On the existence of weak solutions in the context of multidimensional incompressible fluid dynamics

Robert Lasarzik

submitted: April 22, 2021

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: robert.lasarzik@wias-berlin.de

On the existence of weak solutions in the context of multidimensional incompressible fluid dynamics

Robert Lasarzik

Abstract

We define the concept of energy-variational solutions for the Navier–Stokes and Euler equations. This concept is shown to be equivalent to weak solutions with energy conservation. Via a standard Galerkin discretization, we prove the existence of energy-variational solutions and thus weak solutions in any space dimension for the Navier–Stokes equations. In the limit of vanishing viscosity the same assertions are deduced for the incompressible Euler system. Via the selection criterion of maximal dissipation we deduce well-posedness for these equations.

Contents

1   Introduction

The Navier–Stokes and Euler equations are the standard models for incompressible fluid dynamics. Both are a recurrent tools in computational fluid dynamics for weather forecast, micro fluidic devices [26] or industrial processes like steel production [1]. There exists a vast literature concerning the Navier–Stokes and Euler equations. In case of the Navier–Stokes equation, we only mention here the existence proof for *weak solutions* in three dimension by Leray [22] and the *weak-strong uniqueness* result due to Serrin [25]. In the context of the Euler equations, the existence of weak solutions in any space dimension is already known (see [9]) also fulfilling the energy inequality (see [10]). This result was proven via the convex integration technique. This technique grants the existence of infinitely many and also non-physical weak solutions. Additionally, it was proven for the Navier–Stokes equations via similar techniques that there exist infinitely many weak solutions that do not fulfill the energy

inequality [5]. But what is lacking in the literature so far is an existence result for the Navier–Stokes equations in space dimensions larger than four and a physically motivated selection criterion that provides well-posedness for weak solutions, which are nowadays well-accepted. The present article provides a remedy for these shortcomings by introducing *energy-variational solutions*. As the name already suggests, this notion of generalized solutions is based on a *variation of the underlying energy-dissipation principle*. The definition is very similar to dissipative solutions but it is more selective such that these solutions are actually equivalent to weak solutions with energy inequality in the case of the considered Navier–Stokes and Euler equations.

Dissipative solutions were proposed by P.-L. Lions [23, Sec. 4.4] in the context of the Euler equations. The current author applied this concept in the context of nematic liquid crystals [17] and nematic electrolytes [2]. It was observed that natural discretizations complying with the properties of the system, like energetic or entropic principles, as well as algebraic restrictions converge naturally to a dissipative solution instead of a measure valued solution (see [2] and [19] for details). In comparison to measure-valued solutions, the degrees of freedom are heavily reduced and no defect measures occur, which are especially difficult to approximate. The relative energy inequality, which is at the heart of the dissipative and energy-variational solution concept is also a recurrent tool in PDE theory to prove for instance weak-strong uniqueness [18], stability of stationary states [17], convergence to singular limits [13], or to design optimal control schemes [19]. An advantage in comparison to distributional or measure-valued solutions is that the solution set inherits the convexity of the energy and dissipation functional, which permits to define appropriate uniqueness criteria [20].

The definition of energy-variational solutions follows a similar idea as the definition of dissipative solutions, both rely on the so-called relative energy inequality, which compares the solution to smooth test functions fulfilling the PDE only approximately. But the relative energy inequality for energy-variational solutions is refined such that the resulting inequality becomes an equality for smooth solutions. The nonlinear-convective terms are not only estimated by the relative energy but included in the underlying dissipation potential. Still the properties of the relative energy inequality remain present, it is preserved for sequences converging in the weak topologies of the associated natural energy and dissipation spaces. Thus reformulating the weak solution as an energy-variational solution has the advantage that no strong convergence is needed in order to pass to the limit in this formulation. The existence result only relies on standard constructive proofs, *i.e.*, a Galerkin discretization in the case of the Navier–Stokes equations and the vanishing viscosity limit in the case of the Euler equations.

Since the energy and dissipation functionals in the considered cases are convex, the set of energy-variational solutions is convex and weakly closed. This allows to identify selection criteria in order to select the physically relevant solution. Following the ideas of [3, 7, 8, 20], we propose the selection principle of maximal dissipation. This says that the physically relevant solution dissipates energy at the highest rate, hence minimizes the energy. This principle becomes even more apparent in thermodynamical consistent systems, where the maximized dissipation implies maximal entropy (see for instance [14] and [6, Sec. 9.7]).

In [20], the set of dissipative solutions together with the energy functional is identified as a suitable convex structure on which such a minimization problem can be defined. The resulting maximally dissipative solution is indeed well-posed in the sense of Hadamard. The result of the article at hand applies this technique to energy-variational solutions such that the selected solution actually is a weak solution, which is nowadays well-accepted. Via the selection criterion of maximal dissipation, we may select a unique weak solution. This corresponding result may be formulated as well-posedness for maximal dissipative weak solutions.

It is worth noticing that in the framework of the relative energy inequality it is possible to pass to

the limit in the quadratic convection term without any strong compactness argument. Only arguments from the direct method of the calculus of variations are needed. It is possible to pass to the limit in the quadratic term, since it is dominated by the energy in the relative energy inequality. This provides a new tool for the existence of energy-variational and thus weak solutions to nonlinear evolution equations. Usually compact embeddings and *a priori* estimates of the time derivative are used to infer strong convergence via some Aubin–Lions argument (compare to [28]). Depending on the strategy of such an existence proof, often an *a priori* estimate for a fractional time-derivative is deduced. These ingredients are irrelevant in the present proof, since it only relies on weak convergence in natural spaces and the weakly-lower semi-continuity of the underlying energy and dissipation functionals. The proposed technique seems to be very powerful and easily adapted to other systems of PDEs. Hence, this gives hope that the new approach may allows to prove the existence of energy-variational and thus weak solutions to some PDE systems, where this seems to be out of reach with other available techniques. This includes multidimensional conservation laws [3], liquid crystals [18], heat-conducting complex fluids [21], or GENERIC systems in general (see [14] and [20]).

Plan of the paper: After providing some notation and preliminaries in Section 2.1, the different solution concepts of weak, energy-variational and minimal energy-variational solutions are defined in Section 2.2. Then, we state the main Theorems in Section 2.3 and prove them afterwards (see Section 3).

2 Definitions and main theorems

2.1 Preliminaries

Before, we provide the definitions and main results, we collect some notation and preliminary results.

Notations: Throughout this paper, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$. The space of smooth solenoidal functions with compact support is denoted by $\mathscr{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^d)$. By $L_\sigma^2(\Omega)$ and $H_{0,\sigma}^1(\Omega)$ we denote the closure of $\mathscr{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^d)$ with respect to the norm of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. Note that $L_\sigma^2(\Omega)$ can be characterized by $L_\sigma^2(\Omega) = \{\boldsymbol{v} \in L^2(\Omega) | \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \partial\Omega\}$, where the first condition has to be understood in the distributional sense and the second condition in the sense of the trace in $H^{-1/2}(\partial\Omega)$. The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; the duality pairing is denoted by $\langle \cdot, \cdot \rangle$ and the L^2-inner product by $(\cdot, \cdot)$. The symmetric part of a matrix is given by $\boldsymbol{A}_{\text{sym}} := \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$ for $\boldsymbol{A} \in \mathbb{R}^{d \times d}$. For the product of two matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d \times d}$, we observe

$$\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A} : \boldsymbol{B}_{\text{sym}}, \quad \text{if } \boldsymbol{A}^T = \boldsymbol{A}.$$

Furthermore, it holds $\boldsymbol{a} \otimes \boldsymbol{b} : \boldsymbol{A} = \boldsymbol{a} \cdot \boldsymbol{A}\boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ and hence $\boldsymbol{a} \otimes \boldsymbol{a} : \boldsymbol{A} = \boldsymbol{a} \cdot \boldsymbol{A}\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{A}_{\text{sym}}\boldsymbol{a}$. We define $\phi \in \tilde{\mathscr{C}}([0,T])$ by $\phi \in \mathscr{C}^1([0,T])$ with $\phi \geq 0$, and $\phi' \leq 0$ on $[0,T]$ as well as $\phi(0) = 1$ and $\phi(T) = 0$.

