

## **A generalized Haus master equation model for mode-locked class-B lasers**

Michel Nizette<sup>1</sup>, Andrei G. Vladimirov<sup>2</sup>

submitted: April 29, 2021

<sup>1</sup> Département de Physique  
Faculté des Sciences  
Université Libre de Bruxelles  
CP 231, Campus Plaine  
B-1050 Bruxelles  
Belgium  
E-Mail: michel.nizette@gmail.com

<sup>2</sup> Weierstrass Institute  
Mohrenstr. 39  
10117 Berlin  
Germany  
E-Mail: andrei.vladimirov@wias-berlin.de

No. 2840  
Berlin 2021



---

*2020 Mathematics Subject Classification.* 37N20, 78M34, 78M35.

*2010 Physics and Astronomy Classification Scheme.* 42.65.-k, 42.65.R, 42.60.F, 42.60.G.

*Key words and phrases.* Haus master equation, mode-locking, class B laser, multiscale method.

The work of M. N. was supported by Fonds de la Recherche Scientifique–FNRS. The work of A. G. V. was supported by the Deutsche Forschungsgemeinschaft (DFG-RSF project No.445430311).

Edited by  
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)  
Leibniz-Institut im Forschungsverbund Berlin e. V.  
Mohrenstraße 39  
10117 Berlin  
Germany

Fax: +49 30 20372-303  
E-Mail: [preprint@wias-berlin.de](mailto:preprint@wias-berlin.de)  
World Wide Web: <http://www.wias-berlin.de/>

# A generalized Haus master equation model for mode-locked class-B lasers

Michel Nizette, Andrei G. Vladimirov

## Abstract

Using the multiscale technique we develop a generalized version of the class-B Haus mode-locking model that accounts for both the slow gain response to the averaged value of the field intensity and the fast gain dynamics on the scale comparable to the pulse duration. We show that unlike the standard class-B Haus mode-locked model, our model is able to describe not only Q-switched instability of the fundamental mode-locked regime, but also the appearance of harmonic mode-locked regimes with the increase of the pump power.

## 1 Introduction

The first successful generation of a stable continuous train of passively mode-locked laser pulses was reported more than thirty years ago, in a dye laser with a saturable dye [1]. From then on, passive mode-locking has been recognized as a very powerful technique for generating high-quality picosecond and sub-picosecond laser pulses with fast repetition rates, and ongoing efforts are still devoted to the design of improved devices producing shorter and shorter pulses.

Owing to the early theoretical analyses of New [2] and Haus [3, 4], the essential physical process responsible for passive mode-locking is well understood. It relies on the presence of a mechanism for loss saturation at high laser power, typically implemented as a saturable absorber medium in the optical cavity. As illustrated in Fig. 1, if the absorber saturates faster than the amplifier medium on arrival

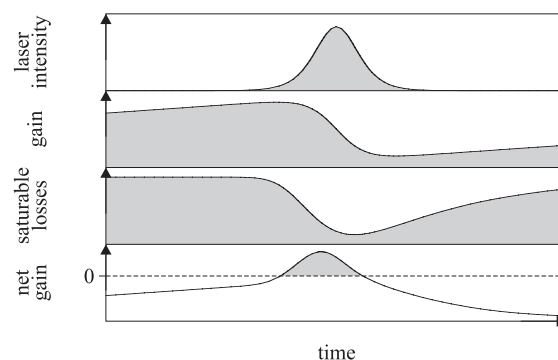


Figure 1: Qualitative depiction of a typical pulse amplification cycle in a mode-locked laser.

of a pulse, a short temporal window of positive net gain is opened, enabling the pulse amplification necessary to compensate for round-trip losses. The net gain then quickly becomes negative again as a consequence of the saturation of the amplifier medium, combined to the relaxation of the absorber back to its unsaturated state if the latter is fast enough. This process favors emission in the form of narrow pulses.

This basic physical picture is supported and has been confirmed many times by numerical and analytical studies of passively mode-locked laser models of various complexities [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. From the early work cited above [3, 4] originated a highly successful, universally adopted model for pulse amplification and shaping that is now known as Haus master equation [15]. It is a

partial differential equation that describes the temporal evolution of the pulse profile with successive round-trips. Coupled to appropriate rate equations for the gain and absorber dynamics, it provides a model that is able to reproduce and quantify the pulse amplification scenario described above.

The same mechanisms that are at play in passive mode-locking are also responsible for a detrimental, but practically unavoidable physical effect known as Q-switched instability [16, 17, 18, 19, 20, 21, 22] of a mode-locked regime. This refers to an instability characterized by the build-up of a modulation of the laser response with a period that typically extends over several round-trips. This process is the hallmark of class-B lasers, where the photon lifetime in the cavity is comparable to or shorter than the gain response time. It can be understood qualitatively as an alternation between two states characterized by different levels of absorber saturation. Initially, the laser field in the cavity is weak and the absorber is nearly unsaturated. The gain then builds up under the action of pumping for as long as the absorber can inhibit the amplification of radiation. Eventually, losses are overcome by gain and strong laser emission turns on, saturating the absorber and gradually depleting the gain medium back to a sub-threshold value. At this point, the laser field is not amplified any more and decays. The cycle then starts over.

Understanding the conditions of appearance of Q-switching in a mode-locked laser and determining experimental conditions under which it can be avoided is of primary importance for the successful generation of regular trains of mode-locked pulses. Theoretical studies based on Haus master equation have been very useful to this end [17, 23, 24, 18, 21]. However, the ability of the model to predict Q-switching requires some particular care in the formulation of the rate equation for the gain. The successful formulation is one where the gain response is assumed much slower than the cavity round-trip time (consistently with the time scale relationship that defines a class-B laser) and responds only to an average value of laser intensity over time.

Unfortunately, this Q-switching-enabled variant of the gain equation, being insensitive to the pulse temporal intensity profile within a round-trip, is unable to describe the fast gain depletion-recovery cycle that accompanies successive pulse amplifications. Therefore, it cannot account for the gain contribution to pulse shaping, which constitutes a serious shortcoming of the model. The situation is not too bad if the absorber is fast enough to follow the pulse intensity profile adiabatically (the so-called fast-absorber case). This occurs, for example, when the saturable losses are provided via additive-pulse mode locking [25] or Kerr lensing [26]. Then, losses saturate during pulse amplification and desaturate immediately afterwards, which suffices to create the short window of net gain needed for mode locking; the contribution from the gain medium is not essential to the process (see Fig. 2a).

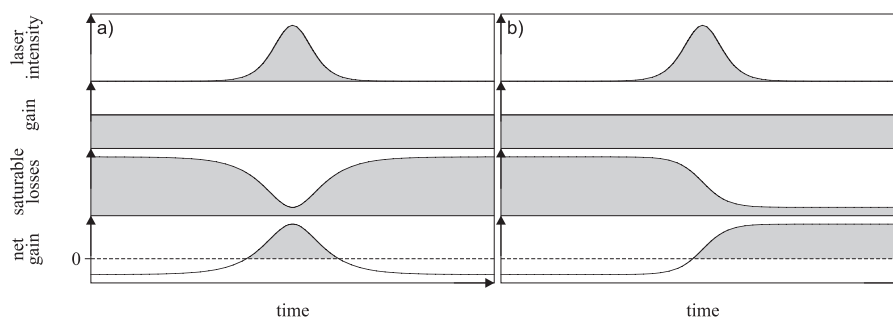


Figure 2: Qualitative depiction of a pulse amplification cycle in a mode-locked laser, in the absence of gain dynamics, and for a) a fast absorber, b) a slow absorber.

However, the shortcoming mentioned above is critical if the absorber relaxes on a time scale much longer than the pulse duration (the slow-absorber case). This situation occurs, for example, in a dye

[27] or semiconductor [6] absorber. In that case, the absorber remains saturated for a while after the passing of a pulse so that, in the absence of a significant depletion of the gain medium, the net gain would remain positive for a time interval significantly longer than the pulse duration (see Fig. 2b). This scenario is incompatible with the simple idea that the net gain remains positive for only a short time, thereby favoring narrow pulses. It is also certainly questionable from a physical point of view, as a positive net gain outside of the temporal boundaries of the pulse, and the amplification of the residual field that would result, is generally thought to favor a detrimental build-up of a macroscopic field, leading to the eventual destruction of the mode locking pattern. This phenomenon is known as background instability [11].

The problem has been partly identified in [6], where it has been stated that for very slow gain and absorber media, absorber saturation alone cannot explain the mode locking of a semiconductor laser. The full absorber saturation-and-recovery cycle, they argued, should be taken into account in the description of the amplification mechanism, implying that a semiconductor laser cannot be analyzed under the usual slow-absorber approximation that consists in neglecting the slow recovery processes on a time scale comparable to the pulse duration. Their non-approximate treatment of the problem indeed solved the issue at hand and led to a physically consistent picture of the pulse amplification cycle. However, their theory still lacks the ability to account for a fast response of the gain medium, which we regard as a limitation of the model rather than a physical reality. Numerical studies based on models more sophisticated than Haus master equation indeed suggest that the gain medium does respond on the pulse time scale, even in class-B lasers [11]. In our view, any accurate modeling of the core amplification mechanism should account for this, if only because a correct description of the net gain evolution cycle is essential for the reliable prediction of such very important information as conditions of background stability.

Owing to its mathematical simplicity, Haus master equation has been tremendously useful in understanding passive mode locking. Unfortunately, as argued above, the choice of a particular formulation of the rate equation for the gain dynamics conditions the ability or inability of the model to describe such important phenomena as Q-switching or the gain contribution to pulse shaping. The model originally put forth by Haus to explain mode locking in the case of slow absorbers [4] is valid for class-A lasers but lacks the ability to predict Q-switching in class-B lasers, and conversely, the Q-switching-enabled variant used in [21] does not reveal any gain dynamics on the pulse time scale. To our knowledge, no formulation so far accounts for both effects simultaneously, and researchers have traditionally been forced to choose one variant or the other depending on which effect was foremost in the focus of their study, or to resort to more complex models if both were equally important.

The purpose of this work is to overcome this shortcoming. Using an approach based on multiscale expansion we derive a model for the gain dynamics that generalizes the conventional formulations and which, coupled to Haus master equation, describes both Q-switching and the gain depletion-recovery cycle in a satisfactory way. The benefit is that a single version of Haus model allows the study of both phenomena. Unlike the extended Haus master equations obtained empirically in [28] the gain evolution on the slow time scale is included as an additional equation, rather than boundary condition. Furthermore, one can note that the discussion in [28] is incomplete since it ignores the previous models of soliton [29] and passive [3, 21] mode-locking, which include slow gain evolution equations.

In this introductory section, we stated the problem in a non-technical way. We make our objective more precise in Sec. 2 by introducing the two conventional versions of the Haus model discussed above and the underlying assumptions, as well as examining their respective limitations from a more mathematical point of view. We define the classifications into class-A and class-B lasers and into fast and slow

absorbers in terms of relative magnitudes of model parameters, and state the corresponding usual approximations. We pinpoint one of the assumptions of the model as the source of its shortcomings. We then derive, in Sec. 3, a new model for the gain dynamics with the limiting assumption relaxed, which is our main result. We only give the outline of the method; the calculations themselves are too lengthy to be included in the main body of this paper and are relegated to appendices. In Sec. 4, we check the validity of the new model formulation with the help of numerical simulations, and emphasize its ability to predict both Q-switching and a fast depletion-relaxation cycle of the gain medium. Conclusions are given in Sec. 5.

