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Abstract

We consider a fluid model including viscoelastic and viscoplastic effects. The state is given by the fluid velocity and an internal stress tensor that is transported along the flow with the Zaremba–Jaumann derivative. Moreover, the stress tensor obeys a nonlinear and nonsmooth dissipation law as well as stress diffusion. We prove the existence of global-in-time weak solutions satisfying an energy inequality under general Dirichlet conditions for the velocity field and Neumann conditions for the stress tensor.

1 Introduction

In this article we investigate the equations of motion that describe the flow of a viscoelastoplastic fluid with stress diffusion modeled in the following way. On a time interval $(0,T)$ and a bounded domain $\Omega \subset \mathbb{R}^3$, we consider the system of equations

$$\begin{cases} \rho \mathrm{D}_t V - \operatorname{div}\big(\eta_1 S + 2\mu D(V) - PI\big) = F & \text{in } \Omega \times (0,T), \\ \operatorname{div} V = 0 & \text{in } \Omega \times (0,T), \\ \overset{\triangledown}{S} + \partial \mathcal{P}(S) - \gamma \Delta S = \eta_2 D(V) & \text{in } \Omega \times (0,T). \end{cases} \tag{1.1}$$

Here the first two equations describe the flow of an incompressible fluid with Eulerian velocity field $V\colon \Omega \times (0,T) \to \mathbb{R}^3$ and pressure field $P\colon \Omega \times (0,T) \to \mathbb{R}$ affected by a prescribed external force $F\colon \Omega \times (0,T) \to \mathbb{R}^3$. The relevant Cauchy stress tensor $\mathbb{T} = \eta_1 S + 2\mu D(V) - PI$ consists of the classical term $2\mu D(V) - PI$ for Newtonian fluids and an extra stress tensor

$$S\colon \Omega \times (0,T) \to \mathbb{R}^{3\times 3}_\delta := \big\{ M \in \mathbb{R}^{3\times 3} \,\big|\, M = M^\top,\ \operatorname{Tr} M = 0 \big\},$$

which satisfies the additional evolution equation $(1.1)_3$ and is thus subject to a special transport encoded in $\overset{\triangledown}{S}$ along the velocity field V, a nonlinear dissipation law via $\partial\mathcal{P}(S)$, and another diffusion process. Here ρ, η_1, η_2, μ and γ denote positive constants, and $D(V) := \frac{1}{2}(\nabla V + \nabla V^\top)$ denotes the symmetric rate-of-strain tensor.

System (1.1) is complemented by boundary and initial conditions. The former are given by

$$V = g, \qquad \mathrm{n}\cdot\nabla S = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

which means that the fluid velocity at the boundary coincides with some prescribed function $g\colon \partial\Omega \times (0,T) \to \mathbb{R}^3$ and that S has vanishing normal derivative. The initial conditions are

$$V(\cdot,0) = V_0, \qquad S(\cdot,0) = S_0 \qquad \text{in } \Omega. \tag{1.3}$$

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For a fluid velocity V, the material derivative is given by

$$\mathrm{D}_t A := \partial_t A + V \cdot \nabla A,$$

and as an objective derivative of a tensor S we use the Zaremba–Jaumann derivative

$$\overset{\triangledown}{S} := \mathrm{D}_t S + S\,W(V) - W(V)\,S = \partial_t S + V \cdot \nabla S + S\,W(V) - W(V)\,S,$$

also called co-rotational derivative, where $W(V) := \frac{1}{2}(\nabla V - \nabla V^\top)$. Note that this choice of the objective derivative is not canonical and there are different ways to define objective derivatives for tensors. However, the choice made here is commonly used in geodynamics (cf. [MDM02, Ger07]) and comes along with special features that are essential for our the mathematical analysis, see below.

The mathematical study of viscoelastic fluids with different choices of the objective derivatives (including the upper and lower convected Maxwell derivatives) started in the middle 1980s, see e.g. [JRS85, ReR86, RHN87, CoS91, Ren00]. Because of the strong nonlinearities arising from the objective derivatives, a first global existence result was only established years later in [LiM00] based on the Zaremba–Jaumann derivative and a linear dissipation law $\partial\mathcal{P}(S) = aS$ with $a > 0$. More recently, the more difficult case of a Maxwell fluid with $\mu = 0$ (and without stress diffusion, i.e. $\gamma = 0$) has also been considered, see [CL*19] and references therein.

For more general nonlinear situations there is a series of works involving implicitly defined stress-strain relations of the type $G(S, D(V)) = 0$, see [BG*12] and the references in the recent survey [BMR20]. Viscoelastic fluids have a constitutive relation of rate-type, i.e. they involve suitable convective derivatives of the strain tensor $D(V)$ or of the stress tensor, as in our equation $(1.1)_3$. The treatment of such nonlinearities is possible by using the recently introduced regularization of stress diffusion, i.e. $\gamma > 0$, as first illustrated in [BM*18] for a simplified model replacing the tensor evolution by a scalar problem. We refer to [MP*18] for a careful thermodynamical modeling of such viscoelastic fluids and to [BBM21], where a large data global existence result for weak solutions was obtained for a one-parameter family of convected tensor derivatives including the (simpler) case of the Zaremba–Jaumann rate.

Our work is in a similar spirit as the latter one, but it generalizes the conventional linear or quadratic stress-strain relation by allowing in $(1.1)_3$ for a general subdifferential

$$S \mapsto \partial\mathcal{P}(S) = \left\{ A \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_\delta) \;\middle|\; \mathcal{P}(\widetilde{S}) \geq \mathcal{P}(S) + \int_\Omega A\colon (\widetilde{S} - S)\,\mathrm{d}x \text{ for all } \widetilde{S} \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_\delta) \right\}$$

of a general dissipation potential $\mathcal{P}$, meaning that $\mathcal{P} : \mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_\delta) \to [0,\infty]$ is convex, lower semicontinuous and satisfies $\mathcal{P}(0) = 0$. Such nonsmooth dissipation potentials are important for viscoelastoplastic fluid models that are used in geodynamics for the defomation of rocks in lithospheric plates, namely

$$\mathcal{P}(S) = \int_\Omega \mathfrak{P}(S(x))\,\mathrm{d}x \quad \text{with } \mathfrak{P}(S) = \begin{cases} \frac{a}{2}|S|^2 & \text{for } |S| \leq \sigma_{\text{yield}}, \\ \infty & \text{for } |S| > \sigma_{\text{yield}}, \end{cases}$$

see [MDM02, Ger07]. In the context of geodynamics, it is also crucial to allow for nontrivial boundary data $g \neq 0$ in (1.2), because often the prescribed drifts of tectonic plates act as boundary data for the specific region of interest.

Main results. In Section 4, see Theorem 4.4, we first provide a global existence result for weak solutions for (1.1)–(1.3) under the additional assumption that the dissipation potential $\mathcal{P}$ belongs to

$C^{1,1}(L^2(\Omega;\mathbb{R}^{3\times3}_\delta))$, which means that $S \mapsto \partial\mathcal{P}(S)$ is monotone and globally Lipschitz continuous. For this we decompose the velocity field into $V = v+w$, where w satisfies the boundary data in (1.2), namely $w = g$ on $\partial\Omega$ and solves a suitable linear Stokes equation. Here we rely on corresponding results for the Navier-Stokes equations with time-dependent boundary data developed in [FGS06, FKS11a, FKS11b].

In a final step we treat the case of a general dissipation potential $\mathcal{P}$ by applying the above result to smooth Moreau envelopes $\mathcal{P}_\varepsilon$, see (2.1), and treat the limit $\varepsilon \to 0$ in an even weaker formulation in form of an evolutionary variational inequality, see Theorem 4.9.

Strategy. The basic features of the model include, of course, all the difficulties known for the three-dimensional Navier–Stokes equations such that we cannot expect better solutions than Leray–Hopf solutions for the velocity component V. For $\gamma > 0$, the equation $(1.1)_3$ for the stress tensor S is a (semilinear) parabolic equation with linear source term $D(V)$, but, crucially, also coupled nonlinearly to V via the Zaremba–Jaumann derivative $\overset{\triangledown}{S}$.

In our analysis we essentially exploit the fact that for sufficiently smooth functions V and S satisfying $\mathrm{n}\cdot V = 0$ on $\partial\Omega$ we have the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|S(t,x)|^2\,\mathrm{d}x = \int_\Omega \partial_t S : S\,\mathrm{d}x = \int_\Omega \mathrm{D}_t S : S\,\mathrm{d}x = \int_\Omega \overset{\triangledown}{S} : S\,\mathrm{d}x. \tag{1.4}$$

Exploiting this identity and assuming for the moment that $V = 0$ on $\partial\Omega$, one can show that smooth solutions satisfy the energy-dissipation balance

$$\int_\Omega \Big(\frac{\rho}{2}|V(t)|^2 + \frac{\eta_1}{2\eta_2}|S(t)|^2\Big)\,\mathrm{d}x + \int_0^t\int_\Omega \Big(2\mu|D(V)|^2 + \frac{\eta_1}{\eta_2}S : \partial\mathcal{P}(S) + \frac{\eta_1\gamma}{\eta_2}|\nabla S|^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega \Big(\frac{\rho}{2}|V_0|^2 + \frac{\beta}{2}|S_0|^2\Big)\,\mathrm{d}x + \int_0^t\int_\Omega V\cdot F\,\mathrm{d}x\,\mathrm{d}t. \tag{1.5}$$

For the more general case with nontrivial boundary data, we refer to (3.32). We clearly see how the quadratic energy consisting of the kinetic energy and an elastic energy associated with S can be changed by the external force F and is dissipated by three mechanisms: (i) a direct fluid viscosity given by $\mu > 0$, (ii) a stress dissipation encoded in the dissipation potential $\mathcal{P}$, and (iii) the stress diffusion associated with $\gamma > 0$.

To simplify the notation we will fix (without loss of generality) two constants and choose $\rho = 1$ and $\eta_1 = \eta_2 = \eta$ subsequently. With this choice the quadratic energy is simply given as one half of the L^2 norm of (V,S).

Our construction of solutions for smooth $\mathcal{P}$ is based on a Galerkin approximation that is manufactured in such a way that the suitably generalized version for $g \not\equiv 0$ of the above energy estimate still holds. Exploiting the standard energy estimates carefully and relying on the compactness arguments of Aubin–Lions type for V and S allows us to pass to the weak limit even in the critical terms $V\cdot\nabla S$ and $SW(V)$. Of course, in the limit the energy-dissipation balance (1.5) will only survive as an energy-dissipation inequality. In Corollary 4.5 we use this to provide conditions on the boundary data g and the forcing F that guarantee that the total energy is bounded uniformly in time.

When approaching the nonsmooth dissipation potentials $\mathcal{P}$ via their Moreau envelopes $\mathcal{P}_\varepsilon$, we lose the compactness for S^ε because $\partial\mathcal{P}_\varepsilon(S^\varepsilon)$ cannot be controlled. Nevertheless, we are able to pass to the limit in the critical terms of the form $S^\varepsilon\nabla V^\varepsilon$ by integration by parts and relying on the boundary conditions, see Lemma 4.11. We again obtain an energy-dissipation inequality of the type (1.5), where $\int_\Omega \partial\mathcal{P}(S) : S\,\mathrm{d}x$ is replaced by the smaller term $\mathcal{P}(S)$.

2 Preliminaries

Here, we specify general notations and introduce some concepts and tools required for the subsequent analysis.

General notations. For two vectors $a, b \in \mathbb{R}^3$ we denote their inner product by $a \cdot b = a_j b_j$ and their tensor product by $a \otimes b$ with $(a \otimes b)_{jk} = a_j b_k$. Here and in the following we use Einstein's summation convention and implicitly sum over repeated indices from 1 to 3. The inner product of two tensors $A, B \in \mathbb{R}^{3\times 3}$ is denoted by $A : B = A_{jk} B_{jk}$. Moreover, $A^\top$ and $\mathrm{Tr}\, A$ denote the transpose and the trace of A. We further set $a \otimes b : A = (a \otimes b) : A = a_j A_{jk} b_k$ and, if $C \in \mathbb{R}^{3\times 3}$ is a third tensor, $AB : C = (AB) : C = A_{jk} B_{k\ell} C_{j\ell}$.

Usually, $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $T \in (0, \infty]$. Points (x, t) in the space-time cylinder $\Omega \times (0, T)$, consist of a spatial variable $x \in \Omega$ and a time variable $t \in (0, T)$. For a sufficiently regular function u, we denote its partial derivatives in time and space by $\partial_t u$ and $\partial_j u$, $j = 1, 2, 3$, respectively. The symbols ∇ and Δ denote (spatial) gradient and Laplace operator. If v is a vector-valued function, we let $\mathrm{div}\, v = \partial_j v_j$ denote its divergence and set $v \cdot \nabla u = v_j \partial_j u$. Symmetric and antisymmetric parts of $\nabla v = (\partial_k v_j)$ are given by

$$D(v) := \frac{1}{2}(\nabla v + \nabla v^\top), \qquad W(v) := \frac{1}{2}(\nabla v - \nabla v^\top),$$

respectively. If S is a tensor-valued function, its divergence $\mathrm{div}\, S$ is given by $(\mathrm{div}\, S)_j = \partial_k S_{jk}$. If T is another tensor-valued function, we define $\nabla T : \nabla S = \partial_\ell T_{jk} \partial_\ell S_{jk}$ and $v \cdot \nabla T : S = v_j (\partial_j T_{k\ell}) S_{k\ell}$.

Function spaces. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $A \in \{\Omega, \overline{\Omega}\}$. Then the class $\mathrm{C}^k(A)$ consists of all k-times continuously differentiable (real-valued) functions on A, and $\mathrm{C}^k_0(A)$ contains all compactly supported functions in $\mathrm{C}^k(A)$. By $\mathrm{L}^q(\Omega)$ with $q \in [1, \infty]$ we denote the classical Lebesgue spaces with corresponding norm $\|\cdot\|_q$, and $\mathrm{H}^k(\Omega)$ with $k \in \mathbb{N}$ denotes the L^2-based Sobolev space of order k, equipped with the norm $\|\cdot\|_{k,2}$. Moreover, $\mathrm{H}^1_0(\Omega)$ contains all elements of $\mathrm{H}^1(\Omega)$ with vanishing boundary trace, and $\mathrm{H}^{1/2}(\partial\Omega)$ denotes the class of boundary traces of functions from $\mathrm{H}^1(\Omega)$. By $\mathrm{H}^{-1}(\Omega)$ and $\mathrm{H}^{-1/2}(\partial\Omega)$ we denote the dual spaces of $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^{1/2}(\partial\Omega)$, respectively, where we use the distributional duality pairing.

The norm of a Banach space X is denoted by $\|\cdot\|_X$, and the same symbol is used for the norms of X^3 and $X^{3\times 3}$. When the dimension is clear from the context, we simply write X instead of X^3 or $X^{3\times 3}$. Moreover, X' denotes the dual space of X, and $\mathrm{C}^{1,1}(X)$ is the set of all continuously Fréchet differentiable functions $X \to \mathbb{R}$ with globally Lipschitz continuous derivative.

For an interval $I \subset \mathbb{R}$, the class $\mathrm{C}^0(I; X)$ consists of all continuous X-valued functions, and $\mathrm{C}_\mathrm{w}(I; X)$ consists of all weakly continuous X-valued functions. The Bochner-Lebesque spaces of X-valued functions are denoted by $\mathrm{L}^q(I; X)$ for $q \in [1, \infty]$, and $\mathrm{L}^q_\mathrm{loc}(I; X)$ denotes the class of all functions that belong to $\mathrm{L}^q(J; X)$ for all compact subintervals $J \subset I$. When $I = (0, T)$, we set $\mathrm{C}^0(0, T; X) = \mathrm{C}^0(I; X)$ and $\mathrm{L}^q(0, T; X) = \mathrm{L}^q(I; X)$. For functions A on $\Omega \times I$ we use the shorthand $A(t) := A(\,\cdot\,, t)$.

