mathbb{X}_\mathfrak{r}$ a Voronoi tessellation and denote by $G_a := G(x_a)$ the Voronoi cell corresponding to x_a with diameter d_a with $\mathfrak{A}_{1,a} := \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_a)$. Let $\tilde{\Phi}_0 \in C^\infty(\mathbb{R};[0,1])$ be monotone decreasing with $\tilde{\Phi}_0' > -\frac{4}{\mathfrak{r}}$, $\tilde{\Phi}_0(x) = 1$ if $x \leq 0$ and $\tilde{\Phi}_0(x) = 0$ for $x \geq \frac{\mathfrak{r}}{2}$. We define on $\mathbb{R}^d$ the Lipschitz functions*

$$\tilde{\Phi}_a(x) := \tilde{\Phi}_0\left(\operatorname{dist}\left(x, G_a\right)\right) \quad \textit{and} \quad \Phi_a(x) := \tilde{\Phi}_a(x)\left(\sum_b \tilde{\Phi}_b(x)\right)^{-1}. \tag{4.1}$$

Lemma 2.20 implies

$$\forall x\in\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_a):\quad \#\left\{b\,:\,x\in\mathfrak{A}_{1,b}\right\}\leq\left(\frac{4d_a}{\mathfrak{r}}\right)^d \tag{4.2}$$

and thus (4.1) yields for some C depending only on $\tilde{\Phi}_0$ that

$$|\nabla\Phi_a|\leq Cd_a^d\quad\text{and}\quad\forall k:\ |\nabla\Phi_k|\,\chi_{\mathfrak{A}_{1,a}}\leq Cd_a^d\,. \tag{4.3}$$

Definition 4.3 (Weak Neighbors)**.** Under the Assumption 4.2, two points $x_a,x_b\in\mathbb{X}_{\mathfrak{r}}$ are called to be weakly connected (or weak neighbors), written $a\sim\sim b$ or $x_a\sim\sim x_b$ if $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_a)\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_b)\neq\emptyset$. For $\mathbf{Q}\subset\mathbb{R}^d$ open we say $\mathfrak{A}_{1,a}\sim\sim\mathbf{Q}$ if $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathfrak{A}_{1,a})\cap\mathbf{Q}\neq\emptyset$. We then define

$$\mathbb{X}_{\mathfrak{r}}(\mathbf{Q}):=\left\{x_a\in\mathbb{X}_{\mathfrak{r}}:\ \mathfrak{A}_{1,a}\sim\sim\mathbf{Q}\neq\emptyset\right\}\,,\quad \mathbf{Q}^{\sim\sim}:=\bigcup_{\mathfrak{A}_{1,a}\sim\sim\mathbf{Q}}\mathfrak{A}_{1,a}\,. \tag{4.4}$$

In view of Assumption 4.2 we bound δ_Δ by $\mathfrak{r}>0$ and recall (3.1). As announced in the introduction, we apply Corollary 3.7 for $n\in\mathbb{N}$ (we study mostly $n=1$ and $n=2$ in the following) to obtain a complete covering of $\partial\mathbf{P}$ by balls $\mathbb{B}_{\tilde{\rho}_n(p_i^n)}(p_i^n)$, $(p_i^n)_{k\in\mathbb{N}}$, where $\tilde{\rho}_n(p):=2^{-5}\rho_n(p)$. Recalling (3.2)–(3.3) we define with $\tilde{\rho}_{n,i}:=\tilde{\rho}_n(p_i^n)$, $\hat{\rho}_{n,i}:=\hat{\rho}_{n,3}(p_i^n)$ and

$$A_{1,i}^n:=\mathbb{B}_{\tilde{\rho}_{n,i}}(p_i^n)\,,\quad A_{2,i}^n:=\mathbb{B}_{3\tilde{\rho}_{n,i}}(p_i^n)\,,\quad A_{3,i}^n:=\mathbb{B}_{\hat{\rho}_{n,i}}(p_i^n)\,,\quad B_{n,i}:=\mathbb{B}_{\frac{1}{8}\tilde{\rho}_{n,i}}(p_i^n)\,, \tag{4.5}$$

where we recall the construction of $\mathfrak{r}_{n,\alpha,i}$ and $y_{n,\alpha,i}$ in (1.16)–(1.17) and note that $\mathbb{B}_{\tilde{\rho}_{n,i}}(p_i^n)\supset\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})$ independent from α.

Lemma 4.4. *For $n\in\mathbb{N}$, $\alpha\in[0,1]$ and any two balls $A_{1,i}^n\cap A_{1,j}^n\neq\emptyset$ either $A_{1,i}^n\subset A_{2,j}^n$ or $A_{1,j}^n\subset A_{2,i}^n$ and*

$$A_{1,i}^n\cap A_{1,j}^n\neq\emptyset\quad\Rightarrow\quad\mathbb{B}_{\frac{1}{2}\tilde{\rho}_{n,i}}(p_i)\subset A_{2,j}^n\text{ and }\mathbb{B}_{\frac{1}{2}\tilde{\rho}_{n,j}}(p_j)\subset A_{2,i}^n\,. \tag{4.6}$$

Furthermore, there exists a constant C depending only on the dimension d and some $\hat{d}\in[0,d]$ such that

$$\forall k\qquad\#\left\{j\,:\,A_{1,j}^n\cap A_{1,i}^n\neq\emptyset\right\}+\#\left\{j\,:\,A_{2,j}^n\cap A_{2,i}^n\neq\emptyset\right\}\leq C\,, \tag{4.7}$$

$$\forall x\qquad\#\left\{j\,:\,x\in A_{1,j}^n\right\}+\#\left\{j\,:\,x\in A_{2,j}^n\right\}\leq C+1\,, \tag{4.8}$$

$$\forall x\qquad\#\left\{j\,:\,x\in\overline{\mathbb{B}_{\hat{\rho}_{n,j}}(p_j)}\right\}<C\big(1+M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}(x)\big)^{n\hat{d}}\,. \tag{4.9}$$

Finally, there exist non-negative functions $\phi_{n,0}$ and $(\phi_{n,i})_{k\in\mathbb{N}}$ independent from α such that for $k\geq1$: $\operatorname{supp}\phi_{n,i}\subset A_{1,i}^n$, $\phi_{n,i}|_{B_{n,j}}\equiv0$ for $k\neq j$. Further, $\phi_{n,0}\equiv0$ on all $B_{n,i}$ and on $\partial\mathbf{P}$ and $\sum_{k=0}^\infty\phi_{n,i}\equiv1$ and there exists C depending only on d such that for all $k\in\mathbb{N}$ it holds

$$x\in A_{1,i}^n\qquad\Rightarrow\qquad\forall j\in\mathbb{N}\cup\{0\}:\quad|\nabla\phi_{n,j}(x)|\leq C\tilde{\rho}_{n,i}^{-1}\,. \tag{4.10}$$

Remark 4.5. We usually can improve $\hat{d}$ to at least $\hat{d}=d-1$. To see this assume $\partial\mathbf{P}$ is flat on the scale of δ. Then all points p_i lie on a $d-1$-dimensional plane and we can thus improve the argument in the following proof to $\hat{d}=d-1$.

Proof. (4.6) follows from $(3.5)_2$. For improved readability we drop the indeces n and α.

Let $k \in \mathbb{N}$ be fixed. By construction in Corollary 3.7, every $A_{1,j}$ with $A_{1,j} \cap A_{1,k} \neq \emptyset$ satisfies $\tilde{\rho}_j \geq \frac{1}{2}\tilde{\rho}_k$ and hence if $A_{1,j} \cap A_{1,k} \neq \emptyset$ and $A_{1,i} \cap A_{1,k} \neq \emptyset$ we find $|p_j - p_i| \geq \frac{1}{4}\tilde{\rho}_k$ and $|p_j - p_k| \leq 3\tilde{\rho}_k$. This implies (4.7)–(4.8) for $A_{1,j}$ and the statement for $A_{2,j}$ follows analogously.

For two points p_i, p_j such that $x \in A_{3,i} \cap A_{3,j}$ it holds due to the triangle inequality $|p_i - p_j| \leq \max\left\{\frac{1}{4}\hat{\rho}_i, \frac{1}{4}\hat{\rho}_j\right\}$. Let $\mathbb{X}(x) := \left\{p_i \in \mathbb{X} : \, x \in \overline{\mathbb{B}_{\frac{1}{8}\hat{\rho}_i}(p_i)}\right\}$ and choose $\tilde{p}(x) = \tilde{p} \in \mathbb{X}(x)$ such that $\delta_{\mathsf{m}} := \delta(\tilde{p})$ is maximal. Then $\mathbb{X}(x) \subset \mathbb{B}_{\frac{1}{4}\delta_{\mathsf{m}}}(\tilde{p})$ and every $p_i \in \mathbb{X}(x)$ satisfies $\delta_{\mathsf{m}} > \delta_i > \frac{1}{3}\delta_{\mathsf{m}}$. Correspondingly, $\tilde{\rho}_i > \frac{1}{3}\delta_{\mathsf{m}} 2^{-5} \tilde{M}^{-n}_{\frac{\delta_i}{8}} > \frac{1}{3}\delta_{\mathsf{m}} 2^{-5} \tilde{M}^{-n}_{\frac{3\delta_m}{8}}$ for all such p_i. In view of (3.5) this lower local bound of $\tilde{\rho}_i$ implies a lower local bound on the mutual distance of the p_i. Since this distance is proportional to $\delta_{\mathsf{m}} \tilde{M}^{-n}_{\frac{3\delta_m}{8}}$, this implies (4.9) with $\hat{d} = d$. This is by the same time the upper estimate on $\hat{d}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be symmetric, smooth, monotone on $(0, \infty)$ with $\phi' \leq 2$ and $\phi = 0$ on $(1, \infty)$. For each k we consider a radially symmetric smooth function $\hat{\phi}_k(x) := \phi\left(\frac{|x-p_k|^2}{\tilde{\rho}_k}\right)$ and an additional function $\tilde{\phi}_0(x) = \mathrm{dist}(x, \partial\mathbf{P} \cup \bigcup_k B_{n,k})$. In a similar way we modify $\tilde{\phi}_k := \hat{\phi}_k \, \mathrm{dist}\left(x, \bigcup_{j\neq k} B_{n,j}\right)$ such that $\tilde{\phi}_k|_{B_{n,j}} \equiv 0$ for $j \neq k$. Then we define $\phi_k := \tilde{\phi} / \left(\tilde{\phi}_0 + \sum_j \tilde{\phi}_j\right)$. Note that by construction of $\mathfrak{r}_k$ and y_k we find $\phi_k|_{B_k} \equiv 1$ and $\sum_{k\geq 1} \phi_k \equiv 1$ on $\partial\mathbf{P}$.

Estimate (4.10) follows from (4.7). □

4.2 Extensions preserving the Gradient norm via (δ, M)-Regularity of $\partial\mathbf{P}$

By Lemma 2.3 in case $n = 1$ there exist local extension operator

$$\mathcal{U}_{n,i} : \; W^{1,p}\big(\mathbf{P} \cap A^n_{3,i}\big) \to W^{1,p}\Big(\mathbb{B}_{\frac{1}{8}\rho_{n,i}}(p^n_i) \backslash \mathbf{P}\Big) \hookrightarrow W^{1,p}\big(A^n_{2,i} \backslash \mathbf{P}\big) \tag{4.11}$$

which is linear continuous with bounds

$$\|\nabla \mathcal{U}_{n,i} u\|_{L^p(A^n_{2,i} \backslash \mathbf{P})} \leq 2M_{n,i} \|\nabla u\|_{L^p\left(A^n_{3,i} \cap \mathbf{P}\right)}, \tag{4.12}$$

$$\|\mathcal{U}_{n,i} u\|_{L^p(A^1_{2,i} \backslash \mathbf{P})} \leq \|u\|_{L^p\left(A^1_{3,i} \cap \mathbf{P}\right)}. \tag{4.13}$$

Of course, higher $n > 1$ are always valid, but the result becomes worse, as we will see. However, in case $\partial\mathbf{P}$ is locally always in the upper half plane, the case $n = 0$ is also valid, improving the estimates of the extension operators significantly. This phenomenon is acknowledged through the Definition 1.9 of the extension order.

Definition 4.6. Using Notation 1.10 for every $\mathbf{Q} \subset \mathbb{R}^d$ let

$$\begin{aligned} \mathcal{U}_{n,\alpha,\mathbf{Q}} : C^1\left(\overline{\mathbf{P} \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}\right) &\to C^1\left(\overline{\mathbf{Q} \backslash \mathbf{P}}\right), \\ u &\mapsto \chi_{\mathbf{Q} \backslash \mathbf{P}} \sum_{i \neq 0} \sum_a \Phi_a \left(\phi_{n,i} \left(\mathcal{U}_{n,i}(u - \tau_{n,\alpha,i} u) + \tau_{n,\alpha,i} u - \mathcal{M}_a u\right) + \mathcal{M}_a u\right). \end{aligned} \tag{4.14}$$

Due to the defintions, we find

$$\tau_{n,\alpha,i} \mathcal{M}_a u = \mathcal{M}_a u \,. \tag{4.15}$$

Lemma 4.7. *Let $\mathbf{P} \subset \mathbb{R}^d$ be a Lipschitz domain (i.e. locally (δ, M)-regular) with δ_Δ bounded by $\mathfrak{r} > 0$ and let Assumption 1.8 hold and let $\hat{d}$ be the constant from (4.9). Then for every bounded open $\mathbf{Q} \subset \mathbb{R}^d$ with $\mathbb{B}_{10\mathfrak{r}}(0) \subset \mathbf{Q}$ and $1 \le r < p$ the linear operator*

$$\mathcal{U}_{n,\alpha,\mathbf{Q}} : W^{1,p}\left(\mathbf{P} \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right) \to W^{1,r}(\mathbf{Q})$$

is continuous and writing

$$f_{\alpha,n,\hat{d}}(M,\cdot) := \left(\left(1 + M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{n\hat{d}}\left(1 + M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^r\left(1 + M_{[\tilde{\rho}_n],\mathbb{R}^d}\right)^{\alpha(d-1)}\right)^{\frac{p}{p-r}}$$

the operator $\mathcal{U}_{n,\alpha,\mathbf{Q}}$ satisfies for some C not depending on $\mathbf{P}$

$$\begin{aligned}
⨍_{\mathbf{Q}} \left|\nabla\left(\mathcal{U}_{n,\alpha,\mathbf{Q}} u\right)\right|^r \le{}& C\left(⨍_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})} f_{\alpha,n,\hat{d}}(M)\right)^{r\frac{p-r}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}} |\nabla u|^p\right)^{\frac{r}{p}} \\
&+ C\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_a \chi_{\mathfrak{A}_{1,a}}\left|\sum_{i\neq 0}\rho_{1,i}^{-1}\chi_{A_{1,i}}\left(\tau_{n,\alpha,i}u - \mathcal{M}_a u\right)\right|^r && (4.16)\\
&+ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left|\sum_{l=1}^d\sum_{a:\,\partial_l\Phi_a>0}\sum_{b:\,\partial_l\Phi_b<0}\frac{\partial_l\Phi_a\left|\partial_l\Phi_b\right|}{D_{l+}^\Phi}\left(\mathcal{M}_a u - \mathcal{M}_b u\right)\right|^r && (4.17)
\end{aligned}$$

$$⨍_{\mathbf{Q}} \left|\mathcal{U}_{n,\alpha,\mathbf{Q}} u\right|^r \le C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left(1 + M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{\frac{p\hat{d}}{p-r}}\right)^{\frac{p-r}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}}|u|^p\right)^{\frac{r}{p}}, \tag{4.18}$$

where

$$D_{l+}^\Phi := \sum_{a\neq 0:\,\partial_l\Phi_a<0}\left|\partial_l\Phi_a\right| . \tag{4.19}$$

Remark. Since the covering $A_{1,i}$ is locally finite we find

$$\left|\sum_{i\neq 0}\rho_{1,i}^{-1}\chi_{A_{1,i}}\left(\tau_{n,\alpha,i}u - \mathcal{M}_a u\right)\right|^r \le \sum_{i\neq 0}\rho_{1,i}^{-r}\chi_{A_{1,i}}\left|\tau_{n,\alpha,i}u - \mathcal{M}_a u\right|^r .$$

4.3 Extensions preserving the Symmetric Gradient norm via (δ, M)-Regularity of $\partial\mathbf{P}$

By Lemmas 3.4 and 2.7 in case $n = 2$ the local extension operator

$$\mathcal{U}_{n,k} : W^{1,p}\big(\mathbf{P} \cap A_{3,k}^n\big) \to W^{1,p}\Big(\mathbb{B}_{\frac{1}{8}\rho_{n,k}}(p_k^n)\setminus\mathbf{P}\Big) \hookrightarrow W^{1,p}\big(A_{2,k}^n\setminus\mathbf{P}\big) \tag{4.20}$$

is linear continuous with bounds

$$\left\|\nabla^s\mathcal{U}_{n,k}u\right\|_{L^p(\mathbb{B}_{\frac{1}{8}\rho_{n,k}}(p_k^n)\setminus\mathbf{P})} \le C\tilde{M}_{n,k}^2\left\|\nabla^s u\right\|_{L^p\left(A_{3,k}^n\cap\mathbf{P}\right)} . \tag{4.21}$$

Like in Section 4.2 lower values of n are possible, acknowledged by Definition 1.9 of symmetric extension order.