## 2 Conventional model formulations

In order to understand how two different versions of Haus partial-differential equation model arise for different types of lasers and why both fall short of including all the relevant physics, it is helpful to review briefly their derivations from more fundamental principles (see also Appendix A of [21] for a more detailed treatment in the particular case of class-B lasers). Our starting point is a difference-differential model for passive mode locking due to Haus [6]:

$$a(T+r) - a(T) = \frac{1}{2} \left( d^2 \frac{d^2}{dT^2} + g - q - k \right) a(T) \quad (1a)$$

$$\frac{dg}{dT} = \gamma_g (g_{\max} - g) - s_g g a^2, \quad (1b)$$

$$\frac{dq}{dT} = \gamma_q (q_{\max} - q) - s_q q a^2, \quad (1c)$$

where  $T$  represents time,  $a$  denotes the instantaneous amplitude of the laser field, and  $g$  and  $q$  stand respectively for the gain and saturable losses per round-trip. The parameters  $g_{\max}$  and  $q_{\max}$  denote their unsaturated values. The time difference  $r$  represents the cold cavity round-trip time (that is, the round-trip time for a weak fluctuation of the field inside the cavity at transparency). The parameter  $d$  introduces an intrinsic limit of the narrowness of the pulse due to the finite bandwidth of the optical cavity. It has the dimension of a time, and provides an estimate of a lower bound for the pulse duration. It is typically significantly shorter than the round-trip time  $r$ .  $k$  represents the linear cavity losses, so that the net gain per round-trip is given by  $g - q - k$ . Finally,  $\gamma_g$  and  $\gamma_q$  are the relaxation rates of the gain and absorber media, and  $s_g$  and  $s_q$  are saturation coefficients. Note that although the difference-differential model (1) is free from the limitations of the partial-differential Haus models discussed below it also has an important drawback: the smoothness of its solution is reduced each round trip number. Physically this means that the high frequency perturbations of the solution grow with the round trips.

### 2.1 Field equation

Haus partial-differential master equation is easily derived as a limit of the difference-differential field equation (1a) for a large cavity bandwidth and a weak net gain. To this end, we introduce a formal smallness parameter  $\varepsilon$ , in terms of which we define the scales of the gain and absorber variables  $g$

and  $q$  and of the pulse duration measure  $d$  by performing the following substitutions in the model (1):

$$g \rightarrow \varepsilon^2 g, \quad g_{\max} \rightarrow \varepsilon^2 g_{\max}, \quad (2a)$$

$$q \rightarrow \varepsilon^2 q, \quad q_{\max} \rightarrow \varepsilon^2 q_{\max}, \quad (2b)$$

$$k \rightarrow \varepsilon^2 k, \quad (2c)$$

$$d \rightarrow \varepsilon d. \quad (2d)$$

Note that  $\varepsilon$  has no physical meaning; it is used simply as an order parameter in the forthcoming limit calculations and will disappear from the resulting final equations. We further introduce a two-scale expansion for the time variable:

$$\frac{d}{dT} = \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}, \quad (3)$$

in terms of which the delayed variable  $a(T + r)$  in the field equation (1a) can be expressed as

$$a(T + r) = a(t + r, \tau + \varepsilon^2 r) \simeq a(t + r, \tau) + \varepsilon^2 r \frac{\partial a}{\partial \tau}(t + r, \tau). \quad (4)$$

Substituting the scaling (2) and then the expansions (3) and (4) into the difference-differential equation (1a) and keeping corrections only to the order of  $\varepsilon^2$  gives

$$a(t + r) - a(t) + \varepsilon^2 r \frac{\partial a}{\partial \tau}(t + r) = \frac{1}{2} \varepsilon^2 \left( d^2 \frac{\partial^2}{\partial t^2} + g - q - k \right) a(t). \quad (5)$$

Finally, equating separately the coefficients of like powers of  $\varepsilon$  on either side of Eq. (5) gives the following two equations:

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + g - q - k \right) a, \quad a(t + r) = a(t). \quad (6)$$

The first one is the Haus partial-differential master equation, and the second one provides a periodic boundary condition for it.

Note how Eq. (6) involves the two different time scales  $t$  and  $\tau$  as independent variables. The validity of this two-dimensional representation of time relies on the property of quasi-continuous pulse evolution between successive round-trips. This means that the temporal profile of the pulse inside the cavity varies little from one round-trip to the next, as a consequence of the weak net gain assumption. The difference between the amplitudes of two successive emitted copies of the pulse in the left-hand side of Eq. (1a) then appears in Eq. (6) approximated as a continuous derivative, where  $\tau$  thus represents the slow time variable in terms of which the pulse evolution and shaping processes are described (or, more generally, any process that takes place over several round-trips). In contrast, the role of the fast time variable  $t$  is to express the instantaneous configuration of the field in the cavity at a particular stage of its evolution, as well as any other process that occurs on the time scale of the round-trip or faster. The periodic boundary condition in Eq. (6) reflects the approximate periodicity of the pulse train over a few round-trips. Eq. (6) is the simplest possible formulation of Haus master equation. Extensions exist that account for the complex nature of the field amplitude  $a$  (to include phase dynamics) and other physical effects (such as group velocity dispersion, Kerr effect, or linewidth enhancement factors) [15]. However the specific problem addressed in this paper does not require such extensions (nor does it preclude their use).

## 2.2 Absorber equation

The gain  $g$  and absorber  $q$  must be described by their own evolution equations in order to provide a closed dynamical system together with Haus master equation (6). The absorber equation is obtained trivially by substituting Eqs. (2) and (3) into Eq. (1c) and then retaining only the leading-order terms in  $\varepsilon$ , which gives an equation that is formally identical to Eq. (1c), but with  $T$  replaced with  $t$ :

$$\frac{\partial q}{\partial t} = \gamma_q (q_{\max} - q) - s_q q a^2. \quad (7)$$

Because Eq. (7) is linear in  $q$ , its most general solution is the sum of a periodic contribution with the same period as  $a^2$  and an exponentially decaying term that lasts only for a few round-trips. With the restriction to solutions in which the transient contribution has already died out, we can thus impose on Eq. (7) the same periodic boundary condition as on Eq. (6) for their compatibility as a coupled system:

$$q(t + r) = q(t). \quad (8)$$

Two cases are usually distinguished about the time scales involved in Eq. (7). The first is that of a fast absorber, which refers to a situation where the absorber relaxes on a time scale much shorter than the pulse duration:  $\gamma_q^{-1} \ll d$ . This time scale relationship allows the adiabatic elimination of the absorber variable  $q$  as an explicit function of the field intensity  $a^2$ , which is achieved by setting the left-hand side of the absorber rate equation (7) to zero and solving it for  $q$ . This yields

$$q = \frac{q_{\max}}{1 + \gamma_q^{-1} s_q a^2}. \quad (9)$$

The other distinguished case is that of a slow absorber, when the absorber relaxes on a time scale much longer than the pulse duration, but comparable to the round-trip time or shorter:  $d \ll \gamma_q^{-1} \lesssim r$ . This situation often justifies the neglecting of the relaxation term  $\gamma_q (q_{\max} - q)$  in Eq. (7) during the absorber depletion stage, so that the explicit solution for  $q$  is now a function of the cumulated field energy up to time  $t$ :

$$q = q_1 \exp\left(-s_q \int_{t_1}^t dt a^2\right), \quad (10)$$

where  $q_1$  represents the absorber state just before the pulse arrival, and  $t_1$  is the corresponding instant in time. Eq. (10) holds for the duration of a pulse; after that, the neglected relaxation process takes over as the laser field vanishes. Note that for both fast and slow absorbers, the relaxation process is assumed to occur on a time scale not longer than the round-trip time, and is consistently described in Eq. (7) in terms of the fast time variable  $t$  rather than the slow time variable  $\tau$ .

## 2.3 Gain equation

For the gain medium, a similar dichotomic classification based on the relaxation rate  $\gamma_g$  exists, but the reference time scale is different. A laser for which the gain relaxation takes place on a time scale comparable to the round-trip time or slower ( $\gamma_g^{-1} \lesssim r$ ) is called a class-A laser. In contrast, a laser whose gain medium relaxes over many round-trips ( $\gamma_g^{-1} \gg r$ ) is called a class-B laser. This classification will be used to determine which time variable ( $t$  or  $\tau$ ) is involved in the description of the gain relaxation process. Unlike the absorber rate equation, we shall see that there is no single formulation of the rate equation for the gain that will handle both cases, so one must choose from the outset which kind of laser is involved.



### 2.3.1 Gain equation for a class-A laser

In a class-A laser, the gain response time is not especially long compared to the round-trip time. The derivation of the gain rate equation thus does not require any particular assumption on the relaxation and saturation rates  $\gamma_g$  and  $s_g$ , and is entirely parallel to that of the absorber rate equation (7) and of its boundary condition (8). We obtain:

$$\frac{\partial g}{\partial t} = \gamma_g (g_{\max} - g) - s_g g a^2 \quad g(t+r) = g(t). \quad (11)$$

Together, Eqs. (6), (7), (8) and (11) form a closed system for the field, absorber, and gain medium in a class-A laser.

The validity of Eq. (11) does not extend to class-B lasers, however. A simple argument for this is that a correct class-B laser model should reduce, in the absence of fast mode-locking dynamics, to the classical pair of rate equations that describes a single-mode emission [17, 18], namely:

$$r \frac{da}{d\tau} = \frac{1}{2} (g - q - k) a, \quad (12a)$$

$$\frac{dg}{d\tau} = \gamma_g (g_{\max} - g) - s_g g a^2. \quad (12b)$$

But while dropping the dependence in the fast time variable  $t$  in Eq. (6) gives the correct rate equation (12a) for the field, dropping it in Eq. (11) gives a simple algebraic equation from which the gain can be solved as a function of the field intensity:

$$g = \frac{g_{\max}}{1 + \gamma_g^{-1} s_g a^2}, \quad (13)$$

instead of the expected rate equation (12b). In fact, so far as  $t$ -independent solutions are considered, the explicit expression (9) for  $q$  holds no matter whether the absorber is fast or slow, so Eqs. (9) and (13) can be both substituted into the field rate equation (12a) to give a single closed rate equation for the field amplitude  $a$ :

$$r \frac{da}{d\tau} = \frac{1}{2} \left( \frac{g_{\max}}{1 + \gamma_g^{-1} s_g a^2} - \frac{q_{\max}}{1 + \gamma_q^{-1} s_q a^2} - k \right) a. \quad (14)$$

It is a known fact from elementary dynamical system theory that oscillations cannot arise from a single ordinary differential equation such as Eq. (14). This provides evidence (in the particular case of single-mode emission) that Q-switching cannot arise from the class-A formulation (11) of the gain rate equation.

