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Value at Risk approach to producer's best response in electricity market with uncertain demand

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Abstract

We deal with several sources of uncertainty in electricity markets. The independent system operator (ISO) maximizes the social welfare using chance constraints to hedge against discrepancies between the estimated and real electricity demand. We find an explicit solution of the ISO problem, and use it to tackle the problem of a producer. In our model, production as well as income of a producer are determined based on the estimated electricity demand predicted by the ISO, that is unknown to producers. Thus, each producer is hedging against the uncertainty of prediction of the demand using the value-at-risk approach. To illustrate our results, a numerical study of a producer's best response given a historical distribution of both estimated and real electricity demand is provided.

1 Introduction

In this paper, we deal with the problem which arises on the deregulated electricity markets. Since nowadays the electricity cannot be effectively stored, one of the goals is to balance the aggregated demand and the supply from several producers over a short time period. In particular, we focus on the day-ahead market where the producers offer electricity deliveries a day before the real demand is observed. The producers provide the bidding curves stating the price for a particular electricity quantity that they are able to deliver. Since in the real world the aggregated demand is not precisely known and thus is uncertain, we incorporate it into our models as a random variable leading to stochastic optimization problems. An Independent System Operator (ISO) collects the bidding curves (bids) from several producers and according to the an estimated distribution of the aggregated demand it computes the production quantities to be dispatched. Under our settings, the ISO goal is to maximize social welfare while satisfying the demand with a high probability. In our problems, we do not react on past outcomes of the random variables, which would correspond to the 'wait-and-see' approach, but we hedge against future unknown outcomes, so our decisions must be made 'here-and-know' and thus are not random but deterministic.

We focus on modelling the various ways the ISO and the individual producers are facing the uncertainty of (future) electricity demand in the day-ahead market. We have two main points in this respect. First, the way the ISO and the producers manage the uncertain demand should reflect their different roles. The ISO has to balance the supply and the demand with high reliability, whereas a producer is more focused on the reliability of the profit. To this end, we model the ISO problem with a chance constraint for the demand satisfaction, and we introduce a chance-constrained problem of a producer where the upper quantile of the random profit is maximized.

Second, we argue that the uncertainty of the demand has to be modelled using distinct random variables. Indeed, in real markets producers are bidding earlier than the ISO is clearing the market. Although this time difference may be relatively small, it can nevertheless force producers to use, e.g., a

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less precise weather forecast to compute their optimal bids. Even if this was not the case, it is reasonable to assume that different tools and heuristics used by competing market participants to model this uncertainty yield different results. To simplify the notation, we use one random variable for the ISO and another one for all the producers; note that one may easily generalize the model by using an individual random variable for each producer.

To be able to provide a numerical study, we had to estimate the probability distribution of the future electricity demand from the point of view of the ISO as well as producers. We were unable to provide realistic estimates, since both the data and methods used are publicly unavailable. Thus we used publicly available data limited to point estimates and real observations of the demand available for several weeks, and fit the parameters of the lognormal distribution. We are aware that this is a simplifying approach and in practice we would need a more sophisticated method for the demand forecasting such as time series analysis, cf. [10, 11], which can take into account seasonality effects and external factors affecting the demand, such as weather.

General risky design equilibrium problems with stochastic elements were investigated by [18, 25]. Recently, [32] proposed a new stochastic-programming market-clearing mechanism to optimize predispatch quantities given the probability distribution of the random demand and the costs of real-time deviations. The previously proposed stochastic real-time clearing formulation in which generation capacity, demand and transmission line capacity are considered as random, has been extended in [33] by employing the social surplus function which induces penalties between day-ahead and real-time quantities.

To deal with uncertain (random) demand, we employ the chance constrained formulations of the problems of the ISO and each producer. Chance constrained problems (CCP), a standard tool of stochastic optimization, cf. [22, 28], are usually used to get optimal solutions which are highly reliable with respect to stochastic parts of the optimization problems under uncertainty. Recent progress in this area includes sequential algorithm based on an exact penalty [13], optimality conditions and regularization [1, 2], or new quantile cuts for MINLP reformulations [31]. First and second order differentiability results under elliptically symmetric distributions, which can be directly employed in standard NLP solvers, have been derived in [34]. A chance-constrained economic dispatch model was presented by [12]. The model integrates energy storage and high renewable penetration to satisfy renewable portfolio requirements. In our case, the stochastic problem of each producer is related to the Value at Risk problem which was elaborated by several previous works, see, e.g., [15, 26, 27, 30]. However, due to the specific structure of the producer problems, we are able to derive and solve a nonlinear programming equivalent using the demand distribution function with decision dependent arguments which is, according to our best knowledge, the first attempt in the area of CCP.

From the modelling point of view, the above described problem leads to a multi-leader-common-follower problem, where the producers are considered as the leaders and the ISO is viewed as a common follower. Showing existence of solution of such a bi-level problem is typically difficult. Even if convexity is assumed at both levels, is no more satisfied once the upper-level pay-off functions are composed with the solution map of the lower-level problem (the ISO). For a specific setting all equilibria may be found analytically, see e.g. [7, 8], assuming, however, that the demand is deterministic. Alternatively, one may model electricity markets as supply function equilibrium (SFE), a concept introduced in [23] that naturally generalizes market models of both Cournot and Bertrand. Modelling the competition of producers by Nash equilibrium, the profit maximizing support functions (i.e. inverses of bid functions) are smooth functions described by differential equations. This general model of market competition has been well adapted to the particular situation of electricity markets, see, e.g. [5, 20] and the references therein. Note, however, that such an approach is orthogonal to the value-at-risk approach used here. Indeed, supply functions determined by maximisation of (expected) profit are in no relation to optimal bid functions of producers using value-at-risk. As far as we know, an analytical solution to such a market is yet to be determined; our work may be considered as the first step in this direction.

To take into account the uncertainty in the electricity demand better, we simplified the model of a pay-as-clear day-ahead market in several aspects. Next, we will discuss our main assumptions:

1 To focus more on competition amongst producers, we don't model individual consumers. They may be probably included in a more detailed model by following, e.g., [21].

2 We consider one specific time period day-ahead (a quarter of an hour), which is independent of other time periods. Thus, the model would have to be considerably adapted to incorporate other market participants, e.g., market speculators selling and purchasing contracts for different periods in the day.

3 Transmission constraints are not taken into account; in other words model is formulated at a single node of transmission network. Such constraints substantially complicate the analysis of the problem, see, e.g. the discussion in [20] where radial transmission network with local demand shocks is analysed, and existence and uniqueness of supply-function equilibrium in two-node networks is shown.