Definition 4.8. Using the notation of Definition 1.13 let

$$\mathcal{U}_{n,\alpha,\mathbf{Q}}:\, C^1\Big(\overline{\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}\Big)\to C^1\Big(\overline{\mathbf{Q}\backslash\mathbf{P}}\Big)\,,$$
$$u\mapsto\chi_{\mathbf{Q}\backslash\mathbf{P}}\sum_k\sum_a\Phi_a\left(\phi_{n,k}\left(\mathcal{U}_{n,k}\big(u-\tau^{\mathfrak{s}}_{n,\alpha,k}u\big)+\tau^{\mathfrak{s}}_{n,\alpha,k}u-\mathcal{M}^{\mathfrak{s}}_a\right)+\mathcal{M}^{\mathfrak{s}}_a u\right) \tag{4.22}$$

where $\mathcal{U}_{n,k}$ are the extension operators on $A^n_{3,k}$ given by the symmetric extension order of $\mathbf{P}$.

By definition we verify $\nabla^s\left(u-\tau^{\mathfrak{s}}_{n,\alpha,i}u\right)=\nabla^s u$ as well as

$$\fint_{\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})}(\nabla-\nabla^s)\left(u-\tau^{\mathfrak{s}}_{n,\alpha,i}u\right)=0\,,\qquad\fint_{\mathbb{B}_{\mathfrak{r}_{n,\alpha,i}}(y_{n,\alpha,i})}\left(u-\tau^{\mathfrak{s}}_{n,\alpha,i}u\right)=0$$

and similarly for $\mathcal{M}^{\mathfrak{s}}_a u$. Furthermore, it holds

$$\tau^{\mathfrak{s}}_{n,\alpha,i}\mathcal{M}^{\mathfrak{s}}_a u=\mathcal{M}^{\mathfrak{s}}_a u\,. \tag{4.23}$$

Lemma 4.9. *Let $\mathbf{P}\subset\mathbb{R}^d$ be a locally (δ,M)-regular open set with delta bounded by $\mathfrak{r}>0$ and let Assumption 1.8 hold and let $\hat{d}$ be the constant from (4.9). Then for every bounded open $\mathbf{Q}\subset\mathbb{R}^d$, $1\le r<p$ the operator*

$$\mathcal{U}_{n,\mathbf{Q}}:\,W^{1,p}\left(\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right)\to W^{1,r}\left(\mathbf{Q}\right)$$

is linear, well defined and with

$$f^{\mathfrak{s}}_{\alpha,n,\hat{d}}(M,\cdot\,):=\left(\left(1+M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{\hat{d}}\left(1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^{2r}\left(1+M_{[\tilde{\rho}_n],\mathbb{R}^d}\right)^{\alpha(d-1)}\right)^{\frac{p}{p-r}}$$

satisfies

$$\fint_{\mathbf{Q}}\left|\nabla^s\left(\mathcal{U}_{2,\mathbf{Q}}u\right)\right|^r\le C\left(\fint_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})}f^{\mathfrak{s}}_{\alpha,n,\hat{d}}(M)\right)^{r\frac{p-r}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}}|\nabla^s u|^p\right)^{\frac{r}{p}}$$
$$+C\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}\sum_a\chi_{\mathfrak{A}_{1,a}}\left|\sum_{i\neq0}\rho^{-1}_{1,i}\chi_{A_{1,i}}\left(\tau^{\mathfrak{s}}_{n,\alpha,i}u-\mathcal{M}^{\mathfrak{s}}_a u\right)\right|^r \tag{4.24}$$
$$+\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left|\sum_{l=1}^d\sum_{a:\,\partial_l\Phi_a>0}\sum_{b:\,\partial_l\Phi_b<0}\frac{\partial_l\Phi_a\left|\partial_l\Phi_b\right|}{D^{\Phi}_{l+}}\left(\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u\right)\right|^r \tag{4.25}$$
$$\fint_{\mathbf{Q}}|\mathcal{U}_{\mathbf{Q}}u|^r\le C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left(1+M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{\frac{2p\hat{d}}{p-r}}\right)^{\frac{p-r}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}}|u|^p\right)^{\frac{r}{p}}\,, \tag{4.26}$$

where D^{Φ}_{l+} is given by (4.19)

4.4 Support

Theorem 4.10. *For both operators given in (4.14) and (4.22) the following holds: For every bounded open set $\mathbf{Q}$ with $0\in\mathbf{Q}$ and $n_0,n_1\in\mathbb{N}$ let*

$$\forall M>1:\qquad\tilde{\mathbf{Q}}_M:=\bigcup_{x_a\in\mathbb{X}_{\mathfrak{r}}\cap M\mathbf{Q}}\mathbb{B}_{\mathfrak{r}}(G_a)\,.$$

If the mesoscopic regularity function $\tilde{f}$ of $\mathbf{P}$ satisfies $\tilde{f}(D) \leq C D^{-\frac{d-1}{\alpha}+\beta}$ for some $C > 0$, $\alpha \in (0,1)$ and $\beta > 1$ then there exists almost surely $M_0 > 1$ such that for every $M > M_0$ it holds $\tilde{\mathbf{Q}}_M \subset \mathbb{B}_{M^\alpha}(M\mathbf{Q})$.

Proof. We consider two balls $\mathbb{B}_r(0) \subset \mathbf{Q} \subset \mathbb{B}_R(0)$ with $r > 0$.

We write $\mathbf{Q}_M := M\mathbf{Q}$ and $\mathbb{B}_{M,\alpha,\mathbf{Q}} := \mathbb{B}_{M^\alpha}(\mathbf{Q}_M)$ for $\alpha \in (0,1)$ with $\mathbb{B}^{\complement}_{M,\alpha,\mathbf{Q}} := \mathbb{R}^d \backslash \mathbb{B}_{M,\alpha,\mathbf{Q}}$. For $k \in \mathbb{N}$ we introduce

$$\mathbf{Q}_{M,k} := \{x \in \mathbf{Q}_M : \operatorname{dist}(x, \partial\mathbf{Q}_M) \in [k,k)\}$$

and find

$$\mathbb{P}\Big(\tilde{\mathbf{Q}}_M \subset \mathbb{B}_{M,\alpha,\mathbf{Q}}\Big) = 1 - \sum_k \mathbb{P}\Big(\exists x_a \in \mathbf{Q}_{M,k} \cap \mathbb{X}_r : \mathbb{B}_{\mathfrak{r}}(G_a) \cap \mathbb{B}^{\complement}_{M,\alpha,\mathbf{Q}} \neq \emptyset\Big).$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\Big(\exists x_a \in \mathbf{Q}_{M,k} \cap \mathbb{X}_r : \mathbb{B}_{\mathfrak{r}}(G_a) \cap \mathbb{B}^{\complement}_{M,\alpha,\mathbf{Q}} \neq \emptyset\Big) \\
&\qquad \leq \mathbb{P}\Big(\exists x_a \in \mathbf{Q}_{M,k} \cap \mathbb{X}_r : \mathbb{B}_{2d_a}(x_a) \cap \mathbb{B}^{\complement}_{M,\alpha,\mathbf{Q}} \neq \emptyset\Big) \\
&\qquad \leq C\partial\mathbf{Q}_M \mathbb{P}\left(d_a > \frac{k}{2} + M^\alpha\right) \\
&\qquad \leq C M^{d-1}\left(\frac{k}{2} + M^\alpha\right)^{-\left(\frac{d-1}{\alpha}+\beta_1+\beta_2\right)} \leq C M^{-\beta_1}\left(\frac{k}{2}\right)^{-\beta_2}
\end{aligned}$$

where C depends only on the minimal mutual distance of the points, i.e. $\mathfrak{r}$, and the shape of $\mathbf{Q}$. Now, since $\beta > 1$ we can choose $\beta_2 > 1$ and find

$$\mathbb{P}\Big(\tilde{\mathbf{Q}}_M \subset \mathbb{B}_{M,\alpha,\mathbf{Q}}\Big) \geq 1 - C M^{-\beta_1}\,.$$

Since the right hand side converges to 1 as $M \to \infty$, we can conclude. □

4.5 Proof of Lemmas 4.7 and 4.9

Lemma 4.11. *Let α_i, u_i, $i = 1 \dots n$, be a family of real numbers such that $\sum_i \alpha_i = 0$ and let $\alpha_+ := \sum_{i:\,\alpha_i>0} \alpha_i$. Then*

$$\sum_i \alpha_i u_i = \sum_{i:\,\alpha_i>0} \sum_{j:\,\alpha_j<0} \frac{\alpha_i \left|\alpha_j\right|}{\alpha_+} \left(u_i - u_j\right)\,.$$

Proof.

$$\begin{aligned}
\sum_i \alpha_i u_i &= \sum_{i:\,\alpha_i>0} \alpha_i u_i + \sum_{j:\,\alpha_j<0} \alpha_j u_j \\
&= \sum_{i:\,\alpha_i>0} \alpha_i \sum_{j:\,\alpha_j<0} \frac{-\alpha_j}{\alpha_+} u_i + \sum_{j:\,\alpha_j<0} \alpha_j \sum_{i:\,\alpha_i>0} \frac{\alpha_i}{\alpha_+} u_j \\
&= \sum_{i:\,\alpha_i>0} \sum_{j:\,\alpha_j<0} \frac{\alpha_i \left|\alpha_j\right|}{\alpha_+} \left(u_i - u_j\right)\,.
\end{aligned}$$

□

Proof of Lemma 4.7. For improved readability, we drop the indeces n and α in the following.

We prove Lemma 4.7, i.e. (4.16) as (4.18) can be derived in a similar but shorter way. Lemma 4.9 can be proved in a similar way with some inequalities used below being replaced by the "symmetrized" counterparts. We will make some comments towards this direction in Step 4 of this proof.

For shortness of notation (and by abuse of notation) we write

$$\fint_{\mathbf{P}\cap\mathbf{Q}} g := \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}} g\,, \qquad \fint_{\mathbf{Q}\backslash\mathbf{P}} g := \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}} g$$

and similar for integrals over $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}$ and $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\backslash\mathbf{P}$. For simplicity of notation, we further drop the index 1 in the subsequent calculations.

We introduce the quantities

$$\tilde{M}_{\tilde{\rho},i} := M_{\tilde{\rho}(p_i)}(p_i)\,, \qquad \tilde{M}_{\delta,1,i} := M_{\frac{1}{8}\delta(p_i)}(p_i)\,, \qquad \tilde{M}_{\delta,2,i} := M_{\frac{3}{8}\delta(p_i)}(p_i)$$

note that $\tilde{\rho}_i \leq \frac{1}{8}\delta_i$ as well as $\sqrt{4M_i^2+2} \leq 2\tilde{M}_i$. Writing

$$\begin{aligned} u_i &:= \mathcal{U}_i\left(u-\tau_i u\right)+\tau_i u && \text{on } A_{2,i} \\ u_{i,a} &:= \mathcal{U}_i\left(u-\tau_i u\right)+\tau_i u-\mathcal{M}_a u && \text{on } A_{2,i}\cap\mathfrak{A}_{1,a} \end{aligned}$$

on $A_{2,i}$, The integral over $\nabla\left(\mathcal{U}_{\mathbf{Q}}u\right)$ can be estimated via

$$\fint_{\mathbf{Q}\backslash\mathbf{P}} \left|\nabla\left(\mathcal{U}_{\mathbf{Q}}u\right)\right|^r \leq C_r\left(I_1+I_2+I_3\right) \tag{4.27}$$

$$I_1 := \fint_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\sum_a \Phi_a\phi_i\nabla u_{i,a}\right|^r, \qquad I_2 := \fint_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\sum_a u_{i,a}\Phi_a\nabla\phi_i\right|^r,$$

$$I_3 := \fint_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\sum_a u_{i,a}\phi_i\nabla\Phi_a\right|^r. \tag{4.28}$$

Step 1: Using (1.14) and $\nabla u_{i,a} = \nabla u_i$ as well as $\sum_a \Phi_a = 1$ we conclude

$$\begin{aligned} I_1 = \fint_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\phi_i\nabla u_i\right|^r &\leq \fint_{\mathbf{Q}\backslash\mathbf{P}}\sum_{i\neq 0}\phi_i\left|\nabla u_i\right|^r \leq \fint_{\mathbf{Q}\backslash\mathbf{P}}\sum_{i\neq 0}\chi_{A_{1,i}}\left|\nabla u_i\right|^r \\ &\leq C\sum_{i\neq 0}\fint_{\mathbf{Q}}\chi_{A_{2,i}}\left|\nabla u_i\right|^r \leq C\sum_{i\neq 0}\tilde{M}_{\delta,2,i}^r\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\chi_{A_{3,i}}\left|\nabla u\right|^r . \end{aligned}$$

It only remains to estimate $\sum_i \chi_{A_{3,i}}(x)$. After a Hölder estimate and using $\tilde{M}_{\delta,2,i} \leq 1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}$ on $A_{3,i}$, we obtain

$$\begin{aligned} \sum_{i\neq 0}\tilde{M}_{\delta,2,i}^r\fint_{\mathbf{Q}\cap\mathbf{P}}\chi_{A_{3,i}}\left|\nabla u\right|^r &\leq \fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\sum_{i\neq 0}\chi_{A_{3,i}}\left(1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^r\left|\nabla u\right|^r \\ &\leq \left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left(\sum_{i\neq 0}\chi_{A_{3,i}}\right)^{\frac{p}{p-r}}\left(1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^{\frac{rp}{p-r}}\right)^{\frac{p-r}{p}}\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left|\nabla u\right|^p\right)^{\frac{r}{p}}. \end{aligned} \tag{4.29}$$

Step 2: Concerning I_2, we first observe that for each $j\neq 0$ it holds

$$\chi_{A_{1,j}\backslash\mathbf{P}}u_{j,a}\nabla\phi_0+\chi_{A_{1,j}\backslash\mathbf{P}}\sum_{i\neq 0}u_{j,a}\nabla\phi_i=0\,. \tag{4.30}$$

We use $\sum_{j\in\mathbb{N}}\chi_{A_{1,j}}\geq\chi_{A_{1,i}}$ for every $i\in\mathbb{N}$ together with (4.30) and (4.7) to obtain

$$\begin{aligned}\fint_{\mathbf{Q}\backslash\mathbf{P}}\Phi_a\left|\sum_{i\neq 0}u_{i,a}\nabla\phi_i\right|^r &\leq C\fint_{\mathbf{Q}\backslash\mathbf{P}}\Phi_a\left|\sum_{j\neq 0}\chi_{A_{1,j}}\sum_{i\neq 0}\left(u_{i,a}-u_{j,a}\right)\nabla\phi_i+u_{j,a}\nabla\phi_0\right|^r\\ &\overset{(4.7)}{\leq} C\fint_{\mathbf{Q}\backslash\mathbf{P}}\Phi_a\left(\sum_{j\neq 0}\chi_{A_{1,j}}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\left|u_{i,a}-u_{j,a}\right|^r\left|\nabla\phi_i\right|^r+\left|\sum_{j\neq 0}\chi_{A_{1,j}}u_{j,a}\nabla\phi_0\right|^r\right)\,.\end{aligned}$$

Note that

$$\forall a,b,i,j:\qquad u_{i,a}-u_{j,a}=u_{i,b}-u_{j,b}=u_i-u_j\,. \tag{4.31}$$

Furthermore u_i and u_j are defined on $A_{2,i}$ and $A_{2,j}$ respectively and $u_i=u_j$ on $\mathbb{B}_{\mathfrak{r}_j}(p_j)$ and $\mathbb{B}_{\mathfrak{r}_i}(p_i)$ because of (4.6). Furthermore, both functions can be extended from $A_{2,i}$ and $A_{2,j}$ to $\tilde u_i$ and $\tilde u_j$ on $\mathbb{B}_{4\tilde\rho_i}(p_i)$ and $\mathbb{B}_{4\tilde\rho_j}(p_j)$ respectively using Lemma 2.1 such that for some C independent from i,j

$$k=i,j:\quad\left\|\nabla\tilde u_k\right\|_{L^r\left(\mathbb{B}_{4\tilde\rho_k}(p_k)\right)}\leq C\left\|\nabla\tilde u_k\right\|_{L^r\left(A_{2,k}\right)}\,.$$

