Automatic control via thermostats of a hyperbolic Stefan problem with memory

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Abstract. A hyperbolic Stefan problem based on the linearized Gurtin–Pipkin heat conduction law is considered. Temperature and free boundary are controlled by a thermostat acting on the boundary. This feedback control is based on temperature measurements performed by real thermal sensors located into the domain containing the two–phase system and/or at its boundary. Three different types of thermostats are analyzed: ideal switch, relay switch, and Preisach hysteresis operator. The resulting models lead to formulate integrodifferential hyperbolic Stefan problems with nonlinear and nonlocal boundary conditions. In all the cases, existence results are proved. Uniqueness is also shown, unless in the situation corresponding to the ideal switch.

1. Introduction

Consider a two–phase system which occupies a bounded domain $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ at any time $t \in [0,T]$ $(T > 0)$. Letting $Q_T := \Omega \times (0,T)$, we denote by $\vartheta : Q_T \to \mathbf{R}$ the relative temperature (rescaled in order $\vartheta = 0$ be the critical temperature at which the two phases can coexist) and by $\chi : Q_T \to [0,1]$ the concentration of the more energetic phase (e.g., water in a water–ice system). Within the framework of the study of memory effects in heat conduction phenomena, the following integrodifferential model has been proposed (cf. [4])

$$\partial_t(\varphi_0\vartheta + \varphi * \vartheta + \psi * \chi) - k * \Delta\vartheta = f \quad \text{in } Q_T, \tag{1.1}$$

$$\chi \in H(\vartheta) \quad \text{in } Q_T, \tag{1.2}$$

where φ_0 is a positive constant, φ, ψ, $k : [0,+\infty) \to \mathbf{R}$ are smooth relaxation (or memory) kernels, and $*$ indicates the time convolution product on $(0,t)$. Besides, $f : Q_T \to \mathbf{R}$ is a known function which depends both on the heat supply and on the past histories of ϑ and χ up to $t = 0$ (supposed to be given), while H stands for the Heaviside graph, that is,

$$H(s) = \begin{cases} \{0\} & \text{if } s < 0, \\ [0,1] & \text{if } s = 0, \\ \{1\} & \text{if } s > 0. \end{cases}$$

System (1.1–2) endowed with suitable initial and boundary conditions produces a hyperbolic Stefan problem provided that $k(0) > 0$ (cf. [4, Section 2]). In the case of homogeneous Dirichlet boundary conditions, this problem has been already investigated in some detail, proving both strong and weak well–posedness among other things (see [3–6]). Here we are interested in analyzing the heat exchange at the boundary $\Gamma := \partial\Omega$ under the influence of a thermostat, and to this aim we have to deal with a boundary condition of the third type. Then, by taking the initial condition

$$(\varphi_0\vartheta)(\cdot,0) = e_0 \quad \text{in } \Omega \tag{1.3}$$

(the datum e_0 expressing an enthalpy density), we are going to play with the following relation

$$-k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \vartheta_{\mathrm{e}}) \quad \text{on } \Sigma_T := \Gamma \times (0,T), \tag{1.4}$$

where $\vartheta_{\mathbf{n}}$ is the outward normal derivative of ϑ on Γ, α denotes a positive constant, $\vartheta_{\mathbf{\Gamma}}$ is the trace of ϑ on the boundary, and ϑ_{e} represents the external temperature coupled with another term depending on the prescribed past history of ϑ up to $t = 0$. As we shall see, ϑ_{e} plays a crucial role in the control problems described below.

Suppose that we are able to measure the temperature ϑ by a real system of thermal sensors placed in some fixed positions inside the body or on its surface. This fact can be made schematic by assuming that the quantity

$$\mathcal{M}(\vartheta)(t) := \fint_{\Omega_0} \vartheta(x,t)\omega_I(x)dx + \fint_{\Gamma_0} \vartheta_{\mathbf{\Gamma}}(y,t)\omega_S(y)d\Gamma \tag{1.5}$$

is known at any time $t \in [0,T]$. Here $\Omega_0 \subset \Omega$ and $\Gamma_0 \subset \Gamma$ are the involved sets, with positive Lebesgue domain and surface measures, respectively. Moreover, the notation $\fint$ is used to indicate the mean value, and $\omega_I : \Omega_0 \to [0,+\infty)$, $\omega_S : \Gamma_0 \to [0,+\infty)$ are weight functions determined by the characteristics of the sensors.

To control the evolution of the free boundary, a thermostat device acts modifying ϑ_{e} on account of $\mathcal{M}(\vartheta)$. According to [11] (see also [8, 12]), the behavior of the thermostat is described by

$$\vartheta_{\mathrm{e}}(y,t) = u(t)\vartheta_A(y,t) + \vartheta_B(y,t), \qquad (y,t) \in \Sigma_T, \tag{1.6}$$

$u : [0,T] \to \mathbf{R}$ being a heating (or cooling) device whose dynamics obeys

$$\beta u' + u = \mathcal{W}(\mathcal{M}(\vartheta)) + \vartheta_C \quad \text{ in } [0,T], \tag{1.7}$$

$$u(0) = u_0. \tag{1.8}$$

The functions ϑ_A, $\vartheta_B : \Sigma_T \to \mathbf{R}$ and $\vartheta_C : [0,T] \to \mathbf{R}$ are given (with a sign property for ϑ_A), β is a positive parameter, $\mathcal{W}$ models the action of the thermostat, and $u_0 \in \mathbf{R}$. We have to specify the operator $\mathcal{W}$ yet. Here, referring to [7–8, 11–14], we are going to consider three different cases $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$.

(A) *Simple switch*
A critical time–dependent value $\varrho(t)$ provides a *jump discontinuity* so that, for $r \in C^0([0,T])$ and $t \in [0,T]$,

$$\mathcal{W}_1(r)(t) := \mathcal{H}(r(t) - \varrho(t)) = \begin{cases} +1 & \text{if } r(t) < \varrho(t), \\ w(t) & \text{if } r(t) = \varrho(t), \\ -1 & \text{if } r(t) > \varrho(t), \end{cases} \tag{1.9}$$

where $w \in L^\infty(0,T)$ fulfills $-1 \le w(t) \le 1$ for a.a. $t \in (0,T)$. Obviously, $\varrho \in C^0([0,T])$ is fixed inside the thermostat. As we will postulate later, the selection of w is purely *random* and $-\mathcal{H}$ is nothing but the opposite of the maximal graph resulting from the sign function.

(B) *Relay switch*

In this case there are two thresholds $\varrho_L,\ \varrho_U \in C^0([0,T])$ with $\varrho_L < \varrho_U$ and such that, for any $r \in C^0([0,T])$, $\mathcal{W}_2(r)$ changes its value at time t_c from -1 to $+1$ or vice versa according to the rule

$$\mathcal{W}_2(r)(t_c) := \begin{cases} +1 & \text{if } r(t_c) = \varrho_L(t_c) \text{ and } \mathcal{W}_2(r)(t) = -1 \ \textit{just before,} \\ -1 & \text{if } r(t_c) = \varrho_U(t_c) \text{ and } \mathcal{W}_2(r)(t) = +1 \ \textit{just before.} \end{cases} \tag{1.10}$$

The meaning of *just before* is made precise later (see Section 4), as well as the definition of $\mathcal{W}_2(r)$ in the case when r coincides with one of the two quantities ϱ_L or ϱ_U on an open subset of $[0,T]$.

(C) *Hysteresis operator of Preisach type*

To introduce $\mathcal{W}_3$, we partly follow [13, Chapter IV]. For $r \in C([0,T])$ and for any pair $(\varrho_1,\varrho_2) \in \mathbf{R}^2$ satisfying $\varrho_1 < \varrho_2$, we set

$$\mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)(0) := \begin{cases} +1 & \text{if } r(0) \le \varrho_1, \\ \varsigma(\varrho_1,\varrho_2) & \text{if } \varrho_1 < r(0) < \varrho_2, \\ -1 & \text{if } r(0) \ge \varrho_2 \end{cases}$$

with $\varsigma : (\varrho_1,\varrho_2) \mapsto \varsigma(\varrho_1,\varrho_2) \in \{-1,+1\}$ being a given Borel measurable function. In addition, if $t \in (0,T]$ we let

$$\mathcal{T}_t := \{\tau \in \ (0,t] \ : \ \ r(\tau) = \varrho_1 \ \text{ or } \ r(\tau) = \varrho_2\}$$

and

$$\mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)(t) := \begin{cases} \mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)(0) & \text{if } \mathcal{T}_t = \emptyset, \\ +1 & \text{if } \mathcal{T}_t \ne \emptyset \text{ and } r(\max \mathcal{T}_t) = \varrho_1, \\ -1 & \text{if } \mathcal{T}_t \ne \emptyset \text{ and } r(\max \mathcal{T}_t) = \varrho_2. \end{cases}$$

One can easily check that the functions $z = \mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)$ are continuous on the right in $[0,T)$ and have finite total variation on $[0,T]$, i.e., $z \in C^0_r([0,T)) \cap BV(0,T)$. We are thus led to consider the mapping $\mathcal{H}_{(\varrho_1,\varrho_2)}(\cdot,\varsigma) : C^0([0,T]) \to C^0_r([0,T)) \cap BV(0,T)$ which is called *delayed relay operator* (compare with (B)). Now, if μ is a nonnegative Borel measure on the plane $\mathcal{P} := \{(\varrho_1,\varrho_2) \in \mathbf{R}^2 \ : \ \ \varrho_1 < \varrho_2\}$, the associated Preisach operator is specified by

$$\mathcal{W}_3(r)(t) := \int_{\mathcal{P}} \mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)(t)\, d\mu(\varrho_1,\varrho_2). \tag{1.11}$$

The main properties of such transformation will be recalled in Section 5.

Next, let us come to our control problems. They can be roughly formulated saying that we are looking for a triplet $(\vartheta, \mathcal{X}, u)$ fulfilling (1.1–4), (1.6–8) with $\mathcal{M}$ prescribed by (1.5) and $\mathcal{W} \equiv \mathcal{W}_j$, $j = 1, 2, 3$. Problems of this kind have been studied by several authors (see, e.g., [7, 8, 11, 12] and the references therein). Nevertheless, the present paper reports the first attempt to investigate the thermostat control of a hyperbolic Stefan problem with memory effects.

It is convenient to put the feedback control problems we have just described in a more general form (see [8]). Indeed, looking at (1.7–8), it is easy to observe that u is given by a Volterra operator, namely

$$u(t) = \int_0^t e^{-(t-\tau)/\beta} \left(\mathcal{W}_j(\mathcal{M}(\vartheta))(\tau) + \vartheta_C(\tau)\right) d\tau + u_0 e^{-t/\beta} \tag{1.12}$$

for any $t \in [0, T]$ and for $j = 1, 2, 3$. On account of (1.12), equation (1.6) can be written as

$$\vartheta_{\mathrm{e}} = \mathcal{F}[\mathcal{W}_j(\mathcal{M}(\vartheta))] \quad \text{on } \Sigma_T, \tag{1.13}$$

where, for any $r \in L^2(0, T)$,

$$\mathcal{F}[r](y, t) := \int_0^t E(y, t, \tau) r(\tau) d\tau + E_0(y, t), \qquad (y, t) \in \Sigma_T, \tag{1.14}$$

with (in the special case of (1.12))

$$E(\cdot, t, \tau) = e^{-(t-\tau)/\beta} \vartheta_A(\cdot, t),$$
$$E_0(\cdot, t) = \left(\int_0^t e^{-(t-\tau)/\beta} \vartheta_C(\tau) d\tau + u_0 e^{-t/\beta}\right) \vartheta_A(\cdot, t) + \vartheta_B(\cdot, t)$$

almost everywhere on Γ, t and τ both varying in $[0, T]$. Consequently, the feedback control problems reduce to hyperbolic integrodifferential Stefan problems with a nonlinear and nonlocal boundary condition. More precisely, for $j = 1, 2, 3$ we shall deal with

Problem ($\mathbf{P}_j$). Find a pair (ϑ, χ) satisfying (1.1–3) and

$$-k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \mathcal{F}[\mathcal{W}_j(\mathcal{M}(\vartheta))]) \quad \text{on } \Sigma_T. \tag{1.15}$$

Taking advantage of the fixed–point techniques used in [11] (see also [8, 12]), we prove the existence of a solution to (P_1). Uniqueness is not expected in this framework, due to the random behavior of the model. Regarding (P_2), the inductive argument developed in [11] allows to show existence and uniqueness. In the case of (P_3), we can apply the Schauder fixed–point theorem to derive existence of solutions. Besides, under further restrictions on the measure μ and for $\omega_S \equiv 0$ in (1.5) (no boundary measurements) we deduce uniqueness via suitable contracting estimates.

