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Robert Lasarzik

Abstract

We define the concept of energy-variational solutions for the Navier–Stokes and Euler equations and prove their existence in any space dimension. It is shown that the concept of energy-variational solutions enjoys several desirable properties. Energy-variational solutions are not only known to exist and coincide with local strong solutions, but the operator, mapping the data to the set of energy-variational solutions, is additionally continuous and all restrictions and all concatenations of energy-variational solutions are energy-variational solutions again. Finally, different selection criteria for these solutions are discussed.

Keywords: Existence, Navier–Stokes, Euler, incompressible, fluid dynamics, generalized solutions.

MSC 2020: 35D99, 35Q30, 35Q31, 76D05

1 Introduction

The Navier–Stokes and Euler equations are the standard models for incompressible fluid dynamics. Both are recurrent tools in computational fluid dynamics for weather forecast, micro fluidic devices [31] or industrial processes like steel production [1]. There exists a vast literature concerning the Navier–Stokes and Euler equations. In case of the Navier–Stokes equation, we only mention here the existence proof for *weak solutions* in three dimension by Leray [25] and the *weak-strong uniqueness* result due to Serrin [30]. In the context of the Euler equations, the existence of weak solutions in any space dimension is already known for special initial data (see [11]) also fulfilling the energy inequality (see [12]). This result was proven via the convex integration technique. This technique grants the existence of infinitely many and also non-physical weak solutions. Additionally, it was proven for the Navier–Stokes equations via similar techniques that there exist infinitely many weak solutions that do not fulfill the energy inequality [6]. But what is lacking in the literature so far is an existence result for the Navier–Stokes equations in space dimensions larger than four and for the Euler equation with general initial data. Revisiting the previously introduced dissipative solutions for the equations of incompressible fluid dynamics, we refine this concept by introducing *energy-variational solutions*. As the name already suggests, this notion of generalized solutions is based on a *variation of the underlying energy-dissipation principle*. The relative energy inequality can be seen as a variation of the energy-dissipation principle with respect to sufficiently regular functions.

Dissipative solutions were proposed by P.-L. Lions [26, Sec. 4.4] in the context of the Euler equations. The current author applied this concept in the context of nematic liquid crystals [20] and nematic electrolytes [3]. It was observed that natural discretizations complying with the properties of the system, like energetic or entropic principles, as well as algebraic restrictions converge naturally to a dissipative solution instead of a measure valued solution (see [3] and [22] for details). In comparison to measure-valued solutions, the degrees of freedom are heavily reduced and no defect measures occur, which are especially difficult to approximate. The relative energy inequality, which is at the heart of the dissipative and energy-variational solution concept is also a recurrent tool in PDE theory to prove for instance weak-strong uniqueness [21], stability of stationary states [20], convergence to singular limits [14], or to design optimal control schemes [22]. An advantage in comparison to distributional or measure-valued solutions is that the solution set inherits the convexity of the energy and dissipation functional, which permits to define appropriate uniqueness criteria [23].

The definition of energy-variational solutions follows a similar idea as the definition of dissipative solutions, both rely on the so-called relative energy inequality, which compares the solution to smooth test functions fulfilling the PDE only approximately. But the relative energy inequality for energy-variational solutions is refined such that the resulting inequality becomes an equality for smooth solutions. The nonlinear-convective terms are not only estimated by the relative energy but included in the underlying dissipation potential. Furthermore, the relative energy inequality holds for all given intervals $(s,t) \subset [0,T]$. This is achieved by introducing an additional variable E in time. The difference of E and the energy $\mathscr{E}(\boldsymbol{v})$ measures the discrepancy between weak and strong convergence in every point in time. If the auxiliary variable E coincides with the energy $\mathscr{E}$ a.e. in $(0,T)$, the weak formulation is fulfilled. By introducing this additional variable the solution concept has the semi-flow property, every concatenation and restriction of the solutions to a sub or super time interval is an energy-variational solution again. Still the properties of the relative energy inequality remain present, it is preserved for sequences converging in the weak topologies of the associated natural energy and dissipation spaces. Thus in comparison to standard weak solutions, energy-variational solutions have the advantage that no strong convergence is needed in order to pass to the limit in this formulation. Only Helly's selection principle is used in order to infer the existence of the additional defect variable. The existence result only relies on standard constructive proofs, *i.e.*, a Galerkin discretization in the case of the Navier–Stokes equations and the vanishing viscosity limit in the case of the Euler equations.

Since the energy and dissipation functionals in the considered cases are convex, the set of energy-variational solutions is convex and weakly* closed. This allows to identify selection criteria in order to select the physically relevant solution. Following the ideas of [4, 8, 9, 23], we propose the selection principle of maximal dissipation. This says that the physically relevant solution dissipates energy at the highest rate, hence minimizes the energy in every point in time. This principle becomes even more apparent in thermodynamical consistent systems, where the maximized dissipation implies maximal entropy (see for instance [15] and [7, Sec. 9.7]).

In [23], the set of dissipative solutions together with the time integral of the energy functional is identified as a suitable convex structure on which such a minimization problem can be defined. The resulting maximally dissipative solution is indeed well-posed in the sense of Hadamard. The result of the article at hand applies this technique to energy-variational solutions and the selected unique solution for a strictly convex functional of the variables inherits the semi-flow property. In comparison to such general selection criteria via an convex function integrated in time, we consider the minimization at finitely-many points-in-time. Interestingly, this point-wise minimization is well defined for energy-variational solutions and implies additional regularity. In the finitely-many point-in-time the auxiliary variable E coincides with the energy $\mathscr{E}$ and the solution is right-continuous with respect to the strong topology in $L^2(\Omega)$.

It is worth noticing that in the framework of energy-variational solutions it is possible to pass to the limit in the quadratic convection term without any strong compactness argument. Only arguments from the direct method of the calculus of variations are needed. It is possible to pass to the limit in the quadratic term, using an additional variable, which catches the difference between weak and strong convergence of the energy. Usually compact embeddings and *a priori* estimates of the time derivative are used to infer strong convergence via some Aubin–Lions argument (compare to [33]). These ingredients are irrelevant in the present proof, since it only relies on weak convergence in natural spaces and the weakly-lower semi-continuity of the underlying energy and dissipation functionals. The proposed technique seems to be very powerful and easily adapted to other systems of PDEs. Hence, this gives hope that the new approach may allows to prove the existence of energy-variational to some PDE systems. This includes multidimensional conservation laws [4], liquid crystals [21], heat-conducting complex fluids [24], or GENERIC systems in general (see [15] and [23]).

Plan of the paper: After providing some notation and preliminaries in Section 2.1, the different solution concepts of weak and energy-variational solutions are defined in Section 2.2. Then, we state the main Theorems in Section 2.3 and prove them afterwards (see Section 3).

2 Definitions and main theorems

2.1 Preliminaries

Before, we provide the definitions and main results, we collect some notation and preliminary results.

Notations: Throughout this paper, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$. The space of smooth solenoidal functions with compact support is denoted by $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d)$. By $L^2_\sigma(\Omega)$ and $H^1_{0,\sigma}(\Omega)$ we denote the closure of $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d)$ with respect to the norm of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. Note that $L^2_\sigma(\Omega)$ can be characterized by $L^2_\sigma(\Omega) = \{\boldsymbol{v} \in L^2(\Omega) | \nabla\cdot \boldsymbol{v} = 0$ in $\Omega, \boldsymbol{n}\cdot\boldsymbol{v} = 0$ on $\partial\Omega\}$, where the first condition has to be understood in the distributional sense and the second condition in the sense of the trace in $H^{-1/2}(\partial\Omega)$. By $\boldsymbol{n}$, we denote the outer normal vector of Ω.

The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; the duality pairing is denoted by $\langle\cdot,\cdot\rangle$ and the L^2-inner product by $(\cdot,\cdot)$. The total variation of a function $E : \mathbb{R} \to \mathbb{R}$ is given by $|E|_{\mathsf{TV}(0,T)} = \sup_{0<t_0<\ldots<t_n<T} \sum_{k=1}^N |E(t_{k-1}) - E(t_k)|$ where the supremum is taken over all finite partitions of the interval $[0,T]$. We denote the space of all functions of bounded variations on $[0,T]$ by $\mathrm{BV}([0,\mathrm{T}])$.

Note that the total variation of a monotone decreasing nonnegative function only depends on the initial value, *i.e.,*

$$\|E\|_{\mathsf{TV}(0,T)} = \sup_{0<t_0<\ldots<t_n<T} \sum_{k=1}^N |E(t_{k-1}) - E(t_k)| \leq E(0) - E(T) \leq E(0)\,.$$

The symmetric part of a matrix is given by $\boldsymbol{A}_{\mathsf{sym}} := \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T)$ for $\boldsymbol{A} \in \mathbb{R}^{d\times d}$. For the product of two matrices $\boldsymbol{A},\boldsymbol{B} \in \mathbb{R}^{d\times d}$, we observe

$$\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A} : \boldsymbol{B}_{\mathsf{sym}}\,, \quad \text{if } \boldsymbol{A}^T = \boldsymbol{A}\,.$$

Furthermore, it holds $\boldsymbol{a}\otimes\boldsymbol{b} : \boldsymbol{A} = \boldsymbol{a}\cdot\boldsymbol{A}\boldsymbol{b}$ for $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d$, $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ and hence $\boldsymbol{a}\otimes\boldsymbol{a} : \boldsymbol{A} = \boldsymbol{a}\cdot\boldsymbol{A}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{A}_{\mathsf{sym}}\boldsymbol{a}$. By $(A)_{\mathsf{sym},-}$ we denote the negative semi-definite part of the symmetric part of the matrix $A \in \mathbb{R}^{d\times d}$. By I, we denote the identity matrix in $\mathbb{R}^{d\times d}$ and by $\mathbb{R}_+ := [0,\infty)$ the positive real numbers.

The following lemma provides the connection between the almost everywhere pointwise formulation of an inequality with the weak one.

Lemma 2.1. Let $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$ with $g \geq 0$ a.e. in $(0,T)$. Then the two inequalities

$$-\int_0^T \phi'(t)g(t)\,\mathrm{d}t + \int_0^T \phi(t)f(t)\,\mathrm{d}t \leq 0$$

for all $\phi \in \mathscr{C}_c^1((0,T))$ with $\phi \geq 0$ for all $t \in (0,T)$ and

$$g(t) - g(s) + \int_s^t f(\tau)\,\mathrm{d}\tau \leq 0 \quad \text{for a.e. } t,\, s \in (0,T) \tag{1}$$

are equivalent.

Moreover, there exists a function $h \in \mathrm{BV}([0,\mathrm{T}])$ such that $h = g$ a.e. in $(0,T)$ and the inequality (1) holds for every $s,t \in (0,T)$ with g replaced by h.

Proof. Since this is a rather standard lemma, only a short comment on the proof is provided. For the if-direction, one may argues by inserting an approximating sequence of the indicator function $\chi_{[s,t]}$ for ϕ. To infer the only-if-direction, we sum up the second inequality for any partition $0 < t_1 < \ldots < t_N < T$ of $[0,T]$ to infer

$$\sum_{n=0}^{N-1} \phi(\xi_k)[g(t_{n+1}) - g(t_n)] + \sum_{n=0}^{N-1} \phi(\xi_k) \int_{t_n}^{t_{n+1}} f(\tau)\,\mathrm{d}\tau \leq 0 \quad \text{with } \xi_n \in (t_n, t_{n+1})\,.$$

Passing to the limit in the partition, gives the integral in the sense of Stieltjes (cf. [27, Chap. 8, Sec. 6]). An integration-by-parts in the first term implies the first inequality in Lemma 2.1.

From inequality (1), we infer that the function $t \mapsto g(t) + \int_0^t f(s)\,\mathrm{d}s$ is a monotone function a.e. on $(0,T)$. Redefining g on a set of measure zero gives a function h that such that $t \mapsto h(t) + \int_0^t f(s)\,\mathrm{d}s$ is monotone, thus $\mathrm{BV}([0,\mathrm{T}])$, which implies since $\int_0^t f(s)\,\mathrm{d}s$ is absolutely continuous that $h \in \mathrm{BV}([0,\mathrm{T}])$.

□

Additionally, we use a lemma that provides the lower semi-continuity of convex functionals.

Lemma 2.2. Let $A \subset \mathbb{R}^{d+1}$ be a bounded open set and $f : A \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $d,n,m \geq 1$, a measurable nonnegative function such that $f(\boldsymbol{y},\cdot,\cdot)$ is lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^m$ for a.e. $\boldsymbol{y} \in A$, and f is convex in the last entry. For sequences $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$, $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$, and functions $\boldsymbol{u} \in L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$ and $\boldsymbol{v} \in L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$ with

$$\boldsymbol{u}_k \to \boldsymbol{u} \quad \text{a.e. in } A \quad \text{and} \quad \boldsymbol{v}_k \rightharpoonup \boldsymbol{v} \quad \text{in } L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$$

it holds

$$\liminf_{k\to\infty} \int_A f(\boldsymbol{y},\boldsymbol{u}_k(\boldsymbol{y}),\boldsymbol{v}_k(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} \geq \int_A f(\boldsymbol{y},\boldsymbol{u}(\boldsymbol{y}),\boldsymbol{v}(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}\,.$$

The proof of this assertion can be found in [18].

The following property of $\mathrm{BV}([0,\mathrm{T}])$-functions can for instance be found in [17].

Lemma 2.3. Let $E : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation, $E \in \mathrm{BV}([0,\mathrm{T}])$. Then E is continuous up to a countable subset of $(0,T)$ and the left- and right-limits are uniquely defined in every interior point, *i.e.*,

$$E(t-) = \lim_{s\nearrow t} E(s) \quad E(t+) = \lim_{s\searrow t} E(s) \quad \text{for all } t \in (0,T)$$

and with one-sided limits at the end points. The usual choice are the so-called “cadlag” (continuity a droit limit a gauche) representations by defining $E(t) := E(t+)$.

Lemma 2.4 (Weak and strong continuity)**.** Let $\boldsymbol{v} \in \mathscr{C}_w([0,T];L^2_\sigma(\Omega))$ and $\mathscr{E} : L^2_\sigma(\Omega) \to [0,\infty)$ be given by $\mathscr{E}(\boldsymbol{v}) := \frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)}$. Assume that there exists a function $E \in \mathrm{BV}([0,\mathrm{T}])$ such that $E(s) \geq \mathscr{E}(\boldsymbol{v}(s))$ for all $s \in [0,T]$ and additionally that there exists a $t_0 \in [0,T]$ such that $E(t_0) = \mathscr{E}(\boldsymbol{v}(t_0))$ then the function $\boldsymbol{v}$ is right-hand continuous in t_0 with respect to the strong topology. If t_0 is a continuity point of E, $\boldsymbol{v}$ is continuous in t_0 with respect to the strong topology.

Proof. The left-hand side continuity of E in t_0, the weakly-lower semi continuity of the functional $\mathscr{E}$ together with the weak convergence of $\boldsymbol{v}(s)$ to $\boldsymbol{v}(t_0)$ as $s \searrow t_0$ leads to the chain of inequalities

$$E(t_0) = \lim_{s\searrow t_0} E(s) \geq \liminf_{s\searrow t_0} \mathscr{E}(\boldsymbol{v}(s)) \geq \mathscr{E}(\boldsymbol{v}(t_0)) = E(t_0)\,.$$

From the uniform convexity of $L^2(\Omega)$ together with the weak convergence, we infer that $\lim_{s\searrow t_0}\boldsymbol{v}(s) = \boldsymbol{v}(t_0)$ in the strong topology of $L^2_\sigma(\Omega)$. The same chain of inequalities holds for the right-hand limit $s \nearrow t_0$, in the case that E is continuous in t_0.

□

2.2 Definitions

First we recall the Navier–Stokes and Euler equations,

$$\begin{aligned} \partial_t \boldsymbol{v} + \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{v} &= 0 \qquad \text{in } \Omega\times(0,T)\,, \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 \qquad \text{in } \Omega\,, \\ \nu(I-\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{v} = 0 \quad \text{and} \quad \boldsymbol{n}\cdot\boldsymbol{v} &= 0 \qquad \text{on } \partial\Omega\times(0,T)\,. \end{aligned} \tag{2}$$

By writing the boundary conditions in this way, the system incorporates the Navier–Stokes system with no-slip conditions for $\nu > 0$ and the Euler equations for $\nu = 0$. Indeed, for $\nu > 0$, the tangential and normal part of the velocity field vanish such that this is equivalent to $\boldsymbol{v} = 0$ on $\partial\Omega\times(0,T)$. For the case of $\nu = 0$, *i.e.*, no friction, only the normal component vanishes on the boundary. The underlying natural energy and dissipation spaces are given by $\mathbb{X} = L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega))$ for $\nu > 0$ and $\mathbb{X}_0 = L^\infty(0,T;L^2_\sigma(\Omega))$ for $\nu = 0$ and the space of test-functions is given by $\mathbb{Y} = \mathbb{X}\cap L^2(0,T;H^2(\Omega))\cap L^1(0,T;W^{1,\infty}(\Omega))\cap H^1(0,T;(L^2_\sigma(\Omega))^*)$ for $\nu > 0$ and $\mathbb{Y}_0 = \mathbb{X}_0\cap L^1(0,T;W^{1,\infty}(\Omega))\cap H^1(0,T;(L^2_\sigma(\Omega))^*)$ for $\nu = 0$. The space $\mathbb{Y}$ is chosen smooth enough such that the Stokes operator (for $\nu > 0$) and the convection term map $\mathbb{Y}$ to $L^1(0,T;(L^2_\sigma(\Omega))^*)$. The right-hand side $\boldsymbol{f}$ is assumed to be in $\mathbb{Z}$, where $\mathbb{Z} := L^2(0,T;H^{-1}(\Omega)) + L^1(0,T;L^2(\Omega))$ for $\nu > 0$ and $\mathbb{Z}_0 := L^1(0,T;L^2(\Omega))$ for $\nu = 0$.