4 The aggregated electricity demand is considered to be in-elastic. The model may be generalized in this respect by following the direction of [3], thus modelling the linear elasticity of the demand.

5 We assume that the producers and the ISO are able to estimate the probability distribution of the demand in each step of the modelling.

6 We model the production costs using convex quadratic functions which is quite common as a reasonable simplification in the analysis of equilibria in electricity markets, see, e.g., [7, 17, 19, 21]. Such approximation captures well, at least qualitatively, the increasing marginal costs of electricity production.

7 We limit producers to bid functions that are convex and quadratic, following again, e.g. [7, 17, 19, 21], thus obtaining approximation that is convenient for further mathematical analysis. In real markets, however, producers typically bid piecewise-linear functions.

Some of the above limitations may possibly be overcome in the future, whereas other seem to be inevitable to facilitate the analysis below (in particular, quadratic cost and bid functions, see the points 6 and 7 above, lead to the statement of Theorem 4.7).

The paper is organized as follows. In Section 2, the basic notation, assumptions and market settings are introduced. Section 3 is focused on the optimal dispatch problem of the ISO. Section 4 deals with the chance constrained profit maximization problem of a producer. Numerical study using the real data from the French electricity market is proposed in Section 5. Section 6 concludes the paper.

2 Notation and problem setting

First we summarize the basic hypothesis that are considered in this work: we consider a pay-as-clear electricity market with $N > 1$ producers; we only consider producers, that is the demand of consumers is aggregated. Finally, the transmission network is not taken into account, thus also thermal losses and "local demand" are omitted.

By $\delta > 0$ we denote the (aggregated) electricity demand, $\mathcal{N} = \{1,\ldots,N\}$ is the set of producers, and $q_i \geq 0$ represents the non-negative production quantity of the i-th producer. Considering $q \in \mathbb{R}^N_+$ we use $q_{-i} \in \mathbb{R}^{N-1}_+$ to denote the vector $(q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_N)$, and the same convention is used also for other vectors hereinafter. For $i \in \mathcal{N}$ we use $a_i, b_i \geq 0$ to denote the coefficients of the i-th producer's bid $a_iq_i + b_iq_i^2$ and $A_i \geq 0, B_i > 0$ to denote the coefficients of the true production cost function $A_iq_i + B_iq_i^2$. We use $\mathbb{R}_{++} = \mathbb{R}_+ \setminus \{0\}$.

For a one-dimensional random variable X on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, we denote its distribution by μ which is defined as

$$\mu_X(A) := \mathbb{P}(\omega \in \Omega | X(\omega) \in A)$$

for all Borel measurable subsets $A \subseteq \mathbb{R}$. This distribution induces the distribution function

$$F_X(x) := \mu_X((-\infty,x)),\ \forall\, x \in \mathbb{R},$$

the inverse of which is the quantile function F_X^{-1} defined by

$$F_X^{-1}(t) := \inf\{x | F_X(x) \geq t\}.$$

We say that a measurable real function f_X is a density of X, if

$$\mu_X(A) = \int_{z\in A} f_X(z)dz$$

for all Borel measurable subsets $A \subseteq \mathbb{R}$ or, equivalently, if

$$F_X(x) = \int_{-\infty}^{x} f_X(z)dz,\ \forall\, x \in \mathbb{R}.$$

3 Problem of the ISO

Each producer provides the ISO with a quadratic bid $a_iq_i + b_iq_i^2$. The ISO thus have knowledge of the bid vectors $a = (a_1,\cdots,a_N) \in \mathbb{R}^N_+$ and $b = (b_1,\cdots,b_N) \in \mathbb{R}^N_{++}$, however, the ISO is not aware of the true production cost parameters $A_i, B_i, i = 1,\ldots,N$. Thus, knowing only the bid vectors, the ISO computes the production quantity to be dispatched to the producers $q = (q_1,\ldots,q_N) \in \mathbb{R}^N_+$ to maximize the so-called social welfare, see e.g. [9, 21]. Assuming, moreover, that also the aggregated demand $\delta > 0$ is given, the problem ISO(a,b,δ) reads

$$\text{ISO}(a,b,\delta) \qquad \min_{q} \sum_{i\in\mathcal{N}} a_iq_i + b_iq_i^2$$
$$\text{s.t.} \begin{cases} 0 \leq q_i,\ \forall i \in \mathcal{N}, \\ \sum_{i\in\mathcal{N}} q_i = \delta, \end{cases}$$

Note that the market clearing price $\lambda(a,b,\delta)$ corresponds to the Lagrange multiplier of the demand satisfaction constraint in ISO(a,b,δ). The following result is fundamental for this work.

Theorem 3.1. *Let $\delta > 0$. Then for any $(a,b) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++}$, the market clearing price $\lambda(a,b,\delta)$ and optimal production quantities $q(a,b,\delta)$ in problem ISO(a,b,δ) are the solutions of the system of equations*

$$\sum_{k=1}^{N} \left(\frac{\lambda - a_k}{2b_k}\right)^+ = \delta, \tag{1}$$

and

$$q_i = \left(\frac{\lambda - a_i}{2b_i}\right)^+, i \in \mathcal{N}, \tag{2}$$

in variables (λ, q).

Proof. See a more general statement of [8, Theorem 2.1]. □

The fact that $\lambda(a,b,\delta)$ is well defined by (1) may be seen from the following remark.

Remark 3.2. *Consider the setting of Theorem 3.1. If we moreover assume, without loss of generality, that* $a_i \leq a_j$ *for* $i < j$*, we may restate (1) equally as*

$$\lambda(a,b,\delta) = \min_{k=1,..,N} \frac{1}{\sum_{j=1}^{k} \frac{1}{2b_j}} \left[\delta + \sum_{j=1}^{k} \frac{a_j}{2b_j}\right]. \tag{3}$$

For details see [8, Remark 3]. In this article, however, we will not assume any ordering of producers.