### 2.3.2 Gain equation for a class-B laser

A different equation for the gain is therefore required for a proper description of Q-switching in a class-B laser. In order to account for the slowness of the gain relaxation and saturation processes, we must supplement the scaling (2) with the following substitution relations:

$$\gamma_g \rightarrow \varepsilon^2 \gamma_g, \quad s_g \rightarrow \varepsilon^2 s_g. \quad (15)$$

Substituting Eqs. (2) and (15) into Eq. (1b) then gives

$$\frac{dg}{dT} = \varepsilon^2 [\gamma_g (g_{\max} - g) - s_g g a^2]. \quad (16)$$

The structure of Eq. (16) justifies the application of an averaging method [30]. This consists in taking the gain variable  $g$  as independent of the fast time variable  $t$ , expressing the time derivative in the left-hand side in terms of the slow time variable  $\tau$  (using the relation  $\tau = \varepsilon^2 T$ ), and averaging the right-hand side over one period in  $t$ . We thus obtain:

$$\frac{dg}{d\tau} = \gamma_g (g_{\max} - g) - s_g g r^{-1} \int_0^r dt a^2. \quad (17)$$

The form (17) of the gain equation was used successfully in [21] to predict Q-switching. For particular solutions independent of the fast time  $t$ , Eq. (17) reduces to the correct single-mode rate equation (12b), so Eq. (17) passes the simple validity test that the class-A gain equation (11) did not pass. However, Eq. (17) involves only the mean intensity over one round-trip, so according to it, the gain cannot respond to fast field variations. Although consistent with the assumption that the gain medium is much slower, this introduces a serious new limitation in the model. Indeed, consider a mode-locked class-B laser with a slow absorber. In view of Eq. (10), the net gain  $g - q - k$  during the passing of a pulse is given by

$$g - q - k = g - q_1 \exp\left(-s_q \int_{t_1}^t dt a^2\right) - k, \quad (18)$$

where  $g$  and  $q_1$  are independent of  $t$ . This expression is a monotonously increasing function of  $t$ , in agreement with Fig. 2b, and consistently with the fact that the only dynamical process taken into account by Eq. (18) is the absorber saturation, which only contributes to a gradual increase of the net gain. As argued in the introduction, the monotonous net gain evolution is not confirmed (at least for common operating conditions) by numerical simulations of the delay differential equation (DDE) mode-locked laser model, which suggest to the contrary that some fast dynamics of the gain medium does play a significant role in shaping the net gain profile, even in class-B lasers [11].

## 2.4 Origin of the model shortcomings

We have shown on mathematical grounds that both the Q-switching-enabled gain model (17) and the class-A gain model (11) suffer shortcomings when applied to class B lasers: while Eq. (11) is unable to predict Q-switching, Eq. (17) is unable to describe the fast response of the gain medium to the passing of a pulse. The two models in fact miss part of the physics for opposite reasons: in Eq. (17), the gain medium is not fast enough to follow the fast intensity variations, whereas in Eq. (11) it is not slow enough to endow the system with the necessary inertia to develop slow oscillations

Since both Eqs. (11) and (17) are obtained as limits of the more general Eq. (1b), the key to obtaining a unified model capable of describing both phenomena is to drop some of the scaling assumptions (2), (15). To identify which ones can be retained and which are to be relaxed, we note that part of the success of the more complex model studied in [11] stems from its extended validity into the regime of strong amplification that typically holds in semiconductor lasers. This observation suggests reconsidering the appropriateness of the weak-gain assumption (2a). A strong enough pumping of the gain medium may indeed be required to compensate for its slow responsiveness to intensity variations and create a modulation the gain profile of sufficient depth to induce a non-negligible contribution to pulse shaping. Moreover, both the pumping rate and the lasing threshold usually influence the range of variation of the gain, which hints at the need to drop the weak cavity loss assumption (2c) too. In the next section, we show that a multiple-scale analysis based on a set of assumptions weakened along those lines does indeed yield a model for the gain dynamics with the desired properties.

### 3 Improved model formulation

Before proceeding, we note that the derivation of difference-differential model (1) from fundamental principles already incorporates an assumption of weak gain and losses [6] (as is manifest from their linearity in  $g$ ,  $q$ , and  $k$ ). One may therefore legitimately question its appropriateness as a starting point for an analysis that is intended to retain validity for a larger range of gain than the classical theories presented in Sec. 2. To settle this point, we consider also the DDE model used in [11], which holds for arbitrary gain and losses:

$$\left(1 + d \frac{d}{dT}\right) a(T + R) = K^{\frac{1}{2}} \exp\left[\frac{1}{2}(g - q)\right] a(T), \quad (19a)$$

$$\frac{dg}{dT} = \gamma_g (g_{\max} - g) - s_g [\exp(g) - 1] \exp(-q) a^2, \quad (19b)$$

$$\frac{dq}{dT} = \gamma_q (q_{\max} - q) - s_q [1 - \exp(-q)] a^2, \quad (19c)$$

where  $A(T)$  is the electric field envelope at the entrance of the absorber medium,  $G(T)$  and  $Q(T)$  are gain and loss introduced by the amplifying and absorber sections, respectively, and  $T$  is time.  $R$  is the cold cavity round trip time,  $D$  is the inverse spectral filtering width, and  $K$  is the attenuation factor per cavity round trip. The parameters  $g_{\max}$  and  $q_{\max}$  describe the unsaturated gain and absorption, while  $s_g$  and  $s_q$  are the saturation factors of the corresponding sections. For simplicity we have omitted in Eqs. (19) the linewidth enhancement factors introduced in [11] to describe semiconductor lasers and assumed that  $A$  is real. However, all the calculations below can be trivially generalized to the complex case when the linewidth enhancement factors are present.

In view of Eq. (19a), the quantity  $\ln(K)$  represents the gain above cavity threshold. Let us introduce new variable  $n$ , such as

$$n \left(1 - \frac{n}{4}\right) = g + \ln(K) \quad (20)$$

Close to the threshold we can consider the following scaling

$$n \rightarrow \varepsilon n, \quad (21a)$$

$$q \rightarrow \varepsilon^2 q, \quad q_{\max} \rightarrow \varepsilon^2 q_{\max}, \quad (21b)$$

$$d \rightarrow \varepsilon d, \quad (21c)$$

$$\gamma_g \rightarrow \varepsilon^2 \gamma_g, \quad s_g \rightarrow \varepsilon^2 s_g \quad (21d)$$

with small  $\varepsilon$ . Eq. (21a) can be viewed as a weakened form of the low-gain assumption (2a), as it allows the gain variations to cover a larger range (on the order of  $\varepsilon$  instead of  $\varepsilon^2$ ). Also, we allow arbitrarily large values of the pumping term  $g_{\max}$  and of the linear losses  $k$ . In all other respects, the scaling (21) is identical to Eqs. (2) and (15) together. Substituting the expression for  $g$  obtained from Eq. (20) together with the relation  $R = r - \varepsilon D$  and then Eqs. (21) into Eqs. (19) and keeping only the highest-order terms in  $\varepsilon$  gives:

$$a(T + r) - a(T) = \frac{1}{2} \left( \varepsilon^2 d^2 \frac{d^2}{dT^2} + \varepsilon n - \varepsilon^2 q \right) a(T) + O(\varepsilon^3), \quad (22a)$$

$$\frac{dn}{dT} = \varepsilon [P_g - \varepsilon \bar{\gamma}_g n - (S_g + \varepsilon \bar{s}_g n) a^2] + O(\varepsilon^3), \quad (22b)$$

$$\frac{dq}{dT} = \gamma_q (q_{\max} - q) - s_q q a^2 + O(\varepsilon^2), \quad (22c)$$

In order to understand how two different versions of Haus partial-differential equation model arise for different types of lasers and why both fall short of including all the relevant physics, it is helpful to review briefly their derivations from more fundamental principles (see where

$$P_g = \gamma_g [g_{\max} + \ln(K)], \quad \bar{\gamma}_g = \gamma_g - \frac{1}{2}P_g, \quad S_g = (K^{-1} - 1) s_g, \quad \bar{s}_g = \frac{1}{2}(3K^{-1} - 1)s_g. \quad (23)$$

Similarly to the DDE model (19) in view of the field equation (1a), the quantity

$$n = g - k \quad (24)$$

represents the gain above cavity threshold. Following our considerations given above, we seek an asymptotic limit of the difference-differential equations (1) with the scaling (21) for  $\varepsilon$  small. One can easily check that this leads a system equivalent to Eqs. (22) up to  $O(\varepsilon^2)$  corrections. This level of accuracy is sufficient to justify all calculations in this paper, which establishes the equivalence of the models (1) and (19) in the limit (21).

The equations (22) have been used to derive a generalized class B laser version of the Haus master equations. Note that since all quantities in Eq. (21) involved in the absorber equation (1c) is identical to Eqs. (2) and (15), the asymptotic absorber equation (7) and its boundary condition (8) thus remain valid in the limit considered here, so Eq. (1c) does not require any further analysis. From now on, we focus all our efforts on dealing with the remaining field equation (22a) and gain equation (22b). Our method now proceeds in two steps. The first one is a strict (though not straightforward) application of multiple-scale analysis. Because the calculations are too lengthy to be presented in the body of this section, they are relegated to Appendix A. The result is a system of equations equivalent to Eqs. (22) for small  $\varepsilon$ , which are listed as Eqs. (B.2)–(B.7) and commented on in the self-contained Appendix B.

The asymptotic equations (B.2)–(B.7) can be useful in their own right as they would provide a good starting point for an all-analytical bifurcation study of mode-locked class-B lasers (though such an endeavour falls outside of the scope of the present paper and will be left for future work). They are not the final result of the present analysis, however. Keeping in mind that our goal is to find a generalized gain model to be coupled to Haus master equation (6), we note that Eqs. (B.2)–(B.7) present the drawback of not involving the physical field and gain variables directly. The relations between those and the asymptotic dynamical variables are in fact rather complicated. As a second step in the analysis, therefore, we perform various transformations (given in Appendix C) to recast Eqs. (B.2)–(B.7) into a much more physically transparent form. This procedure is not a strict application of asymptotic analysis as it involves inhomogeneous transformations (i.e., the summing of quantities proportional to distinct powers of  $\varepsilon$ ). Nevertheless, it satisfies our goal by yielding Haus master equation coupled to a new model for the gain dynamics. Namely, we obtain

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + n - q \right) a, \quad a(t+r) = a(t), \quad (25)$$

which are just Eq. (6) expressed in terms of the gain above threshold  $n$  together with Eqs. (7),(8), and the system

$$\frac{\partial n}{\partial t} = S_g \left( r^{-1} \int_0^r a^2 dt - a^2 \right), \quad (26a)$$

$$\frac{d\bar{n}}{d\tau} = P_g - \bar{\gamma}_g \bar{n} - (S_g + \bar{s}_g \bar{n}) r^{-1} \int_0^r a^2 dt, \quad (26b)$$

where  $\bar{n} = r^{-1} \int_0^r n dt$  represents the average gain over one round-trip and the parameters  $P_g$ ,  $\bar{\gamma}_g$ ,  $S_g$ , and  $\bar{s}_g$  are defined by Es. (23).

