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Well-posedness for a class of phase-field systems modeling prostate cancer growth with fractional operators and general nonlinearities

In memory of Professor Claudio Baiocchi
with great admiration and moving memories

Pierluigi Colli[1], Gianni Gilardi[1], Jürgen Sprekels[2]

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[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
gianni.gilardi@unipv.it

[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
and
Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

This paper deals with a general system of equations and conditions arising from a mathematical model of prostate cancer growth with chemotherapy and antiangiogenic therapy that has been recently introduced and analyzed (see [P. Colli et al., Mathematical analysis and simulation study of a phase-field model of prostate cancer growth with chemotherapy and antiangiogenic therapy effects, Math. Models Methods Appl. Sci. **30** (2020), 1253–1295]). The related system includes two evolutionary operator equations involving fractional powers of selfadjoint, nonnegative, unbounded linear operators having compact resolvents. Both equations contain nonlinearities and in particular the equation describing the dynamics of the tumor phase variable has the structure of a Allen–Cahn equation with double-well potential and additional nonlinearity depending also on the other variable, which represents the nutrient concentration. The equation for the nutrient concentration is nonlinear as well, with a term coupling both variables. For this system we design an existence, uniqueness and continuous dependence theory by setting up a careful analysis which allows the consideration of nonsmooth potentials and the treatment of continuous nonlinearities with general growth properties.

1 Introduction

In the paper [23] the following initial and boundary value problem

$$\partial_t\varphi - \lambda\Delta\varphi + 2\varphi(1-\varphi)f(\varphi,\sigma,u) = 0 \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\partial_t\sigma - \eta\Delta\sigma + \gamma_h\sigma + (\gamma_c-\gamma_h)\sigma\varphi = S_h + (S_c-S_h)\varphi - s\varphi \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\partial_t p - D\Delta p + \gamma_p p = \alpha_h + (\alpha_c-\alpha_h)\varphi \qquad \text{in } \Omega\times(0,T), \tag{1.3}$$

$$\varphi = 0, \quad \partial_\nu\sigma = \partial_\nu p = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1.4}$$

$$\varphi(0)=\varphi_0, \quad \sigma(0)=\sigma_0 \quad \text{and} \quad p(0)=p_0 \qquad \text{in } \Omega, \tag{1.5}$$

has been introduced and analytically studied. Here Ω is a bounded domain in $\mathbb{R}^3$ with a smooth boundary Γ, ∂_ν denotes the normal derivative on Γ, and $T>0$ is some final time. Moreover, the nonlinearity f is defined by

$$f(\varphi,\sigma,u) := M\big[1-2\varphi-3\big(m(\sigma)-m_{ref}u\big)\big],$$
$$\text{where} \quad m(\sigma) := m_{ref}\Big(\frac{\rho+A}{2} + \frac{\rho-A}{\pi}\,\arctan\frac{\sigma-\sigma_l}{\sigma_r}\Big) \tag{1.6}$$

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for some given constants M, m_{ref}, ρ, A, σ_l and σ_r. The symbols λ, η and D appearing in (1.1)–(1.3) denote positive diffusion coefficients, and γ_i, S_i, and α_i, with $i = c$ or $i = h$, are given constants as well, while u and s are prescribed functions. Finally, φ_0, σ_0 and p_0 are given initial data.

The above system models a prostate cancer growth with chemotherapy, where the physical variables φ, σ and p denote the relative amount of tumor and the concentrations of nutrient and of the PSA released by the cells, respectively. In fact, the model describes the tumor dynamics using the phase field φ, whose evolution is ruled by (1.1): this equation accounts for the transitions from the value $\varphi \approx 0$ in the host tissue to $\varphi \approx 1$ in the tumor. The dynamics of the nutrient concentration σ is governed by the reaction-diffusion equation (1.2), while the concentration p of PSA in the prostatic tissue obeys the diffusive equation (1.3) with right-hand side depending linearly on φ.

By looking at (1.1) and (1.6), about the term $2\varphi(1-\varphi)f(\varphi,\sigma,u)$ we note that the common factor $2\varphi(1-\varphi)$ is on one hand multiplied by $M(1-2\varphi)$ to render the derivative of the double-well potential $\varphi \mapsto M\varphi^2(1-\varphi)^2$, and on the other by $-3M\bigl(m(\sigma) - m_{ref}u\bigr)$, where the term $m(\sigma)$ describes the tumor net proliferation rate depending on the nutrient. In the definition of $m(\sigma)$, the values ρ and A stand for constant proliferation and apoptosis (i.e., programmed cell death) indices, and σ_r and σ_l are a reference and a threshold value for the nutrient concentration, respectively. The positive constant m_{ref} scales the function u that represents the tumor-inhibiting effect of a cytotoxic drug. When $|m(\sigma) - m_{ref}u|$ is sufficiently small, the function $2\varphi(1-\varphi)f(\varphi,\sigma,u)$ is a double-well potential with local minima at $\varphi = 0$ and $\varphi = 1$. Within this range, low values of the nutrient concentration (or large values of u) produce a lower energy level in the healthy tissue ($\varphi = 0$) than in the tumoral tissue ($\varphi = 1$). The opposite is true for high values of the nutrient concentration (or low values of u).

As for (1.2), we point out that γ_h, γ_c are positive constants that represent the nutrient uptake rate in the healthy and cancerous tissue, respectively; the coefficients S_h and S_c are the nutrient supply rates in the respective tissues; s is a given function yielding the reduction in nutrient supply caused by antiangiogenic therapy. In the model, S_h, S_c, and s are all nonnegative and s satisfies the constraint $s \le S_c$. Both healthy and cancerous prostatic cells release PSA, although tumor cells do so at a much larger rate: by (1.3) the PSA is assumed to diffuse through the prostatic tissue and to decay naturally at rate γ_p. The constants α_h and α_c in (1.3) denote, respectively, the tissue PSA production rate of healthy and malignant cells. About the boundary conditions (1.4) we emphasize that they are very natural, since we assume that the domain Ω is large enough in order that the prostatic tumor has not yet reached the boundary, whence $\varphi = 0$ on $\Gamma \times (0,T)$. On the other hand, for the concentrations σ and p no flux is assumed across the boundary, whence Neumann homogeneous boundary conditions are prescribed for them.

More details on the model and a large list of concerned references can be found in [23] and also in the twin paper [24], dedicated to the study of optimal control problems in which the functions u and s, related to cytotoxic and antiangiogenic therapies, act as controls in the system. We point out that the complete system (1.1)–(1.5) is designed to describe the growth of a prostatic tumor under the influence of therapies and it turns out to be a phase-field model. In recent years the phase-field (or diffuse interface) method has been extensively used to describe tumor growth in the computational and mathematical literature (see, e.g., [8, 9, 11, 13, 15, 26, 32, 34, 37, 39, 40, 44, 46, 47, 49, 54–56]). Indeed, tumor growth has become an important issue in applied mathematics and a significant number of models has been introduced and discussed, with numerical simulations as well, in connection and comparison with the behavior of other special materials: one may also see [5, 7, 12, 16, 20, 25, 27, 28, 31, 33, 35, 36, 42, 48, 50, 52, 53].

The basic reference [23] contains, in particular, a mathematical study of the well-posedness of the problem (1.1)–(1.5) that is based on a fixed-point approach to equations (1.1) and (1.2). The argumentation makes use of the boundedness property for φ, namely this phase variable is assumed (and then shown) to remain between the values 0 and 1, i.e., in the right physical range, during the evolution. In this paper, we aim to significantly generalize system (1.1)–(1.5) by replacing the elliptic operators $v \mapsto -\lambda\Delta v$ in (1.1) and $v \mapsto -\eta\Delta v + \gamma_h v$ in (1.2) by operators of fractional type and introducing nonlinear variants of the structural functions appearing in the system, especially of the double-well potential hidden in (1.1) and (1.6). For our purposes, it is convenient to replace the variable φ acting in (1.1)–(1.5) by $(1+\varphi)/2$, in order to let the 'new' φ take the admissible values in the interval $[-1,1]$. Note that this change does not affect the structure, since it implies only a rescaling in the equations. Moreover, we decide that in the sequel the third equation (1.3) can be forgotten: indeed, since our aim is essentially to provide a general theory for well-posedness, the (even generalized) third equation can be immediately solved once that φ is known. Thus, the system we are interested in is the following (with different notation with respect to the above one)

$$\partial_t\varphi + A^{2\rho}\varphi + F'(\varphi) = h(\varphi)\big(m(\sigma) - m_0 u\big), \tag{1.7}$$

$$\partial_t\sigma + B^{2\tau}\sigma + \gamma(\varphi)\sigma = \kappa(\varphi) - S\varphi, \tag{1.8}$$

$$\varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0, \tag{1.9}$$

where $A^{2\rho}$ and $B^{2\tau}$, with $\rho > 0$ and $\tau > 0$, denote fractional powers of the selfadjoint, monotone, and unbounded linear operators A and B, respectively, which are densely defined in the Hilbert space $H := L^2(\Omega)$ and have compact resolvents. Notice that the boundary conditions are implicit in the definition of the operators. Moreover, F is a potential of double-well type; h, m, γ, and κ, are real functions defined in the whole of $\mathbb{R}$; m_0 is a constant. Finally, u and S are given functions on $\Omega \times (0,T)$, and φ_0 and σ_0 are prescribed initial data as before. Well-known and important examples of F are the so-called *classical double-well potential* and the *logarithmic potential*, defined by the formulas

$$F_{reg}(r) := c_0\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.10}$$

$$F_{log}(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2\,, & r \in (-1,1) \\ 2\ln(2) - c_1\,, & r \in \{-1,1\} \\ +\infty\,, & r \notin [-1,1] \end{cases}\,, \tag{1.11}$$

respectively, where c_0 and c_1 are positive constants. Other significant potentials are the following

$$F_{sing}(r) := \begin{cases} \dfrac{r^2}{1-r} - c_2 r^2\,, & r \in (-\infty,1) \\ +\infty\,, & r \in [1,+\infty) \end{cases}\,, \tag{1.12}$$

$$F_{2obs}(r) := I_{[0,1]}(r) - c_3 r^2, \quad r \in \mathbb{R}, \tag{1.13}$$

where $I_{[0,1]}$ is the indicator function of the interval $[0,1]$ and c_2 and c_3 are positive constants. We recall that the indicator function $I_X : \mathbb{R} \to (-\infty,+\infty]$ of the generic subset $X \subset \mathbb{R}$ is defined by $I_X(r) := 0$ if $r \in X$ and $I_X(r) := +\infty$ otherwise. Note that the potential $F_{sing}(r)$ blows up as r approaches 1 (tumorous phase) while may become largely negative for negative values of r (no problem to go down to the healthy phase).

In cases like (1.13), F is not differentiable in the endpoints of its domain, so that the derivative F' appearing in (1.7) is meaningless and has to be suitably replaced. Namely, we split F as $F = \widehat{\beta} + \widehat{\pi}$, where $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is convex and lower semicontinuous (e.g., the indicator function $I_{[0,1]}$ for F_{2obs}) and $\widehat{\pi}$ is a smooth perturbation ($\widehat{\pi}(r) = -c_3 r^2$, $r \in \mathbb{R}$, in (1.13)). Accordingly, we replace F' by $\beta + \pi$, where $\beta := \partial\widehat{\beta}$ is the subdifferential of $\widehat{\beta}$ and π is the derivative of $\widehat{\pi}$, and read (1.7) as a differential inclusion. In general, we can rewrite the equation or inclusion (1.7) as a variational inequality involving $\widehat{\beta}$ rather than β. Actually, we will do this in the following.

We analyze the system (1.7)–(1.9) by proving the existence of solutions in a large set of assumptions for the data of system and using a double approximation based on the regularization of nonlinearities and a Faedo–Galerkin discretization. Then, in a more focused framework for nonlinearities we prove a continuous dependence result by dealing with very sharp estimates in the proofs. Of course, our analysis takes advantage of well-established approaches for the study of parabolic systems and, in this respect, we would like to recall the pioneering and important contribution given by Claudio Baiocchi [1, 2], a master of mathematics and excellent teacher for at least two of the authors of this manuscript.

All in all, we point out that the theory developed in this paper offers a different approach to well-posedness with respect to the one in [23], since in our general setting with fractional operators no L^∞-estimate is proved for the components of the solution (in particular, not for φ), but we are able to show anyway existence and uniqueness of the solution, by exploiting in a very careful way the shape of nonlinearities in the system.