Now, we turn our attention to the ISO problem with demand given by a positive random variable D^{ISO} on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Since we deal with one particular part of the day (a quarter of an hour), we are not using any time index. We assume that the main goal of the ISO is to establish equilibrium between supply and demand with great reliability to avoid high costs associated with the supply failure. Such a problem may be formulated as a chance constrained problem where a probability $p \in]0,1[$ is prescribed to satisfy the demand:

$$\text{SD-ISO}(a,b) \qquad \begin{aligned} &\min_{q} \sum_{i\in\mathcal{N}} a_i q_i + b_i q_i^2 \\ &\text{s.t.} \begin{cases} 0 \leq q_i, \ \forall i \in \mathcal{N}, \\ \mathbb{P}\left[\sum_{i\in\mathcal{N}} q_i \geq D^{ISO}\right] \geq p. \end{cases} \end{aligned} \tag{4}$$

As the individual chance constraint above has a structure of the so-called separable (random) right-hand side, cf. [28], one can easily reformulate it using the quantile function of D^{ISO}, thus obtaining a deterministic constraint. Consequentially, the explicit solution of ISO(a,b,δ) stated in Theorem 3.1 remains valid even for SD-ISO(a,b) provided δ is replaced by $F^{-1}_{D^{ISO}}(p)$, i.e. SD-ISO(a,b) = ISO$(a,b,F^{-1}_{D^{ISO}}(p))$.

4 Problem of producer

In this section, we illuminate the point of view of a particular producer $i \in \mathcal{N}$ supposing that its true production cost function is given by $A_i q_i + B_i q_i^2$ with known $A_i \geq 0$ and $B_i > 0$. Producer $i \in \mathcal{N}$ then aims to maximize his profit function $\pi_i(a,b,\delta)$

$$\pi_i(a,b,\delta) = (\lambda(a,b,\delta) - A_i)\, q_i(a,b,\delta) - B_i\, q_i(a,b,\delta)^2 \tag{5}$$

with respect to his decision variables $a_i \geq 0, b_i > 0$, with the remaining bid coefficients (a_{-i}, b_{-i}) fixed. Furthermore, we assume that the electricity demand δ is not known when the producer's bid is submitted to the ISO. Instead, we consider stochastic demand given by a positive random variable D on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a probability density function $f(\delta)$ specified below. We stress that the producers and the ISO can represent demand with random variables having different distributions since the ISO can use more recent information to predict the demand for the considered time period in the next day.

Now, producer i can solve the following chance-constrained problem, where the profit m_i that can be reached with a probability $p_i \in]0,1[$ with respect to the random demand D is maximized:

$$P_i(a_{-i}, b_{-i}, p_i) \qquad \max_{a_i, b_i, m_i} \; m_i \quad \text{s.t.} \begin{cases} \mathbb{P}\left[\pi_i(a_i, a_{-i}, b_i, b_{-i}, D) \geq m_i\right] \geq p_i, \\ a_i, b_i, m_i \geq 0. \end{cases} \tag{6}$$

Note that this formulation is related to the value at risk (VaR), where analogous chance constraint is imposed on random losses resulting from investments on financial markets, cf. [15, 30]. Alternatively, one may consider losses above the quantile leading to the measure known as conditional value at risk (CVaR), see [26, 27].

Next we show that problem $P_i(a_{-i}, b_{-i}, p_i)$ is well-posed.

Theorem 4.1. *For any $i \in \mathcal{N}$, $a_{-i} \in \mathbb{R}^{N-1}_{+}$, $b_{-i} \in \mathbb{R}^{N-1}_{++}$ and $p_i \in]0,1[$ there exists a solution to $P_i(a_{-i}, b_{-i}, p_i)$.*

Before proving the above theorem we show several auxiliary lemmas. First, we continuously extend the profit function $\pi_i(a,b,\delta)$ to $b \in \mathbb{R}^N_+$ such that $b_i = 0$ and $b_{-i} \in \mathbb{R}^{N-1}_{++}$.

Lemma 4.2. *Let $\delta > 0$, $a_{-i} \in \mathbb{R}^{N-1}_{+}$, $b_{-i} \in \mathbb{R}^{N-1}_{++}$ and $a_i \in \mathbb{R}_+$. Denote $\widetilde{\mathcal{N}} = \{k \in \mathcal{N} : a_k < a_i\}$, and $(\tilde{a}, \tilde{b})$ bid coefficients of producers in $\widetilde{\mathcal{N}}$. Moreover, let*

$$\tilde{q}_i = \delta - \sum_{j \in \widetilde{\mathcal{N}}} \frac{a_i - a_j}{2b_j}$$

and $\tilde{\lambda} = \lambda(\tilde{a}, \tilde{b}, \delta)$. Then it holds

$$\lim_{b_i \to 0+} \pi_i(a_i, a_{-i}, b_i, b_{-i}, \delta) = \begin{cases} 0 & \text{if} \quad a_i \geq \tilde{\lambda}, \\ (a_i - A_i)\tilde{q}_i - B_i \tilde{q}_i^{\,2} & \text{if} \quad a_i < \tilde{\lambda}. \end{cases} \tag{7}$$

Proof. For the sake of this proof we order producers as in Remark 3.2. We show that

$$\lim_{b_i \to 0+} \lambda(a_i, a_{-i}, b_i, b_{-i}, \delta) = \min\{a_i, \tilde{\lambda}\}. \tag{8}$$

This follows directly from (3) since for any $k \geq i$ it holds

$$\lim_{b_i \to 0+} \frac{\delta + \sum_{j=1}^{k} \frac{a_j}{2b_j}}{\sum_{j=1}^{k} \frac{1}{2b_j}} = a_i.$$

Further we prove that $\lim_{b_i \to 0+} q_i(a_i, a_{-i}, b_i, b_{-i}, \delta) = \max\{\tilde{q}_i, 0\}$. Indeed, for any $b_i > 0$ we have $q_i(a_i, a_{-i}, b_i, b_{-i}, \delta) = \delta - \sum_{j \neq i} q_j(a_i, a_{-i}, b_i, b_{-i}, \delta)$ with production quantities q_j given by (2), then it suffices to calculate the limit using (8). The assertion of the lemma then follows directly from (5) using both established limits and observing that $\tilde{q}_i \leq 0$ if and only if $a_i \geq \tilde{\lambda}$. $\square$

Remark 4.3. *Note that for $a_i = \tilde{\lambda}$ we have $\tilde{q}_i = 0$ due to (1), thus the function given by (7) is continuous in a_i. Extending the definition of $\pi_i(a,b,\delta)$ by the above calculated limits we obtain a function that is continuous in (a_i,b_i) on a closed set $\mathbb{R}^2_+$. However, one has to remember the special role of $b_i = 0$. Should the optimal bid function of producer i be such that $b_i = 0$, it has to be interpreted as "limiting" bid function, cf. remarks following equation (15) in [8], since such b_i is not feasible in the problem of the ISO(a,b,δ).*

Let us henceforth denote by $\mu_i(a_{-i},b_{-i},p_i)$ the supremum of the objective function in $P_i(a_{-i},b_{-i},p_i)$. One may immediately establish a lower bound on $\mu_i(a_{-i},b_{-i},p_i)$.