We use the standard notation $(H_0^1(\Omega))^* = H^{-1}(\Omega)$. For $\tilde{\boldsymbol{v}} \in W^{1\infty}(\Omega; \mathbb{R}^d)$, $(\nabla \tilde{\boldsymbol{v}})_{\text{sym},-}$ denotes the largest eigenvalue of the matrix $-(\nabla \tilde{\boldsymbol{v}})_{\text{sym}}$,

$$(\nabla \tilde{\boldsymbol{v}})_{\text{sym},-} = \left(\sup_{|\boldsymbol{a}|=1} - \left(\boldsymbol{a}^T \cdot (\nabla \tilde{\boldsymbol{v}})_{\text{sym}} \boldsymbol{a} \right) \right).$$

By I, we denote the identity matrix in $\mathbb{R}^{d \times d}$ and by $\mathbb{R}_+ := [0, \infty)$ the positive real numbers.

The following lemma provides the connection between the almost everywhere pointwise formulation of an inequality with the weak one.

Lemma 2.1. Let $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$ with $g \geq 0$ a.e. in $(0,T)$. Then the two inequalities

$$-\int_0^T \phi'(t)g(t)\,\mathrm{d}t - g(0) + \int_0^T \phi(t)f(t)\,\mathrm{d}t \leq 0$$

for all $\phi \in \tilde{\mathscr{C}}([0,T])$. and

$$g(t) - g(0) + \int_0^t f(s)\,\mathrm{d}s \leq 0 \quad \text{for a.e. } t \in (0,T) \tag{1}$$

are equivalent. See the notations for the definition of $\tilde{\mathscr{C}}([0,T])$.

See [20, Lemma 2.4] for a proof. Additionally, we use a lemma that provides the lower semi-continuity of convex functionals.

Lemma 2.2. Let $A \subset \mathbb{R}^{d+1}$ be a bounded open set and $f: A \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $d,n,m \geq 1$, a measurable non-negative function such that $f(\boldsymbol{y},\cdot,\cdot)$ is lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^m$ for a.e. $\boldsymbol{y} \in A$, and f is convex in the last entry. For sequences $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$ and $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$ as well as functions $\boldsymbol{u} \in L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$ and $\boldsymbol{v} \in L^1_{\mathrm{loc}}(A;\mathbb{R}^m)$ with

$$\boldsymbol{u}_k \to \boldsymbol{u} \quad \text{a.e. in } A \quad \text{and} \quad \boldsymbol{v}_k \rightharpoonup \boldsymbol{v} \quad \text{in } L^1_{\mathrm{loc}}(A;\mathbb{R}^n)$$

it holds

$$\liminf_{k\to\infty} \int_A f(\boldsymbol{y},\boldsymbol{u}_k(\boldsymbol{y}),\boldsymbol{v}_k(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} \geq \int_A f(\boldsymbol{y},\boldsymbol{u}(\boldsymbol{y}),\boldsymbol{v}(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}.$$

The proof of this assertion can be found in [16].

2.2 Definitions

First we recall the Navier–Stokes and Euler equations,

$$\partial_t \boldsymbol{v} + \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega \times (0,T),$$
$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega, \tag{2}$$
$$\nu(I - \boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{v} = 0 \quad \text{and} \quad \boldsymbol{n}\cdot\boldsymbol{v} = 0 \quad \text{on } \partial\Omega \times (0,T).$$

By writing the boundary conditions in this way, the system incorporates the Navier–Stokes system with no-slip conditions for $\nu > 0$ and the Euler equations for $\nu = 0$. Indeed, for $\nu > 0$, the tangential and normal part of the velocity field vanish such that this is equivalent to $\boldsymbol{v} = 0$ on $\partial\Omega \times (0,T)$. For the case of $\nu = 0$, *i.e.*, no friction, only the normal component vanishes on the boundary. The underlying natural energy and dissipation spaces are given by $\mathbb{X}_\nu = L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega))$ for $\nu > 0$ and $\mathbb{X}_\nu = L^\infty(0,T;L^2_\sigma(\Omega))$ for $\nu = 0$ and the space of test-function is given by $\mathbb{Y}_\nu = H^2(\Omega) \cap H^1_{0,\sigma}(\Omega) \cap L^d(\Omega)$ for $\nu > 0$ and $\mathbb{Y}_\nu = W^{1,\infty}(\Omega) \cap H^1_{0,\sigma}(\Omega)$ for $\nu = 0$. The space $\mathbb{Y}_\nu$ is chosen smooth enough such that the Stokes operator (for $\nu > 0$) and the convection term map $\mathbb{Y}_\nu$ to $L^2_\sigma(\Omega)$. This is obvious for the Stokes operator. In case of the convection term, we observe for $\nu > 0$ that $H^2(\Omega) \hookrightarrow W^{1,2d/(d-2)}(\Omega)$ such that Hölder's inequality implies

$$\|(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\|_{L^2(\Omega)} \leq \|\tilde{\boldsymbol{v}}\|_{L^d(\Omega)}\|\nabla\tilde{\boldsymbol{v}}\|_{L^{2d/(d-2)}(\Omega)} \leq \|\tilde{\boldsymbol{v}}\|_{\mathbb{Y}_\nu}^2 \quad \text{for } \nu > 0.$$

The case of $\nu = 0$ follows by similar arguments. The right-hand side $\boldsymbol{f}$ is assumed to be in $\mathbb{Z}_\nu$, where $\mathbb{Z}_\nu := L^2(0,T;H^{-1}(\Omega)) \oplus L^1(0,T;L^2(\Omega))$ for $\nu > 0$ and $\mathbb{Z}_\nu := L^1(0,T;L^2(\Omega))$ for $\nu = 0$.

Definition 2.3 (weak solution)**.** A function $\boldsymbol{v}$ is called a weak solution, if $\boldsymbol{v} \in \mathbb{X}_{\nu}$ fulfills the energy inequality

$$\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2(\Omega)}\Big|_0^t + \int_0^t \nu\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \int_0^t \langle \boldsymbol{f},\boldsymbol{u}\rangle\,\mathrm{d}s \quad \text{for a.e. } t \in (0,T) \tag{3}$$

and the weak formulation

$$-\int_0^T \int_\Omega \boldsymbol{v}\partial_t\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \int_\Omega (\nu\nabla\boldsymbol{v}:\nabla\boldsymbol{\varphi} - (\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,\mathrm{d}t + \int_\Omega \boldsymbol{v}_0\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}\boldsymbol{x} \tag{4}$$

for every $\boldsymbol{\varphi} \in \mathscr{C}_c^1([0,T)) \otimes \mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^d)$.