We further need spaces of solenoidal vector fields and of symmetric deviatoric tensor fields. The corresponding classes of smooth functions on Ω are given by

$$\mathrm{C}^\infty_{0,\sigma}(\Omega) := \big\{\varphi \in \mathrm{C}^\infty_0(\Omega)^3 \;\big|\; \mathrm{div}\, \varphi = 0\big\},$$
$$\mathrm{C}^\infty_{0,\delta}(\overline{\Omega}) := \big\{\psi \in \mathrm{C}^\infty_0(\overline{\Omega})^{3\times 3} \;\big|\; \psi = \psi^\top,\ \mathrm{Tr}\, \psi = 0\big\}.$$

We further set

$$\mathrm{C}^\infty_{0,\sigma}(\Omega\times I) := \big\{\Phi \in \mathrm{C}^\infty_0(\Omega\times I)^3 \;\big|\; \operatorname{div}\Phi = 0\big\},$$
$$\mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times I) := \big\{\Psi \in \mathrm{C}^\infty_0(\overline{\Omega}\times I)^{3\times 3} \;\big|\; \Psi = \Psi^\top,\ \operatorname{Tr}\Psi = 0\big\}.$$

We define the associated L^2 spaces on Ω by

$$\mathrm{L}^2_\sigma(\Omega) := \big\{v \in \mathrm{L}^2(\Omega)^3 \;\big|\; \operatorname{div} v = 0,\ v|_{\partial\Omega}\cdot \mathrm{n} = 0\big\} = \overline{\mathrm{C}^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2},$$
$$\mathrm{L}^2_\delta(\Omega) := \big\{S \in \mathrm{L}^2(\Omega)^{3\times 3} \;\big|\; S = S^\top,\ \operatorname{Tr} S = 0\big\} = \overline{\mathrm{C}^\infty_{0,\delta}(\overline{\Omega})}^{\|\cdot\|_2}.$$

Here the condition $v|_{\partial\Omega}\cdot\mathrm{n} = 0$ in the definition of $\mathrm{L}^2_\sigma(\Omega)$ has to be understood in a weak sense; see [Gal11, Theorem III.2.3] for example. We further introduce the corresponding Sobolev spaces

$$\mathrm{H}^1_{0,\sigma}(\Omega) := \big\{v \in \mathrm{H}^1_0(\Omega)^3 \;\big|\; \operatorname{div} v = 0\big\} = \overline{\mathrm{C}^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{1,2}},$$
$$\mathrm{H}^1_\delta(\Omega) := \big\{S \in \mathrm{H}^1(\Omega)^{3\times 3} \;\big|\; S = S^\top,\ \operatorname{Tr} S = 0\big\} = \overline{\mathrm{C}^\infty_{0,\delta}(\overline{\Omega})}^{\|\cdot\|_{1,2}}.$$

We can now define the solution spaces

$$\mathrm{LH}_T := \mathrm{L}^\infty(0,T;\mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)^3)$$

for the fluid velocity and

$$\mathrm{X}_T := \mathrm{L}^\infty(0,T;\mathrm{L}^2_\delta(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)^{3\times 3})$$

for the stress tensor. Observe that LH_T is the classical Leray–Hopf class for weak solutions to the Navier–Stokes equations, and X_T is the analog for semilinear parabolic equations taking values in deviatoric tensor fields.

Dissipation potentials. Let H be a Hilbert space with scalar product $(\cdot,\cdot)_\mathrm{H}$. We call a function $\mathcal{P}\colon \mathrm{H}\to[0,\infty]$ a *dissipation potential* if $\mathcal{P}$ is convex and lower semicontinuous with $\mathcal{P}(0) = 0$. We denote by $\partial\mathcal{P}$ the subgradient of $\mathcal{P}$, i.e.

$$\partial\mathcal{P}(S) = \big\{\tau \in \mathrm{H} \;\big|\; (\tau, \tilde{S} - S)_\mathrm{H} + \mathcal{P}(S) \leq \mathcal{P}(\tilde{S}) \ \text{ for all } \tilde{S}\in\mathrm{H}\big\}$$

for $S\in\mathrm{H}$. Observe that, by definition, $\partial\mathcal{P}(S) = \emptyset$ if $\mathcal{P}(S) = +\infty$. If $\partial\mathcal{P}(S) = \{\tau\}$ for some $\tau\in\mathrm{H}$, we identify the set $\partial\mathcal{P}(S)$ with its unique element τ. In this case, we call τ the (Gâteaux) differential of $\mathcal{P}$ at S.

Given a general dissipation potential $\mathcal{P}$, we denote by $\{\mathcal{P}_\varepsilon\}_{\varepsilon\in(0,1]}$ the family of *Moreau envelopes* of $\mathcal{P}$ [Mor65], that is, we let

$$\mathcal{P}_\varepsilon(S) = \inf_{S'\in\mathrm{H}} \left(\frac{1}{2\varepsilon}\|S - S'\|^2_\mathrm{H} + \mathcal{P}(S')\right), \quad \varepsilon > 0. \tag{2.1}$$

Then, $\mathcal{P}_\varepsilon : \mathrm{H}\to[0,\infty)$ is again a dissipation potential and is Fréchet differentiable with Lipschitz continuous differential $\partial\mathcal{P}_\varepsilon$ with Lipschitz constant $1/\varepsilon$ (see [BaC17, Section 12.4] for example). In particular, if $\mathcal{P}$ is given by an integral in the form

$$\mathcal{P}(S) = \int_\Omega \mathfrak{P}(S(x))\,\mathrm{d}x \quad \text{for } S\in\mathrm{H} = \mathrm{L}^2(\Omega),$$

then its Moreau envelope has the form $\mathcal{P}_\varepsilon(S) = \int_\Omega \mathfrak{P}_\varepsilon(S(x))\,\mathrm{d}x$.

Moreover, since $S = 0$ is a minimum of $\mathcal{P}_\varepsilon$, the Lipschitz continuity yields

$$\|\partial\mathcal{P}_\varepsilon(S)\|_{\mathrm{H}} = \|\partial\mathcal{P}_\varepsilon(S) - \partial\mathcal{P}_\varepsilon(0)\|_{\mathrm{H}} \le \varepsilon^{-1}\|S - 0\|_{\mathrm{H}} = \varepsilon^{-1}\|S\|_{\mathrm{H}}.$$

By the definition in (2.1), we have $0 \le \mathcal{P}_\varepsilon(S) \le \mathcal{P}(S)$ and hence

$$\limsup_{\varepsilon\to 0} \int_0^T \mathcal{P}_\varepsilon(S(t))\,\mathrm{d}t \le \int_0^T \mathcal{P}(S(t))\,\mathrm{d}t \quad \text{for all } S \in \mathrm{L}^2(0,T;\mathrm{H}). \tag{2.2}$$

We will need the following version of the classical approximation properties of the Moreau envelope [BaC17, Rou13].

Lemma 2.1. *Let $\mathcal{P}$ be a dissipation potential. Then the family of Moreau envelopes $\{\mathcal{P}_\varepsilon\}$ of $\mathcal{P}$ satisfies the inequality*

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{P}_\varepsilon(S_\varepsilon(t))\,\mathrm{d}t \ge \int_0^T \mathcal{P}(S(t))\,\mathrm{d}t \tag{2.3}$$

whenever $S_\varepsilon \rightharpoonup S$ in $\mathrm{L}^2(0,T;\mathrm{H})$.

Proof. Let $\delta > \varepsilon > 0$. Then, by definition, $\mathcal{P}_\varepsilon \ge \mathcal{P}_\delta$. Hence

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{P}_\varepsilon(S_\varepsilon(t))\,\mathrm{d}t \ge \liminf_{\varepsilon\to 0} \int_0^T \mathcal{P}_\delta(S_\varepsilon(t))\,\mathrm{d}t \ge \int_0^T \mathcal{P}_\delta(S(t))\,\mathrm{d}t. \tag{2.4}$$

The second step follows from the fact that for $\delta > 0$ the functional

$$F_\delta : \mathrm{L}^2(0,T;\mathrm{H}) \ni S \mapsto \int_0^T \mathcal{P}_\delta(S(t))\,\mathrm{d}t$$

is convex and continuous, and thus weakly lower semicontinuous. The convexity of F_δ is inherited from the convexity of $\mathcal{P}_\delta$, while continuity follows from standard theory on Nemytskii operators (see e.g. [Rou13, Theorem 1.43]) together with the growth condition $0 \le \mathcal{P}_\delta(S) \le \|S\|_{\mathrm{H}}^2/(2\delta)$, which is a consequence of the definition of the Moreau envelope.

To show the assertion, it now remains to prove that

$$\lim_{\delta\to 0} \int_0^T \mathcal{P}_\delta(S(t))\,\mathrm{d}t = \int_0^T \mathcal{P}(S(t))\,\mathrm{d}t. \tag{2.5}$$

By [BaC17, Proposition 12.33 (ii)], $\mathcal{P}_\delta(S(t)) \to \mathcal{P}(S(t))$ for a.e. $t \in (0,T)$. The nonnegativity of $\mathcal{P}_\delta$ and Beppo Levi's monotone convergence imply the identity (2.5). Together with (2.4) the desired assertion (2.3) follows. □

3 Decomposition of the velocity field

To show existence of a weak solution to the system (1.1)–(1.3), we decompose the velocity and pressure fields into two parts, $V = v + w$ and $P = p + \mathfrak{q}$, where $(w, \mathfrak{q})$ is a solution to the Stokes initial-value problem with boundary data $w = g$

$$\begin{cases} \partial_t w - \operatorname{div}\big(2\mu D(w) - \mathfrak{q}I\big) = \widetilde{F} & \text{in } \Omega\times(0,T), \\ \operatorname{div} w = 0 & \text{in } \Omega\times(0,T), \\ w = g & \text{on } \partial\Omega\times(0,T), \\ w(\cdot,0) = w_0 & \text{in } \Omega \end{cases} \tag{3.1}$$

for some functions $\widetilde{F}$ and w_0, and v satisfies the remaining system with homogeneous boundary data $v=0$

$$\begin{cases} \partial_t v + (v{+}w)\cdot\nabla(v{+}w) - \operatorname{div}\big(\eta S + 2\mu D(v) - pI\big) = f & \text{in } \Omega\times(0,T), \\ \operatorname{div} v = 0 & \text{in } \Omega\times(0,T), \\ \partial_t S + (v{+}w)\cdot\nabla S + SW(v{+}w) - W(v{+}w)S & \\ \qquad + \partial\mathcal{P}(S) - \gamma\Delta S - \eta D(v{+}w) = 0 & \text{in } \Omega\times(0,T), \\ v = 0, \quad \mathrm{n}\cdot\nabla S = 0 & \text{on } \partial\Omega\times(0,T), \\ v(\cdot,0) = v_0, \quad S(\cdot,0) = S_0 & \text{in } \Omega \end{cases} \tag{3.2}$$

with $f = F - \widetilde{F}$ and $v_0 = V_0 - w_0$. In this way, we have decomposed the question of existence of solutions to (1.1)–(1.3) into two separate problems.

In this section we prescribe a class of vector fields w such that the modified system (3.2) has a solution, see Assumption 3.1. In the following section, see Lemma 4.2, we establish a class of admissible functions g such that the Stokes problem (3.1) admits solutions w belonging to this class. The existence of solutions for the original system (1.1)–(1.3) for the case of smooth potentials $\mathcal{P}$ is then stated in Theorem 4.4.

The above decomposition method is a common way to treat inhomogeneous boundary data $g \neq 0$, and was successfully used to show existence of weak solutions to the classical Navier–Stokes initial-value problem in different configurations, see [FGS06, FKS10, FKS11a, FKS11b] for example.

For showing the existence of weak solutions to the modified system (3.2) we make the following assumptions. Let $0 < T \le \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\mathcal{P}\colon \mathrm{L}^2_\delta(\Omega) \to [0,\infty)$ be a dissipation potential, which satisfies $\mathcal{P} \in \mathrm{C}^{1,1}(\mathrm{L}^2_\delta(\Omega))$. For the data we assume

$$\begin{aligned} &v_0 \in \mathrm{L}^2_\sigma(\Omega), \quad S_0 \in \mathrm{L}^2_\delta(\Omega), \quad f = f_0 + \operatorname{div} f_1, \\ &f_0 \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^2(\Omega)^3), \qquad f_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\Omega)^{3\times3}). \end{aligned} \tag{3.3}$$

Moreover, the extension function w is assumed to have the following properties.

Assumption 3.1. The function w satisfies

$$w \in \mathrm{L}^4_{\mathrm{loc}}([0,T);\mathrm{L}^4(\Omega)^3), \qquad \nabla w \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\Omega)^{3\times3}),$$

and one of the following three properties:

(a) $w \in \mathrm{L}^s_{\mathrm{loc}}([0,T);\mathrm{L}^r(\Omega)^3)$ for some $r \in (3,\infty)$, $s \in (2,\infty)$ with $\frac{2}{s} + \frac{3}{r} = 1$,

(b) $w \in \mathrm{C}^0(0,T;\mathrm{L}^3(\Omega)^3)$ with $\|w\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^3(\Omega))} \le \alpha$ for $\alpha > 0$ sufficiently small,

(c) $w \in \mathrm{C}^0(0,T;\mathrm{L}^3(\Omega)^3)$ and for all $t \in (0,T)$ we have

$$\forall\, v \in \mathrm{H}^1_{0,\sigma}(\Omega): \quad \left|\int_\Omega w(t)\otimes v : \nabla v \,\mathrm{d}x\right| \le \frac{\mu}{2}\|\nabla v\|_2^2, \tag{3.4}$$

$$\forall\, S \in \mathrm{H}^1_\delta(\Omega): \quad \left|\int_\Omega w(t)\cdot\nabla S : S \,\mathrm{d}x\right| \le \frac{\gamma}{2}\|S\|_{1,2}^2, \tag{3.5}$$

where μ and γ are the constants appearing in (1.1).

Remark 3.2. Note that condition (a) cannot directly be generalized to the case $r = 3$, $s = \infty$, which is treated in (b) and (c). Moreover, the smallness of the extension w required in (b) naturally transfers to a smallness assumption on the associated boundary data g. In contrast, although condition (c) is a direct consequence of (b), it does not require such a condition. As we shall see at the beginning of Section 4.1, we can find an extension w satisfying (c) without imposing a smallness assumption on g.

The aim of this section is to show existence to problem (3.2) in the following sense. Recall that in the present situation $\partial\mathcal{P}$ is a well-defined (single-valued) function.

Definition 3.3. We call a couple (v, S) a *weak solution to* (3.2) if $(v, S) \in \mathrm{LH}_{T'} \times \mathrm{X}_{T'}$ for all $0 < T' < T$, if $v|_{\partial\Omega\times(0,T)} = 0$ and if the identities

$$\begin{aligned}\int_0^T \int_\Omega \big[-v\cdot\partial_t\Phi - (v{+}w)\otimes(v{+}w) : \nabla\Phi + \eta S : \nabla\Phi + \mu\nabla v : \nabla\Phi \big] \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_\Omega \big[f_0\cdot\Phi - f_1 : \nabla\Phi\big] \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega v_0\cdot\Phi(\cdot,0)\,\mathrm{d}x,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\int_0^T \int_\Omega \big[-S\cdot\partial_t\Psi + (v{+}w)\cdot\nabla S : \Psi + \big(SW(v{+}w) - W(v{+}w)S\big) : \Psi \\ + \partial\mathcal{P}(S) : \Psi + \gamma\nabla S : \nabla\Psi - \eta D(v{+}w) : \Psi\big] \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega S_0 : \Psi(\cdot,0)\,\mathrm{d}x\end{aligned} \tag{3.7}$$

hold for all $\Phi \in \mathrm{C}^\infty_{0,\sigma}(\Omega\times[0,T))$ and $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T))$.