The rate equation (26b) is similar to the conventional gain rate equation (17) for a class-B laser presented in Sec. 2, in that it involves the field intensity averaged over one round-trip time, and is not sensitive to the details of the mode-locked emission pattern. Those two equations would in fact be completely equivalent, were it not for the presence of a different saturation coefficient  $\bar{s}_g$  in Eq. (26b) and for the fact that Eq. (26b) describes only the mean value of the gain over one round-trip. Based on the knowledge of the evolution of this mean value, Eq. (26a) determines the full depletion-recovery cycle of the gain. The formulation (25) of Haus master equation does not involve the  $t$ -independent solution  $\bar{n}$  of Eq. (26b) directly, but the  $t$ -dependent solution  $n$  of Eq. (26a) that averages to  $\bar{n}$ . In that sense, Eqs. (26) extend Eq. (17) by accounting for the fast gain dynamics on a time scale comparable to the duration of a mode-locked pulse while retaining on average the slow dynamics of Eq. (17) responsible for Q-switching.

The correction (23) to the saturation coefficient finds its justification in the multiple-scale expansion of the delayed term  $a(T - r)$  in Eq. (22a). According to the calculations of Appendix A, limiting the expansion to the first derivative as in Eq. (4) is not valid anymore in the limit (21). The second derivative does play a role in the analysis, and leads to a contribution to the net gain that is found to be equivalent to an effective decrease of the gain saturability.

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + g - q - k \right) a, \quad a(t+r) = a(t), \quad (27)$$

$$\frac{\partial q}{\partial t} = q_0 - \gamma_q q - \bar{s}_g q a^2, \quad q(t+r) = q(t), \quad (28)$$

$$\frac{\partial g}{\partial t} = r^{-1} \int_0^r a^2 dt - a^2, \quad (29)$$

$$\frac{d\bar{g}}{d\tau} = g_0 - \bar{\gamma}_g \bar{g} - k^{-1} \bar{g} r^{-1} \int_0^r a^2 dt, \quad (30)$$

where  $\bar{g} = r^{-1} \int_0^r g dt$ ,  $k = 2 \frac{1-K}{3-K}$ ,  $g_0 = P_g + k\gamma_g$ ,  $q_0 = \gamma_q q_{\max}$ , and  $\bar{s}_g = s_q/S_g$ . Note finally that by introducing new variable  $\tilde{g} = g - \bar{g}$  we can rewrite Eq. (27) in the form

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + \tilde{g} + \bar{g} - q - k \right) a, \quad a(t+r) = a(t), \quad (31)$$

while the equations (28)-(30) remain unchanged except for the replacement  $g \rightarrow \tilde{g}$ :

$$\frac{\partial \tilde{g}}{\partial t} = r^{-1} \int_0^r a^2 dt - a^2. \quad (32)$$

then one can easily see that the extended Haus model becomes equivalent to the conventional class-B haus model (6), (7), and (17) if we set  $\tilde{g} = 0$  in (31) and abandon Eq. (32).

## 4 Numerical results

An algorithm for solving the formulation of Haus model that incorporates the gain equations (29) and (30) should proceed as follows. An initial condition for this problem is the profile  $a(t, \tau_0)$  of the field

in the cavity at  $\tau = \tau_0$ , which is periodic in  $t$  with period  $r$ , together with the single value  $\bar{g}(\tau_0)$  of the mean gain at  $\tau = \tau_0$ . The absorber depletion-recovery profile  $q(t, \tau_0)$  is then determined from Eq. (28) with the periodic boundary condition (8), and the gain depletion-recovery profile  $n(t, \tau_0)$  is computed from Eq. (26a). Since any solution of Eq. (29) automatically has the same period as  $a^2$ , the periodicity of  $g(t, \tau_0)$  does not have to be imposed explicitly. However, Eq. (29) determines its solutions only up to an arbitrary additive constant, which is to be fixed by the integral condition  $\bar{g} = r^{-1} \int_0^r g dt$ . The knowledge of  $q(t, \tau_0)$  and  $g(t, \tau_0)$  then provides enough data to compute the field profile  $a(t, \tau_0 + dt)$  one time step  $dt$  later, using Haus master equation (27) together with its periodic boundary condition. Likewise, the mean gain  $\bar{g}(\tau_0 + dt)$  one step later is computed from Eq. (30). Starting data is then available for the next integration step.

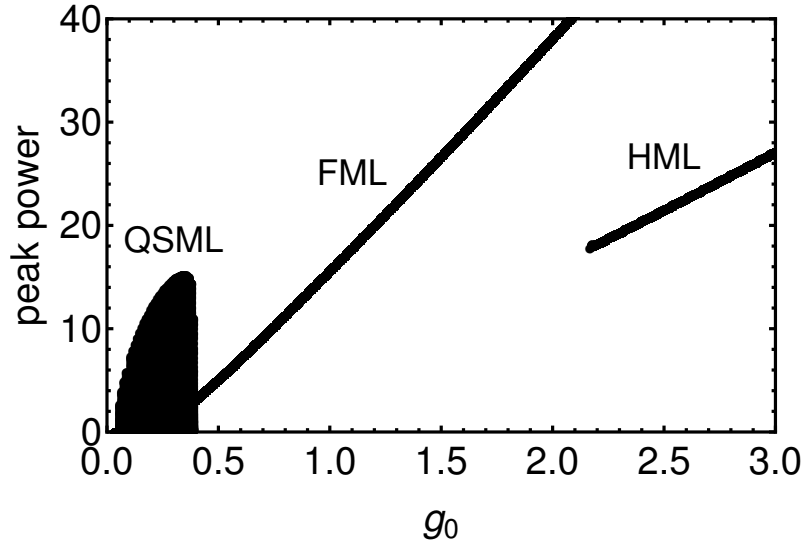


Figure 3: Pulse peak power  $a^2$  as a function of the pump parameter  $g_0$  obtained using the generalized Haus model, Eqs. (27)-(30). QSML, FML, and HML denote Q-switched, fundamental, and harmonic (with two pulses per cavity round trip) mode-locking regimes, respectively. Parameter values are:  $r = 2.5$ ,  $k = 0.519$ ,  $q_0 = 1.0$ ,  $\gamma_g = 7.5 \cdot 10^{-3}$ ,  $\gamma_q = 0.2$ ,  $s_q = 7.0$ ,  $d = 0.02$ .

We have solved the generalized Haus model (27)-(30) numerically using the split-step method with 1024 Fourier modes. The resulting bifurcation diagram presenting the evolution of pulse peak power  $a^2$  with the increase of the pump parameter  $g_0$  is shown in Fig. 3. It is seen that apart from the fundamental mode-locked (FML) regime with a single pulse per cavity round trip, this model can demonstrate harmonic mode-locking (HML) regime with two pulses per cavity round trip time, as well as Q-switched mode-locking (QSML) regime with periodically oscillating pulse peak power corresponding to a cloud of points in Fig. 3. The slow time evolution of the pulse peak power of the QSML regime is shown in Fig. 4(a) together with the fast time evolution of the intensities of the FML and HML regimes, see Figs. 4(b) and 4(c), respectively.

Bifurcation diagram similar to that shown in Fig. 3, but calculated using the conventional class-B Haus model, which neglects the fast gain variation on the pulse width timescale is shown in Fig. 5. This model can be obtained by by setting  $\tilde{g} = 0$  in Eq. (31) and abandoning Eq. (32). It is seen that although the conventional model describes the Q-switched and fundamental mode-locking regimes rather well, it fails to describe the appearance of harmonic mode-locking regime, which emerges with the increase of the pump parameter  $g_0$ . Furthermore, the conventional model predicts slightly slower growth of the pulse peak power with  $g_0$  and broader mode-locked pulses than the generalized Haus model (27)-(30).

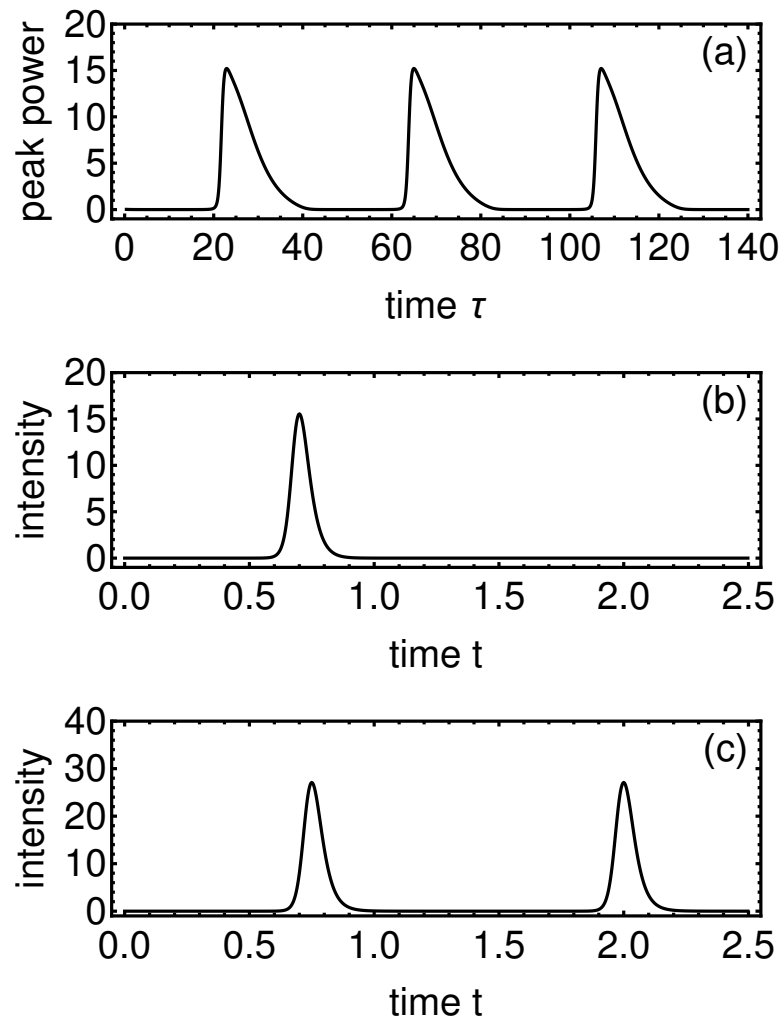


Figure 4: Time traces obtained by numerical integration of the generalized Haus model, Eqs. (27)-(30). (a) – Q-switched mode-locking, pulse peak power as a function of slow time  $\tau$ . (b) – Fundamental mode-locked regime, intensity  $a^2$  as a function of the fast time  $t$ . (c) – Harmonic mode-locking regime with two pulses per cavity round trip, intensity  $a^2$  as a function of the fast time  $t$ . Other parameters are the same as in Fig. 3

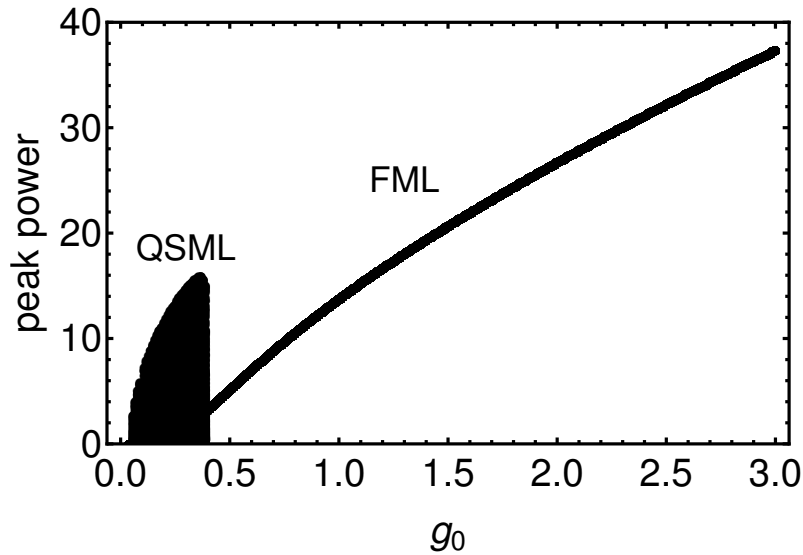


Figure 5: Bifurcation diagram similar to that shown in Fig. 3, but obtained with conventional class-B Haus model. QSML and FML denote Q-switched and fundamental mode-locking regimes, respectively. Parameter values are the same as in Fig. 3.