To the energy $\mathscr{E} : L^2_\sigma(\Omega) \to \mathbb{R}$ given by $\mathscr{E}(\boldsymbol{v}) = \frac{1}{2}\|\boldsymbol{v}\|^2_{L^2_\sigma(\Omega)}$, we define the relative energy $\mathscr{R} : L^2_\sigma(\Omega)\times L^2_\sigma(\Omega) \to \mathbb{R}_+$ by

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\,, \tag{3a}$$

and the system operator $\mathscr{A}_\nu : \mathbb{Y} \to L^1(0,T;(L^2_\sigma(\Omega))^*)$ via

$$\langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\cdot\rangle = \langle \partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} - \nu\Delta\tilde{\boldsymbol{v}} - \boldsymbol{f},\cdot\rangle\,, \tag{3b}$$

which has to be understood in a weak sense, at least with respect to space.

Note that the system operator does not include boundary conditions, since they are encoded in the underlying spaces. This may change for different boundary conditions.

Definition 2.5. We consider a regularity weight $\mathcal{K} : L^2_\sigma(\Omega) \to [0,\infty]$ and define its domain in a standard way via $\mathcal{D}(\mathcal{K}) := \{\tilde{\boldsymbol{v}} \in \mathbb{X} | \mathcal{K}(\tilde{\boldsymbol{v}}) \in L^1(0,T)_+\}$. We assume that there is a fine enough topology on $\mathcal{D}(\mathcal{K})$ such that $\mathcal{K}$ is continuous and $\mathcal{C}^1([0,T];\mathcal{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d))$ is dense in $\mathcal{D}(\mathcal{K})$ with respect to this topology. We always assume that $\mathcal{K}(0) = 0$.

The form $\mathcal{K}$ is called admissible for $\nu > 0$ if the relative form $\mathcal{W}_\nu : \mathbb{X} \times \mathbb{Y} \to L^1(0,T)$ given by

$$\mathcal{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \nu\|\nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} - \int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + \mathcal{K}(\tilde{\boldsymbol{v}})\mathcal{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) \tag{4}$$

is nonnegative for all $\boldsymbol{v} \in \mathbb{X}$ and all $\tilde{\boldsymbol{v}} \in \mathbb{Y} \cap \mathcal{D}(\mathcal{K})$. Similarly, the form $\mathcal{K}_0$ is called admissible for $\nu = 0$ if the relative form $\mathcal{W}_0 : \mathbb{X}_0 \times \mathbb{Y}_0 \to L^1(0,T)$ given by

$$\mathcal{W}_0(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \int_\Omega (\boldsymbol{v}-\tilde{\boldsymbol{v}})^T \cdot (\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \mathcal{K}_0(\tilde{\boldsymbol{v}})\mathcal{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}). \tag{5}$$

is nonnegative for all $\boldsymbol{v} \in \mathbb{X}_0$ and all $\tilde{\boldsymbol{v}} \in \mathbb{Y}_0 \cap \mathcal{D}(\mathcal{K}_0)$.

Example 2.6. The standard example for a choice for $\mathcal{K}$ are the usual Serrin-type norms:

$$\mathcal{K}(\tilde{\boldsymbol{v}}) = \mathcal{K}^{s,r}_\nu(\tilde{\boldsymbol{v}}) = c\|\tilde{\boldsymbol{v}}\|^s_{L^r(\Omega)} \quad \text{for } \frac{2}{s} + \frac{d}{r} = 1 \tag{6}$$

with $r \in (d,\infty)$ and $s \in (2,\infty)$. Indeed Hölder's, Gagliardo–Nirenberg's, and Young's inequality provide the estimate for $\nu > 0$

$$\begin{aligned}\left|\int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\right| &\le \|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^p(\Omega)}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)} \\ &\le c_p\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^{(1-\alpha)}_{L^2(\Omega)}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^{(1+\alpha)}_{L^2(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)} \\ &\le \frac{\nu}{2}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + c\|\tilde{\boldsymbol{v}}\|^{2/(1-\alpha)}_{L^{2p/(p-2)}(\Omega)}\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)},\end{aligned} \tag{7a}$$

where α is chosen according to Gagliardo–Nirenberg's inequality by

$$\alpha = d(p-2)/2p \quad \text{for} \quad d \le 2p/(p-2).$$

In the case of $\nu = 0$, we may choose $\mathcal{K}_0(\tilde{\boldsymbol{v}}) := \|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{\mathcal{C}(\overline{\Omega})}$ in order to estimate

$$((\boldsymbol{v}-\tilde{\boldsymbol{v}})\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}});(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}) \le 2\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{\mathcal{C}(\overline{\Omega})}\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}. \tag{7b}$$

The estimate (7) imply that $\mathcal{W}_\nu$ is nonnegative.

Remark 2.1. In contrast to previous publications, we want the form $\mathcal{K}$ to be general and not specifically chosen. This makes the solution concept of energy-variational solutions more selective and especially allows the vanishing viscosity limit in the proof of Theorem 2.14. The form $\mathcal{K}$ is chosen in a way that the relative form $\mathcal{W}_\nu$ is nonnegative, convex, and weakly-lower semi-continuous. Indeed, since $\mathcal{W}_\nu$ is quadratic in $\boldsymbol{v}$ and nonnegative, it is a standard matter to prove the convexity of the mapping $\boldsymbol{v} \mapsto \mathcal{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}})$. The mapping $\boldsymbol{v} \mapsto \mathcal{W}_\nu(\boldsymbol{v}|\cdot)$ is continuous in the strong topology in $H^1_{0,\sigma}(\Omega)$ and $L^2_\sigma(\Omega)$ for $\nu > 0$ and $\nu = 0$, respectively. Thus this mapping is weakly-lower semi-continuous (see for instance [13, Chap. 1, Cor. 2.2]).

The assumption on the topology on $\mathcal{D}(\mathcal{K})$ and the continuity of $\mathcal{K}$ is of a technical nature. Instead of the choice $\mathcal{D}(\mathcal{K}) = L^2(0,T;L^\infty(\Omega)\cap H^1_{0,\sigma}(\Omega))$ and $\mathcal{K}(\tilde{\boldsymbol{v}}) = c\|\tilde{\boldsymbol{v}}\|^2_{L^\infty(\Omega)}$ we rather chose the finer topology of $\mathscr{D}(\mathscr{K}) = L^2(0,T;\mathscr{C}_0(\Omega)\cap H^1_{0,\sigma}(\Omega))$ and $\mathscr{K}(\tilde{\boldsymbol{v}}) = c\|\tilde{\boldsymbol{v}}\|_{\mathscr{C}(\Omega)}$ in the case $\nu > 0$. It would also be possible to choose some intermediate separable space allowing jumps (see [29, Example 1.4.10]). For $\nu = 0$, we choose instead of $\mathscr{D}(\mathscr{K}) = L^1(0,T;W^{1,\infty}(\Omega)\cap L^2_\sigma(\Omega))$ and $\mathscr{K}(\tilde{\boldsymbol{v}}) = 2\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{L^\infty(\Omega)}$ the finer topology of $\mathscr{D}(\mathscr{K}) = L^1(0,T;\mathscr{C}^1_0(\Omega)\cap L^2_\sigma(\Omega))$ and $\mathscr{K}(\tilde{\boldsymbol{v}}) = 2\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{\mathscr{C}(\overline{\Omega})}$. These finer choices allow us to use the approximation property by density arguments. But also the case of the coarser topologies and associated L^∞-norms could be made rigorous by an adapted method.

Definition 2.7 (energy-variational solution)**.** A pair $(\boldsymbol{v},E)$ is called an energy-variational solution if $(\boldsymbol{v},E)\in\mathbb{X}\cap\mathscr{C}_w([0,T];L^2_\sigma(\Omega))\times\mathrm{BV}([0,\mathrm{T}])$ and $E(t)\geq\mathscr{E}(\boldsymbol{v}(t))$ for all $t\in[0,T]$, and for all admissible forms $\mathscr{K}:\mathbb{X}\supset\mathscr{D}(\mathscr{K})\to L^1(0,T)_+$ according to Definition 2.5, the relative energy inequality

$$
\begin{aligned}
&\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t)) + E(t) - \mathscr{E}(\boldsymbol{v}(t))\\
&\qquad + \int_s^t \mathscr{W}_\nu(\boldsymbol{v},\tilde{\boldsymbol{v}}) + \langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) + E - \mathscr{E}(\boldsymbol{v})\right]\mathrm{d}\tau\\
&\qquad\qquad \leq \mathscr{R}(\boldsymbol{v}(s)|\tilde{\boldsymbol{v}}(s)) + E(s-) - \mathscr{E}(\boldsymbol{v}(s))
\end{aligned}
\tag{8}
$$

holds for all $t > s \in [0,T]$ and for all $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$. The initial value $\boldsymbol{v}(0) = \boldsymbol{v}_0$ is attained in the weak sense.

Remark 2.2 (Reformulation)*.* Inserting the definition of the system operator $\mathscr{A}_\nu$, we infer the reduced relative energy inequality

$$
\left[E - (\boldsymbol{v},\tilde{\boldsymbol{v}})\right]\Big|_{s-}^{t} + \int_s^t \nu\,(\nabla\boldsymbol{v};\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}) + (\boldsymbol{v}\otimes\boldsymbol{v};\nabla\tilde{\boldsymbol{v}}) + (\boldsymbol{v},\partial_t\tilde{\boldsymbol{v}}) + \mathscr{K}(\tilde{\boldsymbol{v}})\left[\mathscr{E}(\boldsymbol{v}) - E\right]\mathrm{d}\tau \leq 0 \tag{9}
$$

for all $t > s \in [0,T]$ and for all $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$.

Remark 2.3 (Properties of energy-variational solutions)*.* By the Definition 2.7 it is immediately clear that any energy-variational solution on an interval $[0,T]$ is also an energy-variational solution on any sub interval (s,t) for all $s,\, t\in(0,T)$. Furthermore, let $(\boldsymbol{v}^1,\xi^1)$ be an energy-variational solution on the interval $(0,t)$ and $(\boldsymbol{v}^2,E^2)$ an energy-variational solution on the interval (t,T) with $\boldsymbol{v}^2(t) = \boldsymbol{v}^1(t)$ and $E^2(t)\leq E^1(t)$. Then the concatenation of $(\boldsymbol{v}^1,E^1)$ by $(\boldsymbol{v}^2,E^2)$, *i.e.*, the function $(\boldsymbol{v},E)$ given by

$$
\begin{cases}
(\boldsymbol{v}(t),E(t)) = (\boldsymbol{v}^1(t),E^1(t)) & \text{for } t\in[0,t)\\
(\boldsymbol{v}(t),E(t)) = (\boldsymbol{v}^2(t),E^2(t)) & \text{for } t\in[t,T]
\end{cases}
$$

is again an energy-variational solution on $[0,T]$. This is the new key ingredient in comparisson to the previously introduced dissipative solutions [23].

Remark 2.4 (Comparison to dissipative solutions)*.* Another difference of the proposed energy-variational solution framework in comparison to dissipative solutions lies in the definition of the relative form $\mathscr{W}_\nu$. In dissipative solution concepts, the terms in the relative dissipation were only estimated from below by zero (see [26] and [23]). The new insight is that these terms in $\mathscr{W}_\nu$ can be kept and do not have to be estimated. This also leads to the fact that the relative energy inequality is actually an equality for smooth solutions. Indeed in this case the energy inequality (10) is an equality and thus also the relative energy inequality becomes an equality. Furthermore, for the regularity weight $\mathscr{K}$, we allow a family of functions. Finally, the introduction of the auxiliary variable E allows to write down the relative energy inequality before applying the Gronwall argument and formulating the relative energy inequality on any sub-interval of $[0,T]$.

Corollary 2.8 (Refinement of dissipative solutions). Let $(\boldsymbol{v},E)$ be an energy-variational solution according to Definition 2.7 with $E(0)=\mathscr{E}(\boldsymbol{v}_0)$. Then $\boldsymbol{v}$ is a dissipative solution, i.e. for an admissible $\mathscr{K}\geq 0$ according to Definition 2.5, it holds that

$$\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t))+\int_0^t \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle e^{\int_s^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}\tau \leq \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0))e^{\int_0^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}s},$$

for a.e. $t\in(0,T)$ and for all $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$. This is the definition according to Lions (see [26, Sec. 4.4]). This implies that in the case $E(0)=\mathscr{E}(\boldsymbol{v}_0)$, energy-variational solutions fulfill the so-called weak-strong uniqueness property. If a strong solution exists locally-in-time, every energy-variational solution coincides with this strong solution as long as the latter exists.

Proof. Let $(\boldsymbol{v},E)$ be a energy-variational solution according to Definition 2.7. From the condition on the initial values, we infer

$$\begin{aligned}\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t))+E(t)-\mathscr{E}(\boldsymbol{v}(t))+\int_0^t \mathscr{W}_\nu(\boldsymbol{v},\tilde{\boldsymbol{v}})+\langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}\tau&\\ \leq \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0))+\int_0^t \mathscr{K}(\tilde{\boldsymbol{v}})[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]\,\mathrm{d}\tau&\end{aligned}$$

for a.e. $t\in(0,T)$ and for all $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$. Gronwall's inequality and the property that $E-\mathscr{E}(\boldsymbol{v})\geq 0$ as well as $\mathscr{W}_\nu\geq 0$, implies the assertion. □

Definition 2.9 (weak solution). A function $\boldsymbol{v}$ is called a weak solution with the strong energy inequality if $\boldsymbol{v}\in\mathbb{X}$ fulfills the strong energy inequality

$$\frac{1}{2}\|\boldsymbol{v}\|_{L^2(\Omega)}^2\Big|_s^t+\int_s^t \nu\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2\,\mathrm{d}\tau\leq\int_s^t\langle\boldsymbol{f},\boldsymbol{v}\rangle\,\mathrm{d}\tau\quad\text{for a.e. } s<t\in(0,T) \tag{10}$$

and the weak formulation

$$-\int_0^T\int_\Omega \boldsymbol{v}\partial_t\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\int_\Omega(\nu\nabla\boldsymbol{v}:\nabla\boldsymbol{\varphi}-(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\int_0^T\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,\mathrm{d}t+\int_\Omega\boldsymbol{v}_0\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}\boldsymbol{x} \tag{11}$$

for every $\boldsymbol{\varphi}\in\mathscr{C}_c^1([0,T))\otimes\mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^d)$.

2.3 Main results

The main results of the paper at hand are the following.

Proposition 2.10. Let $\boldsymbol{v}$ be a weak solution according to Definition 2.9. Then there exists an energy-variational solution $(\boldsymbol{u},E)$ according to Definition 2.7 such that $\boldsymbol{v}=\boldsymbol{u}$ a.e. in $\Omega\times(0,T)$ with $E(t)=\mathscr{E}(\boldsymbol{u}(t))$ for a.e. $t\in(0,T)$.

Proposition 2.11. Let $(\boldsymbol{v},E)\in\mathbb{X}\cap\mathscr{C}_w([0,T];L^2_\sigma(\Omega))\times\mathrm{BV}([0,\mathrm{T}])$ be an energy-variational solution solution according to Definition 2.7. Assume that the regularity measure $\mathscr{K}$ is homogeneous of rank one, *i.e.,* $\mathscr{K}(\alpha\tilde{\boldsymbol{v}})=\alpha\mathscr{K}(\tilde{\boldsymbol{v}})$ for all $\alpha\in[0,\infty)$.

Then it holds that

$$E\Big|_q^r+\int_q^r \nu\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2-\langle\boldsymbol{f},\boldsymbol{v}\rangle\,\mathrm{d}\tau\leq 0\quad\text{for a.e. } q<r\in(s,t) \tag{12a}$$

and

$$\int_s^t (\nu (\nabla \boldsymbol{v}, \nabla \tilde{\boldsymbol{v}}) - (\boldsymbol{v} \otimes \boldsymbol{v}; \nabla \tilde{\boldsymbol{v}}) - (\boldsymbol{v}, \partial_t \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} \rangle) \, \mathrm{d}\tau \in B\left(0, \int_s^t \mathscr{K}(\tilde{\boldsymbol{v}}) [E - \mathscr{E}(\boldsymbol{v})] \, \mathrm{d}\tau\right) \tag{12b}$$

for all $\tilde{\boldsymbol{v}} \in \mathscr{C}_c^1((s,t); \mathscr{C}_{0,\sigma}^\infty(\Omega))$ and all $s < t \in (0,T)$. Here $B(0,r)$ denotes the ball in $\mathbb{R}$ around 0 with radius r.