Let us spend some words on the use of fractional operators, which in recent years provided a challenging subject for mathematicians: they have been successfully utilized in many different situations, and a wide literature already exists about equations and systems with fractional terms. For an overview of recent contributions, we refer to the papers [14,18] and [10], which offer to the interested reader a number of suggestions to expand the knowledge of the field. Moreover, we underline that the authors of the present paper already investigated systems with fractional operators in the papers [15–17, 19–22], in particular studying another class of tumor growth models [15,20] inspired by [40] and the related contributions [11,13,34,50]. In our approach here, we adopt the same setting for fractional operators, that are defined through the spectral theory. This framework includes, in particular, powers of a second-order elliptic operator, and other operators like, e.g., fourth-order ones or systems involving the Stokes operator. A precise definition for our fractional operators $A^{2\rho}$, $B^{2\tau}$ along with their properties follows in Section 2 below. About the use of fractional operators in a physiological framework, we notice that some components in tumor development, such as immune cells, exhibit an anomalous diffusion dynamics (as it observed in experiments [29]), and other components, like nutrient concentration, are possibly governed by different fractional or non-fractional flows. We conclude by arguing that fractional operators are becoming more and more implemented in the field of biological applications and related reaction-diffusion equations (cf, e.g., [3, 8, 29, 30, 38, 41, 43, 52, 53, 57]).

The paper is organized as follows. As for Section 2, we state precisely the problem as well as the assumptions and the well-posedness results. Section 3 contains the proof of the continuous dependence result. The approximation of the problem via regularization of nonlinearities and introduction of the discrete problem is carried out in Section 4, while the existence of solutions is shown in Section 5 by performing a limit procedure on the regularized problem.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth, with outward unit normal vector field ν

on $\Gamma := \partial\Omega$. Moreover, ∂_ν stands for the corresponding normal derivative. We use the notation

$$H := L^2(\Omega) \tag{2.1}$$

and denote by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ without indices the standard norm and inner product of H. On the contrary, for a generic Banach space X, its norm is denoted by $\|\cdot\|_X$, with the following exceptions: a special notation is used for the norms in the spaces V_A^ρ and V_B^τ introduced below, and the norm in any L^p space is denoted by $\|\cdot\|_p$ for $1 \le p \le +\infty$. Now, we start introducing our assumptions. As for the operators, we postulate that

$$A : D(A) \subset H \to H \quad \text{and} \quad B : D(B) \subset H \to H \quad \text{are unbounded, monotone,}$$
$$\text{selfadjoint linear operators with compact resolvents.} \tag{2.2}$$

This assumption implies that there are sequences $\{\lambda_j\}$ and $\{\lambda'_j\}$ of eigenvalues and orthonormal sequences $\{e_j\}$ and $\{e'_j\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j, \quad \text{and} \quad (e_i, e_j) = (e'_i, e'_j) = \delta_{ij}, \quad \text{for } i, j = 1, 2, \ldots \tag{2.3}$$

such that

$$0 \le \lambda_1 \le \lambda_2 \le \ldots \quad \text{and} \quad 0 \le \lambda'_1 \le \lambda'_2 \le \ldots \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty, \tag{2.4}$$

$$\{e_j\}_{j\in\mathbb{N}} \text{ and } \{e'_j\}_{j\in\mathbb{N}} \text{ are complete systems in } H. \tag{2.5}$$

By the same assumption, the powers of A and B for an arbitrary positive real exponent are well defined. Indeed, we can set

$$V_A^\rho := D(A^\rho) = \Big\{ v \in H : \sum_{j=1}^\infty |\lambda_j^\rho (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.6}$$

$$A^\rho v = \sum_{j=1}^\infty \lambda_j^\rho (v, e_j) e_j \quad \text{for } v \in V_A^\rho, \tag{2.7}$$

the series being convergent in the strong topology of H, due to the properties (2.6) of the coefficients. We endow V_A^ρ with the (graph) norm and inner product

$$\|v\|_{A,\rho}^2 := (v, v)_{A,\rho} \quad \text{and} \quad (v, w)_{A,\rho} := (v, w) + (A^\rho v, A^\rho w) \quad \text{for } v, w \in V_A^\tau. \tag{2.8}$$

This makes V_A^ρ a Hilbert space. In the same way, starting from (2.2)–(2.5) for B, we can set

$$V_B^\tau := D(B^\tau), \quad \text{with the norm } \|\cdot\|_{B,\tau} \text{ associated to the inner product}$$
$$(v, w)_{B,\tau} := (v, w) + (B^\tau v, B^\tau w) \quad \text{for } v, w \in V_B^\tau. \tag{2.9}$$

From now on, we assume:

$$\rho \text{ and } \tau \text{ are fixed positive real numbers.} \tag{2.10}$$

For the other ingredients of our system, we postulate the following properties:

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper and l.s.c. with}$$
$$\widehat{\beta}(0) = 0 \quad \text{and} \quad \lim_{|r|\nearrow+\infty} \widehat{\beta}(r) = +\infty\,. \tag{2.11}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative.} \tag{2.12}$$

$$m : \mathbb{R} \to \mathbb{R} \quad \text{is continuous and bounded, and} \quad m_0 \in \mathbb{R}\,. \tag{2.13}$$

$$\alpha_h\,,\, \alpha_\gamma\,,\, \alpha_\kappa \in [1,+\infty), \quad p_h \in (1,+\infty) \quad \text{and} \quad p_\gamma\,,\, q_\gamma \in [2,+\infty) \ \text{satisfy}$$
$$\frac{1}{p_\gamma} + \frac{1}{q_\gamma} = \frac{1}{2} \quad \text{and} \quad p_0 := \frac{\max\{\alpha_\gamma p_\gamma, 2\alpha_\kappa\}}{\alpha_h + 1} > 1\,. \tag{2.14}$$

$$V_A^\rho \subset L^{\alpha_h p_h}(\Omega) \cap L^{p_h'}(\Omega) \cap L^{\alpha_\gamma p_\gamma}(\Omega) \cap L^{2\alpha_\kappa}(\Omega) \quad \text{and} \quad V_B^\tau \subset L^{q_\gamma}(\Omega)$$
$$\text{with continuous embeddings}\,. \tag{2.15}$$

$$h,\, \gamma,\, \kappa : \mathbb{R} \to \mathbb{R} \quad \text{are continuous and satisfy the growth conditions}$$
$$|h(r)| \le C_0\, |r|^{\alpha_h} + C_1\,, \quad |\gamma(r)| \le C_0\, |r|^{\alpha_\gamma} + C_1\,,$$
$$\text{and} \quad |\kappa(r)| \le C_0\, |r|^{\alpha_\kappa} + C_1\,, \quad \text{for every } r \in \mathbb{R}\,, \tag{2.16}$$
$$|h(r)|^2 \le C_2\, \widehat{\beta}(r) + C_3 \quad \text{for every } r \in \mathbb{R}\,. \tag{2.17}$$

In (2.16)–(2.17), $C_i,\ i = 0, \ldots, 3$, are given positive constants and p_h' in (2.15) is the conjugate exponent of p_h.

Remark 2.1. We notice that assumptions (2.11)–(2.12) are fulfilled by all of the important potentials, in particular by the ones in (1.10)–(1.13). About F_{sing} in (1.12), we point out that we can take the related $\widehat{\beta}$ and $\widehat{\pi}$ as

$$\widehat{\beta}(r) = \begin{cases} d\, r^2 + \dfrac{r^2}{1-r}\,, & r \in (-\infty, 1) \\ +\infty\,, & r \in [1, +\infty) \end{cases} \,, \quad \widehat{\pi}(r) = -(c_2 + d) r^2, \quad r \in \mathbb{R}, \tag{2.18}$$

for any nonnegative choice of the coefficient d.

Remark 2.2. In the case when $A^{2\rho}$ and $B^{2\tau}$ are second-order elliptic operators with homogeneous Dirichlet and Neumann boundary conditions, respectively, and the above functions $h,\ \gamma,\ \kappa$ and $F := \widehat{\beta} + \widehat{\pi}$ represent those appearing in problem (1.1)–(1.5), then $V_A^\rho = H_0^1(\Omega)$, $V_B^\tau = H^1(\Omega)$ and $\alpha_h = 2$, $\alpha_\gamma = \alpha_\kappa = 1$, so that one can take, e.g., $p_h = 2$ and $p_\gamma = q_\gamma = 4$ in order to satisfy (2.14) and (2.15) (since $H^1(\Omega) \subset L^6(\Omega) \subset L^4(\Omega)$) as well as (2.16). Moreover, with this choice, (2.17) also holds since $\widehat{\beta}$ is a fourth order polynomial.

However, it is clear that the present framework is much more general. Nontrivial situations are given in the examples below.

Example 2.3. The domains V_A^ρ and V_B^τ of the operators A^ρ and B^τ are embedded in $L^5(\Omega)$ and $L^4(\Omega)$, respectively, and $\widehat{\beta} + \widehat{\pi}$ is the potential (1.11) or (1.13) (with effective domain of $\widehat{\beta}$ restricted to $[-1, 1]$). Then an admissible choice of the exponents is the following: $\alpha_h = 3$, $\alpha_\gamma = 5/4$, $\alpha_\kappa = 5/2$, $p_h = 4/3$ and $p_\gamma = q_\gamma = 4$. We have indeed: $\alpha_h p_h = p_h' = 4$ and $p_0 = 5/4$. Concerning the potential F_{sing} in (1.12), we have to take $\widehat{\beta}$ and $\widehat{\pi}$ as in (2.18): then, in view of (2.17), in this setting we can just let $\alpha_h = 1$ sharp.

Example 2.4. We modify the previous example by assuming that V_A^ρ and V_B^τ are embedded in $L^5(\Omega)$ and $L^6(\Omega)$, respectively, and take $\alpha_h = \alpha_\gamma = \alpha_\kappa = 5/4$. Then, an admissible choice of the exponents of the L^p spaces is given by $p_h = 2$ and $p_\gamma = q_\gamma = 4$, as one immediately sees. We notice that this example is compatible with the further assumptions we introduce later on (see the forthcoming Remark 2.6).

We set, for convenience,

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \pi := \widehat{\pi}' . \tag{2.19}$$

Moreover, we term $D(\widehat{\beta})$ and $D(\beta)$ the effective domains of $\widehat{\beta}$ and β, respectively. Notice that β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ (we refer, e.g., to [4, 6] for definitions and properties of maximal monotone operators and subdifferentials of convex functions).

At this point, we can state the problem under investigation. We give a weak formulation of the equations (1.7)–(1.8) and present (1.7) as a variational inequality. For the data, we make the following assumptions:

$$u \in L^2(0,T;L^\infty(\Omega)) \quad \text{and} \quad S \in L^2(0,T;L^\infty(\Omega)) . \tag{2.20}$$

$$\varphi_0 \in V_A^\rho \quad \text{with} \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega) \quad \text{and} \quad \sigma_0 \in V_B^\tau . \tag{2.21}$$

Then, we set

$$Q := \Omega \times (0,T) \tag{2.22}$$

and look for a pair (φ, σ) satisfying

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho), \tag{2.23}$$

$$\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\tau), \tag{2.24}$$

$$\widehat{\beta}(\varphi) \in L^\infty(0,T;L^1(\Omega)), \tag{2.25}$$

and solving the system

$$\begin{aligned}
&\int_\Omega \partial_t\varphi(t)(\varphi(t)-v) + \bigl(A^\rho\varphi(t), A^\rho(\varphi(t)-v)\bigr) + \int_\Omega \widehat{\beta}(\varphi(t)) + \int_\Omega \pi(\varphi(t))(\varphi(t)-v) \\
&\le \int_\Omega h(\varphi(t))\bigl(m(\sigma(t)) - m_0\, u(t)\bigr)(\varphi(t)-v) + \int_\Omega \widehat{\beta}(v) \\
&\quad \text{for every } v \in V_A^\rho \text{ and for a.a. } t \in (0,T),
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
&\int_\Omega \partial_t\sigma(t)\, v + \bigl(B^\tau\sigma(t), B^\tau v\bigr) + \int_\Omega \gamma(\varphi(t))\,\sigma(t)\, v \\
&= \int_\Omega \kappa(\varphi(t))\, v - \int_\Omega S(t)\varphi(t)\, v \quad \text{for every } v \in V_B^\tau \text{ and for a.a. } t \in (0,T),
\end{aligned} \tag{2.27}$$

$$\varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0 . \tag{2.28}$$

The last integral in (2.26) has to be read as $+\infty$ if $\widehat{\beta}(v) \notin L^1(\Omega)$, of course. We also remark that all the other integrals involving nonlinearities are meaningful. Indeed, π is Lipschitz continuous and, by combining (2.23) and (2.24) with our assumptions on the structure and the data, one can show that (similarly as in the proof of the forthcoming estimates (5.8)–(5.10))

$$\gamma(\varphi)\,\sigma\,, \kappa(\varphi) \in L^\infty(0,T;H), \tag{2.29}$$

$$h(\varphi)\bigl(m(\sigma) - m_0 u\bigr)\,\varphi \in L^2(0,T;L^{p_0}(\Omega)), \tag{2.30}$$

$$h(\varphi)\bigl(m(\sigma) - m_0 u\bigr) \in L^2(0,T;L^{p_h}(\Omega)), \tag{2.31}$$

and we observe that every test function v in (2.26) belongs to $L^{p_h'}(\Omega)$ by (2.15). Finally, we stress that (2.26) and (2.27) are equivalent to their time-integrated variants with time dependent test functions.