Time dependence of the net gain parameter  $\eta = g - q - k$  (see Eq. (27)) and field intensity  $a^2$  on the fast time  $t$  is shown in Figs. 6 and 7 for the generalized and conventional class-B Haus models, respectively. It is seen from Fig. 6(a) that in the generalized class-B Haus model with the pump parameter  $g_0 = 1.0$  the net gain window corresponds to a short time interval when the pulse intensity is large. For larger pumps [see Fig. 6(b) corresponding to  $g_0 = 3.0$ ], however, positive net gain appears before the pulse triggering the so-called *leading-edge instability* [11], which eventually gives rise to a harmonic mode-locking regime. Unlike the generalized model, the conventional class-B Haus model does not demonstrate the development of leading edge instability with the increase of the pump parameter, see Figs. 7(a) and (b). Furthermore, since the fast gain saturation is absent in the conventional model, the net gain window is limited by the absorber saturation only. This is why the pulse widths obtained with the conventional model (see Figs. 7(a) and (b)) are broader than those shown in Fig. 6.

## 5 Conclusions

Although, unlike the DDE mode-locking model, the Haus master equations are based on low gain and loss approximation, which limits the parameter range of their validity, they are widely used and serve as an efficient tool for the analysis of mode-locked devices, such as e.g. fiber and solid state lasers. This is not only due to the availability of well developed tools for analytical and numerical analysis of nonlinear PDEs, but also because of the possibility of straightforward inclusion of the group velocity dispersion into the master equations. On the contrary, the inclusion of the chromatic dispersion into the DDE mode-locking models is not that straightforward, see [31, 32]. Another limitation of the partial differential master equations, is that unlike difference-differential Haus model (1), the development of adequate PDE models of mode-locked class B lasers require a careful formulation of the equations describing gain dynamics on different time scales. In particular, while the class A version of the Haus master equations (6), (7), and (11) is not applicable to describe mode-locking in class B lasers and fails to describe Q-switched instability of the mode-locked regime, the class B model (6), (7), and (17),



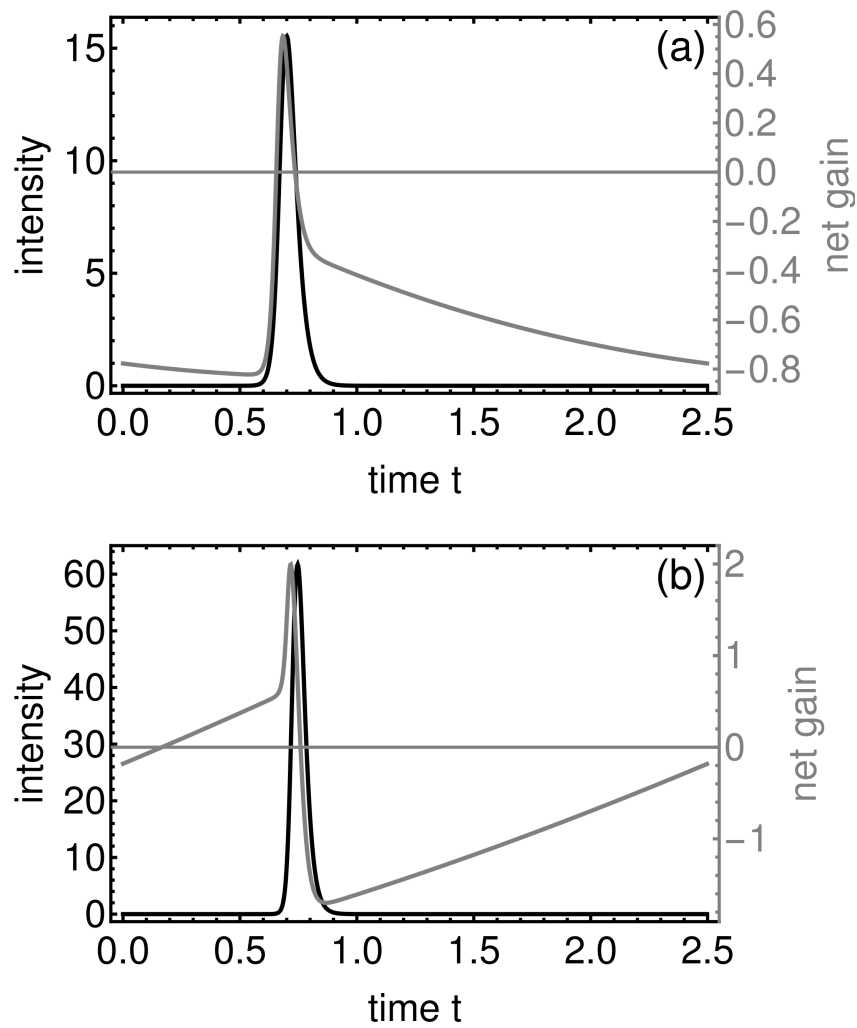


Figure 6: Pulse amplitude  $a^2$  (black line) and net gain  $\eta$  (gray line) obtained with the generalized Haus model (27)-(30) as functions of the fast time  $t$ . (a)  $g_0 = 1.0$ ; (b)  $g_0 = 3.0$ . Other parameters are the same as in Fig. 3.

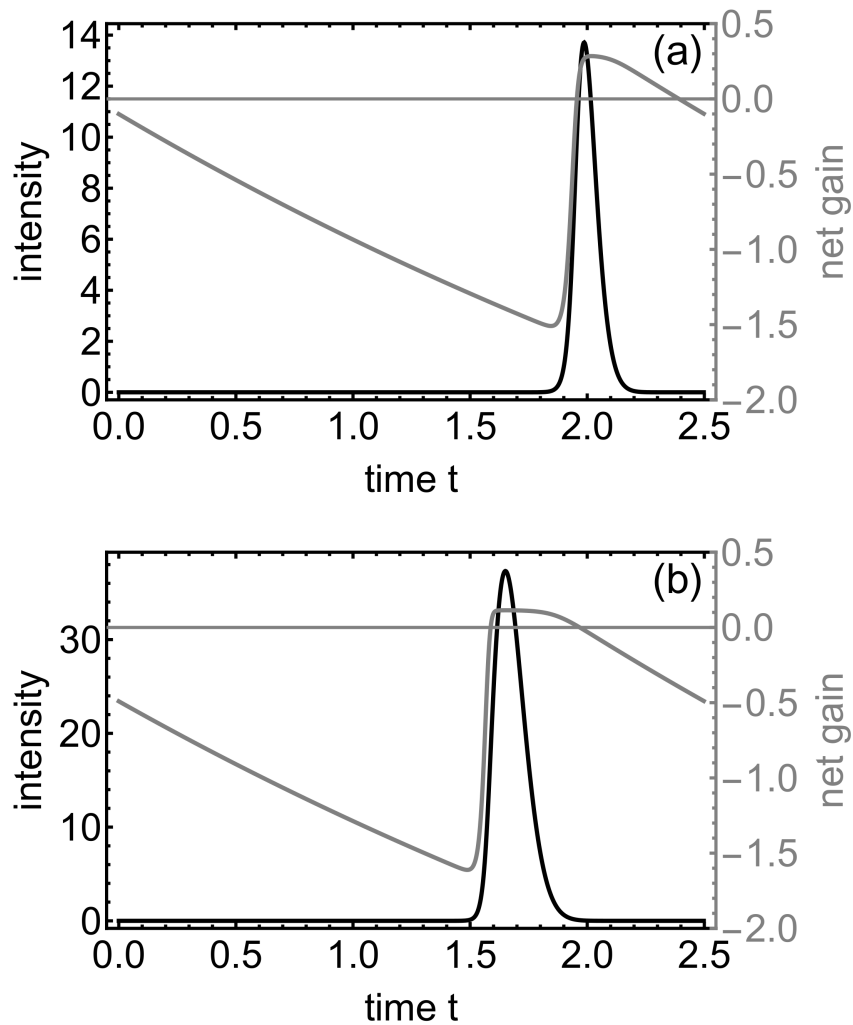


Figure 7: The same as Fig. 6 but obtained with the conventional class-B Haus model. (a) –  $g_0 = 1.0$ ; (b) –  $g_0 = 3.0$ . Other parameters are the same as in Fig. 3.

which accounts for the slow time scale evolution of the gain, is capable of describing Q-switched mode-locking regime, but fails to predict a transition to harmonic mode-locking regimes with the increase of the pumping parameter. On the other hand, the solution of the difference-differential Haus model (1), which is free from these limitations, loses smoothness with increasing round trip number and hence exhibits an instability at large frequencies. Here, using asymptotic expansions, we have derived a generalized version of the Haus PDE model including the equations for the slow and fast scale gain evolution, instead of the time dependent boundary condition proposed in [28]. Our numerical simulations indicate that our generalized model allows to describe both the Q-switched mode-locking and the development of harmonic mode-locking regimes.

## Appendix A Multiple-scale analysis of Eqs. (22)

We first introduce a multiple-scale expansion for time. By analogy with Eq. (3), we name the fastest time scale  $t$  and the slower ones  $\tau_1$  and  $\tau_2$ :

$$\frac{d}{dT} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + O(\varepsilon^3). \quad (\text{A.1})$$

The delayed variable  $a(T+r)$  in the field equation (22a) can be expressed in terms of the various time scales in the expansion (A.1) as follows:

$$\begin{aligned} a(T+r) &= a(t+r, \tau_1 + \varepsilon r, \tau_2 + \varepsilon^2 r) \\ &= \left[ 1 + \varepsilon r \frac{\partial}{\partial \tau_1} + \varepsilon^2 \left( \frac{r^2}{2} \frac{\partial^2}{\partial \tau_1^2} + r \frac{\partial}{\partial \tau_2} \right) \right] a(t+r, \tau_1, \tau_2) + O(\varepsilon^3). \end{aligned} \quad (\text{A.2})$$

We further expand:

$$a = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + O(\varepsilon^3), \quad n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + O(\varepsilon^3). \quad (\text{A.3})$$

Substituting Eqs. (A.1) and (A.2) and then Eq. (A.3) into Eqs. (22) and equating the coefficients of like powers of  $\varepsilon$  separately leads to a hierarchy of linear problems. A study of their solvability conditions will provide a set of equations equivalent to Eqs. (22) in the limit of small  $\varepsilon$ .