Lemma 4.4. *For any $i \in \mathcal{N}$, $(a_{-i},b_{-i}) \in \mathbb{R}^{2N-2}_+$, and $p_i \in]0,1[$ there exists $a_i,b_i \geq 0$ such that $\mathbb{P}[\pi_i(a,b,D) \geq 0] \geq p_i$, thus $\mu_i(a_{-i},b_{-i},p_i) \geq 0$.*

Proof. For any $b_i > 0$ we observe that $\lambda(a,b,\delta) \leq a_i$ implies $\pi_i(a,b,\delta) = 0$ using (2). It thus suffices to find a_i high enough such that $\mathbb{P}[\lambda(a,b,D) \leq a_i] = p_i$. To this end observe that for a_i high enough, $\lambda(a,b,\delta)$ does not depend on a_i, see equation (3). □

Thus we may further assume $\mu_i(a_{-i},b_{-i},p_i) > 0$ and focus only on feasible points of $P_i(a_{-i},b_{-i},p_i)$ such that $\pi_i(a,b,D) > 0$ for almost all realizations of the random demand. To this end we define the probability function as follows

$$\rho_i(a,b,m_i) = \begin{cases} \mathbb{P}[\pi_i(a,b,D) \geq m_i] & \text{if } m_i > 0, \\ \lim\limits_{\widetilde{m}_i \to 0+} \mathbb{P}[\pi_i(a,b,D) \geq \widetilde{m}_i] & \text{if } m_i = 0, \end{cases} \tag{9}$$

Note that for $m_i = 0$ condition $\pi_i(a,b,\delta) \geq m_i$ allows for $q_i(a,b,\delta) = 0$, and so bid vectors a,b such that $\lambda(a,b,\delta) \leq a_i$ should also be taken into account. As we are interested only in positive profits, cf. Lemma 4.4, we may treat the case of $m_i = 0$ separately thus simplifying the statement of Theorem 4.7. Using the probability function, we define a set-valued mapping which corresponds to the set of feasible solutions as

$$X_i(a_{-i},b_{-i},p_i) = \left\{(a_i,b_i,m_i) \in \mathbb{R}^3_+ : \rho_i(a,b,m_i) \geq p_i\right\}. \tag{10}$$

Further we show that the feasible set of problem $P_i(a_{-i},b_{-i},p_i)$ may be restricted to $X_i(a_{-i},b_{-i},p_i)$.

Lemma 4.5. *Let $i \in \mathcal{N}$, $a_{-i} \in \mathbb{R}^{N-1}_+$, $b_{-i} \in \mathbb{R}^{N-1}_{++}$ and $p_i \in]0,1[$, and let $Y \subset \mathbb{R}^3_+$ be the set of feasible points of $P_i(a_{-i},b_{-i},p_i)$. Then, $X_i(a_{-i},b_{-i},p_i)$ is a closed subset of Y, and for any $(a_i,b_i,m_i) \in Y$ such that $m_i > 0$ it holds that $(a_i,b_i,m_i) \in X_i(a_{-i},b_{-i},p_i)$.*

Proof. Set $X_i(a_{-i},b_{-i},p_i)$ is closed due to continuity of $\rho_i(a,b,m_i)$. Further, consider $(a_i,b_i,m_i) \in \mathbb{R}^3_+$. If $m_i > 0$, directly from the definition we have

$$(a_i,b_i,m_i) \in X_i(a_{-i},b_{-i},p_i)$$

if and only if $(a_i,b_i,m_i) \in Y$. To complete the proof for the case of $m_i = 0$, it suffices to observe $\rho_i(a,b,0) \leq \mathbb{P}[\pi_i(a,b,D) \geq 0]$ due to continuity of the profit function π_i. □

The last auxiliary lemma follows.

Lemma 4.6. *Let $i \in \mathcal{N}$, $a_{-i} \in \mathbb{R}_+^{N-1}$, $b_{-i} \in \mathbb{R}_{++}^{N-1}$ and $p_i \in]0,1[$. Denote*

$$S(m_i) = \{(a_i,b_i) \in \mathbb{R}_+^2 : (a_i,b_i,m_i) \in X_i(a_{-i},b_{-i},p_i)\}.$$

Then, for any $m_i > 0$ set $S(m_i)$ is compact. Further, $S(y) \supset S(x)$ for $x > y$.

Proof. From the definition of $X_i(a_{-i},b_{-i},p_i)$ we have

$$S(m_i) = \{(a_i,b_i) \in \mathbb{R}_+^2 : \rho_i(a,b,m_i) \geq p_i\}.$$

Thus the monotonicity of $S(m_i)$ is directly due to monotonicity of $\rho_i(a,b,m_i)$ in m_i, see (9).

Next, consider a realization $\delta > 0$ of random demand D and observe that $\pi_i(a,b,\delta) \geq m_i$ implies $q_i(a,b,\delta) > 0$ due to (2), and so $\lambda(a,b,\delta) > a_i$. Using moreover (5), we have also

$$\frac{\lambda(a,b,\delta)^2}{2b_i} \geq \lambda(a,b,\delta)\, q_i(a,b,\delta) \geq \pi_i(a,b,\delta).$$

Thus, denoting $\xi(a_i,b_i) = \max\left\{a_i, \sqrt{2m_ib_i}\right\}$, we have

$$S(m_i) \subset \left\{(a_i,b_i) \in \mathbb{R}_+^2 : \mathbb{P}\left[\lambda(a,b,D) \geq \xi(a_i,b_i)\right] \geq p_i\right\}.$$

Then, considering the cumulative distribution function F of D, we may write

$$S(m_i) \subset \left\{(a_i,b_i) \in \mathbb{R}_+^2 : \lambda\left(a,b,F_D^{-1}(1-p_i)\right) \geq \xi(a_i,b_i)\right\},$$

where F_D^{-1} denotes the quantile function of D. Now, we observe that for any $\delta > 0$ it holds, due to (3), $\lambda(a_{-i},b_{-i},\delta) \geq \lambda(a,b,\delta)$. Then, defining

$$\lambda_M = \lambda\left(a_{-i},b_{-i},F_D^{-1}(1-p_i)\right),$$

we conclude $S(m_i) \subset [0,\lambda_M] \times [0,\lambda_M^2/(2m_i)]$ with regards to definition of $\xi(a_i,b_i)$. □