Proposition 2.12. Let $(\boldsymbol{v}, E) \in \mathbb{X} \cap \mathscr{C}_w([0,T]; L_\sigma^2(\Omega)) \times \mathrm{BV}([0,\mathrm{T}])$ be an energy-variational solution according to Definition 2.7. Assume that $E(\tau) = \mathscr{E}(\boldsymbol{v}(\tau))$ for a.e. $\tau \in (s,t)$, with $s < t \in (0,T)$. Then it holds that

$$\frac{1}{2} \|\boldsymbol{v}\|_{L^2(\Omega)}^2 \Big|_q^r + \int_q^r \nu \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 - \langle \boldsymbol{f}, \boldsymbol{v} \rangle \, \mathrm{d}\tau \le 0 \quad \text{for a.e. } q < r \in (s,t) \tag{13a}$$

and

$$\int_s^t (\nu (\nabla \boldsymbol{v}, \nabla \tilde{\boldsymbol{v}}) - (\boldsymbol{v} \otimes \boldsymbol{v}; \nabla \tilde{\boldsymbol{v}}) - (\boldsymbol{v}, \partial_t \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} \rangle) \, \mathrm{d}\tau = 0 \tag{13b}$$

for all $\tilde{\boldsymbol{v}} \in \mathscr{C}_c^1((s,t); \mathscr{C}_{0,\sigma}^\infty(\Omega))$.

Remark 2.5. If the auxiliary variable E coincides the the actual energy $\mathscr{E}(\boldsymbol{v})$, then the Navier–Stokes equation is fulfilled in the weak sense.

Proposition 2.13. Let $(\boldsymbol{v}, E)$ be an energy-variational solution according to Definition 2.7. Then the relative energy inequality (8) of Definition 2.7 can be equivalently written as

$$\partial_t (E(t) - (\boldsymbol{v}(t), \tilde{\boldsymbol{v}})) + \nu (\nabla \boldsymbol{v}(t); \nabla \boldsymbol{v}(t) - \nabla \tilde{\boldsymbol{v}}) + (\boldsymbol{v}(t) \otimes \boldsymbol{v}(t); \nabla \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}(t), \boldsymbol{v}(t) - \tilde{\boldsymbol{v}} \rangle \\ + \mathscr{K}(\tilde{\boldsymbol{v}}) \left(\frac{1}{2} \|\boldsymbol{v}(t)\|_{L^2(\Omega)}^2 - E(t) \right) \le 0 . \tag{14}$$

for a.e. $t \in (0,T)$ and all $\tilde{\boldsymbol{v}} \in \mathscr{C}_{0,\sigma}^\infty(\Omega)$. The time derivatives of E has to be understood in the usual $\mathrm{BV}([0,\mathrm{T}])$ sense, the sense of Radon measures and the time-derivative of $\boldsymbol{v}$ in the weak sense.

Theorem 2.14 (Main result)**.** Let $\Omega \subset \mathbb{R}^d$ for $d \ge 2$ be a bounded Lipschitz domain, $\nu \ge 0$. Let $\mathscr{R}$, $\mathscr{W}_\nu$, $\mathscr{K}$, and $\mathscr{A}_\nu$ be given as above in (3) and let $\mathscr{K}$ fulfill Definition 2.5 with $E(0) = \mathscr{E}(\boldsymbol{v}_0)$.

Then there exists at least one energy-variational solution $\boldsymbol{v} \in \mathbb{X}$ to every $\boldsymbol{v}_0 \in L_\sigma^2(\Omega)$ and $\boldsymbol{f} \in \mathbb{Z}$ in the sense of Definition 2.7. The set of solutions $\mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f}) \subset \mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$ consists of the pairs $(\boldsymbol{v}, E)$ being an energy-variational solution according to Definition 2.7 to a given initial-value $\boldsymbol{v}_0 \in L_\sigma^2(\Omega)$ and right-hand side $\boldsymbol{f} \in \mathbb{Z}$. The set $\mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ is convex and weakly*-closed in the topology of $\mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$. Moreover, a set $A \subset \mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ with $\sup_{(\boldsymbol{v},E) \in A} E(0) < \infty$ is compact in this topology.

Additionally, the set-valued mapping $\mathscr{S} : L_\sigma^2(\Omega) \times \mathbb{Z} \to \mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$, which maps $(\boldsymbol{v}_0, \boldsymbol{f})$ to the solution set consisting of elements $(\boldsymbol{v}, E) \in \mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ is continuous in the set valued sense, *i.e.,* if $(\boldsymbol{v}_0^n, \boldsymbol{f}^n) \to (\boldsymbol{v}_0, \boldsymbol{f})$ in $L_\sigma^2(\Omega) \times \mathbb{Z}$, then the associated solutions sets $\mathscr{S}(\boldsymbol{v}_0^n, \boldsymbol{f}^n)$ converge to $\mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ in the Kuratowski sense with respect to the topology induced by the weak*-convergence in $\mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$.

This means, that to every element $(\boldsymbol{v}, E) \in \mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ and every sequence $(\boldsymbol{v}_0^n, \boldsymbol{f}^n) \to (\boldsymbol{v}_0, \boldsymbol{f})$, we may construct a sequence $\{(\boldsymbol{v}^n, E^n)\}$ such that

$$(\boldsymbol{v}^n, E^n) \to (\boldsymbol{v}, E) \quad \text{in } \mathbb{X} \times \mathrm{BV}([0,\mathrm{T}]) \, .$$

(which is referred to as lower semi-continuity), and if there exists a sequence $(\boldsymbol{v}_0^n, \boldsymbol{f}^n) \to (\boldsymbol{v}_0, \boldsymbol{f})$ and a sequence $(\boldsymbol{v}^n, E^n) \overset{*}{\rightharpoonup} (\boldsymbol{v}, E)$ with $(\boldsymbol{v}^n, E^n) \in \mathscr{S}(\boldsymbol{v}_0^n, \boldsymbol{f}^n)$ then $(\boldsymbol{v}, E) \in \mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ (which is referred to es upper semi-continuity).

Remark 2.6 (Initial condition)*.* The initial condition will in fact be attained in the strong sense. Indeed, since the function E can be decomposed into the sum of a monotonously decreasing function and a absolutely continuous function such that $\lim_{s \searrow 0} E(s) \leq E(0)$. Additionally, $\boldsymbol{v}$ is weakly continuous and $\mathscr{E}$ weakly-lower semi-continuous and $E(0) = \mathscr{E}(\boldsymbol{v}(0))$. Together, we infer in the same way as in the proof of Lemma 2.4 that $\lim_{s \searrow 0} \|\boldsymbol{v}(s) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0$.

Remark 2.7*.* In the case of $d = 2, 3$ or 4, the existence of weak solutions to the Navier–Stokes equations is well known (see for instance [33]). Due to Proposition 2.10, this also proves the existence of energy-variational solutions. The new result of the preceding theorem is expanding the existence of energy-variational solutions to any space dimension.

The above definition seems to be suitable to define reasonable selection criteria for solutions. By a designed strictly convex functional one may selects a unique suitable solution. The next proposition even guarantees some kind of well-posedness.

In the following, we denote the set $\mathfrak{S}^{t_0}(\boldsymbol{v}_0, E_0, \boldsymbol{f})$ as the set of energy-variational solutions $(\boldsymbol{v}, E) \in \mathscr{S}(\boldsymbol{v}_0, \boldsymbol{f})$ according to Definition 2.7 for a given value $\boldsymbol{v}(t_0) = \boldsymbol{v}_0$ and right-hand side $\boldsymbol{f}$ such that it holds $E(t_0-) = E_0$.

Proposition 2.15. Let the assumptions of Theorem 2.14 be fulfilled. Let the functional $J : [0,T] \times L^2_\sigma(\Omega) \times [0,\infty)$ be measurable in the first variable and continuous, strictly convex and coercive in the second two variables. We consider the Nemitzkii mapping $(\boldsymbol{v}, E) \mapsto \int_0^T J(t, \boldsymbol{v}, E)\,\mathrm{d}t$ that is continuous with respect to the strong topology and weakly-lower semi-continuous with respect to the weak topology (cf. [13, Chap. I, Cor. 2.2]) on $L^2(0,T; L^2_\sigma(\Omega)) \times L^p(0,T)$ for $p \in [1,\infty)$. Then there exists a unique minimizer $(\boldsymbol{v}^*, E^*)$ of the optimization problem

$$\min_{(\boldsymbol{v},E) \in \mathfrak{S}^{t_0}(\boldsymbol{v}_0, \mathscr{E}(\boldsymbol{v}_0), \boldsymbol{f})} \int_0^T J(t, \boldsymbol{v}, E)\,\mathrm{d}t \tag{15}$$

and the optimization problem is well-posed in the sense that

$$\arg\min_{(\boldsymbol{v},E) \in \mathfrak{S}^0(\boldsymbol{v}_0^n, E_0^n, \boldsymbol{f}^n)} \int_0^T J(t, \boldsymbol{v}, E)\,\mathrm{d}t \overset{*}{\rightharpoonup} \arg\min_{(\boldsymbol{v},E) \in \mathfrak{S}^0(\boldsymbol{v}_0, \mathscr{E}(\boldsymbol{v}_0), \boldsymbol{f})} \int_0^T J(t, \boldsymbol{v}, E)\,\mathrm{d}t$$

in $\mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$ for initial values $\boldsymbol{v}_0^n \to \boldsymbol{v}_0$ in $L^2_\sigma(\Omega)$ and right-hand sides $\boldsymbol{f}^n \to \boldsymbol{f}$ in $\mathbb{Z}$. Note that the minimizer is unique such that the above convergence is actually a convergence of a singleton and not a set. The sequence $\{E_0^n\} \subset [\mathscr{E}(\boldsymbol{v}_0), \infty)$ is such there exists a constant $C > 0$ with $E_0^n - \mathscr{E}(\boldsymbol{v}_0) \leq C\|\boldsymbol{v}_0^n - \boldsymbol{v}_0\|^2_{L^2(\Omega)}$.

The minimizer inherits the semi-flow property in the sense that for a solution $(\boldsymbol{v}, E)$ with

$$(\boldsymbol{v}, E) \in \arg\min_{(\tilde{\boldsymbol{v}}, \tilde{E}) \in \mathfrak{S}^0(\boldsymbol{v}_0, E_0, \boldsymbol{f})} \int_0^T J(t, \tilde{\boldsymbol{v}}, \tilde{E})\,\mathrm{d}t$$

it holds that for the minimizer $(\boldsymbol{u}, F)$ fulfilling

$$(\boldsymbol{u}, F) \in \arg\min_{(\tilde{\boldsymbol{v}}, \tilde{E}) \in \mathfrak{S}^{t_0}(\boldsymbol{v}(t_0), E(t_0), \boldsymbol{f})} \int_{t_0}^T J(t, \tilde{\boldsymbol{v}}, \tilde{E})\,\mathrm{d}t$$

that $(\boldsymbol{v}(t), E(t)) = (\boldsymbol{u}(t), F(t))$ for a.e. $t \in [t_0, T]$.

But the above selection criteria are quite restrictive, since they only allow functionals J to be defined on the whole time domain, *i.e.,* integrated in time. As the next result shows, it seems to be desirable to define point-wise selection criteria, which is possible due to the fact that the solutions are well-defined point-wise in time and not only point-wise almost everywhere.

Proposition 2.16. Let the assumptions of Theorem 2.14 be fulfilled. For any given finite set of points $\{t_1,\dots,t_N\} \subset [0,T]$, there exists an energy-variational solutions $(\boldsymbol{v},E)$ according to Definition 2.7 such that

$$E(t_j) = \mathscr{E}(\boldsymbol{v}(t_j)) \quad \text{for all } j \in \{1,\dots,N\}\,.$$

Additionally it holds that $\boldsymbol{v}$ is left-continuous with respect to the strong topology in every point t_j with $j \in \{1,\dots,N\}$.

The above result only holds in finitely many points. Assuming that it would hold in every point, we end up with the Definition 2.17 of minimal energy-variational solutions.

Definition 2.17 (Minimal energy-variational solution)**.** A pair $(\boldsymbol{v},E)$ is called a minimal energy-variational solution if $(\boldsymbol{v},E)$ is an energy-variational solution according to Definition 2.7 with

$$E(t) \leq \bar{E}(t)$$

for all $t \in [0,T]$ and all energy-variational solutions $(\bar{\boldsymbol{v}},\bar{E}) \in \mathbb{X}$ according to Definition 2.7 for a given initial value $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$ and right-hand side $\boldsymbol{f} \in \mathbb{Z}$ with $E(t-) = \bar{E}(t-)$ and $\boldsymbol{v}(t) = \bar{\boldsymbol{v}}(t)$.

In this article, we do not prove the existence of solutions according to Definition 2.17. We only propose these solutions as a reasonable concept and show that a solution fulfilling this definition is also a weak solution.

Proposition 2.18. Assume that a minimal energy-variational solution $(\boldsymbol{v},E)$ according to Definition 2.17 exists. Then it holds that $E(t) = \mathscr{E}(\boldsymbol{v}(t))$ such that this solution is actually a weak solution according to Definition 2.9, which is a consequence of Proposition 2.12.

We remark again that we do not claim to have proven the existence of weak solutions in any space dimension.

Remark 2.8*.* We note that the Definition 2.17 is well defined. Due to the fact that the function E is a $\mathrm{BV}([0,\mathrm{T}])$ function and thus the limit $\lim_{s\searrow t} E(s) = E(t+) = E(t)$ exists and is unique for all $t \in [0,T)$. Note that due to the inequality (38), only negative jumps with $\lim_{s\searrow t} E(s) = E(t+) = E(t) \leq \lim_{s\nearrow t} E(s) = E(t-)$ are allowed. The increasing contribution to E are only due to the right-hand side $\boldsymbol{f}$ and thus, by construction absolutely continuous. Note also that the pointwise minimization in the case of $t = 0$ immediately implies that $E(0) = 1/2\|\boldsymbol{v}_0\|^2_{L^2(\Omega)}$, since the relative-energy inequality is automatically fulfilled for $t = s = 0$.

The selection criteria can be understood in the way that the selected solution fulfills a minimization problem in every point in time, for every $t \in [0,t_0]$ the solution $(\boldsymbol{v},E)$ solves the minimization problem

$$(\boldsymbol{v}(t),E(t)) = \operatorname{arg\,min}_{\substack{(\boldsymbol{u},F)\in\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})\\ (\boldsymbol{u}(t),F(t-))=(\boldsymbol{v}(t),E(t-))}} F(t)\,. \tag{16}$$

Remark 2.9 (Selection criterion)*.* The proposed selection criterion relies on the insight that a physically relevant solution dissipates energy at the highest rate (see [8] or [9]). This leads to a minimized energy (compare the energy inequality (10), which is formally an equality). In a thermodynamical consistent system, the energy would be constant, but the maximized dissipation leads to a maximized entropy (see [15] for instance). This criterion was introduced as the entropy rate admissibility criterion [9]. There are different works on the entropy rate admissibility criterion applied to different systems. For instance, in the case of scalar conservation laws it was shown that this criterion coincides with the Oleinik-E condition and thus the usual entropy admissibility criterion for solutions with finitely many shocks (see [9] or [7, Thm. 9.7.2] for the result). Since this criterion was proven to select the physically relevant solution in these scarcely available examples of nonlinear PDEs that are well understood, it may also does this for more involved systems (like the ones we consider here).

3 Proofs of the main theorems

3.1 Energy-variational and weak solutions

First, we show that the velocity $\boldsymbol{v}$ of a weak solution is an energy-variational solution.

Proof of Proposition 2.10. Let $\boldsymbol{v}$ be a weak solution to the Navier–Stokes and Euler equations (2) with strong energy inequality for $\nu \geq 0$.