Since $\tilde u_i=\tilde u_j$ on $\mathbb{B}_{\mathfrak{r}_j}(p_j)$ and $\mathbb{B}_{\mathfrak{r}_i}(p_i)$ we chose $k(i,j)$ such that for $\tilde M_{k(i,j)}=1+\min\left\{M_{\tilde\rho,i},M_{\tilde\rho,j}\right\}$ and it holds by the Poincaré inequality (2.13), the microscopic regularity α and the estimate (3.4)

$$\int_{A_{1,i}\cap A_{1,j}}\left|u_{i,a}-u_{j,a}\right|^r\left|\nabla\phi_i\right|^r\leq C\rho_i^{-r}\int_{A_{1,k(i,j)}}\left|\tilde u_i-\tilde u_j\right|^r\leq C\tilde M_{k(i,j)}^{\alpha(d-1)}\int_{A_{2,k(i,j)}}\left|\nabla\left(\tilde u_i-\tilde u_j\right)\right|^r\,.$$

We obtain with microscopic regularity α, the finite covering (4.8) and the proportionality (3.5) that

$$\begin{aligned}\fint_{\mathbf{Q}\backslash\mathbf{P}}\sum_a\Phi_a\chi_{A_{1,j}}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\left|u_{i,a}-u_{j,a}\right|^r\left|\nabla\phi_i\right|^r &=\fint_{\mathbf{Q}\backslash\mathbf{P}}\chi_{A_{1,j}}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\left|\tilde u_i-\tilde u_j\right|^r\left|\nabla\phi_i\right|^r\\ &\leq\frac{C}{|\mathbf{Q}|}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\tilde M_{k(i,j)}^{\alpha(d-1)}\int_{A_{2,j}}\left|\nabla\left(\tilde u_i-\tilde u_j\right)\right|^r\\ &\leq\frac{C}{|\mathbf{Q}|}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\tilde M_{k(i,j)}^{\alpha(d-1)}\left(\int_{A_{2,i}}\left|\nabla\tilde u_i\right|^r+\int_{A_{2,j}}\left|\nabla\tilde u_j\right|^r\right)\\ &\leq\frac{C}{|\mathbf{Q}|}\sum_{i:\,A_{1,i}\cap A_{1,j}\neq\emptyset}\left(\int_{A_{3,i}\cup A_{3,j}}\tilde M^r_{[\frac18\delta],\mathbb{R}^d}\left(1+M_{[\tilde\rho],\mathbb{R}^d}\right)^{\alpha(d-1)}\left|\nabla u\right|^r\right)\,.\end{aligned}$$

Next we estimate from (4.10)

$$\begin{aligned}&\fint_{\mathbf{Q}\backslash\mathbf{P}}\sum_a\Phi_a\left|\sum_{j\neq 0}\chi_{A_{1,j}}u_{j,a}\nabla\phi_0\right|^r\\ &\qquad\leq C\fint_{\mathbf{Q}\backslash\mathbf{P}}\sum_a\Phi_a\left(\sum_{j\neq 0}\rho_j^{-r}\chi_{A_{1,j}}\left|\mathcal{U}_j\left(u-\tau_j u\right)\right|^r+\left|\sum_{j\neq 0}\rho_j^{-1}\chi_{A_{1,j}}\left(\tau_j u-\mathcal{M}_a u\right)\right|^r\right)\,.\end{aligned}$$

Using once more Assumption 1.8 and

$$\nabla\mathcal{U}_j\left(u-\tau_j u\right)=\nabla\left(\mathcal{U}_j\left(u-\tau_j u\right)+\tau_j u\right)=\nabla u_j \tag{4.32}$$

and $\sum_a \Phi_a = 1$ we infer from (2.13)

$$C\fint_{\mathbf{Q}\setminus\mathbf{P}}\sum_a\Phi_a\sum_{j\neq 0}\rho_j^{-r}\chi_{A_{1,j}}\left|\mathcal{U}_j\left(u-\tau_j u\right)\right|^r\leq\frac{C}{|\mathbf{Q}|}\sum_{j\neq 0}\left(1+M_{\tilde{\rho},j}\right)^{\alpha(d-1)}\int_{A_{2,j}}\left|\nabla u_j\right|^r .$$

Now we make use of the extension estimate (1.14) to find

$$\int_{A_{2,j}}\left|\nabla u_j\right|^r\leq C M_{\delta,1,j}^r\int_{A_{3,j}\cap\mathbf{P}}|\nabla u|^r$$

which in total implies for $f_{12}(M)=\left(1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^{\frac{rp}{p-r}}\left(1+M_{[\tilde{\rho}],\mathbb{R}^d}\right)^{\frac{p\alpha(d-1)}{p-r}}$

$$\begin{aligned}I_1+I_2\leq C&\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left(\sum_{i\neq 0}\chi_{A_{3,i}}\right)^{\frac{p}{p-r}}f_{12}(M)\right)^{\frac{p-r}{p}}\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}|\nabla u|^p\right)^{\frac{r}{p}}\\&+C\fint_{\mathbf{Q}\setminus\mathbf{P}}\sum_a\Phi_a\left|\sum_{j\neq 0}\rho_j^{-1}\chi_{A_{1,j}}\left(\tau_j u-\mathcal{M}_a u\right)\right|^r .\end{aligned}$$

Making use of (4.9) we find

$$\left|\sum_{i\neq 0}\chi_{A_{3,i}}\right|\leq\left(1+M_{[\frac{3\delta}{8},\frac{\delta}{8}],\mathbb{R}^d}\right)^{\hat{d}},$$

and it only remains to estimate I_3.

Step 3: We observe with help of $\sum_a\nabla\Phi_a=0$ and $\sum_{i\neq 0}\phi_i=\phi_0$ that

$$\sum_{i\neq 0}\sum_a u_{i,a}\phi_i\nabla\Phi_a=\sum_{i\neq 0}u_i\phi_i\sum_a\nabla\Phi_a+\sum_a\mathcal{M}_a u\nabla\Phi_a=\sum_a\mathcal{M}_a u\nabla\Phi_a .$$

and Lemma 4.11 yields

$$\begin{aligned}I_3&=\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\phi_0\sum_a\mathcal{M}_a u\nabla\Phi_a\right|^r\\&\leq\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\sum_{l=1}^d\sum_{a:\,\partial_l\Phi_a>0}\sum_{b:\,\partial_l\Phi_b<0}\frac{\partial_l\Phi_a\left|\partial_l\Phi_b\right|}{D_{l+}^{\Phi}}\left(\mathcal{M}_a u-\mathcal{M}_b u\right)\right|^r .\end{aligned}$$

Step 4: Concerning the proof of Lemma 4.9 we follow the above lines with the following modifications. We use the Nitsche extension operators. Hence, instead of (1.14) we use (1.15). The local extended functions are called

$$\begin{aligned}u_i&:=\mathcal{U}_i\left(u-\tau_i^{\mathfrak{s}}u\right)+\tau_i^{\mathfrak{s}}u&&\text{on }A_{2,i}\\u_{i,a}&:=\mathcal{U}_i\left(u-\tau_i^{\mathfrak{s}}u\right)+\tau_i^{\mathfrak{s}}u-\mathcal{M}_a^{\mathfrak{s}}u&&\text{on }A_{2,i}\cap\mathfrak{A}_{1,a}\end{aligned}$$

and (4.31) remains valid. We find it worth mentioning that $\nabla^s\left(\tau_i^{\mathfrak{s}}u-\mathcal{M}_a^{\mathfrak{s}}u\right)=0$ and hence

$$\nabla^s\left(\phi_i\Phi_a u_{i,a}\right)=\frac{1}{2}\left(\nabla(\phi_i\Phi_a)\otimes u_{i,a}+u_{i,a}\otimes\nabla(\phi_i\Phi_a)\right)+\phi_i\Phi_a\nabla^s\mathcal{U}_{2,i}\left(u-\tau_i^{\mathfrak{s}}u\right) .$$

We furthermore replace Lemma 2.1 by Lemma 2.6 and the Poincaré inequality (2.13) by (2.23). Finally we observe that (4.32) is replaced by

$$\nabla^s \mathcal{U}_j \left(u - \tau_j u\right) = \nabla^s \left(\mathcal{U}_j \left(u - \tau_j u\right) + \tau_j u\right) = \nabla^s u_j$$

□

4.6 Traces on (δ, M)-Regular Sets, Proof of Theorem 1.7

Proof. We use the covering of $\partial\mathbf{P}$ by $B_i := A^1_{1,i}$ and set $\tilde\rho_i := \tilde\rho_{1,i}$, $\hat\rho_i := \hat\rho_{i,5}(p^1_k)$ and write $M_i = M_{\hat\rho_i}(p^1_k)$, $\hat B_i := \mathbb{B}_{\hat\rho_i}(p^1_k)$. Due to Lemma 2.5 we find locally

$$\left\|\mathcal{T}u\right\|_{L^{p_0}(\partial\mathbf{P}\cap B_k)} \leq C_{p_0,p_0}\tilde\rho_k^{-\frac{1}{p_0}}\sqrt{4M_k^2+2}^{\frac{1}{p_0}+1}\left\|u\right\|_{W^{1,p_0}\left(\hat B_k\right)}\,. \tag{4.33}$$

We thus obtain

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\left|\sum_k \phi_k\mathcal{T}_k u\right|^r \leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac14}(\mathbf{Q})\cap\partial\mathbf{P}}\sum_k \chi_{B_k}\tilde\rho_k^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\left(\frac{1}{|\mathbf{Q}|}\sum_k\int_{\mathbb{B}_{\frac14}(\mathbf{Q})\cap\partial\mathbf{P}}\chi_{B_k}\tilde\rho_k\left|\mathcal{T}_k u\right|^{p_0}\right)^{\frac{r}{p_0}}$$

which yields by the uniform local bound of the covering, $\tilde\eta$ defined in Lemma 3.12, twice the application of (3.10) and (4.33)

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\left|\sum_k \phi_k\mathcal{T}_k u\right|^r \leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\rho_{5,\mathbb{R}^d}^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\cdot \cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}}\sum_k\chi_{\hat B_k}\sqrt{4M_k^2+2}^{\frac{1}{p_0}+1}\left(|\nabla u|^{p_0}+|u|^{p_0}\right)\right)^{\frac{r}{p_0}}.$$

With Hölders inequality and replacing M_k by $M_{[\frac{1}{32}\delta],\mathbb{R}^d}$, the last estimate leads to (1.12). The second estimate goes analogue since the local covering by $A_{2,k}$ is finite. □

5 The Issue of Connectedness

Remark 5.1*.* The following Lemmas 5.2 and 5.3 also hold with τ_i and $\mathcal{M}_a$ replaced by $\tau_i^{\mathfrak{s}}$ and $\mathcal{M}_a^{\mathfrak{s}}$ respectively.

Lemma 5.2. *Under Assumptions 1.8, 4.2 let* $\left(f_j\right)_{j\in\mathbb{N}}$ *be non-negative and have support* $\mathrm{supp} f_j \supset$

$\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_j)$ *and let* $\sum_{j\in\mathbb{N}} f_j \equiv 1$. *Writing* $\mathbb{X}(\mathbf{Q}) := \{x_j:\ \mathrm{supp} f_j \cap \mathbf{Q} \neq \emptyset\}$, *and*

$$F^1_{s,\iota}(\mathbf{Q}) := \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}\cap\mathbb{R}^d_3} |\tilde\rho_{\mathbb{R}^d}|^{-\frac{sr}{s-r}}\,\tilde M^{2-\iota}\right)^{\frac{s-r}{s}}$$

$$F^2_{s,\tilde s,\iota}(\mathbf{Q}) := \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \tilde M^{\frac{(\iota-2)(\tilde s-r)}{r(s-\tilde s)}}\right)^{r\frac{s-\tilde s}{\tilde s s}}$$

$$F^3_s(\mathbf{Q},u) := \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}} \sum_{i\neq 0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\,|\tau_i u-\mathcal{M}_a u|^s\right)^{\frac{r}{s}}$$

$$F^{3,\mathfrak{s}}_s(\mathbf{Q},u) := \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}} \sum_{i\neq 0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\,|\tau^{\mathfrak{s}}_i u-\mathcal{M}^{\mathfrak{s}}_a u|^s\right)^{\frac{r}{s}}$$

for every $l=1,\dots d$ *and* $r<\tilde s<s$ *it holds*

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_a\sum_{i\neq 0}\rho_{1,i}^{-r}\chi_{A_{1,i}}\chi_{\mathfrak{A}_{1,a}}\,|\tau_{n,\alpha,i}u-\mathcal{M}_a u|^r \leq \begin{cases} F^1_{s,2}(\mathbf{Q})\,F^3_s(\mathbf{Q}) \\ F^1_{s,d}(\mathbf{Q})\,F^2_{s,\tilde s,d}(\mathbf{Q})\,F^3_s(\mathbf{Q},u)\end{cases},$$

and

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_a\sum_{i\neq 0}\rho_{1,i}^{-r}\chi_{A_{1,i}}\chi_{\mathfrak{A}_{1,a}}\,\left|\tau^{\mathfrak{s}}_{n,\alpha,i}u-\mathcal{M}^{\mathfrak{s}}_a u\right|^r \leq \begin{cases} F^1_{s,2}(\mathbf{Q})\,F^{3,\mathfrak{s}}_s(\mathbf{Q}) \\ F^1_{s,d}(\mathbf{Q})\,F^2_{s,\tilde s,d}(\mathbf{Q})\,F^{3,\mathfrak{s}}_s(\mathbf{Q},u)\end{cases}.$$

Proof. We find from Hölder's and Jensen's inequality

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{i\neq 0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_a \rho_{1,i}^{-r}\frac{|\partial_l\phi_i|}{D_{l+}}f_a\,|\tau_i u-\mathcal{M}_a u|^r \\ \leq \begin{cases} F^1_{s,2}(\mathbf{Q})\,F^3_s(\mathbf{Q}) \\ F^1_{s,d}(\mathbf{Q})\,F^2_{s,\tilde s,d}(\mathbf{Q})\,F^3_s(\mathbf{Q})\end{cases}.\end{aligned}$$

The second part follows accordingly. □

Lemma 5.3. *Under Assumptions 1.8, 4.2 for every* $l=1,\dots d$ *and* $\tilde\alpha>0$ *it holds*

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left|\sum_{k:\,\partial_l\Phi_k>0}\ \sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left(\mathcal{M}_k u-\mathcal{M}_j u\right)\right|^r \\ \leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_{j:\,\partial_l\Phi_j<0} d_j^{\frac{\tilde\alpha s+drs}{s-r}}\chi_{\nabla\Phi_j\neq 0}\right)^{\frac{s}{s-r}}\right)^{\frac{s-r}{s}}\cdot\dots \\ \dots\cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\ \sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{-\tilde\alpha\frac{s}{r}}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\,|\mathcal{M}_k u-\mathcal{M}_j u|^s\right)^{\frac{r}{s}},\end{aligned}$$

with the similar formula holding for $\mathcal{M}_\bullet$ *replaced by* $\mathcal{M}^{\mathfrak{s}}_\bullet$.