For instance, the former is equivalent to

$$\int_Q \partial_t\varphi(\varphi-v) + \int_0^T \bigl(A^\rho\varphi(t), A^\rho(\varphi(t)-v(t))\bigr)\,dt + \int_Q \widehat{\beta}(\varphi) + \int_Q \pi(\varphi)(\varphi-v)$$
$$\leq \int_Q h(\varphi)\bigl(m(\sigma)) - m_0\,u\bigr)(\varphi-v) + \int_Q \widehat{\beta}(v) \quad \text{for every } v\in L^2(0,T;V_A^\rho), \tag{2.32}$$

with the same warning as above for the last term. Also in this inequality and in the analogous equation for σ, all the integral are meaningful due to (2.23)–(2.25) and (2.29)–(2.31). Here is our existence result.

Theorem 2.5. *Let the assumptions* (2.11)–(2.17) *on the structure of the system and* (2.20)–(2.21) *on the data be fulfilled. Then there exists at least one pair (φ,σ) satisfying* (2.23)–(2.25) *and solving problem* (2.26)–(2.28)*. Moreover, such a solution can be constructed that satisfies the estimate*

$$\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)} + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\tau)} + \|\widehat{\beta}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \leq \overline{C}_1, \tag{2.33}$$

with a constant $\overline{C}_1$ that depends only on the structure of the system, the norms of the data corresponding to (2.20)–(2.21)*, and T.*

In order to prove uniqueness and continuous dependence, we have to reinforce our assumptions on the structure. Namely, we make the following postulates:

$$p_{h,1}\,,\ q_{h,1}\,,\ p_{h,2}\,,\ q_{h,2}\,,\ p_{\gamma,1}\,,\ q_{\gamma,1}\,,\ r_{\gamma,1}\,,\ p_{\gamma,2}\,,\ q_{\gamma,2}\,,\ p_\kappa\,,\ q_\kappa \in [2,+\infty) \quad \text{satisfy}$$
$$\frac{1}{p_{h,1}} + \frac{1}{q_{h,1}} = \frac{1}{p_{h,2}} + \frac{1}{q_{h,2}} = \frac{1}{p_{\gamma,1}} + \frac{1}{q_{\gamma,1}} + \frac{1}{r_{\gamma,1}}$$
$$= \frac{1}{p_{\gamma,2}} + \frac{1}{q_{\gamma,2}} = \frac{1}{p_\kappa} + \frac{1}{q_\kappa} = \frac{1}{2}\,. \tag{2.34}$$

$$V_A^\rho \subset L^{p_*}(\Omega) \quad \text{and} \quad V_B^\tau \subset L^{q_*}(\Omega) \quad \text{with continuous embeddings, where}$$
$$p_* := \max\{p_{h,1}(\alpha_h-1)\,,\ q_{h,1}\,,\ \alpha_h p_{h,2}\,,\ p_{\gamma,1}(\alpha_\gamma-1)\,,\ q_{\gamma,1}\,,\ \alpha_\gamma p_{\gamma,2}\,,\ p_\kappa\,,\ q_\kappa\}$$
$$\text{and} \quad q_* := \max\{q_{h,2}, r_{\gamma,1}, q_{\gamma,2}\}\,. \tag{2.35}$$

$$m \text{ is Lipschitz continuous}\,. \tag{2.36}$$

$$h,\ \gamma \text{ and } \kappa \text{ are of class } C^1 \text{ and satisfy}$$
$$|h'(r)| \leq C_0'\,|r|^{\alpha_h-1} + C_1'\,, \quad |\gamma'(r)| \leq C_0'\,|r|^{\alpha_\gamma-1} + C_1'\,,$$
$$\text{and} \quad |\kappa'(r)| \leq C_0'\,|r|^{\alpha_\kappa-1} + C_1'\,, \quad \text{for every } r\in\mathbb{R}, \tag{2.37}$$

where C_0' and C_1' are given constants. We notice that the inequalities (2.37) imply both (2.16) and the inequality

$$|h(r)-h(s)| \leq \bigl(C_0'\max\{|r|^{\alpha_h-1}, |s|^{\alpha_h-1}\} + C_1'\bigr)|r-s| \quad \text{for every } r,s\in\mathbb{R}\,, \tag{2.38}$$

as well as its analogues for γ and κ.

Remark 2.6. The assumptions (2.34)–(2.35) look very complicated. However, they are satisfied in a number of situations. One is that of the system (1.1)–(1.5) described in the Introduction, as one immediately sees by also accounting for Remark 2.2. A nontrivial case is given by Example 2.4, where we recall that $V_A^\rho \subset L^5(\Omega)$, $V_B^\sigma \subset L^6(\Omega)$ and $\alpha_h = \alpha_\gamma = \alpha_\kappa = 5/4$. Indeed, an admissible choice of the new parameters is the following: $p_{\gamma,1} = 15/2$, $q_{\gamma,1} = 5$, $r_{\gamma,1} = 6$, and all of the other exponents are taken as 4.

Here is our result.

Theorem 2.7. *Besides the hypotheses of Theorem 2.5, assume that* (2.34)–(2.37) *are satisfied. Then the solution to problem* (2.26)–(2.28) *is unique. Moreover, given a constant M, let u_i, S_i and $\varphi_{0,i}$, $i=1,2$, be two choices of u, S and φ_0, respectively, and (φ_i,σ_i) be corresponding solutions, and assume that*

$$\|u_i\|_{L^2(0,T;L^\infty(\Omega))}\,,\ \|S_i\|_{L^2(0,T;L^\infty(\Omega))}\,,\ \|\varphi_i\|_{L^\infty(0,T;V_A^\rho)}\,,\ \|\sigma_i\|_{L^\infty(0,T;V_B^\tau)}\leq M \tag{2.39}$$

for $i=1,2$. Then the estimate

$$\begin{aligned}&\|\varphi_1-\varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V_A^\rho)}+\|\sigma_1-\sigma_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V_B^\tau)}\\&\leq \overline{C}_2\big(\|u_1-u_2\|_{L^2(0,T;L^\infty(\Omega))}+\|S_1-S_2\|_{L^2(0,T;L^\infty(\Omega))}+\|\varphi_{0,1}-\varphi_{0,2}\|\big)\end{aligned} \tag{2.40}$$

holds true with a constant $\overline{C}_2$ that only depends on the structure of our system, T, and M.

The remainder of the paper is organized as follows. The uniqueness and continuous dependence result is proved in Section 3, while the existence of a solution is established in the last Section 5 and is prepared by the study of the approximating problem introduced in Section 4.

Throughout the paper, we make a wide use of the Hölder inequality and of the elementary Young inequality

$$ab\leq \delta a^2+\frac{1}{4\delta}\,b^2 \quad \text{for every } a,b\in\mathbb{R} \text{ and } \delta>0\,. \tag{2.41}$$

Moreover, we use the notation

$$Q_t:=\Omega\times(0,t) \quad \text{for } t\in(0,T), \tag{2.42}$$

so that $Q=Q_T$. Finally, we state a general rule concerning the constants we are going to follow. We always use a small-case italic c without subscripts for different constants that may only depend the structure of our system (i.e., the operators A^ρ and B^τ, the shape of the nonlinearities, the parameters that appear in our assumptions), the final time T and the properties of the data involved in the statements at hand. In particular, the values of such constants do not depend on the regularization parameter ε we introduce in Section 4. Symbols like c_δ (e.g., with $\delta=\varepsilon$) denote constants that depend on the parameter δ, in addition. It has to be clear that the values of c and c_δ might change from line to line and even within the same formula or chain of inequalities. In contrast, we use different symbols (e.g., capital letters like C_i in (2.16)) for precise values of constants we want to refer to.

3 Continuous dependence and uniqueness

This section is devoted to the proof of the uniqueness and the continuous dependence stated in Theorem 2.7. More precisely, we prove just the continuous dependence, since uniqueness follows as a consequence. We pick two choices of the data as in the statement, the corresponding solutions and a constant M satisfying (2.39). We set for convenience

$$u:=u_1-u_2\,,\ \ S:=S_1-S_2\,,\ \ \varphi_0:=\varphi_{0,1}-\varphi_{0,2}\,,\ \ \varphi:=\varphi_1-\varphi_2 \text{ and } \sigma:=\sigma_1-\sigma_2\,. \tag{3.1}$$

We also set for brevity

$$\varphi_*:=\max\{|\varphi_1|,|\varphi_2|\} \quad \text{pointwise} \tag{3.2}$$

and denote by D the maximum of the norms of the embeddings (2.15) and (2.35). We assume that $D \geq 1$ without loss of generality. At this point, we are ready to start. According to our general rule, we use the same symbols c and c_δ (where $\delta > 0$ is chosen later on) for (possibly) different constants, as explained at the end of the previous section. In this proof, the values of c and c_δ are allowed to depend on M, in addition. We write (2.26) at the time s for (φ_1, σ_1) and (φ_2, σ_2) and test the inequality obtained by $\varphi_2(s)$ and $\varphi_1(s)$, respectively. Then, we sum up, integrate over $(0,t)$ with respect to s and add the same quantity $\int_{Q_t} |\varphi|^2$ to both sides. Since the terms involving $\widehat{\beta}$ cancel each other, we obtain that

$$\begin{aligned}
&\frac{1}{2}\,\|\varphi(t)\|^2 + \int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds \\
&\leq \frac{1}{2}\,\|\varphi_0\|^2 + \int_{Q_t} |\varphi|^2 - \int_{Q_t} \bigl(\pi(\varphi_1) - \pi(\varphi_2)\bigr)\varphi \\
&\quad + \int_{Q_t} \bigl\{h(\varphi_1)\bigl(m(\sigma_1) - m_0 u_1\bigr) - h(\varphi_2)\bigl(m(\sigma_2) - m_0 u_2\bigr)\bigr\}\varphi\,. \qquad (3.3)
\end{aligned}$$

By the Young inequality and the Lipschitz continuity of π, we immediately see that

$$-\int_{Q_t} \bigl(\pi(\varphi_1) - \pi(\varphi_2)\bigr)\varphi \leq c \int_{Q_t} |\varphi|^2\,. \qquad (3.4)$$

For the next term, we owe to both the assumptions of Theorem 2.5 and the supplementary assumptions on the structure (in particular, to (2.38)). We have that

$$\begin{aligned}
&\int_{Q_t} \bigl\{h(\varphi_1)\bigl(m(\sigma_1) - m_0 u_1\bigr) - h(\varphi_2)\bigl(m(\sigma_2) - m_0 u_2\bigr)\bigr\}\varphi \\
&\leq \int_{Q_t} |h(\varphi_1) - h(\varphi_2)|\,|m(\sigma_1) - m_0 u_1|\,|\varphi| \\
&\quad + \int_{Q_t} |h(\varphi_2)|\,|m(\sigma_1) - m(\sigma_2) - m_0 u|\,|\varphi| \qquad (3.5)
\end{aligned}$$

and we estimate the last two integrals, separately. We have that

$$\begin{aligned}
&\int_{Q_t} |h(\varphi_1) - h(\varphi_2)|\,|m(\sigma_1) - m_0 u_1|\,|\varphi| \leq c \int_{Q_t} (\varphi_*^{\alpha_h - 1} + 1)(1 + |u_1|)|\varphi|^2 \\
&\leq c \int_0^t (\|(\varphi_*(s))^{\alpha_h - 1}\|_{p_{h,1}} + 1)(1 + \|u_1(s)\|_\infty)\|\varphi(s)\|_{q_{h,1}}\,\|\varphi(s)\|\,ds \\
&\leq \delta \int_0^t (\|(\varphi_*(s))^{\alpha_h - 1}\|_{p_{h,1}}^2 + 1)\|\varphi(s)\|_{q_{h,1}}^2\,ds + c_\delta \int_0^t (1 + \|u_1(s)\|_\infty^2)\|\varphi(s)\|^2\,ds \\
&\leq \delta\, D^2 (\|\varphi_*\|_{L^\infty(0,T;L^{p*}(\Omega))}^{2(\alpha_h - 1)} + 1) \int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta \int_0^t (1 + \|u_1(s)\|_\infty^2)\|\varphi(s)\|^2\,ds \\
&\leq \delta\, D^2\, D^{2(\alpha_h - 1)} (\|\varphi_*\|_{L^\infty(0,T;V_A^\rho)}^{2(\alpha_h - 1)} + 1) \int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds \\
&\quad + c_\delta \int_0^t (1 + \|u_1(s)\|_\infty^2)\|\varphi(s)\|^2\,ds \\
&\leq \delta\, D^{2\alpha_h} (M^{2(\alpha_h - 1)} + 1) \int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta \int_0^t (1 + \|u_1(s)\|_\infty^2)\|\varphi(s)\|^2\,ds\,. \qquad (3.6)
\end{aligned}$$