In order to prove these results, we first need a careful analysis of the well–posedness of the Stefan problem (1.1–4). Also, we have to state Problems (P_j), $j = 1, 2, 3,$ in a more rigorous way. Let us present the plan of the paper. In Section 2, we give a precise formulation of (1.1–4) and of the related results, which are shown and fully detailed in Sections 6–8. In addition, we establish the crucial properties of the operators $\mathcal{M}$ and $\mathcal{F}$. The subsequent Sections 3, 4, 5 are devoted to Problems (P_1), (P_2), (P_3), respectively. In each of these sections, the feedback control problem is settled and discussed up to the proof of the related main result.

2. Preliminaries

Here and in the sequel $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is an open, bounded, and connected set with a smooth boundary Γ (for instance, Γ of class $\mathcal{C}^2$). We put $\mathrm{H} := L^2(\Omega)$ and $\mathrm{V} := H^1(\Omega)$, recalling that $\mathrm{V} \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{V}'$ with dense and compact injections provided H is identified with its dual space H'. The duality pairing between V' and V and the scalar product in H are both denoted by $\langle \cdot, \cdot \rangle$, while $(\cdot, \cdot)$ stands for the scalar product in H^N. The norm either in H or in H^N is simply indicated by $\|\cdot\|$. Also, let $\|\cdot\|_\Gamma$ be the norm in $L^2(\Gamma)$.

We remind that $v_{\mathbf{\Gamma}}$ stands for the trace of a function $v \in \mathrm{V}$ on Γ, the normal derivative on Γ being represented by $v_{\mathbf{n}}$ whenever v is regular enough. Moreover, prime specifies the derivatives of functions depending only on time, whereas the position $(1 * z)(\cdot, t) := \int_0^t z(\cdot, s) ds$ holds for any $t \in [0, T]$ and any $z \in L^1(0, T; \mathrm{V}')$.

Now, we introduce the assumptions on the data (cf. [4, Section 2]). Assume that

$$\varphi \in W^{1,1}(0, T), \tag{2.1}$$

$$\psi \in W^{2,1}(0, T), \quad \psi(0) > 0, \tag{2.2}$$

$$k \in W^{2,1}(0, T), \quad k(0) > 0, \tag{2.3}$$

$$f \in W^{1,1}(0, T; \mathrm{H}), \tag{2.4}$$

$$e_0 \in \mathrm{V}, \tag{2.5}$$

$$\vartheta_{\mathrm{e}} \in H^1(0, T; L^2(\Gamma)), \tag{2.6}$$

$$\vartheta_{\mathrm{e}}(\cdot, 0) = \varphi_0^{-1} e_{0\mathbf{\Gamma}} \quad \text{a.e. on } \Gamma. \tag{2.7}$$

Consequently, accounting for the identity $k * \vartheta = k(0)(1 * \vartheta) + k' * (1 * \vartheta)$, we can rigorously formulate the hyperbolic Stefan problem (1.1–4).

Problem (SP). Find the pair (ϑ, χ) such that $\vartheta \in L^\infty(0, T; \mathrm{V}) \cap W^{1,\infty}(0, T; \mathrm{H})$ and $\chi \in L^\infty(0, T; \mathrm{H})$ satisfy

$$1 * \Delta\vartheta \in L^\infty(0, T; \mathrm{H}), \tag{2.8}$$

$$\vartheta_{\mathbf{n}} \in L^2(\Sigma_T), \tag{2.9}$$

$$\vartheta_{\mathbf{\Gamma}} \in H^1(0,T;L^2(\Gamma)), \tag{2.10}$$

$$\partial_t(\varphi_0\vartheta + \varphi * \vartheta + \psi * \chi) - k * \Delta\vartheta = f \quad \text{a.e. in } Q_T, \tag{2.11}$$

$$\chi \in H(\vartheta) \quad \text{a.e. in } Q_T, \tag{2.12}$$

$$(\varphi_0\vartheta)(\cdot,0) = e_0 \quad \text{a.e. in } \Omega, \tag{2.13}$$

$$-k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \vartheta_{\mathrm{e}}) \quad \text{a.e. on } \Sigma_T, \tag{2.14}$$

where $Q_t := \Omega \times (0,t)$ and $\Sigma_t := \Gamma \times (0,t)$ for $t \in [0,T]$.

Remark 2.1. Note that $\chi \in L^\infty(Q_T)$ thanks to (2.12). Owing to (2.8), the normal trace $\vartheta_{\mathbf{n}}$ has a a meaning in $H^{-1}(0,T;H^{-1/2}(\Gamma))$ and (2.9) asserts that is even a square integrable function. By (2.10) it turns out that (2.14) holds almost everywhere on Σ_T.

Existence and uniqueness for (SP) are given by

Theorem 2.1. *Let (2.1–7) hold. Then Problem (SP) has a unique solution* (ϑ,χ). *Moreover, there exists a positive constant* Λ_1 *such that*

$$\begin{aligned} &\|\vartheta\|_{L^\infty(0,T;\mathrm{V})\cap W^{1,\infty}(0,T;\mathrm{H})} + \|1 * \Delta\vartheta\|_{L^\infty(0,T;\mathrm{H})} + \|\vartheta_{\mathbf{\Gamma}}\|_{H^1(0,T;L^2(\Gamma))} \\ &+ \|\vartheta_{\mathbf{n}}\|_{L^2(\Sigma_T)} \le \Lambda_1 \left\{1 + \|f\|_{W^{1,1}(0,T;\mathrm{H})} + \|e_0\|_{\mathrm{V}} + \|\vartheta_{\mathrm{e}}\|_{H^1(0,T;L^2(\Gamma))}\right\}, \end{aligned} \tag{2.15}$$

this constant depending only on Ω, Γ, T, φ_0, $k(0)$, α, *and on the norms* $\|\varphi\|_{W^{1,1}(0,T)}$, $\|\psi\|_{W^{2,1}(0,T)}$, $\|k\|_{W^{2,1}(0,T)}$.

A further useful result concerns the Lipschitz continuity of the map associating ϑ_{e} with the solution component ϑ, namely

Theorem 2.2. *Let (2.1–5) hold. Consider* ϑ_{e}^i, $i = 1, 2$, *fulfilling (2.6–7) and denote by* (ϑ_i,χ_i) *the corresponding solution to Problem (SP). Then there exists a positive constant* Λ_2 *such that for any* $t \in [0,T]$ *one has*

$$\begin{aligned} &\|\vartheta_1 - \vartheta_2\|_{C^0([0,t];\mathrm{H})} + \|1 * (\vartheta_1 - \vartheta_2)\|_{C^0([0,t];\mathrm{V})} \\ &+ \|(\vartheta_1 - \vartheta_2)_{\mathbf{\Gamma}}\|_{L^2(\Sigma_t)} \le \Lambda_2 \left\|\vartheta_{\mathrm{e}}^1 - \vartheta_{\mathrm{e}}^2\right\|_{L^2(\Sigma_t)}. \end{aligned} \tag{2.16}$$

Moreover, Λ_2 *depends only on* T, φ_0, $\psi(0)$, $k(0)$, α, $\|\varphi\|_{W^{1,1}(0,T)}$, $\|\psi\|_{W^{2,1}(0,T)}$, *and* $\|k\|_{W^{2,1}(0,T)}$.

The last part of this section is devoted to establish some basic properties of the already introduced operators $\mathcal{M}$, defined by (1.5), and $\mathcal{F}$, taken as in (1.14) for suitable functions $E(y,t,\tau)$, $E_0(y,t)$ (cf. (2.18) below) not necessarily coinciding with those of the Introduction. We require that

$$\omega_I \in L^2(\Omega_0;[0,+\infty)), \quad \omega_S \in L^2(\Gamma_0;[0,+\infty)) \tag{2.17}$$

and that

$$E,\, E_t, E_\tau \in L^2(\Gamma \times (0,T)^2), \quad E_0 \in H^1(0,T;L^2(\Gamma)), \tag{2.18}$$

noticing that E_t, E_τ represent the partial derivatives of E with respect to the *time* variables. We also set the compatibility condition (cf. (1.13) and (2.7))

$$\varphi_0 E_0(\,\cdot\,,0) = e_{0\mathbf{\Gamma}} \quad \text{ a.e. on } \Gamma. \tag{2.19}$$

Here are two statements concerning $\mathcal{M}$ and $\mathcal{F}$, respectively.

Proposition 2.1. *Under the assumption (2.17), $\mathcal{M} : \mathrm{V} \to \mathbf{R}$ is linear and continuous and can be naturally extended to spaces of functions from $(0,T)$ to V. In particular, for all v_1, $v_2 \in \mathrm{V}$ there holds*

$$|\mathcal{M}(v_1) - \mathcal{M}(v_2)| \le \|\omega_I\|_{L^2(\Omega_0)} \|v_1 - v_2\| + \|\omega_S\|_{L^2(\Gamma_0)} \|(v_1 - v_2)_{\mathbf{\Gamma}}\|_{\Gamma}\,. \tag{2.20}$$

Moreover, whenever $v \in L^\infty(0,T;\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H})$ and $v_{\mathbf{\Gamma}} \in H^1(0,T;L^2(\Gamma))$, then $\mathcal{M}(v)$ belongs to $C^{0,1/2}([0,T])$ and satisfies

$$|\mathcal{M}(v)(t) - \mathcal{M}(v)(\tau)| \le \Lambda_3 \left\{ \|v_t\|_{L^2(Q_T)} + \|\partial_t v_{\mathbf{\Gamma}}\|_{L^2(\Sigma_T)} \right\} |t-\tau|^{1/2} \tag{2.21}$$

for any pair $(t,\tau) \in [0,T]^2$, where, for instance, $\Lambda_3 = \|\omega_I\|_{L^2(\Omega_0)} + \|\omega_S\|_{L^2(\Gamma_0)}$.

Proof. As (2.20) is a trivial consequence of the definition of $\mathcal{M}$, it suffices to check (2.21). Letting $t \le \tau$, from (2.20) it turns out that

$$\begin{aligned} &|\mathcal{M}(v)(t) - \mathcal{M}(v)(\tau)| = |\mathcal{M}(v(\,\cdot\,,t)) - \mathcal{M}(v(\,\cdot\,,\tau))| \\ &\le \Lambda_3 \int_t^\tau \left(\|v_t(\,\cdot\,,s)\| + \|\partial_t v_{\mathbf{\Gamma}}(\,\cdot\,,s)\|_{\Gamma} \right) ds \end{aligned}$$

and the Hölder inequality allows us to infer (2.21). $\square$

Proposition 2.2. *Under the assumptions (2.18–19), $\mathcal{F}$ is a continuous operator from $L^\infty(0,T)$ to $H^1(0,T;L^2(\Gamma))$ such that*

$$\mathcal{F}[r](\,\cdot\,,0) = \varphi_0^{-1} e_{0\mathbf{\Gamma}} \quad \textit{a.e. on } \Gamma, \ \forall\, r \in L^\infty(0,T).$$

Moreover, one can determine a constant Λ_4, depending only on T, $\|E\|_{L^2(\Gamma\times(0,T)^2)}$, $\|E_t\|_{L^2(\Gamma\times(0,T)^2)}$, $\|E_0\|_{H^1(0,T;L^2(\Gamma))}$, on the $L^2(\Sigma_T)$ norm of the (diagonal) trace function $(y,t) \mapsto E(y,t,t)$, and on the quantity $\max_{0\le t\le T} \|E(\,\cdot\,,t,\,\cdot\,)\|_{L^2(\Sigma_T)}$, such that

$$\|\mathcal{F}[r]\|_{H^1(0,T;L^2(\Gamma))} \le \Lambda_4 \left\{ 1 + \|r\|_{L^\infty(0,T)} \right\}, \tag{2.22}$$

$$\|(\mathcal{F}[r_1] - \mathcal{F}[r_2])(t)\|_{\Gamma} \le \Lambda_4 \|r_1 - r_2\|_{L^2(0,t)} \tag{2.23}$$

for any $t \in [0,T]$ *and for all* r, r_1, $r_2 \in L^\infty(0,T)$.