Proof. We observe with help of (4.3) and with Lemma 3.17.2)

$$\forall x: \quad \sup_k |\partial_l \Phi_k|(x) \leq \sup\left\{ |\nabla\Phi_k(x)| \,:\, x \in \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_k) \right\} \leq C \sup\left\{ d_k^d \,:\, x \in G_k \right\}, \tag{5.1}$$

$$\sup_{x \in \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_j)} |\partial_l \Phi_j|(x) \leq C d_j^d . \tag{5.2}$$

We write

$$I := \frac{1}{|\mathbf{Q}|} \int_{\mathbf{P}\cap\mathbf{Q}} \left| \sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{\partial_l \Phi_k \, |\partial_l \Phi_j|}{D_{l+}^{\Phi}} (2-\phi_0)\left(\mathcal{M}_k u - \mathcal{M}_j u\right) \right|^r$$

and find

$$\begin{aligned} I &\leq C \frac{1}{|\mathbf{Q}|} \int_{\mathbf{P}\cap\mathbf{Q}} \sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{|\partial_l \Phi_k|^r \, |\partial_l \Phi_j|}{D_{l+}^{\Phi}} |\mathcal{M}_k u - \mathcal{M}_j u|^r \\ &\leq C \frac{1}{|\mathbf{Q}|} \int_{\mathbf{P}\cap\mathbf{Q}} \left(\sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{d_j^{\alpha\frac{s}{s-r}} |\partial_l \Phi_k|^{\frac{sr}{s-r}} |\partial_l \Phi_j|}{D_{l+}^{\Phi}} \right)^{\frac{s-r}{s}} \cdot \ldots \\ &\qquad \ldots \cdot \left(\sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{d_j^{-\alpha\frac{s}{r}} |\partial_l \Phi_j|}{D_{l+}^{\Phi}} |\mathcal{M}_k u - \mathcal{M}_j u|^s \right)^{\frac{r}{s}} . \end{aligned}$$

Now we make use of (5.1) and once more of Lemma 3.17.2) to obtain for the first bracket on the right hand side an estimate of the form

$$|\partial_l \Phi_k|^{\frac{sr}{s-r}} |\partial_l \Phi_j| \leq |\partial_l \Phi_k| \, |\partial_l \Phi_k|^{\frac{sr}{s-r}-1} |\partial_l \Phi_j| \leq C \, |\partial_l \Phi_k| \, d_j^{d\frac{sr-s+r}{s-r}} d_j^d \leq C \, |\partial_l \Phi_k| \, d_j^{d\frac{sr}{s-r}} ,$$

which implies

$$\begin{aligned} \sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{d_j^{\alpha\frac{s}{s-r}} |\partial_l \Phi_k|^{\frac{sr}{s-r}} |\partial_l \Phi_j|}{D_{l+}^{\Phi}} &\leq C \sum_{k:\,\partial_l\Phi_k>0} \sum_{j:\,\partial_l\Phi_j<0} \frac{d_j^{\alpha\frac{s}{s-r}} d_j^{\frac{dsr}{s-r}} |\partial_l \Phi_k|}{D_{l+}^{\Phi}} \\ &\leq C \sum_{j:\,\partial_l\Phi_j<0} d_j^{\alpha\frac{s}{s-r}} d_j^{\frac{dsr}{s-r}} \chi_{\nabla\Phi_j \neq 0} , \end{aligned}$$

where we used $\sum |\partial_l \Phi_k| = D_{l+}^{\Phi}$. From Hölder's inequality the Lemma follows. □

6 Sample Geometries

6.1 Delaunay Pipes for a Matern Process

For two points $x, y \in \mathbb{R}^d$, we denote

$$P_r(x,y) := \left\{ y + z \in \mathbb{R}^d : \, 0 \leq z \cdot (x-y) \leq |x-y|^2 , \, \left| z - z \cdot (x-y) \frac{x-y}{|x-y|} \right| < r \right\} ,$$

the cylinder (or pipe) around the straight line segment connecting x and y with radius $r > 0$.

Recalling Example 2.48 we consider a Poisson point process $\mathbb{X}_{\mathrm{pois}}(\omega) = (x_i(\omega))_{i\in\mathbb{N}}$ with intensity λ (recall Example 2.48) and construct a hard core Matern process $\mathbb{X}_{\mathrm{mat}}$ by deleting all points with a mutual distance smaller than $d\mathfrak{r}$ for some $\mathfrak{r} > 0$ (refer to Example 2.49). From the remaining point process $\mathbb{X}_{\mathrm{mat}}$ we construct the Delaunay triangulation $\mathbb{D}(\omega) := \mathbb{D}(X_{\mathrm{mat}}(\omega))$ and assign to each $(x, y) \in \mathbb{D}$ a random number $\delta(x, y)$ in $(0, \mathfrak{r})$ in an i.i.d. manner from some probability distribution $\delta(\omega)$. We finally define

$$\mathbf{P}(\omega) := \bigcup_{(x,y)\in\mathbb{D}(\omega)} P_{\delta(x,y)}(x, y) \bigcup_{x\in\mathbb{X}_{\mathrm{mat}}} \mathbb{B}_{\frac{\mathfrak{r}}{2}}(x)$$

the family of all pipes generated by the Delaunay grid “smoothed” by balls with the fix radius $\mathfrak{r}$ around each point of the generating Matern process.

Since the Matern process is mixing and δ is mixing, Lemma 2.34 yields that the whole process is still ergodic. We start with a trivial observation.

Corollary 6.1. *The microscopic regularity of $\mathbf{P}$ is $\alpha = 0$ (Def. 1.8) and it holds $\hat{d} = d - 1$ in Lemma 4.4. Furthermore both the extension order and the symmetric extension order are $n = 0$.*

Proof. This follows from the fact that $\partial\mathbf{P}$ can be locally represented as a graph in the upper half space with $\mathbf{P}$ filling the lower half space. □

Lemma 6.2. *For the Voronoi tessellation $(G_a)_{a\in\mathbb{N}}$ corresponding to $\mathbb{X}_{\mathrm{mat}}$ holds*

$$\mathbb{P}(d_a \geq D) \leq \exp\left(-\lambda \left|\mathbb{S}^{d-1}\right| (4D)^d \left(1 - e^{-\lambda\left|\mathbb{S}^{d-1}\right|(d\mathfrak{r})^d}\right)\right).$$

Proof. For the underlying Poisson point process $\mathbb{X}_{\mathrm{pois}}$ it holds for the void probability inside a ball $\mathbb{B}_R(x)$

$$\mathbb{P}(\mathbb{X}_{\mathrm{pois}}(\mathbb{B}_R(x)) = 0) = \mathbb{P}_{R,0} := e^{-\lambda\left|\mathbb{S}^{d-1}\right|R^d}.$$

The probability for a point $x \in \mathbb{X}_{\mathrm{pois}}$ to be removed is thus $1 - \mathbb{P}_{d\mathfrak{r},0}$ and is i.i.d distributed among points of $\mathbb{X}_{\mathrm{pois}}$. The total probability to not find any point of $\mathbb{X}_{\mathrm{mat}}$ is thus given by not finding a point of $\mathbb{X}_{\mathrm{pois}}$ plus the probability that all points of $\mathbb{X}_{\mathrm{pois}}$ are removed, i.e.

$$\begin{aligned}\mathbb{P}(\mathbb{X}_{\mathrm{mat}}(\mathbb{B}_R(x)) = 0) &= \sum_{n=0}^{\infty} e^{-\lambda|A|} \frac{\lambda^n |A|^n}{n!} (1 - \mathbb{P}_{d\mathfrak{r},0})^n \\ &= \exp\left(-\lambda |A| + \lambda |A| (1 - \mathbb{P}_{d\mathfrak{r},0})\right) = e^{-\lambda|A|\left(1-\mathbb{P}_{d\mathfrak{r},0}\right)}.\end{aligned}$$

From here one concludes. □

Remark 6.3. The family of balls $\mathbb{B}_{\mathfrak{r}}(x)$ can also be dropped from the model. However, this would imply we had to remove some of the points from $\mathbb{X}_{\mathrm{mat}}$ for the generation of the Voronoi cells. This would cause technical difficulties which would not change much in the result, as the probability for the size of Voronoi cells would still decrease exponentially.

Lemma 6.4. *$\mathbb{X}_{\mathrm{mat}}$ is a point process for $\mathbf{P}(\omega)$ that satisfies Assumption 4.2 and $\mathbf{P}$ is isotropic cone mixing for $\mathbb{X}_{\mathrm{mat}}$ with exponentially decreasing $f(R) \leq Ce^{-R^d}$ and it holds $n = 0$ and $\alpha = 0$. Furthermore, assume there exists $C_\delta, a_\delta > 0$ such that $\mathbb{P}(\delta(x, y) < \delta_0) \leq C_\delta e^{-a_\delta \frac{1}{\delta_0}}$, then $\mathbb{P}(\tilde{M} > M_0) \leq Ce^{-aM_0}$ for some $C, a > 0$. If $\mathbb{P}(\delta(x, y) < \delta_0) \leq C_\delta \delta_0^\beta$ then for every $R \in (0, \infty)$ it holds*

$$\mathbb{E}\left(M^R_{[\frac{\delta}{2}],\mathbb{R}^d}\right) + \mathbb{E}\left(\tilde{\delta}^{-(\beta+d-1)}_{\mathbb{R}^d}\right) < C\mathbb{E}(|x - y|), \tag{6.1}$$

where $\mathbb{E}(|x - y|)$ is the expectation of the length of pipes.

Proof. Isotropic cone mixing: For $x,y\in 2d\mathfrak{r}\mathbb{Z}^d$ the events $\left(x+[0,1]^d\right)\cap\mathbb{X}_{\mathrm{mat}}$ and $\left(y+[0,1]^d\right)\cap\mathbb{X}_{\mathrm{mat}}$ are mutually independent. Hence

$$\mathbb{P}\big(\left(k2dr\,[-1,1]^d\right)\cap\mathbb{X}_{\mathrm{mat}}=\emptyset\big)\leq\mathbb{P}\big([-1,1]^d\cap\mathbb{X}_{\mathrm{mat}}=\emptyset\big)^{k^d}.$$

Hence the open set $\mathbf{P}$ is isotropic cone mixing for $\mathbb{X}=\mathbb{X}_{\mathrm{mat}}$ with exponentially decaying $f(R)\leq Ce^{-R^d}$.

Estimate on the distribution of M: By definition of the Delaunay triangulation, two pipes intersect only if they share one common point $x\in\mathbb{X}_{\mathrm{mat}}$.

Given three points $x,y,z\in\mathbb{X}_{\mathrm{mat}}$ with $x\sim y$ and $x\sim z$, the highest local Lipschitz constant on $\partial\left(P_{\delta(x,y)}(x,y)\cup P_{\delta(x,z)}(x,z)\right)$ is attained in

$$\tilde{x}=\arg\max\left\{|x-\tilde{x}|\,:\,\tilde{x}\in\partial P_{\delta(x,y)}(x,y)\cap\partial P_{\delta(x,z)}(x,z)\right\}.$$

It is bounded by

$$\max\left\{\arctan\left(\frac{1}{2}\sphericalangle\left((x,y),(x,z)\right)\right),\,\frac{1}{\delta(x,y)},\,\frac{1}{\delta(x,z)}\right\},$$

where $\alpha:=\sphericalangle\left((x,y),(x,z)\right)$ in the following denotes the angle between (x,y) and (x,z), see Figure 4. If d_x is the diameter of the Voronoi cell of x, we show that a necessary (but not sufficient) condition that the angle α can be smaller than some α_0 is given by

$$d_x\geq C\frac{1}{\sin\alpha_0},\tag{6.2}$$

where $C>0$ is a constant depending only on the dimension d. Since for small α we find $M\approx\frac{1}{\sin\alpha}$, and since the distribution for d_x decays subexponentially, also the distribution for M at the junctions of two pipes decays subexponentially. However, inside the pipes, we find $\Delta(p)=2\delta(x,y)$ and hence $\delta_\Delta(p)=\delta(x,y)$. Due to the cylindric structure, we furthermore find essential boundedness of M. This also implies $\alpha=n=0$ inside the pipes. At the junction of Balls and pipes we find $\partial\mathbf{P}$ to be in the upper half of the local plane approximation and hence also here $\alpha=n=0$ can be chosen (see also Remarks 2.4 and 2.8).

Concerning the expectation of $M_{[\frac{\delta}{2}],\mathbb{R}^d}$ and $\delta_{\mathbb{R}^d}$, we only have to accound for the pipes by the above argumentation since the other contribution to M is exponentially distributed. In particular, we find for one single pipe $P_{\delta(x,y)}(x,y)$ that

$$\int_{P_{\delta(x,y)}(x,y)}\delta_{\mathbb{R}^d}^{-\alpha-d+1}\leq C\,|x-y|\,\delta(x,y)^{-\alpha},$$

and hence (6.1) due to the independence of length and diameter. It thus remains to proof (6.2).

Proof of (6.2): Given an angle $\alpha>0$ and $x\in\mathbb{X}_{\mathrm{mat}}$ we derive a lower bound for the diameter of $G(x)$ such that for two neighbors y,z of x it can hold $\sphericalangle\left((x,y),(x,z)\right)\leq\alpha$. With regard to Figure 4, we assume $|x-y|\geq|x-z|$.

Writing $d_x:=d(x)$ the diameter of $G(x)$ and $\tilde{\alpha}=\sphericalangle\left((x,z),(z,y)\right)$, w.l.o.g let $y=(d_1+d_2,0,\dots,0)$, where $d_1+d_2<d_x$ and $d_1=|y-z|\cos\tilde{\alpha}$. Hence we can assume
$z=(d_2,-|y-z|\sin\tilde{\alpha},0\dots0)$ and in what follows, we focus on the first two coordinates only. The boundaries between the cells x and z and x and y lie on the planes

$$h_{xz}(t)=\frac{1}{2}z+t\begin{pmatrix}|y-z|\sin\tilde{\alpha}\\ d_2\end{pmatrix},\quad h_{xy}(s)=\frac{1}{2}y+s\begin{pmatrix}0\\1\end{pmatrix}$$

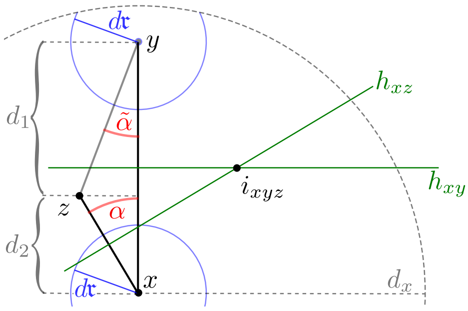

Figure 4: Sketch of the proof of Lemma 6.4 and estimate (6.2).

respectively. The intersection of these planes has the first two coordinates

$$i_{xyz} := \left(\frac{d_1+d_2}{2}, -\frac{1}{2}|y-z|\sin\tilde\alpha + \frac{1}{2}\frac{d_1 d_2}{|y-z|\sin\tilde\alpha}\right) .$$

Using the explicit form of d_2, the latter point has the distance

$$|i_{xyz}|^2 = \frac{1}{4}|y-z|^2 + \frac{1}{4}d_2^2 + \frac{1}{4}\frac{d_2^2\cos^2\tilde\alpha}{\sin^2\tilde\alpha}$$

to the origin $x=0$. Using $|y-z|\sin\tilde\alpha = |z|\sin\alpha$ and $d_2 = |y| - |z|\cos\alpha$ we obtain

$$|i_{xyz}|^2 = \frac{1}{4}\left(|y-z|^2\left(1+\frac{(|y|-|z|\cos\alpha)^2\cos^2\tilde\alpha}{|z|^2\sin^2\alpha}\right) + (|y|-|z|\cos\alpha)^2\right) .$$

Given y, the latter expression becomes small for $|y-z|$ small, with the smallest value being $|y-z| = d\mathfrak{r}$. But then

$$\cos^2\tilde\alpha = 1-\sin^2\tilde\alpha = 1-\frac{(|z|\sin\alpha)^2}{|y-z|^2}$$

and hence the distance becomes

$$|i_{xyz}|^2 = \frac{1}{4}\left((d\mathfrak{r})^2\left(1+\frac{(|y|-|z|\cos\alpha)^2\left((d\mathfrak{r})^2+|z|^2\sin^2\alpha\right)}{(d\mathfrak{r})^2|z|^2\sin^2\alpha}\right) + (|y|-|z|\cos\alpha)^2\right) .$$

We finally use $|y| = |z|\cos\alpha - \sqrt{(d\mathfrak{r})^2 - |z|^2\sin^2\alpha}$ and obtain

$$|i_{xyz}|^2 = \frac{1}{4}\left((d\mathfrak{r})^2\left(1+\frac{\left((d\mathfrak{r})^4-|z|^4\sin^4\alpha\right)}{(d\mathfrak{r})^2|z|^2\sin^2\alpha}\right) + \left((d\mathfrak{r})^2-|z|^2\sin^2\alpha\right)\right) .$$

The latter expression now needs to be smaller than d_x. We observe that the expression on the right hand side decreases for fixed α if $|z|$ increases.