For a test function $\tilde{\boldsymbol{v}} \in \mathbb{Y}$, we find by testing the system operator $\mathscr{A}_\nu(\tilde{\boldsymbol{v}})$ by $\phi\tilde{\boldsymbol{v}}$ with $\phi \in \mathscr{C}_c^1([0,T))$ and standard calculations that

$$\int_0^T \phi \, \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \tilde{\boldsymbol{v}} \rangle \, \mathrm{d}t = \\ - \int_0^T \phi' \frac{1}{2} \|\tilde{\boldsymbol{v}}(t)\|_{L^2(\Omega)}^2 \, \mathrm{d}t + \int_0^T \phi \left(\nu \|\nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 - \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} \rangle \right) \mathrm{d}t - \phi(0) \frac{1}{2} \|\tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2 . \quad (17)$$

Testing again the system operator $\mathscr{A}_\nu(\tilde{\boldsymbol{v}})$ by $\phi\boldsymbol{v}$ and choosing $\boldsymbol{\varphi}$ to be $\phi\tilde{\boldsymbol{v}}$ in (11) with $\phi \in \mathscr{C}_c^1([0,T))$ (or approximate it appropriately), we find

$$- \int_0^T \phi' \int_\Omega \boldsymbol{v} \cdot \tilde{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \phi \left(\int_\Omega (2\nu \nabla \boldsymbol{v} : \nabla \tilde{\boldsymbol{v}} - (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}} \cdot \nabla) \tilde{\boldsymbol{v}} \cdot \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t \\ = \int_0^T \phi \, \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} \rangle \, \mathrm{d}t + \phi(0) \int_\Omega \boldsymbol{v}_0 \cdot \tilde{\boldsymbol{v}}(0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \phi \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} + \boldsymbol{v} \rangle \, \mathrm{d}t . \quad (18)$$

Reformulating (10) by Lemma 2.1, adding (17), as well as subtracting (18), let us deduce that

$$- \int_0^T \phi' \frac{1}{2} \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 \, \mathrm{d}t + \nu \int_0^T \phi \|\nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 \, \mathrm{d}t - \phi(0) \frac{1}{2} \|\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2 \\ - \int_0^T \phi \left(\int_\Omega ((\tilde{\boldsymbol{v}} \cdot \nabla) \tilde{\boldsymbol{v}} \cdot \boldsymbol{v} - (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \tilde{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t + \int_0^T \phi \, \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \, \mathrm{d}t \leq 0 \quad (19)$$

for all $\phi \in C_c^1([0,T])$ with $\phi \geq 0$ a.e. on $(0,T)$. We adopt some standard manipulations using the skew-symmetry of the convective term in the last two arguments and the fact that $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are divergence free, to find

$$- \int_\Omega ((\tilde{\boldsymbol{v}} \cdot \nabla) \tilde{\boldsymbol{v}} \cdot \boldsymbol{v} - (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \tilde{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{x} = - \int_\Omega ((\boldsymbol{v} \cdot \nabla)(\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}} \cdot \nabla) \tilde{\boldsymbol{v}} \cdot (\boldsymbol{v} - \tilde{\boldsymbol{v}})) \, \mathrm{d}\boldsymbol{x}$$

$$= -\int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}$$

for $\nu > 0$ and

$$-\int_\Omega ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}-(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\boldsymbol{v}-\tilde{\boldsymbol{v}})^T\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} \tag{20}$$

$$\int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\otimes\tilde{\boldsymbol{v}}):\nabla\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}+\int_\Omega(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}$$

$$= \int_\Omega (\boldsymbol{v}-\tilde{\boldsymbol{v}})^T\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} \tag{21}$$

for $\nu = 0$. Inserting this into (19), adding as well as subtracting $\mathscr{K}_\nu(\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})$, we conclude

$$-\int_0^T \phi'\frac{1}{2}\|\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T \phi\left[\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}})+\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle-\mathscr{K}(\tilde{\boldsymbol{v}})\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\mathrm{d}s \le 0$$

for every function $\tilde{\boldsymbol{v}} \in \mathbb{Y}\cap\mathscr{D}(\mathscr{K})$ and all $\phi \in C^1_c((0,T))$ with $\phi \ge 0$ a.e. on $(0,T)$. After applying Lemma 2.1, we may choose the variable E such that $E \ge \mathscr{E}(\boldsymbol{v})$ a.e. in $(0,T)$ and (8) is fulfilled everywhere. □

Proof of Proposition 2.11. We assume that $(\boldsymbol{v},E) \in \mathbb{X}_\nu \cap \mathscr{C}_w([0,T];L^2_\sigma(\Omega)) \times \mathrm{BV}([0,\mathrm{T}])$ is an energy-variational solution according to Definition 2.7. Firstly, we observe that the relative energy inequality (8) with $\tilde{\boldsymbol{v}} = 0$ gives the energy inequality (12a).

Secondly, we infer from the relative energy inequality (8) and Lemma 2.1 that

$$-\int_s^t \phi'[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]+\phi\mathscr{K}(\tilde{\boldsymbol{v}})[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]\,\mathrm{d}\tau + \int_s^t \phi\,[\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}})+\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle]\,\mathrm{d}\tau \le 0$$

for all $\phi \in C^1_c(s,t)$ with $\phi \ge 0$ a.e. on (s,t). The Definition of $\mathscr{W}_\nu$ implies

$$\begin{aligned}&-\int_s^t \phi'[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]+\phi\mathscr{K}(\tilde{\boldsymbol{v}})\,[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]\,\mathrm{d}\tau\\&+\int_s^t \phi\left[\nu\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}-(((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}})+\mathscr{K}(\tilde{\boldsymbol{v}})\,\mathbb{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}\tau \le 0\end{aligned} \tag{22a}$$

for $\nu > 0$ and

$$\begin{aligned}&-\int_s^t \phi'[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]+\phi\mathscr{K}_0(\tilde{\boldsymbol{v}})\,[\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+E-\mathscr{E}(\boldsymbol{v})]\,\mathrm{d}t\\&+\int_s^t \phi\,[((\boldsymbol{v}-\tilde{\boldsymbol{v}})\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}}),(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}})+\mathscr{K}_0(\tilde{\boldsymbol{v}})\,\mathbb{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})+\langle\mathscr{A}_0(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle]\,\mathrm{d}\tau \le 0\end{aligned} \tag{22b}$$

for $\nu = 0$. For the system operator $\mathscr{A}_\nu$, we find

$$\int_s^t \phi \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle \,\mathrm{d}\tau = \int_s^t \phi' \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ + \int_s^t \phi \left[(\partial_t \tilde{\boldsymbol{v}}, \boldsymbol{v}) + \nu (\nabla \tilde{\boldsymbol{v}}, \nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}) - ((\tilde{\boldsymbol{v}}\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}\tau$$

for all $\phi \in \mathscr{C}^1_c((s,t))$ with $\phi \geq 0$ a.e. on (s,t). Inserting this into (22), we may deduce

$$-\int_s^t \phi' \left[\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)} - (\boldsymbol{v},\tilde{\boldsymbol{v}}) + E - \mathscr{E}(\boldsymbol{v})\right] + \phi \mathscr{K}(\tilde{\boldsymbol{v}})\left[E-\mathscr{E}(\boldsymbol{v})\right]\mathrm{d}\tau \\ + \int_s^t \phi\left[\nu(\nabla\boldsymbol{v},\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}) + (\boldsymbol{v}\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}}),\nabla\tilde{\boldsymbol{v}}) + (\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}) - \langle \boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}\tau \leq 0\,. \quad (23)$$

Again the skew-symmetry of the trilinear form in the last two entries is used. Choosing $\tilde{\boldsymbol{v}} = \alpha\tilde{\boldsymbol{u}}$ and multiplying the inequality by $1/\alpha$ for $\alpha > 0$, we find

$$\frac{1}{\alpha}\left(-\int_s^t \phi'\left[\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)} + E - \mathscr{E}(\boldsymbol{v})\right]\mathrm{d}\tau + \int_s^t \phi\left[\nu\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} - \langle\boldsymbol{f},\boldsymbol{v}\rangle\right]\mathrm{d}\tau\right) \\ + \int_s^t (\boldsymbol{v},\partial_t(\phi\tilde{\boldsymbol{u}}))\,\mathrm{d}\tau - \int_s^t \phi\left[\nu(\nabla\boldsymbol{v},\nabla\tilde{\boldsymbol{u}}) - ((\boldsymbol{v}\otimes\boldsymbol{v}),\nabla\tilde{\boldsymbol{u}}) - \langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle\right]\mathrm{d}\tau \\ \leq \int_s^t \phi\mathscr{K}(\tilde{\boldsymbol{u}})(E-\mathscr{E}(\boldsymbol{v}))\,\mathrm{d}\tau\,. \quad (24)$$

Note that the term $((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}})$ vanishes since $\boldsymbol{v}$ is solenoidal. Additionally, it is used that $\mathscr{K}$ is homogeneous of rank one. For $\alpha\to\infty$ the first line in (24) vanishes and in the resulting inequality we may observe that $\tilde{\boldsymbol{u}}$ occurs linearly such that by inserting $\tilde{\boldsymbol{u}}$ as well as $-\tilde{\boldsymbol{u}}$, we receive two inequalities,

$$-\int_s^t \phi\mathscr{K}(\tilde{\boldsymbol{u}})(E-\mathscr{E}(\boldsymbol{v}))\,\mathrm{d}\tau \\ \leq \int_s^t (\boldsymbol{v},\partial_t(\tilde{\boldsymbol{u}}\phi))\,\mathrm{d}t - \int_s^t \phi\left[\nu(\nabla\boldsymbol{v},\nabla\tilde{\boldsymbol{u}}) - ((\boldsymbol{v}\otimes\boldsymbol{v}),\nabla\tilde{\boldsymbol{u}}) - \langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle\right]\mathrm{d}\tau \\ \leq \int_s^t \phi\mathscr{K}(\tilde{\boldsymbol{u}})(E-\mathscr{E}(\boldsymbol{v}))\,\mathrm{d}\tau\,. \quad (25)$$

By defining $\tilde{\boldsymbol{v}} = -\phi\tilde{\boldsymbol{u}}$, we may observe the formulation (12b).

□

Proof of Proposition 2.12. This proof is very similar to the previous proof. All arguments in the previous proof up to the inequality (23) are independent of the rank-1-homogeneity of $\mathscr{K}$. Thus choosing $\tilde{\boldsymbol{v}} = \alpha\tilde{\boldsymbol{u}}$ in (23) and multiplying by $1/\alpha$ implies (24) with $E \equiv \mathscr{E}(\boldsymbol{v})$. Thus for $\alpha\to\infty$, we infer due to the linearity of the test function $\tilde{\boldsymbol{u}}$ the relation (12b) with $E\equiv\mathscr{E}(\boldsymbol{v})$, which is nothing else than (13b). □

Proof of Proposition 2.13. We may consider the reduced relative energy inequality (9) for $t = t+h$ and $s = t$ and multiply the inequality by $1/h$. Taking the limit $h \searrow 0$ in the resulting inequality, we infer that

$$\frac{1}{h}\int_t^{t+h} \nu(\nabla\boldsymbol{v};\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}) + (\boldsymbol{v}\otimes\boldsymbol{v};\nabla\tilde{\boldsymbol{v}}) + (\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}) - \mathscr{K}(\tilde{\boldsymbol{v}})\left[E-\mathscr{E}(\boldsymbol{v})\right]\mathrm{d}\tau \longrightarrow \\ \nu(\nabla\boldsymbol{v}(t);\nabla\boldsymbol{v}(t)-\nabla\tilde{\boldsymbol{v}}(t)) + (\boldsymbol{v}(t)\otimes\boldsymbol{v}(t);\nabla\tilde{\boldsymbol{v}}(t)) + (\partial_t\tilde{\boldsymbol{v}}(t),\boldsymbol{v}(t)) - \mathscr{K}(\tilde{\boldsymbol{v}}(t))\left[E(t)-\mathscr{E}(\boldsymbol{v}(t))\right]$$

for a.e. $t\in(0,T)$ since all appearing terms are Lebesgue integrable.

Since E is a $\mathrm{BV}([0,\mathrm{T}])$ function, its derivative is a Radon measure (see [27, Chap. 8, Sec. 8], or [17, Thm. 2.13]). For the mixed term in the relative energy, we infer by the product rule for weak derivatives that

$$\lim_{h\searrow 0}\frac{1}{h}\left[(\boldsymbol{v}(t+h),\tilde{\boldsymbol{v}}(t+h))-(\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))\right]=\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))=(\partial_t\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))+(\boldsymbol{v}(t),\partial_t\tilde{\boldsymbol{v}}(t)) \tag{26}$$

for a.e. $t\in(0,T)$. Note that the weak derivative $\partial_t\boldsymbol{v}$ exists. Indeed, by choosing the regularity weight $\mathscr{K}(\tilde{\boldsymbol{v}}):=\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{\mathscr{C}(\overline{\Omega})}$, which is homogeneous of rank one, we may find by Proposition 2.11 that (12b) and especially (25) in the proof of Proposition 2.11 holds. In inequality (25), we may define the linear form $\boldsymbol{l}:\mathscr{C}_0^1((0,T);H^1_{0,\sigma}(\Omega)\cap\mathscr{C}_0^1(\Omega;\mathbb{R}^d))\to\mathbb{R}$ by

$$\begin{aligned}\langle\boldsymbol{l},\tilde{\boldsymbol{v}}\rangle:=&\int_0^T(\boldsymbol{v},\partial_t\tilde{\boldsymbol{v}})\,\mathrm{d}t\\ \le&\int_0^T\nu(\nabla\boldsymbol{v};\nabla\tilde{\boldsymbol{v}})-(\boldsymbol{v}\otimes\boldsymbol{v};\nabla\tilde{\boldsymbol{v}})-\langle\boldsymbol{f},\tilde{\boldsymbol{v}}\rangle+\mathscr{K}(\tilde{\boldsymbol{v}})[\mathscr{E}(\boldsymbol{v})-E]\,\mathrm{d}t\\ \le&-\frac{\nu}{2}\|\boldsymbol{v}\|^2_{L^2(0,T;H^1_{0,\sigma}(\Omega))}+\frac{\nu}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(0,T;H^1_{0,\sigma}(\Omega))}+\|\boldsymbol{v}\|^2_{L^\infty(0,T;L^2_\sigma(\Omega))}\|\tilde{\boldsymbol{v}}\|_{L^1(0,T;W^{1,\infty})}\\ &+\|\boldsymbol{f}\|_{\mathbb{Z}}\left(\|\boldsymbol{v}\|_{L^\infty(0,T;L^2_\sigma(\Omega))}+\|\boldsymbol{v}\|_{L^2(0,T;H^1_{0,\sigma}(\Omega))}\right)+\|\tilde{\boldsymbol{v}}\|_{L^1(0,T;\mathscr{C}_0^1(\Omega))}\|E-\mathscr{E}(\boldsymbol{v})\|_{L^\infty(0,T)}\,.\end{aligned}$$

By Hahn-Banach's theorem [5, Thm. 1.1], there exists an element

$$\partial_t\boldsymbol{v}\in(L^1(0,T;\mathscr{C}_0^1(\Omega))\cap L^2(0,T;H^1_{0,\sigma}(\Omega)))^*\subset L^2(0,T;(H^1_{0,\sigma}(\Omega)\cap W^{2,p}(\Omega))^*)$$

for some $p>d$, which agrees with the definition of the weak solution.

Thus, for a.e. $t\in(0,T)$, we may identify

$$\lim_{h\searrow 0}\frac{1}{h}\left[E(t+h)-E(t)-(\boldsymbol{v}(t+h),\tilde{\boldsymbol{v}}(t+h))+(\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))\right]=\frac{\mathrm{d}}{\mathrm{d}t}\left[E(t)-(\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))\right]$$

which is well-defined a.e. in $(0,T)$ in the sense of Radon measures (see [27, 17]). Note that the pointwise inequality is equivalent to the inequality in the distributional sense, *i.e.,*

$$\begin{aligned}&-\int_0^T\phi'\left[E(t)-(\boldsymbol{v}(t),\tilde{\boldsymbol{v}}(t))\right]\mathrm{d}t\\ &\quad+\int_0^T\phi\left[\nu\left(\nabla\boldsymbol{v}(t);\nabla\boldsymbol{v}(t)-\nabla\tilde{\boldsymbol{v}}(t)\right)+(\boldsymbol{v}(t)\otimes\boldsymbol{v}(t);\nabla\tilde{\boldsymbol{v}}(t))+(\partial_t\tilde{\boldsymbol{v}}(t),\boldsymbol{v}(t))\right]\mathrm{d}t\\ &\quad\quad-\int_0^T\phi\,\mathscr{K}(\tilde{\boldsymbol{v}}(t))\left[E(t)-\mathscr{E}(\boldsymbol{v}(t))\right]\mathrm{d}t\le 0\end{aligned}$$

for all $\phi\in\mathscr{C}_c^\infty(0,T)$ with $\phi\ge 0$ on $(0,T)$.

□

3.2 Existence of energy-variational solutions

In order to prove existence of energy-variational solutions, we pass to the limit in the relative energy inequality. Therefore, we do not need any strong compactness arguments, which are essential in existence proofs for weak solutions to nonlinear PDEs. The formulation of the relative energy inequality allows to pass to the limit only relying on weakly-lower semi-continuity of the associated functionals and Helly's selection principle.

Proof of Theorem 2.14. The proof is based on the usual Galerkin approximation together with standard weak convergence techniques. We divide the proof in different steps.