### A.1 $O(\varepsilon^0)$ problem

The  $O(\varepsilon^0)$  problem is:

$$a_0(t+r) - a_0(t) = 0, \quad (\text{A.4a})$$

$$\frac{\partial n_0}{\partial t} = 0, \quad (\text{A.4b})$$

and gives a periodic boundary condition for the field amplitude  $a_0$  and the information that the leading-order gain component  $n_0$  does not vary on the fastest time scale  $t$ .

## A.2 $O(\varepsilon^1)$ problem

The  $O(\varepsilon^1)$  problem is:

$$a_1(t+r) - a_1(t) = \left( -r \frac{\partial}{\partial \tau_1} + \frac{1}{2} n_0 \right) a_0, \quad (\text{A.5a})$$

$$\frac{\partial n_1}{\partial t} = -\frac{\partial n_0}{\partial \tau_1} + P_g - S_g a_0^2. \quad (\text{A.5b})$$

The validity of the expansions (A.3) for all times requires that  $a_1$  and  $n_1$  be bounded functions of  $t$ . This imposes the vanishing of the right-hand side of Eq. (A.5a) and the vanishing of the average of the right-hand side of Eq. (A.5b) over all  $t$ , which leads to the following solvability conditions:

$$r \frac{\partial a_0}{\partial \tau_1} = \frac{1}{2} n_0 a_0, \quad (\text{A.6a})$$

$$\frac{\partial n_0}{\partial \tau_1} = P_g - S_g \tilde{a}^2, \quad (\text{A.6b})$$

where  $\tilde{a}$  is the quadratic average field amplitude over one round-trip:

$$\tilde{a}(\tau_1, \tau_2) = \sqrt{r^{-1} \int_0^r dt a_0^2(t, \tau_1, \tau_2)}. \quad (\text{A.7})$$

The quantity  $\hat{a}$  defined by

$$a_0 = \tilde{a} \hat{a} \quad (\text{A.8})$$

thus represents the emission pattern normalized so that

$$r^{-1} \int_0^r dt \hat{a}^2 = 1. \quad (\text{A.9})$$

It will be advantageous to express the field variable  $a_0$  everywhere in terms of the decomposition (A.8), as we will find  $\tilde{a}$  and  $\hat{a}$  to be governed separately by their own evolution equations. First, the  $O(\varepsilon^0)$  equation (A.4a) for the field translates to a periodic boundary condition for  $\hat{a}$ :

$$\hat{a}(t+r) = \hat{a}(t). \quad (\text{A.10})$$

Next, substituting Eq. (A.8) into the solvability condition (A.6a) for the field gives

$$r \left( \frac{\partial \tilde{a}}{\partial \tau_1} \hat{a} + \tilde{a} \frac{\partial \hat{a}}{\partial \tau_1} \right) = \frac{1}{2} n_0 \tilde{a} \hat{a}. \quad (\text{A.11})$$

Multiplying both sides of Eq. (A.11) by  $\hat{a}$ , integrating over  $t$ , and using the normalization condition (A.9) further yields

$$r \frac{\partial \tilde{a}}{\partial \tau_1} = \frac{1}{2} n_0 \tilde{a}, \quad (\text{A.12})$$

which, together with the solvability condition (A.6b) for the gain, defines a closed system for the average field amplitude  $\tilde{a}$  and leading-order gain  $n_0$ . This system is a conservative oscillator that describes Q-switching and admits the first integral

$$H(\tau_2) = (2r)^{-1} n_0^2 + S_g \tilde{a}^2 - P_g \ln(P_g^{-1} S_g \tilde{a}^2). \quad (\text{A.13})$$

The slow evolution of the Q-switching energy  $H$  on the slowest time scale  $\tau_2$  is as yet undetermined.

Substituting Eq. (A.12) for  $\tilde{a}$  back into Eq. (A.11) gives

$$\frac{\partial \hat{a}}{\partial \tau_1} = 0, \quad (\text{A.14})$$

which means that the normalized emission pattern  $\hat{a}$  does not vary on the time scale  $\tau_1$  of Q-switching. Finally, substituting the solvability conditions (A.6) and the decomposition (A.8) back into the  $O(\varepsilon^1)$  problem (A.5), we obtain

$$a_1(t+r) - a_1(t) = 0, \quad (\text{A.15a})$$

$$\frac{\partial n_1}{\partial t} = S_g (1 - \hat{a}^2) \tilde{a}^2. \quad (\text{A.15b})$$

Eq. (A.15a) is a periodic boundary condition for  $a_1$ . Eq. (A.15b) is an evolution equation for the small gain correction on the fast time scale  $t$ . Because its right-hand side averages to zero over one round-trip, its solutions  $n_1$  are periodic with period  $r$ .

Finally, by multiplying both sides of Eq. (A.15b) by  $n_1$  and integrating over one period in  $t$ , we find an identity that will be useful later on:

$$\int_0^r dt n_1 = \int_0^r dt \hat{a}^2 n_1. \quad (\text{A.16})$$

### A.3 $O(\varepsilon^2)$ problem

The  $O(\varepsilon^2)$  problem is:

$$a_2(t+r) - a_2(t) = \left( -r \frac{\partial}{\partial \tau_1} + \frac{1}{2} n_0 \right) a_1 + \left[ -\frac{r^2}{2} \frac{\partial^2}{\partial \tau_1^2} - r \frac{\partial}{\partial \tau_2} + \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + n_1 - q \right) \right] a_0, \quad (\text{A.17a})$$

$$\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial \tau_1} - 2S_g a_0 a_1 - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + s_g a_0^2 \right) n_0. \quad (\text{A.17b})$$

The following calculations involve many variable changes and make heavy use of the field amplitude decomposition (A.8), of the normalization condition (A.9), of the Q-switching oscillator equations (A.6b) and (A.12), and of which variable is independent of which time scale. For the sake of concision, we omit from now on any reference to those in most places where they are invoked.

Boundedness of  $a_2$  and  $n_2$  in  $t$  requires the vanishing of the right-hand side of Eq. (A.17a) and of the average of the right-hand side of Eq. (A.17b) over all  $t$ , leading to the solvability conditions

$$\left( r \frac{\partial}{\partial \tau_1} - \frac{1}{2} n_0 \right) a_1 - \frac{1}{2} n_1 \tilde{a} \hat{a} = \left[ -\frac{r^2}{2} \frac{\partial^2}{\partial \tau_1^2} - r \frac{\partial}{\partial \tau_2} + \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} - q \right) \right] \tilde{a} \hat{a} \quad (\text{A.18a})$$

$$\frac{\partial}{\partial \tau_1} r^{-1} \int_0^r dt n_1 + 2S_g \tilde{a} r^{-1} \int_0^r dt \hat{a} a_1 = - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + s_g \tilde{a}^2 \right) n_0 \quad (\text{A.18b})$$

Those can be rewritten as

$$\left( r \frac{\partial a'}{\partial \tau_1} - \frac{1}{2} n' \right) \hat{a} = \left[ -r \tilde{a}^{-1} \frac{\partial \tilde{a}}{\partial \tau_2} + \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} - q \right) \right] \hat{a}, \quad (\text{A.19a})$$

$$\frac{\partial n'}{\partial \tau_1} + 2S_g \tilde{a}^2 \tilde{a}' = - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + \bar{s}_g \tilde{a}^2 + \frac{1}{2} n_0 \frac{\partial}{\partial \tau_1} \right) n_0, \quad (\text{A.19b})$$

in terms of the effective saturation coefficient  $\bar{s}_g$  given in Eq. (23) and of the auxiliary variables  $a'$ ,  $\bar{a}'$ ,  $n'$ , and  $\bar{n}'$  defined as

$$a' = \tilde{a}^{-1} \hat{a}^{-1} a_1, \quad \bar{a}' = r^{-1} \int_0^r dt \hat{a}^2 a', \quad (\text{A.20a})$$

$$n' = n_1 - \frac{r}{2} (P_g - S_g \tilde{a}^2) - \frac{1}{4} n_0^2, \quad \bar{n}' = r^{-1} \int_0^r dt n'. \quad (\text{A.20b})$$

Multiplying both sides of Eq. (A.19a) by  $\hat{a}$ , integrating over one period in  $t$ , and using the identity (A.16) yields, together with Eq. (A.19b), the following linear  $t$ -independent inhomogeneous system for two unknowns  $\tilde{a}'$  and  $\tilde{n}'$ :

$$r \frac{\partial \tilde{a}'}{\partial \tau_1} - \frac{1}{2} \tilde{n}' = -r \tilde{a}^{-1} \frac{\partial \tilde{a}}{\partial \tau_2} - \frac{1}{2} r^{-1} \int_0^r dt \hat{a}^2 q, \quad (\text{A.21a})$$

$$\frac{\partial \tilde{n}'}{\partial \tau_1} + 2S_g \tilde{a}^2 \tilde{a}' = - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + \bar{s}_g \tilde{a}^2 + \frac{1}{2} n_0 \frac{\partial}{\partial \tau_1} \right) n_0, \quad (\text{A.21b})$$

where we have defined

$$\tilde{a}' = \bar{a}', \quad (\text{A.22a})$$

$$\tilde{n}' = \bar{n}' - d^2 r^{-1} \int_0^r dt \left( \frac{\partial \hat{a}}{\partial t} \right)^2. \quad (\text{A.22b})$$

Fredholm's solvability condition requires that the right-hand side of the system (A.21) be orthogonal to the solutions of the adjoint homogeneous problem. After a few calculations, we find that this condition can be written in matrix form as

$$\int_0^{T_1} d\tau_1 \begin{bmatrix} -2(P_g - S_g \tilde{a}^2) & n_0 \end{bmatrix} \begin{bmatrix} -r \tilde{a}^{-1} \frac{\partial \tilde{a}}{\partial \tau_2} - \frac{1}{2} r^{-1} \int_0^r dt \hat{a}^2 q \\ - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + \bar{s}_g \tilde{a}^2 + \frac{1}{2} n_0 \frac{\partial}{\partial \tau_1} \right) n_0 \end{bmatrix} = 0, \quad (\text{A.23})$$

where  $T_1$  denotes the period of the Q-switching oscillator as defined by Eqs. (A.6b) and (A.12). Using the expression (A.13) for the Q-switching energy  $H$ , Eq. (A.23) simplifies to

$$r \frac{dH}{d\tau_2} = T_1^{-1} \int_0^{T_1} d\tau_1 \left[ (P_g - S_g \tilde{a}^2) r^{-1} \int_0^r dt \hat{a}^2 q - (\bar{\gamma}_g + \bar{s}_g \tilde{a}^2) n_0^2 \right]. \quad (\text{A.24})$$

Eq. (A.24) determines the slow evolution of the Q-switching oscillation cycle. Its right-hand side involves the normalized emission pattern  $\hat{a}$ , whose evolution is as yet undetermined. We therefore now need an equation for  $\hat{a}$ .