Proof. In view of (1.14) and (2.18), it is clear that

$$\partial_t(\mathcal{F}[r])(y,t) = E(y,t,t)r(t) + \int_0^t E_t(y,\tau,t)r(\tau)d\tau + \partial_t E_0(y,t)$$

for a.a. $(y,t) \in \Sigma_T$. Hence, observing that

$$\begin{aligned}\int_0^T \left\| \int_0^t E_t(\cdot,t,\tau)r(\tau)d\tau \right\|_\Gamma^2 dt &\leq \int_0^T \|r\|^2_{L^2(0,t)} \int_0^t \|E_t(\cdot,t,\tau)\|^2_\Gamma \, d\tau \, dt \\ &\leq T\,\|r\|^2_{L^\infty(0,T)} \|E_t\|^2_{L^2(\Gamma\times(0,T)^2)},\end{aligned}$$

it is a standard matter to recover (2.22). Regarding (2.23), by the Hölder inequality we have that

$$\begin{aligned}\|(\mathcal{F}[r_1]-\mathcal{F}[r_2])(t)\|_\Gamma &\leq \int_0^t \|E(\cdot,t,\tau)\|_\Gamma \, |(r_1-r_2)(\tau)| d\tau \\ &\leq \|E(\cdot,t,\cdot)\|_{L^2(\Sigma_t)} \|r_1-r_2\|_{L^2(0,t)}\end{aligned}$$

and easily achieve the proof. □

3. Problem (P$_1$): simple switch

As we have already mentioned in the Introduction (see (A)), we prescribe that at the times when r reaches the threshold ϱ, then $\mathcal{W}_1(r)$ takes some value between -1 and $+1$ in a purely random way. We assume that

$$\varrho \in C^0([0,T]) \tag{3.1}$$

and consider the set–valued convexification of the ranges in (1.9) putting

$$\mathcal{H}(s) := \begin{cases} \{+1\} & \text{if } s < 0, \\ [-1,1] & \text{if } s = 0, \\ \{-1\} & \text{if } s > 0, \end{cases} \tag{3.2}$$

so that $-\mathcal{H} : \mathbf{R} \to 2^{\mathbf{R}}$ is a maximal monotone graph. Now, let $\mathcal{W}_1$ be the multivalued operator from $C^0([0,T])$ to $L^\infty(0,T)$ defined by

$$w \in \mathcal{W}_1(r) \quad \text{if} \quad w(t) \in \mathcal{H}(r(t)-\varrho(t)) \quad \text{for a.a. } t \in (0,T). \tag{3.3}$$

It is straightforward to verify that $\mathcal{W}_1(r)$ is nonempty for any $r \in C^0([0,T])$, since the existence of a measurable selection w (obviously bounded) is ensured by the continuity of r and ϱ.

Hence, Problem (P_1) can be precisely formulated as

Problem ($\mathbf{P}_1$). Find $\vartheta \in L^\infty(0,T;\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H})$, $\chi \in L^\infty(0,T;\mathrm{H})$, and $z \in L^\infty(0,T)$ satisfying (2.8–13) and

$$- k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \mathcal{F}[z]) \quad \text{a.e. on } \Sigma_T, \tag{3.4}$$

$$z \in \mathcal{W}_1(\mathcal{M}(\vartheta)). \tag{3.5}$$

The result we are going to prove is

Theorem 3.1. *Let (2.1–5), (2.17–19), and (3.1–2) hold. Then there exists a solution to Problem (P_1).*

Proof. We introduce the set

$$Y_T := \left\{\eta \in H^1(0,T;L^2(\Gamma)) \ : \ \eta(\,\cdot\,,0) = \varphi_0^{-1} e_{0\mathbf{\Gamma}} \quad \text{a.e. on } \Gamma\right\} \tag{3.6}$$

and the mapping

$$S_1 : Y_T \to 2^{Y_T}, \qquad S_1(\eta) := \{\mathcal{F}[w]\,,\ w \in \mathcal{W}_1(\mathcal{M}(\vartheta(\eta)))\}\,, \tag{3.7}$$

where

$$(\vartheta(\eta), \chi(\eta)) \ \text{ represents the solution of (SP) with } \ \vartheta_{\mathrm{e}} = \eta. \tag{3.8}$$

These positions are plainly justified by Theorem 2.1, Propositions 2.1–2, and (3.3). Moreover, from (2.22) and (3.2) it follows that

$$\|\xi\|_{H^1(0,T;L^2(\Gamma))} \le 2\Lambda_4 \quad \forall\ \xi \in S_1(\eta),\ \ \forall\ \eta \in Y_T. \tag{3.9}$$

Thus, setting

$$U_T := \left\{\eta \in Y_T \ : \quad \|\eta\|_{H^1(0,T;L^2(\Gamma))} \le 2\Lambda_4\right\}, \tag{3.10}$$

we have that S_1 is a multivalued operator from U_T to U_T. Thanks to (3.4–5), (3.8), and (2.14), we realize that any fixed point of S_1 makes the pair in (3.8) solve (P_1). Therefore, to show Theorem 3.1 we just need to find $\eta \in U_T$ such that $\eta \in S_1(\eta)$.

Reasoning as in [8], we use the Glicksberg fixed–point theorem (see [9]). This tool works for set–valued mappings under suitable convexity, compactness, and closure hypotheses. Let us omit the statement here and check carefully the assumptions in our frame.

Denote by $H^1_w(0,T;L^2(\Gamma))$ the space $H^1(0,T;L^2(\Gamma))$ endowed with the weak topology. It is a locally convex topological vector space. Due to the boundedness property in (3.10), U_T is a nonempty, convex, and compact subset of $H^1_w(0,T;L^2(\Gamma))$. In addition, U_T is also sequentially compact. For an arbitrary $\eta \in U_T$ we claim that

$S_1(\eta)$ is nonempty (because of (3.7) and (3.3)) and convex. Indeed, if $\xi_1, \xi_2 \in S_1(\eta)$, then there are $w_1, w_2 \in L^\infty(0,T)$ such that $\xi_i = \mathcal{F}[w_i]$ and $w_i \in \mathcal{W}_1(\mathcal{M}(\vartheta(\eta)))$ for $i = 1,2$. On the other hand, by construction $\mathcal{W}_1(\mathcal{M}(\vartheta(\eta)))$ is convex so that $\lambda w_1 + (1-\lambda) w_2 \in \mathcal{W}_1(\mathcal{M}(\vartheta(\eta)))$ whenever $0 \le \lambda \le 1$. As $\mathcal{F}$ is an affine transformation, we conclude that $\mathcal{F}[\lambda w_1 + (1-\lambda) w_2] = \lambda \xi_1 + (1-\lambda)\xi_2 \in S_1(\eta)$ for any $\lambda \in [0,1]$.

It remains to deduce that the graph of S_1,

$$G(S_1) := \{(\eta, \xi) : \ \eta \in U_T, \ \ \xi \in S_1(\eta)\}, \tag{3.11}$$

is closed in the product space $(H^1_w(0,T;L^2(\Gamma)))^2$. To this aim, take a Moore–Smith sequence $\{(\eta_a, \xi_a)\}_{a \in \mathcal{A}}$, $\mathcal{A}$ being a directed set, fulfilling $(\eta_a, \xi_a) \in G(S_1)$ for any $a \in \mathcal{A}$ and

$$\lim_{a \in \mathcal{A}} \eta_a = \eta_\infty, \qquad \lim_{a \in \mathcal{A}} \xi_a = \xi_\infty$$

for some pair $(\eta_\infty, \xi_\infty) \in (H^1(0,T;L^2(\Gamma)))^2$. Owing to (3.9–10), $\{(\eta_a, \xi_a)\}_{a \in \mathcal{A}}$ is bounded, hence weakly sequentially compact, in $(H^1(0,T;L^2(\Gamma)))^2$. Therefore, we can extract a subsequence $\{(\eta_j, \xi_j)\}_{j \in \mathbf{N}}$ such that

$$\eta_j \to \eta_\infty, \quad \xi_j \to \xi_\infty \quad \text{weakly in } H^1(0,T;L^2(\Gamma)) \tag{3.12}$$

as $j \nearrow \infty$. Remark that η_∞ and ξ_∞ are both in U_T. Since $\xi_j \in S_1(\eta_j)$, we can fix $z_j \in L^\infty(0,T)$ satisfying $\xi_j = \mathcal{F}[z_j]$ and $z_j \in \mathcal{W}_1(\mathcal{M}(\vartheta(\eta_j)))$ for any $j \in \mathbf{N}$. By (3.2–3) we have that $\|z_j\|_{L^\infty(0,T)} \le 1$, and consequently a subsequence of $\{z_j\}$ admits a weak star limit z_∞, i.e.,

$$z_j \to z_\infty \quad \text{weakly star in } L^\infty(0,T). \tag{3.13}$$

On account of (3.12–13) and Proposition 2.2, the (strong and weak) continuity of $\mathcal{F}$ implies that $\xi_\infty = \mathcal{F}[z_\infty]$. At this point, to obtain $(\eta_\infty, \xi_\infty) \in G(S_1)$ it suffices to prove that (cf. (3.7–8))

$$z_\infty \in \mathcal{W}_1(\mathcal{M}(\vartheta(\eta_\infty))). \tag{3.14}$$

By virtue of (3.10), (2.15) and (2.12) there exist ϑ_∞ and χ_∞ such that, possibly taking subsequences,

$$\vartheta(\eta_j) \to \vartheta_\infty \quad \text{weakly star in } L^\infty(0,T;\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H}), \tag{3.15}$$

$$1 * \Delta\vartheta(\eta_j) \to 1 * \Delta\vartheta_\infty \quad \text{weakly star in } L^\infty(0,T;\mathrm{H}), \tag{3.16}$$

$$\vartheta_{\mathbf{\Gamma}}(\eta_j) \to \vartheta_{\infty\mathbf{\Gamma}} \quad \text{weakly in } H^1(0,T;L^2(\Gamma)), \tag{3.17}$$

$$\vartheta_{\mathbf{n}}(\eta_j) \to \vartheta_{\infty\mathbf{n}} \quad \text{weakly in } L^2(\Sigma_T), \tag{3.18}$$

$$\chi(\eta_j) \to \chi_\infty \quad \text{weakly star in } L^\infty(Q_T) \tag{3.19}$$

as $j \nearrow \infty$. The convergence (3.15) and the generalized Ascoli theorem entail

$$\vartheta(\eta_j) \to \vartheta_\infty \quad \text{strongly in } C^0([0,T];\mathrm{H}). \tag{3.20}$$

With the help of (3.12), (3.15–20), and (2.1–3) it is not difficult to infer that $(\vartheta_\infty, \chi_\infty)$ solves Problem (SP) for $\vartheta_{\mathrm{e}} = \eta_\infty$, so that (by Theorem 2.1 and (3.8)) $\vartheta_\infty = \vartheta(\eta_\infty)$. In fact, from (3.19–20) it results that

$$\iint_{Q_T} \chi(\eta_j)\vartheta(\eta_j) \to \iint_{Q_T} \chi_\infty \vartheta_\infty, \tag{3.21}$$

and then (3.21) and the maximal monotonicity of H (as induced operator in $L^2(Q_T)$) ensure that (see, e.g., [2, Lemma 1.3, p. 42]) χ_∞ and ϑ_∞ fulfill the nonlinear condition (2.12). Now, (3.15), (3.17), and Proposition 2.1 enable us to conclude that $\{\mathcal{M}(\vartheta(\eta_j))\}_{j\in\mathbf{N}}$ is a bounded and equicontinuous (cf. (2.21)) subset of $C^0([0,T])$, and thus $\mathcal{M}(\vartheta(\eta_j))$ converges to $\mathcal{M}(\vartheta(\eta_\infty))$ not only weakly star in $L^\infty(0,T)$ but strongly in $C^0([0,T])$. Finally, referring to (3.13), (3.2–3) and arguing as above, one can exploit the maximal monotonicity of $-\mathcal{H}$ to get (3.14).