Remark 2.1. The previous definition differs from the usual definition of weak solutions to the Navier–Stokes equations, since no regularity for the (fractional) time-derivative of $\boldsymbol{v}$ is assumed such that we formulated the time-derivative in a weak sense. This is somehow also the difference in comparison to previous existence proofs, where a bound on the (fractional) time derivative together with a compact embedding was used to apply some Lions–Aubin argument, in order to pass to the limit in the nonlinear term in the weak formulation. In this article, no strong convergence is needed. The convergence in the nonlinear term can be deduced, since it is dominated by the energy in the formulation of the relative energy inequality.

We define the relative energy $\mathscr{R}$ by

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \frac{1}{2}\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}, \tag{5a}$$

the relative dissipation $\mathscr{W}_\nu$ by

$$\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \nu\|\nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} - \int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + \mathscr{K}_\nu(\tilde{\boldsymbol{v}})\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}, \tag{5b}$$

for $\nu > 0$, where the regularity measure $\mathscr{K}_\nu$ is given by

$$\mathscr{K}_\nu(\tilde{\boldsymbol{v}}) = \mathscr{K}_\nu^{s,r}(\tilde{\boldsymbol{v}}) = c\|\tilde{\boldsymbol{v}}\|^s_{L^r(\Omega)} \quad \text{for } \frac{2}{s} + \frac{d}{r} = 1, \tag{5c}$$

where c can be calculated according to the estimate (6a) below. In the case $\nu = 0$, the relative dissipation $\mathscr{W}_0$ is given by

$$\mathscr{W}_0(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \int_\Omega (\boldsymbol{v}-\tilde{\boldsymbol{v}})^T \cdot (\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \mathscr{K}_0(\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}). \tag{5d}$$

The regularity measure changes to $\mathscr{K}_0(\tilde{\boldsymbol{v}}) = 2\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{L^\infty(\Omega)}$, where $(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}$ denotes the largest eigenvalue of the matrix $-(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}$ (see 2.1). Finally, the solution operator $\mathscr{A}_\nu$ is given by

$$\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\cdot\rangle = \langle\partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} - \nu\Delta\tilde{\boldsymbol{v}} - \boldsymbol{f},\cdot\rangle, \tag{5e}$$

which has to be understood in a weak sense, at least with respect to space.

Note that the solution operator does not include boundary conditions, since they are encoded in the underlying spaces. This may changes for different boundary conditions.

Remark 2.2. The relative dissipation $\mathscr{W}_\nu$ is chosen in a way that it is nonnegative, convex, and weakly-lower semi-continuous. Indeed Hölder's, Gagliardo–Nirenberg's, and Young's inequality provide the estimate for $\nu > 0$

$$\left| \int_\Omega ((\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot \nabla)(\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot \tilde{\boldsymbol{v}} \mathrm{d}\boldsymbol{x} \right| \leq \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^p(\Omega)} \|\nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)} \|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)}$$

$$\leq c_p \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^{(1-\alpha)} \|\nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^{(1+\alpha)} \|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)} \tag{6a}$$

$$\leq \frac{\nu}{2} \|\nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + c \|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)}^{2/(1-\alpha)} \frac{1}{2} \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 ,$$

where α is chosen according to Gagliardo–Nirenberg's inequality by

$$\alpha = d(p-2)/2p \quad \text{for} \quad d \leq 2p/(p-2).$$

In the case of $\nu = 0$, we may estimate

$$((\boldsymbol{v} - \tilde{\boldsymbol{v}}) \otimes (\boldsymbol{v} - \tilde{\boldsymbol{v}}); (\nabla \tilde{\boldsymbol{v}})_{\mathrm{sym}}) \leq 2 \|(\nabla \tilde{\boldsymbol{v}})_{\mathrm{sym},-}\|_{L^\infty(\Omega)} \frac{1}{2} \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 . \tag{6b}$$

The estimate (6) imply that $\mathscr{W}_\nu$ is non-negative. Since $\mathscr{W}_\nu$ is quadratic in $\boldsymbol{v}$ and non-negative, it is a standard matter to prove the convexity of the mapping $\boldsymbol{v} \mapsto \mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}})$. The mapping $\boldsymbol{v} \mapsto \mathscr{W}_\nu(\boldsymbol{v}|\cdot)$ is continuous in the strong topology in $H^1_{0,\sigma}(\Omega)$ and $L^2_\sigma(\Omega)$ for $\nu > 0$ and $\nu = 0$, respectively. Thus this mapping is weakly-lower semi-continuous (see for instance [12, Chap. 1, Cor. 2.2]).

Definition 2.4 (energy-variational solution). A function $\boldsymbol{v}$ is called an energy-variational solution, if $\boldsymbol{v} \in \mathbb{X}_\nu$ and the relative energy inequality

$$\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t)) + \int_0^t \left(\mathscr{W}_\nu(\boldsymbol{v},\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}(t)), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \right) e^{\int_s^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \mathrm{d}\tau} \mathrm{d}s \leq \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0)) e^{\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \mathrm{d}s} \tag{7}$$

holds for a.e. $t \in (0,T)$ and for all $\tilde{\boldsymbol{u}} \in \mathscr{C}^1([0,T];\mathbb{Y}_\nu)$.

Remark 2.3 (Properties of energy-variational solutions). An energy-variational solution fulfills certain standard properties of generalized solutions concepts. If a strong solution exists locally-in-time, every energy-variational solution coincides with this strong solution as long as the latter exists. This is the so-called weak-strong uniqueness property. On the other hand, if the energy-variational solution enjoys sufficient regularity, than it is again a unique strong solution. An advantage of this formulation in comparisson to the standard weak formulation is that the set of energy-variational solutions is by its definition weakly sequentially closed in the weak topology of the natural energy and dissipation spaces (see [20]).

Remark 2.4 (Comparison to dissipative solutions). The difference of the proposed energy-variational solution framework in comparison to dissipative solutions lies in the definition of the relative dissipation $\mathscr{W}_\nu$. In dissipative solution concepts, the terms in the relative dissipation were only estimated from below by zero (see [23] and [20]). The new insight is that these terms in $\mathscr{W}_\nu$ can be kept and don't have to be estimated. This also leads to the fact that the relative energy inequality is actually an equality for smooth solutions. Indeed in this case the energy inequality (3) is an equality and thus also the relative energy inequality. Especially, the energy-variational solution concept is independent of the choice of the regularity measure $\mathscr{K}_\nu$.

Definition 2.5 (minimal energy-variational solution). A function $\boldsymbol{u}$ is called a minimal energy-variational solution, if $\boldsymbol{u} \in \mathbb{X}$ is the solution of the following optimization problem

$$\min_{\boldsymbol{u} \in \mathbb{X}} \int_0^T \mathscr{E}(\boldsymbol{u}(t)) \mathrm{d}t \quad \text{such that } \boldsymbol{u} \text{ is an energy-variational solution according to Definition 2.4.}$$

Remark 2.5 (Selection criterion). The proposed selection criterion relies on the insight that a physically relevant solution dissipates energy the most (see [7] or [8]). This leads to a minimized energy (compare the energy inequality (3), which is formally an equality). In a thermodynamical consistent system, the energy would be constant, but the maximized dissipation leads to a maximized entropy (see [14] for instance). This criterion was introduced as the entropy rate admissibility criterion [8]. There are different works on the entropy rate admissibility criterion applied to different systems. For instance, in the case of scalar conservation laws it was shown that this criterion coincides with the Oleinik-E condition and thus the usual entropy admissibility criterion for solutions with finitely many shocks (see [8] or [6, Thm. 9.7.2] for the result). Since this criterion was proven to select the physically relevant solution in these scarcely available examples of nonlinear PDEs that are well understood, it may also does this for more involved systems (like the ones we consider here).