Observe that (3.6) and (3.7) are obtained by multiplying $(3.2)_1$ and $(3.2)_3$ by the respective test functions and formally integrating by parts. In particular, (3.6) is in accordance with the notion of weak solutions for the classical Navier–Stokes problem, since we take divergence-free test functions and omit the pressure term. Moreover, since $(3.2)_3$ is an equation of tensors in $\mathbb{R}^{3\times3}_\delta$, it suffices to use symmetric deviatoric test functions in the weak formulation of $(3.2)_3$. Due to this choice, we can replace $D(v{+}w)$ with $\nabla(v{+}w)$ in (3.7), subsequently.

3.1 Approximate solutions

At first, we construct a sequence of approximate solutions to (3.2). To this end, we first introduce suitable basis functions.

Lemma 3.4. *There exists a sequence $(\varphi_k) \subset \mathrm{C}^\infty_{0,\sigma}(\Omega)$, which is an orthonormal basis of $\mathrm{L}^2_\sigma(\Omega)$, such that for all $\Phi \in \mathrm{C}^\infty_{0,\sigma}(\Omega\times[0,T))$ and all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $\gamma_1,\dots,\gamma_N \in \mathrm{C}^1_0([0,T))$ such that*

$$\max_{t\in[0,T]} \Big\| \sum_{k=1}^N \gamma_k(t)\varphi_k - \Phi \Big\|_{\mathrm{C}^2(\Omega)} + \max_{t\in[0,T]} \Big\| \sum_{k=1}^N \partial_t\gamma_k(t)\varphi_k - \partial_t\Phi \Big\|_{\mathrm{C}^1(\Omega)} < \varepsilon. \tag{3.8}$$

Proof. See [Gal00, Lemma 2.3]. □

Lemma 3.5. *There exists a sequence $(\psi_k) \subset \mathrm{C}^\infty_{0,\delta}(\overline{\Omega})$, which is an orthonormal basis of $\mathrm{L}^2_\delta(\Omega)$, such that for all $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T))$ and all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $\widetilde{\gamma}_1,\dots,\widetilde{\gamma}_N \in \mathrm{C}^1_0([0,T))$ such that*

$$\max_{t\in[0,T]} \Big\| \sum_{k=1}^N \widetilde{\gamma}_k(t)\psi_k - \Psi \Big\|_{\mathrm{C}^2(\Omega)} + \max_{t\in[0,T]} \Big\| \sum_{k=1}^N \partial_t\widetilde{\gamma}_k(t)\psi_k - \partial_t\Psi \Big\|_{\mathrm{C}^1(\Omega)} < \varepsilon. \tag{3.9}$$

Proof. One can follow the proof of [Gal00, Lemma 2.3]. □

Remark 3.6. Observe that (φ_k) is a basis of $\mathrm{H}^1_{0,\sigma}(\Omega)$, and (ψ_k) is a basis of $\mathrm{H}^1_\delta(\Omega)$. To see this, consider $\varphi \in \mathrm{C}^\infty_{0,\sigma}(\Omega)$ and let $\Phi \in \mathrm{C}^\infty_{0,\sigma}(\Omega \times [0,T))$ such that $\Phi(\cdot,0) = \varphi$. Let $\varepsilon > 0$ and $\gamma_1,\dots,\gamma_N$ as in Lemma 3.4. Then

$$\Big\|\sum_{k=1}^N \gamma_k(0)\varphi_k - \varphi\Big\|_{\mathrm{C}^2(\Omega)} \leq \max_{t\in[0,T]} \Big\|\sum_{k=1}^N \gamma_k(t)\varphi_k - \Phi(\cdot,t)\Big\|_{\mathrm{C}^2(\Omega)} < \varepsilon.$$

Since $\mathrm{C}^\infty_{0,\sigma}(\Omega)$ is dense in $\mathrm{H}^1_{0,\sigma}(\Omega)$ and Ω is bounded, this shows the claim for (φ_k). Taking $\psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega})$ and $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega} \times [0,T))$ with $\Psi(\cdot,0) = \psi$ instead, we can use Lemma 3.5 to conclude the statement for (ψ_k) in the same way.

With these bases at hand, we now construct a sequence of approximate solutions in the following way. For $k \in \mathbb{N}$ we call (v,S) an approximate solution (of order k) if there exist $\alpha_r,\ \beta_r \in \mathrm{C}^1(0,T_k)$, $r = 1,\dots,k$, such that

$$v(x,t) = v_k(x,t) = \sum_{r=1}^k \alpha_r(t)\varphi_r(x), \qquad S(x,t) = S_k(x,t) = \sum_{r=1}^k \beta_r(t)\psi_r(x), \tag{3.10}$$

and for all $\ell \in \{1,\dots,k\}$ the pair (v,S) satisfies

$$\begin{aligned}\int_\Omega \big[\partial_t v\cdot\varphi_\ell - (v{+}w)\otimes(v{+}w) : \nabla\varphi_\ell + \eta S : \nabla\varphi_\ell + \mu\nabla v : \nabla\varphi_\ell\big]\,\mathrm{d}x&\\ = \int_\Omega f_0\cdot\varphi_\ell\,\mathrm{d}x - \int_\Omega f_1 : \nabla\varphi_\ell\,\mathrm{d}x&,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\int_\Omega \big[\partial_t S\cdot\psi_\ell + (v{+}w)\cdot\nabla S : \psi_\ell + SW(v{+}w) : \psi_\ell - W(v{+}w)S : \psi_\ell&\\ +\partial\mathcal{P}(S) : \psi_\ell + \gamma\nabla S : \nabla\psi_\ell - \eta\nabla(v{+}w) : \psi_\ell\big]\,\mathrm{d}x = 0&\end{aligned} \tag{3.12}$$

in $(0,T)$ and

$$\int_\Omega v(0)\cdot\varphi_\ell\,\mathrm{d}x = \int_\Omega v_0\cdot\varphi_\ell\,\mathrm{d}x, \qquad \int_\Omega S(0) : \psi_\ell\,\mathrm{d}x = \int_\Omega S_0 : \psi_\ell\,\mathrm{d}x. \tag{3.13}$$

The existence of approximate solutions is guaranteed by the following result.

Lemma 3.7. *For all $k \in \mathbb{N}$ there exists an approximate solution $(v,S) = (v_k,S_k)$, which satisfies the energy equalities*

$$\begin{aligned}&\frac12\|v(t)\|_2^2 + \mu\|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))}\\ &\quad= \frac12\|v(0)\|_2^2 + \int_0^t\int_\Omega \big[f_0\cdot v - f_1 : \nabla v + w\otimes(v{+}w) : \nabla v - \eta S : \nabla v\big]\,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned} \tag{3.14a}$$

$$\begin{aligned}&\frac12\|S(t)\|_2^2 + \gamma\|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \int_0^t\int_\Omega \partial\mathcal{P}(S) : S\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad= \frac12\|S(0)\|_2^2 + \int_0^t\int_\Omega \big[-w\cdot\nabla S : S + \eta\nabla(v{+}w) : S\big]\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{3.14b}$$

for all $t \in (0,T)$. Moreover, for $0 < T' < T$, there exists a constant $M_{T'} > 0$, only depending on the data and T' but independent of k and $\mathcal{P}$, such that

$$\sup_{t\in(0,T')}\Big(\|v(t)\|_2^2+\|S(t)\|_2^2\Big)+\int_0^{T'}\Big(\|v(t)\|_{1,2}^2+\|S(t)\|_{1,2}^2+\mathcal{P}(S)\Big)\,\mathrm{d}t\le M_{T'}, \tag{3.15}$$

so that the sequence $(v_k,S_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{LH}_{T'}\times\mathrm{X}_{T'}$.

Remark 3.8. The energy balances of the kinetic energy and of the stored elastic energy are expressed in (3.14) in two separate equations. By summation we obtain the total energy equality

$$\begin{aligned}&\frac12\|v(t)\|_2^2+\frac12\|S(t)\|_2^2+\mu\|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))}+\gamma\|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))}\\&+\int_0^t\int_\Omega\partial\mathcal{P}(S):S\,\mathrm{d}x\,\mathrm{d}\tau=\frac12\|v(0)\|_2^2+\frac12\|S(0)\|_2^2\\&\quad+\int_0^t\int_\Omega\big[f_0\cdot v-f_1:\nabla v+w\otimes(v{+}w):\nabla v-w\cdot\nabla S:S+\eta\nabla w:S\big]\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{3.16}$$

Proof. We reduce the equations (3.11)–(3.13) to an initial-value problem for the coefficient function $(\alpha,\beta)=(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)$. For simplicity, let $(\cdot)'=\frac{\mathrm{d}}{\mathrm{d}t}$ denote the time derivative. Due to orthogonality properties of the two bases (φ_k) and (ψ_k), we then obtain

$$\begin{aligned}&\alpha'(t)=F^1(\alpha(t),\beta(t),t),\quad \beta'(t)=F^2(\alpha(t),\beta(t)),\\&\alpha_\ell(0)=\int_\Omega v_0\cdot\varphi_\ell\,\mathrm{d}x,\qquad \beta_\ell(0)=\int_\Omega S:\psi_\ell\,\mathrm{d}x\qquad(\ell=1,\dots,k),\end{aligned}\tag{3.17}$$

where $F^j=(F^j_1,\dots,F^j_k)$, $j=1,2$, with

$$\begin{aligned}F^1_\ell(\alpha,\beta,t)=&\sum_{r,s=1}^n\alpha_r\alpha_s\int_\Omega\varphi_r\otimes\varphi_s:\nabla\varphi_\ell\,\mathrm{d}x+\sum_{r=1}^k\alpha_r\int_\Omega\big(w\otimes\varphi_r+\varphi_r\otimes w\big):\nabla\varphi_\ell\,\mathrm{d}x\\&+\int_\Omega w\cdot\nabla w\cdot\varphi_\ell\,\mathrm{d}x-\eta\sum_{r=1}^k\beta_r\int_\Omega\psi_r:\nabla\varphi_\ell\,\mathrm{d}x-\mu\sum_{r=1}^k\alpha_r\int\nabla\varphi_r:\nabla\varphi_\ell\,\mathrm{d}x\\&+\int_\Omega f_0(\cdot,t)\cdot\varphi_\ell-f_1(\cdot,t):\nabla\varphi_\ell\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}F^2_\ell(\alpha,\beta)=&-\sum_{r,s=1}^k\alpha_r\beta_s\int\varphi_r\cdot\nabla\psi_s:\psi_\ell\,\mathrm{d}x-\sum_{r=1}^k\beta_r\int w\cdot\nabla\psi_s:\psi_\ell\,\mathrm{d}x\\&-\sum_{r,s=1}^k\beta_r\alpha_s\int_\Omega\big[\psi_rW(\varphi_r)-W(\varphi_r)\psi_s\big]:\psi_\ell\,\mathrm{d}x-\int_\Omega\big[\psi_rW(w)-W(w)\psi_s\big]:\psi_\ell\,\mathrm{d}x\\&-\int_\Omega\partial\mathcal{P}(\sum_{r=1}^k\alpha_r\psi_r):\psi_\ell\,\mathrm{d}x-\gamma\sum_{r=1}^k\beta_r\int_\Omega\nabla\psi_r:\nabla\psi_\ell\,\mathrm{d}x\\&+\eta\sum_{r=1}^k\alpha_r\int_\Omega\nabla\varphi_r:\psi_\ell\,\mathrm{d}x+\eta\int_\Omega\nabla w:\psi_\ell\,\mathrm{d}x\end{aligned}$$

for $\ell = 1, \ldots, k$. In particular, since $\partial\mathcal{P}$ is Lipschitz continuous by assumption, (3.17) is an initial-value problem with right-hand side $F = (F^1, F^2)$ that satisfies a local Lipschitz condition. By the Picard–Lindelöf theorem we thus obtain a unique local solution (α, β) to (3.17). Let $(0, T_k) \subset (0, T)$ denote its maximal existence interval. Then (v, S) defined by (3.10) satisfy (3.11)–(3.12) in $(0, T_k)$ and (3.13).

To conclude the energy equalities (3.14a) and (3.14b) for all $t \in (0, T_k)$, we multiply (3.11) by α_ℓ and (3.12) by β_ℓ, sum over $\ell = 1, \ldots, k$ and integrate over the time interval $(0, t)$. This leads to the two equalities (3.14) by employing the identities

$$\int_\Omega v \otimes (v{+}w) : \nabla v \,\mathrm{d}x = \frac{1}{2}\int_\Omega (v{+}w) \cdot \nabla |v|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} (v{+}w) \cdot \mathrm{n} |v|^2 \,\mathrm{d}\sigma = 0,$$
$$\int_\Omega v \cdot \nabla S : S \,\mathrm{d}x = \frac{1}{2}\int_\Omega v \cdot \nabla |S|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} v \cdot \mathrm{n} |S|^2 \,\mathrm{d}\sigma = 0,$$

which hold due to $v = 0$ on $\partial\Omega \times (0, T_k)$.

Next we show that $T_k = T$. To this end, we add (3.14a) and (3.14b) to obtain (3.16) and further estimate the right-hand side of (3.16) for $t \in (0, T_k)$. The initial terms can be estimated with Bessel's inequality as

$$\|v(0)\|_2^2 \le \|v_0\|_2^2, \qquad \|S(0)\|_2^2 \le \|S_0\|_2^2. \tag{3.18}$$

For the linear terms, we employ a combination of Hölder's and Young's inequalities to obtain

$$\int_0^t \int_\Omega f_0 \cdot v \,\mathrm{d}x \,\mathrm{d}\tau \le c_0(\varepsilon) \|f_0\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \varepsilon \|v\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))}, \tag{3.19}$$
$$\int_0^t \int_\Omega f_1 : \nabla v \,\mathrm{d}x \,\mathrm{d}\tau \le c_1(\varepsilon) \|f_1\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \varepsilon \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))}, \tag{3.20}$$
$$\int_0^t \int_\Omega \nabla w : S \,\mathrm{d}x \,\mathrm{d}\tau \le c_2(\varepsilon) \|\nabla w\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \varepsilon \|S\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))}, \tag{3.21}$$
$$\int_0^t \int_\Omega (w \otimes w) : \nabla v \,\mathrm{d}x \,\mathrm{d}\tau \le c_3(\varepsilon) \|w\|^4_{\mathrm{L}^4(\Omega\times(0,t))} + \varepsilon \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \tag{3.22}$$

for any $\varepsilon > 0$. Next we address the nonlinear terms, where we need to discuss the different cases in Assumption 3.1.