On the other hand, we can resolve $|z|(y) = |y|\cos\alpha - \sqrt{|y|^2\sin^2\alpha + (d\mathfrak{r})^2}$. From the conditions $|y| \le d_x$ and $|i_{xyz}| \le d_x$, we then infer (6.2). □

Theorem 6.5. *Assuming* $\mathbb{E}\left(\delta^{-s-d}+\delta^{1+s-2d}\right)^{\frac{p}{p-s}} < \infty$ *and using the notation of Lemma 5.2 the above constructed* $\mathbf{P}$ *has the property that for* $1 \le r < s < p$ *there almost surely exists* $C>0$ *such that for every* $n\in\mathbb{N}$ *and every* $u \in W^{1,p}_{0,\partial(n\mathbf{Q})}(\mathbf{P}\cap n\mathbf{Q})$

$$\left(\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}\cap n\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{-\tilde\alpha\frac{s}{r}}|\partial_l\Phi_j|}{D^{\Phi}_{l+}}|\mathcal{M}_k u - \mathcal{M}_j u|^s\right)^{\frac{r}{s}} + F^3_s(n\mathbf{Q},u)$$
$$\le C\left(\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}\cap n\mathbf{Q}}|\nabla u|^p\right)^{\frac{r}{p}} ,$$

and for every $u \in \mathbf{W}^{1,p}_{0,\partial(n\mathbf{Q})}(\mathbf{P}\cap n\mathbf{Q})$

$$\left(\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}\cap n\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{-\tilde\alpha\frac{s}{r}}\left|\partial_l\Phi_j\right|}{D^{\Phi}_{l+}}\left|\mathcal{M}^{\mathfrak{s}}_k u-\mathcal{M}^{\mathfrak{s}}_j u\right|^s\right)^{\frac{r}{s}}+F^{3,\mathfrak{s}}_s(n\mathbf{Q},u)$$
$$\leq C\left(\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}\cap n\mathbf{Q}}|\nabla^s u|^p\right)^{\frac{r}{p}}.$$

Lemma 6.6. *For every bounded open set* $\mathbf{Q}$ *with* $0\in\mathbf{Q}$ *and* $n_0,n_1\in\mathbb{N}$ *let*

$$\forall M>1:\qquad \tilde{\mathbf{Q}}_{M,n_0,n_1}:=\bigcup_{\substack{x_a\in\mathbb{X}_{\mathrm{mat}}\\ \mathbb{B}_{n_0 d_a}(x_a)\cap M\mathbf{Q}\neq\emptyset}}\mathbb{B}_{n_1 d_a}(x_a)\,.$$

Then for fixed n_0 *and* n_1 *there almost surely exists* $r>0$ *such that for every* $M>1$ *it holds* $\tilde{\mathbf{Q}}_{M,n_0,n_1}\subset Mr\mathbf{Q}$

Proof. There exists $r_0<R$ such that $\mathbb{B}_{r_0}(0)\subset\mathbf{Q}\subset\mathbb{B}_R(0)$ we assume w.l.o.g $\mathbf{Q}=\mathbb{B}_R(0)$. We denote $\mathbf{Q}_M:=M\mathbf{Q}$ and observe that $\frac{|\partial\mathbf{Q}_M|}{|\mathbf{Q}_M|}\leq C\,M^{-1}$ where $|\partial\mathbf{Q}_M|:=\mathcal{H}^{d-1}(\partial\mathbf{Q}_M)$. For

$$\mathbf{Q}_{M,a,b}:=\left\{x\in\mathbb{R}^d\backslash\mathbf{Q}_M:\,a<\mathrm{dist}(x,\mathbf{Q}_M)<b\right\}\,,$$

we observe that $\#\big(\mathbf{Q}_{M,a,b}\cap\mathbb{X}_{\mathrm{mat}}\big)\leq CM^{d-1}(b-a)$ due to the minimal mutual distance. The probability that at least one $x\in\mathbf{Q}_{M,a,b}\cap\mathbb{X}_{\mathrm{mat}}$ satisfies $\mathbb{B}_{n_0 d(x)}(x)\cap\mathbf{Q}_M\neq\emptyset$ is given by

$$\begin{aligned}\mathbb{P}(\mathbf{Q}_M,a,b):&=\mathbb{P}\big(\exists x\in\mathbf{Q}_{M,a,b}\cap\mathbb{X}_{\mathrm{mat}}:\,\mathbb{B}_{n_0 d(x)}(x)\cap\mathbf{Q}_M\neq\emptyset\big)\\
&=\sum_{k=1}^{\infty}k\,\mathbb{P}\big(k=\#\mathbf{Q}_{M,a,b}\cap\mathbb{X}_{\mathrm{mat}}\big)\,\mathbb{P}\left(d>\frac{a}{n_0}\right)\\
&\leq\mathbb{P}\left(d>\frac{a}{n_0}\right)e^{-\lambda\left|\mathbf{Q}_{M,a,b}\right|}\sum_{k=1}^{\infty}\frac{\lambda^k\left|\mathbf{Q}_{M,a,b}\right|^k}{(k-1)!}=\mathbb{P}\left(d>\frac{a}{n_0}\right)\lambda\left|\mathbf{Q}_{M,a,b}\right|\,.\end{aligned}$$

Now let $r>0$ and observe $\left|\mathbf{Q}_{M,a,b}\right|\leq C\,(b-a)\,\left(b+MR\right)^{d-1}$ while $\mathbb{P}\Big(d>\frac{a}{n_0}\Big)\leq Ce^{-\alpha\,a^d}$. Then the probability that there exists $x\in\mathbb{X}_{\mathrm{mat}}\backslash\mathbf{Q}_{rM}$ such that $\mathbb{B}_{n_0 d(x)}(x)\cap\mathbf{Q}_M$ is smaller than

$$\begin{aligned}&\sum_{k=0}^{\infty}\mathbb{P}(\mathbf{Q}_M,(r-1)M+k,(r-1)M+k+1)\\
&\qquad=\sum_{k=0}^{\infty}\mathbb{P}\left(d>\frac{(r-1)M+k}{n_0}\right)\lambda\left((rM+k+1)^d-(rM+k)^d\right)\\
&\qquad\leq e^{-\alpha((r-1)M)^d}(rM)^d\sum_{k=0}^{\infty}e^{-\alpha k^d}\lambda(k+2)^d\,,\end{aligned}$$

and the right hand side tends uniformly to 0 as $r\to\infty$. □

Proof of Theorem 6.5. In what follows, we will mostly perform the calculations for $\tau^{\mathfrak{s}}_i$ and $\mathcal{M}^{\mathfrak{s}}_a$ since these calculations are more involved and drop n except for the last Step 4.

We first estimate the difference $|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_b^{\mathfrak{s}}u|$ for two directly neighbored points $x_a \sim x_b$ of the Delaunay grid. These are connected through a cylindric pipe

$$P_{\delta,a,b} = P(x_a, x_b, \delta(a,b)) := \operatorname{conv}\left(\mathbb{B}_{\delta(a,b)}(x_a) \cup \mathbb{B}_{\delta(a,b)}(x_b)\right)$$

with round ends and of thickness $\delta(a,b)$ and total length $|x_a - x_b| + 2\delta(a,b) < 2\,|x_a - x_b|$ and we first introduce the new averages in the spirit of (2.27)

$$\mathcal{M}_a^{\delta}u := \fint_{\mathbb{B}_\delta(x_a)} u\,, \qquad \mathcal{M}_a^{\mathfrak{s},\delta}u(x) := \overline{\nabla_{a,\delta}^{\perp}}u(x-a) + \fint_{\mathbb{B}_\delta(x_a)} u\,.$$

As for (4.15) and (4.23) we obtain

$$\mathcal{M}_{a_1}^{\delta_1}\mathcal{M}_{a_2}^{\delta_2}u = \mathcal{M}_{a_2}^{\delta_2}u\,, \qquad \mathcal{M}_{a_1}^{\mathfrak{s},\delta_1}\mathcal{M}_{a_2}^{\mathfrak{s},\delta_2}u = \mathcal{M}_{a_2}^{\mathfrak{s},\delta_2}u\,.$$

For every $i, a \in \mathbb{N}$ with $p_i \in \mathbb{B}_{\mathfrak{r}}(G_a)$ there exists almost surely $a_i \in \mathbb{N}$ such that p_i and x_{a_i} are connected in $\mathbf{P}$ through a straight line segment (i.e. p_i lies on the boundary of one of the pipes emerging at x_{a_i} or in $\mathbb{B}_{\mathfrak{r}}(x_{a_i})$) and

$$\begin{aligned}|\tau_i u - \mathcal{M}_a u|^s &\leq 2^s\left(|\tau_i u - \mathcal{M}_{a_i}u|^s + |\mathcal{M}_{a_i}u - \mathcal{M}_a u|^s\right)\,,\\ |\tau_i^{\mathfrak{s}}u(x) - \mathcal{M}_a^{\mathfrak{s}}u(x)|^s &\leq 2^s\left(\left|\tau_i^{\mathfrak{s}}u(x) - \mathcal{M}_{a_i}^{\mathfrak{s}}u(x)\right|^s + \left|\mathcal{M}_{a_i}^{\mathfrak{s}}u(x) - \mathcal{M}_a^{\mathfrak{s}}u(x)\right|^s\right)\,.\end{aligned}$$

The second term is of “mesoscopic type”, while the first term is of local type. We will study both types of terms separately.

Step 1: Using (2.28)–(2.29), we observe for neighbors $a \sim b$

$$\begin{aligned}|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_b^{\mathfrak{s}}u|^s &\leq \sum_{k=a,b}\left|\mathcal{M}_k^{\mathfrak{s}}u - \mathcal{M}_k^{\mathfrak{s},\delta(a,b)}u\right|^s + \left|\mathcal{M}_a^{\mathfrak{s},\delta(a,b)}u - \mathcal{M}_b^{\mathfrak{s},\delta(a,b)}u\right|^s\\ &\leq CF_s^{\mathfrak{s},1}(x,\delta(a,b))\left(|x-x_a|^s + |x-x_b|^s\right)\\ &\qquad\left(\|\nabla^s u\|^s_{L^s(\mathbb{B}_{\frac{\mathfrak{r}}{16}}(\{x_a,x_b\}))} + |x_a - x_b|^{2s}\,\|\nabla^s u\|^s_{L^s(P_{\delta,a,b})}\right)\,. \qquad (6.3)\end{aligned}$$

where

$$F_s^{\mathfrak{s},q}(x,\delta) := \left(\delta^{-d} + \delta^{-s-d} + \delta^{1+s-2d}\right)^q\,. \qquad (6.4)$$

Step 2: For reasons that we will encounter below, we define

$$\mathrm{I}_\alpha := \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{a:\,\mathbb{B}_{4d_a}(x_a)\cap\mathbf{Q}\neq\emptyset}\chi_{\mathbb{B}_{\mathfrak{r}}(G_a)}\sum_{\substack{b:\,\mathbb{B}_{4d_b}(x_b)\cap\mathbf{Q}\neq\emptyset\\ d_b\leq d_a,\,|x_a-x_b|\leq 3d_a}} d_a^{-\alpha\frac{s}{r}}\,|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_b^{\mathfrak{s}}u|^s\,.$$

Assume $\chi_{\mathbb{B}_{\mathfrak{r}}(G_a)}\chi_{A_{1,i}} \not\equiv 0$. Then it holds $p_i \in \mathbb{B}_{2d_{a_i}}(x_{a_i})$ which implies

$$\begin{aligned}&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}\sum_{i\neq 0}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\left|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_{a_i}^{\mathfrak{s}}u\right|^s\\ &\qquad\leq \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}\sum_{x_a\in\mathbb{X}(\mathbf{Q})}\sum_{\substack{x_b\in\mathbb{X}_{\mathrm{mat}}\\ \mathbb{B}_{2d_b}(x_b)\cap\mathbb{B}_{\mathfrak{r}}(G_a)\neq\emptyset}}\sum_{i:\,x_{a_i}=x_b} f_a\chi_{A_{1,i}}\,|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_b^{\mathfrak{s}}u|^s\\ &\qquad\leq \frac{1}{|\mathbf{Q}|}C\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}\sum_{x_b\in\mathbb{X}_{\mathrm{mat}}}\sum_{\substack{x_a\in\mathbb{X}(\mathbf{Q})\\ \mathbb{B}_{2d_b}(x_b)\cap\mathbb{B}_{\mathfrak{r}}(G_a)\neq\emptyset}}\chi_{\mathbb{B}_{\mathfrak{r}}(G_a)}\,|\mathcal{M}_a^{\mathfrak{s}}u - \mathcal{M}_b^{\mathfrak{s}}u|^s\,.\end{aligned} \qquad (6.5)$$

Hence, we encounter the conditions $\mathbb{B}_{\mathfrak{r}}(G_a)\cap\mathbf{Q}\neq\emptyset$ and $\mathbb{B}_{2d_b}(x_b)\cap\mathbb{B}_{\mathfrak{r}}(G_a)\neq\emptyset$ as well as

$$|x_a-x_b|\leq 3\max\{d_a,d_b\}\ .$$

In particular, we conclude the symmetric condition

$$\mathbb{B}_{4d_a}(x_a)\cap\mathbf{Q}\neq\emptyset\,,\quad \mathbb{B}_{4d_a}(x_b)\cap\mathbf{Q}\neq\emptyset\,,\qquad \mathbb{B}_{2d_a}(x_a)\cap\mathbb{B}_{2d_b}(x_b)\neq\emptyset$$

and

$$\text{R.H.S of (6.5)}\leq \mathrm{I}_0\,. \tag{6.6}$$

Similarly

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{a:\,\partial_l\Phi_a>0}\sum_{b:\,\partial_l\Phi_b<0}\frac{d_b^{-\alpha\frac{s}{r}}\,|\partial_l\Phi_b|}{D^{\Phi}_{l+}}\,|\mathcal{M}_a u-\mathcal{M}_b u|^s\leq I_\alpha\,. \tag{6.7}$$

Step 3: We now derive an estimate for I_α. For pairs (a,b) with $d_b\leq d_a$, $|x_a-x_b|\leq 3d_a$ let $y_{a,b}:=\big(y_1,\dots,y_{n(a,b)}\big)$ be a discrete path on the Delaunay grid of $\mathbb{X}_{\mathrm{mat}}$ with length smaller than $2\,|x_a-x_b|$ (this exists due to [28]) that connects x_a and x_b. By the minimal mutual distance of points, this particularly implies that $n(a,b)\leq 6d_a/2\mathfrak{r}$ and the path lies completely within $\mathbb{B}_{4.5\,d_a}(x_a)$. Because

$$\begin{aligned}|\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u|^s&\leq n(a,b)^s\sum_{k=1}^{n(a,b)-1}\left|\mathcal{M}^{\mathfrak{s}}_{y_k}u-\mathcal{M}^{\mathfrak{s}}_{y_{k+1}}u\right|^s\\&\leq 6d_a^s/2\mathfrak{r}\sum_{k=1}^{n(a,b)-1}\left|\mathcal{M}^{\mathfrak{s}}_{y_k}u-\mathcal{M}^{\mathfrak{s}}_{y_{k+1}}u\right|^s\end{aligned}$$

it holds with (6.3)

$$\begin{aligned}|(\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u)\,(x)|^s\leq Cd_a^s\int_{\mathbb{B}_{6d_a}(x_a)}&\left(\sum_{e\sim f}F^{\mathfrak{s},1}_s(\delta(e,f))\,(|x-x_e|^s+|x-x_f|^s)\,\cdot\right.\\&\left.\cdot\,|x_e-x_f|^{2s}\left(\chi_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_e)}+\chi_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_f)}+d_a^{s-1}\chi_{P_{\delta,e,f}}\right)\right)|\nabla^s u|^s\end{aligned}$$

We make use of $|x-x_e|^s\leq 2^s\left(|x-x_a|^s+|x_a-x_e|^s\right)\leq 2^s\left(|x-x_a|^s+d_a^s\right)$ and $|x_e-x_f|^{2s}\leq Cd_a^{2s}$ and $B_{e,f}:=\mathbb{B}_{\frac{\mathfrak{r}}{16}}(\{x_e,x_f\})\cup P_{\delta,e,f}$ to find

$$|(\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u)\,(x)|^s\leq Cd_a^{4s}\int_{\mathbb{B}_{6d_a}(x_a)}\left(\sum_{e\sim f}F^{\mathfrak{s},1}_s(\delta(e,f))\,(|x-x_a|^s+d_a^s)\;\chi_{B_{e,f}}\right)|\nabla^s u|^s\ .$$

In the integrals I_α, any of the integrals $\int\chi_{\mathbb{B}_{\mathfrak{r}}(G_a)}\,|\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u|^s$ has $|x-x_a|<2d_a$ and we can use an estimate of the form

$$|(\mathcal{M}^{\mathfrak{s}}_a u-\mathcal{M}^{\mathfrak{s}}_b u)\,(x)|^s\leq Cd_a^{5s}\int_{\mathbb{B}_{6d_a}(x_a)}\left(\sum_{e\sim f}F^{\mathfrak{s},1}_s(\delta(e,f))\chi_{B_{e,f}}\right)|\nabla^s u|^s\ .$$

With this estimate, and using

$$\#\left\{b:\,\mathbb{B}_{4d_b}(x_b)\cap\mathbf{Q}\neq\emptyset,\,d_b\leq d_a,\,|x_a-x_b|\leq 3d_a\right\}\leq Cd_a^d$$

the integral I_α can be controlled through

$$I_\alpha \leq \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}} \sum_{a:\,\mathbb{B}_{4d_a}(x_a)\cap\mathbf{Q}\neq\emptyset} d_a^{2d+5s-\alpha\frac{s}{r}}\chi_{\mathbb{B}_{6d_a}(x_a)}\left(\sum_{e\sim f} F_s^{\mathfrak{s},1}(\delta(e,f))\chi_{B_{e,f}}\right)|\nabla^s u|^s \,.$$