The other integral is estimated this way (as for p_h, recall (2.14)–(2.15)):

$$\begin{aligned}
&\int_{Q_t} |h(\varphi_2)|\,|m(\sigma_1)-m(\sigma_2)-m_0 u|\,|\varphi| \\
&\le c\int_{Q_t} (|\varphi_2|^{\alpha_h}+1)|\sigma|\,|\varphi| + c\int_{Q_t} (|\varphi_2|^{\alpha_h}+1)|u|\,|\varphi| \\
&\le c\int_0^t (\||\varphi_2(s)|^{\alpha_h}\|_{p_{h,2}}+1)\|\sigma(s)\|_{q_{h,2}}\,\|\varphi(s)\|\,ds \\
&\quad + c\int_0^t (\||\varphi_2(s)|^{\alpha_h}\|_{p_h}+1)\|u(s)\|_\infty\,\|\varphi(s)\|_{p_h'}\,ds \\
&\le \delta\int_0^t \|\sigma(s)\|_{q_{h,2}}^2\,ds + c_\delta(\||\varphi_2|^{\alpha_h}\|^2_{L^\infty(0,T;L^{p_{h,2}}(\Omega))}+1)\int_0^t\|\varphi(s)\|^2\,ds \\
&\quad + \delta\int_0^t \|\varphi(s)\|^2_{p_h'}\,ds + c_\delta\,(\||\varphi_2|^{\alpha_h}\|^2_{L^\infty(0,T;L^{p_h}(\Omega))}+1)\int_0^t\|u(s)\|_\infty^2\,ds \\
&\le \delta\,D^2\int_0^t \|\sigma(s)\|^2_{B,\tau}\,ds + c_\delta(\|\varphi_2\|^{2\alpha_h}_{L^\infty(0,T;L^{\alpha_h p_{h,2}}(\Omega))}+1)\int_0^t\|\varphi(s)\|^2\,ds \\
&\quad + \delta\,D^2\int_0^t \|\varphi(s)\|^2_{A,\rho}\,ds + c_\delta(\|\varphi_2\|^{2\alpha_h}_{L^\infty(0,T;L^{\alpha_h p_h}(\Omega))}+1)\,\|u\|^2_{L^2(0,T;L^\infty(\Omega))} \qquad (3.7)\\
&\le \delta\,D^2\int_0^t \|\sigma(s)\|^2_{B,\tau}\,ds + c_\delta\int_0^t\|\varphi(s)\|^2\,ds \\
&\quad + \delta\,D^2\int_0^t \|\varphi(s)\|^2_{A,\rho}\,ds + c_\delta\,\|u\|^2_{L^2(0,T;L^\infty(\Omega))}\,. \qquad (3.8)
\end{aligned}$$

By collecting (3.3)–(3.8), we conclude that

$$\begin{aligned}
&\frac12\,\|\varphi(t)\|^2 + \int_0^t \|\varphi(s)\|^2_{A,\rho}\,ds \\
&\le \frac12\,\|\varphi_0\|^2 + \delta\big\{D^{2\alpha_h}(M^{2(\alpha_h-1)}+1)+D^2\big\}\int_0^t\|\varphi(s)\|^2_{A,\rho}\,ds \\
&\quad + c_\delta\,\|u\|^2_{L^2(0,T;L^\infty(\Omega))} + c_\delta\int_0^t(1+\|u_1(s)\|^2_\infty)\|\varphi(s)\|^2\,ds \\
&\quad + \delta D^2\int_0^t\|\sigma(s)\|^2_{B,\tau}\,ds\,, \qquad (3.9)
\end{aligned}$$

and we observe that the function $s\mapsto\|u_1(s)\|^2_\infty$ belongs to $L^1(0,T)$ and that its norm is bounded by M^2. Now, we write (2.27) at the time s for both solutions, test the difference by $\sigma(s)$, integrate over $(0,t)$ and add the same quantity $\int_{Q_t}|\sigma|^2$ to both sides. We obtain that

$$\begin{aligned}
&\frac12\,\|\sigma(t)\|^2 + \int_0^t\|\sigma(s)\|^2_{B,\tau}\,ds \\
&= \int_{Q_t}|\sigma|^2 - \int_{Q_t}\big(\gamma(\varphi_1)\sigma_1-\gamma(\varphi_2)\sigma_2\big)\sigma \\
&\quad + \int_{Q_t}\big(\kappa(\varphi_1)-\kappa(\varphi_2)\big)\sigma - \int_{Q_t}(S_1\varphi_1-S_2\varphi_2)\sigma\,. \qquad (3.10)
\end{aligned}$$

We estimate the last three terms, separately, by accounting for the analogues of (2.38) for γ and κ. As for the first one, we have that

$$
\begin{aligned}
&-\int_{Q_t} \big(\gamma(\varphi_1)\sigma_1 - \gamma(\varphi_2)\sigma_2\big)\sigma = -\int_{Q_t} \big((\gamma(\varphi_1)-\gamma(\varphi_2))\sigma_1 + \gamma(\varphi_2)\sigma\big)\sigma \\
&\le c\int_{Q_t} (\varphi_*^{\alpha_\gamma-1}+1)|\varphi|\,|\sigma_1|\,|\sigma| + c\int_{Q_t} (|\varphi_2|^{\alpha_\gamma}+1)\,|\sigma|^2 \\
&\le c\int_0^t (\|(\varphi_*(s))^{\alpha_\gamma-1}\|_{p_{\gamma,1}}+1)\,\|\varphi(s)\|_{q_{\gamma,1}}\,\|\sigma_1(s)\|_{r_{\gamma,1}}\,\|\sigma(s)\|\,ds \\
&\quad + c\int_0^t (\|(\varphi_2(s))^{\alpha_\gamma}\|_{p_{\gamma,2}}+1)\,\|\sigma(s)\|_{q_{\gamma,2}}\,\|\sigma(s)\|\,ds \\
&\le \delta\int_0^t (\|\varphi_*(s)\|_{p_*}^{2(\alpha_\gamma-1)}+1)\,\|\varphi(s)\|_{q_{\gamma,1}}^2\,ds + c_\delta\int_0^t \|\sigma_1(s)\|_{r_{\gamma,1}}^2\,\|\sigma(s)\|^2\,ds \\
&\quad + \delta\int_0^t \|\sigma(s)\|_{q_{\gamma,2}}^2\,ds + c_\delta\int_0^t (\|\varphi_2(s)\|_{p_{\gamma,2}\alpha_\gamma}^{2\alpha_\gamma}+1)\,\|\sigma(s)\|^2\,ds \\
&\le \delta\,D^{2\alpha_\gamma}\int_0^t (\|\varphi_*(s)\|_{A,\rho}^{2(\alpha_\gamma-1)}+1)\,\|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta\,D^2\int_0^t \|\sigma_1(s)\|_{B,\tau}^2\,\|\sigma(s)\|^2\,ds \\
&\quad + \delta\,D^2\int_0^t \|\sigma(s)\|_{B,\tau}^2\,ds + c_\delta\,D^{2\alpha_\gamma}\int_0^t (\|\varphi_2(s)\|_{A,\rho}^{2\alpha_\gamma}+1)\,\|\sigma(s)\|^2\,ds \\
&\le \delta\,D^{2\alpha_\gamma}\,(M^{2(\alpha_\gamma-1)}+1)\int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta\int_0^t \|\sigma(s)\|^2\,ds \\
&\quad + \delta\,D^2\int_0^t \|\sigma(s)\|_{B,\tau}^2\,ds + c_\delta\int_0^t \|\sigma(s)\|^2\,ds\,. \qquad (3.11)
\end{aligned}
$$

We estimate the next term in this way:

$$
\begin{aligned}
&\int_{Q_t} \big(\kappa(\varphi_1)-\kappa(\varphi_2)\big)\sigma \le c\int_{Q_t} (\varphi_*^{\alpha_\kappa-1}+1)\,|\varphi|\,|\sigma| \\
&\le c\int_0^t (\|(\varphi_*(s))^{\alpha_\kappa-1}\|_{p_\kappa}+1)\,\|\varphi(s)\|_{q_\kappa}\,\|\sigma(s)\|\,ds \\
&\le \delta\int_0^t (\|(\varphi_*(s))^{\alpha_\kappa-1}\|_{p_\kappa}^2+1)\,\|\varphi(s)\|_{q_\kappa}^2\,ds + c_\delta\int_0^t \|\sigma(s)\|^2\,ds \\
&\le \delta\,D^{2\alpha_\kappa}\int_0^t (\|\varphi_*(s)\|_{A,\rho}^{2(\alpha_\kappa-1)}+1)\,\|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta\int_0^t \|\sigma(s)\|^2\,ds \\
&\le \delta\,D^{2\alpha_\kappa}\,(M^{2(\alpha_\kappa-1)}+1)\int_0^t \|\varphi(s)\|_{A,\rho}^2\,ds + c_\delta\int_0^t \|\sigma(s)\|^2\,ds\,. \qquad (3.12)
\end{aligned}
$$

Finally, we observe that

$$
\begin{aligned}
&-\int_{Q_t} (S_1\varphi_1 - S_2\varphi_2)\sigma = -\int_{Q_t} \big(S\,\varphi_1 + S_2\,\varphi\big)\sigma \\
&\le \int_{Q_t} |S|\,|\varphi_1|\,|\sigma| + \int_{Q_t} |S_2|\,|\varphi|\,|\sigma| \\
&\le c\int_0^t \|S(s)\|_\infty\,\|\varphi_1(s)\|\,\|\sigma(s)\|\,ds + c\int_0^t \|S_2(s)\|_\infty\,\|\varphi(s)\|\,\|\sigma(s)\|\,ds
\end{aligned}
$$

$$\leq c \int_0^T \|S(s)\|_\infty^2 \, ds + c \int_0^t \|\sigma(s)\|^2 \, ds$$
$$+ c \int_0^t \|\varphi(s)\|^2 \, ds + c \int_0^t \|S_2(s)\|_\infty^2 \, \|\sigma(s)\|^2 \, ds \,. \tag{3.13}$$

By collecting (3.10)–(3.13) and rearranging, we conclude that

$$\frac{1}{2} \|\sigma(t)\|^2 + \int_0^t \|\sigma(s)\|_{B,\tau}^2 \, ds$$
$$\leq \delta \left\{ D^{2\alpha_\gamma} \left(M^{2(\alpha_\gamma - 1)} + 1 \right) + D^{2\alpha_\kappa} \left(M^{2(\alpha_\kappa - 1)} + 1 \right) \right\} \int_0^t \|\varphi(s)\|_{A,\rho}^2 \, ds$$
$$+ \delta \, D^2 \int_0^t \|\sigma(s)\|_{B,\tau}^2 \, ds + c \, \|S\|_{L^2(0,T;L^\infty(\Omega))}^2$$
$$+ c \int_0^t \|\varphi(s)\|^2 \, ds + c_\delta \int_0^t (1 + \|S_2(s)\|_\infty^2) \, \|\sigma(s)\|^2 \, ds \,, \tag{3.14}$$

where we observe that the function $s \mapsto \|S_2(s)\|_\infty^2$ belongs to $L^1(0,T)$ and that its norm is bounded by M^2. At this point, we add (3.9) and (3.14) to each other, choose δ small enough and apply the Gronwall lemma. This yields (2.40) with a constant that has the same dependence as the constant $\overline{C}_2$ in the statement. This completes the proof.

4 Approximation

In this section, we deal with an approximation of problem (2.26)–(2.28) depending on the parameter $\varepsilon \in (0,1)$ and solve it by a Faedo–Galerkin scheme. We first introduce the Moreau–Yosida regularizations $\widehat{\beta}_\varepsilon$ and β_ε of $\widehat{\beta}$ and β at the level $\varepsilon > 0$ (see, e.g., [6, p. 28 and p. 39]), and we recall that β_ε is the derivative of $\widehat{\beta}_\varepsilon$ and is Lipschitz continuous. We also remark that the following properties hold true (the first inequality being due to (2.11)):

$$0 \leq \widehat{\beta}_{\varepsilon''}(r) \leq \widehat{\beta}_{\varepsilon'}(r) \leq \widehat{\beta}(r) \quad \text{if } 0 < \varepsilon' \leq \varepsilon''$$
$$\text{and} \quad \lim_{\varepsilon \searrow 0} \widehat{\beta}_\varepsilon(r) = \widehat{\beta}(r), \quad \text{for every } r \in \mathbb{R} \,. \tag{4.1}$$

The approximating problem, whose unknown is the pair $(\varphi_\varepsilon, \sigma_\varepsilon)$, is obtained by replacing $\widehat{\beta}$ by $\widehat{\beta}_\varepsilon$ in (2.26)–(2.28) and approximating the other nonlinearities by smoother functions. Moreover, we also regularize the data u and S. However, since $\widehat{\beta}_\varepsilon$ is differentiable, the variational inequality (i.e., the analogue of (2.26)) can be replaced by an equation involving the derivative β_ε of $\widehat{\beta}_\varepsilon$. As for the other approximating nonlinearities, we require that they are bounded and Lipschitz continuous and converge to the original ones uniformly on every compact interval of $\mathbb{R}$. Moreover, inequalities that are analogous to (2.16)–(2.17) should be satisfied by the regularized functions uniformly with respect to ε. A possible construction of such approximations is based on the lemma stated below. It is clear that the second assumption in (4.7) is empty if $D(\widehat{\beta}) = \mathbb{R}$. In this case, K_3 can be any positive constant.