One may chooses different selection criteria. All results also hold, in case that the function $\int_0^T \mathscr{E}(\cdot)\,\mathrm{d}t$ is replaced by any other strictly convex function on $\mathbb{X}_\nu$.

2.3 Main results

The main results of the paper at hand are the following

Proposition 2.6. Let $\boldsymbol{v} \in \mathbb{X}$. Then $\boldsymbol{v}$ is an energy-variational solution solution according to Definition 2.4 if and only if it is a weak solution according to Definition 2.3.

Remark 2.6 (Comparison to measure-valued solutions). In the case of the Euler equations ($\nu = 0$ in (2)), measure-valued solutions are well-known since the seminal work of DiPerna nad Majda [11]. It was already observed in [4, Prop. 2] that the expectation of the oscillation measure of the generalized Young measure associated to a measure-valued solution is indeed a dissipative solution due to Lions [23, Sec. 4.4] (compare to [4]). The same assertion holds true for the proposed energy-variational formulation and thus, also weak solutions.

Theorem 2.7. Let $\Omega \subset \mathbb{R}^d$ for $d \geq 2$ be a bounded Lipschitz domain, $\nu \geq 0$. Let $\mathscr{R}$, $\mathscr{W}_\nu$, $\mathscr{K}_\nu$, and $\mathscr{A}_\nu$ be given as above in (5).

Then there exists at least one energy-variational solution $\boldsymbol{v} \in \mathbb{X}_\nu$ to every $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$ and $\boldsymbol{f} \in \mathbb{Z}_\nu$ in the sense of Definition 2.4 and thus also a weak solution according to Definition (2.3).

Remark 2.7. In the case of $d = 2, 3$ or 4, the existence of weak solutions to the Navier–Stokes equations is well known (see for instance [28]). Due to Proposition 2.6, this also proves the existence of energy-variational solutions. The new result of the preceding theorem is expanding these existence results to any space dimension. Additionally, the technique of the proof is essentially new, since is only relies on the reformulation of the problem and weak convergence arguments.

Theorem 2.8. Let $\Omega \subset \mathbb{R}^d$ for $d \geq 2$ be a bounded Lipschitz domain, $\nu \geq 0$. Let $\mathscr{R}$, $\mathscr{W}_\nu$, $\mathscr{K}_\nu$, and $\mathscr{A}_\nu$ be given as above in (5).

Then there exists a unique minimal energy-variational solution $\boldsymbol{v} \in \mathbb{X}_\nu$ to every $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$ and $\boldsymbol{f} \in \mathbb{Z}_\nu$ in the sense of Definition 2.5 and the minimal energy-variational solution depends continuously on the initial datum and the right-hand side in the following sense: If $(\boldsymbol{v}_0^n, \boldsymbol{f}^n) \to (\boldsymbol{v}_0, \boldsymbol{f})$ in $L^2_\sigma(\Omega) \times \mathbb{Z}_\nu$, then to every $n \in \mathbb{N}$, there exists a minimal energy-variational solution $\boldsymbol{v}^n \in \mathbb{X}_\nu$ and it holds $\boldsymbol{v}^n \overset{*}{\rightharpoonup} \boldsymbol{v}$ in $\mathbb{X}_\nu$. Especially, this minimal energy-variational solution is indeed a weak solution according to Definition 2.3.

Remark 2.8. The continuous dependence result is rather weak, it only holds in the weak topology. This means that small differences in the initial value or right hand side may lead to large differences due to oscillations. This is not surprising, if one thinks about turbulence in fluids. Nevertheless, this is the first well-posedness result for the Navier–Stokes and Euler systems involving weak solutions.

3 Proofs of the main theorems

3.1 Equivalence of weak and energy-variational solutions

First, we show that weak solutions are equivalent to energy-variational solutions. The if-direction is very similar to the proof in [20].

Proof of Proposition 2.6. Let $\boldsymbol{v}$ be a weak solution to the Navier–Stokes and Euler equations (2) with energy inequality according to Definition 2.3 for $\nu \geq 0$.

For a test function $\tilde{\boldsymbol{v}} \in \mathscr{C}^1([0,T];\mathbb{Y}_\nu)$, we find by testing the solution operator $\mathscr{A}_\nu(\tilde{\boldsymbol{v}})$ by $\phi\tilde{\boldsymbol{v}}$ with $\phi \in \mathscr{C}_c^1((0,T))$ and standard calculations that

$$\int_0^T \phi\,\langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}t =$$
$$-\int_0^T \phi'\frac{1}{2}\|\tilde{\boldsymbol{v}}(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t + \int_0^T \phi\left(\nu\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2 - \langle\boldsymbol{f},\tilde{\boldsymbol{v}}\rangle\right)\mathrm{d}t - \phi(0)\frac{1}{2}\|\tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2. \quad (8)$$

Testing again the solution operator $\mathscr{A}_\nu(\tilde{\boldsymbol{v}})$ by $\phi\boldsymbol{v}$ and choosing $\boldsymbol{\varphi}$ to be $\phi\tilde{\boldsymbol{v}}$ in (4) with $\phi \in \mathscr{C}_c^1((0,T))$ (or approximate it appropriately), we find

$$-\int_0^T \phi'\int_\Omega \boldsymbol{v}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^T \phi\int_\Omega (2\nu\nabla\boldsymbol{v}:\nabla\tilde{\boldsymbol{v}} - (\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$= \int_0^T \phi\,\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}\rangle\,\mathrm{d}t + \phi(0)\int_\Omega \boldsymbol{v}_0\cdot\tilde{\boldsymbol{v}}(0)\,\mathrm{d}\boldsymbol{x} + \int_0^T \phi\,\langle\boldsymbol{f},\tilde{\boldsymbol{v}}+\boldsymbol{v}\rangle\,\mathrm{d}t. \quad (9)$$