Hence, S_1 has a closed graph in $(H^1_w(0,T;L^2(\Gamma)))^2$. The Glicksberg theorem now applies and yields the existence of at least one solution to (P_1) . □

4. Problem (P$_2$): relay switch

The response of the thermostat in this case ((B) in the Introduction) needs to be better specified following [11, Section 5]. First of all, consider the two critical functions

$$\varrho_L,\ \varrho_U \in C^0([0,T]), \tag{4.1}$$

which are supposed to satisfy

$$\varrho_U(t) - \varrho_L(t) \geq \delta > 0 \quad \forall\, t \in [0,T], \tag{4.2}$$

for some fixed bound δ. Then, in view of (1.10), (1.15) and (1.3–5), we can assume, for instance, that

$$\mathcal{M}(e_0) = \fint_{\Omega_0} e_0\omega_I + \fint_{\Gamma_0} e_{0\boldsymbol{\Gamma}}\omega_S \leq \varphi_0\varrho_L(0), \tag{4.3}$$

and the relay is initially switched on, namely

$$\mathcal{W}_2(\mathcal{M}(\vartheta))(0) = +1. \tag{4.4}$$

Here the notion of solution for (P_2) is made precise .

Problem (P$_2$). Find $\vartheta \in L^\infty(0,T;\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H})$, $\chi \in L^\infty(0,T;\mathrm{H})$, $z \in L^\infty(0,T)$ and a finite sequence $\{t_h\}_{h=0}^m$ of *switching times* with

$$0 =: t_0 < t_1 < \ldots < t_m = T$$

satisfying (2.8–13), (4.4),

$$-\,k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \mathcal{F}[z]) \quad \text{a.e. on } \Sigma_T, \tag{4.5}$$

$$z(t) = (-1)^h \quad \text{if } t \in [t_h, t_{h+1}[, \tag{4.6}$$

$$t_{h+1} \text{ is exactly the infimum of the set } \{T\} \cup K_{h+1}, \tag{4.7}$$

where

$$K_{h+1} := \left\{ t \in (t_h, T] \ : \ \mathcal{M}(\vartheta)(t) = \begin{cases} \varrho_U(t) & \text{if } h \text{ is even,} \\ \varrho_L(t) & \text{if } h \text{ is odd} \end{cases} \right\}$$

for $h = 0, \ldots, m-1$.

Remark 4.1. Note that K_{h+1} may be empty for some h and in this case $t_{h+1} = T$ and $m = h+1$. On the other hand, as the difference $\mathcal{M}(\vartheta) - \varrho_L$ is uniformly continuous in $[0,T]$ (see Proposition 2.1 and (4.1)), the switching times are at most finitely many owing to (4.2).

In analogy with [11, Theorem 5.1], we have

Theorem 4.1. *Let (2.1–5), (2.17–19), (4.1–3) hold. Then there exists a unique solution to Problem (P$_2$).*

Before proving Theorem 4.1, we establish a preliminary result about the number of switching times. The next assertion improves that of Remark 4.1, taking advantage of the fact that there is a modulus of continuity for $\mathcal{M}(\vartheta)$ uniform with respect to any admissible ϑ.

Lemma 4.1. *Under the same hypotheses as in Theorem 4.1, for*

$$w \in B := \left\{ r \in L^\infty(0,T) \ : \ \|r\|_{L^\infty(0,T)} \leq 1 \right\}$$

let $(\vartheta(w), \chi(w))$ be the unique solution to (SP) with $\vartheta_{\mathrm{e}} = \mathcal{F}[w]$. Then there exists some number $\gamma > 0$ such that, for any $w \in B$ and for all times t_L, $t_U \in [0,T]$ fulfilling

$$\mathcal{M}(\vartheta(w))(t_L) = \varrho_L(t_L), \quad \mathcal{M}(\vartheta(w))(t_U) = \varrho_U(t_U), \tag{4.8}$$

there holds

$$|t_L - t_U| \geq \gamma,$$

and consequently the function $\mathcal{M}(\vartheta(w))$ commutes at most $[T/\gamma]$ times between the threshold functions ϱ_L and ϱ_U ($[T/\gamma]$ denoting the integer part of T/γ).

Proof. Since w lies in B, thanks to (2.22), (2.15), and (2.21) there is a constant Λ_5 such that

$$|\mathcal{M}(\vartheta(w))(t_L) - \mathcal{M}(\vartheta(w))(t_U)| \leq \Lambda_5 |t_L - t_U|^{1/2}, \tag{4.9}$$

where Λ_5 depends only on Λ_1, Λ_3, Λ_4, $\|f\|_{W^{1,1}(0,T;\mathrm{H})}$, and $\|e_0\|_{\mathrm{V}}$. Recalling (4.1), let $\gamma_1 > 0$ be such that

$$|\varrho_L(t) - \varrho_L(\tau)| \leq \frac{\delta}{2} \quad \text{whenever} \quad t,\, \tau \in [0,T], \;\; |t - \tau| < \gamma_1.$$

Therefore, from (4.8–9) it may happen that either $|t_L - t_U| \geq \gamma_1$ or

$$\Lambda_5 |t_L - t_U|^{1/2} \geq |\varrho_L(t_L) - \varrho_U(t_U)| \geq \varrho_U(t_U) - \varrho_L(t_U) - |\varrho_L(t_L) - \varrho_L(t_U)| \geq \frac{\delta}{2}$$

because of (4.2). Hence we can choose $\gamma := \min\left\{\gamma_1, \delta^2/(2\Lambda_5)^2\right\}$. Concerning the last part of the statement we also remind (2.13) and (4.3), which imply that, provided $t_1 < T$, there is some $\tau_0 \in [t_0, t_1)$ with $\mathcal{M}(\vartheta(w))(\tau_0) = \varrho_L(\tau_0)$. □

Proof of Theorem 4.1. We basically reproduce the inductive argument devised in [11, Proof of Theorem 5.1]. On account of (4.4), we begin picking

$$w_0(t) = +1 \quad \forall\, t \in [0,T]$$

and taking the triplet $(\vartheta(w_0), \chi(w_0), w_0)$, where $(\vartheta(w_0), \chi(w_0))$ solves Problem (SP) when $\vartheta_{\mathrm{e}} = \mathcal{F}[w_0]$ (cf. Theorem 2.1). Consider the set

$$D_1 := \{t \in (0,T] \;:\; \mathcal{M}(\vartheta(w_0))(t) = \varrho_U(t)\}.$$

If $D_1 \equiv \emptyset$, then $(\vartheta(w_0), \chi(w_0), w_0)$ provides the unique solution to (P_2). Otherwise, $t_1 := \inf D_1$ is a minimum due to the continuity of $\mathcal{M}(\vartheta(w_0))$ and Lemma 4.1 yields $t_1 \geq \gamma$. Further, we define

$$w_1(t) := \begin{cases} w_0(t) & \text{if } t \in [0, t_1), \\ -1 & \text{if } t \in [t_1, T], \end{cases}$$

the solution $(\vartheta(w_1), \chi(w_1))$ of (SP) with $\vartheta_{\mathrm{e}} = \mathcal{F}[w_1]$, and the set

$$D_2 := \{t \in (t_L, T] \;:\; \mathcal{M}(\vartheta(w_1))(t) = \varrho_L(t)\}.$$

Reasoning as above, we conclude that $(\vartheta(w_1), \chi(w_1), w_1)$ gives the unique solution to (P_2) unless $D_2 \neq \emptyset$. In this alternative situation we introduce $t_2 := \inf D_2$ and

$$w_2(t) := \begin{cases} w_1(t) & \text{if } t \in [0, t_2), \\ +1 & \text{if } t \in [t_2, T]. \end{cases}$$

Similarly, we realize that $t_2 \in D_2$ and $t_2 \geq 2\gamma$. Then we can start again by considering $(\vartheta(w_2), \chi(w_2), w_2)$. Proceeding by induction, it is clear that there exist $m \in \mathbf{N}$, a triplet $(\vartheta(w_m), \chi(w_m), w_m)$, and a sequence of switching times $\{t_h\}_{h=0}^m$ such that $m \leq T/\gamma$, $t_m = T$, and the triplet represented by $\vartheta = \vartheta(w_m)$, $\chi = \chi(w_m)$, $z = w_m$ uniquely satisfies (2.8–13) and (4.4–6). □

Remark 4.2. As one can easily observe, our analysis covers the degenerate cases when $\mathcal{M}(\vartheta)$ just touches one of the thresholds ϱ_L, ϱ_U or coincides with one of them on a time interval.

5. Problem ($\mathbf{P}_3$): hysteresis operator of Preisach type

The case (C) of the Introduction is characterized by position (1.11) for suitable choices of the measure μ and of the two–valued function ς. The latter has to fulfill

$$\varsigma : \mathcal{P} \to \{-1, 1\} \text{ is Borel measurable.} \tag{5.1}$$

Regarding the former, in order to prove an existence result for Problem (P_3) (which is stated precisely below) we require that

$$\mu \text{ is a nonnegative Borel measure with bounded density,} \tag{5.2}$$

$$\mu(\{\varrho_1\} \times \mathbf{R}) = \mu(\mathbf{R} \times \{\varrho_2\}) = 0 \quad \forall\, (\varrho_1, \varrho_2) \in \mathbf{R}^2. \tag{5.3}$$

Within this framework, the operator $\mathcal{W}_3$ enjoys two important properties.

Proposition 5.1. *Under the assumptions (5.1–3), there hold*

$$\|\mathcal{W}_3(r)\|_{L^\infty(0,T)} \leq \mu(\mathcal{P}) < +\infty \quad \forall\, r \in C^0([0,T]), \tag{5.4}$$

$$\mathcal{W}_3 \;\; \textit{is strongly continuous from} \;\; C^0([0,T]) \;\; \textit{to} \;\; C^0([0,T]). \tag{5.5}$$

Moreover, denoting by ℓ the bidimensional Lebesgue measure, if

$$\mu(A) \leq \Lambda_\mu\, \ell(A) \quad \textit{for all Lebesgue measurable sets } A \subset \mathcal{P}, \tag{5.6}$$

for some constant Λ_μ, then there exists a constant Λ_6, depending only on $\mu(\mathcal{P})$ and Λ_μ, such that, for all r_1, $r_2 \in C^0([0,T])$,

$$|(\mathcal{W}_3(r_1) - \mathcal{W}_3(r_2))(t)| \leq \Lambda_6\, \|r_1 - r_2\|_{C^0([0,t])} \quad \forall\, t \in [0,T]. \tag{5.7}$$

Proof. As $\left|\mathcal{H}_{(\varrho_1,\varrho_2)}(r,\varsigma)\right| \leq 1$ a.e. in $(0,T)$ (see the Introduction, above (1.11)), (5.4) is a straightforward consequence of (5.2). Concerning (5.5), we just notice that it follows from (5.3), referring to [14, Section IV.3, Theorem 3.2] for details. The stronger

Lipschitz continuity (5.7) is ensured by assumption (5.6) (which implies (5.3), of course) thanks to, e.g., [13, formula (1.14) and Proposition 8]. More general conditions yielding (5.7) are examined in [14, Section IV.3]. □

After these preliminaries, let us come to the formulation of (P_3).