Lemma 4.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous and define $\psi^\varepsilon, \psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by the formulas*

$$\psi^\varepsilon(r) := \psi(r) \quad \textit{if } |r| \leq 1/\varepsilon \,, \quad \psi^\varepsilon(r) := \psi((\operatorname{sign} r)/\varepsilon) \quad \textit{if } |r| > 1/\varepsilon$$
$$\textit{and} \quad \psi_\varepsilon(r) := \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} \psi^\varepsilon(s) \, ds \quad \textit{for } r \in \mathbb{R}. \tag{4.2}$$

Then, ψ_ε is bounded and Lipschitz continuous on $\mathbb{R}$, and it holds that

$$\psi_\varepsilon \to \psi \quad \textit{uniformly on every bounded interval of } \mathbb{R}. \tag{4.3}$$

Moreover, if

$$|\psi(r)| \le K_0\,|r|^\alpha + K_1 \quad \textit{for every } r \in \mathbb{R} \textit{ and some positive constants } \alpha,\, K_0 \textit{ and } K_1\,, \tag{4.4}$$

then there are constants $\widehat{K}_0$ and $\widehat{K}_1$ depending only on α, K_0 and K_1 such that

$$|\psi_\varepsilon(r)| \le \widehat{K}_0\,|r|^\alpha + \widehat{K}_1 \quad \textit{for every } r \in \mathbb{R} \textit{ and } \varepsilon \in (0,1)\,. \tag{4.5}$$

Finally, also assume that

$$|\psi(r)|^2 \le K_2\,\widehat{\beta}(r) + K_3 \quad \textit{for every } r \in \mathbb{R}, \tag{4.6}$$

where K_2 and K_3 are constants satisfying

$$K_2 \ge 1 \quad \textit{and} \quad K_3 \ge \sup_{|r-r_0|\le\delta} |\psi(r)|^2 \tag{4.7}$$

for every end-point r_0 of $D(\widehat{\beta})$ (if any) and some $\delta > 0$. Then, the function $\widetilde{\psi_\varepsilon} : \mathbb{R} \to \mathbb{R}$ defined by setting

$$\widetilde{\psi_\varepsilon}(r) := \max\bigl\{-(K_2\,\widehat{\beta}_\varepsilon(r) + K_3)^{1/2}, \min\{\psi_\varepsilon(r), (K_2\,\widehat{\beta}_\varepsilon(r) + K_3)^{1/2}\}\bigr\} \tag{4.8}$$

is bounded and Lipschitz continuous on $\mathbb{R}$ and satisfies

$$|\widetilde{\psi_\varepsilon}(r)|^2 \le K_2\,\widehat{\beta}_\varepsilon(r) + K_3 \quad \textit{for every } r \in \mathbb{R} \textit{ and } \varepsilon \in (0,1), \tag{4.9}$$

as well as the analogues of (4.3) *and* (4.5)*.*

Proof. Clearly, ψ_ε is of class C^1 and its derivative is given by $(1/(2\varepsilon)(\psi^\varepsilon(r+\varepsilon) - \psi^\varepsilon(r-\varepsilon)$. In particular, if $|r| > (1/\varepsilon)+\varepsilon$, we have that $\psi_\varepsilon'(r) = 0$, so that ψ_ε is bounded and Lipschitz continuous on $\mathbb{R}$. Take now any $M > 0$ and $\eta > 0$. Since ψ is uniformly continuous on every bounded interval, we have that $|\psi(r) - \psi(s)| \le \eta$ whenever $\varepsilon \in (0,1)$ is small enough, $|r|, |s| \le M+1$ and $|r-s| \le \varepsilon$. Then, if ε also satisfies $M+1 < 1/\varepsilon$ and r belongs to $[-M, M]$, we conclude that

$$|\psi_\varepsilon(r) - \psi(r)| = \Bigl|\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} \bigl(\psi(s) - \psi(r)\bigr)\,ds\Bigr| \le \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} \eta\,ds = \eta\,.$$

All this proves (4.3). Assume now (4.4). Then, if $|s| \le |r|$, we have that $|\psi(s)| \le K_0\,|s|^\alpha + K_1 \le K_0\,|r|^\alpha + K_1$. Therefore, $\sup_{|s|\le|r|} |\psi(s)| \le K_0\,|r|^\alpha + K_1$, whence also

$$\sup_{|s|\le|r|} |\psi^\varepsilon(s)| \le \sup_{|s|\le|r|} |\psi(s)| \le K_0\,|r|^\alpha + K_1\,.$$

Set now $M_\alpha := \sup_{r\in\mathbb{R}}(|r|+1)^\alpha/(|r|^\alpha+1)$, which is obviously finite. Then, for every $r \in \mathbb{R}$ and $\varepsilon \in (0,1)$, we have that

$$|\psi_\varepsilon(r)| \le \sup_{|s|\le|r|+1} |\psi^\varepsilon(s)| \le K_0(|r|+1)^\alpha + K_1 \le K_0 M_\alpha(|r|^\alpha+1) + K_1\,,$$

so that (4.5) holds with $\widehat{K}_0 = K_0 M_\alpha$ and $\widehat{K}_1 = K_0 M_\alpha + K_1$.

Let us come to the properties of $\widetilde{\psi_\varepsilon}$ under the assumption (4.6). First of all, recall that K_2 and K_3 are positive, that $\widehat{\beta}_\varepsilon$ is nonnegative and locally Lipschitz continuous, and that it tends to $+\infty$ as its argument tends to $\pm\infty$ (as a consequence of the last condition in (2.11)). Thus, it turns out that $\widetilde{\psi_\varepsilon}(r) = \psi_\varepsilon(r)$ for $|r|$ large enough. Moreover, ψ_ε is (globally) Lipschitz continuous. This yields that the function $\widetilde{\psi_\varepsilon}$ it is well defined and Lipschitz continuous. Moreover, $\widetilde{\psi_\varepsilon}$ is bounded in view of the boundedness of ψ_ε. Furthermore, (4.9) trivially follows from the definition of $\widetilde{\psi_\varepsilon}$, and the analogue of (4.5) for $\widetilde{\psi_\varepsilon}$ is a consequence of (4.5) itself, since $|\widetilde{\psi_\varepsilon}| \le |\psi_\varepsilon|$. Let us prove the analogue of (4.3) by assuming that the interior of $D(\widehat{\beta})$ is nonempty (the opposite case is even easier since then $D(\widehat{\beta})$ is a singleton). Take a compact interval I contained in the interior of $D(\widehat{\beta})$. Then, the restriction of $\widehat{\beta}$ to I is continuous. Therefore, by (4.1) and Dini's theorem on monotone convergence, $\widehat{\beta}_\varepsilon$ converges to $\widehat{\beta}$ uniformly in I. By combining this with (4.3), we infer that $\widetilde{\psi_\varepsilon}$ uniformly converges in I to the function

$$I \ni r \mapsto \max\bigl\{-(K_2\,\widehat{\beta}(r) + K_3)^{1/2}, \min\{\psi(r), (K_2\,\widehat{\beta}(r) + K_3)^{1/2}\}\bigr\} = \psi(r)\,,$$

the last equality being due to (4.6). If $D(\widehat{\beta}) = \mathbb{R}$, then the convergence properties under investigation are completely proved. In the opposite case, we consider two situations regarding the compact interval I. In the first one, $I = [a, b]$ is contained in the exterior of $D(\widehat{\beta})$. Then, $\widehat{\beta}_\varepsilon(r)$ tends to $+\infty$ as ε tends to zero for every $r \in I$. Moreover, the convergence is monotone due to (4.1). By applying Dini's theorem to $1/\widehat{\beta}_\varepsilon$ (whose pointwise limit is obviously continuous), we deduce that $\widehat{\beta}_\varepsilon$ is uniformly divergent. Thus, by setting $M := \sup_{a-1\le r\le b+1} |\psi(r)|$, we have that $\widehat{\beta}_\varepsilon(r) \ge M^2$ for every $r \in I$ and $\varepsilon > 0$ small enough. On the other hand, $|\psi_\varepsilon(r)| \le M$ for every $r \in I$ (see the first part of this proof). We thus have for such values of ε (since $K_2 \ge 1$ and $\widehat{\beta}_\varepsilon$ is nonnegative) that

$$|\psi_\varepsilon(r)|^2 \le M^2 \le \widehat{\beta}_\varepsilon(r) \le K_2\,\widehat{\beta}_\varepsilon(r) + K_3 \quad \text{for every } r \in I.$$

Therefore, $\widetilde{\psi_\varepsilon}$ coincides with ψ_ε on I for $\varepsilon > 0$ small enough, and our assertion follows from (4.3). In the other situation, $I = [r_0 - \delta, r_0 + \delta]$ is the δ-neighborhood of an endpoint r_0 of $D(\beta)$ like in (4.7). Then, for every $r \in I$, we have that $|\psi_\varepsilon(r)|^2 \le K_3 \le K_2\,\widehat{\beta}_\varepsilon(r) + K_3$, whence also $\widetilde{\psi_\varepsilon}(r) = \psi_\varepsilon(r)$, so that our assertion follows from (4.3) also in this case. Since every compact interval of $\mathbb{R}$ is the union of $n \le 3$ intervals of the previous type, the uniform convergence property we have claimed is completely proved. □

We use the above lemma to introduce the approximating nonlinearities. We choose $K_0 = C_0$ and $K_1 = C_1$, the constants appearing in (2.16), and let α take the values α_h, α_γ, and α_κ, according to the functions we want to define. Finally, by recalling (2.17), we set $K_2 := \max\{C_2, 1\}$ and, if $D(\widehat{\beta}) \ne \mathbb{R}$, we choose $K_3 \ge C_3$ in order to satisfy (4.7) with $\psi = h$ (in particular, (4.6) holds with $\psi = h$ as a consequence of (2.17)). Then, we agree that

$$\begin{aligned}&m_\varepsilon, \gamma_\varepsilon, \text{ and } \kappa_\varepsilon, \text{ are defined as } \psi_\varepsilon \text{ with } \psi = m, \gamma, \kappa, \text{ respectively,}\\ &\quad\text{and } h_\varepsilon \text{ is defined as } \widetilde{\psi_\varepsilon} \text{ with } \psi = h\,.\end{aligned} \tag{4.10}$$

Finally, we replace the data u and S by approximating data u_ε and S_ε satisfying

$$\begin{aligned}&u_\varepsilon \in L^\infty(Q) \quad \text{and} \quad S_\varepsilon \in L^\infty(Q) \quad \text{with}\\ &|u_\varepsilon| \le |u|\,, \quad |S_\varepsilon| \le |S|\,, \quad u_\varepsilon \to u \quad \text{and} \quad S_\varepsilon \to S \quad \text{a.e. in } Q\,.\end{aligned} \tag{4.11}$$

To fulfill these conditions, one can set, e.g., $u_\varepsilon := \max\{-1/\varepsilon, \min\{u, 1/\varepsilon\}\}$, and analogously define S_ε. Therefore, the problem we consider is the following:

$$\int_\Omega \partial_t\varphi_\varepsilon(t)\, v + \bigl(A^\rho\varphi_\varepsilon(t), A^\rho v\bigr) + \int_\Omega (\beta_\varepsilon + \pi)(\varphi_\varepsilon(t))\, v$$
$$= \int_\Omega h_\varepsilon(\varphi_\varepsilon(t))\bigl(m_\varepsilon(\sigma_\varepsilon(t)) - m_0\, u_\varepsilon(t)\bigr)\, v$$
$$\text{for every } v \in V_A^\rho \text{ and for a.a. } t \in (0,T), \tag{4.12}$$

$$\int_\Omega \partial_t\sigma_\varepsilon(t)\, v + \bigl(B^\tau\sigma_\varepsilon(t), B^\tau v\bigr) + \int_\Omega \gamma_\varepsilon(\varphi_\varepsilon(t))\, \sigma_\varepsilon(t)\, v$$
$$= \int_\Omega \kappa_\varepsilon(\varphi_\varepsilon(t))\, v - \int_\Omega S_\varepsilon(t)\varphi_\varepsilon(t)\, v$$
$$\text{for every } v \in V_B^\tau \text{ and for a.a. } t \in (0,T), \tag{4.13}$$

$$\varphi_\varepsilon(0) = \varphi_0 \quad \text{and} \quad \sigma_\varepsilon(0) = \sigma_0\,. \tag{4.14}$$

Remark 4.2. We stress once more that (4.12) is equivalent to the variational inequalities obtained by replacing the nonlinearities with their approximations defined above in both (2.26) and (2.32). Moreover, (4.12) and (4.13) are also equivalent to their time-integrated versions with test functions taken in $L^2(0,T;V_A^\rho)$ and $L^2(0,T;V_B^\tau)$, respectively.