Reformulating (3) by Lemma 2.1, adding (8), as well as subtracting (9), let us deduce that

$$-\int_0^T \phi'\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2\,\mathrm{d}t + \nu\int_0^T \phi\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2\,\mathrm{d}t - \phi(0)\frac{1}{2}\|\boldsymbol{v}_0-\tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2$$
$$-\int_0^T \phi\int_\Omega ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v} - (\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^T \phi\,\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}t \leq 0 \quad (10)$$

for all $\phi \in \tilde{C}([0,T])$. Note that $\tilde{C}([0,T]) \subset \mathrm{clos}_{\mathscr{C}^1((0,T))\cap\mathscr{C}([0,T])}(\mathscr{C}_c^1((0,T)))$. We adopt some standard manipulations using the skew-symmetry of the convective term in the last two arguments and the fact that $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are divergence free, to find

$$-\int_\Omega ((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v} - (\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega ((\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot(\boldsymbol{v}-\tilde{\boldsymbol{v}}))\,\mathrm{d}\boldsymbol{x}$$
$$= -\int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}$$

for $\nu > 0$ and

$$-\int_\Omega ((\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}} \cdot \boldsymbol{v} - (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \tilde{\boldsymbol{v}}) \, d\boldsymbol{x} = \int_\Omega (\boldsymbol{v} - \tilde{\boldsymbol{v}})^T \cdot (\nabla \tilde{\boldsymbol{v}})_{\text{sym}} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \, d\boldsymbol{x}$$

$$\int_\Omega ((\boldsymbol{v} - \tilde{\boldsymbol{v}}) \otimes \tilde{\boldsymbol{v}}) : \nabla \tilde{\boldsymbol{v}} \, d\boldsymbol{x} + \int_\Omega (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{v}} \cdot (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \, d\boldsymbol{x}$$

$$= \int_\Omega (\boldsymbol{v} - \tilde{\boldsymbol{v}})^T \cdot (\nabla \tilde{\boldsymbol{v}})_{\text{sym}} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \, d\boldsymbol{x}$$

for $\nu = 0$.

Inserting this into (10), adding as well as subtracting $\mathscr{K}_\nu(\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})$ and replacing ϕ by $\varphi e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds}$ (or approximate it appropriately), we conclude

$$-\int_0^T \varphi' \frac{1}{2} \|\boldsymbol{v}(t) - \tilde{\boldsymbol{v}}(t)\|_{L^2(\Omega)}^2 e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds} \, dt - \frac{1}{2} \|\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2$$

$$+ \int_0^T \varphi \left(\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \right) e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, dt} \, ds \leq 0$$

for every smooth function $\tilde{\boldsymbol{v}} \in \mathscr{C}^1([0,T]; \mathbb{Y}_\nu)$ and all $\varphi \in \tilde{\mathscr{C}}([0,T])$. Lemma 2.1 and multiplying the resulting inequality by $e^{\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds}$ implies (7).

Now, we assume that $\boldsymbol{v} \in \mathbb{X}_\nu$ is an energy-variational solution according to Definition 2.4. Multiplying the relative energy inequality (7) by $e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds}$ and applying Lemma (2.1), we find

$$-\int_0^T \phi' \mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds} \, dt + \int_0^T \phi \left(\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \right) e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds} \, dt$$

$$- \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0)) \leq 0 \quad (11)$$

for all $\phi \in \tilde{\mathscr{C}}([0,T])$. Via defining $\varphi(t) = \phi(t) e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds}$ and the product rule $\varphi'(t) = (\phi'(t) - \phi(t)\mathscr{K}_\nu(\tilde{\boldsymbol{v}}(t)))e^{-\int_0^t \mathscr{K}_\nu(\tilde{\boldsymbol{v}}) \, ds}$ we find

$$-\int_0^T \varphi' \mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) \, dt + \int_0^T \varphi \left(\mathscr{W}_\nu(\boldsymbol{v}|\tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle - \mathscr{K}_\nu(\tilde{\boldsymbol{u}})\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) \right) \, dt - \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0)) \leq 0$$

for all $\varphi \in \tilde{\mathscr{C}}([0,T])$. Applying again Lemma 2.1 and the Definition of $\mathscr{W}_\nu$, we observe

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})\Big|_0^t + \int_0^t \nu \|\nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 - (((\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot \nabla)(\boldsymbol{v} - \tilde{\boldsymbol{v}}), \tilde{\boldsymbol{v}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \, ds \leq 0 \quad (12a)$$

for $\nu > 0$ and a.e. $t \in (0,T)$ as well as

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})\Big|_0^t + \int_0^t ((\boldsymbol{v} - \tilde{\boldsymbol{v}}) \otimes (\boldsymbol{v} - \tilde{\boldsymbol{v}}), (\nabla \tilde{\boldsymbol{v}})_{\text{sym}}) + \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \, ds \leq 0 \quad (12b)$$

for $\nu = 0$ and for a.e. $t \in (0,T)$. For the solution operator $\mathscr{A}_\nu$, we find

$$\int_0^t \langle \mathscr{A}_\nu(\tilde{\boldsymbol{v}}), \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \, ds$$

$$= -\frac{1}{2} \|\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 \Big|_0^t + \int_0^t (\partial_t \tilde{\boldsymbol{v}}, \boldsymbol{v}) + \nu (\nabla \tilde{\boldsymbol{v}}, \nabla \boldsymbol{v} - \nabla \tilde{\boldsymbol{v}}) - ((\tilde{\boldsymbol{v}} \cdot \nabla)(\boldsymbol{v} - \tilde{\boldsymbol{v}}), \tilde{\boldsymbol{v}}) - \langle \boldsymbol{f}, \boldsymbol{v} - \tilde{\boldsymbol{v}} \rangle \, ds.$$

Inserting this into (12), we may deduce

$$\frac{1}{2}\|\boldsymbol{v}\|_{L^2(\Omega)}^2\Big|_0^t - (\boldsymbol{v},\tilde{\boldsymbol{v}})\Big|_0^t + \int_0^t \nu\,(\nabla\boldsymbol{v},\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}) + (\boldsymbol{v}\otimes(\boldsymbol{v}-\tilde{\boldsymbol{v}}),\nabla\tilde{\boldsymbol{v}}) + (\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}) - \langle\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}s \leq 0$$

for a.e. $t \in (0,T)$. Again the skew-symmetry of the trilinear form in the last two entries is used. Choosing $\tilde{\boldsymbol{v}} = \alpha\tilde{\boldsymbol{u}}$ and multiplying the inequality by $1/\alpha$ for $\alpha > 0$, we find

$$\frac{1}{\alpha}\left(\frac{1}{2}\|\boldsymbol{v}\|_{L^2(\Omega)}^2\Big|_0^t + \int_0^t \nu\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2 - \langle\boldsymbol{f},\boldsymbol{v}\rangle\right)$$
$$- (\boldsymbol{v},\tilde{\boldsymbol{u}})\Big|_0^t - \int_0^t \nu\,(\nabla\boldsymbol{v},\nabla\tilde{\boldsymbol{u}}) - ((\boldsymbol{v}\otimes\boldsymbol{v}),\nabla\tilde{\boldsymbol{u}}) - (\boldsymbol{v},\partial_t\tilde{\boldsymbol{u}}) - \langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle\,\mathrm{d}s \leq 0 \quad (13)$$

for a.e. $t \in (0,T)$. Note that the term $((\boldsymbol{v}\cdot\nabla)\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}})$ vanishes since $\boldsymbol{v}$ is solenoidal. For $\alpha\to\infty$ the first line in (13) vanishes and in the resulting inequality we may observe that $\tilde{\boldsymbol{u}}$ occurs linearly such that by inserting $\tilde{\boldsymbol{u}}$ as well as $-\tilde{\boldsymbol{u}}$, we receive an equality,

$$- (\boldsymbol{v},\tilde{\boldsymbol{u}})\Big|_0^t - \int_0^t \nu\,(\nabla\boldsymbol{v},\nabla\tilde{\boldsymbol{u}}) - ((\boldsymbol{v}\otimes\boldsymbol{v});\nabla\tilde{\boldsymbol{u}}) - (\boldsymbol{v},\partial_t\tilde{\boldsymbol{u}}) - \langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle\,\mathrm{d}s = 0.$$

for a.e. $t \in (0,T)$. Multiplying this resulting equation by ϕ' with $\phi \in \mathscr{C}_c^\infty([0,T))$ and integrating over $(0,T)$, we may observe via integration-by-parts and defining $\boldsymbol{\varphi} = \phi\tilde{\boldsymbol{u}}$ the weak formulation (4).