Step 1, Galerkin approximation: Since the space $H^1_{0,\sigma}(\Omega)$ is separable and the space of smooth solenoidal functions with compact support, $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d)$, is dense in $H^1_{0,\sigma}(\Omega)$, there exists a Galerkin scheme of $H^1_{0,\sigma}(\Omega)$, *i.e.*, $\{W_n\}_{n\in\mathbb{N}}$ with $\operatorname{clos}_{H^1_{0,\sigma}(\Omega)}(\lim_{n\to\infty} W_n) = H^1_{0,\sigma}(\Omega)$. Let $P_n : L^2_\sigma(\Omega) \longrightarrow W_n$ denote the $L^2_\sigma(\Omega)$-orthogonal projection onto W_n. The approximate problem is then given as follows: Find an absolutely continuous solution $\boldsymbol{v}^n$ with $\boldsymbol{v}^n(t) \in W_n$ for all $t \in [0,T]$ solving the system

$$(\partial_t \boldsymbol{v}^n + (\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n, \boldsymbol{w}) + \nu\,(\nabla\boldsymbol{v}^n;\nabla\boldsymbol{w}) = \langle \boldsymbol{f},\boldsymbol{w}\rangle\,, \quad \boldsymbol{v}^n(0) = P_n\boldsymbol{v}_0 \quad \text{for all } \boldsymbol{w}\in W_n\,. \tag{27}$$

A classical existence theorem (see Hale [16, Chapter I, Theorem 5.2]) provides, for every $n\in\mathbb{N}$, the existence of a maximal extended solution to the above approximate problem (27) on an interval $[0,T_n)$ in the sense of Carathéodory.

Step 2, A priori estimates: It can be deduce that $T_n = T$ for all $n\in\mathbb{N}$ if the solution undergoes no blow-up. With the standard *a priori* estimates, we can exclude blow-ups and thus deduce global-in-time existence. Testing (27) by $\boldsymbol{v}^n$, we derive the standard energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^n\|^2_{L^2(\Omega)} + \nu\|\nabla\boldsymbol{v}^n\|^2_{L^2(\Omega)} = \langle \boldsymbol{f},\boldsymbol{v}^n\rangle\,. \tag{28}$$

For $\boldsymbol{f}\in\mathbb{Z} = L^2(0,T;H^{-1}(\Omega))\oplus L^1(0,T;L^2(\Omega))$ for $\nu>0$, the right-hand side can be estimated appropriately. Indeed, there exist two functions $\boldsymbol{f}_1\in L^2(0,T;H^{-1}(\Omega))$ and $\boldsymbol{f}_2\in L^1(0,T;L^2(\Omega))$ such that we may estimate with Hölder's, Young's, and Poincaré's inequality that

$$\langle \boldsymbol{f},\boldsymbol{v}^n\rangle \le \frac{\nu}{2}\|\nabla\boldsymbol{v}^n\|^2_{L^2(\Omega)} + \frac{C}{2\nu}\|\boldsymbol{f}_1\|^2_{H^{-1}(\Omega)} + \|\boldsymbol{f}_2\|_{L^2(\Omega)}\left(\|\boldsymbol{v}^n\|^2_{L^2(\Omega)}+1\right). \tag{29}$$

Inserting this into (28) allows to apply a version of Gronwall's Lemma in order to infer that $\{\boldsymbol{v}^n\}$ is bounded and thus weakly* compact in $\mathbb{X}$ such that there exists a $\boldsymbol{v}\in\mathbb{X}$ with

$$\boldsymbol{v}^n \overset{*}{\rightharpoonup} \boldsymbol{v} \quad \text{in } \mathbb{X}\,. \tag{30}$$

From (28), we observe

$$\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^n\|^2_{L^2(\Omega)}\right|\mathrm{d}t \le 2\int_0^T \nu\|\nabla\boldsymbol{v}^n\|^2_{L^2(\Omega)} + |\langle \boldsymbol{f},\boldsymbol{v}^n\rangle|\,\mathrm{d}t$$

and by the boundedness of the sequence $\{\boldsymbol{v}^n\}$ in $\mathbb{X}$ as well as (29) that the sequence of functions on $[0,T$, $\{\|\boldsymbol{v}^n\|^2_{L^2(\Omega)}\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}([0,\mathrm{T}])$. By Helly's selection principle, we may infer that there exists a function $E\in\mathrm{BV}([0,\mathrm{T}])$ such that

$$\frac{1}{2}\|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)} \to E(t) \quad \text{for all } t\in(0,T)\,. \tag{31}$$

Step 3, Discrete relative energy inequality: In order to show the convergence to energy-variational solutions, we derive a discrete version of the relative energy inequality. Assume $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$. Adding (28) and (27) tested with $-P_n\tilde{\boldsymbol{v}}$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^n\|^2_{L^2(\Omega)} + \nu\,(\nabla\boldsymbol{v}^n;\nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}) = \langle \boldsymbol{f},\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle + (\partial_t\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}})\,. \tag{32}$$

For the system operator $\mathscr{A}_\nu$, we observe that

$$\langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle = \\ (\partial_t P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|P_n\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \nu\,(\nabla P_n\tilde{\boldsymbol{v}}, \nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) - \langle \boldsymbol{f}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle\,.$$

Adding to as well as subtracting from (32) the term $\langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle$ leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \Big(\nu\|\nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle\Big) \\ = \Big(((\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n)\Big)\,. \tag{33}$$

By some algebraic transformations, we find

$$((\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) \\ = (((\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) \\ + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}})\,. \tag{34}$$

For the first term on the right-hand side of (34), we observe

$$\nu\|\nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} - (((\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) = \mathscr{W}_\nu(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) - \mathscr{K}(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})\,.$$

For the second term on the right-hand side of (34), we find with an integration-by-parts (or the usual skew-symmetry in the second two variables of the trilinear convection term) that

$$((P_n\tilde{\boldsymbol{v}}\cdot\nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) = 0\,.$$

In order to find the discrete version of the relative energy inequality, the term $\mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}|P_n\tilde{\boldsymbol{v}})$ is added and subtracted to (33) and the resulting equality is integrated over (s,t) such that

$$\mathscr{R}(\boldsymbol{v}^n(t)|P_n\tilde{\boldsymbol{v}}(t)) + \int_s^t [\mathscr{W}_\nu(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle - \mathscr{K}(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})]\,\mathrm{d}s \\ = \mathscr{R}(\boldsymbol{v}^n(s)|P_n\tilde{\boldsymbol{v}}(s)) \tag{35}$$

for a.e. $s, t \in (0,T)$ and $\nu > 0$.

Step 4, Passage to the limit: Via Lemma 2.1, the equality (35) may be relaxed to an inequality and written as

$$-\int_0^T \phi'\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})\,\mathrm{d}s \\ + \int_0^T \phi\,[\mathscr{W}_\nu(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle - \mathscr{K}(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})]\,\mathrm{d}s \le 0$$

for all $\phi \in C^1_c(0,T)$ with $\phi \ge 0$ a.e. on $(0,T)$. Since $\mathscr{C}^1([0,T];\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d))$ is also dense in $\mathbb{Y}$, we may observe the strong convergence of the projection P_n, *i.e.*,

$$\|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;H^1_{0,\sigma}(\Omega))} + \|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{d/2}(\Omega))} \to 0 \quad \text{as } n\to\infty \quad \text{for all } \tilde{\boldsymbol{v}} \in \mathbb{Y}\cap\mathscr{D}(\mathscr{K})\,. \tag{36}$$

This together with (30) allows to pass to the limit in the second term via the weakly-lower semi-continuity of the convex functional $\mathscr{W}_\nu$ (see Lemma 2.2 and Remark 2.1). Since $\boldsymbol{v}^n$ only occurs linearly in the term including $\mathscr{A}_\nu$ on the left-hand side, we may also pass to the limit in this term. Indeed, the time derivative may be interchanged with the projection P_n such that

$$(\partial_t P_n \tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}) = (P_n \partial_t \tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}) = (\partial_t \tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}) \,,$$

where it was used that P_n is an orthogonal projection. This together with (36) imply that the consistency error vanishes, *i.e.*,

$$\begin{aligned}
&\int_0^T \phi \, \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}) - \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle \, \mathrm{d}s \\
&\quad = \nu \int_0^T \phi \, (\nabla \tilde{\boldsymbol{v}} - \nabla P_n \tilde{\boldsymbol{v}}; \nabla \boldsymbol{v}^n - \nabla P_n \tilde{\boldsymbol{v}})) \, \mathrm{d}s \\
&\qquad + \int_0^T \phi \, (((\tilde{\boldsymbol{v}} - P_n \tilde{\boldsymbol{v}}) \cdot \nabla)\tilde{\boldsymbol{v}} + (P_n \tilde{\boldsymbol{v}} \cdot \nabla)(\tilde{\boldsymbol{v}} - P_n \tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}) \, \mathrm{d}s \\
&\quad \leq \nu \|\phi\|_{L^\infty(\Omega)} \|\nabla \tilde{\boldsymbol{v}} - \nabla P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))} \|\nabla \boldsymbol{v}^n - \nabla P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + \|\phi\|_{L^\infty(\Omega)} \|\tilde{\boldsymbol{v}} - P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{d/2}(\Omega))} \|\nabla \tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^{2d/(d-2)}(\Omega))} \|\boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} \\
&\qquad + \|\phi\|_{L^\infty(\Omega)} \|P_n \tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^d(\Omega))} \|\nabla \tilde{\boldsymbol{v}} - \nabla P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))} \|\boldsymbol{v}^n - P_n \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} \,.
\end{aligned}$$

Weak convergence of $\boldsymbol{v}^n$ in $L^2(0,T;H^1_{0,\sigma}(\Omega))$ implies that the norms of $\boldsymbol{v}^n$ on the right-hand side are bounded independent of n. Note that dimension $d = 2$ is excluded at this point. But the proof can also be adapted to dimension two. The strong convergence (36) allows to pass to the limit on the right-hand side, which vanishes. The strong convergence of the projection P_n to the identity on $L^2_\sigma(\Omega)$ as $n \to \infty$ allows to pass to the limit in the initial values, too. Finally, we observe from (30), (31), and (36) that

$$\mathscr{R}(\boldsymbol{v}^n | P_n \tilde{\boldsymbol{v}}) \to \mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) + E - \mathscr{E}(\boldsymbol{v}) \quad \text{a.e. in } (0,T)\,,$$

for a.e. $t \in (0,T)$. As a consequence, we observe that the relative energy inequality (8) holds in the limit a.e. in $(0,T)$. Now choosing $t = T$ and $s = 0$ in (8) as well as $\mathscr{K}(\tilde{\boldsymbol{v}}) = 2\|(\nabla \tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{\mathscr{C}(\overline{\Omega})}$ from (25) we find similar to the proof of Proposition 2.13 that

$$\begin{aligned}
-\int_0^T \langle \partial_t \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle \, \mathrm{d}t &= -(\boldsymbol{v}(T), \tilde{\boldsymbol{v}}(T)) + (\boldsymbol{v}_0, \tilde{\boldsymbol{v}}(0)) + \int_0^T (\partial_t \tilde{\boldsymbol{v}}, \boldsymbol{v}) \, \mathrm{d}t \\
&\leq \int_0^T \nu(\nabla \boldsymbol{v}, \nabla \tilde{\boldsymbol{v}}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} \rangle - (E - \mathscr{E}(\boldsymbol{v}))\, \mathscr{K}(\tilde{\boldsymbol{v}}) \, \mathrm{d}t
\end{aligned} \tag{37}$$

The right-hand side is known to be bounded. We observe

$$\begin{aligned}
&\int_0^T \nu(\nabla \boldsymbol{v}, \nabla \tilde{\boldsymbol{v}}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \tilde{\boldsymbol{v}} \rangle - (E - \mathscr{E}(\boldsymbol{v}))\, \mathscr{K}(\tilde{\boldsymbol{v}}) \, \mathrm{d}t \\
&\qquad \leq \nu \|\nabla \boldsymbol{v}\|_{L^2(\Omega \times (0,T))} \|\nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega \times (0,T))} + \|\boldsymbol{v}\|^2_{L^\infty(0,T;L^2(\Omega))} \|\tilde{\boldsymbol{v}}\|_{L^1(0,T;W^{1,\infty}(\Omega))} \\
&\quad + \|\boldsymbol{f}\|_{\mathbb{Z}} \Big(\nu \|\nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega \times (0,T))} + \|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega))} \Big) + 2\|E - \mathscr{E}(\boldsymbol{v})\|_{L^\infty(0,T)} \|\tilde{\boldsymbol{v}}\|_{L^1(0,T;\mathscr{C}^1_0(\Omega))} \,.
\end{aligned}$$

On the left-hand side of the inequality (37), the definition of the weak-time derivative appears. On the right-hand side, the terms depending on $\boldsymbol{v}$ and E are bounded. The existence of such an element $\partial_t \boldsymbol{v}$ can be deduced by Hahn-Banach's theorem similar to the proof of Proposition 2.13 on page 15. Taking the supremum over all test functions, we observe that

$$\|\partial_t \boldsymbol{v}\|_{L^2(0,T;((H^1_{0,\sigma}(\Omega) \cap W^{2,p}(\Omega))^*)} = \sup_{\substack{\tilde{\boldsymbol{v}} \in L^2(0,T;H^1_{0,\sigma}(\Omega) \cap W^{2,p}(\Omega)), \\ \|\tilde{\boldsymbol{v}}\|_{L^2(0,T;H^1_{0,\sigma}(\Omega) \cap W^{2,p}(\Omega))} = 1}} -\langle \partial_t \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle \leq C \quad \text{for } p > d\,.$$

From a standard lemma, we infer since $L^2_\sigma(\Omega)$ is reflexive that

$$\mathscr{C}_w([0,T];(H^1_{0,\sigma}(\Omega)\cap W^{2,p}(\Omega))^*)\cap L^\infty(0,T;L^2_\sigma(\Omega))\subset\mathscr{C}_w([0,T];L^2_\sigma(\Omega))$$

and from this that $\boldsymbol{v}\in\mathscr{C}_w([0,T];L^2_\sigma(\Omega))$ (see [32]). Thus the pointwise evaluation in (8) is well-defined.

Step 5, Vanishing viscosity limit $\nu\to 0$: Now, we focus on the case $\nu=0$. Therefore, we consider a sequence $\{(\boldsymbol{v}^\nu,E^\nu)\}_{\nu\in(0,1)}$ of energy-variational solutions to the Navier-Stokes equations according to Theorem 2.14 for $\nu\to 0$. These solutions fulfill Definition 2.7 with $\mathscr{W}_\nu$ given by (4). Inserting $\tilde{\boldsymbol{v}}=0$ in this definition, we find the usual energy estimate

$$E^\nu(t)-E^\nu(s)+\int_s^t \nu\|\nabla\boldsymbol{v}^\nu\|^2_{L^2(\Omega)}-\langle\boldsymbol{f},\boldsymbol{v}^\nu\rangle\,\mathrm{d}\tau\le 0 \tag{38}$$

for a.e. s, $t\in(0,T)$ such that with the usual estimates of the right-hand side, *i.e.*, (29) with $\boldsymbol{f}_1=0$ (Note that $\mathbb{Z}_0=L^1(0,T;L^2(\Omega))$), we find by (38) for $s=0$ and using $E\ge\mathscr{E}(\boldsymbol{v})$ that

$$\|\boldsymbol{v}^\nu(t)\|^2_{L^2(\Omega)}+\int_0^t \nu\|\nabla\boldsymbol{v}^\nu\|^2_{L^2(\Omega)}\,\mathrm{d}s\le\|\boldsymbol{v}_0\|^2_{L^2(\Omega)}+\int_0^t\|\boldsymbol{f}_2\|_{L^2(\Omega)}\left(\|\boldsymbol{v}^\nu\|^2_{L^2(\Omega)}+1\right)\mathrm{d}s\,.$$

Via Gronwall's lemma we infer $\boldsymbol{f}$, $\boldsymbol{v}_0$ and $E(0)$-dependent bounds on $\boldsymbol{v}$ in $\mathbb{X}$. Note that in the current case $\mathscr{E}(\boldsymbol{v}_0)=E(0)$. Thus, we deduce the weak convergence of a subsequence in the energy space, *i.e.,*

$$\boldsymbol{v}^\nu\overset{*}{\rightharpoonup}\boldsymbol{v}\quad\text{in }\mathbb{X}_0$$

with $\mathbb{X}_0$ as given above by $\mathbb{X}_0:=L^\infty(0,T;L^2_\sigma(\Omega))$.