Proof. of Theorem 4.1. If $\mu_i(a_{-i},b_{-i},p_i) = 0$ then a maximizer of $P_i(a_{-i},b_{-i},p_i)$ exists due to Lemma 4.4. Thus we may further assume $\mu_i(a_{-i},b_{-i},p_i) > 0$ and, consequently, restrict our attention to subset $X_i(a_{-i},b_{-i},p_i)$ of the feasible points of $P_i(a_{-i},b_{-i},p_i)$ due to Lemma 4.5. Further, since super-level sets of criterion function m_i are compact, see Lemma 4.6, and the criterion function is continuous, see Lemma 4.2 and Remark 4.3, the maximum of $P_i(a_{-i},b_{-i},p_i)$ is attained. □

4.1 Problem reformulation

In this subsection, we reformulate problem $P_i(a_{-i},b_{-i},p_i)$ to facilitate the numerical experiment in Section 5. First, we analyse what values $\delta > 0$ of the demand yield $\pi_i(a,b,\delta) \geq m_i$. To this end we define functions $\lambda_1(a_i,b_i,m_i), \lambda_2(a_i,b_i,m_i)$ for all possible values of $(a_i,b_i,m_i) \in \mathbb{R}_+^3$ in the following way:

1 if $2b_i > B_i$ then $\lambda_2(a_i,b_i,m_i) = +\infty$,

$$\lambda_1(a_i,b_i,m_i) = a_i + \frac{b_i}{B_i - 2b_i}\left(a_i - A_i - \sqrt{(a_i - A_i)^2 - 4m_i(B_i - 2b_i)}\right),$$

2 if $2b_i = B_i$ and $a_i > A_i$ then $\lambda_1(a_i,b_i,m_i) = a_i + \frac{B_i m_i}{a_i - A_i}$, $\lambda_2(a_i,b_i,m_i) = +\infty$,

3 if $2b_i < B_i$ and $a_i \geq A_i + 2\sqrt{m_i(B_i - 2b_i)}$ then $\lambda_1(a_i,b_i,m_i)$ is as in the case (a) and

$$\lambda_2(a_i,b_i,m_i) = a_i + \frac{b_i}{B_i - 2b_i}\left(a_i - A_i + \sqrt{(a_i - A_i)^2 - 4m_i(B_i - 2b_i)}\right),$$

4 otherwise $\lambda_1(a_i,b_i,m_i) = +\infty$ and $\lambda_2(a_i,b_i,m_i) = -\infty$.

Then we may reformulate the probability function ρ_i.

Theorem 4.7. *Let $i \in \mathcal{N}$, $a_{-i} \in \mathbb{R}_+^{N-1}$, $b_{-i} \in \mathbb{R}_{++}^{N-1}$, and $(a_i,b_i,m_i) \in \mathbb{R}_+^3$ it holds*

$$\rho_i(a,b,m_i) = \mathbb{P}\left[\lambda(a,b,D) \in [\lambda_1(a_i,b_i,m_i), \lambda_2(a_i,b_i,m_i)]\right]. \tag{11}$$

Proof. For any (a_i,b_i,m_i) such that $b_i > 0$, realization $\delta > 0$ of demand D and

$$\pi_i(a,b,\delta) \geq m_i > 0 \tag{12}$$

we necessarily have $q_i(a,b,\delta) > 0$ and so using (2) also

$$q_i(a,b,\delta) = \frac{\lambda(a,b,\delta) - a_i}{2b_i} \tag{13}$$

and $\lambda(a,b,\delta) > a_i$. Then, denoting $\alpha = a_i - A_i$, $\beta = 2b_i - B_i$, and

$$\Lambda(a_i,b_i,m_i) = \left\{l \in \mathbb{R} : l > a_i, \beta l^2 - 2l(a_i\beta - \alpha b_i) + a_i(A_i\beta - \alpha B_i) - 4b_i^2 m_i \geq 0\right\}, \tag{14}$$

and substituting (13) into (5), we may reformulate inequality (12) equivalently as $\lambda(a,b,\delta) \in \Lambda(a_i,b_i,m_i)$. Evaluating functions $\inf\{\Lambda(a_i,b_i,m_i)\}$ and $\sup\{\Lambda(a_i,b_i,m_i)\}$, with the discriminant of the left-hand side of the inequality in (14) being

$$4b_i^2(\alpha^2 + 4m_i\beta),$$

one may establish that

$$\lambda_1(a_i,b_i,m_i) = \inf\{\Lambda(a_i,b_i,m_i)\}$$

and

$$\lambda_2(a_i,b_i,m_i) = \sup\{\Lambda(a_i,b_i,m_i)\},$$

thus showing (11). Further, observe that all arguments above are valid also for $b_i = 0$ due to continuity of profit π_i with respect to $b_i \to 0+$, see Lemma 4.2. Finally, one can continuously extend the previous results to the case of $m_i = 0$. Indeed, the above consideration are valid provided $m_i > 0$, then we may restrict the feasible set of problem $P(a_{-i},b_{-i},p_i)$ to $X_i(a_{-i},b_{-i},p_i)$ as discussed in Lemma 4.5, and eventually use the fact that $\rho_i(a,b,m_i)$ is defined by limit for $m_i = 0$, see equation (9). □

Inspired by equation (1) we define

$$\delta_{1,2}(a,b,m_i) = \sum_{k=1}^{N}\left(\frac{\lambda_{1,2}(a_i,b_i,m_i) - a_k}{2b_k}\right)^+, \tag{15}$$

and conclude this section with a corollary playing a key role in the numerical experiment.

Corollary 4.8. *Let $i \in \mathcal{N}$, $a_{-i} \in \mathbb{R}_+^{N-1}$, $b_{-i} \in \mathbb{R}_{++}^{N-1}$, and $(a_i, b_i, m_i) \in \mathbb{R}_+^3$ and denote by F_D the distribution function of the demand D . Then either the optimal profit in $P_i(a_{-i}, b_{-i}, p_i)$ is zero, or $P_i(a_{-i}, b_{-i}, p_i)$ may be equivalently formulated as*

$$\begin{aligned} \max_{a_i, b_i, m_i} \quad & m_i \\ \text{s.t.} \quad & \begin{cases} F_D(\delta_2(a, b, m_i)) - F_D(\delta_1(a, b, m_i)) \geq p_i, \\ a_i, b_i, m_i \geq 0. \end{cases} \end{aligned} \tag{16}$$

Proof. Using Theorem 4.7, it suffices to observe that the probability function can be expressed as

$$\rho_i(a, b, m_i) = \int_{\delta_1(a,b,m_i)}^{\delta_2(a,b,m_i)} f(\delta)\, d\delta.$$

□

5 Numerical study

In the numerical study, we apply the above derived results to real data from the French electricity market. We derive the optimal bidding curves for five artificial producers and report the corresponding optimal dispatch quantities which are provided by the ISO. We employ the real data to estimate the distribution of the random demand. Note that these estimates are different for the producers and the ISO as discussed below. Our goal is also to illustrate that the reformulations obtained in the previous section lead to problems which can be solved by standard software tool such as Matlab.