Theorem 4.3. *Under the same assumptions as in Theorem 2.5, the approximating problem* (4.12)–(4.14) *has a unique solution* $(\varphi_\varepsilon, \sigma_\varepsilon)$ *satisfying the analogues of the regularity conditions* (2.23)–(2.24).

The remainder of the section is devoted to the proof of this theorem. As for uniqueness, we can apply Theorem 2.7, since β_ε has the same properties as β, and the functions m_ε, h_ε, γ_ε and κ_ε are bounded and have bounded derivatives, so that all of the assumptions that are needed are satisfied by the approximating nonlinearities. Hence, we just have to prove the existence of a solution. To this end, we introduce a discrete problem depending on the parameter $n \in \mathbb{N}$ by means of a Faedo–Galerkin scheme. Then, we solve it and then take the limit of its solution as n tends to infinity.

The discrete problem. We recall that e_j and e'_j, $j = 1, 2, \dots$, are the eigenfunctions of the operators A and B, respectively. For every integer $n \geq 1$ we set

$$V_A^{\rho,n} := \operatorname{span}\{e_1, \dots, e_n\} \quad \text{and} \quad V_B^{\tau,n} := \operatorname{span}\{e'_1, \dots, e'_n\} \tag{4.15}$$

and look for a pair (φ_n, σ_n) enjoying the regularity

$$\varphi_n \in H^1(0,T;V_A^{\rho,n}) \quad \text{and} \quad \sigma_n \in H^1(0,T;V_B^{\tau,n}) \tag{4.16}$$

and solving the following problem

$$\int_\Omega \partial_t\varphi_n(t)\, v + \bigl(A^\rho\varphi_n(t), A^\rho v\bigr) + \int_\Omega (\beta_\varepsilon + \pi)(\varphi_n(t))\, v$$
$$= \int_\Omega h_\varepsilon(\varphi_n(t))\bigl(m_\varepsilon(\sigma_n(t)) - m_0\, u_\varepsilon(t)\bigr)\, v$$
$$\text{for every } v \in V_A^{\rho,n} \text{ and for a.a. } t \in (0,T), \tag{4.17}$$

$$\int_\Omega \partial_t\sigma_n(t)\, v + \bigl(B^\tau\sigma_n(t), B^\tau v\bigr) + \int_\Omega \gamma_\varepsilon(\varphi_n(t))\,\sigma_n(t)\, v$$
$$= \int_\Omega \kappa_\varepsilon(\varphi_n(t))\, v - \int_\Omega S_\varepsilon(t)\varphi_n(t)\, v$$
$$\text{for every } v \in V_B^{\tau,n} \text{ and for a.a. } t \in (0,T), \tag{4.18}$$

$$(\varphi_n(0), v) = (\varphi_0, v) \quad \text{and} \quad (\sigma_n(0), v) = (\sigma_0, v)$$
$$\text{for every } v \in V_A^{\rho,n} \text{ and } v \in V_B^{\tau,n}, \text{ respectively.} \tag{4.19}$$

Even though φ_n and σ_n obviously depend on ε as well, we do not stress this in the notation. We observe that (4.19) simply means that

$$\varphi_n(0) = \sum_{j=1}^n (\varphi_0, e_j)e_j \quad \text{and} \quad \sigma_n(0) = \sum_{j=1}^n (\sigma_0, e'_j)e'_j\,, \tag{4.20}$$

since $\varphi_n(0) \in V_A^{\rho,n}$ and $\sigma_n(0) \in V_B^{\tau,n}$. Notice that this implies that

$$\|\varphi_n(0)\| \le \|\varphi_0\|\,, \quad \|\sigma_n(0)\| \le \|\sigma_0\|\,,$$
$$\|\varphi_n(0)\|_{A,\rho} \le \|\varphi_0\|_{A,\rho} \quad \text{and} \quad \|\sigma_n(0)\|_{B,\tau} \le \|\sigma_0\|_{B,\tau}\,. \tag{4.21}$$

Indeed, we have for instance that

$$\|A^\rho\varphi(0)\|^2 = \Bigl\|\textstyle\sum_{i=1}^n (\varphi_0, e_j)A^\rho e_j\Bigr\|^2 = \Bigl\|\sum_{i=1}^n (\varphi_0, e_j)\lambda_j^\rho e_j\Bigr\|^2$$
$$\le \Bigl\|\textstyle\sum_{i=1}^\infty (\varphi_0, e_j)\lambda_j^\rho e_j\Bigr\|^2 = \Bigl\|\sum_{i=1}^\infty (\varphi_0, e_j)A^\rho e_j\Bigr\|^2 = \|A^\rho\varphi_0\|^2\,.$$

The discrete problem has a unique solution, as we see at once. By (4.16), the unknowns have to be expanded as

$$\varphi_n(t) = \sum_{j=1}^n y_j(t)e_j \quad \text{and} \quad \sigma_n(t) = \sum_{j=1}^n z_j(t)e'_j\,,$$

with some coefficients $y_j, z_j \in H^1(0,T)$. Therefore, the true unknowns are the vectors $y := (y_1,\dots,y_n)$ and $z := (z_1,\dots,z_n)$. Since it is sufficient to take $v = e_i$ and $v = e'_i$ with $i = 1,\dots,n$ in (4.17) and (4.18), respectively, and since the eigenvectors satisfy (2.3), the variational equations (4.17) and (4.18) in terms of y and z become

$$y'_i(t) + \lambda_i^{2\rho}\, y_i(t) + \Psi_{1,i}(t, y(t), z(t)) = \Psi_{2,i}(t, y(t), z(t))$$
$$\text{for } i = 1,\dots,n \text{ and for a.a. } t \in (0,T), \tag{4.22}$$

$$z'_i(t) + (\lambda'_i)^{2\tau}\, z_i(t) + \Psi_{3,i}(t, y(t), z(t)) = \Psi_{4,i}(t, y(t), z(t))$$
$$\text{for } i = 1,\dots,n \text{ and for a.a. } t \in (0,T), \tag{4.23}$$

where the Carathéodory functions $\Psi_{k,i} : (0,T) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are defined by

$$\Psi_{1,i}(t,\bar y,\bar z) := \int_\Omega (\beta_\varepsilon + \pi)\Bigl(\textstyle\sum_{j=1}^n \bar y_j e_j\Bigr)e_i\,,$$
$$\Psi_{2,i}(t,\bar y,\bar z) := \int_\Omega h_\varepsilon\Bigl(\textstyle\sum_{j=1}^n \bar y_j e_j\Bigr)\Bigl[m_\varepsilon\Bigl(\sum_{j=1}^n \bar z_j e'_j\Bigr) - m_0\, u_\varepsilon(t)\Bigr]\, e_i\,,$$
$$\Psi_{3,i}(t,\bar y,\bar z) := \int_\Omega \gamma_\varepsilon\Bigl(\textstyle\sum_{j=1}^n \bar y_j e_j\Bigr)\sum_{j=1}^n \bar z_j e'_j\, e'_i\,,$$
$$\Psi_{4,i}(t,\bar y,\bar z) := \int_\Omega \kappa_\varepsilon\Bigl(\textstyle\sum_{j=1}^n \bar y_j e_j\Bigr)e'_i - \int_\Omega S_\varepsilon(t)\sum_{j=1}^n \bar y_j e_j\, e'_i\,,$$

for $\bar{y}, \bar{z} \in \mathbb{R}^n$ and for a.a. $t \in (0,T)$. Hence, we have obtained a system of $2n$ ordinary differential equations in $2n$ unknowns. Since all of the functions $\beta_\varepsilon, \ldots, \kappa_\varepsilon$ are Lipschitz continuous and h_ε, m_ε and γ_ε are even bounded, as well as u_ε and S_ε, the functions $\Psi_{k,i}$ are Lipschitz continuous with respect to $(\bar{y}, \bar{z}) \in \mathbb{R}^{2n}$ uniformly with respect to t. Since (4.19) (or (4.20)) provides an initial condition for (y, z), we conclude that the discrete problem has a unique solution with the regularity specified by (4.16).

Now that the discrete problem is solved, we can start estimating. According our general rule, the symbol c_ε denotes (possibly different) constants that are allowed to depend on the structure, the data, T, and ε, but not on n.

First a priori estimate. We write (4.17) at the time s and choose $v = \partial_t \varphi_n(s) \in V_A^{\rho,n}$. Then, we integrate with respect to s over the interval $(0,t)$ with $t \in (0,T]$. Moreover, we add the same quantity $\frac{1}{2}\|\varphi_n(t)\|^2 - \frac{1}{2}\|\varphi_n(0)\|^2 = \int_{Q_t} \varphi_n \partial_t \varphi_n$ to both sides. After rearranging terms, we obtain the identity

$$\begin{aligned} &\int_{Q_t} |\partial_t \varphi_n|^2 + \frac{1}{2}\, \|\varphi_n(t)\|_{A,\rho}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) \\ &= \frac{1}{2}\, \|\varphi_n(0)\|_{A,\rho}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(0)) \\ &\quad + \int_{Q_t} \bigl(\varphi_n - \pi(\varphi_n) + h_\varepsilon(\varphi_n)\bigl(m_\varepsilon(\sigma_n) - m_0 u_\varepsilon\bigr)\bigr) \partial_t \varphi_n \,. \end{aligned} \tag{4.24}$$

All of the terms on the left-hand side are nonnegative, and the whole right-hand side can be estimated by the quantity

$$c_\varepsilon\, \|\varphi_n(0)\|_{A,\rho}^2 + \frac{1}{2} \int_0^t |\partial_t \varphi_n|^2 + c_\varepsilon \int_{Q_t} |\varphi_n|^2 + c_\varepsilon \,,$$

since $\widehat{\beta}_\varepsilon$ grows quadratically at infinity, π is Lipschitz continuous on $\mathbb{R}$, and the functions h_ε, m_ε, and u_ε, are bounded. Moreover, we can owe to (4.21). We thus obtain the estimate

$$\|\varphi_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho)} \le c_\varepsilon \,. \tag{4.25}$$

Second a priori estimate. Similarly, we test (4.18) written at the time s by $\partial_t \sigma_n(s)$ and integrate over $(0,t)$. Also in this case, we add the same quantity to both sides. We thus obtain that

$$\begin{aligned} &\int_{Q_t} |\partial_t \sigma_n|^2 + \frac{1}{2}\, \|\sigma_n(t)\|_{B,\tau}^2 \\ &= \frac{1}{2}\, \|\sigma_n(0)\|_{B,\tau}^2 + \int_{Q_t} \bigl(\sigma_n - \gamma_\varepsilon(\varphi_n)\sigma_n + \kappa_\varepsilon(\varphi_n)\bigr) \partial_t \sigma_n - \int_{Q_t} S_\varepsilon\, \varphi_n\, \partial_t \sigma_n \,. \end{aligned} \tag{4.26}$$

Using (4.21) for σ_n, we can estimate the first term on the right-hand side. Since γ_ε and κ_ε are bounded, as well as S_ε, the whole right-hand side is thus bounded by

$$c_\varepsilon + \frac{1}{2} \int_{Q_t} |\partial_t \sigma_n|^2 + c_\varepsilon \int_{Q_t} |\sigma_n|^2 + c_\varepsilon \int_0^t \|\varphi_n(s)\|^2 \, ds \,.$$

By accounting for (4.25), and applying the Gronwall lemma, we conclude that

$$\|\sigma_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V_B^\tau)} \le c_\varepsilon \,. \tag{4.27}$$