Problem (P_3). Find $\vartheta \in L^\infty(0,T;\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H})$ and $\chi \in L^\infty(0,T;\mathrm{H})$ satisfying (2.9–13) and

$$-k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \mathcal{F}[\mathcal{W}_3(\mathcal{M}(\vartheta))]) \quad \text{a.e. on } \Sigma_T. \tag{5.8}$$

By exploiting just (5.5) (and not (5.7)), we are able to show existence for (P_3). More precisely, we have

Theorem 5.1. *Let (2.1–5), (2.17–19), (5.1–3) hold. Then there exists a solution to Problem (P_3).*

Proof. Let us first put Problem (P_3) in a convenient fixed–point setting by introducing the operator

$$S_3 : C^0([0,T]) \to C^0([0,T]), \quad S_3(r) := \mathcal{M}(\vartheta(r)), \tag{5.9}$$

where

$$(\vartheta(r), \chi(r)) \text{ is the unique solution to (SP) with } \vartheta_{\mathrm{e}} = \mathcal{F}[\mathcal{W}_3(r)]. \tag{5.10}$$

On account of Propositions 2.1–2 and Theorem 2.1, one can readily check that S_3 is well defined. Besides, observe that solving (P_3) is equivalent to finding a fixed point for S_3. By means of (5.4), (2.22), (5.10), (2.15), and (2.21) we deduce that $S_3(C^0([0,T])) \subset C^{0,1/2}([0,T])$ and we are able to find a constant Λ_7, depending on $\mu(\mathcal{P})$ and on the same quantities as Λ_5 does (cf. (4.9)), such that

$$\|S_3(r)\|_{C^{0,1/2}([0,T])} \leq \Lambda_7 \quad \forall\, r \in C^0([0,T]). \tag{5.11}$$

Hence, due to the Ascoli theorem, S_3 maps $C^0([0,T])$ (which is nonempty, closed, and convex) into a relatively compact subset of $C^0([0,T])$. Therefore, if we prove the continuity of S_3, then we achieve the proof by the Schauder theorem. To this aim, let $\{r_j\}_{j\in\mathbf{N}}$ be a sequence converging to some r in $C^0([0,T])$. Then (5.5), (2.23), (5.10), (2.16), (2.20), and (5.9) allow us to infer, step by step, that $\mathcal{W}_3(r_j) \to \mathcal{W}_3(r)$ in $C^0([0,T])$, $\mathcal{F}[\mathcal{W}_3(r_j)] \to \mathcal{F}[\mathcal{W}_3(r)]$ in $C^0([0,T];L^2(\Gamma))$, $\vartheta(r_j) \to \vartheta(r)$ in $C^0([0,T];\mathrm{H})$ and $\vartheta_{\mathbf{\Gamma}}(r_j) \to \vartheta_{\mathbf{\Gamma}}(r)$ in $L^2(0,T;L^2(\Gamma))$, and finally

$$S_3(r_j) \to S_3(r) \quad \text{in } L^2(0,T) \tag{5.12}$$

as $j \nearrow \infty$, all convergences being strong. On the other hand, in view of (5.11), by compactness there are a subsequence $\{j_h\}_{h\in\mathbf{N}}$ and an element w such that

$$S_3(r_{j_h}) \to w \quad \text{in } C^0([0,T]) \tag{5.13}$$

as $h \nearrow \infty$. Combining (5.12) and (5.13), the uniqueness of the limit entails $w = S_3(r)$ and

$$S_3(r_j) \to S_3(r) \quad \text{in } C^0([0,T])$$

for the whole sequence, that is the desired conclusion. □

Remark 5.1. In the study of Problem (P_1) we could not argue this way since the operator $\mathcal{W}_1$ was set–valued and, further, without strong continuity properties.

Under additional assumptions on μ and on the location of the thermostat sensors, we can also prove uniqueness.

Theorem 5.2. *Let (2.1–5), (2.17–19), (5.1–3), (5.6), and*

$$\omega_S \equiv 0 \tag{5.14}$$

hold. Then Problem (P_3) admits a unique solution.

Proof. By contradiction assume that there are two pairs (ϑ_i, χ_i), $i = 1, 2$, solving (P_3). By comparing (5.8) and (2.14), we put $\vartheta_{\mathrm{e}}^i = \mathcal{F}[\mathcal{W}_3(\mathcal{M}(\vartheta_i))]$, $i = 1, 2$, and use (2.23), (5.7), and (2.20) to obtain

$$\begin{aligned}
\left\|(\vartheta_{\mathrm{e}}^1 - \vartheta_{\mathrm{e}}^2)(t)\right\|_\Gamma^2 &\le \int_0^t \left|\Lambda_4(\mathcal{W}_3(\mathcal{M}(\vartheta_1)) - \mathcal{W}_3(\mathcal{M}(\vartheta_2)))(s)\right|^2 ds \\
&\le (\Lambda_4\Lambda_6)^2 \int_0^t \|\mathcal{M}(\vartheta_1) - \mathcal{M}(\vartheta_2)\|_{C^0([0,s])}^2 \, ds \\
&\le \left(\Lambda_4\Lambda_6 \|\omega_I\|_{L^2(\Omega_0)}\right)^2 \int_0^t \|\vartheta_1 - \vartheta_2\|_{C^0([0,s];\mathrm{H})}^2 \, ds
\end{aligned}$$

for any $t \in [0,T]$. Now, we can invoke (2.16) and get

$$\left\|(\vartheta_{\mathrm{e}}^1 - \vartheta_{\mathrm{e}}^2)(t)\right\|_\Gamma^2 \le \Lambda_8 \int_0^t \left\|(\vartheta_{\mathrm{e}}^1 - \vartheta_{\mathrm{e}}^2)(s)\right\|_\Gamma^2 ds \qquad \forall\, t \in [0,T],$$

with, for instance, $\Lambda_8 = \left(\Lambda_2\Lambda_4\Lambda_6 \|\omega_I\|_{L^2(\Omega_0)}\right)^2 T$. Then the Gronwall lemma enables us to establish that $\vartheta_{\mathrm{e}}^1 = \vartheta_{\mathrm{e}}^2$ a.e. in Σ_T, whence the thesis is an outcome of Theorem 2.1. □

Remark 5.2. The uniqueness theorem works in a restricted framework essentially because of (5.14). We do not know whether or not this restriction can be removed, having at our disposal only an inequality like (5.7) for the mapping $\mathcal{W}_3$. In fact, the reader may check that if (5.7) was replaced by

$$\left|(\mathcal{W}_3(r_1) - \mathcal{W}_3(r_2))(t)\right| \le \Lambda_6 \left|(r_1 - r_2)(t)\right|, \tag{5.15}$$

or more generally by

$$\|(\mathcal{W}_3(r_1) - \mathcal{W}_3(r_2))\|_{L^2(0,t)} \le \Lambda_6 \,|(r_1 - r_2)|_{L^2(0,t)}\,, \tag{5.16}$$

then the above argument could be suitably modified in order to apply to the standard situation where $\omega_S \in L^2(\Gamma_0; [0, +\infty))$. On the other hand, a Preisach hysteresis operator is not expected to enjoy properties like (5.15) or (5.16).

6. Auxiliary parabolic problems

The last part of the paper is concerned with the proof of Theorems 2.1 and 2.2. To sum up, we have to show the well–posedness of Problem (SP) and, in particular, to derive estimates (2.15) and (2.16). Existence of a solution is proved passing through a *parabolic regularization* of (SP). The aim of this section is that of preparing some technical results for two related problems, one linear and the other nonlinear.

Therefore, we fix a parameter $\varepsilon > 0$ (subject to tend to 0 elsewhere) and, letting the data φ, F, ϑ_0, ϑ_{e} fulfill

$$\varphi \in H^1(0,T), \tag{6.1}$$

$$F \in H^1(0,T;\mathrm{H}), \quad F(\,\cdot\,,0) \in \mathrm{V}, \tag{6.2}$$

$$\vartheta_0 \in \mathrm{V}, \quad \Delta\vartheta_0 \in \mathrm{V}, \tag{6.3}$$

$$\vartheta_{\mathrm{e}} \in W^{2,1}(0,T;L^2(\Gamma)), \tag{6.4}$$

$$-\,\varepsilon\vartheta_{0\mathbf{n}} = \alpha(\vartheta_{0\mathbf{\Gamma}} - \vartheta_{\mathrm{e}}(\,\cdot\,,0)) \quad \text{a.e. on } \Gamma, \tag{6.5}$$

we consider

Problem (LP)$_\varepsilon$. Find $\vartheta \in C^1([0,T];\mathrm{V}) \cap H^2(0,T;\mathrm{H})$ satisfying

$$\Delta\vartheta \in H^1(0,T;\mathrm{H}), \tag{6.6}$$

$$\vartheta_{\mathbf{n}} \in C^1([0,T];L^2(\Gamma)), \tag{6.7}$$

$$\partial_t(\varphi_0\vartheta + \varphi * \vartheta) - \varepsilon\Delta\vartheta - k * \Delta\vartheta = F \quad \text{a.e. in } Q_T, \tag{6.8}$$

$$\vartheta(\,\cdot\,,0) = \vartheta_0 \quad \text{a.e. in } \Omega, \tag{6.9}$$

$$-\,\varepsilon\vartheta_{\mathbf{n}} - k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \vartheta_{\mathrm{e}}) \quad \text{a.e. on } \Sigma_T. \tag{6.10}$$

Regarding this linear problem, we can state

Theorem 6.1. *Under the assumptions (2.3) and (6.1–5), Problem (LP)$_\varepsilon$ has a unique solution.*

Proof. Taking (6.10) into account, let us rewrite equation (6.8) as

$$\varphi_0\langle\vartheta_t, v\rangle + \varepsilon(\nabla\vartheta, \nabla v) + \alpha\int_\Gamma \vartheta_{\mathbf{\Gamma}} v_{\mathbf{\Gamma}} + \langle\mathcal{R}(\vartheta), v\rangle$$
$$= \langle F, v\rangle + \alpha\int_\Gamma \vartheta_{\mathrm{e}} v_{\mathbf{\Gamma}} \quad \forall\, v \in \mathrm{V}, \text{ a.e. in } (0,T), \tag{6.11}$$

where the mapping $\mathcal{R}$ is specified by

$$\langle\mathcal{R}(\zeta), v\rangle := \langle\partial_t(\varphi * \zeta), v\rangle + (k * \nabla\zeta, \nabla v)$$

for any $\zeta \in L^2(0,T;\mathrm{V})$. Recalling the well–known formulas

$$a * b = a(0)(1 * b) + a_t * 1 * b, \qquad \partial_t(a * b) = a(0)b + a_t * b, \tag{6.12}$$

which hold whenever they make sense, by (6.1) and (2.3) one easily verifies that $\mathcal{R}$ is linear and continuous from $C^0([0,T];\mathrm{V}) \cap H^1(0,T;\mathrm{H})$ to $L^2(0,T;\mathrm{H}) + W^{1,1}(0,T;\mathrm{V}')$. Then, an application of [1, Teorema 6.1] allows us to deduce that there exists one and only one function $\vartheta \in C^0([0,T];\mathrm{V}) \cap H^1(0,T;\mathrm{H})$ solving the Cauchy problem (6.11), (6.9). From (6.11) we plainly recover (6.8) in the sense of distributions, and so the condition

$$\varepsilon\Delta\vartheta + k * \Delta\vartheta \in L^2(0,T;\mathrm{H}) \tag{6.13}$$

follows. Thanks to the classical theory of Volterra integral equations, it turns out that $\Delta\vartheta \in L^2(0,T;\mathrm{H})$. Thus, (6.13) ensures the validity of (6.8) and of the equality for the traces, that is,

$$-\varepsilon\vartheta_{\mathbf{n}} - k * \vartheta_{\mathbf{n}} = \alpha(\vartheta_{\mathbf{\Gamma}} - \vartheta_{\mathrm{e}}) \quad \text{in } L^2(0,T;H^{-1/2}(\Gamma)). \tag{6.14}$$

Moreover, owing to (6.4) and to the regularity of ϑ, (6.14) implies

$$\vartheta_{\mathbf{n}} \in C^0([0,T];L^2(\Gamma)) \tag{6.15}$$

and consequently (6.10) is fulfilled.