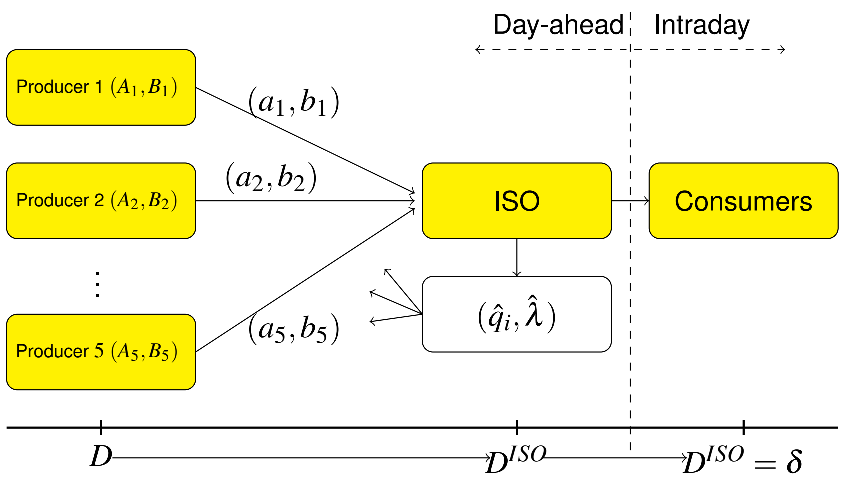

Figure 1: Day-ahead electricity market schema

The sequence of steps from the perspective of producer $i \in \mathcal{N}$: estimate the future demand distribution D; use it to solve producer's optimization problem, thus obtaining bidding parameters a_i, b_i to be submitted to the ISO; after the clearing process of the ISO using D^{ISO}, the producer obtains the dispatch order and the payment. The sequence of steps from the perspective of the ISO: generate a forecast of the demand distribution D^{ISO}; obtain the bids a, b from the producers; use these data

Table 1: Input data [2]: point estimates of the demand at various stages ($\hat{D}_t$ used by producers, $\hat{D}_t^{ISO}$ used by ISO), observed consumptions D_t^{ISO} and clearing prices

Day (2017) from 10:00 am to 10:15 am	Demand estimate (GW) $\hat{D}_t$	Demand estimate (GW) $\hat{D}_t^{ISO}$	Observed consumption (GW) D_t^{ISO}	Clearing price (EUR/MWh)
3-Jan	87.8	84.7	84.323	74.65
4-Jan	86.5	86.5	86.270	72.85
5-Jan	86.8	84.6	84.406	73.03
10-Jan	81.5	80.8	80.521	84.59
11-Jan	80.6	79.0	79.443	82.57
12-Jan	78.1	76.6	77.020	86.71
17-Jan	88.1	87.6	88.211	129.33
18-Jan	91.7	92.2	92.751	111.75
19-Jan	93.8	92.8	93.120	94.00
24-Jan	90.2	90.5	90.544	151.07
25-Jan	91.5	90.9	90.821	151.29
26-Jan	90.9	92.6	93.100	126.09
31-Jan	74.5	73.1	72.964	89.80
1-Feb	72.1	71.6	71.780	79.11
2-Feb	70.3	70.3	70.746	58.55
7-Feb	74.6	75.0	75.081	NA
8-Feb	75.9	76.2	76.322	73.02
9-Feb	79.1	79.7	79.522	70.97
14-Feb	73.2	73.5	73.618	64.67
15-Feb	71.4	69.9	70.256	58.80
16-Feb	71.0	70.0	74.800	61.90
21-Feb	68.6	67.9	68.023	54.59
22-Feb	67.3	67.0	67.627	53.00
23-Feb	67.1	67.8	68.586	43.36
28-Feb	69.8	72.2	72.433	49.61

to clear the market day-ahead; announce the dispatch and pay the producers according to the clearing price. In the case that the demand realization $D^{ISO} = \delta$ doesn't match the planned supply, the difference is then compensated in the intraday market. The situation is outlined in Figure 1.

We will now introduce a naive approach to estimating parameters of D and D^{ISO}. Table 1 contains the point estimates and the real data observed between January 3 and February 28, 2017. These days correspond to Tuesdays, Wednesdays, and Thursdays; we wanted to avoid Mondays and Fridays when the demand can differ considerably. The table contains the point estimates $\hat{D}_t$ of the demand used by producers, the point estimates $\hat{D}_t^{ISO}$ used by the ISO, the real consumptions on the day D_t^{ISO}, and the clearing prices. We assume that the point estimates are i.i.d. realizations of the demand forecasts for the next day. We are aware that this is a highly simplifying assumption and in practice we would need more sophisticated approach as we have discussed in Introduction. Based on the observations, we have estimated the parameters of the lognormal distribution, cf. Table 2. Note that our approach is not

[2]Data has been taken from `http://www.rte-france.com/fr/eco2mix/eco2mix-consommation`.

Table 2: Parameters estimates of the lognormal distribution

Parameters	$\hat{\mu}$	$\hat{\sigma}^2$	exp. value	MSPE
Producer(s)	4.3623	0.0123	78.92	77.01
ISO	4.3672	0.0119	79.29	75.01

limited to this particular probability distribution but we can use any distribution with positive support, e.g. Gamma or inverse-Gaussian.