Limit. By virtue of (4.25)–(4.27) we can find subsequences (still labeled with the index n for simplicity) that converge to some limits in the weak or weak star topologies associated with the estimates. Since both V_A^ρ and V_B^τ are compactly embedded in H (due to assumption (2.2)), by recalling, e.g., [51, Sect. 8, Cor. 4], we conclude that there is a pair $(\varphi_\varepsilon, \sigma_\varepsilon)$ satisfying the analogues of the regularity conditions (2.23)–(2.24) such that, at least for a subsequence of n,

$$\begin{aligned} \varphi_n \to \varphi_\varepsilon \quad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \\ & \text{and strongly in } C^0([0,T];H), && (4.28) \\ \sigma_n \to \sigma_\varepsilon \quad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_B^\tau) \\ & \text{and strongly in } C^0([0,T];H), && (4.29) \end{aligned}$$

We now show that such a pair solves the approximating problem. Namely, we consider the time integrated version mentioned in Remark 4.2. Take any integer $\bar n \geq 1$ and any $v \in L^2(0,T;V_A^{\rho,\bar n})$. Then, if the (selected) index n satisfies $n \geq \bar n$, then $v(t) \in V_A^{\rho,n}$ for a.a. $t \in (0,T)$ and we can use $v(t)$ as a test function in (4.17). After integration over $(0,T)$, we obtain that

$$\int_Q \partial_t \varphi_n \, v + \int_0^T \bigl(A^\rho \varphi_n(t), A^\rho v(t) \bigr) \, dt + \int_Q (\beta_\varepsilon + \pi)(\varphi_n) \, v = \int_Q h_\varepsilon(\varphi_n) \bigl(m_\varepsilon(\sigma_n) - m_0 \, u_\varepsilon \bigr) \, v \, ,$$

and we can let n tend to infinity. Since all of the nonlinearities are Lipschitz continuous and h_ε and m_ε, as well as u_ε, are even bounded, we conclude that

$$\begin{aligned} & \int_Q \partial_t \varphi_\varepsilon \, v + \int_0^T \bigl(A^\rho \varphi_\varepsilon(t), A^\rho v(t) \bigr) \, dt + \int_Q (\beta_\varepsilon + \pi)(\varphi_\varepsilon) \, v \\ & = \int_Q h_\varepsilon(\varphi_\varepsilon) \bigl(m_\varepsilon(\sigma_\varepsilon) - m_0 \, u_\varepsilon \bigr) \, v \, . \end{aligned} \tag{4.30}$$

Since $\bar n$ is arbitrary, this equality holds for every step function v with values in the union $V_A^{\rho,\infty}$ of the spaces $V_A^{\rho,\bar n}$. By recalling that $V_A^{\rho,\infty}$ is dense in V_A^ρ, an easy density argument shows that (4.30) holds for every $v \in L^2(0,T;V_A^\rho)$.

Concerning the equations for σ_n and σ_ε, we observe that $\gamma_\varepsilon(\varphi_n)$ converges to $\gamma_\varepsilon(\varphi)$ a.e. in Q, since γ_ε is continuous. On the other hand, γ_ε is bounded. Hence, we infer that $\gamma_\varepsilon(\varphi_n)$ also converges to $\gamma_\varepsilon(\varphi)$ in the weak star topology of $L^\infty(Q)$, and combining this with the strong convergence given by (4.29), we conclude that $\gamma_\varepsilon(\varphi_n)\sigma_n$ converges to $\gamma_\varepsilon(\varphi)\sigma_\varepsilon$ weakly in $L^2(0,T;H)$. Finally, $\kappa_\varepsilon(\varphi_n)$ converges to $\kappa_\varepsilon(\varphi_\varepsilon)$ strongly in $C^0([0,T];H)$ by Lipschitz continuity. By arguing as for the previous equation, and using a similar density property related to the spaces $V_B^{\tau,n}$, we conclude that $(\varphi_\varepsilon, \sigma_\varepsilon)$ solves the integrated version of (4.13) with test functions taken in $L^2(0,T;V_B^\tau)$ as well. Finally, $\varphi_n(0)$ and $\sigma_n(0)$ converge to $\varphi_\varepsilon(0)$ and $\sigma_\varepsilon(0)$ in H by (4.28)–(4.29). On the other hand, (4.20) implies that $\varphi_n(0)$ and $\sigma_n(0)$ converge to φ_0 and σ_0, respectively, in the same topology. We infer that the initial conditions (4.14) are satisfied and conclude that $(\varphi_\varepsilon, \sigma_\varepsilon)$ actually solves problem (4.12)–(4.14).

5 Existence

In this section, we prove Theorem 2.5. We start from the solution $(\varphi_\varepsilon, \sigma_\varepsilon)$ to the approximate problem and take the limit as ε tends to zero. To perform this project, we have to prove some a priori estimates. In particular, we show that $(\varphi_\varepsilon, \sigma_\varepsilon)$ satisfies the analogue of (2.33) with a constant whose dependence is the same as that of $\overline{C}_1$ in Theorem 2.5 (in particular, it is independent of ε), since the symbol c we use always stands for (possibly different) constants independent of ε, according to our general rule. It follows that this estimate is kept in the limit as ε goes to zero and that the last sentence of Theorem 2.5 is proved as well. Hence, we do not return to this point in the sequel.

First a priori estimate. We claim that from (4.24), by a limit procedure as $n \nearrow \infty$, it is possible to derive the following inequality

$$\begin{aligned}&\int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \frac{1}{2}\, \|\varphi_\varepsilon(t)\|^2_{A,\rho} + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t)) \\ &\leq \frac{1}{2}\, \|\varphi_0\|^2_{A,\rho} + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) + \int_{Q_t} \bigl(\varphi_\varepsilon - \pi(\varphi_\varepsilon) + h_\varepsilon(\varphi_\varepsilon)\bigl(m_\varepsilon(\sigma_\varepsilon) - m_0\, u_\varepsilon\bigr)\bigr)\, \partial_t \varphi_\varepsilon\,. \end{aligned} \tag{5.1}$$

Indeed, by (4.28) and the weak lower semicontinuity of norms we infer that

$$\begin{aligned}&\int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \frac{1}{2}\, \|\varphi_\varepsilon(t)\|^2_{A,\rho} \leq \liminf_{n\nearrow\infty} \int_{Q_t} |\partial_t \varphi_n|^2 + \liminf_{n\nearrow\infty} \frac{1}{2}\, \|\varphi_n(t)\|^2_{A,\rho} \\ &\quad \leq \liminf_{n\nearrow\infty} \left(\int_{Q_t} |\partial_t \varphi_n|^2 + \frac{1}{2}\, \|\varphi_n(t)\|^2_{A,\rho} \right), \end{aligned}$$

since, $\varphi_n, \varphi_\varepsilon$ being weakly continuous from $[0,T]$ to V_A^ρ, for all $t \in [0,T]$ it occurs that $\varphi_n(t)$ weakly converges to $\varphi_\varepsilon(t)$ in V_A^ρ. Besides, note that if a sequence v_n converges to v strongly in H, then

$$\int_\Omega \widehat{\beta}_\varepsilon(v_n) \to \int_\Omega \widehat{\beta}_\varepsilon(v) \ \text{ as } \ n \nearrow \infty.$$

This is due to the mean value theorem in the integral form, which gives

$$\int_\Omega \widehat{\beta}_\varepsilon(v_n) - \int_\Omega \widehat{\beta}_\varepsilon(v) = \int_\Omega \left(\int_0^1 \beta_\varepsilon(v + s(v_n - v))(v_n - v) ds \right),$$

and to the Lipschitz continuity of β_ε. Hence, the terms $\int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))$ and $\int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(0))$ in (4.24) converge to the respective ones $\int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))$ and $\int_\Omega \widehat{\beta}_\varepsilon(\varphi_0)$ in (5.1). Moreover, we point out that

$$\frac{1}{2}\, \|\varphi_n(0)\|^2_{A,\rho} \leq \frac{1}{2}\, \|\varphi_0\|^2_{A,\rho}$$

thanks to (4.21), and that in the last term of (4.24) we can pass to the limit by strong-weak convergence in $L^2(0,T;H)$ (cf. (4.28)–(4.29)). Thus, (5.1) is completely verified.

All of the terms on the left-hand side of (5.1) are nonnegative. As for the ones on the right-hand side, we recall (4.1) for the second one and use the Schwarz and Young inequality for the volume integral. Moreover, we notice that (4.2) immediately yields that ψ_ε is uniformly bounded if ψ is bounded, so that m_ε is uniformly bounded since m is so. Furthermore, by recalling (4.10), we account for the inequality (4.9) applied to h_ε. Finally, we owe to the inequality $|u_\varepsilon| \leq |u|$ a.e. in Q (see (4.11)). Then, the right-hand side of (5.1) is estimated from above by

$$\begin{aligned}&c + \frac{1}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 \, + \, c \int_{Q_t} \bigl(1 + |\varphi_\varepsilon|^2\bigr) + c \int_{Q_t} \bigl(1 + |u|^2\bigr)\, |h_\varepsilon(\varphi_\varepsilon)|^2 \\ &\leq \frac{1}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 \, + \, c \int_0^t \|\varphi_\varepsilon(s)\|^2 \, ds + c \int_0^t \bigl(1 + \|u(s)\|^2_\infty\bigr) \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(s))\, ds + c\,. \end{aligned}$$

Since the function $s \mapsto \|u(s)\|_\infty^2$ belongs to $L^1(0,T)$ by (2.20), we can apply the Gronwall lemma and conclude that

$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)} + \|\widehat{\beta}_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,. \tag{5.2}$$

Second a priori estimate. Similarly, as for the derivation of (5.1), from (4.26) and (4.28)–(4.29) it is straightforward to deduce the inequality

$$\int_{Q_t} |\partial_t\sigma_\varepsilon|^2 + \frac{1}{2}\,\|\sigma_\varepsilon(t)\|_{B,\tau}^2 \le \frac{1}{2}\,\|\sigma_0\|_{B,\tau}^2 + \int_{Q_t} \big(\sigma_\varepsilon - \gamma_\varepsilon(\varphi_\varepsilon)\,\sigma_\varepsilon + \kappa_\varepsilon(\varphi_\varepsilon) - S_\varepsilon\varphi_\varepsilon\big)\partial_t\sigma_\varepsilon\,. \tag{5.3}$$

We treat the nontrivial terms on the right-hand side, separately. To do this, we owe to our assumptions (2.14)–(2.16) and to their consequences given by Lemma 4.1 applied with $\psi = \gamma$ and $\psi = \kappa$. Moreover, we also account for (5.2) already established. We have, by virtue of Hölder's and Young's inequalities, that

$$\begin{aligned} &-\int_{Q_t} \gamma_\varepsilon(\varphi_\varepsilon)\,\sigma_\varepsilon\,\partial_t\sigma_\varepsilon \le \int_0^t \|\gamma_\varepsilon(\varphi_\varepsilon(s))\|_{p_\gamma}\,\|\sigma_\varepsilon(s)\|_{q_\gamma}\,\|\partial_t\sigma_\varepsilon(s)\|_2\,ds \\ &\le \frac{1}{6}\int_{Q_t} |\partial_t\sigma_\varepsilon|^2 + c\int_0^t \|\gamma_\varepsilon(\varphi_\varepsilon(s))\|_{p_\gamma}^2\,\|\sigma_\varepsilon(s)\|_{q_\gamma}^2\,ds \\ &\le \frac{1}{6}\int_{Q_t} |\partial_t\sigma_\varepsilon|^2 + c\int_0^t \|\sigma_\varepsilon(s)\|_{B,\tau}^2\,ds\,, \end{aligned} \tag{5.4}$$

where in the last estimate we have employed (2.15) and (5.2) to see that

$$\|\gamma_\varepsilon(\varphi_\varepsilon(s))\|_{p_\gamma}^2 \le c\,\|\varphi_\varepsilon(s)\|_{\alpha_\gamma p_\gamma}^{2\alpha_\gamma} + c \le c\,\|\varphi_\varepsilon(s)\|_{A,\rho}^{2\alpha_\gamma} + c \le c \quad \text{for a.a. } s \in (0,T). \tag{5.5}$$

Now, we treat the next term on the right-hand side of (5.3). By applying the Young inequality, we immediately obtain that

$$\int_{Q_t} \kappa_\varepsilon(\varphi_\varepsilon)\,\partial_t\sigma_\varepsilon \le \frac{1}{6}\int_{Q_t} |\partial_t\sigma_\varepsilon|^2 + c\int_0^t \|\kappa_\varepsilon(\varphi_\varepsilon(s))\|_2^2\,ds\,.$$

On the other hand, we have that

$$\int_0^t \|\kappa_\varepsilon(\varphi_\varepsilon(s))\|_2^2\,ds \le c\int_0^t \|\varphi_\varepsilon(s)\|_{2\alpha_\kappa}^{2\alpha_\kappa}\,ds + c \le c\int_0^t \|\varphi_\varepsilon(s)\|_{A,\rho}^{2\alpha_\kappa}\,ds + c \le c\,. \tag{5.6}$$