It only remains to infer the further smoothness of ϑ. Observe that $u := \partial_t\vartheta$ formally satisfies (see (6.12))

$$\varphi_0\langle u_t, v\rangle + \varepsilon(\nabla u, \nabla v) + \alpha\int_\Gamma u_{\mathbf{\Gamma}} v_{\mathbf{\Gamma}} + \langle\mathcal{R}(u), v\rangle = \langle F_t - \varphi'\vartheta_0, v\rangle$$
$$-k(\nabla\vartheta_0, \nabla v) + \alpha\int_\Gamma (\partial_t\vartheta_{\mathrm{e}}) v_{\mathbf{\Gamma}} \quad \forall\, v \in \mathrm{V}, \text{ a.e. in } (0,T) \tag{6.16}$$

in addition to

$$u(\,\cdot\,,0) = \varphi_0^{-1}(F(\,\cdot\,,0) - \varphi(0)\vartheta_0 + \varepsilon\Delta\vartheta_0) \quad \text{a.e. in } \Omega. \tag{6.17}$$

We note that (6.17) has been obtained by reading (6.8) at the initial time and using (6.9). Due to (6.1–4), the abstract result in [1, Teorema 6.1] applies to (6.16–17) as well. Therefore, this initial value problem has one and only one solution, say $\widetilde{u}$, belonging to $C^0([0,T];\mathrm{V}) \cap H^1(0,T;\mathrm{H})$. Besides, arguing as before (cf. (6.13–15)) we achieve that

$$\Delta\widetilde{u} \in L^2(0,T;\mathrm{H}) \quad \text{and} \quad \widetilde{u}_{\mathbf{n}} \in C^0([0,T];L^2(\Gamma)) \tag{6.18}$$

(here (6.5) plays a role). At this point, it is not difficult to check that $\vartheta \equiv \vartheta_0 + 1 * \widetilde{u}$ whence $\vartheta \in C^1([0,T];\mathrm{V}) \cap H^2(0,T;\mathrm{H})$ and $\widetilde{u} \equiv u \equiv \partial_t\vartheta$. Since (6.18) and (6.3) entail (6.6–7), the proof is completed. □

Next, we show a similar result for a nonlinear version of $(\mathrm{LP})_\varepsilon$. More precisely, let G be a (possibly nonlinear and nonlocal) operator such that

$$G : H^1(0,t;\mathrm{H}) \to H^1(0,t;\mathrm{H}) \qquad \forall\, t \in (0,T], \tag{6.19}$$

$$G[\zeta](\,\cdot\,,0) \in \mathrm{V} \quad \text{whenever} \quad \zeta \in H^1(0,T;\mathrm{H}) \quad \text{and} \quad \zeta(\,\cdot\,,0) \in \mathrm{V}. \tag{6.20}$$

A constant Λ_9 is supposed to exist in order that

$$\|G[\zeta]\|_{L^2(0,T;\mathrm{H})} \leq \Lambda_9, \tag{6.21}$$

$$\|G[\zeta_1] - G[\zeta_2]\|_{L^2(0,t;\mathrm{H})} \leq \Lambda_9\, \|\zeta_1 - \zeta_2\|_{L^2(0,t;\mathrm{H})} \tag{6.22}$$

for any $t \in (0,T]$ and for all $\zeta,\ \zeta_1,\ \zeta_2 \in H^1(0,T;\mathrm{H})$. In this setting we introduce

Problem $(\mathrm{NP})_\varepsilon$. Find $\vartheta \in C^1([0,T];\mathrm{V}) \cap H^2(0,T;\mathrm{H})$ satisfying (6.6–7), (6.9–10), and

$$\partial_t(\varphi_0\vartheta + \varphi * \vartheta) - \varepsilon\Delta\vartheta - k * \Delta\vartheta = G[\vartheta] \quad \text{a.e. in } Q_T. \tag{6.23}$$

We are still able to prove

Theorem 6.2. *Let (2.3), (6.1), and (6.3–5) hold. Moreover, let G be as in (6.19–22). Then there exists a unique solution to Problem $(NP)_\varepsilon$.*

Proof. Once more we exploit a fixed–point technique. Taking $\zeta \in H^1(0,T;\mathrm{H})$ with $\zeta(\,\cdot\,,0) \in \mathrm{V}$, due to (6.19–20) and Theorem 6.1 we can consider the function

$$\vartheta = \vartheta(\zeta) \in C^1([0,T];\mathrm{V}) \cap H^2(0,T;\mathrm{H}) \quad \text{solving} \quad (\mathrm{LP})_\varepsilon \quad \text{for} \quad F = G[\zeta]. \tag{6.24}$$

Multiply the corresponding equation (6.8) by ϑ and integrate by parts in space and time over Q_t, $t \in (0,T]$. With the help of (6.9–10) and (6.12) one easily obtains

$$\frac{\varphi_0}{2}\|\vartheta(\,\cdot\,,t)\|^2 + \varepsilon\iint_{Q_t}|\nabla\vartheta|^2 + \frac{k(0)}{2}\|(1 * \nabla\vartheta)(\,\cdot\,,t)\|^2 + \alpha\iint_{\Sigma_t}|\vartheta_{\mathbf{\Gamma}}|^2 = \frac{\varphi_0}{2}\|\vartheta_0\|^2 + \sum_{j=1}^{4} I_j(t), \tag{6.25}$$

where

$$I_1(t) := -\iint_{Q_t} (\varphi(0)\vartheta + \varphi' * \vartheta)\vartheta, \qquad I_2(t) := -\int_0^t (\nabla(k' * 1 * \vartheta)(\cdot\,, s), \nabla\vartheta(\cdot\,, s))ds$$

$$I_3(t) := \alpha \iint_{\Sigma_t} \vartheta_e \vartheta_{\mathbf{\Gamma}}, \qquad I_4(t) := \iint_{Q_t} G[\zeta]\vartheta.$$

To estimate $I_1(t)$ we have recourse to the Young inequality for the convolution product, namely

$$\|a * b\|_{L^r(0,T;X)} \le \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)} \qquad \forall\ a \in L^p(0,T),\ \ b \in L^q(0,T;X), \quad (6.26)$$

letting X denote a real Banach space and $1 \le p,\ q,\ r \le \infty$ fulfill $1/r = (1/p)+(1/q)-1$. It is straightforward to get

$$|I_1(t)| \le \left\{|\varphi(0)| + \|\varphi'\|_{L^1(0,T)}\right\} \int_0^t \|\vartheta(\cdot\,, s)\|^2 ds. \qquad (6.27)$$

By (6.12) it is a standard matter to verify that

$$\begin{aligned} I_2(t) = &-((k' * 1 * \nabla\vartheta)(\cdot\,, t), (1 * \nabla\vartheta)(\cdot\,, t)) \\ &+ \int_0^t \left(k'(0)\|(1 * \nabla\vartheta)(\cdot\,, s)\|^2 + ((k'' * 1 * \nabla\vartheta)(\cdot\,, s), (1 * \nabla\vartheta)(\cdot\,, s))\right) ds, \end{aligned}$$

from which we derive

$$|I_2(t)| \le \frac{k(0)}{4}\|(1 * \nabla\vartheta)(\cdot\,, t)\|^2 + \Lambda_{10}\int_0^t \|(1 * \nabla\vartheta)(\cdot\,, s)\|^2 ds, \qquad (6.28)$$

where Λ_{10} is a positive constant only depending on $k(0)$, T, and $\|k\|_{W^{2,1}(0,T)}$. Note that here (6.26) has been used twice, the first time for $r = \infty$ and $p = q = 2$. Concerning $I_3(t)$ and $I_4(t)$, we have that

$$|I_3(t)| + |I_4(t)| \le \frac{\alpha}{2}\iint_{\Sigma_t} \left(|\vartheta_{\mathbf{\Gamma}}|^2 + |\vartheta_e|^2\right) + \frac{1}{2}\int_0^t \left(\|G[\zeta](\cdot\,, s)\|^2 + \|\vartheta(\cdot\,, s)\|^2\right)ds. \quad (6.29)$$

Combining (6.27–29) with (6.25), then applying the Gronwall lemma, one finds a constant Λ_{11} (whose dependences are clear) such that

$$\begin{aligned} &\|\vartheta\|^2_{C^0([0,t];\mathrm{H})} + \varepsilon\|\nabla\vartheta\|^2_{L^2(0,t;\mathrm{H}^N)} + \|1 * \nabla\vartheta\|^2_{C^0([0,t];\mathrm{H}^N)} + \|\vartheta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_t)} \\ &\le \Lambda_{11}\left\{\|\vartheta_0\|^2 + \|\vartheta_e\|^2_{L^2(\Sigma_t)} + \|G[\zeta]\|^2_{L^2(0,t;\mathrm{H})}\right\}, \end{aligned} \qquad (6.30)$$

this inequality obviously holding for any $t \in [0,T]$.

Besides estimate (6.30), we need a higher order estimate which is obtained by multiplying equation (6.23) by $-\Delta\vartheta$ (this is admissible owing to (6.6)) and integrating over $\Omega \times (0,t)$, with $t \in (0,T]$. In this case, (6.9–10) and the Green formula help us to infer that

$$\frac{\varphi_0}{2}\|\nabla\vartheta(\cdot,t)\|^2 + \varepsilon\int_0^t \|\Delta\vartheta(\cdot,s)\|^2 ds + \frac{k(0)}{2}\|(1*\Delta\vartheta)(\cdot,t)\|^2$$
$$= \frac{\varphi_0}{2}\|\nabla\vartheta_0\|^2 + \sum_{j=5}^{8} I_j(t), \tag{6.31}$$

where

$$I_5(t) := \iint_{\Sigma_t} (\varphi_0\partial_t\vartheta_{\mathbf{\Gamma}} + \varphi(0)\vartheta_{\mathbf{\Gamma}} + \varphi' * \vartheta_{\mathbf{\Gamma}})\vartheta_{\mathbf{n}},$$
$$I_6(t) := -\int_0^t (\nabla(\varphi(0)\vartheta + \varphi' * \vartheta)(\cdot,s), \nabla\vartheta(\cdot,s))ds,$$
$$I_7(t) := -\int_0^t \langle\Delta(k' * 1 * \vartheta)(\cdot,s)\Delta\vartheta(\cdot,s)\rangle ds, \qquad I_8(t) := -\iint_{Q_t} G[\zeta]\Delta\vartheta.$$

As $\alpha\partial_t\vartheta_{\mathbf{\Gamma}} = -\varepsilon\partial_t\vartheta_{\mathbf{n}} - k(0)\vartheta_{\mathbf{n}} - k' * \vartheta_{\mathbf{n}} + \alpha\partial_t\vartheta_{\mathrm{e}}$ a.e. on Σ_T because of (6.10) and (6.7), playing on $I_5(t)$ with (6.26) and the elementary Young inequality, it is not difficult to get

$$I_5(t) \le -\frac{\varepsilon\varphi_0}{2\alpha}\|\vartheta_{\mathbf{n}}(\cdot,t)\|_\Gamma^2 + \frac{\varepsilon\varphi_0}{2\alpha}\|\vartheta_{0\mathbf{n}}\|_\Gamma^2 - \frac{k(0)\varphi_0}{2\alpha}\|\vartheta_{\mathbf{n}}\|_{L^2(\Sigma_t)}^2$$
$$+\Lambda_{12}\left\{\int_0^t \|\vartheta_{\mathbf{n}}\|_{L^2(\Sigma_s)}^2 ds + \|\vartheta_{\mathrm{e}}\|_{L^2(\Sigma_t)}^2 + \|\vartheta_{\mathbf{\Gamma}}\|_{L^2(\Sigma_t)}^2\right\} \tag{6.32}$$

for some constant Λ_{12} which depends exactly on $k(0)$, φ_0, α, $\|k'\|_{L^2(0,T)}$, $|\varphi(0)|$, and $\|\varphi'\|_{L^1(0,T)}$. On the other hand, arguing as in the deduction of (6.27–28), we are led to

$$|I_6(t)| \le (|\varphi(0)| + \|\varphi'\|_{L^1(0,T)})\int_0^t \|\nabla\vartheta(\cdot,s)\|^2 ds, \tag{6.33}$$
$$|I_7(t)| \le \frac{k(0)}{4}\|(1*\Delta\vartheta)(\cdot,t)\|^2 + \Lambda_{10}\int_0^t \|(1*\Delta\vartheta)(\cdot,s)\|^2 ds. \tag{6.34}$$

It remains to point out that

$$|I_8(t)| \le \frac{\varepsilon}{2}\int_0^t \|\Delta\vartheta(\cdot,s)\|^2 ds + \frac{1}{2\varepsilon}\int_0^t \|G[\zeta](\cdot,s)\|^2 ds. \tag{6.35}$$

Now, we estimate the right hand side of (6.31) with the aid of (6.32–35) and also of (6.30). Moving the negative terms on the left hand side and invoking the Gronwall lemma, we realize that

$$
\begin{aligned}
&\|\nabla\vartheta\|^2_{C^0([0,t];\mathrm{H}^N)} + \varepsilon\|\Delta\vartheta\|^2_{L^2(0,t;\mathrm{H})} + \|1*\Delta\vartheta\|^2_{C^0([0,t];\mathrm{H})} \\
&+ \varepsilon\,\|\vartheta_{\mathbf{n}}\|^2_{C^0([0,t];L^2(\Gamma))} + \|\vartheta_{\mathbf{n}}\|^2_{L^2(\Sigma_t)} \\
&\le \Lambda_{13}\left\{\|\vartheta_0\|^2_{\mathrm{V}} + \varepsilon\,\|\vartheta_{0\mathbf{n}}\|^2_\Gamma + \|\vartheta_{\mathrm{e}}\|^2_{H^1(0,t;L^2(\Gamma))} + (1+1/\varepsilon)\,\|G[\zeta]\|^2_{L^2(0,t;\mathrm{H})}\right\} \qquad (6.36)
\end{aligned}
$$

for any $t\in[0,T]$, the constant Λ_{13} being independent of ε. Moreover, on account of (6.30), (6.36), (2.3), and (6.24), a comparison in (6.8) yields an analogous bound for $\|\vartheta_t\|^2_{L^2(0,t;\mathrm{H})}$.