$\square$

3.2 Existence of energy-variational solutions

In order to prove the existence of weak solutions, we use a novel technique. By passing to the limit in the relative energy inequality, we do not need any strong compactness arguments, which was essential in previous proofs to pass to the limit in the nonlinear term. Usually an *a priori* estimate of the (fractional) time-derivative is needed in order to apply some Aubin-Lions compactness argument. This is circumvented by the formulation of the relative energy inequality and only relying on weakly-lower semi-continuity of the associated functionals.

Proof of Theorem 2.7. The proof is based on the usual Galerkin approximation together with standard weak convergence techniques. We divide the proof in different steps

Step 1, Galerkin approximation: Since the space $H_{0,\sigma}^1(\Omega)$ is separable and the space of smooth solenoidal functions with compact support, $\mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^d)$, is dense in $H_{0,\sigma}^1(\Omega)$, there exists a Galerkin scheme of $H_{0,\sigma}^1(\Omega)$, *i.e.*, $\{W_n\}_{n\in\mathbb{N}}$ with $\mathrm{clos}_{H_{0,\sigma}^1(\Omega)}(\lim_{n\to\infty}W_n) = H_{0,\sigma}^1(\Omega)$. Let $P_n : L_\sigma^2(\Omega) \longrightarrow W_n$ denote the $L_\sigma^2(\Omega)$-orthogonal projection onto W_n. The approximate problem is then given as follows: Find an absolutely continuous solution $\boldsymbol{v}^n$ with $\boldsymbol{v}^n(t) \in W_n$ for all $t \in [0,T]$ solving the system

$$(\partial_t\boldsymbol{v}^n + (\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n,\boldsymbol{w}) + \nu\,(\nabla\boldsymbol{v}^n;\nabla\boldsymbol{w}) = \langle\boldsymbol{f},\boldsymbol{w}\rangle\,, \quad \boldsymbol{v}^n(0) = P_n\boldsymbol{v}_0 \quad \text{for all } \boldsymbol{w} \in W_n\,. \quad (14)$$

A classical existence theorem (see Hale [15, Chapter I, Theorem 5.2]) provides, for every $n \in \mathbb{N}$, the existence of a maximal extended solution to the above approximate problem (14) on an interval $[0,T_n)$ in the sense of Carathéodory.

Step 2, A priori estimates: It can be deduce that $T_n = T$ for all $n \in \mathbb{N}$, if the solution undergoes no blow-up. With the standard *a priori* estimates, we can exclude blow-ups and thus deduce global-in-time existence. Testing (14) by $\boldsymbol{v}^n$, we derive the standard energy estimate

$$\frac{1}{2}\|\boldsymbol{v}^n\|_{L^2(\Omega)}^2 + \nu \int_0^t \|\nabla \boldsymbol{v}^n\|_{L^2(\Omega)}^2 \, \mathrm{d}s = \frac{1}{2}\|P_n\boldsymbol{v}_0\|_{L^2(\Omega)}^2 + \int_0^t \langle \boldsymbol{f}, \boldsymbol{v}^n \rangle \, \mathrm{d}s. \quad (15)$$

For $\boldsymbol{f} \in \mathbb{Z}_\nu = L^2(0,T;H^{-1}(\Omega)) \oplus L^1(0,T;L^2(\Omega))$ for $\nu > 0$, the right-hand side can be estimated appropriately. Indeed, there exist two functions $\boldsymbol{f}_1 \in L^2(0,T;H^{-1}(\Omega))$ and $\boldsymbol{f}_2 \in L^1(0,T;L^2(\Omega))$ such that we may estimate with Hölder's, Young's, and Poincaré's inequality that

$$\langle \boldsymbol{f}, \boldsymbol{v}^n \rangle \leq \frac{\nu}{2}\|\nabla \boldsymbol{v}^n\|_{L^2(\Omega)}^2 + \frac{C}{2\nu}\|\boldsymbol{f}_1\|_{H^{-1}(\Omega)}^2 + \|\boldsymbol{f}_2\|_{L^2(\Omega)} \left(\|\boldsymbol{v}^n\|_{L^2(\Omega)}^2 + 1 \right). \quad (16)$$

Inserting this into (15) allows to apply a Version of Gronwall's Lemma in order to infer that $\{\boldsymbol{v}^n\}$ is bounded and thus weakly* compact in $\mathbb{X}_\nu$ such that there exists a $\boldsymbol{v} \in \mathbb{X}_\nu$ with

$$\boldsymbol{v}^n \overset{*}{\rightharpoonup} \boldsymbol{v} \quad \text{in } \mathbb{X}_\nu. \quad (17)$$

Step 3, Discrete relative energy inequality: In order to show the convergence to energy-variational solutions, we derive a discrete version of the relative energy inequality. Assume $\tilde{\boldsymbol{v}} \in C^1([0,T];\mathbb{Y}_\nu)$. Adding (15) and (14) tested with $-P_n\tilde{\boldsymbol{v}}$ (and integrated in time), we find

$$\frac{1}{2}\|\boldsymbol{v}^n\|_{L^2(\Omega)}^2 + \nu \int_0^t (\nabla \boldsymbol{v}^n; \nabla \boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}) \, \mathrm{d}s =$$
$$\frac{1}{2}\|P_n\boldsymbol{v}_0\|_{L^2(\Omega)}^2 + \int_0^t \langle \boldsymbol{f}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}} \rangle + (\partial_t \boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((\boldsymbol{v}^n \cdot \nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) \, \mathrm{d}s. \quad (18)$$

For the Solution operator $\mathscr{A}_\nu$, we observe that

$$\langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}} \rangle =$$
$$(\partial_t P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) - \partial_t \frac{1}{2}\|P_n\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + (\nabla P_n\tilde{\boldsymbol{v}}, \nabla \boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) - \langle \boldsymbol{f}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}} \rangle.$$

Adding to as well as subtracting from (18) the term $\int_0^t \langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}} \rangle \, \mathrm{d}s$ leads to

$$\frac{1}{2}\|\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 \Big|_0^t + \int_0^t \left(\nu \|\nabla \boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + \langle \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}} \rangle \right) \mathrm{d}s$$
$$= \int_0^t \left(((\boldsymbol{v}^n \cdot \nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n) \right) \mathrm{d}s.$$

By some algebraic transformations, we find

$$((\boldsymbol{v}^n \cdot \nabla)\boldsymbol{v}^n, P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n)$$
$$= (((\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) \cdot \nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}})$$
$$+ ((P_n\tilde{\boldsymbol{v}} \cdot \nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}} \cdot \nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}). \quad (19)$$

For the first term on the right-hand side of (19), we observe

$$\nu \|\nabla \boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 - (((\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) \cdot \nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) = \mathscr{W}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) - \mathscr{K}(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}).$$

For the second term on the right-hand side of (19), we find with an integration-by-parts (or the usual skew-symmetry in the second two variables of the trilinear convection term) that

$$((P_n\tilde{\boldsymbol{v}}\cdot\nabla)(\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}), P_n\tilde{\boldsymbol{v}}) + ((P_n\tilde{\boldsymbol{v}}\cdot\nabla)P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) = 0.$$

In order to find the discrete version of the relative energy inequality, the term $\mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}|P_n\tilde{\boldsymbol{v}})$ is added and subtracted such that applying a version of Gronwall's lemma implies

$$\mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})e^{-\int_0^t \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau} + \int_0^t \left(\mathscr{W}_\nu(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) + \langle\mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle\right)e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s$$
$$\leq \mathscr{R}(P_n\boldsymbol{v}_0|P_n\tilde{\boldsymbol{v}}(0)) \quad (20)$$

for a.e. $t \in (0,T)$ and $\nu > 0$.