Finally, by using the inequality $|S_\varepsilon| \le |S|$ a.e. in Q (see (4.11)), we can write

$$-\int_{Q_t} S_\varepsilon\varphi_\varepsilon\,\partial_t\sigma_\varepsilon \le \frac{1}{6}\int_{Q_t} |\partial_t\sigma_\varepsilon|^2 + c\int_{Q_t} |S|^2\,|\varphi_\varepsilon|^2$$

and, due to assumption (2.20) for S and to (5.2) once more,

$$\int_{Q_t} |S|^2\,|\varphi_\varepsilon|^2 \le \int_0^t \|S(s)\|_\infty^2\,\|\varphi_\varepsilon(s)\|^2\,ds \le c\,.$$

By combining all the inequalities just obtained with (5.3) and applying the Gronwall lemma, we conclude that

$$\|\sigma_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\tau)} \le c\,. \tag{5.7}$$

Estimates of the nonlinear terms. By recalling (5.5), and repeating the arguments that led to (5.6) without time integration, we see on account of (5.2) that

$$\|\gamma_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^{p_\gamma}(\Omega))} \le c \quad \text{and} \quad \|\kappa_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;H)} \le c\,. \tag{5.8}$$

By combining with (5.7) and recalling (2.14), we deduce that

$$\|\gamma_\varepsilon(\varphi_\varepsilon)\sigma_\varepsilon\|_{L^\infty(0,T;H)} \le c\,. \tag{5.9}$$

Since the terms involving h_ε are a little more complicated, we prepare an auxiliary estimate. We recall (2.14) for the definition of p_0. Take any $w \in V_A^\rho$ and notice that

$$|h_\varepsilon(w)|^{p_h} \le c\,|w|^{\alpha_h p_h} + c \quad \text{and} \quad (|w|\,|h_\varepsilon(w)|)^{p_0} \le c\,|w|^{(\alpha_h+1)p_0} + c \quad \text{a.e. in } \Omega\,,$$

thanks to the assumption on h given by (2.16) and Lemma 4.1 applied with $\psi = h$. By accounting for (2.15), we deduce that

$$\begin{aligned}
&\|h_\varepsilon(w)\|_{p_h}^{p_h} \le c\,\|w\|_{p_h\alpha_h}^{\alpha_h p_h} + c \le c\,\|w\|_{A,\rho}^{\alpha_h p_h} + c \quad \text{and} \\
&\|w\,h_\varepsilon(w)\|_{p_0}^{p_0} \le c\,\|w\|_{\max\{\alpha_\gamma p_\gamma, 2\alpha_\kappa\}}^{\max\{\alpha_\gamma p_\gamma, 2\alpha_\kappa\}} + c \le c\,\|w\|_{A,\rho}^{\max\{\alpha_\gamma p_\gamma, 2\alpha_\kappa\}} + c\,.
\end{aligned}$$

By applying this with $w = \varphi_\varepsilon(t)$ for a.a. $t \in (0,T)$ and recalling (5.2), we deduce that

$$\|h_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^{p_h}(\Omega))} \le c \quad \text{and} \quad \|\varphi_\varepsilon\,h_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^{p_0}(\Omega))} \le c\,.$$

Since m_ε is uniformly bounded (as already observed) and u_ε is bounded in $L^2(0,T;L^\infty(\Omega))$ by (4.11) and (2.20), we conclude that

$$\begin{aligned}
&\|h_\varepsilon(\varphi_\varepsilon)(m_\varepsilon(\sigma_\varepsilon) - m_0 u_\varepsilon)\|_{L^2(0,T;L^{p_h}(\Omega))} \\
&\quad + \|h_\varepsilon(\varphi_\varepsilon)(m_\varepsilon(\sigma_\varepsilon) - m_0 u_\varepsilon)\varphi_\varepsilon\|_{L^2(0,T;L^{p_0}(\Omega))} \le c\,.
\end{aligned} \tag{5.10}$$

Conclusion. We are ready to take the limit of $(\varphi_\varepsilon, \sigma_\varepsilon)$ as ε tends to zero. More precisely, in considering the approximating problem, we replace (4.30) (equivalent to (4.12)) and (4.13) with the equivalent variational inequality and integrated variational equation, respectively. We thus start from

$$\begin{aligned}
&\int_Q \partial_t\varphi_\varepsilon(\varphi_\varepsilon - v) + \int_0^T \bigl(A^\rho\varphi_\varepsilon(t), A^\rho(\varphi_\varepsilon(t) - v(t))\bigr)\,dt + \int_Q \widehat{\beta}_\varepsilon(\varphi_\varepsilon) + \int_Q \pi(\varphi_\varepsilon)(\varphi_\varepsilon - v) \\
&\le \int_Q h_\varepsilon(\varphi_\varepsilon)\bigl(m_\varepsilon(\sigma_\varepsilon)) - m_0\,u_\varepsilon\bigr)(\varphi_\varepsilon - v) + \int_Q \widehat{\beta}_\varepsilon(v) \quad \text{for every } v \in L^2(0,T;V_A^\rho)\,,
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
&\int_Q \partial_t\sigma_\varepsilon\,v + \int_0^T \bigl(B^\tau\sigma_\varepsilon(t), B^\tau v\bigr)\,dt + \int_Q \gamma_\varepsilon(\varphi_\varepsilon)\,\sigma_\varepsilon\,v \\
&= \int_Q \kappa_\varepsilon(\varphi_\varepsilon)\,v - \int_\Omega S_\varepsilon\varphi_\varepsilon\,v \quad \text{for every } v \in L^2(0,T;V_B^\tau)\,,
\end{aligned} \tag{5.12}$$

as well as the initial conditions (4.14). By (5.2) and (5.7), and accounting for standard weak, weak star and strong compactness results (see, e.g., [51, Sect. 8, Cor. 4] for the latter), we see that there exists a strictly decreasing subsequence $\varepsilon_n \searrow 0$ such that the corresponding convergence holds true. However, we still write ε, at least for a while, to simplify the notation. Namely, we have that

$$\begin{aligned} \varphi_\varepsilon \to \varphi \quad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho), \\ & \quad \text{strongly in } C^0([0,T];H) \text{ and a.e. in } Q, \qquad (5.13) \\ \sigma_\varepsilon \to \sigma \quad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_B^\tau), \\ & \quad \text{strongly in } C^0([0,T];H) \text{ and a.e. in } Q, \qquad (5.14) \end{aligned}$$

for some pair (φ, σ) satisfying (2.23)–(2.24). It follows that such a pair fulfills the initial conditions (2.28) and that

$$\pi(\varphi_\varepsilon) \to \pi(\varphi) \quad \text{strongly in } C^0([0,T];H), \tag{5.15}$$

because π is Lipschitz continuous. Moreover, since p_γ, p_h and p_0 are larger than 1, m_ε is uniformly bounded, and the estimates (5.8)–(5.10) hold true, we also have that

$$\gamma_\varepsilon(\varphi_\varepsilon)\sigma_\varepsilon \to \zeta_1 \qquad \text{weakly star in } L^\infty(0,T;H), \tag{5.16}$$

$$\kappa_\varepsilon(\varphi_\varepsilon) \to \zeta_2 \qquad \text{weakly star in } L^\infty(0,T;H), \tag{5.17}$$

$$h_\varepsilon(\varphi_\varepsilon)(m_\varepsilon(\sigma_\varepsilon) - m_0 u_\varepsilon) \to \zeta_3 \qquad \text{weakly in } L^2(0,T;L^{p_h}(\Omega)), \tag{5.18}$$

$$h_\varepsilon(\varphi_\varepsilon)(m_\varepsilon(\sigma_\varepsilon) - m_0 u_\varepsilon)\varphi_\varepsilon \to \zeta_4 \qquad \text{weakly in } L^2(0,T;L^{p_0}(\Omega)), \tag{5.19}$$

$$m_\varepsilon(\sigma_\varepsilon) \to \zeta_5 \qquad \text{weakly star in } L^\infty(Q), \tag{5.20}$$

for a suitable subsequence and some limiting functions ζ_i. On the other hand, γ_ε converges to γ uniformly on every compact interval of $\mathbb{R}$ thanks to Lemma 4.1 applied with $\psi = \gamma$. Combining this with the almost everywhere convergence given by (5.13)–(5.14), we deduce that

$$\gamma_\varepsilon(\varphi_\varepsilon)\sigma_\varepsilon \to \gamma(\varphi)\sigma \quad \text{a.e. in } Q\,.$$

We infer that (see, e.g., [45, Lemme 1.3, p. 12]) $\zeta_1 = \gamma(\varphi)\sigma$ a.e. in Q. By analogously arguing for the other nonlinear terms appearing in (5.17)–(5.20) (and recalling (4.11) for u_ε), we also conclude that

$$\zeta_2 = \kappa(\varphi)\,, \quad \zeta_3 = h(\varphi)\big(m(\sigma) - m_0 u\big)\,, \quad \zeta_4 = \zeta_3\,\varphi \quad \text{and} \quad \zeta_5 = m(\varphi)\,,$$

and that these limits can also be understood in the sense of convergence a.e. in Q. Furthermore, as $p_0 > 1$, the convergence in (5.19) also holds in the strong topology of $L^1(Q)$. It is also clear that (4.11), (2.20) and (5.13) imply that $S_\varepsilon\varphi_\varepsilon$ converges to $S\varphi$, e.g., weakly in $L^2(0,T;H)$ (by boundedness in this space and convergence a.e. in Q). For the last term of (5.11) we note that

$$\int_Q \widehat{\beta}_\varepsilon(v) \le \int_Q \widehat{\beta}(v) \quad \text{for every } v \in L^2(0,T;V_A^\rho)$$

by (4.1). Then, all of the above ensures that we can take the limit in all of the terms of (5.11)–(5.12) but the second and third ones on the left-hand side of the variational inequality. In particular, we obtain the time-integrated version of (2.27) with test functions v in $L^2(0,T;V_B^\tau)$. For the first of the terms of (5.11) we still have to consider, we have that

$$\int_0^T \big(A^\rho\varphi(t), A^\rho(\varphi(t) - v(t))\big)\,dt \le \liminf_{\varepsilon\searrow 0} \int_0^T \big(A^\rho\varphi_\varepsilon(t), A^\rho(\varphi_\varepsilon(t) - v(t))\big)\,dt\,, \tag{5.21}$$

by the semicontinuity of the norm and the weak convergence of $A^\rho\varphi_\varepsilon$ to $A^\rho\varphi$ in $L^2(0,T;H)$ ensured by (5.13). For the other one, we are going to derive the inequality

$$\int_\Omega \widehat{\beta}(\varphi(t)) \le \liminf_{\varepsilon\searrow 0} \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t)) \quad \text{for a.a. } t \in (0,T)\,. \tag{5.22}$$

Notice that its right-hand side (as a function of t) is bounded by (5.2). In particular, the requirement $\widehat{\beta}(\varphi) \in L^\infty(0,T;L^1(\Omega))$ (see (2.25)) is fulfilled once (5.22) is established, and we also have that

$$\int_Q \widehat{\beta}(\varphi) \le \liminf_{\varepsilon \searrow 0} \int_Q \widehat{\beta}_\varepsilon(\varphi_\varepsilon)\,, \tag{5.23}$$

by Fatou's lemma applied to the functions $t \mapsto \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t))$. It is clear that (5.21)–(5.23) are understood for the subsequence $\{\varepsilon_n\}$ selected in relation to (5.13)–(5.20). In proving (5.22), we use this subsequence in the notation, for clarity. Moreover, we account for the properties (4.1) of $\widehat{\beta}_\varepsilon$. Let us start. Since $\varepsilon_n < \varepsilon_m$ whenever $n > m$, we have that

$$\widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}(t)) \le \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}(t)) \quad \text{a.e. in } Q, \text{ for every } n > m,$$

whence also

$$\widehat{\beta}_{\varepsilon_m}(\varphi) = \lim_{n\to\infty} \widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}) = \liminf_{n\to\infty} \widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}) \le \liminf_{n\to\infty} \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}) \quad \text{a.e. in } Q,$$

since $\widehat{\beta}_{\varepsilon_m}$ is continuous. On the other hand, we have that

$$\widehat{\beta}(\varphi) = \lim_{m\to\infty} \widehat{\beta}_{\varepsilon_m}(\varphi) \quad \text{a.e. in } Q. \tag{5.24}$$

We infer that

$$\widehat{\beta}(\varphi(t)) \le \liminf_{n\to\infty} \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}(t)) \quad \text{a.e. in } \Omega, \quad \text{for a.a. } t \in (0,T), \tag{5.25}$$

and (5.22) follows from Fatou's lemma. At this point, we can owe to (5.23) and let ε tend to zero in (5.11) as well to obtain (2.32) with test functions v in $L^2(0,T;V_A^\rho)$. We conclude that (φ,σ) actually solves problem (2.26)–(2.28), and the proof of Theorem 2.5 is complete.

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