As already discussed in the Introduction, we further construct the producers' estimate of D based on point estimates $\hat{D}_t$ that are published by the ISO. Such naive method is necessary to illustrate our approach since one may not use private data and/or models of producers. For a producer, the parameters of the lognormal distribution of D are estimated using the pairs $\hat{D}_t$, $\hat{D}_t^{ISO}$. The expected value corresponds to the sample mean of the producer demand estimate $\hat{D}$, i.e.

$$\hat{\mathbb{E}}[\hat{D}] = \frac{1}{T}\sum_{t=1}^{T}\hat{D}_t,$$

whereas the variance represents the Mean Square Prediction Error (MSPE) which is estimated as a sum of the sample variance of $\hat{D}$

$$\widehat{\mathrm{Var}}(\hat{D}) = \frac{1}{T-1}\sum_{t=1}^{T}\left(\hat{D}_t - \hat{\mathbb{E}}[\hat{D}]\right)^2,$$

and the mean square error

$$\mathrm{MSE}(\hat{D}^{ISO},\hat{D}) = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{D}_t^{ISO} - \hat{D}_t\right)^2,$$

i.e.

$$\mathrm{MSPE}(\hat{D}^{ISO},\hat{D}) = \widehat{\mathrm{Var}}(\hat{D}) + \mathrm{MSE}(\hat{D}^{ISO},\hat{D}).$$

The values of the parameters $\hat{\mu},\hat{\sigma}^2$ for a producer are then obtained by the following arithmetic operations valid for the lognormal distribution using the desirable expected value and variance

$$\hat{\mu} = \ln\left(\frac{(\hat{\mathbb{E}}[\hat{D}])^2}{\sqrt{\mathrm{MSPE}(\hat{D}^{ISO},\hat{D}) + (\hat{\mathbb{E}}[\hat{D}])^2}}\right), \tag{17}$$

$$\hat{\sigma}^2 = \ln\left(1 + \frac{\mathrm{MSPE}(\hat{D}^{ISO},\hat{D})}{(\hat{\mathbb{E}}[\hat{D}])^2}\right). \tag{18}$$

Analogous approach is used to estimate the parameters of D^{ISO} with the pairs of observations $\hat{D}_t^{ISO}$, D_t^{ISO} in the place of $\hat{D}_t$, $\hat{D}_t^{ISO}$, see Table 2.

Considering five producers, we will solve problem (6) for each producer given the initial values of the bidding coefficients a_i,b_i and the production cost coefficients A_i,B_i, see Table 3. Note that producer 1 is considered as a largest one with the smallest linear cost coefficient and the highest quadratic one, whereas producer 5 is the smallest with corresponding cost curve. We will employ the reformulated form of the problem of producer, see (16), assuming $p_i = 0.9$ for all i. The problems are solved by *fmincon* procedure by the sequential quadratic programming algorithm available in Matlab. As the starting point for the algorithm, the bidding coefficients a_i,b_i are selected. We consider the following approaches:

1 compute $\hat{a}_i, \hat{b}_i$ given (a_{-i}, b_{-i}) for all $i = 1, .., N$, i.e. all producers perform optimization independently,

2 compute only $\hat{a}_3, \hat{b}_3$ given (a_{-3}, b_{-3}), i.e. only one producer optimizes its profit,

3 compute $\hat{a}_i, \hat{b}_i$ given $\hat{a}_1, ...\hat{a}_{i-1}, a_{i+1}, ..., a_N$, $\hat{b}_1, ...\hat{b}_{i-1}, b_{i+1}, ..., b_N$ for all $i = 1, .., N$, i.e. producer i is given the optimal bids of producers $1, ..., i-1$.

Table 3: Starting values of the coefficients (a_i, b_i) and results of the producer $(\hat{a}_i, \hat{b}_i, \hat{m}_i)$ and the ISO problems $(\hat{q}_i, \hat{\lambda})$

Approach	Producer	1	2	3	4	5
	A_i	23.20	34.10	36.00	34.50	51.30
	B_i	0.69	0.62	0.51	0.72	0.35
	a_i	24.20	35.10	37.00	35.50	52.30
	b_i	0.79	0.72	0.61	0.82	0.45
1.	$\hat{a}_i$	24.40	34.92	37.44	35.80	53.45
	$\hat{b}_i$	0.82	0.63	0.63	0.83	0.38
	$\hat{m}_i$	446.28	274.76	242.58	198.07	34.79
	$\hat{q}_i$	21.36	19.47	17.28	14.20	7.71
	$\hat{\lambda}$	59.27				
2.	$\hat{a}_i$	24.20	35.10	37.44	35.50	52.30
	$\hat{b}_i$	0.79	0.72	0.63	0.82	0.45
	$\hat{m}_i$			242.58		
	$\hat{q}_i$	22.45	17.06	17.59	14.74	8.19
	$\hat{\lambda}$	59.67				
3.	$\hat{a}_i$	24.40	35.75	40.99	35.83	54.04
	$\hat{b}_i$	0.82	0.73	0.53	0.82	0.35
	$\hat{m}_i$	446.28	240.74	250.72	208.76	42.01
	$\hat{q}_i$	21.87	16.74	17.99	14.75	8.69
	$\hat{\lambda}$	60.09				

The optimal bids $\hat{a}_i, \hat{b}_i$ are then reported to the ISO, which produces the dispatch orders $\hat{q}_i$ and the clearing price $\hat{\lambda}$ by solving SD-ISO(a, b) problem (4), see Table 3. Although the delivered bids are all different in all cases, the clearing price is stable.

We also investigated the development of optimal values of parameters for producer 3 with respect to the changes of the probabilistic level $p_i \in [0.5, 0.99]$, see Figure 2. Note that a_{-3} and b_{-3} are hold fixed and that the optimal values from the previous iteration are used as the starting points for the update. We can observe that the behaviour is quite stable for b_3, whereas a_3, m_3 rapidly change for high probabilistic level p_3.

Sensitivity analysis of optimal value m_3 was performed also with respect to the its production cost parameters A_3, B_3, see Figure 3. The development of the profit m_3 with respect to the producer's bid coefficients a_2, b_2 can be found in Figure 4. In both cases, the profit is stable, it increases with increasing b_2 and decreases with increasing B_3.