Denoting

$$\begin{aligned}
f(\omega) &:= \sum_a d_a^{2d+5s-\alpha\frac{s}{r}}\chi_{\mathbb{B}_{6d_a}(x_a)}\,,\\
f(\omega,\mathbf{Q}) &:= \sum_{a:\,\mathbb{B}_{4d_a}(x_a)\cap\mathbf{Q}\neq\emptyset} d_a^{2d+5s-\alpha\frac{s}{r}}\chi_{\mathbb{B}_{6d_a}(x_a)}\,,\\
g(\omega) &:= \sum_{e\sim f} F_s^{\mathfrak{s},1}(\delta(e,f))\chi_{B_{e,f}}\,,
\end{aligned}$$

and using $u\equiv 0$ outside $\mathbf{Q}$, we observe

$$I_\alpha \leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}} f(\omega,\mathbf{Q})^{\frac{p}{p-s}}\,g(\omega)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}|\nabla^s u|^p\right)^{\frac{s}{p}}\,.$$

Step 4: Since every quantity related to the distribution of d_a is distributed exponentially, we can be very generous with this variable. We observe

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_a f_a\sum_{i\neq 0}\frac{|\partial_l\phi_i|}{D_{l+}}\left|\tau_i^\mathfrak{s}u-\mathcal{M}_{a_i}^\mathfrak{s}u\right|^s \leq \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{i\neq 0}\chi_{A_{1,i}}\left|\tau_i^\mathfrak{s}u-\mathcal{M}_{a_i}^\mathfrak{s}u\right|^s$$

but for every fixed x (and using that $x\in\mathbb{B}_{2d_{a_i}}(x_{a_i})$) using again Jensens inequality

$$\int_{A_{1,i}}\left|\tau_i^\mathfrak{s}u(x)-\mathcal{M}_{a_i}^\mathfrak{s}u(x)\right|^s \leq C\int_{\mathbb{B}_{\mathfrak{r}_i}(y_i)}\left(\left|\nabla\left(u-\mathcal{M}_{a_i}^\mathfrak{s}u\right)\right|^s d_{a_i}^s+\left|u-\mathcal{M}_{a_i}^\mathfrak{s}u\right|^s\right)\,.$$

Having this in mind, we may sum over all y_i to find

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_a f_a\sum_{i\neq 0}\frac{|\partial_l\phi_i|}{D_{l+}}\left|\tau_i^\mathfrak{s}u-\mathcal{M}_{a_i}^\mathfrak{s}u\right|^s\\
&\qquad\leq \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}} C\sum_a\chi_{2d_a}\sum_{b\sim a} d_a^s\,\chi_{P(x_a,x_b,\delta(a,b))}\left(|\nabla\left(u-\mathcal{M}_a^\mathfrak{s}u\right)|^s+|u-\mathcal{M}_a^\mathfrak{s}u|^s\right)\,.
\end{aligned}$$

With the splitting $u-\mathcal{M}_a^\mathfrak{s}u = u-\mathcal{M}_a^{\mathfrak{s},\delta(a,b)}u+\mathcal{M}_a^{\mathfrak{s},\delta(a,b)}u-\mathcal{M}_a^\mathfrak{s}u$ and Lemmas 2.18 and 2.17 it follows with $F_s^{\mathfrak{s},1}$ from (6.4)

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_a f_a\sum_{i\neq 0}\frac{|\partial_l\phi_i|}{D_{l+}}\left|\tau_i^\mathfrak{s}u-\mathcal{M}_{a_i}^\mathfrak{s}u\right|^s\\
&\quad\leq \frac{1}{|\mathbf{Q}|}C\sum_a\sum_{b\sim a}F_s^{\mathfrak{s},1}(\delta(a,b))\left(\|\nabla u\|^s_{L^s(\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_a)\cup\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_b))}+(2d_a)^s\|\nabla u\|^s_{L^s(P(x_a,x_b,\delta(a,b)))}\right)d_a^{d+s}
\end{aligned}$$

by a restructuration, the right hand side is bounded by

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}} C\left(\sum_a\chi_{2d_a}d_a^{3s+d}\right)g(\omega)\,|\nabla^s u|^s\\
&\qquad\qquad\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}} f_1(\omega)^{\frac{p}{p-s}}Bg(\omega)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}}|\nabla^s u|^p\right)^{\frac{s}{p}}\,,
\end{aligned}$$

where

$$f_1(\omega) := \sum_a \chi_{2d_a} \left(2d_a\right)^{s+d}$$

Step 4: We can replace in the above calculations $\mathbf{Q}$ by $n\mathbf{Q}$. By Lemma 6.6 we can extend $f(\omega, n\mathbf{Q})$ to $f(\omega)|_{Rn\mathbf{Q}}$ for some fixed $R > 1$ and on $R\mathbf{Q}$ we can use standard ergodic theory. Hence, the expressions in δ and d_a converge to a constant as $n \to \infty$ provided

$$\mathbb{E}\Big((f_1 g)^{\frac{p}{p-s}} + (fg)^{\frac{p}{p-s}}\Big) < \infty\,. \tag{6.8}$$

However, f, f_1 and g are stationary by definition and f and g or f_1 and g are independent. Since f and f_1 clearly have finite expectation by the exponential distribution of d_a and Lemma 3.18, we only mention that due to the strong mixing of δ and its independence from the distribution of connections

$$\mathbb{E}(g^{\frac{p}{p-s}}) \leq \mathbb{E}\left(\sum_{e\sim f} \chi_{B_{e,f}}\right) \mathbb{E}\Big(\big(\delta^{-s-d} + \delta^{1+s-2d}\big)^{\frac{p}{p-s}}\Big)$$

and thus (6.8) holds. □

The work [28] which we used in the last proof also opens the door to demonstrate the following result which will be used in part III of this series to prove regularity properties of the homogenized equation.

Theorem 6.7. *For fixed* $y_0 \in \mathbb{X}_{\mathrm{mat}}$ *and every* $\tilde{y} \in \mathbb{X}_{\mathrm{mat}}$ *let* $P(y_0, \tilde{y}) = (y_0, y_1(\tilde{y}), \dots, y_N(\tilde{y}))_{N\in\mathbb{N}}$ *with* $y_N(\tilde{y}) = \tilde{y}$ *be the shortest path of points in* $\mathbb{X}_{\mathrm{mat}}$ *connecting* y_0 *and* $\tilde{y}$ *in* $\mathbf{P}$ *and having length* $L(y_0, \tilde{y})$. *Then there exists*

$$\begin{aligned}\gamma_{y_0,\tilde{y}}:\ [0, L(y_0,\tilde{y})] \times \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0) &\to \mathbf{P}\\ (t,z) &\mapsto \gamma_{y_0,\tilde{y}}(t,z)\end{aligned}$$

such that $\gamma_{y_0,\tilde{y}}(t,\cdot)$ *is invertible for every* t *and* $\left\|\partial_t \gamma_{y_0,\tilde{y}}\right\|_\infty \leq 2$. *For* $R > 1$ *let*

$$N_{y_0,R}(x) := \#\left\{\tilde{y} \in \mathbb{B}_R(y_0) \cap \mathbb{X}_{\mathrm{mat}}:\ \exists t:\ x \in \gamma_{y_0,\tilde{y}}\Big(t, \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\Big)\right\}\,.$$

Then there exists $C > 0$ *such that for every* y_0 *it holds*

$$N_{y_0,R}(x) \leq C\left(R^d - \left(\frac{x}{2}\right)^d\right) \quad \text{for } |x - y_0| < 2R\,, \qquad N_{y_0,R}(x) = 0 \quad \text{else.}$$

Proof. The function $\gamma_{y_0,\tilde{y}}$ consists basically of pipes connecting $y_i(\tilde{y})$ with $y_{i+1}(\tilde{y})$ that conically become smaller within the ball $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(y_i(\tilde{y}))$ before entering the pipe and vice versa in $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(y_{i+1}(\tilde{y}))$. Defining

$$N_{y_0,r,R}(x) := \#\left\{\tilde{y} \in (\mathbb{B}_R(y_0)\backslash\mathbb{B}_r(y_0)) \cap \mathbb{X}_{\mathrm{mat}}:\ \exists t:\ x \in \gamma_{y_0,\tilde{y}}\Big(t, \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\Big)\right\}$$

[28] implies $N_{y_0,r,R}(x) = 0$ for all $|x - y_0| > 2R$ but also due to the minimal mutual distance $N_{y_0,r,R}(x) \leq CR^{d-1}(R - r)$, where C depends only on $\mathfrak{r}$ and d.

Hence writing $\lfloor x \rfloor := \min\{n \in \mathbb{N}:\ n + 1 > x\}$ we can estimate for every $K \in \mathbb{N}$

$$N_{y_0,K}(x) \leq \sum_{k=0}^{K-1} N_{y_0,k,k+1}(x) \leq C \sum_{k=\left\lfloor \frac{x}{2}\right\rfloor}^{K-1} (k+1)^{d-1} \leq C\left(K^d - \left\lfloor\frac{x}{2}\right\rfloor^d\right)\,.$$

□

We close this section by proving Theorem 1.16.

Proof of Theorem 1.16. The statement on the support is provided by Theorem 4.10 and the fact that we restrict to functions with support in $m\mathbf{Q}$. Hence in the following we can apply all cited results to $\mathbb{B}_{m^{1-\beta}}(m\mathbf{Q})$ instead of $m\mathbf{Q}$. According to Lemmas 4.7 and 5.2–5.3 and to Theorem 6.5 we need only need to ensure

$$\mathbb{E}\left(\delta^{-s-d}+\delta^{1+s-2d}\right)^{\frac{p}{p-s}}+\mathbb{E}\left(1+M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^r+\mathbb{E}\left|\tilde{\rho}_{\mathbb{R}^d}\right|^{-\frac{sr}{s-r}}<\infty\,,$$

since d_a is distributed exponentially and the corresponding terms are bounded as long as $r\neq s\neq p$. We note that the exponential distribution of M allows us to restrict to the study of δ and $\tilde{\rho}$.

According to Lemma 6.4 it is sufficient that $\max\left\{\frac{p(s+d)}{p-s},\frac{p(2d-s-1)}{p-s}\right\}\leq\beta$ and $\frac{sr}{s-r}\leq\beta+d-1$. □

6.2 Boolean Model for the Poisson Ball Process

The following argumentation will be strongly based on the so called void probability. This is the probability $\mathbb{P}_0(A)$ to not find any point of the point process in a given open set A and is given by (2.49) i.e. $\mathbb{P}_0(A):=e^{-\lambda|A|}$. The void probability for the ball process is given accordingly by

$$\mathbb{P}_0(A):=e^{-\lambda\left|\overline{\mathbb{B}_1(A)}\right|}\,,\qquad \overline{\mathbb{B}_1(A)}:=\left\{x\in\mathbb{R}^d\;:\;\operatorname{dist}(x,A)\leq 1\right\}\,,$$

which is the probability that no ball intersects with $A\subset\mathbb{R}^d$.

Theorem 6.8. *Let $\mathbf{P}\left(\omega\right):=\bigcup_i B_i(\omega)$ (or $\mathbf{P}\left(\omega\right):=\mathbb{R}^d\backslash\bigcup_i B_i(\omega)$) and define*

$$\tilde{\delta}(x):=\min\left\{\delta(\tilde{x})\;:\;\tilde{x}\in\partial\mathbf{P}\text{ s.t. }x\in\mathbb{B}_{\frac{1}{8}\delta(\tilde{x})}(\tilde{x})\right\}\,,$$

where $\min\emptyset:=0$ for convenience. Then $\partial\mathbf{P}$ is almost surely locally (δ,M) regular and for every $\gamma<1$, $\beta<d+2$ it holds

$$\mathbb{E}\left(\delta^{-\gamma}\right)+\mathbb{E}\left(\tilde{\delta}^{-\gamma-1}\right)+\mathbb{E}\left(\tilde{M}_{[0]}^{\beta}\right)<\infty\,.$$

Furthermore, it holds $\hat{d}\leq d-1$ and $\alpha=0$ in inequalities (4.9) and (4.4). Furthermore the extension order and symmetric extension order are both $n=0$. If $\mathbf{P}\left(\omega\right):=\mathbb{R}^d\backslash\overline{\bigcup_i B_i(\omega)}$ the above holds with α replaced by 1 and with extension order $n=1$ and symmetric extension order $n=2$.

Remark 6.9*.* We observe that the union of balls has better properties than the complement.

Proof. We study only $\mathbf{P}\left(\omega\right):=\bigcup_i B_i(\omega)$ since $\mathbb{R}^d\backslash\overline{\bigcup_i B_i}(\omega)$ is the complement sharing the same boundary. Hence, in case $\mathbf{P}(\omega)=\mathbb{R}^d\backslash\overline{\bigcup_i B_i}(\omega)$, all calculations remain basically the same. However, in the first case, it is evident that $\alpha=0$ and $n=0$ because the geometry has only cusps and no dendrites and we refer to Remarks 2.4 and 2.8.

In what follows, we use that the distribution of balls is mutually independent. That means, given a ball around $x_i\in\mathbb{X}_{\mathrm{pois}}$, the set $\mathbb{X}_{\mathrm{pois}}\backslash\{x_i\}$ is also a Poisson process. W.l.o.g. , we assume $x_i=x_0=0$ with $B_0:=\mathbb{B}_1(0)$. First we note that $p\in\partial B_0\cap\partial\mathbf{P}$ if and only if $p\in\partial B_0\backslash\mathbf{P}$, which holds with probability $\mathbb{P}_0(\mathbb{B}_1(p))=\mathbb{P}_0(B_0)$. This is a fixed quantity, independent from p.

Now assuming $p\in\partial B_0\backslash\mathbf{P}$, the distance to the closest ball besides B_0 is denoted

$$r(p)=\operatorname{dist}(p,\partial\mathbf{P}\backslash\partial B_0)$$

with a probability distribution

$$\mathbb{P}_{\rm dist}(r):=\mathbb{P}_0(\mathbb{B}_{1+r}(p))/\mathbb{P}_0(\mathbb{B}_1(p))\,.$$

It is important to observe that ∂B_0 is r-regular in the sense of Lemma 2.24. Another important feature in view of Lemma 3.2 is $r(p)<\Delta(p)$. In particular, $\delta(p)>\frac{1}{2}r(p)$ and ∂B_0 is $(\delta,1)$-regular in case $\delta<\sqrt{\frac{1}{2}}$. Hence, in what follows, we will derive estimates on $r^{-\gamma}$, which immediately imply estimates on $\delta^{-\gamma}$.

Estimate on γ: A lower estimate for the distribution of $r(p)$ is given by

$$\mathbb{P}_{\rm dist}(r):=\mathbb{P}_0(\mathbb{B}_{1+r}(p))/\mathbb{P}_0(\mathbb{B}_1(p))\approx 1-\lambda\left|\mathbb{S}^{d-1}\right|r\,. \tag{6.9}$$

This implies that almost surely for $\gamma<1$

$$\limsup_{n\to\infty}\frac{1}{(2n)^d}\int_{(-n,n)^d\cap\partial\mathbf{P}}r(p)^{-\gamma}\,\mathrm{d}\mathcal{H}^{d-1}(p)<\infty\,,$$

i.e. $\mathbb{E}(\delta^{-\gamma})<\infty$.

Intersecting balls: Now assume there exists x_i, $i\neq 0$ such that $p\in\partial B_i\cap\partial B_0$. W.l.o.g. assume $x_i=x_1:=(2x,0,\dots,0)$ and $p=\left(\sqrt{1-x^2},0,\dots,0\right)$. Then

$$\delta(p)\leq\delta_0(p):=2\sqrt{1-x^2}$$

and p is at least $M(p)=\frac{x}{\sqrt{1-x^2}}$-regular. Again, a lower estimate for the probability of r is given by (6.9) on the interval $(0,\delta_0)$. Above this value, the probability is approximately given by $\lambda\left|\mathbb{S}^{d-1}\right|\delta_0$ (for small δ_0i.e. $x\approx 1$). We introduce as a new variable $\xi=1-x$ and obtain from $1-x^2=\xi(1+x)$ that

$$\delta_0\leq C\xi^{\frac{1}{2}}\quad\text{and}\quad M(p)\leq C\xi^{-\frac{1}{2}}\,. \tag{6.10}$$

No touching: At this point, we observe that M is almost surely locally finite. Otherwise, we would have $x=1$ and for every $\varepsilon>0$ we had $x_1\in\mathbb{B}_{2+\varepsilon}(x_0)\backslash\mathbb{B}_{2-\varepsilon}(x_0)$. But

$$\mathbb{P}_0(\mathbb{B}_{2+\varepsilon}(x_0)\backslash\mathbb{B}_{2-\varepsilon}(x_0))\approx 1-\lambda 2\left|\mathbb{S}^{d-1}\right|\varepsilon\;\to\;1\qquad\text{as }\varepsilon\to 0\,.$$

Therefore, the probability that two balls "touch" (i.e. that $x=1$) is zero. The almost sure local boundedness of M now follows from the countable number of balls.