Case $s < \infty$: We use part (a) of Assumption 3.1 with $r > 3$ and define $p \in (2, 6)$ via $1/p = 1/2 - 1/r$. With $\theta = 3/2 - 3/p = 3/r = 1 - 2/s \in (0, 1)$ the Gagliardo–Nirenberg inequality gives

$$\|v(t)\|_p \le c_4 \|v(t)\|_2^{1-\theta} \|\nabla v(t)\|_2^\theta,$$
$$\|S(t)\|_p \le c_5 \|S(t)\|_2^{1-\theta} \|\nabla S(t)\|_2^\theta + c_6 \|S(t)\|_2$$

for all $t \in [0, T_k]$, so that Hölder's and Young's inequalities lead to

$$\begin{aligned}\int_0^t \int_\Omega w \otimes v : \nabla v \,\mathrm{d}x\,\mathrm{d}\tau &\le \int_0^t \|w\|_r \|v\|_p \|\nabla v\|_2 \,\mathrm{d}\tau \le c_7 \int_0^t \|w\|_r \|v\|_2^{1-\theta} \|\nabla v\|_2^{1+\theta} \,\mathrm{d}\tau \\ &\le \varepsilon \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + c_8(\varepsilon) \int_0^t \|w\|_r^s \|v\|_2^2 \,\mathrm{d}\tau,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\int_0^t \int_\Omega w \cdot \nabla S : S \,\mathrm{d}x\,\mathrm{d}\tau &\le \int_0^t \|w\|_r \|\nabla S\|_2 \|S\|_p \,\mathrm{d}\tau \\ &\le c_9 \int_0^t \|w\|_r \Big(\|\nabla S\|_2^{1+\theta} \|S\|_2^{1-\theta} + \|\nabla S\|_2 \|S\|_2 \Big) \,\mathrm{d}\tau \\ &\le \varepsilon \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + c_{10}(\varepsilon) \int_0^t \big(\|w\|_r^s + \|w\|_r^2\big) \|S\|_2^2 \,\mathrm{d}\tau.\end{aligned} \tag{3.24}$$

Estimating the right-hand side of (3.16) with (3.18)–(3.24) and choosing $\varepsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}&\|v\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|S(t)\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \\ &\quad + \int_0^t \int_\Omega \partial\mathcal{P}(S) : S \,\mathrm{d}x\,\mathrm{d}t \le c_{11} \Big(\|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|f_1\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \\ &\qquad + \|\nabla w\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|w\|^4_{\mathrm{L}^4(\Omega\times(0,t))} + \int_0^t \big(\|w\|_r^s + \|w\|_r^2\big)\big(\|v\|_2^2 + \|S\|_2^2\big) \,\mathrm{d}\tau \Big).\end{aligned}$$

We can now add the squared norm of v and S in $\mathrm{L}^2(\Omega \times (0,t))$ to both sides of the inequality. An application of Gronwall's inequality then leads to

$$\begin{aligned}&\|v\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|S\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|v\|^2_{\mathrm{L}^2(0,t;\mathrm{H}^1(\Omega))} + \|S\|^2_{\mathrm{L}^2(0,t;\mathrm{H}^1(\Omega))} \\ &\quad + \int_0^t \mathcal{P}(S) \,\mathrm{d}t \le c_{12} \big(\|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|f_1\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \\ &\qquad + \|\nabla w\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|w\|^4_{\mathrm{L}^4(\Omega\times(0,t))} \big) \exp\Big(c_{13} \int_0^t \big(\|w\|_r^s + \|w\|_r^2\big) \,\mathrm{d}\tau \Big)\end{aligned} \tag{3.25}$$

in the case $s < \infty$.

<u>Case $s = \infty$:</u> We can apply (3.4) and (3.5) to obtain

$$\int_0^t \int_\Omega w \otimes v : \nabla v \,\mathrm{d}x\,\mathrm{d}\tau \le \frac{\mu}{2} \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))}, \tag{3.26}$$

$$\int_0^t \int_\Omega w \cdot \nabla S : S \,\mathrm{d}x\,\mathrm{d}\tau \le \frac{\gamma}{2} \big(\|S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \big). \tag{3.27}$$

Using (3.26) and (3.27) instead of (3.23) and (3.24), the argument from above leads to

$$\begin{aligned}&\|v\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|S\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2(\Omega))} + \|v\|^2_{\mathrm{L}^2(0,t;\mathrm{H}^1(\Omega))} + \|S\|^2_{\mathrm{L}^2(0,t;\mathrm{H}^1(\Omega))} \\ &\quad + \int_0^t \mathcal{P}(S) \,\mathrm{d}t \le c_{14} \big(\|v_0\|_2^2 + \|S_0\|_2^2 + \|f_0\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|f_1\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \\ &\qquad + \|\nabla w\|^2_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega))} + \|w\|^4_{\mathrm{L}^4(\Omega\times(0,t))} \big) \mathrm{e}^{c_{15} t}\end{aligned} \tag{3.28}$$

in the case $s = \infty$.

Now consider general $s \in (2, \infty]$ again. For any $T' \in (0, T_k)$, the right-hand side of (3.25) and (3.28) can be bounded uniformly in $t \in (0, T')$ by a constant $M_{T'} > 0$ that only depends on the data and T'. In particular, $M_{T'}$ is independent of $k \in \mathbb{N}$ and $\mathcal{P}$, and we conclude (3.15) in both cases. By Parseval's identity, this shows

$$\sum_{r=1}^{k} |\alpha_r(t)|^2 + |\beta_r(t)|^2 = \|v(t)\|_2^2 + \|S(t)\|_2^2 \leq M_{T'}$$

for all $t \in (0, T')$. Hence the solution (α, β) does not blow-up at $t = T'$, and we conclude $T_k = T$ together with (3.15). □

Lemma 3.9. *For any $0 < T' < T$, the sequence $(\partial_t v_k, \partial_t S_k)_{k\in\mathbb{N}}$ is bounded in*

$$\mathrm{L}^1\big(0, T'; \big(\mathrm{H}^1_{0,\sigma}(\Omega)\big)'\big) \times \mathrm{L}^{8/7}\big(0, T'; \big(\mathrm{H}^1(\Omega)^{3\times 3}\big)'\big)$$

with

$$\|\partial_t v_k\|_{\mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega))')} \leq M'_{T'},$$
$$\|\partial_t S_k\|_{\mathrm{L}^{8/7}(0,T';(\mathrm{H}^1(\Omega))')} \leq M_{T',\mathcal{P}},$$

where $M'_{T'}$ is independent of $\mathcal{P}$, but $M_{T',\mathcal{P}}$ depends on $\mathcal{P}$.

Proof. Let $k \in \mathbb{N}$ and $\ell \in \{1, \dots, k\}$. For $v = v_k$ we have the interpolation inequality

$$\|v(t)\|_4 \leq c_0 \|v(t)\|_2^{1/4} \|\nabla v(t)\|_2^{3/4},$$

so that from (3.11) we deduce the estimate

$$\begin{aligned}
&\int_\Omega \partial_t v \cdot \varphi_\ell \,\mathrm{d}x \\
&\leq c_1\big(\|v{+}w\|_4^2 \|\nabla\varphi_\ell\|_2 + \|S\|_2\|\nabla\varphi_\ell\|_2 + \|\nabla v\|_2\|\nabla\varphi_\ell\|_2 + \|f_0\|_2\|\varphi_\ell\|_2 + \|f_1\|_2\|\nabla\varphi_\ell\|_2\big) \\
&\leq c_2\big(\|v\|_2^{1/2}\|\nabla v\|_2^{3/2} + \|w\|_4^2 + \|S\|_2 + \|\nabla v\|_2 + \|f_0\|_2 + \|f_1\|_2\big)\|\varphi_\ell\|_{1,2}
\end{aligned}$$

in $(0, T)$. Now let $\varphi \in \mathrm{H}^1_{0,\sigma}(\Omega)^3$. Then $\varphi = \sum_{\ell=0}^{\infty} a_\ell \varphi_\ell$ in $\mathrm{H}^1(\Omega)$ for some sequence $(a_\ell) \subset \mathbb{R}$. Since (φ_ℓ) is orthogonal in $\mathrm{L}^2_\sigma(\Omega)$, in a similar way we obtain

$$\begin{aligned}
\int_\Omega \partial_t v \cdot \varphi \,\mathrm{d}x &= \int_\Omega \partial_t v \cdot \sum_{\ell=1}^{k} \alpha_\ell \varphi_\ell \,\mathrm{d}x \\
&\leq c_3\big(\|v\|_2^{1/2}\|\nabla v\|_2^{3/2} + \|w\|_4^2 + \|S\|_2 + \|\nabla v\|_2 + \|f_0\|_2 + \|f_1\|_2\big)\Big\|\sum_{\ell=1}^{k} \alpha_\ell\varphi_\ell\Big\|_{1,2} \\
&\leq c_4\big(\|v\|_2^{1/2}\|\nabla v\|_2^{3/2} + \|w\|_4^2 + \|S\|_2 + \|\nabla v\|_2 + \|f_0\|_2 + \|f_1\|_2\big)\|\varphi\|_{1,2}.
\end{aligned}$$

Hence we have $\partial_t v(t) \in \big(\mathrm{H}^1_{0,\sigma}(\Omega)\big)'$ for a.a. $t \in (0, T)$. Moreover, with Hölder's inequality and (3.15) we conclude

$$\begin{aligned}
&\int_0^{T'} \|\partial_t v\|_{(\mathrm{H}^1_{0,\sigma}(\Omega))'} \,\mathrm{d}t \\
&\quad\le c_5 \int_0^{T'} \big(\|v\|_2^{1/2}\|\nabla v\|_2^{3/2} + \|w\|_4^2 + \|S\|_2 + \|\nabla v\|_2 + \|f_0\|_2 + \|f_1\|_2\big)\,\mathrm{d}t \\
&\quad\le c_6(T')\bigg(\sup_{t\in(0,T')}\|v(t)\|_2^{1/2}\bigg(\int_0^{T'}\|\nabla v\|_2^2\,\mathrm{d}t\bigg)^{3/4} + \sup_{t\in(0,T')}\|S(t)\|_2 \\
&\qquad\qquad + \bigg(\int_0^{T'}\big(\|w\|_4^4 + \|\nabla v\|_2^2 + \|f_1\|_2^2\big)\,\mathrm{d}t\bigg)^{1/2} + \int_0^{T'}\|f_0\|_2\,\mathrm{d}t\bigg) \\
&\quad\le c_7(T')\big(M_{T'}^{5/4} + M_{T'} + \|w\|^2_{\mathrm{L}^4(\Omega\times(0,T'))} + M_{T'}^{1/2} \\
&\qquad\qquad + \|f_1\|_{\mathrm{L}^2(\Omega\times(0,T'))} + \|f_0\|_{\mathrm{L}^1(0,T';\mathrm{L}^2(\Omega))}\big).
\end{aligned}$$

In the same way as above, additionally employing the interpolation inequality

$$\|S(t)\|_4 \le c_8\|S(t)\|_2^{1/4}\|S(t)\|_{1,2}^{3/4},$$

we can use the basis (ψ_ℓ) to show that $\partial_t S(t) \in \big(\mathrm{H}^1(\Omega)\big)'$ for a.a. $t \in (0,T)$ and

$$\begin{aligned}
&\|\partial_t S\|_{(\mathrm{H}^1(\Omega))'} \\
&\quad\le c_9\big(\|v{+}w\|_4\|\nabla S\|_2 + \|S\|_4\|\nabla(v{+}w)\|_2 + \|\partial\mathcal{P}(S)\|_2 + \|\nabla S\|_2 + \|\nabla(v{+}w)\|_2\big) \\
&\quad\le c_{10}\big(\|v\|_2^{1/4}\|v\|_2^{3/4}\|\nabla S\|_2 + \|w\|_4\|\nabla S\|_2 + \|S\|_2^{1/4}\|S\|_{1,2}^{3/4}\|\nabla(v{+}w)\|_2 \\
&\qquad\qquad + \|S\|_2 + \|\nabla S\|_2 + \|\nabla(v{+}w)\|_2\big),
\end{aligned}$$

where we used that $\partial\mathcal{P}$ is Lipschitz continuous and $\partial\mathcal{P}(S)=0$. This implies

$$\begin{aligned}
&\int_0^{T'} \|\partial_t S\|^{8/7}_{(\mathrm{H}^1(\Omega))'}\,\mathrm{d}t \\
&\quad\le c_{11}\int_0^{T'}\big(\|v\|_2^{2/7}\|v\|_2^{6/7}\|\nabla S\|_2^{8/7} + \|w\|_4^{8/7}\|\nabla S\|_2^{8/7} + \|S\|_2^{2/7}\|S\|_{1,2}^{6/7}\|\nabla(v{+}w)\|_2^{8/7} \\
&\qquad\qquad + \|S\|_2^{8/7} + \|\nabla S\|_2^{8/7} + \|\nabla(v{+}w)\|_2^{8/7}\big)\,\mathrm{d}t \\
&\quad\le c_{12}\sup_{t\in(0,T')}\|v(t)\|_2^{2/7}\bigg(\int_0^{T'}\|v\|_2^2\,\mathrm{d}t\bigg)^{3/7}\bigg(\int_0^{T'}\|\nabla S\|_2^2\,\mathrm{d}t\bigg)^{4/7} \\
&\qquad + c_{13}(T')\bigg(\int_0^{T}\|w\|_4^4\,\mathrm{d}t\bigg)^{2/7}\bigg(\int_0^{T'}\|\nabla S\|_2^2\,\mathrm{d}t\bigg)^{4/7} \\
&\qquad + c_{14}\sup_{t\in(0,T')}\|S(t)\|_2^{2/7}\bigg(\int_0^{T'}\|S\|_{1,2}^2\,\mathrm{d}t\bigg)^{3/7}\bigg(\int_0^{T'}(\|\nabla v\|_2^2 + \|\nabla w\|_2^2)\,\mathrm{d}t\bigg)^{4/7} \\
&\qquad + c_{15}(T')\sup_{t\in(0,T')}\|S(t)\|_2^{2/7} + c_{16}(T')\bigg(\int_0^{T'}\big(\|\nabla S\|_2^2 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big)\,\mathrm{d}t\bigg)^{4/7} \\
&\quad\le c_{17}(T')\big(M_{T'}^{8/7} + M_{T'}^{4/7}(\|\nabla w\|^{8/7}_{\mathrm{L}^2(\Omega\times(0,T'))} + \|w\|^{8/7}_{\mathrm{L}^4(\Omega\times(0,T'))}) \\
&\qquad\qquad + M_{T'}^{1/7} + M_{T'}^{4/7} + \|\nabla w\|^{8/7}_{\mathrm{L}^2(\Omega\times(0,T'))}\big),
\end{aligned}$$

which completes the proof. □

3.2 Existence of weak solutions to the modified system

Based on the previous preparations we establish the existence of weak solutions to (3.2).

Theorem 3.10. *Let v_0, S_0 and f be as in (3.3), let $\mathcal{P} \in \mathrm{C}^{1,1}(\mathrm{L}^2_\delta(\Omega))$ be a dissipation potential, and let w be as in Assumption 3.1. Then there exists a weak solution (v, S) to (3.2) in the sense of Definition 3.3. Additionally, this solution is weakly continuous in $\mathrm{L}^2(\Omega)$, that is,*

$$\forall\, t \in [0,T): \quad \big(v(s), S(s)\big) \rightharpoonup \big(v(t), S(t)\big) \text{ in } \mathrm{L}^2(\Omega) \text{ as } s \to t, \tag{3.29}$$

with $\big(v(0), S(0)\big) = \big(v_0, S_0\big)$, and it satisfies the energy inequalities

$$\begin{aligned} &\frac{1}{2}\|v(t)\|_2^2 + \mu\|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \\ &\qquad \le \frac{1}{2}\|v_0\|_2^2 + \int_0^t\int_\Omega \big[f_0\cdot v - f_1 : \nabla v + w\otimes(v{+}w) : \nabla v - \eta S : \nabla v\big]\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &\frac{1}{2}\|S(t)\|_2^2 + \gamma\|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \int_0^t\int_\Omega \partial\mathcal{P}(S) : S\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad \le \frac{1}{2}\|S_0\|_2^2 + \int_0^t\int_\Omega \big[-w\cdot\nabla S : S + \eta\nabla(v{+}w) : S\big]\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned} \tag{3.31}$$

for all $t \in [0,T)$. In particular, we conclude the total energy inequality

$$\begin{aligned} &\frac{1}{2}\big(\|v(t)\|^2_{\mathrm{L}^2(\Omega)} + \|S(t)\|^2_{\mathrm{L}^2(\Omega)}\big) + \mu\|\nabla v\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))} + \gamma\|\nabla S\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2(\Omega))} \\ &+ \int_0^t\int_\Omega \partial\mathcal{P}(S) : S\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{1}{2}\big(\|v_0\|^2_{\mathrm{L}^2(\Omega)} + \|S_0\|^2_{\mathrm{L}^2(\Omega)}\big) \\ &\quad + \int_0^t\int_\Omega \big[f_0\cdot v - f_1 : \nabla v + w\otimes(v{+}w) : \nabla v - w\cdot\nabla S : S + \eta\nabla w : S\big]\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned} \tag{3.32}$$

for all $t \in [0,T)$.