Using the same estimates as in step *Step 4* for the time derivative, we may deduce that

$$\boldsymbol{v}^\nu\to\boldsymbol{v}\quad\text{in }\mathscr{C}_w([0,T];L^2_\sigma(\Omega))$$

and thus pointwise for all $t\in(0,T)$. Revisiting (38), we infer that the function

$$t\mapsto E^\nu(t)+\int_0^t\nu\|\nabla\boldsymbol{v}^\nu\|^2_{L^2(\Omega)}-\langle\boldsymbol{f},\boldsymbol{v}^\nu\rangle\,\mathrm{d}s$$

is a monotonously non-increasing function and thus a function of bounded variation [27]. Since for $\{\boldsymbol{v}^\nu\}\subset\mathbb{X}$ and $\boldsymbol{f}\in\mathbb{Z}$, the function $\nu\|\nabla\boldsymbol{v}^\nu\|^2_{L^2(\Omega)}-\langle\boldsymbol{f}^\nu,\boldsymbol{v}\rangle$ is integrable such that $\int_0^t\nu\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)}-\langle\boldsymbol{f},\boldsymbol{v}\rangle\,\mathrm{d}s$ is absolutely continuous and thus of bounded variation. The sum and difference of functions of bounded variation are known to be of bounded variation again (see [27, Chap. 8,Thm. 3]), such that$\{E^\nu\}$ is a bounded sequence in $\mathrm{BV}([0,\mathrm{T}])$, where the associated bound on $\|E^\nu\|_{\mathsf{TV}(0,T)}$ again depends on $E(0)=\mathscr{E}(\boldsymbol{v}_0)$ and $\boldsymbol{f}$. Additionally, we may select via Helly's theorem a pointwise converging subsequence

$$\begin{aligned}E^\nu&\overset{*}{\rightharpoonup}E\quad\text{in }\mathrm{BV}([0,\mathrm{T}])\,,\\ E^\nu&\to E\quad\text{pointwise everywhere in }[0,T]\,.\end{aligned}$$

Note that the bound does not depend on ν since the essential *a priori* estimates are independent of ν.

With the usual skew-symmetry in the last two entries of the trilinear form, we find

$$-\left(((\boldsymbol{v}^\nu-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}^\nu-\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\right)=\left((\boldsymbol{v}^\nu-\tilde{\boldsymbol{v}})\otimes(\boldsymbol{v}^\nu-\tilde{\boldsymbol{v}}),(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}\right)\,.$$

This can be used to rewrite the relative energy inequality (8) into

$$\left[E^\nu - (\boldsymbol{v}^\nu,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t + \int_s^t \left[\nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \nu\,(\nabla\tilde{\boldsymbol{v}};\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}})\right]\mathrm{d}\tau$$
$$+\int_s^t \left[\mathscr{W}_0(\boldsymbol{v}^\nu|\tilde{\boldsymbol{v}}) + \langle\mathscr{A}_0(\tilde{\boldsymbol{v}}),\boldsymbol{v}^\nu-\tilde{\boldsymbol{v}}\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^\nu - (\boldsymbol{v}^\nu,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau \le 0 \quad (39)$$

for all $\tilde{\boldsymbol{v}} \in \mathbb{Y}_0\cap\mathscr{D}(\mathscr{K})\cap L^2(0,T;H^2(\Omega))$ and all $\nu > 0$. First, we decrease the number of admissible regularity measures $\mathscr{K}$, in order to make them independent of ν, $\mathscr{K}_0 : \mathbb{Y}_0 \to \mathbb{R}_+$ is chosen such that $\mathscr{W}_0$ given by (5) is convex and weakly lower semi-continuous in $\boldsymbol{v}$ and continuous in $\tilde{\boldsymbol{v}}$ (see Definition 2.7). In the first and last line of (39), we may pass to the limit by the weak* convergence of $\{(\boldsymbol{v}^\nu,E^\nu)\}$ in $\mathbb{X}_0\cap\mathscr{C}_w([0,T];L^2_\sigma(\Omega))\times \mathrm{BV}([0,\mathrm{T}])$, the lower semi-continuity of $\mathscr{W}_0$ and the linear occurrence in all other terms. For the second line, we observe the estimates

$$\int_s^t \left(\nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \nu\,(\nabla\tilde{\boldsymbol{v}};\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}})\right)\mathrm{d}\tau$$
$$\ge \int_s^t \left(\nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \sqrt{\nu}\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\sqrt{\nu}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\right)\mathrm{d}t$$
$$\ge \int_s^t \left(\frac{\nu}{2}\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\,\mathrm{d}s - \frac{\nu}{2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\right)\mathrm{d}\tau$$
$$\ge -\frac{\nu}{2}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega\times(s,t))} \to 0 \qquad \text{as } \nu\to 0.$$

We infer that the relative energy inequality (8) is fulfilled in the limit $\nu\to 0$. This proves the existence of energy-variational solutions to the Euler equations and thus the assertion. In order to allow more general test functions, *i.e.*, $\tilde{\boldsymbol{v}} \in \mathbb{Y}_0\cap\mathscr{D}(\mathscr{K}_0)$ instead of $\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$ one may use usual density arguments and the continuity of $\mathscr{K}_0$ in $\mathbb{Y}_0\cap\mathscr{D}(\mathscr{K}_0)$.

Step 6, Solution set: Let $(\boldsymbol{v},E)$ be an energy-variational solution according to Definition 2.7. First, we observe that E is indeed a function of bounded variation by choosing $\tilde{\boldsymbol{v}}\equiv 0$, similar to (38). Secondly, let $(\boldsymbol{v}^1,E^1)$ and $(\boldsymbol{v}^2,E^2)$ be in $\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$. Then their convex combination $(\boldsymbol{v}^\lambda,E^\lambda) = (\lambda\boldsymbol{v}^1+(1-\lambda)\boldsymbol{v}^2,\lambda E^1+(1-\lambda)E^2)$ for $\lambda\in(0,1)$ is again associated to an energy-variational solution. Indeed, we find by the linearity in E and $\boldsymbol{v}$ and the convexity of $\mathscr{W}_\nu$ in $\boldsymbol{v}$ that

$$\left[E^\lambda - (\boldsymbol{v}^\lambda,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t$$
$$+\int_s^t \left[\mathscr{W}_\nu(\boldsymbol{v}^\lambda,\tilde{\boldsymbol{v}}) + \left\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^\lambda-\tilde{\boldsymbol{v}}\right\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^\lambda - (\boldsymbol{v}^\lambda,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau$$
$$=\lambda\left[E^1 - (\boldsymbol{v}^1,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t$$
$$+\lambda\int_s^t \left[\mathscr{W}_\nu(\boldsymbol{v}^\lambda,\tilde{\boldsymbol{v}}) + \left\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^1-\tilde{\boldsymbol{v}}\right\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^1 - (\boldsymbol{v}^1,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau$$
$$+(1-\lambda)\left[E^2 - (\boldsymbol{v}^2,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t$$
$$+(1-\lambda)\int_s^t \left[\mathscr{W}_\nu(\boldsymbol{v}^\lambda,\tilde{\boldsymbol{v}}) + \left\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^2-\tilde{\boldsymbol{v}}\right\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^2 - (\boldsymbol{v}^2,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau$$
$$\le\lambda\left[E^1 - (\boldsymbol{v}^1,\tilde{\boldsymbol{v}}) + \frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t$$

$$
\begin{aligned}
&+\lambda \int_s^t \left[\mathscr{W}_\nu(\boldsymbol{v}^1,\tilde{\boldsymbol{v}}) + \left\langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^1-\tilde{\boldsymbol{v}}\right\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^1-(\boldsymbol{v}^1,\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau\\
&+(1-\lambda)\left[E^2-(\boldsymbol{v}^2,\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_s^t\\
&+(1-\lambda)\int_s^t \left[\mathscr{W}_\nu(\boldsymbol{v}^2,\tilde{\boldsymbol{v}}) + \left\langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^1-\tilde{\boldsymbol{v}}\right\rangle - \mathscr{K}(\tilde{\boldsymbol{v}})\left(E^2-(\boldsymbol{v}^2,\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right)\right]\mathrm{d}\tau\\
\le{}& 0\,.
\end{aligned}
$$

Finally, we want to show that any sequence $\{\boldsymbol{v}^n,E^n\}_{n\in\mathbb{N}}\subset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$ admits a cluster point in the solution set $\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$. Note that if for a $t_0\in[0,T]$ the sequence $\{E^n(t_0)\}\subset[0,\infty)$ diverges, the same holds for all sequences $\{E^n(s)\}$ with $s\le t_0$ and the relative energy inequality (8) is trivially fulfilled for all $s\le t_0$ in the limit. We only have to prove something if for some t_0 $\sup_{n\in\mathbb{N}}E^n(t_0)<\infty$. Without loss of generality, we assume that $t_0=0$. For any sequence $\{\boldsymbol{v}^n,E^n\}_{n\in\mathbb{N}}\subset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$, we infer from (8) by choosing $\tilde{\boldsymbol{v}}=0$ and $s=0$ the standard *a priori* estimates

$$
E^n(t)+\int_0^t \nu\|\nabla\boldsymbol{v}^n\|^2_{L^2(\Omega)}\,\mathrm{d}s\le E^n(0)+\int_0^t\langle\boldsymbol{f},\boldsymbol{v}^n\rangle\,\mathrm{d}s\,.
$$

From the estimate (29), we infer the *a priori* bounds for $\{(E^n,\boldsymbol{v}^n)\}$ in the space $L^\infty(0,T)\times L^2(0,T;H^1_{0,\sigma}(\Omega))$ in the case $\nu>0$ and $\{E^n\}$ in $L^\infty(0,T)$ for $\nu=0$. From the definition of E, we infer that $E^n\ge 1/2\|\boldsymbol{v}^n\|^2_{L^2(\Omega)}$ on $[0,T]$. This implies that $\{\boldsymbol{v}_n\}$ is bounded in $\mathbb{X}$. Moreover from the inequality (38), we infer the boundedness of the sequence $\{E^n\}$ in $\mathrm{BV}([0,\mathrm{T}])$. Helly's selection principle allows to select an everywhere in $[0,T]$ converging subsequence to some limit $E\in\mathrm{BV}([0,\mathrm{T}])$. Furthermore, we infer $\boldsymbol{v}^n\overset{*}{\rightharpoonup}\boldsymbol{v}$ in $\mathbb{X}$. Due to the additional regularity of the time derivative (37), we even infer $\boldsymbol{v}^n\to\boldsymbol{v}$ in $\mathscr{C}_w([0,T];L^2_\sigma(\Omega))$. Indeed, the strong convergence of E^n and the weak convergence of $\boldsymbol{v}^n$ imply

$$
\begin{aligned}
\lim_{n\to\infty}&\left[E^n-(\boldsymbol{v}^n,\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_{s-}^t\\
&+\lim_{n\to\infty}\int_s^t\langle\mathscr{A}(\tilde{\boldsymbol{v}}),\boldsymbol{v}^n-\tilde{\boldsymbol{v}}\rangle-\mathscr{K}(\tilde{\boldsymbol{v}})\left[E^n-(\boldsymbol{v}^n,\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\mathrm{d}\tau\\
&\qquad=\left[E-(\boldsymbol{v},\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\Big|_{s-}^t\\
&\qquad\quad+\int_s^t\langle\mathscr{A}(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle-\mathscr{K}(\tilde{\boldsymbol{v}})\left[E-(\boldsymbol{v},\tilde{\boldsymbol{v}})+\frac{1}{2}\|\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}\right]\mathrm{d}\tau
\end{aligned}
$$

for all $s<t\in(0,T)$. The weak convergence of $\{\boldsymbol{v}^n\}$ and the weakly lower semi continuity of $\mathscr{W}_\nu$ imply

$$
\liminf_{n\to\infty}\int_s^t\mathscr{W}_\nu(\boldsymbol{v}^n|\tilde{\boldsymbol{v}})\,\mathrm{d}\tau\ge\int_s^t\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}})\,\mathrm{d}\tau
$$

for all $\tilde{\boldsymbol{v}}\in\mathbb{Y}\cap\mathscr{D}(\mathscr{K})$ and for all $s<t\in(0,T)$. This implies that the limit $(\boldsymbol{v},E)$ is again an energy-variational solution according to Definition 2.7. Again the condition $E\ge\mathscr{E}(\boldsymbol{v})$ is fulfilled, due to

$$
E(t)=\lim_{n\to\infty}E^n(t)\ge\liminf_{n\to\infty}\frac{1}{2}\|\boldsymbol{v}^n(t)\|^2_{L^2(\Omega)}\ge\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)}
$$

for all $t\in[0,T]$. We note that $E(0)=\frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega)}$.

Step 7, Continuous dependence: In order to prove the claimed continuity of the set-valued mapping, we need to prove two assertions. Therefore, let

$$
\boldsymbol{f}^n\to\boldsymbol{f}\text{ in }\mathbb{Z}\quad\text{and}\quad\boldsymbol{v}_0^n\to\boldsymbol{v}_0\text{ in }L^2_\sigma(\Omega)\,. \tag{40}
$$

We equip the domain $L^2_\sigma(\Omega)\times\mathbb{Z}$ with the strong topology and the range $\mathbb{X}\times\mathrm{BV}([0,\mathrm{T}])$ with the weak* topology.

Introducing the Kuratowski limits

$$\begin{aligned}\mathrm{Ls}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n) := \Big\{ &(\boldsymbol{v},E)\in\mathbb{X}\times\mathrm{BV}([0,\mathrm{T}]);\\ &\text{there exists a sequence } (\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})\\ &\text{and } (\boldsymbol{v}^n,E^n)\overset{*}{\rightharpoonup}(\boldsymbol{v},E) \text{ such that } (\boldsymbol{v}^n,E^n)\in\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\Big\},\\ \mathrm{Li}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n) := \Big\{ &(\boldsymbol{v},E)\in\mathbb{X}\times\mathrm{BV}([0,\mathrm{T}]);\\ &\text{for all } (\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f}) \text{ there exists } (\boldsymbol{v}^n,E^n)\in\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\\ &\text{such that } (\boldsymbol{v}^n,E^n)\overset{*}{\rightharpoonup}(\boldsymbol{v},E)\Big\}\end{aligned}$$

we need to prove that

$$\mathrm{Ls}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\subset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f}) \text{ and } \mathrm{Li}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\supset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f}),$$

which are called upper and lower semi-continuity, respectively. A set-valued map is said to be continuous if $\mathrm{Ls}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)=\mathrm{Li}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)$.

Step 7.1, Upper semi-continuity: First, we show $\mathrm{Ls}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\subset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$. From the steps 1 to 5, we infer that to every $n\in\mathbb{N}$ there exists an energy-variational solution $(\boldsymbol{v}^n,E^n)$ according to Definition 2.7. By the assumption $E\in\mathrm{BV}([0,\mathrm{T}])$ and $E^n\overset{*}{\rightharpoonup}E$ in $\mathrm{BV}([0,\mathrm{T}])$, we especially infer $\sup_{n\in\mathbb{N}}E^n(0)<\infty$. Since $(\boldsymbol{v}_0^n,\boldsymbol{f}^n)$ is bounded in $L^2_\sigma(\Omega)\times\mathbb{Z}$, we infer the boundedness of the sequence $\{(\boldsymbol{v}^n,E^n)\}_{n\in\mathbb{N}}$ in $\mathbb{X}\times\mathrm{BV}([0,\mathrm{T}])$ essentially in the same way as in *Step 5*. Thus, there exists some $(\boldsymbol{v},E)$ and some subsequence $\{(\boldsymbol{v}^{n_k},E^{n_k})\}$ such that $(\boldsymbol{v}^{n_k},E^{n_k})\overset{*}{\rightharpoonup}(\boldsymbol{v},E)$ in $\mathbb{X}\times\mathrm{BV}([0,\mathrm{T}])$. For every $\psi\in\mathscr{C}^1_c((0,T))$ with $\psi\geq 0$, we observe that

$$\liminf_{n\to\infty}\int_0^T\psi\left(\frac{1}{2}\|\boldsymbol{v}^{n_k}\|^2_{L^2(\Omega)}-E^{n_k}\right)\mathrm{d}t\geq\int_0^T\psi\left(\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)}-E\right)\mathrm{d}t$$

from the strong convergence of $\{E^{n_k}\}$ in $L^1(0,T)$ and the weakly lower semi-continuity of the L^2-norm. Similar, we observe for every fixed $\phi\in\mathscr{C}^1_c((0,T))$ with $\phi\geq 0$, and $\tilde{\boldsymbol{v}}\in\mathbb{Y}$ that

$$\begin{aligned}-\limsup_{n\to\infty}\int_0^T\phi'\left[E^{n_k}-(\boldsymbol{v}^{n_k},\tilde{\boldsymbol{v}})\right]\mathrm{d}t+\liminf_{n\to\infty}\Bigg[&\int_0^T\phi\mathscr{K}(\tilde{\boldsymbol{v}})\left[\frac{1}{2}\|\boldsymbol{v}^{n_k}\|^2_{L^2(\Omega)}-E^{n_k}\right]\mathrm{d}t\\ &\int_0^T\phi\left[\nu\left(\nabla\boldsymbol{v}^{n_k},\nabla\boldsymbol{v}^{n_k}-\nabla\tilde{\boldsymbol{v}}\right)+(\boldsymbol{v}^{n_k}\otimes\boldsymbol{v}^{n_k};\nabla\tilde{\boldsymbol{v}})+(\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}^{n_k})-\langle\boldsymbol{f}^{n_k},\boldsymbol{v}^{n_k}-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}t\Bigg]\\ \geq -\int_0^T\phi'\left[E-(\boldsymbol{v},\tilde{\boldsymbol{v}})\right]\mathrm{d}t&+\int_0^T\phi\mathscr{K}(\tilde{\boldsymbol{v}})\left[\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)}-E\right]\mathrm{d}t\\ &+\int_0^T\phi\left[\nu\left(\nabla\boldsymbol{v},\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\right)+(\boldsymbol{v}\otimes\boldsymbol{v};\nabla\tilde{\boldsymbol{v}})+(\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v})-\langle\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}t\,.\end{aligned}$$

Note that the only difference in this limit in comparison to the proof of *Step 6* is that $\boldsymbol{f}^{n_k}$ converges strongly now, which together with the weak convergence of $\{\boldsymbol{v}^{n_k}\}$ implies the convergence of their product. This implies that the limit $(\boldsymbol{v},E)$ is an energy-variational solution again and therewith $(\boldsymbol{v},E)\in\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$.