Extension to $\tilde\delta$: We again study each ball separately. Let $p\in\partial B_0\backslash\overline{\mathbf{P}}$ with tangent space T_p and normal space N_p. Let $x\in N_p$ and $\tilde p\in\partial B_0$ such that $x\in\mathbb{B}_{\frac{1}{8}\delta(\tilde p)}(\tilde p)$, then also $p\in\mathbb{B}_{\frac{1}{8}\delta(\tilde p)}(\tilde p)$ and $\delta(p)\in(\frac{7}{8},\frac{7}{6})\delta(\tilde p)$ and $\delta(\tilde p)\in(\frac{7}{8},\frac{7}{6})\delta(p)$ by Lemma 2.24. Defining

$$\tilde\delta_i(x):=\min\left\{\delta(\tilde x)\;:\;\tilde x\in\partial B_i\backslash\mathbf{P}\text{ s.t. }x\in\mathbb{B}_{\frac{1}{8}\delta(\tilde x)}(\tilde x)\right\}\,,$$

we find

$$\tilde\delta^{-\gamma}\leq\sum_i\chi_{\tilde\delta_i>0}\tilde\delta_i^{-\gamma}\,.$$

Studying δ_0 on ∂B_0 we can assume $M\leq M_0$ in (3.8) and we find

$$\int_{\mathbf{P}}\chi_{\tilde\delta_0>0}\tilde\delta_0^{-\gamma-1}\leq C\int_{\partial B_0\backslash\mathbf{P}}\delta^{-\gamma}\,.$$

Hence we find

$$\int_{\mathbf{P}} \tilde{\delta}^{-\gamma-1} \leq \sum_i \int_{\mathbf{P}} \chi_{\tilde{\delta}_i>0} \tilde{\delta}_i^{-\gamma-1} \leq \sum_i C \int_{\partial B_i \setminus \mathbf{P}} \delta^{-\gamma} .$$

Estimate on β: For two points $x_i, x_j \in \mathbb{X}_{\mathrm{pois}}$ let $\mathrm{Circ}_{ij} := \partial B_i \cap \partial B_j$ and $\mathbb{B}_{\frac18\tilde{\delta}}(\mathrm{Circ}_{ij}) := \bigcup_{p\in\mathrm{Circ}_{ij}} \mathbb{B}_{\frac18\tilde{\delta}(p)}(p)$. For the fixed ball $B_i = B_0$ we write Circ_{0j} and obtain $|\mathrm{Circ}_{0j}| \leq C\delta_0^d$ with δ_0 from (6.10). Therefore, we find

$$\int_{\mathrm{Circ}_{0j}} (1+M(p))^\beta \leq \delta_0^d (1+M(p))^\beta \leq C\xi^{-\frac12(\beta-d)} .$$

We now derive an estimate for $\mathbb{E}\Big(\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta\Big)$. To this aim, let $q \in (0,1)$.

Then $x \in \mathbb{B}_{2-q^{k+1}}(0)\setminus\mathbb{B}_{2-q^k}(0)$ implies $\xi \geq q^{k+1}$ and

$$\begin{aligned}\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta &\leq C + \sum_{k=1}^\infty \sum_{x_j\in\mathbb{B}_{2-q^{k+1}}(0)\setminus\mathbb{B}_{2-q^k}(0)} \int_{\mathrm{Circ}_{0j}} (1+M(p))^\beta \\ &\leq C + \sum_{k=1}^\infty \sum_{x_j\in\mathbb{B}_{2-q^{k+1}}(0)\setminus\mathbb{B}_{2-q^k}(0)} C\left(q^{k+1}\right)^{-\frac12(\beta-d)}\end{aligned}$$

The only random quantity in the latter expression is $\#\left\{x_j \in \mathbb{B}_{2-q^{k+1}}(0)\setminus\mathbb{B}_{2-q^k}(0)\right\}$. Therefore, we obtain with $\mathbb{E}(\mathbb{X}(A)) = \lambda\,|A|$ that

$$\begin{aligned}\mathbb{E}\left(\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta\right) &\leq C\left(1+\sum_{k=1}^\infty \left(q^k-q^{k+1}\right)\left(q^{k+1}\right)^{-\frac12(\beta-d)}\right) \\ &\leq C\left(1+\sum_{k=1}^\infty \left(q^k\right)^{-\frac12(\beta-d-2)}\right) .\end{aligned}$$

Since the point process has finite intensity, this property carries over to the whole ball process and we obtain the condition $\beta < d+2$ in order for the right hand side to remain bounded.

Estimate on $\hat{d}$: We have to estimate the local maximum number of $A_{3,k}$ overlapping in a single point in terms of $\tilde{M}$. We first recall that $\hat{\rho}(p) \approx 8\tilde{M}(p)\tilde{\rho}(p)$. Thus large discrepancy between $\hat{\rho}$ and $\tilde{\rho}$ occurs in points where $\tilde{M}$ is large. This is at the intersection of at least two balls. Despite these "cusps", the set $\partial\mathbf{P}$ consists locally on the order of $\hat{\rho}$ of almost flat parts. Arguing like in Lemma 4.4 resp. Remark 4.5 this yields $\hat{d} \leq d-1$. □

It remains to verify bounded average connectivity of the Boolean set $\mathbf{P}_\infty$ or its complement. Associated with the connected component $\mathbb{X}_{\mathrm{pois},\infty}$ there is a graph distance

$$\forall x,y \in \mathbb{X}_{\mathrm{pois},\infty} \quad d(x,y) := \inf\left\{l(\gamma) :\ \gamma \text{ path in } \mathbb{X}_{\mathrm{pois},\infty} \text{ from } x \text{ to } y\right\} .$$

Using this distance, we shall rely on the following concept.

Definition 6.10 (Statistical Strech Factor)**.** For $x \in \mathbb{X}_{\mathrm{pois},\infty}$ and $R > \mathfrak{r}$ we denote

$$S(x,R) := \max_{y\in\mathbb{X}_{\mathrm{pois},\infty}\cap\mathbb{B}_R(x)} \frac{d(x,y)}{R}, \qquad S(x) := \sup_{R>\mathfrak{r}} S(x,R),$$

the statistical local strech factor $S(x,R)$ and statistical (global) strech factor $S(x)$.

Lemma 6.11. *There exists $S_0 > 1$ depending only on d and λ such that for $x \in \mathbb{X}_{\mathrm{pois},\infty}$ it holds*

$$\forall S > S_0 : \qquad \mathbb{P}(S(x) > S) \leq \frac{2\mu}{\nu} e^{-\frac{\nu}{2\mu}S} .$$

In order to prove this, we will need the following large deviation result.

Theorem 6.12 (Shape Theorem [29, Thm 2.2])**.** *Let $\lambda > \lambda_c$. Then there exist positive contants μ, ν and k_0 such that the following holds: For every $k > k_0$*

$$\mathbb{P}(S(0,k) > \mu) \leq e^{-\nu k} .$$

Proof of Lemma 6.11. We have

$$\begin{aligned} S(0,k) > \alpha\mu \qquad &\Leftrightarrow \qquad \exists x, y \in \mathbb{B}_k(0) : \quad d(x,y) \geq \alpha\mu k , \\ S(0,\alpha k) > \mu \qquad &\Leftrightarrow \qquad \exists x, y \in \mathbb{B}_{\alpha k}(0) : \quad d(x,y) \geq \alpha\mu k , \end{aligned}$$

i.e.

$$\mathbb{P}(S(0,k) > \alpha\mu) \leq \mathbb{P}(S(0,\alpha k) > \mu) \leq e^{-\frac{\nu}{\mu}(\alpha\mu)k} .$$

One quickly verifies for $k \in \mathbb{N}$ that $S(0,k) \leq S$ and $S(0,k+1) \leq S$ implies $S(0,k+r) \leq 2S$ for all $r \in (0,1)$. Hence we find

$$\mathbb{P}(S(x) > S) \leq \sum_{k\in\mathbb{N}} \mathbb{P}\left(S(0,k) > \frac{S}{2} \right) \leq \sum_{k\in\mathbb{N}} e^{-\frac{\nu}{2\mu}Sk} \leq \frac{2\mu}{\nu} e^{-\frac{\nu}{2\mu}S} .$$

□

While the choice of the points $(p_i)_{i\in\mathbb{N}} \subset \partial\mathbf{P}$ is clearly specified in Section 4.1, there is lots of room in the choice and construction of $\mathbb{X}_{\mathfrak{r}}$. In what follows, we choose $\mathbb{X}_{\mathfrak{r}}$ in the form (2.51). Then we find the following:

Theorem 6.13. *Under the above assumptions on the construction of $\mathbf{P}_\infty$, as well as $p > d$ and using the notation of Lemma 5.2, for every $1 \leq r < s < p$ there almost surely exists $C > 0$ such that for every $n \in \mathbb{N}$ and every $u \in W^{1,p}_{0,\partial(n\mathbf{Q})}(\mathbf{P}_\infty \cap n\mathbf{Q})$*

$$\left(\frac{1}{|n\mathbf{Q}|} \int_{\mathbf{P}_\infty \cap n\mathbf{Q}} \sum_{k:\, \partial_l\Phi_k > 0} \sum_{j:\, \partial_l\Phi_j < 0} \frac{d_j^{-\tilde\alpha\frac{s}{r}} |\partial_l\Phi_j|}{D^\Phi_{l+}} |\mathcal{M}_k u - \mathcal{M}_j u|^s \right)^{\frac{r}{s}} + F^3_s(n\mathbf{Q}, u) \leq C \left(\frac{1}{|n\mathbf{Q}|} \int_{\mathbf{P}_\infty \cap n\mathbf{Q}} |\nabla u|^p \right)^{\frac{r}{p}} ,$$

and for every $u \in \mathbf{W}^{1,p}_{0,\partial(n\mathbf{Q})}(\tilde{\mathbf{P}} \cap n\mathbf{Q})$

$$\left(\frac{1}{|n\mathbf{Q}|} \int_{\mathbf{P}_\infty \cap n\mathbf{Q}} \sum_{k:\, \partial_l\Phi_k > 0} \sum_{j:\, \partial_l\Phi_j < 0} \frac{d_j^{-\tilde\alpha\frac{s}{r}} |\partial_l\Phi_j|}{D^\Phi_{l+}} \left|\mathcal{M}^{\mathfrak{s}}_k u - \mathcal{M}^{\mathfrak{s}}_j u\right|^s \right)^{\frac{r}{s}} + F^{3,\mathfrak{s}}_s(n\mathbf{Q}, u) \leq C \left(\frac{1}{|n\mathbf{Q}|} \int_{\mathbf{P}_\infty \cap n\mathbf{Q}} |\nabla^s u|^p \right)^{\frac{r}{p}} .$$

Lemma 6.14. *Let $\mathbb{X}_{\mathrm{pois}}$ be a Poisson point process with finite intensity. Generate a Voronoi tessellation from $\mathbb{X}_{\mathrm{pois}}$ and for each $x_a \in \mathbb{X}_{\mathrm{pois}}$ let d_a be the diameter of the corresponding Voronoi cell. Then for each $n \in \mathbb{N}$ the following function has finite expectation*

$$f_n := \sum_a \chi_{\mathbb{B}_{nd_a}(x_a)} \,.$$

Note that this statement is not covered by Lemma 3.18 due to the lack of a minimal distance between the points.

Proof. Given the condition $0 \in \mathbb{X}_{\mathrm{pois}}$ we observe

$$\mathbb{E}\Big(\chi_{\mathbb{B}_{nd_0}(0)}\Big)(x) \leq \sum_{k=0}^{\infty} \mathbb{P}(d_0 \in [k,k+1))\chi_{\mathbb{B}_{k+1}(0)}(x)\,.$$

Since $\mathbb{P}(d_0 \in [k,k+1)) \leq e^{-\alpha k}$ for some $\alpha > 0$, we infer

$$\mathbb{E}\Big(\chi_{\mathbb{B}_{nd_0}(0)}\Big)\,(x) \leq Ce^{-\alpha|x|}\,.$$

From here, we conclude with the exponentially in N decreasing probability to find more than N points within $[0,1]^d$:

$$\mathbb{E}\left(\sum_{x_a\in\mathbb{X}_{\mathrm{pois}}\cap[0,1]^d} \chi_{\mathbb{B}_{nd_a}(x_a)}\right)(x) \leq Ce^{-\beta|x|}\,,$$

for some $\beta > 0$. Summing up over all cubes we infer

$$\mathbb{E}(f_n)(0) \leq C\sum_{k\in\mathbb{Z}^d} e^{-\beta|x-k|} \leq C\sum_{N\in\mathbb{N}} N^{d-1}e^{-\beta N} < \infty\,.$$

□

Similar to the proof of Theorem 6.5 it will be necessary to introduce the following quantity for $y \in \mathbb{X}_{\mathrm{pois},\infty}$ based on (2.27):

$$\mathcal{M}^{\mathfrak{s}}_y u(x) := \overline{\nabla^{\perp}_{y,\mathfrak{r}}} u(x-y) + \fint_{\mathbb{B}_{\mathfrak{r}}(y)} u\,.$$

An important property of $\mathcal{M}^{\mathfrak{s}}_y$ is the following.

Lemma 6.15. *Let $y_1, y_2 \in \mathbb{X}_{\mathrm{pois},\infty}$ with $|y_1 - y_2| < 2$ and $\delta := \frac{1}{2}\sup\Big\{r \,:\, \mathbb{B}_r\big(\frac{1}{2}\,(y_1+y_2)\big) \subset \tilde{\mathbf{P}}\Big\}$. Then there exists $f \,:\, \mathbb{B}_1(\{y_1,y_2\}) \to \mathbb{R}$ such that*

$$\left|\mathcal{M}^{\mathfrak{s}}_{y_1}u(x) - \mathcal{M}^{\mathfrak{s}}_{y_2}u(x)\right|^s \leq C\left\|f\nabla^s u\right\|^s_{L^s_{\mathbb{B}_1(\{y_1,y_2\})}}\,,$$
$$\left|\mathcal{M}_{y_1}u(x) - \mathcal{M}_{y_2}u(x)\right|^s \leq C\left\|f\nabla^s u\right\|^s_{L^s_{\mathbb{B}_1(\{y_1,y_2\})}}\,,$$

and

$$\int_{\mathbb{B}_1(\{y_1,y_2\})} |f|^{\frac{sp}{p-s}} \leq C\delta^{\frac{s(p-d)}{p-s}-1}\,. \tag{6.11}$$

Furthermore for some fixed $C > 0$ and for every $y \in \mathbb{X}_{\mathrm{pois},\infty}$

$$\int_{\mathbb{B}_1(y)} \sum_i \chi_{\mathbb{B}_{\tilde{\rho}_i}(p_i)}\left|\tau^{\mathfrak{s}}_i u - \mathcal{M}^{\mathfrak{s}}_y u\right|^s + \sum_{x_a\in\mathbb{X}_{\mathfrak{r}}} \chi_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_a)}\left|\mathcal{M}^{\mathfrak{s}}_a u - \mathcal{M}^{\mathfrak{s}}_y\right|^s \leq C\left\|\nabla^s u\right\|^s_{L^s(\mathbb{B}_1(y))}\,. \tag{6.12}$$

$$\int_{\mathbb{B}_1(y)} \sum_i \chi_{\mathbb{B}_{\tilde{\rho}_i}(p_i)}\left|\tau_i u - \mathcal{M}_y u\right|^s + \sum_{x_a\in\mathbb{X}_{\mathfrak{r}}} \chi_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_a)}\left|\mathcal{M}_a u - \mathcal{M}_y\right|^s \leq C\left\|\nabla^s u\right\|^s_{L^s(\mathbb{B}_1(y))}\,. \tag{6.13}$$

Proof. We only treat the vector valued case, the other is proved similarly using results from Section 2.4. W.l.o.g let $y_1 = y\mathbf{e}_1$ and $y_2 = -y\mathbf{e}_1$. Let $n = \min\left\{n \in \mathbb{N} : \mathbb{B}_{2^{-n}\mathfrak{r}}(0) \subset \tilde{\mathbf{P}}\right\}$, i.e. $\delta \in (2^{-n-1}\mathfrak{r}, 2^{-n}\mathfrak{r})$. Furthermore, let $\alpha_k := 2\mathfrak{r}\sum_{j=1}^{k} 2^{-(n-j)}$ for $k = 1, \dots, n$ and $\alpha_{-k} = -\alpha_k$ with $\alpha_0 = 0$. Using (2.27), for every number $j = -n, \dots, n$ let further

$$\mathcal{M}_j^{\mathfrak{s}} u := \mathcal{M}_{\alpha_j \mathbf{e}_1}^{\mathfrak{s},\mathfrak{r}2^{-(n-|j|)}} .$$