Proof. Let $(v_k, S_k)_{k\in\mathbb{N}}$ be the sequence of approximate solutions in $(0,T)$ from Lemma 3.7. We take an increasing sequence $(T_j) \subset (0,T)$ that converges to $T \in (0,\infty]$. Due to the uniform bounds from (3.15) and Lemma 3.9 for $T' = T_j$, combined with a classical diagonalization argument, we obtain the existence of a subsequence, which we also denote by (v_k, S_k), and a pair (v, S) with $(v,S) \in \mathrm{LH}_{T'} \times \mathrm{X}_{T'}$ for each $T' \in (0,T)$ such that

$$\begin{aligned} v_k &\rightharpoonup v && \text{in } \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)^3), \\ S_k &\rightharpoonup S && \text{in } \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)^{3\times3}), \\ v_k &\overset{*}{\rightharpoonup} v && \text{in } \mathrm{L}^\infty(0,T';\mathrm{L}^2_\sigma(\Omega)), \\ S_k &\overset{*}{\rightharpoonup} S && \text{in } \mathrm{L}^\infty(0,T';\mathrm{L}^2_\delta(\Omega)), \\ \partial_t v_k &\rightharpoonup \partial_t v && \text{in } \mathrm{L}^1(0,T';\big(\mathrm{H}^1_{0,\sigma}(\Omega)\big)'), \\ \partial_t S_k &\rightharpoonup \partial_t S && \text{in } \mathrm{L}^{8/7}(0,T';\big(\mathrm{H}^1(\Omega)^{3\times3}\big)') \end{aligned}$$

as $k \to \infty$. The Aubin–Lions lemma further implies the strong convergence

$$\begin{aligned} v_k &\to v && \text{in } \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega)^3), \\ S_k &\to S && \text{in } \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega)^{3\times3}) \end{aligned}$$

as $k \to \infty$. Let us show that (v, S) is a weak solution to (3.2), that is, that (3.6) and (3.7) are satisfied. To this end, let $\chi \in \mathrm{C}_0^\infty([0,T))$ and let $T' \in (0,T)$ such that $\operatorname{supp}\chi \subset [0,T')$. Fix $\ell \in \mathbb{N}$. Multiply (3.11) and (3.12) by $\chi(t)$, integrate over $(0,T)$ and pass to the limit $k \to \infty$ exploiting the above convergence properties. For example, in view of (3.13), for $k \geq \ell$ we have

$$\begin{aligned}\int_0^T \int_\Omega \partial_t v_k \cdot \varphi_\ell \chi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_\Omega v_k \cdot \varphi_\ell \partial_t \chi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega v_0 \cdot \varphi_\ell \chi(0)\,\mathrm{d}x \\ &\to -\int_0^T \int_\Omega v \cdot \varphi_\ell \partial_t \chi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega v_0 \cdot \varphi_\ell \chi(0)\,\mathrm{d}x.\end{aligned}$$

Employing the strong convergence of (v_k) in $\mathrm{L}^2(\Omega \times (0,T'))$, we further conclude

$$\int_0^T \int_\Omega v_k \otimes v_k : \nabla\varphi_\ell \chi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega v \otimes v : \nabla\varphi_\ell \chi \,\mathrm{d}x\,\mathrm{d}t.$$

Similarly, we can derive

$$\int_0^T \int_\Omega \partial\mathcal{P}(S_k) : \psi_\ell \chi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \partial\mathcal{P}(S) : \psi_\ell \chi \,\mathrm{d}x\,\mathrm{d}t$$

from the Lipschitz continuity of $\partial\mathcal{P}$ and the strong convergence of (S_k) in $\mathrm{L}^2(\Omega \times (0,T'))$. Convergence of the remaining terms can be shown in a similar fashion, and we conclude (3.6) and (3.7) for all Φ and Ψ of the form $\Phi(x,t) = \varphi_\ell(x)\chi(t)$ and $\Psi(x,t) = \psi_\ell(x)\chi(t)$ with $\ell \in \mathbb{N}$. Finally, an approximation argument based on Lemma 3.4 and Lemma 3.5 allows us to pass to general $\Phi \in \mathrm{C}^\infty_{0,\sigma}(\Omega \times [0,T))$ and $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega} \times [0,T))$, respectively. Consequently, (v,S) is a weak solution to (3.2).

Now let us show the energy inequalities (3.30) and (3.31). Similarly to [Gal00, Proof of Theorem 3.1], one can show that $(v_k(t), S_k(t)) \rightharpoonup (v(t), S(t))$ in $\mathrm{L}^2(\Omega)$ as $k \to \infty$ for all $t \in (0,T)$ after possibly modifying the solution (v,S) on a set of measure zero in $(0,T)$. This property and the weak convergence in $\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))$ imply

$$\frac{1}{2}\|v(t)\|_2^2 + \mu\|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \leq \liminf_{k\to\infty}\big(\frac{1}{2}\|v_k(t)\|_2^2 + \mu\|\nabla v_k\|^2_{\mathrm{L}^2(\Omega\times(0,t))}\big), \tag{3.33}$$

$$\frac{1}{2}\|S(t)\|_2^2 + \gamma\|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} \leq \liminf_{k\to\infty}\big(\frac{1}{2}\|S_k(t)\|_2^2 + \gamma\|\nabla S_k\|^2_{\mathrm{L}^2(\Omega\times(0,t))}\big). \tag{3.34}$$

Moreover, the strong convergence $S_k \to S$ in $\mathrm{L}^2(\Omega \times (0,T'))$ and the Lipschitz continuity of $\partial\mathcal{P}$ lead to

$$\lim_{k\to\infty} \int_0^t \int_\Omega \partial\mathcal{P}(S_k) : S_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega \partial\mathcal{P}(S) : S \,\mathrm{d}x\,\mathrm{d}t. \tag{3.35}$$

By construction we further have $\|v(0)\|_2 \leq \|v_0\|_2$ and $\|S(0)\|_2 \leq \|S_0\|_2$ due to Bessel's inequality, and we can directly conclude

$$\lim_{k\to\infty} \int_0^t \int_\Omega \big[f_0 \cdot v_k - f_1 : \nabla v_k\big] \,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega \big[f_0 \cdot v - f_1 : \nabla v\big] \,\mathrm{d}x\,\mathrm{d}t, \tag{3.36}$$

$$\lim_{k\to\infty} \int_0^t \int_\Omega w \otimes w : \nabla v_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega w \otimes w : \nabla v \,\mathrm{d}x\,\mathrm{d}t, \tag{3.37}$$

$$\lim_{k\to\infty} \int_0^t \int_\Omega \eta\nabla w : S_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega \eta\nabla w : S \,\mathrm{d}x\,\mathrm{d}t \tag{3.38}$$

since $f, w\otimes w, \nabla w \in \mathrm{L}^2(\Omega\times(0,T'))$. Moreover, the strong convergence of (S_k) in $\mathrm{L}^2(\Omega\times(0,T'))$ and the weak convergence of (∇v_k) in $\mathrm{L}^2(\Omega\times(0,T'))$ imply

$$\lim_{k\to\infty}\int_0^t\int_\Omega S_k : \nabla v_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega S : \nabla v \,\mathrm{d}x\,\mathrm{d}t. \tag{3.39}$$

For the remaining terms, assume for the moment that $w \in \mathrm{C}_0^\infty(\Omega\times(0,T))$. Then $(w\otimes v_k)$ converges to $w\otimes v$ strongly in $\mathrm{L}^2(\Omega\times(0,T))$. Hence we obtain

$$\lim_{k\to\infty}\int_0^t\int_\Omega w\otimes v_k : \nabla v_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega w\otimes v : \nabla v \,\mathrm{d}x\,\mathrm{d}t. \tag{3.40}$$

For general $w \in \mathrm{L}^s(0,T;\mathrm{L}^r(\Omega))$ we obtain (3.40) by approximating w by elements from $\mathrm{C}_0^\infty(\Omega\times(0,T))$ and exploiting that $\mathrm{LH}_T \hookrightarrow \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ with $1/p = 1/2 - 1/r$, $1/q = 1/2 - 1/s$, so that $2/q + 3/p = 3/2$. Observe that here we use $w \in \mathrm{C}^0(0,T;\mathrm{L}^3(\Omega))$ if $s=\infty$. An analogous argument shows

$$\lim_{k\to\infty}\int_0^t\int_\Omega w\cdot\nabla S_k : S_k \,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega w\cdot\nabla S : S \,\mathrm{d}x\,\mathrm{d}t. \tag{3.41}$$

Finally, we combine (3.33)–(3.41) with the energy equalities (3.14a) and (3.14b) to conclude the energy inequalities (3.30) and (3.31). Moreover, in the same way as for the the classical Navier–Stokes initial-value problem (see [Gal00, Lemma 2.2] for example), the weak solution can be redefined on a set of measure zero such that it is weakly continuous in the sense of (3.29). This finishes the proof of Theorem 3.10. □

Remark 3.11. From the proof of Theorem 3.10 we directly obtain

$$\begin{aligned} &\|v\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|S\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} \\ &\qquad + \|v\|^2_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \|S\|^2_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \int_0^{T'}\mathcal{P}(S)\,\mathrm{d}t \le M_{T'}, \\ &\qquad\qquad \|\partial_t v\|_{\mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega))')} \le M'_{T'}, \\ &\qquad\qquad \|\partial_t S\|_{\mathrm{L}^{8/7}(0,T';(\mathrm{H}^1(\Omega))')} \le M_{T',\mathcal{P}}, \end{aligned}$$

for each $0 < T' < T$, where $M_{T'}$, $M'_{T'}$ and $M_{T',\mathcal{P}}$ are given in Lemma 3.7 and Lemma 3.9.

4 Existence for inhomogeneous boundary data

In this section we show the existence of a solution to problem (1.1)–(1.3) in a suitable sense. Throughout this section, we let $0 < T \le \infty$ and suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\mathrm{C}^{1,1}$-boundary. We further let $\mathcal{P} : \mathrm{L}^2_\delta(\Omega) \to [0,\infty]$ be a dissipation potential as discussed in Section 2. It will be smooth in Section 4.1 and generally nonsmooth in Section 4.2.

For the data we assume that

$$\begin{aligned} &V_0 \in \mathrm{L}^2_\sigma(\Omega), \quad S_0 \in \mathrm{L}^2_\delta(\Omega), \quad F = F_0 + \operatorname{div} F_1, \\ &F_0 \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^2(\Omega)^3), \qquad F_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\Omega)^{3\times 3}). \end{aligned} \tag{4.1}$$

The regularity of g will be specified below.

We will first consider the case of a sufficiently smooth potential $\mathcal{P}$ such that $\partial\mathcal{P}$ is single-valued. Then equation $(1.1)_3$ is a proper differential equation and we can use the notion of weak solutions as in the previous section. Subsequently, we treat the case of a general dissipation potential $\mathcal{P}$, where $\partial\mathcal{P}$ may be multi-valued and $(1.1)_3$ has to be understood as a differential inclusion. Then the notion of weak solutions is no longer available, and we define a generalized solution by means of a variational inequality.

4.1 Weak solutions for smooth potentials

Here we consider the original system (1.1)–(1.3) in the case of a sufficiently smooth potential. More precisely, as in Section 3 we assume that $\mathcal{P}$ is a dissipation potential with

$$\mathcal{P} \in \mathrm{C}^{1,1}(\mathrm{L}^2_\delta(\Omega)).$$

We consider weak solutions to (1.1)–(1.3) in the following sense.

Definition 4.1. We call a couple (V, S) a *weak solution to* (1.1)–(1.3) if

$$V \in \mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega)^3) \cap \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)^3),$$
$$S \in \mathrm{L}^\infty(0,T';\mathrm{L}^2_\delta(\Omega)) \cap \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)^{3\times3}),$$

for all $0 < T' < T$, if $\operatorname{div} V = 0$ and $V|_{\partial\Omega\times(0,T)} = g$, and if the identities

$$\begin{aligned}\int_0^T \int_\Omega \big[-V\cdot\partial_t\Phi - V\otimes V : \nabla\Phi + \eta S : \nabla\Phi + \mu\nabla V : \nabla\Phi\big]\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega F_0\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega F_1 : \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega V_0\cdot\Phi(\cdot,0)\,\mathrm{d}x,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\int_0^T\int_\Omega \big[-S : \partial_t\Psi + V\cdot\nabla S : \Psi + SW(V) : \Psi - W(V)S : \Psi \\ + \partial\mathcal{P}(S) : \Psi + \gamma\nabla S : \nabla\Psi - \eta\nabla V : \Psi\big]\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega S_0 : \Psi(\cdot,0)\,\mathrm{d}x\end{aligned} \tag{4.3}$$

hold for all $\Phi \in \mathrm{C}^\infty_{0,\sigma}(\Omega\times[0,T))$ and $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T))$.

As explained above, we can obtain a weak solution to the problem (1.1)–(1.3) with inhomogeneous Dirichlet boundary values as the sum of a solution to the Stokes initial-value problem (3.1) and a solution to the perturbed problem (3.2) with homogeneous boundary conditions. Since we have shown existence of a weak solution to (3.2) in Theorem 3.10, it remains to address the existence of solutions w to the Stokes initial-value problem (3.1). Observe that in the present situation the forcing term $\widetilde{F}$ and the initial value w_0 in (3.1) are not prescribed by the original problem, whence we have some freedom in their choice. For example, we can simply consider data $\widetilde{F} = w_0 = 0$ and use existing theory for the Stokes initial-value problem with inhomogeneous Dirichlet data (see [Gru01, FGH02, Ray07] for example) to obtain a suitable extension w satisfying Assumption 3.1 in case (a). Proceeding like this for the cases (b) and (c) would require smallness of g. In the following we focus on case (c), which is more general than (b), and show that smallness of g is not necessary if we exploit the freedom we have in the choice of $\widetilde{F}$ and w_0. For this purpose, we use the following lemma.

Lemma 4.2. *Let Ω be a bounded domain with connected $\mathrm{C}^{1,1}$-boundary, $T \in (0, \infty]$ and*

$$g \in \mathrm{L}^\infty(0, T; \mathrm{H}^{1/2}(\partial\Omega)^3), \quad \partial_t g \in \mathrm{L}^\infty(0, T; \mathrm{H}^{-1/2}(\partial\Omega)^3) \tag{4.4a}$$

with

$$\int_{\partial\Omega} g(t) \cdot \mathrm{n} = 0 \text{ for a.a. } t \in (0, T). \tag{4.4b}$$

Then, for each $\alpha > 0$ there exists a function

$$w_\alpha \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega)^3), \quad \partial_t w_\alpha \in \mathrm{L}^\infty(0, T; \mathrm{H}^{-1}(\Omega)^3) \tag{4.5}$$

with $w_\alpha = g$ on $\partial\Omega \times (0, T)$ and $\operatorname{div} w_\alpha = 0$ in $\Omega \times (0, T)$ such that

$$\forall\, v_1, v_2 \in \mathrm{H}_0^1(\Omega): \quad \left| \int_\Omega w_\alpha(t) \otimes v_1 : \nabla v_2 \,\mathrm{d}x \right| \leq \alpha \|\nabla v_1\|_2 \|\nabla v_2\|_2$$

and

$$\begin{aligned} \|w_\alpha(t)\|_{\mathrm{H}^1(\Omega)} &\leq C_1 \|g(t)\|_{\mathrm{H}^{1/2}(\partial\Omega)} \quad \text{for a.a. } t \in (0, T), \\ \|\partial_t w_\alpha\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))} &\leq C_1 \|\partial_t g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{-1/2}(\partial\Omega))} \end{aligned} \tag{4.6}$$

for some constant $C_1 = C_1(\Omega, \alpha) > 0$.