Therefore, thanks to (6.21) there is a constant Λ_{14}, depending only on Λ_9, Ω, Γ, T, φ_0, $k(0)$, α, $\|\varphi\|_{W^{1,1}(0,T)}$, $\|k\|_{W^{2,1}(0,T)}$, $\|\vartheta_0\|^2_{\mathrm{V}}$, $\|\Delta\vartheta_0\|^2_{\mathrm{H}}$, $\|\vartheta_{\mathrm{e}}\|^2_{H^1(0,t;L^2(\Gamma))}$, and ε (we remind that ε is fixed in this section) such that (cf. (6.24))

$$\|\vartheta(\zeta)\|_{H^1(0,T;\mathrm{H})} \le \Lambda_{14} \quad \text{for all } \ \zeta\in H^1(0,T;\mathrm{H}) \ \text{ satisfying } \ \zeta(\,\cdot\,,0)\in \mathrm{V}. \qquad (6.37)$$

Then, if we endow the set

$$X_T := \left\{\zeta\in H^1(0,T;\mathrm{H}) \;:\; \zeta(\,\cdot\,,0)=\vartheta_0 \text{ a.e. in } \Omega,\ \|\zeta\|_{H^1(0,T;\mathrm{H})}\le\Lambda_{14}\right\}$$

with the distance function

$$d_{X_T}(\zeta_1,\zeta_2) := \|\zeta_1-\zeta_2\|_{C^0([0,T];\mathrm{H})}, \qquad \zeta_1,\,\zeta_2\in X_T,$$

it turns out that the operator $\mathcal{N}: z\mapsto\vartheta(z)$ acts from X_T into itself (by virtue of (6.9) and (6.37)) and, due to the weak lower semicontinuity of the norm $\|\cdot\|_{H^1(0,T;\mathrm{H})}$, X_T is a complete metric space. Thus, to prove the theorem it is sufficient to show that either $\mathcal{N}$ or a suitable power of it is a contracting mapping. Letting $\zeta_1,\,\zeta_2\in X_T$ and reasoning as for (6.30), we can easily conclude that

$$\|\mathcal{N}(\zeta_1)-\mathcal{N}(\zeta_2)\|^2_{C^0([0,t];\mathrm{H})} \le \Lambda_{11}\,\|G[\zeta_1]-G[\zeta_2]\|^2_{L^2(0,t;\mathrm{H})} \qquad \forall\, t\in[0,T].$$

Hence, from (6.22) it follows that

$$|d_{X_T}(\mathcal{N}(\zeta_1),\mathcal{N}(\zeta_2))|^2 \le \Lambda_{15}\int_0^t |d_{X_s}(\zeta_1,\zeta_2)|^2\,ds \le \Lambda_{15}\,|d_{X_t}(\zeta_1,\zeta_2)|^2\,t \qquad (6.38)$$

for any $t\in[0,T]$ and for $\Lambda_{15}=\Lambda_{11}(\Lambda_9)^2$, the definition of d_{X_t} being obvious. As it is known, inequalities like (6.38) allow you to determine some $m\in\mathbf{N}$ such that $\mathcal{N}^m$ is a contraction from X_T into itself. Hence an application of the generalized Contracting Mapping Principle ends the matter. □

7. Proof of Theorem 2.1

Since the main aim here is to recover the existence of one solution to Problem (SP), we go directly to implement our approximation procedure in terms of the parameter $\varepsilon > 0$. Then we will derive estimates independent of ε and pass to the limit as $\varepsilon \searrow 0$.

Henceforth, we let (2.1–7) hold. For $0 < \varepsilon \le 1$ consider some regularizing sequences $\{\varphi_\varepsilon\}$, $\{f_\varepsilon\}$, $\{\vartheta_0^\varepsilon\}$, $\{\vartheta_{\rm e}^\varepsilon\}$ fulfilling

$$\varphi_\varepsilon \in H^1(0,T) \quad \forall\, \varepsilon \in (0,1], \tag{7.1}$$

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } W^{1,1}(0,T) \text{ as } \varepsilon \searrow 0, \tag{7.2}$$

$$f_\varepsilon \in H^1(0,T;\mathrm{H}), \quad f_\varepsilon(\,\cdot\,,0) \in \mathrm{V} \quad \forall\, \varepsilon \in (0,1], \tag{7.3}$$

$$f_\varepsilon \to f \quad \text{strongly in } W^{1,1}(0,T;\mathrm{H}) \text{ as } \varepsilon \searrow 0, \tag{7.4}$$

$$e_0^\varepsilon \to e_0 \quad \text{weakly in } \mathrm{V} \text{ as } \varepsilon \searrow 0, \tag{7.5}$$

$$\vartheta_{\rm e}^\varepsilon \in W^{2,1}(0,T;L^2(\Gamma)), \quad \vartheta_{\rm e}^\varepsilon(\,\cdot\,,0) = \varphi_0^{-1} e_{0\mathbf{\Gamma}}^\varepsilon \quad \forall\, \varepsilon \in (0,1], \tag{7.6}$$

$$\vartheta_{\rm e}^\varepsilon \to \vartheta_{\rm e} \quad \text{weakly in } H^1(0,T;L^2(\Gamma)) \text{ as } \varepsilon \searrow 0. \tag{7.7}$$

Moreover, for any $\varepsilon \in (0,1]$ we introduce the solution $\vartheta_0^\varepsilon \in \mathrm{V}$ of the elliptic variational equality

$$\langle \vartheta_0^\varepsilon - \varphi_0^{-1} e_0^\varepsilon, v\rangle + \varepsilon(\nabla \vartheta_0^\varepsilon, \nabla v) + \alpha \int_\Gamma (\vartheta_{0\mathbf{\Gamma}}^\varepsilon - \vartheta_{\rm e}^\varepsilon(\,\cdot\,,0))\, v = 0 \quad \forall\, v \in \mathrm{V}. \tag{7.8}$$

One readily sees that ϑ_0^ε solves the boundary value problem

$$\vartheta_0^\varepsilon - \varepsilon \Delta \vartheta_0^\varepsilon = \varphi_0^{-1} e_0^\varepsilon \quad \text{a.e. in } \Omega, \tag{7.9}$$

$$-\,\varepsilon \vartheta_{0\mathbf{n}}^\varepsilon = \alpha(\vartheta_{0\mathbf{\Gamma}}^\varepsilon - \vartheta_{\rm e}^\varepsilon(\,\cdot\,,0)) \quad \text{a.e. on } \Gamma. \tag{7.10}$$

In addition, choosing the test function $v = \left(\vartheta_0^\varepsilon - \varphi_0^{-1} e_0^\varepsilon\right)/\varepsilon$ in (7.8), thanks to (7.5–6) it is straightforward to infer that

$$\frac{2}{\varepsilon}\left\|\vartheta_0^\varepsilon - \varphi_0^{-1} e_0^\varepsilon\right\|^2 + \|\nabla \vartheta_0^\varepsilon\|^2 + \frac{2\alpha}{\varepsilon}\|\vartheta_{0\mathbf{\Gamma}}^\varepsilon - \vartheta_{\rm e}^\varepsilon(\,\cdot\,,0)\|_\Gamma^2$$
$$\le \left\|\nabla\left(\varphi_0^{-1} e_0^\varepsilon\right)\right\|^2 \le \Lambda_{16}\{1 + \|e_0\|_{\mathrm{V}}\}$$

for any $\varepsilon \in (0,1]$ and for some constant Λ_{16} depending only on φ_0. Consequently, the convergences

$$\vartheta_0^\varepsilon \to \varphi_0^{-1} e_0 \quad \text{weakly in } \mathrm{V} \text{ and strongly in } \mathrm{H} \text{ as } \varepsilon \searrow 0 \tag{7.11}$$

and the boundedness

$$\varepsilon\,\|\vartheta_{0\mathbf{n}}^\varepsilon\|_\Gamma^2 \le (\alpha/2)\Lambda_{16}\{1 + \|e_0\|_{\mathrm{V}}\} \quad \forall\, \varepsilon \in (0,1] \tag{7.12}$$

are entailed by (7.5) and (7.10) (we remind that V is compactly embedded into H).

Next, let us approximate the Heaviside graph H by (cf. [5, Appendix and especially Remark 9.1])

$$H_\varepsilon(s) := \begin{cases} 0 & \text{if } \xi \le 0, \\ \left(3\varepsilon s^2 - 2s^3\right)/\varepsilon^3 & \text{if } 0 < s < \varepsilon, \\ 1 & \text{if } \xi \ge \varepsilon. \end{cases} \tag{7.13}$$

Note that, for any $\varepsilon \in (0,1]$,

$$H_\varepsilon : \mathbf{R} \to [0,1] \quad \text{is a maximal monotone graph in } \mathbf{R}^2, \tag{7.14}$$

$$H_\varepsilon \in C^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R}). \tag{7.15}$$

Then our regularized version of (SP) reads

Problem (SP)$_\varepsilon$. Find $\vartheta_\varepsilon \in C^1([0,T];\mathrm{V}) \cap H^2(0,T;\mathrm{H})$ and $\chi_\varepsilon \in L^\infty(Q_T)$ satisfying

$$\Delta\vartheta_\varepsilon \in H^1(0,T;\mathrm{H}), \tag{7.16}$$

$$\vartheta_{\varepsilon\mathbf{n}} \in C^1([0,T];L^2(\Gamma)), \tag{7.17}$$

$$\chi_\varepsilon \in C^0([0,T];\mathrm{V}) \cap W^{1,\infty}(0,T;\mathrm{H}), \tag{7.18}$$

$$\partial_t(\varphi_0\vartheta_\varepsilon + \varphi_\varepsilon * \vartheta_\varepsilon + \psi * \chi_\varepsilon) - \varepsilon\Delta\vartheta_\varepsilon - k * \Delta\vartheta_\varepsilon = f_\varepsilon \quad \text{a.e. in } Q_T, \tag{7.19}$$

$$\chi_\varepsilon = H_\varepsilon(\vartheta_\varepsilon) \quad \text{a.e. in } Q_T, \tag{7.20}$$

$$\vartheta_\varepsilon(\cdot,0) = \vartheta_0^\varepsilon \quad \text{a.e. in } \Omega, \tag{7.21}$$

$$-\varepsilon\vartheta_{\varepsilon\mathbf{n}} - k * \vartheta_{\varepsilon\mathbf{n}} = \alpha\left(\vartheta_{\varepsilon\boldsymbol{\Gamma}} - \vartheta_e^\varepsilon\right) \quad \text{a.e. on } \Sigma_T. \tag{7.22}$$

Referring to the previous section, now we let

$$G[\zeta] := f_\varepsilon - \partial_t(\psi * H_\varepsilon(\zeta)) = f_\varepsilon - \psi(0)H_\varepsilon(\zeta) - \psi' * H_\varepsilon(\zeta) \tag{7.23}$$

for $\zeta \in H^1(0,T;\mathrm{H})$, and observe that this operator obeys (6.19–22) by virtue of (7.3), (7.15), and (2.2) (obviously, the constant in (6.22) blows up as $1/\varepsilon$ does). Thus, on account of (7.20), Problem (SP)$_\varepsilon$ is nothing but a particular case of Problem (NP)$_\varepsilon$ with G given by (7.23). In view of (7.1), (7.5), and (7.9–10), one checks that Theorem 6.2 applies and therefore Problem (SP)$_\varepsilon$ admits a unique solution $(\vartheta_\varepsilon, \chi_\varepsilon)$.