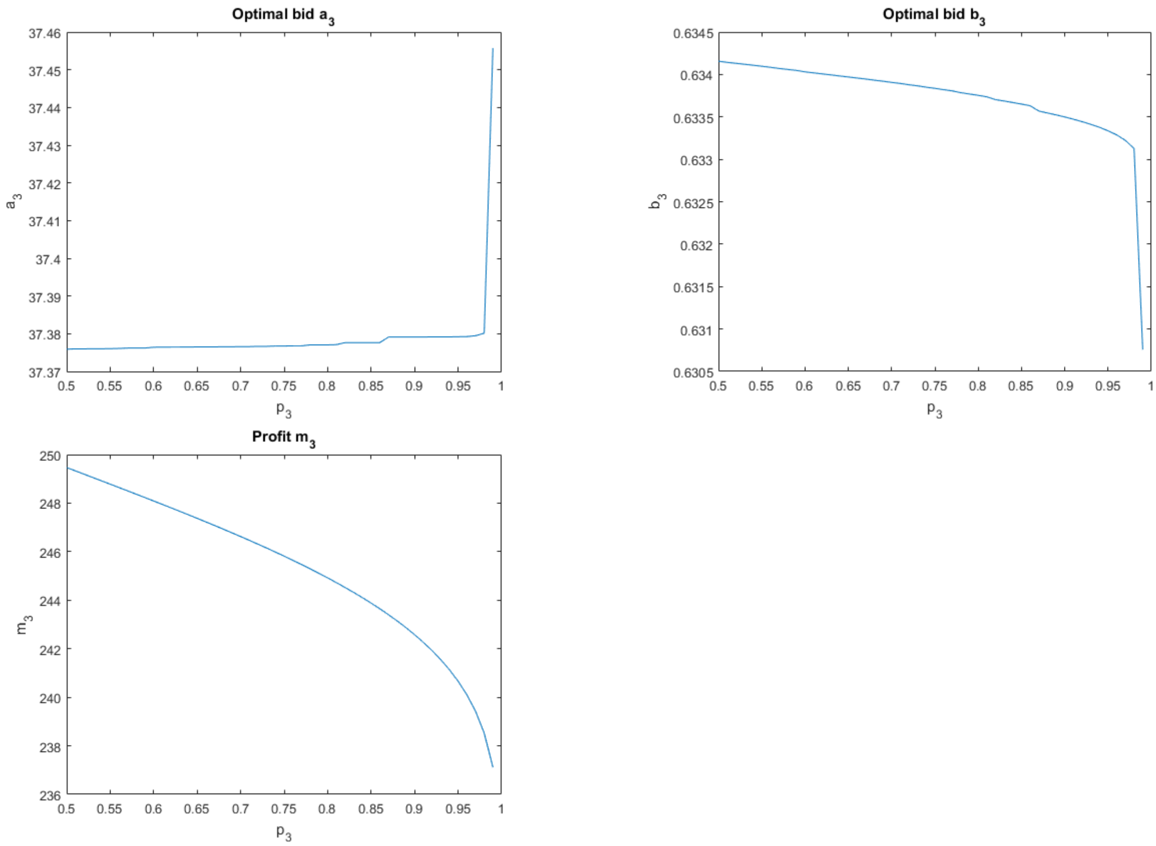

Figure 2: Optimal solutions of problem (6) for producer 3 for different levels $p_i \in [0.5, 0.99]$

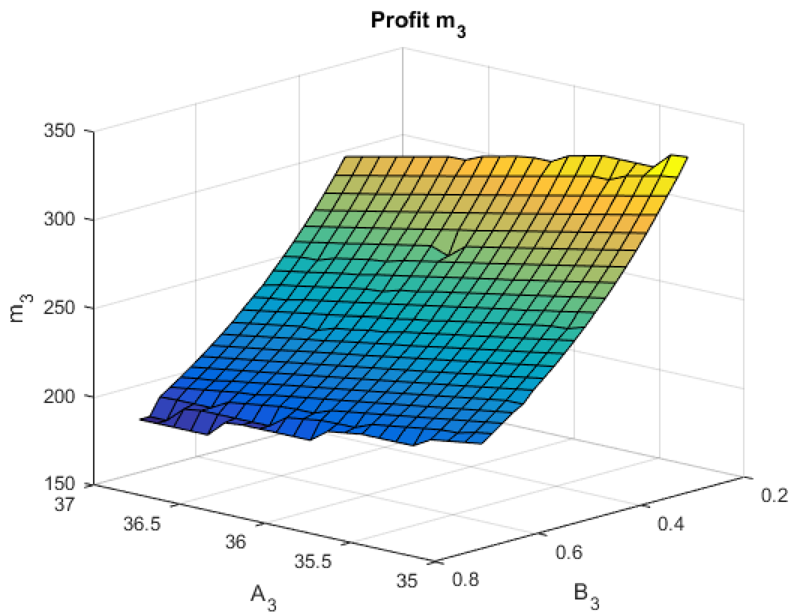

Figure 3: Optimal solutions of problem (6) for producer 3 – sensitivity with respect to the production cost parameters A_3, B_3

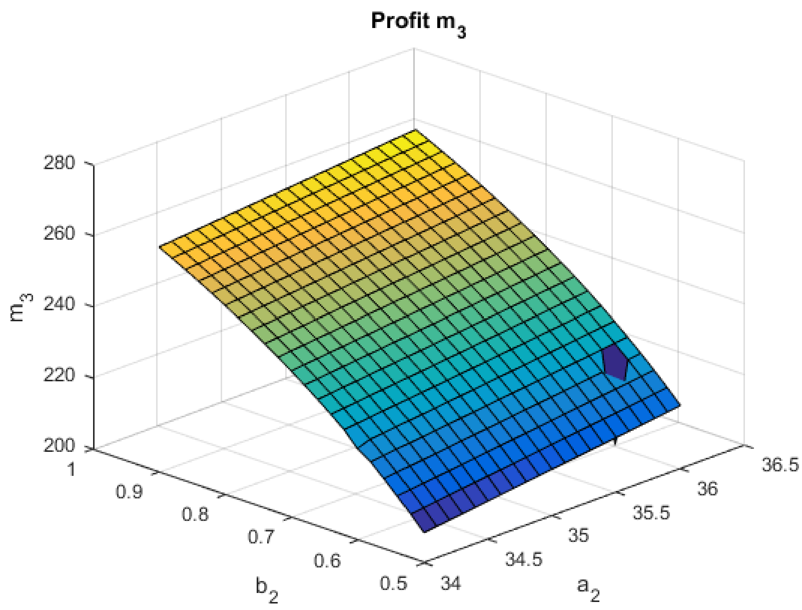

Figure 4: Optimal solutions of problem (6) for producer 3 – sensitivity with respect to the producer's bid coefficients a_2, b_2

6 Conclusions

In this paper, we have investigated two closely connected problems appearing on deregulated electricity markets which are subject to uncertainty. We have focused on the stochastic demand and employed the chance constrained formulations for the problems of the ISO and producers. We have shown that due to the structure of the ISO problem, it is possible to use an earlier result and derive an explicit solution for the production quantities. For each producer, we have formulated a value at risk problem with the maximization of profit which can be reached with certain level of probability. Then, we have derived an explicit reformulation of the probability function which enables to solve the problem using a non-linear programming solver. In the numerical study, we have illustrated our approach using real data from the French market.

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