Step 7.2, Lower semi-continuity: Secondly, we show that $\mathrm{Li}_{(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})}\mathscr{S}(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\supset\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$. Therefore, we have to construct a recovery sequence. For a given $(\boldsymbol{v}_0^n,\boldsymbol{f}^n)\to(\boldsymbol{v}_0,\boldsymbol{f})$, we construct $\bar{\boldsymbol{v}}^n\in L^2(0,T;H^1_{0,\sigma}(\Omega))\cap H^1(0,T;(H^1_{0,\sigma}(\Omega))^*)\cap\mathscr{C}([0,T];L^2_\sigma(\Omega))$ as the solution to the Stokes problem

$$\begin{aligned} \partial_t\bar{\boldsymbol{v}}^n-\nu\Delta\bar{\boldsymbol{v}}^n+\nabla\bar{p}^n&=\boldsymbol{f}^n-\boldsymbol{f} && \text{in } \Omega\times(0,T) \\ \nabla\cdot\bar{\boldsymbol{v}}^n&=0 && \text{in } \Omega\times(0,T) \\ \bar{\boldsymbol{v}}^n&=0 && \text{on } \partial\Omega\times(0,T) \\ \bar{\boldsymbol{v}}^n(0)&=\boldsymbol{v}_0^n-\boldsymbol{v}_0 && \text{in } \Omega\,. \end{aligned} \tag{41}$$

Note that this linear PDE problem can be solved by Lions theorem (see [28, Thm. 11.3]) on linear parabolic problems in the usual weak sense in the space indicated above. Since Lions theorem also guarantees the continuous dependence, we infer that

$$\bar{\boldsymbol{v}}^n\to 0 \quad \text{in } \mathbb{X}\cap\mathscr{C}([0,T];L^2_\sigma(\Omega))\,. \tag{42}$$

Note that in the case $\nu=0$, the problem (41) is solved by the choice $\bar{\boldsymbol{v}}^n(t)=\boldsymbol{v}_0^n-\boldsymbol{v}_0+\int_0^t P(\boldsymbol{f}^n-\boldsymbol{f})\,\mathrm{d}s$, where P denotes the Leray-projection onto solenoidal functions in $L^2(\Omega)$. Now, we may choose $\boldsymbol{v}^n=\boldsymbol{v}+\bar{\boldsymbol{v}}^n$. By construction it holds that

$$\boldsymbol{v}^n(0)=\boldsymbol{v}_0^n \text{ and } \boldsymbol{v}^n\to\boldsymbol{v} \text{ in } \mathbb{X}\cap\mathscr{C}_w([0,T];L^2_\sigma(\Omega))\,.$$

In order to infer a condition on E^n, we consider the relative energy inequality for $\boldsymbol{v}^n$,

$$\begin{aligned} &[E^n-(\boldsymbol{v}^n,\tilde{\boldsymbol{v}})]\Big|_s^t+\int_s^t\mathscr{K}(\tilde{\boldsymbol{v}})\left[\frac12\|\boldsymbol{v}^n\|^2_{L^2(\Omega)}-E^n\right]\mathrm{d}\tau \\ &\quad+\int_s^t\left[\nu(\nabla\boldsymbol{v}^n,\nabla\boldsymbol{v}^n-\nabla\tilde{\boldsymbol{v}})+(\boldsymbol{v}^n\otimes\boldsymbol{v}^n;\nabla\tilde{\boldsymbol{v}})+(\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}^n)-\langle\boldsymbol{f}^n,\boldsymbol{v}^n-\tilde{\boldsymbol{v}}\rangle\right]\mathrm{d}\tau \\ &=[E-(\boldsymbol{v},\tilde{\boldsymbol{v}})]\Big|_s^t+\int_s^t\mathscr{K}(\tilde{\boldsymbol{v}})\left[\frac12\|\boldsymbol{v}\|^2_{L^2(\Omega)}-E\right]\mathrm{d}\tau \\ &\quad+\int_s^t\nu(\nabla\boldsymbol{v},\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}})+(\boldsymbol{v}\otimes\boldsymbol{v};\nabla\tilde{\boldsymbol{v}})+(\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v})-\langle\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}\tau \\ &\quad+\left[\frac12\|\bar{\boldsymbol{v}}^n\|^2_{L^2(\Omega)}-(\bar{\boldsymbol{v}}^n,\tilde{\boldsymbol{v}})\right]\Big|_s^t+\int_s^t\nu(\nabla\bar{\boldsymbol{v}}^n,\nabla\bar{\boldsymbol{v}}^n-\nabla\tilde{\boldsymbol{v}})+(\partial_t\tilde{\boldsymbol{v}},\bar{\boldsymbol{v}}^n)-\langle\boldsymbol{f}^n-\boldsymbol{f},\bar{\boldsymbol{v}}^n-\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}\tau \\ &\quad+\left[\bar{E}^n-\frac12\|\bar{\boldsymbol{v}}^n\|^2_{L^2(\Omega)}\right]\Big|_s^t+\int_s^t\mathscr{K}(\tilde{\boldsymbol{v}})\left[(\boldsymbol{v},\bar{\boldsymbol{v}}^n)+\frac12\|\bar{\boldsymbol{v}}^n\|^2_{L^2(\Omega)}-\bar{E}^n\right]\mathrm{d}\tau \\ &\quad+\int_s^t 2\nu(\nabla\boldsymbol{v},\nabla\bar{\boldsymbol{v}}^n)+(2\bar{\boldsymbol{v}}^n\otimes\boldsymbol{v}+\bar{\boldsymbol{v}}^n\otimes\bar{\boldsymbol{v}}^n;\nabla\tilde{\boldsymbol{v}})-\langle\boldsymbol{f}^n-\boldsymbol{f},\boldsymbol{v}\rangle-\langle\boldsymbol{f},\bar{\boldsymbol{v}}^n\rangle\,\mathrm{d}\tau\,. \end{aligned} \tag{43}$$

Since $(\boldsymbol{v},E)\in\mathscr{S}(\boldsymbol{v}_0,\boldsymbol{f})$, the first two lines on the right-hand side of the previous inequality are non-positive, and for $\bar{\boldsymbol{v}}^n$ as a weak solution to (41), the third line on the right-hand side is zero. Thus it remains to choose $\bar{E}^n$ in such a way that the right-hand side is non-positive. Therefore, we consider the estimate

$$(2\bar{\boldsymbol{v}}^n\otimes\boldsymbol{v}+\bar{\boldsymbol{v}}^n\otimes\bar{\boldsymbol{v}}^n;\nabla\tilde{\boldsymbol{v}})\geq-\mathscr{K}(\tilde{\boldsymbol{v}})\left(\|\boldsymbol{v}\|_{L^2(\Omega)}\|\bar{\boldsymbol{v}}^n\|_{L^2(\Omega)}+\frac12\|\bar{\boldsymbol{v}}^n\|^2_{L^2(\Omega)}\right),$$

in order to infer the conditions

$$\bar{E}^n(t)\geq 2\|\bar{\boldsymbol{v}}^n(t)\|_{L^2(\Omega)}\|\boldsymbol{v}(t)\|_{L^2(\Omega)} \quad \text{for all } t\in[0,T] \tag{44a}$$

and

$$\left[\bar{E}^n - \frac{1}{2}\|\bar{\boldsymbol{v}}^n\|^2_{L^2(\Omega)}\right]\Big|_s^t + \int_s^t 2\nu\left(\nabla \boldsymbol{v}, \nabla \bar{\boldsymbol{v}}^n\right) - \langle \boldsymbol{f}^\nu - \boldsymbol{f}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}, \bar{\boldsymbol{v}}^n\rangle \,\mathrm{d}\tau \le 0 \quad \text{for all } s < t \in [0,T]\,. \tag{44b}$$

One possible choice to define $\bar{E}^n$ would be to set

$$\begin{aligned}\bar{E}^n(0) := \max_{t\in[0,T]} \Big[& 2\|\bar{\boldsymbol{v}}^n\|_{\mathscr{C}([0,T];L^2(\Omega))}\|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega))} - \frac{1}{2}\|\bar{\boldsymbol{v}}^n(t)\|^2_{L^2(\Omega)} \\ & + \int_0^t 2\nu\left(\nabla \boldsymbol{v}, \nabla \bar{\boldsymbol{v}}^n\right) - \langle \boldsymbol{f}^\nu - \boldsymbol{f}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}, \bar{\boldsymbol{v}}^n\rangle \,\mathrm{d}s + \frac{1}{2}\|\bar{\boldsymbol{v}}^n_0\|^2_{L^2(\Omega)}\Big]\end{aligned} \tag{45}$$

and define

$$\bar{E}^n(t) = \bar{E}^n(0) - \frac{1}{2}\|\bar{\boldsymbol{v}}^n_0\|^2_{L^2(\Omega)} + \int_0^t 2\nu\left(\nabla \boldsymbol{v}, \nabla \bar{\boldsymbol{v}}^n\right) - \langle \boldsymbol{f}^\nu - \boldsymbol{f}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}, \bar{\boldsymbol{v}}^n\rangle \,\mathrm{d}s + \frac{1}{2}\|\bar{\boldsymbol{v}}^n(t)\|^2_{L^2(\Omega)}\,.$$

By this choice the conditions (44) are fulfilled and additionally the convergences (42) and (40) as well as the boundedness of $\boldsymbol{v}$ in $\mathbb{X}$ allow to deduce that

$$\bar{E}^n \to 0 \quad \text{everywhere on } [0,T]\,,$$

which implies the assertion.

□

Remark 3.1. The continuous dependence result is presented in a set-theoretic sense via convergence in the Kuratowski sense. These convergences are introduced in [19, Sec. 29]. The connection of Kuratowski convergence of the epigraphs of a sequence of convex functions to Gamma convergence of the associated functions is considered in [10, Thm. 4.16]. The continuity of the set-valued map is consistent with the usual single-valued definition (see [2, Sec. 1.4]).

3.3 Selection criteria and minimization

First, we consider a different way of formulating the continuity of the solution set.

Proposition 3.1. Assume that there exists a sequence $\{(\boldsymbol{v}_0^n, \boldsymbol{f}^n)\} \subset L^2_\sigma(\Omega)\times\mathbb{Z}$ and an element $(\boldsymbol{v}_0, \boldsymbol{f}) \in L^2_\sigma(\Omega)\times\mathbb{Z}$ such that

$$\|\boldsymbol{v}_0^n - \boldsymbol{v}_0\|_{L^2(\Omega)} + \|\boldsymbol{f}^n - \boldsymbol{f}\|_{\mathbb{Z}} \to 0 \text{ as } n\to\infty\,.$$

Then it holds that there exists a sequence $\{E_0^n\}_{n\in\mathbb{N}}$ and $C>0$ such that $E^n \ge \mathscr{E}(\boldsymbol{v}_0^n)$ can be bounded such that $E_0^n - \mathscr{E}(\boldsymbol{v}_0^n) \le C\|\boldsymbol{v}_0^n - \boldsymbol{v}_0\|_{L^2(\Omega)}$

$$\mathfrak{S}^0(\boldsymbol{v}_0^n, E_0^n, \boldsymbol{f}) \xrightarrow{M^*} \mathfrak{S}^0(\boldsymbol{v}_0, \mathscr{E}(\boldsymbol{v}_0), \boldsymbol{f})\,,$$

where the constant C only depends on $\boldsymbol{v}_0$ and the right-hand side $\boldsymbol{f}$.

Remark 3.2 (Mosco convergence). By the superscript M^*, we denote the un-usual Mosco convergence of sets, *i.e.*,

i) for each $(\boldsymbol{v},E) \in \mathfrak{S}^{t_0}(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})$ there exists $\{(\boldsymbol{v}^n,E^n)\} \subset \mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$ such that $(\boldsymbol{v}^n,E^n) \in \mathfrak{S}^{t_0}(\boldsymbol{v}_0^n,E_0^n,\boldsymbol{f})$ with $(\boldsymbol{v}^n,E^n) \overset{*}{\rightharpoonup} (v,E)$ in $\mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$ such that $(\boldsymbol{v}^n,E^n) \to (\boldsymbol{v},E)$ in $L^\infty(0,T;L^2_\sigma(\Omega))\times L^\infty(0,T)$,

ii) for each sequence $\{(\boldsymbol{v}^n,E^n)\} \subset \mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$ with $(\boldsymbol{v}^n,E^n) \in \mathfrak{S}^{t_0}(\boldsymbol{v}_0^n,E_0^n,\boldsymbol{f})$ and $(\boldsymbol{v}^n,E^n) \overset{*}{\rightharpoonup} (\boldsymbol{v},E)$ in $\mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$, it holds $(\boldsymbol{v},E) \in \mathfrak{S}^{t_0}(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})$.

It is similar to the Mosco-convergence of sets, but the weak convergence is replaced by the weak* convergence and the strong convergence of the recovery sequence in a weaker space in our setting.

Proof. In order to define the recovery sequence, we follow the same argument as in Step 7.2 of the proof of Theorem 2.14 on page 23. There the value E_0^n is given in (45).

The second property of the Mosco* convergence follows in the same way as in Step 7.1 of the proof of Theorem 2.14 on page 22. Note that $\{E^n\}$ is such that $E_0^N \to \mathscr{E}(\boldsymbol{v}_0)$ as $n\to\infty$.

□

Proof of Proposition 2.15. Step 1: Existence . First, we observe that the set of energy-variational solutions is convex and weakly*-closed and the functional $(\boldsymbol{v},E) \mapsto \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$ is known to be strictly convex and weakly-lower semi-continuous with respect to the topology in $L^\infty(0,T;L^2_\sigma(\Omega))\times L^\infty(0,T)$ for any $p\in[1,\infty)$, which is coarser than the topology of $\mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$. This allows to infer that the minimization problem (15) has a unique solution, which we call $(\boldsymbol{v},E)$.

Step 2: Continuous dependence. The same argument provides the existence of a unique solution $(\boldsymbol{v}^n,E^n)$ to the minimization problem

$$\min_{(\boldsymbol{v},E)\in\mathfrak{S}^{t_0}(\boldsymbol{v}_0^n,E_0^n,\boldsymbol{f}^n)} \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$$

for every $n\in\mathbb{N}$. Now we observe that due to the compactness of the solution set of energy-variational solutions in the weak* topology and the property ii) of Proposition 3.1, there exists a not relabeled subsequence and an element $(\boldsymbol{u},F) \in \mathfrak{S}^0(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})$ such that $(\boldsymbol{v}^n,E^n) \overset{*}{\rightharpoonup} (\boldsymbol{u},F)$. From the weakly-lower semi-continuity of J, we find

$$\liminf_{n\to\infty} \int_0^t J(t,\boldsymbol{v}^n,E^n)\,\mathrm{d}t \ge \int_0^T J(t,\boldsymbol{u},F)\,\mathrm{d}t \ge \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t\,,$$

where the last inequality follows from the fact that $(\boldsymbol{v},E)$ is the solution to the problem (15). From the point i) of Proposition 3.1, we find that there exists a sequence $(\boldsymbol{u}^n,F^n)$ such that $(\boldsymbol{u}^n,F^n) \in \mathfrak{S}^0(\boldsymbol{v}_0^n,E_0^n,\boldsymbol{f}^n)$ with $(\boldsymbol{u}^n,F^n) \to (\boldsymbol{v},E)$ in $\mathbb{X}\times \mathrm{BV}([0,\mathrm{T}])$. The continuity of the functional J and the inequality $\int_0^T J(t,\boldsymbol{u}^n,F^n)\,\mathrm{d}t \ge \int_0^T J(t,\boldsymbol{v}^n,E^n)\,\mathrm{d}t$, we may conclude that

$$\begin{aligned}\int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T J(t,\boldsymbol{u}^n,F^n)\,\mathrm{d}t \ge \liminf_{n\to\infty}\int_0^T J(t,\boldsymbol{v}^n,E^n)\,\mathrm{d}t \\ \ge \int_0^T J(t,\boldsymbol{u},F)\,\mathrm{d}t \ge \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t\,.\end{aligned}$$

This implies that $\int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t = \int_0^T J(t,\boldsymbol{u},F)\,\mathrm{d}t$ and due to the strict convexity of J, we infer $(\boldsymbol{u},F) = (\boldsymbol{v},E)$ such that also $(\boldsymbol{v}^n,E^n) \overset{*}{\rightharpoonup} (\boldsymbol{v},E)$.