Step 4, Passage to the limit: Via Lemma 2.1, this inequality may be written as

$$-\int_0^T \phi' \mathscr{R}(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}})e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s$$
$$+ \int_0^T \phi\left(\mathscr{W}_\nu(\boldsymbol{v}^n|P_n\tilde{\boldsymbol{v}}) + \langle\mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle\right)e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s \leq \mathscr{R}(P_n\boldsymbol{v}_0|P_n\tilde{\boldsymbol{v}}(0))$$

for all $\phi \in \tilde{C}([0,T])$. Since $\mathscr{C}_{c,\sigma}^\infty(\Omega; \mathbb{R}^d)$ is also dense in $\mathbb{Y}_\nu$, we may observe the strong convergence of the projection P_n, *i.e.*,

$$\|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;H_{0,\sigma}^1(\Omega))} + \|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{d/2}(\Omega))} \to 0 \quad \text{as } n \to \infty \quad \text{for all } \tilde{\boldsymbol{v}} \in \mathscr{C}^1([0,T];\mathbb{Y}_\nu).$$
$$(21)$$

This together with (17) allows to pass to the limit in the first two terms via the weakly-lower semi-continuity of the convex functionals $\mathscr{R}$ and $\mathscr{W}_\nu$ (see Lemma 2.2 and Remark 2.2). Since $\boldsymbol{v}^n$ only occurs linearly in the last term on the left-hand side, we may also pass to the limit in this term. Indeed, the time derivative may be interchanged with the projection P_n such that

$$(\partial_t P_n\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) = (P_n\partial_t\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}) = (\partial_t\tilde{\boldsymbol{v}}, \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}),$$

where it was used that P_n is an orthogonal projection. This together with (21) imply that the consistency error vanishes, *i.e.*,

$$\int_0^T \phi\langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}) - \mathscr{A}_\nu(P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\rangle e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s$$
$$= \nu\int_0^T \phi\left(\nabla\tilde{\boldsymbol{v}} - \nabla P_n\tilde{\boldsymbol{v}}; \nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\right)e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s$$
$$+ \int_0^T \phi\left(((\tilde{\boldsymbol{v}} - P_n\tilde{\boldsymbol{v}})\cdot\nabla)\tilde{\boldsymbol{v}} + (P_n\tilde{\boldsymbol{v}}\cdot\nabla)(\tilde{\boldsymbol{v}} - P_n\tilde{\boldsymbol{v}}), \boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\right)e^{-\int_0^s \mathscr{K}_\nu(P_n\tilde{\boldsymbol{v}})\,\mathrm{d}\tau}\,\mathrm{d}s$$
$$\leq \nu\|\nabla\tilde{\boldsymbol{v}} - \nabla P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))}\|\nabla\boldsymbol{v}^n - \nabla P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \|\tilde{\boldsymbol{v}} - P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{d/2}(\Omega))}\|\nabla\tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^{2d/(d-2)}(\Omega))}\|\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))}$$
$$+ \|P_n\tilde{\boldsymbol{v}}\|_{L^\infty(0,T;L^d(\Omega))}\|\nabla\tilde{\boldsymbol{v}} - \nabla P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^2(\Omega))}\|\boldsymbol{v}^n - P_n\tilde{\boldsymbol{v}}\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))}.$$

Weak convergence of $\boldsymbol{v}^n$ in $L^2(0,T;H_{0,\sigma}^1(\Omega))$ implies that the norms of $\boldsymbol{v}^n$ on the right-hand side are bounded independent of n. The strong convergence (21) allows to pass to the limit on the right-hand side, which vanishes. The strong convergence of the projection P_n to the Identity on $L_\sigma^2(\Omega)$

as $n \to \infty$ allows to pass to the limit in the initial values, too. energy-variational *Step 5, Vanishing viscosity limit:* Now, we focus on the case $\nu = 0$. Therefore, we consider the sequence $\{\boldsymbol{v}^\nu\}_{\nu \in (0,1)}$ of energy-variational solutions to the Navier-Stokes equations according to Theorem 2.7 for $\nu \to 0$. These solutions fulfill Definition 2.4 with $\mathscr{W}_\nu$ given by (5b). Inserting $\tilde{\boldsymbol{v}} = 0$ in this definition, we find the usual energy estimate (3) such that with the usual estimates of the right-hand side, *i.e.*, (16) with $\boldsymbol{f}_1 = 0$ (Note that $\mathbb{Z}_0 = L^\infty(0,T;L^2_\sigma(\Omega))$), we deduce the weak convergence in the energy space, *i.e.*,

$$ \boldsymbol{v}^\nu \overset{*}{\rightharpoonup} \boldsymbol{v} \quad \text{in } \mathbb{X}_0 \,. $$

with $\mathbb{X}_0$ as given above by $\mathbb{X}_0 := L^\infty(0,T;L^2_\sigma(\Omega))$. Now, we need to alter the formulation of the relative energy inequality. Following the steps as in the proof of Proposition 2.6, we observe that $\boldsymbol{v}^\nu$ fulfills the inequality

$$ \mathscr{R}(\boldsymbol{v}^\nu|\tilde{\boldsymbol{v}})\Big|_0^t + \int_0^t \nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} - \int_\Omega ((\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + \langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}}\rangle\,\mathrm{d}s \leq 0 $$

for a.e. $t \in (0,T)$ and all $\tilde{\boldsymbol{v}} \in \mathscr{C}^1([0,T];\mathbb{Y}_\nu)$, where $\mathscr{A}_\nu$ is given by (5e) (see (11) for the preceding inequality). With the usual skew-symmetry in the last two entries of the trilinear form, we find

$$ -(((\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}) = ((\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}})\otimes(\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}}),(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}})\,, $$

and adding and subtracting $\mathscr{K}_0(\tilde{\boldsymbol{v}})\mathscr{R}(\boldsymbol{v}^\nu|\tilde{\boldsymbol{v}})$, Gronwall's lemma, as well as Lemma 2.1 imply

$$ -\int_0^T \phi'\mathscr{R}(\boldsymbol{v}^\nu|\tilde{\boldsymbol{v}})e^{-\int_0^t \mathscr{K}_0(\tilde{\boldsymbol{v}})\,\mathrm{d}s} $$
$$ +\int_0^T \phi\left(\nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)} + \mathscr{W}_0(\boldsymbol{v}^\nu|\tilde{\boldsymbol{v}}) + \langle\mathscr{A}_\nu(\tilde{\boldsymbol{v}}),\boldsymbol{v}^\nu - \tilde{\boldsymbol{v}}\rangle\right)e^{-\int_0^t \mathscr{K}_0(\tilde{\boldsymbol{v}})\,\mathrm{d}s}\,\mathrm{d}t $$
$$ \leq \mathscr{R}(\boldsymbol{v}_0|\tilde{\boldsymbol{v}}(0)) $$

for all $\phi \in \tilde{\mathscr{C}}([0,T])$ and $\tilde{\boldsymbol{v}} \in \mathscr{C}^1([0,T];\mathbb{Y}_0 \cap H^2(\Omega))$ and all $\nu > 0$.