Proof. In [FKS11a, Proposition 5.4] the statement was shown with $\alpha = 1/4$. An adaption of the proof for arbitrary $\alpha > 0$ is straightforward. □

In order to ensure (3.5), it is sufficient to assume smallness of $g \cdot \mathrm{n}$ in a suitable norm.

Lemma 4.3. *In the situation of Lemma 4.2 it holds*

$$\forall\, S \in \mathrm{H}^1(\Omega)^{3\times3}: \quad \left| \int_\Omega w_\alpha(t) \cdot \nabla S : S \,\mathrm{d}x \right| \leq C_2 \|g \cdot \mathrm{n}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\partial\Omega))} \|S\|_{1,2}^2$$

for some constant $C_2 = C_2(\Omega) > 0$.

Proof. We have

$$\int_\Omega w_\alpha(t) \cdot \nabla S : S \,\mathrm{d}x = \frac{1}{2} \int_\Omega w_\alpha(t) \cdot \nabla |S|^2 \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} g(t) \cdot \mathrm{n} |S|^2 \,\mathrm{d}\sigma.$$

Now Hölder's inequality and Sobolev embeddings imply

$$\left| \int_\Omega w_\alpha(t) \cdot \nabla S : S \,\mathrm{d}x \right| \leq c_0 \|g(t) \cdot \mathrm{n}\|_2 \|S\|_{\mathrm{L}^4(\partial\Omega)}^2 \leq c_1 \|g(t) \cdot \mathrm{n}\|_2 \|S\|_{\mathrm{H}^{1/2}(\partial\Omega)}^2.$$

The statement now follows from a standard trace inequality. □

Note that, from a physical point of view, only the case $g \cdot \mathrm{n} = 0$ seems relevant in combination with the Neumann boundary condition $\mathrm{n} \cdot \nabla S = 0$. Lemma 4.3 implies that condition (3.5) is satisfied automatically in this case. Combining this observation with Theorem 3.10 and Lemma 4.2, we now show existence of a weak solution to the original problem (1.1)–(1.3).

Theorem 4.4. *Let V_0, S_0 and F be as in (4.1), let $\mathcal{P} \in \mathrm{C}^{1,1}(\mathrm{L}^2_\delta(\Omega))$ be a dissipation potential, and let g satisfy (4.4a) and $g \cdot \mathrm{n} = 0$. Then there exists a weak solution $(V,S) = (v{+}w, S)$ to (1.1)–(1.3) in the sense of Definition 4.1. Here*

$$w \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)^3), \quad \partial_t w \in \mathrm{L}^\infty(0,T;\mathrm{H}^{-1}(\Omega)^3)$$

and F and V_0 decompose as $F = f + \widetilde{F} = f + \operatorname{div} \widetilde{F}_1$ and $V_0 = v_0 + w_0$ such that (v,S) is the weak solution to (3.2) from Theorem 3.10, and

$$\begin{aligned} \|w\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t w\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{-1}(\Omega))} + \|\widetilde{F}_1\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|w_0\|_{\mathrm{L}^2(\Omega)} \\ \leq C_3\big(\|g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1/2}(\partial\Omega))} + \|\partial_t g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{-1/2}(\partial\Omega))}\big). \end{aligned} \tag{4.7}$$

Additionally, this solution is weakly continuous in $\mathrm{L}^2(\Omega)$, that is,

$$\forall\, t \in [0,T): \quad \big(V(s), S(s)\big) \rightharpoonup \big(V(t), S(t)\big) \text{ in } \mathrm{L}^2(\Omega) \text{ as } s \to t \tag{4.8}$$

with $\big(V(0), S(0)\big) = \big(V_0, S_0\big)$. Moreover, for all $t \in (0,T)$ we have

$$\begin{aligned} \frac{1}{2}\|S(t)\|_2^2 + \gamma \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,t))} + \int_0^t \int_\Omega \partial\mathcal{P}(S) : S \,\mathrm{d}x\,\mathrm{d}\tau \\ \leq \frac{1}{2}\|S_0\|_2^2 + \int_0^t \int_\Omega \eta \nabla V : S \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{4.9}$$

For all $0 < T' < T$ there exists a constant $M_{T'} > 0$, which is independent of $\mathcal{P}$, such that

$$\begin{aligned} \|V\|_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|V\|_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \|\partial_t V\|_{\mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega))')} \\ + \|S\|_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|S\|_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \int_0^{T'} \mathcal{P}(S)\,\mathrm{d}t \leq M_{T'}. \end{aligned} \tag{4.10}$$

Proof. Let $w = w_\alpha$ from Lemma 4.2 with $\alpha = \mu/2$. Then the Aubin–Lions lemma implies $w \in \mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))$, and in virtue of $g \cdot \mathrm{n} = 0$ and Lemma 4.3, we see that Assumption 3.1 (c) is satisfied. From Hölder's inequality and Sobolev embeddings we further conclude the remaining properties of Assumption 3.1. Since $w \in \mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))$, we can define $w_0 := w(\cdot,0) \in \mathrm{L}^2(\Omega)$. Moreover, we set $\widetilde{F} := \partial_t w - \mu\Delta w$. Since every element of $\mathrm{H}^{-1}(\Omega)^n$ can be represented as the divergence of a tensor field from $\mathrm{L}^2(\Omega)^{n\times n}$ (see [Soh01, Lemma 1.6.1] for example), we obtain $\widetilde{F} = \operatorname{div} \widetilde{F}_1$ for some $\widetilde{F}_1 \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^{n\times n})$. We further conclude (4.7) from (4.6). Now we set $f_0 := F_0$, $f_1 := F_1 - \widetilde{F}_1$, $v_0 := V_0 - w_0$, and let (v,S) be the weak solution to (3.2) from Theorem 3.10. Since for all $\Phi \in \mathrm{C}_0^\infty(\Omega \times [0,T))^3$ we have

$$\int_0^T \int_\Omega \big[-w \cdot \partial_t \Phi + \mu \nabla w : \nabla \Phi\big] \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega \widetilde{F}_1 : \nabla\Phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega w_0 \cdot \Phi(\cdot,0)\,\mathrm{d}x,$$

the pair $(V,S) := (v{+}w, S)$ is a weak solution to (1.1)–(1.3) in the sense of Definition 4.1. Finally, (4.8) follows from (3.29) and $w \in \mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))$.

Similarly to the proof of Lemma 4.3, we derive

$$\int_\Omega w_\delta(t) \cdot \nabla S : S \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} g(t) \cdot \mathrm{n}|S|^2 \,\mathrm{d}\sigma = 0$$

since $g \cdot \mathrm{n} = 0$. With this identity, (4.9) directly follows from (3.31). Moreover, we have

$$\begin{aligned}\|w\|_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|w\|_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \|\partial_t w\|_{\mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega))')} \\ \leq c_0(T')\big(\|w\|_{\mathrm{L}^\infty(0,T';\mathrm{H}^1(\Omega))} + \|\partial_t w\|_{\mathrm{L}^\infty(0,T';\mathrm{H}^{-1}(\Omega))}\big) \\ \leq c_1(T')\big(\|g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1/2}(\partial\Omega))} + \|\partial_t g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{-1/2}(\partial\Omega))}\big).\end{aligned}$$

In view of Remark 3.11, this shows (4.10). □

The estimates (4.9) and (4.10) will be needed for passing from smooth Moreau envelopes $\mathcal{P}_\varepsilon$ to nonsmooth potentials $\mathcal{P}$.

In the proof of Theorem 4.4 we chose w in such a way that the nonlinear terms in the total energy inequality (3.32) can be estimated as in (3.4) and (3.5). This enables us to obtain global solutions (V, S) such that the sum of kinetic energy and stored elastic energy

$$\int_\Omega \frac{1}{2}|V(x,t)|^2\,\mathrm{d}x + \int_\Omega \frac{1}{2}|S(x,t)|^2\,\mathrm{d}x$$

remains bounded as $t \to \infty$.

Corollary 4.5. *In the situation of Theorem 4.4 there exists a constant C_4 independent of the data and $T \in (0,\infty]$ such that*

$$\begin{aligned}&\|V\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|\nabla V\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} + \|S\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} \\ &+ \int_0^{T'} \mathcal{P}(S)\,\mathrm{d}t \leq C_4\big(\|V_0\|^2_{\mathrm{L}^2(\Omega)} + \|S_0\|^2_{\mathrm{L}^2(\Omega)} + \|F_0\|^2_{\mathrm{L}^1(0,T';\mathrm{L}^2(\Omega))} + \|F_1\|^2_{\mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))} \\ &\qquad + \|g\|^2_{\mathrm{L}^\infty(0,T';\mathrm{H}^{1/2}(\partial\Omega))} + \|\partial_t g\|^2_{\mathrm{L}^\infty(0,T';\mathrm{H}^{-1/2}(\partial\Omega))} + \|g\|^2_{\mathrm{L}^1(0,T';\mathrm{H}^{1/2}(\partial\Omega))} \\ &\qquad + \|g\|^4_{\mathrm{L}^\infty(0,T';\mathrm{H}^{1/2}(\partial\Omega))} + \|g\|^4_{\mathrm{L}^1(0,T';\mathrm{H}^{1/2}(\partial\Omega))}\big).\end{aligned}$$

for all $0 < T' < T$. In particular, if $T = \infty$ and

$$\begin{aligned}&F_0 \in \mathrm{L}^1(0,\infty;\mathrm{L}^2(\Omega)), && F_1 \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega)),\\ &g \in \mathrm{L}^\infty(0,\infty;\mathrm{H}^{1/2}(\partial\Omega)) \cap \mathrm{L}^1(0,\infty;\mathrm{H}^{1/2}(\partial\Omega)), && \partial_t g \in \mathrm{L}^\infty(0,\infty;\mathrm{H}^{-1/2}(\partial\Omega)),\end{aligned}$$

then $(V,S) \in \mathrm{LH}_T \times \mathrm{X}_T$ for $T = \infty$, so that the energy remains bounded as $t \to \infty$.

Proof. The pair (v, S) satisfies the energy inequality (3.32), and w is constructed in such a way that Assumption 3.1 (c) is satisfied. Therefore, we can proceed as in the proof of Theorem 3.10 to derive estimates (3.19)–(3.22), (3.26), (3.27) in the present situation. Combining these with the energy inequality (3.32), in virtue of the identities $f_0 = F_0$ and $f_1 = F_1 - \widetilde{F}_1$, we conclude

$$\begin{aligned}&\|v\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|\nabla v\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} + \|S\|^2_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} \\ &\quad + \int_0^{T'} \mathcal{P}(S)\,\mathrm{d}t \leq c_0\big(\|v_0\|_2^2 + \|S_0\|_2^2 + \|F_0\|^2_{\mathrm{L}^1(0,T';\mathrm{L}^2(\Omega))} + \|F_1\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} \\ &\qquad + \|\widetilde{F}_1\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} + \|g\|^2_{\mathrm{L}^1(0,T';\mathrm{H}^{1/2}(\partial\Omega))} + \|g\|^4_{\mathrm{L}^4(0,T';\mathrm{H}^{1/2}(\partial\Omega))}\big)\end{aligned}$$

by using Sobolev embeddings. Applying an interpolation argument to the last term and using (4.7), we can further estimate the right-hand side by

$$\begin{aligned}&c_1\big(\|v_0\|_2^2 + \|S_0\|_2^2 + \|F_0\|^2_{\mathrm{L}^1(0,T';\mathrm{L}^2(\Omega))} + \|F_1\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} \\ &\quad + \|g\|^2_{\mathrm{L}^\infty(0,T';\mathrm{H}^{1/2}(\partial\Omega))} + \|\partial_t g\|^2_{\mathrm{L}^\infty(0,T';\mathrm{H}^{-1/2}(\partial\Omega))} + \|g\|^2_{\mathrm{L}^1(0,T';\mathrm{H}^{1/2}(\partial\Omega))} \\ &\quad + \|g\|^4_{\mathrm{L}^1(0,T';\mathrm{H}^{1/2}(\partial\Omega))} + \|g\|^4_{\mathrm{L}^\infty(0,T';\mathrm{H}^{1/2}(\partial\Omega))}\big).\end{aligned}$$

Since $V = v + w$, a combination of the resulting estimate with (4.7) completes the proof. □

4.2 Generalized solutions for nonsmooth potentials

In this subsection, we extend the previous existence result to the case of a nonsmooth convex potential $\mathcal{P}$ possibly featuring an indicator function. More precisely, we will assume that $\mathcal{P}\colon \mathrm{L}^2_\delta(\Omega) \to [0,\infty]$ is a dissipation potential, that is, $\mathcal{P}$ is convex and lower semicontinuous with $\mathcal{P}(0) = 0$. Now it is explicitly allowed that $\mathcal{P}$ is nonsmooth and takes the value $+\infty$. An admissible choice of $\mathcal{P}$ is

$$\mathcal{P}(S) = \iota_K(S) + \tfrac{a}{2}\|S\|^2_{\mathrm{L}^2_\delta(\Omega)} \quad \text{with } a > 0 \tag{4.11}$$

and a nonempty, convex and closed subset $K \subseteq \mathrm{L}^2_\delta(\Omega)$. Here, ι_K denotes the indicator function (from convex analysis) associated with K, that is

$$\iota_K(S) = \begin{cases} 0, & \text{if } S \in K, \\ +\infty, & \text{if } S \notin K. \end{cases} \tag{4.12}$$

With regard to geodynamics applications that include plasticity, we have in mind

$$K = \{S' \in \mathrm{L}^2_\delta(\Omega) : |S'| \le \sigma_{\text{yield}} \text{ a.e. in } \Omega\}, \tag{4.13}$$

which is the elastic domain that is determined by a given yield stress $\sigma_{\text{yield}} > 0$, cf. [MDM02, Ger07]. Clearly, this set enjoys the required properties.

First, we have to formulate an appropriate notion of solution. The following definition adapts the weak solution concept in [Rou13, Chapter 10] involving an evolutionary variational inequality. Roughly speaking, (and assuming sufficient smoothness for the moment) we test (4.3) with $\Psi(t) = \tilde{S}(t) - S(t)$ which allows us to use the estimate $\int_\Omega \partial\mathcal{P}(S(t)) : (\tilde{S}(t) - S(t))\,\mathrm{d}x \le \mathcal{P}(\tilde{S}) - \mathcal{P}(S)$ leading to a variational inequality that avoids the multi-valued function $\partial\mathcal{P}(S)$. The nonlinear term involving the Zaremba–Jaumann derivative can be modified via the identity (1.4) to obtain

$$\begin{aligned}\int_\Omega \overset{\triangledown}{S} : (\tilde{S}-S)\,\mathrm{d}x &= \int_\Omega \Big(\partial_t S : (\tilde{S}-S) + (\overset{\triangledown}{S}-\partial_t S) : \tilde{S}\Big)\,\mathrm{d}x \\ &= \int_\Omega \Big(\partial_t \tilde{S} : (\tilde{S}-S) + (\overset{\triangledown}{S}-\partial_t S) : \tilde{S}\Big)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\tilde{S}-S|^2\,\mathrm{d}x,\end{aligned} \tag{4.14}$$

where $\overset{\triangledown}{S} - \partial_t S = V\cdot\nabla S + SW - WS$. Observe that due to the lack of control of the set $\partial\mathcal{P}(S)$ an Aubin–Lions type compactness result at the level of S will no longer be available. At the same time, owing to the nonlinearities in $\overset{\triangledown}{S}$, a priori estimates implying higher (time) regularity would require a regularization significantly stronger than the linear Laplacian, which we prefer to avoid. In our concept of generalized solutions we therefore drop the boundary term at the final time arising from time integration of (4.14). We can still recover a weak formulation for generalized solutions that are sufficiently regular, provided the dissipation potential enjoys some (mild) approximation property. For details concerning the compatibility of the notions of weak solutions versus generalized solutions, we refer to Lemma 4.10 below. Furthermore, our evolutionary variational inequality encodes the natural energy dissipation inequality for S (see Prop. 4.8).