To this end, we multiply both sides of Eq. (A.21a) by  $\hat{a}$  and subtract it side by side from Eq. (A.19a), obtaining

$$r \hat{a} \frac{\partial \tilde{a}'}{\partial \tau_1} = \left[ -r \frac{\partial}{\partial \tau_2} + \frac{1}{2} \left( d^2 \frac{\partial^2}{\partial t^2} + \hat{n}' + r^{-1} \int_0^r dt \hat{a}^2 q - q \right) \right] \hat{a}, \quad (\text{A.25})$$

where we have defined

$$\hat{a}' = a' - \tilde{a}', \quad (\text{A.26a})$$

$$\hat{n}' = n' - \tilde{n}'. \quad (\text{A.26b})$$

Boundedness of  $\widehat{a}'$  in  $\tau_1$  requires the vanishing of the average of the right-hand side of Eq. (A.25) over all  $\tau_1$ , leading to the following solvability condition:

$$r \frac{\partial \widehat{a}}{\partial \tau_2} = \frac{1}{2} \left[ d^2 \frac{\partial^2}{\partial t^2} + \widehat{n} + T_1^{-1} \int_0^{T_1} d\tau_1 \left( r^{-1} \int_0^r dt \widehat{a}^2 q - q \right) \right] \widehat{a}, \quad (\text{A.27})$$

where we have defined

$$\widehat{n} = T_1^{-1} \int_0^{T_1} d\tau_1 \widehat{n}'. \quad (\text{A.28})$$

Eq. (A.27) provides an equation for  $\widehat{a}$  that depends on  $\widehat{n}$ .

A complementary equation for  $\widehat{n}$  in terms of  $\widehat{a}$  can be derived from the fast gain equation (A.15b) using the relations (A.20b), (A.22b), (A.26b), and (A.28) between the various auxiliary gain variables:

$$\frac{\partial \widehat{n}}{\partial t} = S_g (1 - \widehat{a}^2) T_1^{-1} \int_0^{T_1} d\tau_1 \widetilde{a}^2. \quad (\text{A.29})$$

## Appendix B Asymptotic form of Eqs. (22)

We now summarize the results of the multiple-scale analysis of Appendix A and collect all the obtained asymptotic equations in a single place, renaming the leading-order gain component  $n_0$  as

$$n_0 = \widetilde{n} \quad (\text{B.1})$$

for the sake of notation uniformity. According to Eq. (A.8), the field amplitude  $a$  can be decomposed to leading order as the product of a slow-varying envelope  $\widetilde{a}$  and of a normalized emission pattern  $\widehat{a}$ :

$$a = \widetilde{a} \widehat{a}, \quad (\text{B.2})$$

both governed by their own evolution equations.

Eqs. (A.27) and (A.29) form a system that couples the emission profile  $\widehat{a}$  to some fast gain component  $\widehat{n}$ :

$$r \frac{\partial \widehat{a}}{\partial \tau_2} = \frac{1}{2} \left[ d^2 \frac{\partial^2}{\partial t^2} + \widehat{n} + T_1^{-1} \int_0^{T_1} d\tau_1 \left( r^{-1} \int_0^r dt \widehat{a}^2 q - q \right) \right] \widehat{a}, \quad (\text{B.3a})$$

$$\frac{\partial \widehat{n}}{\partial t} = S_g (1 - \widehat{a}^2) T_1^{-1} \int_0^{T_1} d\tau_1 \widetilde{a}^2, \quad (\text{B.3b})$$

where  $T_1$  is the Q-switching period, to be defined more precisely below. Eq. (B.3a) bears some similarity to Haus master equation (6), while Eq. (B.3b) determines the depletion-and-recovery profile of the gain over one round-trip. In view of Eqs. (A.14) and (A.28),  $\widehat{a}$  and  $\widehat{n}$  do not vary on the Q-switching time scale  $\tau_1$ , so they do not contain any information about Q-switching oscillations. In order to form a well-posed problem, the partial-differential system (B.3) must satisfy some boundary conditions or other constraints. Those are provided by Eqs. (A.9) and (A.10):

$$\widehat{a}(t+r) = \widehat{a}(t), \quad (\text{B.4a})$$

$$r^{-1} \int_0^r dt \widehat{a}^2 = 1. \quad (\text{B.4b})$$

Eq. (B.4a) is a periodic boundary condition for the emission pattern  $\hat{a}$ . The solution  $\hat{n}$  of the fast gain equation (B.3b) is defined up to an additive contribution that depends only on the slowest time scale  $\tau_2$ , and whose value is to be adjusted so that the normalization condition (B.4b) holds for all  $\tau_2$ .

Coupled equations for the slow-varying field amplitude  $\tilde{a}$  and the leading-order gain component  $\tilde{n}$  are provided by Eqs. (A.6b) and (A.12):

$$r \frac{\partial \tilde{a}}{\partial \tau_1} = \frac{1}{2} \tilde{n} \tilde{a}, \quad (\text{B.5a})$$

$$\frac{\partial \tilde{n}}{\partial \tau_1} = P_g - S_g \tilde{a}^2. \quad (\text{B.5b})$$

In view of Eqs. (A.4b) and (A.7),  $\tilde{a}$  and  $\tilde{n}$  do not vary on the fast time scale  $t$ . Eqs. (B.5) define a conservative oscillator whose orbits describe Q-switching cycles. Those are characterized by the values  $H$  of a first integral provided by Eq. (A.13):

$$H = (2r)^{-1} \tilde{n}^2 + S_g \tilde{a}^2 - P_g \ln (P_g^{-1} S_g \tilde{a}^2). \quad (\text{B.6})$$

The Q-switching period  $T_1$  can be computed as the period of  $\tilde{a}$  and  $\tilde{n}$  in  $\tau_1$  according to Eqs. (B.5). The evolution of  $\tilde{a}$  and  $\tilde{n}$  on all time scales, and thus the full Q-switching dynamics, is completely determined by the additional knowledge of the evolution of the Q-switching energy  $H$  on the slowest time scale  $\tau_2$ . It is provided by Eq. (A.24):

$$r \frac{dH}{d\tau_2} = T_1^{-1} \int_0^{T_1} d\tau_1 \left[ (P_g - S_g \tilde{a}^2) r^{-1} \int_0^r dt \tilde{a}^2 q - (\bar{\gamma}_g + \bar{s}_g \tilde{a}^2) \tilde{n}^2 \right], \quad (\text{B.7})$$

where  $\bar{s}_g$  is given by Eq. (23).

Finally, the system (B.2)–(B.7) is closed by its coupling to the absorber equation (7) and its periodic boundary condition (8).

## Appendix C Equivalence of Eqs. (25)–(26) to Eqs. (B.2)–(B.7) in the limit (21)

Eqs. (B.2)–(B.7) involve the three independent time variables  $t$ ,  $\tau_1$ , and  $\tau_2$ . Our first step towards casting them into the form (25)–(26) consists in recombining  $\tau_1$  and  $\tau_2$  into a single slow-time variable  $\tau$ . To this end, our strategy is to propose the following multiple-scale expansion for  $\tau$ :

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau_1} + \varepsilon \frac{\partial}{\partial \tau_2} + O(\varepsilon^2), \quad (\text{C.1})$$

and then devise (by means of educated guesswork) a set of equations in the  $t$  and  $\tau$  variables that admit Eqs. (B.2)–(B.7) as a limit for small  $\varepsilon$ . We then explicitly check the correctness of that limit to establish formally the equivalence of the two formulations. This can be viewed as a multiple scale analysis applied backwards.

First, the equations (B.3) for  $\hat{a}$  and  $\hat{n}$  suggest

$$r \frac{\partial \hat{a}}{\partial \tau} = \frac{1}{2} \varepsilon \left[ d^2 \frac{\partial^2}{\partial t^2} + \hat{n}' + r^{-1} \int_0^r dt \hat{a}^2 q - q \right] \hat{a}, \quad (\text{C.2a})$$

$$\frac{\partial \hat{n}'}{\partial t} = S_g (1 - \hat{a}^2) \tilde{a}^2 \quad (\text{C.2b})$$



as equivalent for small  $\varepsilon$ . It is indeed easily checked that the application of an averaging method to Eq. (C.2a) yields Eq. (B.3a) with the definition (A.28) for  $\hat{n}$  in terms of  $\hat{n}'$ , and that the combination of Eqs. (A.28) and (C.2b) leads to Eq. (B.3b).

Next, Eq. (B.7) for the slow evolution of Q-switching oscillations suggest modifying the conservative oscillator (B.5) as follows:

$$r \frac{\partial \tilde{a}}{\partial \tau} = \frac{1}{2} \left( \tilde{n} - \varepsilon r^{-1} \int_0^r dt \hat{a}^2 q \right) \tilde{a}, \quad (\text{C.3a})$$

$$\frac{\partial \tilde{n}}{\partial \tau} = P_g - S_g \tilde{a}^2 - \varepsilon (\bar{\gamma}_g + \bar{s}_g \tilde{a}^2) \tilde{n}. \quad (\text{C.3b})$$

We now check that Eqs. (B.5)–(B.7) are a limit of Eqs. (C.3) for small  $\varepsilon$  as follows. We substitute the expansions (C.1) and

$$\tilde{a} = \tilde{a}_0 (1 + \varepsilon \tilde{a}') + O(\varepsilon^2), \quad \tilde{n} = \tilde{n}_0 + \varepsilon \tilde{n}' + O(\varepsilon^2) \quad (\text{C.4})$$

into Eqs. (C.3) and equate separately the coefficients of like powers of  $\varepsilon$ , obtaining a hierarchy of problems for the coefficients of the expansions (C.4). The  $O(\varepsilon^0)$  problem is equivalent to Eqs. (B.5). The  $O(\varepsilon^1)$  problem is

$$r \frac{\partial \tilde{a}'}{\partial \tau_1} - \frac{1}{2} \tilde{n}' = -r \tilde{a}_0^{-1} \frac{\partial \tilde{a}_0}{\partial \tau_2} - \frac{1}{2} r^{-1} \int_0^r dt \hat{a}^2 q, \quad (\text{C.5a})$$

$$\frac{\partial \tilde{n}'}{\partial \tau_1} + 2S_g \tilde{a}_0^2 \tilde{a}' = - \left( \frac{\partial}{\partial \tau_2} + \bar{\gamma}_g + \bar{s}_g \tilde{a}^2 \right) \tilde{n}_0, \quad (\text{C.5b})$$

and is formally identical to Eqs. (A.21) with the sole exception of the absence of the last term in Eq. (A.21b). By the same reasoning as in Appendix A, therefore, a solvability condition for Eqs. (C.5) is provided by Eq. (A.23) with the last term of the second element of the column vector removed. Because that term vanishes in the integration over one period in  $\tau_1$ , however, the presence of this term does not matter, and the solvability condition simplifies to Eq. (A.24).