The dissipative term $\nu\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}$ may be estimated from below by zero such that the inequality remains true for all ν without this term. Then passing to the limit with $\nu \to 0$, we observe that $\mathscr{R}$ and $\mathscr{W}_0$ are weakly-lower semi-continuous such that the inequality still holds when passing to the limit in this terms. Concerning the last term on the left-hand side, we observe that $\boldsymbol{v}^\nu$ only occurs linear such that the weak convergence suffices to pass to the limit in the ν-independent terms. For the Laplace operator in the solution operator $\mathscr{A}_\nu$, we observe that

$$ \nu\int_0^t (\nabla\tilde{\boldsymbol{v}};\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}})\,\mathrm{d}s \leq \sqrt{\nu}\|\nabla\boldsymbol{v}^\nu - \nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega\times(0,T))}\sqrt{\nu}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega\times(0,T))} $$
$$ \leq c\sqrt{\nu}\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega\times(0,T))} \to 0 \quad \text{as } \nu \to 0\,. $$

By Lemma (2.1), we find that the relative energy inequality (7) is fulfilled in the limit $\nu \to 0$. This proves the existence of energy-variational solutions to the Euler equations and thus the assertion.

$\square$

Remark 3.1. The proof for the Euler equation is also possible via the Galerkin scheme. But the proof is done here via the standard vanishing viscosity method for convenience. This also provides some insight, why the dissipative formulation is also valuable in the singular limit analysis.

The Galerkin proof can also be seen as a version of Lax theorem. If the scheme is stable with respect to the relative energy inequality, *i.e.*, fulfills (20) and the consistency error vanishes, *i.e.*, $\mathscr{A}_n(\tilde{\boldsymbol{v}}) - \mathscr{A}_n(P_n\tilde{\boldsymbol{v}}) \to 0$ in $\mathbb{X}^*_\nu$ as $n \to \infty$ for smooth functions, then the numerical scheme converges. The discrete solution operator $\mathscr{A}_n$ may be chosen differently.

3.3   Well-posedness of minimal energy-variational solutions

Proof of Theorem 2.8. The assertion is a consequence of [20, Thm. 2.3]. Formally, the considered problem does not fit into the setting of [20], since the relative dissipation $\mathscr{W}_\nu$ is defined differently. But since the proof only uses the convexity and lower-semi continuity of $\mathbb{R}$ and $\mathscr{W}_\nu$ (see Remark 2.2), it can line-by-line be applied to the redefined version of $\mathscr{W}_\nu$. Therefore, we do not copy the proof here.

In short, the convexity and lower semi-continuity of $\mathscr{R}$ and $\mathscr{W}_\nu$ allow to prove the convexity and weak-closedness of the set of the energy-variational solutions. Thus, the minimizer of Definition 2.5 exists and is unique due to the strict convexity of the energy functional. The continuous dependence follows due to the convergence of the relative energy inequality for the given convergences of the right-hand side and the initial value. We refer the reader to [20] or [24, Chapter. 4] for details.

$\square$

References

[1] N. Alia, V. John, and S. Ollila. Revisiting the single-phase flow model for liquid steel ladle stirred by gas. *Applied Mathematical Modelling*, 67:549–556, 2019.

[2] L̀. Baňas, R. Lasarzik, and A. Prohl. Numerical analysis for nematic electrolytes. *Appeared online in IMA J. Numer. Anal.*, 2020.

[3] D. Breit, E. Feireisl, and M. Hofmanová. Dissipative solutions and semiflow selection for the complete Euler system. *Comm. Math. Phys.*, 376(2):1471–1497, 2020.

[4] Y. Brenier, C. De Lellis & L. Székelyhidi, Jr. Weak-strong uniqueness for measure-valued solutions. *Comm. Math. Phys.*, 305(2):351–361, 2011.

[5] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. Math.*, 189(1):101–144, 2019.

[6] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer Berlin, 2016.

[7] C. Dafermos. Maximal dissipation in equations of evolution. *J. Differ. Equ.*, 252(1):567 − 587, 2012.

[8] C. M. Dafermos. The entropy rate admissibility criterion for solutions of hyperbolic conservation laws. *J. Differ. Equ.*, 14(2):202 − 212, 1973.

[9] C. De Lellis, L. Székelyhidi Jr. The Euler equations as a differential inclusion. *Ann. Math.*, 170(3):1417–1436, 2009.

[10] C. De Lellis, L. Székelyhidi Jr. On Admissibility Criteria for Weak Solutions of the Euler Equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[11] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.*, 108(4):667–689, 1987.

[12] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. SIAM, Philadelphia, 1999.

[13] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhäuser, Basel, 2009.

[14] M. Grmela and H. C. Öttinger. Dynamics and thermodynamics of complex fluids. I. Development of a general formalism. *Phys. Rev. E*, 56(6):6620–6632, Dec 1997.

[15] J. Hale. *Ordinary differential equations*. Wiley-Interscience, New York, 1969.

[16] A. Ioffe. On lower semicontinuity of integral functionals. I. *SIAM J. Control Optim.*, 15(4):521–538, 1977.

[17] R. Lasarzik. Dissipative solution to the Ericksen–Leslie system equipped with the Oseen–Frank energy. *Z. Angew. Math. Phy.*, 70(1):8, 2018.

[18] R. Lasarzik. Weak-strong uniqueness for measure-valued solutions to the Ericksen–Leslie model equipped with the Oseen–Frank free energy. *J. Math. Anal. Appl.*, 470(1):36–90, 2019.

[19] R. Lasarzik. Approximation and optimal control of dissipative solutions to the Ericksen–Leslie system. *Numer. Func. Anal. Opt.*, 40(15):1721–1767, 2019.

[20] R. Lasarzik. Maximal dissipative solutions for incompressible fluid dynamics. *arXiv 2001.01512*, 2020.

[21] R. Lasarzik. Analysis of a thermodynamically consistent Navier–Stokes–Cahn–Hilliard model. *arXiv 2997.06607*, 2020.

[22] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63(1):193–248, 1934.

[23] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*. The Clarendon Press, New York, 1996.

[24] T. Roubíček. *Relaxation in optimization theory and variational calculus*, Walter de Gruyter & Co., Berlin, 1997.

[25] J. Serrin. On the interior regularity of weak solutions of the Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[26] L. D. G. Sigalotti, E. Sira, J. Klapp, and L. Trujillo. *Environmental Fluid Mechanics: Applications to Weather Forecast and Climate Change*, pages 3–36. Springer International Publishing, Cham, 2014.

[27] J. Simon. On the existence of the pressure for solutions of the variational Navier–Stokes equations. *J. Math. Fluid Mech.*, 1(3):225–234, 1999.

[28] R. Temam. *The Navier-Stokes equations: Theory and numerical analysis*. American Math. Soc., New York, 1984 (corrected reprint 2001).