Step 3: Semiflow property. Let $(\boldsymbol{v},E) = \arg\min_{\mathfrak{S}^0(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})} \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$ be the minimizer of the minimization problem. Let $(\boldsymbol{v}^1,E^1) = \arg\min_{\mathfrak{S}^{t_0}(\boldsymbol{v}(t_0),E(t_0),\boldsymbol{f})} \int_{t_0}^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$ be the minimizer of the shifted minimization problem. Then it holds for the concatenation

$$\begin{cases} (\boldsymbol{v}^2(t),E^2(t)) = (\boldsymbol{v}(t),E(t)) & \text{for } t \in (0,t_0) \\ (\boldsymbol{v}^2(t),E^2(t)) = (\boldsymbol{v}^1(t),E^1(t)) & \text{for } t \in (t_0,T) \end{cases}$$

that

$$\begin{aligned} \int_{t_0}^T J(t,\boldsymbol{v},E)\,\mathrm{d}t &= \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t - \int_0^{t_0} J(t,\boldsymbol{v},E)\,\mathrm{d}t \\ &\leq \int_0^T J(t,\boldsymbol{v}^2,E^2)\,\mathrm{d}t - \int_0^{t_0} J(t,\boldsymbol{v},E)\,\mathrm{d}t \\ &= \int_0^{t_0} J(t,\boldsymbol{v},E)\,\mathrm{d}t + \int_{t_0}^T J(t,\boldsymbol{v}^1,E^1)\,\mathrm{d}t - \int_0^{t_0} J(t,\boldsymbol{v},E)\,\mathrm{d}t \leq \int_{t_0}^T J(t,\boldsymbol{v}^1,E^1)\,\mathrm{d}t\,, \end{aligned}$$

is an minimizer again. This implies for the restriction

$$(\boldsymbol{v}|_{t_0}^T, E|_{t_0}^T) = \arg\min_{\mathfrak{S}^{t_0}(\boldsymbol{v}(t_0),E(t_0),\boldsymbol{f})} \int_{t_0}^T J(t,\boldsymbol{v},E)\,\mathrm{d}t = (\boldsymbol{v}^1,E^1)\,. \tag{46}$$

On the other hand, let $(\boldsymbol{v}^1,E^1) = \arg\min_{\mathfrak{S}^0(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})} \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$ be the minimizer of the global problem and $(\boldsymbol{v}^2,E^2) = \arg\min_{\mathfrak{S}^{t_0}(\boldsymbol{v}(t_0),E(t_0),\boldsymbol{f})} \int_{t_0}^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$ the minimizer of the local problem then the concatenation

$$\begin{cases} (\boldsymbol{v}(t),E(t)) = (\boldsymbol{v}^1(t),E^1(t)) & \text{for } t \in (0,t_0) \\ (\boldsymbol{v}(t),E(t)) = (\boldsymbol{v}^2(t),E^2(t)) & \text{for } t \in (t_0,T) \end{cases}$$

is a minimal energy-variational solution again, $(\boldsymbol{v},E) \in \arg\min_{\mathfrak{S}^0(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})} \int_0^T J(t,\boldsymbol{v},E)\,\mathrm{d}t$. Indeed, it holds

$$\begin{aligned} \int_0^T J(\boldsymbol{v},E)\,\mathrm{d}t &= \int_0^{t_0} J(\boldsymbol{v}^1,E^1)\,\mathrm{d}t + \int_{t_0}^T J(\boldsymbol{v}^2,E^2)\,\mathrm{d}t \\ &\leq \int_0^{t_0} J(\boldsymbol{v}^1,E^1)\,\mathrm{d}t + \int_{t_0}^T J(\boldsymbol{v}^1,E^1)\,\mathrm{d}t = \int_0^T J(\boldsymbol{v}^1,E^1)\,\mathrm{d}t \end{aligned}$$

since $(\boldsymbol{v}^1,E^1)$ is also an energy-variational solution on (t_0,T) and thus in the admissible set of the minimization problem.

□

Proof of Proposition 2.16. For convenience, the proof is divided into several steps.

Step 1: Local minimization problem. First we consider the optimization problem of minimizing the auxiliary variable E point-wise over the set of energy-variational solutions to a given initial-value $\boldsymbol{v}_0$ and right-hand side $\boldsymbol{f}$. This can be expressed as

$$(\boldsymbol{v},E) = \arg\min_{(\boldsymbol{u},F)\in\mathfrak{S}^0(\boldsymbol{v}_0,\mathscr{E}(\boldsymbol{v}_0),\boldsymbol{f})} E(t_0)\,. \tag{47}$$

We note that point-evaluations are not continuous with respect to the topology of $\mathrm{BV}([0,\mathrm{T}])$-functions. But since in our considered setting jumps can only occur in a jump point s_0 such that $E(s_0) =$

$E(s_0+) \leq E(s_0-)$ the point-evaluations are at least lower-semi continuous. Since the set of energy-variational solutions is convex and closed and the cost-functional is lower semi-continuous, bounded from below, and coercive, there exists a minimum for this problem.

Step 2: Restarting. We observe that for a $t_0 \in (0,T)$ with $E(t_0) > \mathscr{E}(\boldsymbol{v}(t_0))$, we may restart the process in t_0 for any energy-variational solution $(\boldsymbol{v},E) \in \mathbb{X} \times \mathrm{BV}([0,\mathrm{T}])$. If we call $(\bar{\boldsymbol{v}},\bar{E})$ an energy-variational solution on (t_0,T) starting from the initial value $\boldsymbol{v}(t_0)$ then it holds

$$\mathscr{R}(\bar{\boldsymbol{v}}(t)|\tilde{\boldsymbol{v}}(t)) + \bar{E}(t) - \mathscr{E}(\bar{\boldsymbol{v}}(t)) - \int_s^t \mathscr{K}(\tilde{\boldsymbol{v}})\left[\mathscr{R}(\bar{\boldsymbol{v}}|\tilde{\boldsymbol{v}}) + \bar{E} - \mathscr{E}(\bar{\boldsymbol{v}})\right] \mathrm{d}\tau \\ + \int_s^t \mathscr{W}_\nu(\bar{\boldsymbol{v}},\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \bar{\boldsymbol{v}} - \tilde{\boldsymbol{v}} \rangle \,\mathrm{d}\tau \leq \mathscr{R}(\bar{\boldsymbol{v}}(s)|\tilde{\boldsymbol{v}}(s)) + \bar{E}(s-) - \mathscr{E}(\bar{\boldsymbol{v}}(s))$$

for a.e. t, $s \in (t_0,T)$. The concatenation $(\boldsymbol{v}^2,\xi^2)$ defined by

$$\begin{cases} (\boldsymbol{v}^2(t),E^2(t)) = (\boldsymbol{v}(t),E(t)) & \text{for } t \in (0,t_0) \\ (\boldsymbol{v}^2(t),E^2(t)) = (\bar{\boldsymbol{v}}(t),\bar{E}(t)) & \text{for } t \in (t_0,T) \end{cases} \tag{48}$$

is then again an energy-variational solution on $(0,T)$, due to $\boldsymbol{v}^2 \in \mathscr{C}_w([0,T];L^2_\sigma(\Omega))$ and

$$\lim_{t \searrow t_0} E^2(t) \leq \frac{1}{2}\|\boldsymbol{v}(t_0)\|^2_{L^2(\Omega)} < E(t_0)\,.$$

Step 3: Properties of point-wise minimizer. This implies that for a minimizer of the minimization problem (47), it holds that $E(t_0) = \mathscr{E}(\boldsymbol{v}(t_0))$. Otherwise, with the restarting procedure from above, a contradiction to $(\boldsymbol{v},E)$ being a minimum of (47) can be deduced. Assume now that there are two energy-variational solutions $(\boldsymbol{v}^1,E^1)$ and $(\boldsymbol{v}^2,E^2)$ minimizing (47) such that $\boldsymbol{v}^1(t_0) \neq \boldsymbol{v}^2(t_0)$. Due to the convexity of the admissible set and the convexity of the cost functional of (47), we infer that for any $\lambda \in (0,1)$ that also $(\boldsymbol{v}^\lambda,E^\lambda)$ with $\boldsymbol{v}^\lambda = \lambda \boldsymbol{v}^1 + (1-\lambda)\boldsymbol{v}^2$ and $E^\lambda = \lambda E^1 + (1-\lambda)E^2$ is also a minimizer of (47). From the previous restarting procedure, we infer that $E^1(t_0) = \mathscr{E}(\boldsymbol{v}^1(t_0)) = \mathscr{E}(\boldsymbol{v}^2(t_0)) = E^2(t_0)$. But for the convex combination, we infer $\mathscr{E}(\boldsymbol{v}^\lambda(t_0)) < E^\lambda(t_0)$ due to the strict convexity of $\mathscr{E}$. But again from the restarting procedure, we infer that this is not a minimizer from (47) since, we may restart again from $\boldsymbol{v}^\lambda(t_0)$ with $E(t_0) = \mathscr{E}(\boldsymbol{v}^\lambda(t_0))$, which has a smaller energy. Thus the minimum in t_0 is unique. Note that only the value in t_0 is unique but not necessarily the minimizing energy-variational solution.

Step 4: Successive minimization. We consider the sequence of time steps, $\{t_1,\ldots,t_N\} \subset [0,T]$. Starting from $t_{j-1} = t_1$, we may consider the minimization problem

$$(\bar{\boldsymbol{v}}^j,\bar{E}^j) = \arg\min_{(\boldsymbol{u},F)\in\mathfrak{S}^0(\boldsymbol{v}(t_{j-1}),\mathscr{E}(\boldsymbol{v}(t_{j-1})),\boldsymbol{f}(\cdot+t_{j-1}))} \bar{E}(t_j - t_{j-1})\,.$$

Which is solvable such that $\bar{E}^j(t_j - t_{j-1}) = \mathscr{E}(\bar{\boldsymbol{v}}^j(t_j - t_{j-1}))$ according to the previous steps. We choose one of theses minimizing solutions and construct

$$\begin{cases} (\boldsymbol{v}^j(t),E^j(t)) = (\boldsymbol{v}^{j-1}(t),E^{j-1}(t)) & \text{for } t \in (0,t_{j-1}) \\ (\boldsymbol{v}^j(t),E^j(t)) = (\bar{\boldsymbol{v}}^j(t+t_{j-1}),\bar{E}^j(t+t_{j-1})) & \text{for } t \in [t_{j-1},T)\,. \end{cases}$$

After N steps, the solution fulfills $E^N(t_j) = \mathscr{E}(\boldsymbol{v}(t_j))$ for all $j \in \{1,\ldots,N\}$.

From the weak continuity of $\boldsymbol{v} \in \mathscr{C}_w([0,T];L^2_\sigma(\Omega))$, we infer that for every $t_0 \in [0,T)$ it holds

$$\lim_{t \searrow t_0} \boldsymbol{v}(t) \rightharpoonup \boldsymbol{v}(t_0)\,.$$

Due to the regularity of the auxiliary variable $E \in \mathrm{BV}([0,\mathrm{T}])$, we know $\lim_{t\searrow t_0} E(t) = E(t_0)$. Taking this together with $E(t) \geq \mathscr{E}(\boldsymbol{v}(t))$ for all $t \in [0,T]$, we infer like for the initial value in Remark 2.6 that

$$E(t_j) = \lim_{t\searrow t_j} E(t) \geq \liminf_{t\searrow t_j} \mathscr{E}(\boldsymbol{v}(t)) \geq \mathscr{E}(\boldsymbol{v}(t_j)) = E(t_j)\,.$$

This implies that $\lim_{t\searrow t_j} \mathscr{E}(\boldsymbol{v}(t)) = \mathscr{E}(\boldsymbol{v}(t_j))$ such that $\lim_{t\searrow t_j} \boldsymbol{v}(t) \to \boldsymbol{v}(t_j)$ in $L^2_\sigma(\Omega)$ for all $j \in \{1,\dots,N\}$. □

Proof of Proposition 2.18. Assume that there exists a solution in the sense of Definition 2.17. In case there is a point t_0 such that $E(t_0) > \mathscr{E}(\boldsymbol{v}(t_0))$, we may construct an energy-variational solution by restarting in t_0 as in Step 2 of the previous proof. The existence of such an energy-variational solution with smaller auxiliary variable E in t_0 contradicts the Definition of minimal energy-variational solutions according to Definition 2.17. From Proposition 2.12 it follows that $\boldsymbol{v}$ is a weak solution. □

References

[1] N. Alia, V. John, and S. Ollila. Revisiting the single-phase flow model for liquid steel ladle stirred by gas. *Appl. Math. Model.*, 67:549–556, 2019.

[2] J.-P. Aubin, H. Frankowska. *Set-Valued Analysis*. Birkhäuser, Boston, 1990.

[3] L̀. Baňas, R. Lasarzik, and A. Prohl. Numerical analysis for nematic electrolytes. *IMA J. Numer. Anal.*, 41(3):2186–2254, 2020.

[4] D. Breit, E. Feireisl, and M. Hofmanová. Dissipative solutions and semiflow selection for the complete Euler system. *Comm. Math. Phys.*, 376(2):1471–1497, 2020.

[5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, New York, 2011.

[6] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. Math.*, 189(1):101–144, 2019.

[7] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer Berlin, 2016.

[8] C. Dafermos. Maximal dissipation in equations of evolution. *J. Differ. Equ.*, 252(1):567 – 587, 2012.

[9] C. M. Dafermos. The entropy rate admissibility criterion for solutions of hyperbolic conservation laws. *J. Differ. Equ.*, 14(2):202 – 212, 1973.

[10] G. Dal Maso. *An introduction to Γ-convergence.* Birkhäuser, Boston, 1993.

[11] C. De Lellis and L. Székelyhidi Jr. The Euler equations as a differential inclusion. *Ann. Math.*, 170(3):1417–1436, 2009.

[12] C. De Lellis and L. Székelyhidi Jr. On Admissibility Criteria for Weak Solutions of the Euler Equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[13] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. SIAM, Philadelphia, 1999.

[14] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhäuser, Basel, 2009.

[15] M. Grmela and H. C. Öttinger. Dynamics and thermodynamics of complex fluids. I. Development of a general formalism. *Phys. Rev. E*, 56(6):6620–6632, Dec 1997.

[16] J. Hale. *Ordinary differential equations*. Wiley-Interscience, New York, 1969.

[17] M. Heida, R.I.A. Patterson, and D.R.M. Renger. Topologies and measures on the space of functions of bounded variation taking values in a Banach or metric space. *J. Evol. Equ.*, 19(1):111–152, Mar 2019.

[18] A. Ioffe. On lower semicontinuity of integral functionals. I. *SIAM J. Control Optim.*, 15(4):521–538, 1977.

[19] K. Kuratowski. *Topology I.* Academic press, New York, 1966.

[20] R. Lasarzik. Dissipative solution to the Ericksen–Leslie system equipped with the Oseen–Frank energy. *Z. Angew. Math. Phy.*, 70(1):8, 2018.

[21] R. Lasarzik. Weak-strong uniqueness for measure-valued solutions to the Ericksen–Leslie model equipped with the Oseen–Frank free energy. *J. Math. Anal. Appl.*, 470(1):36–90, 2019.

[22] R. Lasarzik. Approximation and optimal control of dissipative solutions to the Ericksen–Leslie system. *Numer. Func. Anal. Opt.*, 40(15):1721–1767, 2019.

[23] R. Lasarzik. Maximally dissipative solutions for incompressible fluid dynamics. *Z. Angew. Math. Phy.*, 73(1), 2022.

[24] R. Lasarzik. Analysis of a thermodynamically consistent Navier–Stokes–Cahn–Hilliard model. *Nonlin. Anal.*, 213:112526, 2021.

[25] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63(1):193–248, 1934.

[26] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1.* The Clarendon Press, New York, 1996.

[27] I. P. Natanson. *Theorie der Funktionen einer reelen Veränderlichen*. Akademie-Verlag, Berlin, 1961.

[28] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*. Springer Berlin, 2004.

[29] T. Roubíček. *Relaxation in optimization theory and variational calculus*, Walter de Gruyter & Co., Berlin, 1997.

[30] J. Serrin. On the interior regularity of weak solutions of the Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[31] L. D. G. Sigalotti, E. Sira, J. Klapp, and L. Trujillo. *Environmental Fluid Mechanics: Applications to Weather Forecast and Climate Change*, pages 3–36. Springer International Publishing, Cham, 2014.

[32] J. Simon. On the existence of the pressure for solutions of the variational Navier–Stokes equations. *J. Math. Fluid Mech.*, 1(3):225–234, 1999.

[33] R. Temam. *The Navier-Stokes equations: Theory and numerical analysis*. American Math. Soc., New York, 1984 (corrected reprint 2001).