Then for $j \geq 0$ we find from Lemma 2.18

$$\begin{aligned}
\left|\mathcal{M}_j^{\mathfrak{s}} u(x) - \mathcal{M}_{j+1}^{\mathfrak{s}} u(x)\right|^s &\leq C\left(\left|\mathcal{M}_{\alpha_j\mathbf{e}_1}^{\mathfrak{s},\mathfrak{r}2^{-(n-j)}} u(x) - \mathcal{M}_{\alpha_{j+1}\mathbf{e}_1}^{\mathfrak{s},\mathfrak{r}2^{-(n-j)}} u(x)\right|^s + \right.\\
&\left. + \left|\mathcal{M}_{\alpha_{j+1}\mathbf{e}_1}^{\mathfrak{s},\mathfrak{r}2^{-(n-j)}} u(x) - \mathcal{M}_{\alpha_{j+1}\mathbf{e}_1}^{\mathfrak{s},\mathfrak{r}2^{-(n-j-1)}} u(x)\right|^s\right)\\
&\leq (\mathfrak{r}2^{-n})^{s-d} 2^{j(s-d)} \left\|\nabla^s u\right\|^s_{L^s\left(\mathrm{conv}\left(\mathbb{B}_{\mathfrak{r}2^{-(n-j)}}(\{\alpha_j\mathbf{e}_1,\alpha_{j+1}\mathbf{e}_1\})\right)\cup\mathbb{B}_{\mathfrak{r}2^{-(n-j-1)}}(\alpha_{j+1}\mathbf{e}_1)\right)}
\end{aligned}$$

Defining

$$\tilde{f}^s := \sum_j (\mathfrak{r}2^{-n})^{s-d} 2^{j(s-d)} \chi_{\mathrm{conv}\left(\mathbb{B}_{\mathfrak{r}2^{-(n-j)}}(\{\alpha_j\mathbf{e}_1,\alpha_{j+1}\mathbf{e}_1\})\right)\cup\mathbb{B}_{\mathfrak{r}2^{-(n-j-1)}}(\alpha_{j+1}\mathbf{e}_1)}$$

and using local finiteness of the covering as well as

$$\left|\mathrm{conv}\left(\mathbb{B}_{\mathfrak{r}2^{-(n-j)}}(\{\alpha_j\mathbf{e}_1, \alpha_{j+1}\mathbf{e}_1\})\right) \cup \mathbb{B}_{\mathfrak{r}2^{-(n-j-1)}}(\alpha_{j+1}\mathbf{e}_1)\right| \leq C\left(\mathfrak{r}^d 2^{-d(n-j)}\right),$$

we find with $\frac{(s-d)p}{p-s} + d = \frac{s(p-d)}{p-s} - 1 = \frac{s(1-d)-p+s}{p-s} = \frac{2s-d-p}{p-s}\,\delta$

$$\begin{aligned}
\int_{\mathbb{B}_1(\{y_1,y_2\})} \left|\tilde{f}\right|^{\frac{sp}{p-s}} &\leq C\sum_j (\mathfrak{r}2^{-n})^{\frac{(s-d)p}{p-s}} 2^{j\frac{(s-d)p}{p-s}} \mathfrak{r}^d 2^{-d(n-j)}\\
&\leq C\delta^{\frac{s(p-d)}{p-s}} \sum_{j=1}^{\ln_2\frac{\mathfrak{r}}{\delta}} 2^{j\frac{s(1-d)}{p-s}} \leq C\delta^{\frac{s(p-d)}{p-s}}\delta^{-1}
\end{aligned}$$

From here we conclude the first part. Inequality (6.12) follows from the fact that $\tilde{\rho}_i \propto \mathfrak{r}_i$ and the disjointness of the balls $\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_a)$ with $\mathbb{B}_{\mathfrak{r}_i}(p_i)$ and Lemma 2.17 with $\mathfrak{r} = const$. ☐

Proof of Theorem 6.13. We work with the enumeration $(p_i)_{i\in\mathbb{N}}$ and $\mathbb{X}_{\mathfrak{r}} = (x_a)_{a\in\mathbb{N}}$ and make use of the underlying point process $\mathbb{X}_{\mathrm{pois}}$: For every $a \in \mathbb{N}$ there exists $y_{x_a} \in \mathbb{X}_{\mathrm{pois}}$ such that $x_a \in \mathbb{B}_1(y_{x_a})$ for every p_i there almost surely exists a unique $y_{p_i} \in \mathbb{X}_{\mathrm{pois}}$ such that $p_i \in \mathbb{B}_1(y_{p_i})$. Due to the minimal mutual distance of points in $\mathbb{X}_{\mathfrak{r}}$, we can conclude the following: Since $p_i \in \mathbb{B}_{\mathfrak{r}}(G_a), \mathbb{B}_{\mathfrak{r}}(x_a) \subset \tilde{P}\cap G_a$ there exists a constant C depending only on $\mathfrak{r}$ and d such that always

$$|y_{p_i} - y_{x_a}| \leq C\, d_a\,. \tag{6.14}$$

Since

$$|\tau_i^{\mathfrak{s}} u - \mathcal{M}_a^{\mathfrak{s}} u|^s \leq 3\left(\left|\tau_i^{\mathfrak{s}} u - \mathcal{M}_{y_{p_i}}^{\mathfrak{s}} u\right|^s + \left|\mathcal{M}_{y_{x_a}}^{\mathfrak{s}} u - \mathcal{M}_a^{\mathfrak{s}} u\right|^s + \left|\mathcal{M}_{y_{x_a}}^{\mathfrak{s}} u - \mathcal{M}_{y_{p_i}}^{\mathfrak{s}} u\right|^s\right)$$

we find

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}} \sum_{i\neq 0} \sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a \frac{|\partial_l \phi_i|}{D_{l+}} |\tau_i^{\mathfrak{s}} u - \mathcal{M}_a^{\mathfrak{s}} u|^s \leq I_1 + I_2 + I_3$$

where we provide and estimate I_1, I_2 and I_3 as follows: First, we observe there exists n_0 such that $n_0\mathfrak{r} > 1$. Then with help of (6.12)

$$\begin{aligned} I_1 &:= \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{i\neq 0}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\left|\tau_i^\mathfrak{s} u - \mathcal{M}^\mathfrak{s}_{y_{p_i}}u\right|^s \\ &\le \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a \sum_{y_b\in\mathbb{X}_{\mathrm{pois},\infty}}\sum_{p_i\in\partial\mathbb{B}_1(y_b)}\frac{|\partial_l\phi_i|}{D_{l+}}\left|\tau_i^\mathfrak{s} u - \mathcal{M}^\mathfrak{s}_{y_b}u\right|^s \\ &\le \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} \chi_{\mathbb{B}_{2d_a}(x_a)} \sum_{y_b\in\mathbb{X}_{\mathrm{pois},\infty}}\sum_{p_i\in\partial\mathbb{B}_1(y_b)}\chi_{\mathbb{B}_{\tilde\rho_i}(p_i)}\left|\tau_i^\mathfrak{s} u - \mathcal{M}^\mathfrak{s}_{y_b}u\right|^s \\ &\le C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\left(\sum_{x_a\in\mathbb{X}(\mathbf{Q})}\chi_{\mathbb{B}_{2d_a}(x_a)}\sum_{y_b\in\mathbb{X}_{\mathrm{pois},\infty}}\chi_{\mathbb{B}_1(y_b)}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}|\nabla u|^p\right)^{\frac{s}{p}}. \end{aligned}$$

Because of Lemmas 3.18 and 6.14 and the exponential decay of probabilities of d_a the first integral on the right hand side is always bounded. Note that (6.12) also implies

$$\begin{aligned} I_2 &:= \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{i\neq 0}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\left|\mathcal{M}^\mathfrak{s}_{y_{x_a}}u - \mathcal{M}^\mathfrak{s}_a u\right|^s \\ &\le \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\, d_a^d\left|\mathcal{M}^\mathfrak{s}_{y_{x_a}}u - \mathcal{M}^\mathfrak{s}_a u\right|^s \\ &\le \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\left(\sum_{y_b\in\mathbb{X}_{\mathrm{pois},\infty}}\sum_{x_a\in\mathbb{X}(\mathbf{Q})\cap\mathbb{B}_1(y_b)} d_a^{2d}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_1}|\nabla u|^p\right)^{\frac{s}{p}}. \end{aligned}$$

Again, the first integral on the right hand side is bounded.

Last, the term

$$I_3 := \int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}}\sum_{i\neq 0}\sum_{x_a\in\mathbb{X}(\mathbf{Q})} f_a\frac{|\partial_l\phi_i|}{D_{l+}}\left|\mathcal{M}^\mathfrak{s}_{y_{x_a}}u - \mathcal{M}^\mathfrak{s}_{y_{p_i}}u\right|^s$$

is the most tricky part.

We find a path $Y(y_{x_a}, y_{p_i}) = (y_1, \dots, y_{n(x_a,p_i)})$ such that $y_1 = y_{x_a}$, $y_{n(x_a,p_i)} = y_{p_i}$ such that y_j, y_{j+1} are neighbors. By our assumptions, for every two points $y, \tilde y \in \mathbb{X}_{\mathrm{pois},\infty}$ with $y - \tilde y < 2\mathfrak{r}$, the convex hull of $\mathbb{B}_\mathfrak{r}(\{y,\tilde y\})$ lies in $\mathbf{P}_\infty$. Hence we iteratively replace sequences $(\dots y_i, y_{i+1}, y_{i+2}, \dots)$ in the path Y by $(\dots y_i, y_{i+2}, \dots)$ if $|y_{i+2} - y_i| < 2\mathfrak{r}$. Hence, w.l.o.g we obtain from (6.14) and the definition of the statistical strech factor

$$n(x_a, p_i) \le 2\frac{\mathrm{Length}Y}{\mathfrak{r}} \le 2\mathfrak{r}^{-1}Cd_aS(y_{x_a}).$$

Therefore, for $y \in \mathbb{X}_{\mathrm{pois},\infty}$ with $\chi_{\mathbb{B}_1(y)}\chi_{G_a} \neq 0$ we observe and the shortest path $Y(x_a, y_{p_i})$ and with Lemma 6.15

$$\begin{aligned} \left|\mathcal{M}^\mathfrak{s}_{y_{x_a}}u - \mathcal{M}^\mathfrak{s}_{y_{p_i}}u\right|^s &\le \left(2\mathfrak{r}^{-1}Cd_aS(y_{x_a})\right)^s\sum_{k=1}^{n(x_a,y_{p_i})-1}\left|\mathcal{M}^\mathfrak{s}_{y_k}u - \mathcal{M}^\mathfrak{s}_{y_{k+1}}u\right|^s \\ &\le \left(2\mathfrak{r}^{-1}Cd_aS(y_{x_a})\right)^s\sum_{k=1}^{n(x_a,y_{p_i})-1}\|f\nabla^s u\|^s_{L^s_{\mathbb{B}_1(\{y_k,y_{k+1}\})}}. \end{aligned}$$

Now all points $y_i \in Y(x_a, y_{p_i})$ lie within a radius of $2Cd_aS(y_{x_a})$ around x_a, wich implies

$$I_3 \leq \int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}} \sum_{x_a\in\mathbb{X}(\mathbf{Q})} \chi_{\mathbb{B}_{2Cd_aS(y_{x_a})}(x_a)} d_a^d f^s \left|\nabla^s u\right|^s$$
$$\leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_\mathfrak{r}} \left(\sum_{x_a\in\mathbb{X}(\mathbf{Q})} \chi_{\mathbb{B}_{2Cd_aS(y_{x_a})}(x_a)} d_a^d f^s\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}} \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_1} |\nabla u|^p\right)^{\frac{s}{p}} .$$

Now, by independence of the respective variables, the constant in front converges to

$$\left(\mathbb{E}\left(\sum_{x_a\in\mathbb{X}(\mathbf{Q})} \chi_{\mathbb{B}_{2Cd_aS(y_{x_a})}(x_a)} d_a^d\right)^{\frac{p}{p-s}} \mathbb{E} f^{\frac{ps}{p-s}}\right)^{\frac{p-s}{p}} .$$

The first term in the product can be estimated with help of Lemma 3.18 and is bounded for every p and s by the exponential distribution of d_a and S. The second term can be estimated similarly. □

A further important property which we will not use in this work, but which is central for part III of this series is the following result.

Theorem 6.16. *Let* $\mathbb{X}_{\mathrm{pois},\infty,\mathfrak{r}} := \left\{x \in \mathbb{X}_{\mathrm{pois},\infty} : \ \forall y \in \mathbb{X}_{\mathrm{pois},\infty}\backslash\{x\}\ |x-y| > \frac{\mathfrak{r}}{8}\right\}$ *be a Matern reduction of the infinite component. For fixed* $y_0 \in \mathbb{X}_{\mathrm{pois},\infty,\mathfrak{r}}$ *and every* $\tilde{y} \in \mathbb{X}_{\mathrm{pois},\infty,\mathfrak{r}}$ *let* $P(y_0,\tilde{y}) = (y_0, y_1(\tilde{y}), \dots, y_N(\tilde{y}))_{N\in\mathbb{N}}$ *with* $y_N(\tilde{y}) = \tilde{y}$ *be the shortest path of points in* $\mathbb{X}_{\mathrm{pois},\infty,\mathfrak{r}}$ *connecting* y_0 *and* $\tilde{y}$ *in* $\mathbf{P}$ *and having length* $L(y_0,\tilde{y})$*. Then there exists*

$$\gamma_{y_0,\tilde{y}} : \ [0, L(y_0,\tilde{y})] \times \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0) \to \mathbf{P}$$
$$(t,z) \mapsto \gamma_{y_0,\tilde{y}}(t,z)$$

such that $\gamma_{y_0,\tilde{y}}(t,\cdot)$ *is invertible for every* t *and* $\left\|\partial_t\gamma_{y_0,\tilde{y}}\right\|_\infty \leq 2$*. For* $R > 1$ *let*

$$N_{y_0,R}(x) := \#\left\{\tilde{y} \in \mathbb{B}_R(y_0) \cap \mathbb{X}_{\mathrm{mat}} : \ \exists t : \ x \in \gamma_{y_0,\tilde{y}}\Big(t, \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\Big)\right\} .$$

Then for every y_0 *there exists almost surely* $C > 0$*,* $S > 0$ *such that it holds*

$$N_{y_0,R}(x) \leq C\left(R^d - \left(\frac{x}{2}\right)^d\right) \quad \text{for } |x-y_0| < SR\,, \qquad N_{y_0,R}(x) = 0 \quad \text{else.}$$

Proof. The function $\gamma_{y_0,\tilde{y}}$ consists basically of pipes connecting $y_i(\tilde{y})$ with $y_{i+1}(\tilde{y})$ that conically become smaller within the ball $\mathbb{B}_{\frac{1}{2}}(y_i(\tilde{y}))$ to fit through the connection between two neighboring balls. Defining

$$N_{y_0,r,R}(x) := \#\left\{\tilde{y} \in (\mathbb{B}_R(y_0)\backslash\mathbb{B}_r(y_0)) \cap \mathbb{X}_{\mathrm{pois},\infty,\mathfrak{r}} : \ \exists t : \ x \in \gamma_{y_0,\tilde{y}}\Big(t, \mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\Big)\right\}$$

we apply Lemma 6.11 instead of [28] implies $N_{y_0,r,R}(x) = 0$ for all $|x-y_0| > SR$ but also due to the minimal mutual distance $N_{y_0,r,R}(x) \leq CR^{d-1}(SR-r)$, where C depends only on $\mathfrak{r}$ and d. From here we follow the proof of Theorem 6.7. □

We close this section and this work by proving Theorem 1.18.

Proof of Theorem 1.18. The statement on the support is provided by Theorem 4.10 and the fact that we restrict to functions with support in $m\mathbf{Q}$. Hence in the following we can apply all cited results to $\mathbb{B}_{m^{1-\beta}}(m\mathbf{Q})$ instead of $m\mathbf{Q}$. According to Lemmas 4.7 and 5.2–5.3 and to Theorem 6.13 we need only need to ensure $p > d$ as well as

$$\mathbb{E}\left(1 + M_{[\frac{1}{8}\delta],\mathbb{R}^d}\right)^{kr} + \mathbb{E}\left|\tilde{\rho}_{\mathbb{R}^d}\right|^{-\frac{sr}{s-r}} < \infty\,,$$

where $k = 1$ for the simple extension case and $k = 2$ for the symmetric extension case. Since d_a is distributed exponentially and the corresponding terms are bounded as long as $r \neq s \neq p$, we observe that we do not have to care about the involved polynomial terms.

According to Theorem 6.8 it is sufficient that $\frac{sr}{s-r} < 2$ (i.e. $\frac{pr}{p-r} < 2$) and $kr < d + 2$. □

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