Due to the nonlinear terms in the Zaremba–Jaumann derivative, in the rigorous formulation of the variational inequality we can only admit test functions enjoying some extra integrability as compared to the typical functional setting for parabolic problems (as used for instance in [Rou13]). Here, we will consider test functions in the space

$$Z_{T'} := \mathrm{H}^1(0,T';\mathrm{L}^2_\delta(\Omega)) \cap \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)) \cap \mathrm{L}^5(0,T';\mathrm{L}^5(\Omega)). \tag{4.15}$$

Definition 4.6 (Generalized solution)**.** Let $T \in (0, \infty]$ and assume, as before, that V_0, S_0 and F are as in (4.1). Let $\mathcal{P}$ be a (general) dissipation potential, and let g satisfy (4.4a) with $g \cdot \mathrm{n} = 0$. We call a couple (V, S) a generalized solution of system (1.1) in $\Omega \times (0, T)$ with boundary conditions (1.2) and initial data (1.3) if for all $T' \in (0, T)$ the following holds: $(V, S) \in \mathrm{LH}_{T'} \times \mathrm{X}_{T'}$ satisfies the weak form (4.2) of the Navier–Stokes equations for the velocity field, $V|_{\partial\Omega\times(0,T)} = g$, and for all $\tilde{S} \in Z_{T'}$

$$
\begin{aligned}
&\int_0^{T'} \int_\Omega \partial_t \tilde{S} : (\tilde{S} - S) + \gamma \nabla S : \nabla(\tilde{S} - S) \,\mathrm{d}x\mathrm{d}t + \int_0^{T'} \Big(\mathcal{P}(\tilde{S}) - \mathcal{P}(S) \Big) \,\mathrm{d}t \\
&+ \int_0^{T'} \int_\Omega V \cdot \nabla S : \tilde{S} + (SW(V) - W(V)S) : \tilde{S} - \eta D(V) : (\tilde{S} - S) \,\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq -\tfrac{1}{2} \|\tilde{S}(0) - S_0\|_2^2.
\end{aligned}
\tag{4.16}
$$

It is easy to see that the terms in (4.16) are well-defined. For the integrals involving the nonlinear terms of the Zaremba–Jaumann rate, this follows from the Sobolev embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^6(\Omega)$, the interpolation

$$
\mathrm{L}^\infty(0, T'; \mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0, T'; \mathrm{L}^6(\Omega)) \hookrightarrow \mathrm{L}^{\frac{10}{3}}(0, T'; \mathrm{L}^{\frac{10}{3}}(\Omega)),
\tag{4.17}
$$

and the generalized Hölder inequality with inverse exponents $\frac{3}{10} + \frac{1}{2} + \frac{1}{5} = 1$.

Remark 4.7*.* In the formulation (4.16) it is crucial that in the integral involving the convective part, only the term $V \cdot \nabla S : \tilde{S}$ occurs, and not $V \cdot \nabla S : (\tilde{S} - S)$, since under the natural regularity hypotheses of S in Def. 4.6, integrability of the term $V \cdot \nabla S : S$ is not ensured.

It is worth noting that, despite the absence of the boundary term at time T' in ineq. (4.16), generalized solutions in the sense of Def. 4.6 obey the natural energy dissipation inequality for S.

Proposition 4.8 (Energy inequality)**.** *Any generalized solution (V, S) in the sense of Definition 4.6 satisfies the partial energy dissipation inequality*

$$
\frac{1}{2}\|S(T')\|_2^2 + \gamma \|\nabla S\|^2_{\mathrm{L}^2(\Omega\times(0,T'))} + \int_0^{T'} \mathcal{P}(S) \,\mathrm{d}\tau \leq \frac{1}{2}\|S_0\|_2^2 + \int_0^{T'} \int_\Omega \eta D(V) : S \,\mathrm{d}x \,\mathrm{d}\tau
\tag{4.18}
$$

for almost all $T' \in (0, T)$.

Proof. First observe that choosing $\tilde{S} \equiv 0$ in (4.16) shows that $\int_0^{T'} \mathcal{P}(S) \,\mathrm{d}\tau < \infty$. Let us further note that since $S \in \mathrm{L}^\infty(0, T'; \mathrm{L}^2(\Omega))$, almost every $T' \in (0, T)$ is a left Lebesgue point of $t \mapsto S(t) \in \mathrm{L}^2(\Omega)$.

Extend now S by zero for $t < 0$ and consider for $\kappa > 0$

$$
S_\kappa(t) = \kappa^{-1} \int_{t-\kappa}^t S(\tau) \,\mathrm{d}\tau.
$$

Further let $\eta \in \mathrm{C}^\infty(\mathbb{R}; [0, 1])$ be nondecreasing, $\eta(t) = 0$ for $t \leq -1$ and $\eta(t) = 1$ for $t \geq 0$, and define $\eta_\delta(t) := \eta_\delta^{(T')}(t) := \eta((t - T')/\delta)$. We then choose in (4.16) the test function $\tilde{S} := \tilde{S}_{\kappa,\delta} := \eta_\delta S_\kappa \in Z_{T'}$, where $\delta \in (0, \delta_*]$ and $\kappa \in (0, \kappa_*]$ are chosen sufficiently small and, in particular, such that $\tilde{S}_{\kappa,\delta}(0) = 0$.

Since $\mathcal{P}$ is convex with $\mathcal{P}(0) = 0$ and $0 \leq \eta_\delta \leq 1$, we can estimate using Jensen's inequality

$$
\int_0^{T'} \mathcal{P}(\tilde{S}_{\kappa,\delta}) \,\mathrm{d}\tau \leq \kappa^{-1} \int_{T'-\delta}^{T'} \eta_\delta(t) \int_{t-\kappa}^t \mathcal{P}(S(\tau)) \,\mathrm{d}\tau \,\mathrm{d}t \leq \kappa^{-1} \int_0^{T'} \mathcal{P}(S) \,\mathrm{d}\tau \cdot \delta,
$$

where we also used the nonnegativity of $\mathcal{P}$. Since $\int_0^{T'}\mathcal{P}(S)\,\mathrm{d}\tau < \infty$, the last line implies that $\lim_{\delta\to 0}\int_0^{T'}\mathcal{P}(\tilde S_{\kappa,\delta})\,\mathrm{d}\tau = 0$ for any $\kappa\in(0,\kappa_*]$.

Let us next turn to the integral involving the time derivative. Using the fact that $\eta_\delta(T')=1$, we find

$$\begin{aligned}\int_0^{T'}\int_\Omega \partial_t\tilde S:(\tilde S - S)\,\mathrm{d}x\,\mathrm{d}t = \frac12\|S_\kappa(T')\|_2^2 - \int_0^{T'}\eta_\delta'(t)\int_\Omega S_\kappa : S\,\mathrm{d}x\,\mathrm{d}t\\ - \int_{T'-\delta}^{T'}\eta_\delta(t)\int_\Omega \partial_t S_\kappa : S\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $S\in\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))$, we easily see that the term in the last line vanishes as $\delta\to 0$. Furthermore, we have the following convergence results, valid for almost all $T'\in(0,T)$:

$$\begin{aligned}&\lim_{\delta\to 0}\int_0^{T'}\eta_\delta'(t)\int_\Omega S_\kappa : S\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega S_\kappa(T'):S(T')\,\mathrm{d}x,\\ &\lim_{\kappa\to 0}\int_\Omega S_\kappa(T'):S(T')\,\mathrm{d}x = \|S(T')\|_2^2,\\ &\lim_{\kappa\to 0}\frac12\|S_\kappa(T')\|_2^2 = \frac12\|S(T')\|_2^2.\end{aligned}$$

Thus, for almost all T' we obtain

$$\lim_{\kappa\to 0}\lim_{\delta\to 0}\int_0^{T'}\int_\Omega \partial_t\tilde S_{\kappa,\delta}:(\tilde S_{\kappa,\delta}-S)\,\mathrm{d}x\,\mathrm{d}t = -\frac12\|S(T')\|_2^2.$$

All remaining integrals in (4.16) involving $\tilde S$ converge to zero as $\delta\to 0$, as long as κ is positive. Thus, sending first $\delta\to 0$ in (4.16) (with $\tilde S = \tilde S_{\kappa,\delta}$), and taking subsequently the limit $\kappa\to 0$, we arrive at (4.18). □

The main purpose of this subsection is to establish the following existence result.

Theorem 4.9 (Existence of generalized solutions to (1.1)–(1.3)). *Let $T\in(0,\infty]$, let Ω be a bounded domain in $\mathbb{R}^3$ with $\mathrm{C}^{1,1}$-boundary, and let $\mathcal{P}$ be a dissipation potential on $\mathrm{L}^2_\delta(\Omega)$. Let V_0, S_0 and F as in (4.1), and let g satisfy (4.4a) and $g\cdot\mathrm{n}=0$. Then there exists a generalized solution (V,S) of the system (1.1) in $\Omega\times(0,T)$ with boundary conditions (1.2) and initial data (1.3) in the sense of Definition 4.6.*

Moreover, this solution satisfies the following energy-dissipation inequality, where $v=V-w$ with w from Theorem 4.4. For a.a. $t\in(0,T)$ we have

$$\begin{aligned}&\int_\Omega \frac12|v(t)|^2+\frac12|S(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega\Big[\mu|\nabla v|^2+\gamma|\nabla S|^2\Big]\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\mathcal{P}(S)\,\mathrm{d}\tau\\ &\le \int_\Omega\frac12|V_0-w(0)|^2+\frac12|S_0|^2\,\mathrm{d}x\\ &\quad+\int_0^t\int_\Omega\Big[F_0\cdot v-(F_1-\widetilde F_1):\nabla v + w\otimes(v+w):\nabla v+\eta D(w):S\Big]\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{4.19}$$

where $\widetilde F = \operatorname{div}\widetilde F_1$ denotes the auxiliary forcing in the Stokes problem for w (cf. (3.1)).

Before turning to the proof of Theorem 4.9, we show that the weak solution from Theorem 4.4 obtained in the case of a smooth potential satisfies a variational inequality and hence is a generalized solution in the sense of Definition 4.6. Moreover, we provide sufficient regularity conditions for (V, S) that allow us to conclude that generalized solutions are already weak solutions in the sense of Definition 4.1. For this purpose, we need an approximation property for the induced potential $\mathscr{P}$ acting on Bochner functions

$$\mathscr{P}(\tilde{S}) := \int_0^{T'} \mathcal{P}(\tilde{S}(t))\,\mathrm{d}t, \qquad \tilde{S} \in \mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega)). \tag{4.20}$$

The approximation condition states:

$$\begin{aligned} &\forall\, \tilde{S} \in \mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega))\ \exists\, (\tilde{S}_n)_{n\in\mathbb{N}} \subset Z_{T'} : \\ &\qquad\qquad \tilde{S}_n \rightharpoonup \tilde{S} \ \text{ in } \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega)) \text{ and } \mathscr{P}(\tilde{S}_n) \to \mathscr{P}(\tilde{S}). \end{aligned} \tag{4.21}$$

This property is certainly satisfied for the plasticity potential $\mathcal{P}$ defined in (4.11)–(4.13).

Lemma 4.10 (Weak versus generalized solutions)**.**
(A) Assume that $\mathcal{P} \in \mathrm{C}^{1,1}(\mathrm{L}^2_\delta(\Omega))$. If (V, S) is a weak solution in the sense of Definition 4.1 that additionally satisfies the partial energy dissipation inequality (4.9), then it is also a generalized solution in the sense of Definition 4.6.
(B) If (V, S) is a generalized solution in the sense of Definition 4.6 with the additional regularity

$$S \in \mathrm{H}^1(0,T';\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T';\mathrm{H}^2(\Omega)) \cap \mathrm{L}^\infty(0,T';\mathrm{L}^\infty(\Omega)) \tag{4.22}$$

for all $T' < T$, and if $\mathcal{P}$ satisfies the approximation property (4.21), then it is also a weak solution in the sense of Definition 4.1, where (4.3) is replaced by

$$\begin{aligned} \int_0^T \int_\Omega \big[-S : \partial_t \Psi + V\cdot\nabla S : \Psi + \big(SW(V) - W(V)S\big) : \Psi \\ + \beta : \Psi + \gamma \nabla S : \nabla \Psi - \eta D(V) : \Psi \big]\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega S_0 : \Psi(0)\,\mathrm{d}x \end{aligned} \tag{4.23}$$

for all $\Psi \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T))$, where $\beta \in \mathrm{L}^2(0,T;\mathrm{L}^2_\delta(\Omega))$ with $\beta(t) \in \partial\mathcal{P}(S(t))$ for a.a. $t \in (0,T)$.

Proof. Part (A): Let (V, S) be a weak solution. We have to show that the evolutionary variational inequality (4.16) holds. Using a standard approximation argument and recalling the weak continuity $S \in \mathrm{C}_\mathrm{w}([0,T);\mathrm{L}^2_\delta(\Omega))$, one can show that the weak equation (4.3) for S on $(0,T)$ implies for all $T' \in (0,T)$ and all $\tilde{S} \in \mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T'])$ the identity

$$\begin{aligned} \int_0^{T'} \int_\Omega \big[-S : \partial_t \tilde{S} + V\cdot\nabla S : \tilde{S} + (SW(V) - W(V)S) : \tilde{S} \\ + \partial\mathcal{P}(S) : \tilde{S} + \gamma \nabla S : \nabla\tilde{S} - \eta D(V) : \tilde{S} \big]\,\mathrm{d}x\,\mathrm{d}t \\ = \int_\Omega S_0 : \tilde{S}(0)\,\mathrm{d}x - \int_\Omega S(T') : \tilde{S}(T')\,\mathrm{d}x. \end{aligned}$$

Subtracting inequality (4.9) we find, upon rearranging terms,

$$\begin{aligned} \int_0^{T'} \int_\Omega [-S : \partial_t \tilde{S} + \partial\mathcal{P}(S) : (\tilde{S}-S) + \gamma\nabla S : \nabla(\tilde{S}-S) + V\cdot\nabla S : \tilde{S} \\ + (SW(V) - W(V)S) : \tilde{S} - \eta D(V) : (\tilde{S}-S)]\,\mathrm{d}x\,\mathrm{d}t \\ \geq -\tfrac{1}{2}\|S_0\|_2^2 + \int_\Omega S_0 : \tilde{S}(0)\,\mathrm{d}x + \tfrac{1}{2}\|S(T')\|_2^2 - \int_\Omega S(T') : \tilde{S}(T')\,\mathrm{d}x. \end{aligned}$$

By the density of $\mathrm{C}^\infty_{0,\delta}(\overline{\Omega}\times[0,T'])$ in $Z_{T'}$, the last inequality continues to hold for all $\tilde S\in Z_{T'}$. The assertion is now obtained by adding $\int_0^{T'}\int_\Omega \partial_t\tilde S:\tilde S\,\mathrm{d}x\mathrm{d}t=\frac12\|\tilde S(T')\|_2^2-\frac12\|\tilde S(0)\|_2^2$, and using the fact that $\frac12\|\tilde S(T')-S(T')\|_2^2\geq 0$ as well as the inequality $\mathcal P(\tilde S(t))-\mathcal P(S(t))\geq\int_\Omega\partial\mathcal P(S(t)):(\tilde S(t){-}S(t))\,\mathrm{d}x$.