The analysis so far establishes that in the limit for small  $\varepsilon$ , the two time scales  $\tau_1$  and  $\tau_2$  can be recombined into a single time variable  $\tau$  by replacing the equations (B.3) for  $\hat{a}$  and  $\hat{n}$  with Eqs. (C.2) and the equations (B.5)–(B.7) for  $\tilde{a}$  and  $\tilde{n}$  with Eqs. (C.3). We now want to cast the two field equations (C.2a) and (C.3a) into a single equation for the combined field amplitude  $a$  given by Eq. (B.2). To this end, we multiply both sides of Eq. (C.2a) by  $\tilde{a}$  and both sides of Eq. (C.3a) by  $\hat{a}$  and add the two resulting equations. Keeping in mind that  $\tilde{a}$  does not depend on  $t$ , we obtain

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( \varepsilon d^2 \frac{\partial^2}{\partial t^2} + n - \varepsilon q \right) a, \quad (\text{C.6})$$

where we have defined

$$n = \tilde{n} + \varepsilon \tilde{n}'. \quad (\text{C.7})$$

Eq. (B.4a) further yields a periodic boundary condition for  $a$ :

$$a(t+r) = a(t). \quad (\text{C.8})$$

An equation for the recombined gain variable  $n$  is then obtained by differentiating both sides of Eq. (C.7) with respect to  $t$ , substituting Eq. (C.2b), and keeping in mind that  $\tilde{n}$  does not depend on  $t$ :

$$\frac{\partial n}{\partial t} = \varepsilon S_g \left( r^{-1} \int_0^r dt a^2 - a^2 \right), \quad (\text{C.9})$$

where we have combined the field decomposition (B.2) and the normalization condition (B.4b) to express  $\tilde{a}^2$  as follows:

$$\tilde{a}^2 = r^{-1} \int_0^r dt a^2. \quad (\text{C.10})$$

Eq. (C.9) alone determines  $n$  only up to an additive contribution that depends only on  $\tau$ . An extra constraint is required to obtain a well-posed problem, and provided by differentiating both sides of Eq. (C.7) with respect to  $\tau$  and integrating over one period in  $t$ , which gives

$$r^{-1} \int_0^r dt \frac{\partial n}{\partial \tau} = P_g - S_g \tilde{a}^2 - \varepsilon (\bar{\gamma}_g + \bar{s}_g \tilde{a}^2) r^{-1} \int_0^r dt \tilde{n} + \varepsilon r^{-1} \int_0^r dt \frac{\partial \tilde{n}'}{\partial \tau} \quad (\text{C.11})$$

Now, note that the equation (C.2b) for  $\hat{n}'$  and the normalization condition (B.4b) imply that  $\hat{n}'$  is periodic in  $t$  with the same period as  $\hat{a}^2$ . A useful relation can be obtained by multiplying both sides of Eq. (C.2a) by  $\hat{a}$  and both sides of Eq. (C.2b) by  $\hat{n}'$  and integrating the two resulting equations over one period in  $t$ . This gives

$$\int_0^r dt \hat{n}' = \int_0^r dt \hat{a}^2 \hat{n}' = d^2 \int_0^r dt \left( \frac{\partial \hat{a}}{\partial t} \right)^2, \quad (\text{C.12})$$

which, in view of Eq. (C.2a), further entails

$$\int_0^r dt \frac{\partial \hat{n}'}{\partial \tau} = O(\varepsilon). \quad (\text{C.13})$$

Substituting the expression (C.10) for  $\tilde{a}^2$ , the expression for  $\tilde{n}$  obtained from Eq. (C.7), and Eq. (C.13) into Eq. (C.11) gives

$$\frac{\partial \bar{n}}{\partial \tau} = P_g - S_g r^{-1} \int_0^r dt a^2 - \varepsilon \left( \bar{\gamma}_g + \bar{s}_g r^{-1} \int_0^r dt a^2 \right) \bar{n} + O(\varepsilon^2), \quad (\text{C.14})$$

where  $\bar{n} = r^{-1} \int_0^r n dt$  represents the average gain over one round-trip.

Finally, neglecting  $O(\varepsilon^2)$  terms in (C.9) and introducing the new variables  $m = n + S_g/\bar{s}_g$  and  $\bar{m} = \bar{n} + S_g/\bar{s}_g$  and in we can rewrite Eqs. (C.6), (C.8), (C.9), and

$$r \frac{\partial a}{\partial \tau} = \frac{1}{2} \left( \varepsilon d^2 \frac{\partial^2}{\partial t^2} + m - \varepsilon q - k \right) a, \quad a(t+r) = a(t), \quad (\text{C.15})$$

$$\frac{\partial m}{\partial t} = \varepsilon S_g \left( r^{-1} \int_0^r dt a^2 - a^2 \right), \quad (\text{C.16})$$

$$\frac{\partial \bar{m}}{\partial \tau} = P_g - S_g r^{-1} \int_0^r dt a^2 - \varepsilon \left( \bar{\gamma}_g + \bar{s}_g r^{-1} \int_0^r dt a^2 \right) \bar{m}, \quad (\text{C.17})$$

Eqs. (C.6), (C.8), (C.9), and (C.14) together are asymptotic to Eqs. (B.2)–(B.7) in the limit for small  $\varepsilon$ . (The  $O(\varepsilon^2)$  corrections in Eq. (C.14) can be safely neglected without invalidating this result.) On the other hand, the same Eqs. (C.6), (C.8), (C.9), and (C.14) can be obtained by carrying out the substitutions (21) and  $\tau \rightarrow \varepsilon^{-1}\tau$  into Eqs. (25)–(26), which proves the equivalence of Eqs. (25)–(26) to Eqs. (B.2)–(B.7) with the scaling (21).

## References

- [1] E. Ippen, C. Shank, and A. Dienes. Passive mode locking of the cw dye laser. *Applied Physics Letters*, 21:348–350, 1972.
- [2] G. New. Pulse evolution in mode-locked quasi-continuous lasers. *IEEE J. Quant. Electron*, 10:115–124, 1974.
- [3] H. Haus. Theory of mode locking with a fast saturable absorber. *Journal of Applied Physics*, 46:3049–3058, 1975.
- [4] H. Haus. Theory of mode locking with a slow saturable absorber. *IEEE J. Quant. Electron*, 11:736–746, 1975.
- [5] E. P. Ippen. Principles of passive mode locking. *Applied Physics B*, 58(3):159–170, 1994.
- [6] J. L. A. Dubbeldam, J. A. Leegwater, and D. Lenstra. Theory of mode-locked semiconductor lasers with finite absorber relaxation times. *Applied Physics Letters*, 70:1938–1940, 1997.
- [7] N. N. Akhmediev, A. Ankiewicz, M. J. Lederer, and B. Luther-Davies. Ultrashort pulses generated by mode-locked lasers with either a slow or a fast saturable-absorber response. *Optics letters*, 23(4):280–282, 1998.
- [8] E. A. Avrutin, J. H. Marsh, and E. L. Portnoi. Monolithic and multi-gigahertz mode-locked semiconductor lasers: constructions, experiments, models and applications. *IEE Proceedings-Optoelectronics*, 147(4):251–278, 2000.
- [9] R Paschotta and U Keller. Passive mode locking with slow saturable absorbers. *Applied Physics B*, 73(7):653–662, 2001.
- [10] A. G. Vladimirov, D. Turaev, and G. Kozyreff. Delay differential equations for mode-locked semiconductor lasers. *Optics Letters*, 29:1221–1223, 2004.
- [11] A. G. Vladimirov and D. Turaev. Model for passive mode locking in semiconductor lasers. *Physical Review A*, 72(3):033808, 2005.
- [12] M. Rossetti, P. Bardella, and I. Montrosset. Modeling passive mode-locking in quantum dot lasers: A comparison between a finite-difference traveling-wave model and a delayed differential equation approach. *IEEE Journal of Quantum Electronics*, 47(5):569–576, 2011.
- [13] S. Wang, A. Docherty, B. S. Marks, and C. R. Menyuk. Comparison of numerical methods for modeling laser mode locking with saturable gain. *JOSA B*, 30(11):3064–3074, 2013.
- [14] J. Javaloyes. Cavity light bullets in passively mode-locked semiconductor lasers. *Physical Review Letters*, 116(4):043901, 2016.
- [15] H. A. Haus. Mode-locking of lasers. *IEEE Journal of Selected Topics in Quantum Electronics*, 6:1173–1185, 2000.
- [16] H. Haus. Parameter ranges for cw passive mode locking. *IEEE journal of Quantum Electronics*, 12(3):169–176, 1976.

- [17] F. X. Kaertner, L. R. Brovelli, D. Kopf, M. Kamp, I. G. Calasso, and U. Keller. Control of solid state laser dynamics by semiconductor devices. *Optical Engineering*, 34(7):2024–2036, 1995.
- [18] C. Hönninger, P. Paschotta, F. Morier-Genoud, M. Moser, and U. Keller. Q-switching stability limits of continuous-wave passive mode locking. *JOSA B*, 16:46–56, 1999.
- [19] T. R. Schibli, E. R. Thoen, F. X. Kärtner, and E. P. Ippen. Suppression of q-switched mode locking and break-up into multiple pulses by inverse saturable absorption. *Applied Physics B*, 70(1):S41–S49, 2000.
- [20] D. Rachinskii, A. Vladimirov, U. Bandelow, B. Hüttl, and R. Kaiser. Q-switching instability in a mode-locked semiconductor laser. *JOSA B*, 23:663–670, 2006.
- [21] T. Kolokolnikov, M. Nizette, T. Erneux, N. Joly, and S. Bielawski. The q-switching instability in passively mode-locked lasers. *Physica D*, 219:13–21, 2006.
- [22] I. Kudelin, S. Sugavanam, and M. Chernysheva. Pulse-onset dynamics in a bidirectional mode-locked fibre laser via instabilities. *Communications Physics*, 3(1):1–10, 2020.
- [23] F. X. Kärtner, I. D. Jung, and U. Keller. Soliton mode-locking with saturable absorbers. *IEEE Journal of Selected Topics in Quantum Electronics*, 2(3):540–556, 1996.
- [24] F. X. Kärtner, J. Aus Der Au, and U. Keller. Mode-locking with slow and fast saturable absorbers—what’s the difference? *IEEE Journal of Selected Topics in Quantum Electronics*, 4(2):159–168, 1998.
- [25] E. P. Ippen, H. Haus, and L. Y. Liu. Additive pulse modelocking. *J. Opt*, 6:1736–1745, 1989.
- [26] D. Spence, P. Kean, and W. Sibbett. 60-fsec pulse generation from a self-mode-locked ti:sapphire laser. *Optics Letters*, 16:42–44, 1991.
- [27] E. Arthurs, D. Bradley, and A. Roddie. Buildup of picosecond pulse generation in passively mode-locked rhodamine dye lasers. *Applied Physics Letters*, 23:88–90, 1973.
- [28] J. Hausen, K. Lüdge, S. V. Gurevich, and J. Javaloyes. How carrier memory enters the haus master equation of mode-locking. *Optics Letters*, 45(22):6210–6213, 2020.
- [29] M. J. Lederer, B. Luther-Davies, H. H. Tan, C. Jagadish, N. N. Akhmediev, and J. M. Soto-Crespo. Multipulse operation of a ti: sapphire laser mode locked by an ion-implanted semiconductor saturable-absorber mirror. *JOSA B*, 16(6):895–904, 1999.
- [30] J. K. Kevorkian and J. D. Cole. *Multiple scale and singular perturbation methods*, volume 114. Springer Science & Business Media, 2012.
- [31] A. Pimenov, S. Slepneva, G. Huyet, and A. G. Vladimirov. Dispersive time-delay dynamical systems. *Physical Review Letters*, 118(19):193901, 2017.
- [32] A. Pimenov, S. Amiranashvili, and A. G. Vladimirov. Temporal cavity solitons in a delayed model of a dispersive cavity ring laser. *Mathematical Modelling of Natural Phenomena*, 15:47, 2020.