Part (B): For a generalized solution (V,S) with the smoothness as in (4.22) we have

$$\beta:=-\overset{\triangledown}{S}+\gamma\Delta S+\eta D(V)\quad\in\mathrm{L}^2([0,T'];\mathrm{L}^2_\delta(\Omega)). \tag{4.24}$$

We further note that, as a consequence of (4.16), $\mathscr P(S)=\int_0^{T'}\mathcal P(S)\,\mathrm{d}t<\infty$.

Using the Zaremba–Jaumann identity (1.4) and the definition of β we find

$$\begin{aligned}\int_0^{T'}\!\!\int_\Omega\Big(V\cdot\nabla S:\tilde S+\big(SW(V)-W(V)S\big):\tilde S\,\mathrm{d}x\,\mathrm{d}t&=\int_0^{T'}\!\!\int_\Omega(\overset{\triangledown}{S}{-}\partial_tS):\tilde S\,\mathrm{d}x\,\mathrm{d}t\\ =\int_0^{T'}\!\!\int_\Omega(\overset{\triangledown}{S}{-}\partial_tS):(\tilde S{-}S)\,\mathrm{d}x\,\mathrm{d}t&=\int_0^{T'}\!\!\int_\Omega\big(-\partial_tS-\beta+\gamma\Delta S+\eta D(V)\big):(\tilde S{-}S)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Inserting this identity into (4.16), we are left with the variational inequality

$$\int_0^{T'}\!\!\int_\Omega\Big((\partial_t\tilde S{-}\partial_tS):(\tilde S{-}S)-\beta:(\tilde S{-}S)\Big)\,\mathrm{d}x\,\mathrm{d}t+\mathscr P(\tilde S)-\mathscr P(S)\geq-\tfrac12\|\tilde S(0)-S_0\|_2^2 \tag{4.25}$$

for all $\tilde S\in Z_{T'}$, where we recall the definition of $\mathscr P$ in (4.20). Given $\tilde R\in Z_{T'}$ with $\tilde R(0)=0=\tilde R(T')$, we choose in (4.25) the test function $\tilde S=S+\tilde R$, which by (4.22) lies in $Z_{T'}$ and moreover satisfies $\tilde S(0)=S_0$, to infer

$$\mathscr P(S{+}\tilde R)-\mathscr P(S)\geq\int_0^{T'}\!\!\int_\Omega\beta:\tilde R\,\mathrm{d}x\,\mathrm{d}t \tag{4.26}$$

for all such $\tilde R$.

We assert that by means of an approximation argument, ineq. (4.26) can be extended to general $\tilde R\in Z_{T'}$, not necessarily vanishing at the boundary of $(0,T')$. To see this, we pick a sequence $\{\theta_j\}\subset C_0^\infty((0,T'))$ with $0\leq\theta_j\leq\theta_{j+1}\leq 1$ for all $j\in\mathbb N$ and such that $\lim_{j\to\infty}\theta_j(t)=1$ for all $t\in(0,T')$. We then infer from ineq. (4.26) for general $\tilde R\in Z_{T'}$

$$\int_0^{T'}\big(\mathcal P(S{+}\theta_j\tilde R)-\mathcal P(S)\big)\,\mathrm{d}t\geq\int_0^{T'}\theta_j(t)\int_\Omega\beta:\tilde R\,\mathrm{d}x\,\mathrm{d}t. \tag{4.27}$$

Since $\mathcal P$ is convex, we have for every $\theta=\theta_j(t)\in[0,1]$

$$\mathcal P(S{+}\theta\tilde R)-\mathcal P(S)=\mathcal P\big(\theta(S{+}\tilde R)+(1-\theta)S\big)-\mathcal P(S)\leq\theta\mathcal P(S{+}\tilde R)-\theta\mathcal P(S).$$

Inserting this inequality into (4.27) gives

$$\int_0^{T'}\theta_j(t)\mathcal P(S{+}\tilde R)\,\mathrm{d}t-\int_0^{T'}\theta_j(t)\mathcal P(S)\,\mathrm{d}t\geq\int_0^{T'}\theta_j(t)\int_\Omega\beta:\tilde R\,\mathrm{d}x\,\mathrm{d}t.$$

Invoking the monotone convergence theorem for the first term in the last line and using dominated convergence for the remaining two time integrals, we can take the limit $j\to\infty$ in the last inequality and arrive at (4.26) for general $\tilde R\in Z_{T'}$.

Thanks to the approximation property (4.21) of $\mathscr{P}$, we can further extend (4.26) to general $R \in \mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega))$. Indeed, letting $\tilde{S} := S + R \in \mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega))$, property (4.21) provides us with a sequence $(\tilde{S}_n) \subset Z_{T'}$ such that $\tilde{S}_n \rightharpoonup S + R$ in $\mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega))$ and $\mathscr{P}(\tilde{S}_n) \to \mathscr{P}(S+R)$. Hence, inserting $\tilde{R} = \tilde{R}_n := \tilde{S}_n - S$ in (4.26) and passing to the limit $n \to \infty$ we obtain

$$\mathscr{P}(S{+}R) \geq \mathscr{P}(S) + \int_0^{T'}\int_\Omega \beta : R \,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } R \in \mathrm{L}^2(0,T';\mathrm{L}^2_\delta(\Omega)).$$

But this is exactly the definition of $\beta \in \partial\mathscr{P}(S)$, and the special definition of $\mathscr{P}$ in terms of $\mathcal{P}$ (cf. (4.20)) implies $\beta(t) \in \partial\mathcal{P}(S(t))$ a.e. on $(0,T')$.

The definition of β in (4.24) implies the desired weak equation (4.23). □

The following auxiliary result presents the crucial idea for passing to the limit $\varepsilon \to 0$ in the nonlinear terms arising from the Zaremba–Jaumann derivative. In the case of nonsmooth potentials we do not have compactness of (S_ε) in $\mathrm{L}^2(\Omega{\times}(0,T'))$, thus we need to show that weak convergence is sufficient.

Lemma 4.11. *Let $V_\varepsilon = (V_i^\varepsilon)$ and $S_\varepsilon = (S_{jk}^\varepsilon)$ satisfy the conditions*

$$V_\varepsilon \to V \text{ in } \mathrm{L}^2(\Omega{\times}(0,T')), \qquad V_\varepsilon \rightharpoonup V \text{ and } S_\varepsilon \rightharpoonup S \text{ in } \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega)),$$

and let $V_\varepsilon|_{\partial\Omega} = g \in \mathrm{L}^2(0,T';\mathrm{L}^2(\partial\Omega))$ be fixed. Further assume that $\|S_\varepsilon\|_{\mathrm{L}^{\frac{10}{3}}(\Omega\times(0,T'))} \leq C$.

Then, for all $i,j,k,l \in \{1,2,3\}$ we have

$$\lim_{\varepsilon\to 0}\int_0^{T'}\int_\Omega S_{ij}^\varepsilon \partial_k V_l^\varepsilon \,\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^{T'}\int_\Omega S_{ij}\partial_k V_l\,\psi\,\mathrm{d}x\,\mathrm{d}t \ \text{ for all } \psi \in \mathrm{L}^5(\Omega{\times}(0,T')). \tag{4.28}$$

Proof. We first show that (4.28) holds for $\psi \in C^1(\overline{\Omega}\times[0,T'])$. In this case, integration by parts with respect to the spatial variable gives

$$\int_0^{T'}\int_\Omega S_{ij}^\varepsilon\partial_k V_l^\varepsilon\,\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^{T'}\int_{\partial\Omega} S_{ij}^\varepsilon g_l \mathrm{n}_k\,\psi\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^{T'}\int_\Omega \partial_k\big(S_{ij}^\varepsilon\,\psi\big)V_l^\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

where we have already exploited the boundary conditions $V_\varepsilon = g$ on $\partial\Omega$.

Due to continuity of the trace operator from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\partial\Omega)$, we have $S_\varepsilon \rightharpoonup S$ in $\mathrm{L}^2(0,T';\mathrm{L}^2(\partial\Omega))$ and can pass to the limit in the first term on the right-hand side. For the last term we use the weak convergence of S_ε to S in $\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))$ as well as the strong convergence of V_ε to V in $\mathrm{L}^2(\Omega{\times}(0,T'))$. Undoing the spatial integration by parts, we obtain the desired result for $\psi \in \mathrm{C}^1(\overline{\Omega}\times[0,T'])$.

The validity of (4.28) for general $\psi \in \mathrm{L}^5(\Omega{\times}(0,T'))$ is now a consequence of the density of $\mathrm{C}^1(\overline{\Omega}\times[0,T'])$ in $\mathrm{L}^5(\Omega{\times}(0,T'))$ and the fact that, by Hölder's inequality (with $\frac{3}{10}+\frac{1}{2}=\frac{4}{5}$), the sequence $\{S_{ij}^\varepsilon\partial_k V_l^\varepsilon\}_\varepsilon$ is ε-uniformly bounded in $\mathrm{L}^{\frac{5}{4}}(\Omega{\times}(0,T'))$. □

We are now in the position to complete the proof of Theorem 4.9 and show existence of generalized solutions to (1.1).

Proof of Theorem 4.9. For $\varepsilon \in (0,1]$ consider the Moreau envelope $\mathcal{P}_\varepsilon$ for $\mathcal{P}$ as introduced in Section 2, and denote by $(V_\varepsilon, S_\varepsilon)$ the weak solution constructed in Theorem 4.4 with $\mathcal{P}$ replaced by $\mathcal{P}_\varepsilon$. By estimate (4.10), we have the ε-uniform bound

$$\begin{aligned}\|V_\varepsilon\|_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|V_\varepsilon\|_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \|\partial_t V_\varepsilon\|_{\mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega))')} \\ + \|S_\varepsilon\|_{\mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))} + \|S_\varepsilon\|_{\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))} + \int_0^{T'}\int_\Omega \mathcal{P}_\varepsilon(S_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \le M_{T'}\end{aligned} \tag{4.29}$$

for all $T' < T$. Hence, there exists a sequence $\varepsilon \to 0$ (not relabeled) and a pair (V,S) such that for all $T' < T$ one has $(V,S) \in \mathrm{LH}_{T'} \times \mathrm{X}_{T'}$ and

$$\begin{aligned}
V_\varepsilon &\rightharpoonup V && \text{in } \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))^3,\\
S_\varepsilon &\rightharpoonup S && \text{in } \mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))^{3\times3},\\
V_\varepsilon &\overset{*}{\rightharpoonup} V && \text{in } \mathrm{L}^\infty(0,T';\mathrm{L}^2_\sigma(\Omega)),\\
S_\varepsilon &\overset{*}{\rightharpoonup} S && \text{in } \mathrm{L}^\infty(0,T';\mathrm{L}^2_\delta(\Omega)),\\
\partial_t V_\varepsilon &\rightharpoonup \partial_t V && \text{in } \mathrm{L}^1(0,T';(\mathrm{H}^1_{0,\sigma}(\Omega)^3)'),\\
V_\varepsilon &\to V && \text{in } \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))^3,
\end{aligned}$$

where, as in the proof of Theorem 3.10, the strong convergence of (V_ε) is obtained from an Aubin–Lions compactness result.

The passage to the limit $\varepsilon \to 0$ in the weak form (4.2) of the equation for the velocity field V_ε follows from standard arguments based on the above convergence properties. As a result, the limiting vector field V satisfies eq. (4.2). Moreover, the fact that $V_\varepsilon|_{\partial\Omega\times(0,T)} = g$ combined with the above convergence properties easily yields $V|_{\partial\Omega\times(0,T)} = g$. Thus, it remains to show that S satisfies inequality (4.16) for all $\tilde S \in Z_{T'}$.

By Lemma 4.10 (A), S_ε satisfies the variational inequality

$$\begin{aligned}\int_0^{T'}\int_\Omega \partial_t\tilde S : (\tilde S - S_\varepsilon) + \gamma\nabla S_\varepsilon : \nabla(\tilde S - S_\varepsilon)\,\mathrm{d}x\mathrm{d}t + \int_0^{T'} \mathcal{P}_\varepsilon(\tilde S) - \mathcal{P}_\varepsilon(S_\varepsilon)\,\mathrm{d}t \\ + \int_0^{T'}\int_\Omega V_\varepsilon\cdot\nabla S_\varepsilon : \tilde S + (S_\varepsilon W(V_\varepsilon) - W(V_\varepsilon)S_\varepsilon) : \tilde S - \eta D(V_\varepsilon) : (\tilde S - S_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ \ge -\tfrac12\|\tilde S(0) - S_0\|_2^2.\end{aligned} \tag{4.30}$$

We will deduce ineq. (4.16) by estimating the $\limsup_{\varepsilon\to0}$ of the left-hand side.

First, the weak convergence $S_\varepsilon \rightharpoonup S$ in $\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))^{3\times3}$ implies that

$$\begin{aligned}\int_0^{T'}\int_\Omega \partial_t\tilde S : (\tilde S - S) + \gamma\nabla S : \nabla(\tilde S - S)\,\mathrm{d}x\,\mathrm{d}t \\ \ge \limsup_{\varepsilon\to0}\int_0^{T'}\int_\Omega \partial_t\tilde S : (\tilde S - S_\varepsilon) + \gamma\nabla S_\varepsilon : \nabla(\tilde S - S_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where we used weak upper semicontinuity of the concave quadratic term.

For the second term on the left-hand side of (4.30), we use the bound

$$\limsup_{\varepsilon\to0}\int_0^{T'}\big(\mathcal{P}_\varepsilon(\tilde S) - \mathcal{P}_\varepsilon(S_\varepsilon)\big)\,\mathrm{d}t \le \int_0^{T'}\big(\mathcal{P}(\tilde S) - \mathcal{P}(S)\big)\,\mathrm{d}t$$

which is consequence of Lemma 2.1 and inequality (2.2).

Further note that $V_\varepsilon \to V$ in $\mathrm{L}^2(0,T';\mathrm{H}^1(\Omega))$ and $S_\varepsilon \rightharpoonup S$ in $\mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$ imply

$$\lim_{\varepsilon\to 0}\int_0^{T'}\int_\Omega V_\varepsilon\cdot\nabla S_\varepsilon : \tilde{S}\,\mathrm{d}x\mathrm{d}t = \int_0^{T'}\int_\Omega V\cdot\nabla S : \tilde{S}\,\mathrm{d}x\,\mathrm{d}t.$$

The term $(S_\varepsilon W(V_\varepsilon) - W(V_\varepsilon)S_\varepsilon) : \tilde{S} + \eta D(V_\varepsilon) : S_\varepsilon$ consists of a finite linear combination of terms handled in Lemma 4.11. This lemma can be applied thanks to the convergence properties of $(V_\varepsilon, S_\varepsilon)$ and the interpolation (4.17) ensuring the boundedness of (S_ε) in $\mathrm{L}^{\frac{10}{3}}(\Omega\times(0,T'))$. Hence we can pass to the limit with all remaining parts in the left-hand side.

The above observations allow us to estimate the $\limsup_{\varepsilon\to 0}$ of the left-hand side of (4.30) above by the left-hand side of inequality (4.16). Hence (V,S) is a generalized solution to (1.1)–(1.3) in the sense of Definition 4.6.

It remains to establish the energy-dissipation inequality (4.19). But this is a simple consequence of the previously established energy-dissipation inequalities. In particular, we note that the boundary extension w constructed for Theorem 4.4 depends only on g and, hence, is independent of the regularization parameter ε. Thus, we can use the energy-dissipation inequality (3.32) for $v_\varepsilon = V_\varepsilon - w$ and S_ε. With the given weak and strong convergences, we can pass to the limit $\varepsilon \to 0$ and obtain the corresponding inequality for the limits $v = V - w$ and S. Adding the energy-dissipation inequality (4.18) from Proposition 4.8 and recalling the fact that $F_0 = f_0$ and $F_1 = f_1 + \widetilde{F}_1$, we arrive at the desired result (4.19). □

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