The second step of the proof consists in proving some *a priori* estimates on ϑ_ε which allow us to pass to the limit in (SP)$_\varepsilon$ as $\varepsilon \searrow 0$ and to get a solution to the limit problem (SP). Let us start by noting that (6.30) and (7.23) directly yield

$$\begin{aligned} &\|\vartheta_\varepsilon\|^2_{C^0([0,T];\mathrm{H})} + \varepsilon\|\nabla\vartheta_\varepsilon\|^2_{L^2(0,T;\mathrm{H}^N)} + \|1 * \nabla\vartheta_\varepsilon\|^2_{C^0([0,T];\mathrm{H}^N)} + \|\vartheta_{\varepsilon\boldsymbol{\Gamma}}\|^2_{L^2(\Sigma_T)} \\ &\le \Lambda_{17}\left\{1 + \|f_\varepsilon\|^2_{L^2(0,T;\mathrm{H})} + \|\vartheta_0^\varepsilon\|^2 + \|\vartheta_\mathrm{e}^\varepsilon\|^2_{L^2(\Sigma_T)}\right\}, \end{aligned} \tag{7.24}$$

where the constant Λ_{17} just depends on Λ_{11}, T, and $\|\psi\|_{W^{1,1}(0,T)}$. Then, multiplying equation (7.19) by $-\Delta\vartheta_\varepsilon$ and integrating over Q_t (with $t \in (0,T]$), we obtain the identity corresponding to (6.31). However, here we deal with $I_8(t)$ in a different way. Indeed, (7.23) and standard integrations by parts enable us to deduce

$$\begin{aligned} I_8(t) = & -\langle (f_\varepsilon - \psi' * H_\varepsilon(\vartheta_\varepsilon))(\cdot,t), (1 * \Delta\vartheta_\varepsilon)(\cdot,t)\rangle \\ & + \int_0^t \langle (\partial_t f_\varepsilon - \psi'(0) H_\varepsilon(\vartheta_\varepsilon) - \psi'' * H_\varepsilon(\vartheta_\varepsilon))(\cdot,s), (1 * \Delta\vartheta_\varepsilon)(\cdot,s)\rangle ds \\ & - \iint_{Q_t} \psi(0) H_\varepsilon'(\vartheta_\varepsilon) \left|\nabla\vartheta_\varepsilon\right|^2 + \iint_{\Sigma_t} \psi(0) (H_\varepsilon(\vartheta_\varepsilon))_{\mathbf{\Gamma}} \vartheta_{\varepsilon\mathbf{n}}. \end{aligned} \tag{7.25}$$

As $H_\varepsilon' \geq 0$ in $\mathbf{R}$, recalling (7.14), (2.2), (6.26), and the elementary Young inequality, from (7.25) it is not difficult to find a constant Λ_{18}, independent of ε, such that (see (6.31–34) for our choice of coefficients)

$$\begin{aligned} I_8(t) \leq & \frac{k(0)}{8} \|(1 * \Delta\vartheta_\varepsilon)(\cdot,t)\|^2 + \frac{k(0)\varphi_0}{4\alpha} \|\vartheta_{\varepsilon\mathbf{n}}\|^2_{L^2(\Sigma_t)} \\ & + \Lambda_{18} \left\{ 1 + \|f_\varepsilon(\cdot,t)\|^2 + \int_0^t (1 + \|\partial_t f_\varepsilon(\cdot,s)\|) \|(1 * \Delta\vartheta_\varepsilon)(\cdot,s)\| ds \right) . \end{aligned} \tag{7.26}$$

Using (7.26) in place of (6.35), we argue as for the derivation of (6.36) even though here we need a generalized version of the Gronwall lemma (like, e.g., the one stated in [1]). This procedure leads to

$$\begin{aligned} & \|\nabla\vartheta_\varepsilon\|^2_{L^\infty(0,T;\mathrm{H}^N)} + \varepsilon \|\Delta\vartheta_\varepsilon\|^2_{L^2(0,T;\mathrm{H})} + \|1 * \Delta\vartheta_\varepsilon\|^2_{L^\infty(0,T;\mathrm{H})} \\ & + \varepsilon \|\vartheta_{\varepsilon\mathbf{n}}\|^2_{L^\infty(0,T;L^2(\Gamma))} + \|\vartheta_{\varepsilon\mathbf{n}}\|^2_{L^2(\Sigma_T)} \\ & \leq \Lambda_{19} \left\{ 1 + \|f_\varepsilon\|^2_{W^{1,1}(0,T;\mathrm{H})} + \|\vartheta_0^\varepsilon\|^2_{\mathrm{V}} + \varepsilon \|\vartheta^\varepsilon_{0\mathbf{n}}\|^2_\Gamma + \|\vartheta^\varepsilon_{\mathrm{e}}\|^2_{H^1(0,T;L^2(\Gamma))} \right\}, \end{aligned} \tag{7.27}$$

with the constant Λ_{19} having the same dependences as Λ_1 does. At this point, the estimates (7.24) and (7.27) entail an analogous bound for $\|\partial_t\vartheta_\varepsilon\|^2_{L^2(0,T;\mathrm{H})}$ via a comparison in (7.19). Then, owing to (7.4), (7.11–12), and (7.7) there exists a constant Λ_{20} such that

$$\begin{aligned} & \|\vartheta_\varepsilon\|_{L^\infty(0,T;\mathrm{V})\cap H^1(0,T;\mathrm{H})} + \sqrt{\varepsilon}\, \|\Delta\vartheta_\varepsilon\|_{L^\infty(0,T;\mathrm{H})} + \|1 * \Delta\vartheta_\varepsilon\|_{L^\infty(0,T;\mathrm{H})} \\ & + \sqrt{\varepsilon}\, \|\vartheta_{\varepsilon\mathbf{n}}\|_{L^\infty(0,T;L^2(\Gamma))} + \|\vartheta_{\varepsilon\mathbf{n}}\|_{L^2(\Sigma_T)} \\ & \leq \Lambda_{20} \left\{ 1 + \|f\|_{W^{1,1}(0,T;\mathrm{H})} + \|e_0\|_{\mathrm{V}} + \|\vartheta_{\mathrm{e}}\|_{H^1(0,T;L^2(\Gamma))} \right\}, \end{aligned} \tag{7.28}$$

where Λ_{20} relies on the same quantities as Λ_1 does.

Since (7.28) and (7.20) hold for any $\varepsilon \in (0,1]$, we can pass to the limit along a subsequence and thus infer the existence of ϑ and χ such that

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly star in } L^\infty(0,T;\mathrm{V}) \text{ and weakly in } H^1(0,T;\mathrm{H}), \tag{7.29}$$

$$\varepsilon\Delta\vartheta_\varepsilon \to 0 \quad \text{strongly in } L^2(Q_T), \tag{7.30}$$

$$1 * \Delta\vartheta_\varepsilon \to 1 * \Delta\vartheta \quad \text{weakly star in } L^\infty(0,T;\mathrm{H}), \tag{7.31}$$

$$\varepsilon\vartheta_{\varepsilon\mathbf{n}} \to 0 \quad \text{strongly in } L^\infty(0,T;L^2(\Gamma)), \tag{7.32}$$

$$\vartheta_{\varepsilon\mathbf{n}} \to \vartheta_{\mathbf{n}} \quad \text{weakly in } L^2(\Sigma_T), \tag{7.33}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } L^\infty(Q_T) \tag{7.34}$$

as $\varepsilon \searrow 0$. In particular, (7.29) implies that (cf. (3.20))

$$\vartheta_\varepsilon \to \vartheta \quad \text{strongly in } C^0([0,T];\mathrm{H}) \text{ as } \varepsilon \searrow 0. \tag{7.35}$$

Convergences (7.29–34) combined with (7.2), (7.4), (7.7), and (7.11) allow us to take the limit in (7.19–22) and recover (2.11–14). The details are either trivial or developed in [4, Appendix]. Therefore, the pair (ϑ,χ) solves Problem (SP). Moreover, it satisfies (7.28) because of the weak star lower semicontinuity of norms, whence (2.15) follows by additionally comparing the terms of (2.11) and (2.14) (refer to (6.12) too).

Finally, in order to complete the proof of Theorem 2.1, we just remark that uniqueness is a consequence of (2.16) coupled with a proper reasoning on equation (2.11) to conclude also for the other variable χ. □

8. Proof of Theorem 2.2

We remind that (ϑ_1,χ_1) and (ϑ_2,χ_2) denote the solutions of (SP) corresponding to the respective data ϑ_{e}^1 and ϑ_{e}^2, which both fulfill (2.6–7). Setting $\Theta := \vartheta_1 - \vartheta_2$ and $\mathcal{X} := \chi_1 - \chi_2$, from (2.11) one can easily derive

$$\psi(0)\mathcal{X} + \psi' * \mathcal{X} = \mathcal{L}(\Theta) \qquad \text{a.e. in } Q_T, \tag{8.1}$$

where

$$\mathcal{L}(\Theta) := -\partial_t(\varphi_0\Theta + \varphi * \Theta) + k * \Delta\Theta. \tag{8.2}$$

It is known that (cf., e.g., [10, Chapter 2, Section 3]) (8.1) can be equivalently rewritten as $\psi(0)\mathcal{X} = \mathcal{L}(\Theta) - \Psi * \mathcal{L}(\Theta)$, that is,

$$\partial_t(\varphi_0\Theta + \varphi * \Theta) - k * \Delta\Theta + \psi(0)\mathcal{X} = -\Psi * \mathcal{L}(\Theta) \qquad \text{a.e. in } Q_T, \tag{8.3}$$

the function $\Psi \in W^{1,1}(0,T)$ being the the unique solution to the integral equation

$$\psi(0)\Psi + \psi' * \Psi = \psi' \quad \text{in } [0,T].$$

We point out that

$$\|\Psi\|_{W^{1,1}(0,T)} \leq \Lambda_{21} \tag{8.4}$$

for some constant Λ_{21} which only depends on $\psi(0)$, T, and $\|\psi'\|_{W^{1,1}(0,T)}$.

Next, multiply equation (8.3) by Θ and integrate in space and time over Q_t, for $t \in (0,T]$. On account of (2.13–14) (observe in particular that $\Theta(\cdot,0) = 0$ a.e. in Ω), by the Green formula and (6.12) it is straightforward to check that

$$\frac{\varphi_0}{2}\|\Theta(\cdot,t)\|^2 + \frac{k(0)}{2}\|(1*\nabla\Theta)(\cdot,t)\|^2 + \alpha\,\|\Theta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_t)} \\ + \psi(0)\iint_{Q_t}\mathcal{X}\Theta = \sum_{j=9}^{11} I_j(t), \tag{8.5}$$

where

$$I_9(t) := \iint_{Q_t} \partial_t(\Psi * (\varphi_0\Theta + \varphi*\Theta) - \varphi*\Theta)\Theta,$$

$$I_{10}(t) := \int_0^t (\nabla((\Psi*k - k)' * 1 * \Theta)(\cdot,s), \nabla\Theta(\cdot,s))ds,$$

$$I_{11}(t) := \alpha\iint_{\Sigma_t}(\Theta_{\mathrm{e}} + \Psi*(\Theta_{\mathbf{\Gamma}} - \Theta_{\mathrm{e}}))\Theta_{\mathbf{\Gamma}},$$

and $\Theta_{\mathrm{e}} := \vartheta^1_{\mathrm{e}} - \vartheta^2_{\mathrm{e}}$. In order to estimate $I_9(t)$, for

$$\partial_t(\Psi*(\varphi_0\Theta + \varphi*\Theta) - \varphi*\Theta) = (\varphi_0\Psi(0) - \varphi(0))\Theta + (\varphi_0\Psi' + \Psi'*\varphi - \varphi')*\Theta,$$

one can exploit (8.4), (2.1), and (6.26) to determine a constant Λ_{22}, whose dependences are obvious, such that

$$|I_9(t)| \leq \Lambda_{22}\int_0^t \|\Theta(\cdot,s)\|^2 ds. \tag{8.6}$$

Observing that $\Psi * k \in W^{2,1}(0,T)$ and comparing $I_{10}(t)$ with $I_2(t)$, we have that (cf. (6.28))

$$|I_{10}(t)| \leq \frac{k(0)}{4}\|(1*\nabla\Theta)(\cdot,t)\|^2 + \Lambda_{23}\int_0^t \|(1*\nabla\Theta)(\cdot,s)\|^2 ds, \tag{8.7}$$

the constant Λ_{23} being dependent on $k(0)$, T, $\|\Psi\|_{W^{1,1}(0,T)}$, and $\|k'\|_{W^{1,1}(0,T)}$. On the other hand, (6.26) and standard inequalities lead to

$$|I_{11}(t)| \leq \frac{\alpha}{2}\,\|\Theta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_t)} + \alpha\,\|\Theta_{\mathrm{e}}\|^2_{L^2(\Sigma_t)} \\ + 2\alpha\,\|\Psi\|^2_{L^2(0,T)}\int_0^t \left(\|\Theta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_s)} + \|\Theta_{\mathrm{e}}\|^2_{L^2(\Sigma_s)}\right) ds. \tag{8.8}$$

Now, we collect (8.5–8) and note that

$$\psi(0)\mathcal{X}\Theta \geq 0 \quad \text{a.e. in } Q_T$$

due to (2.2), (2.12), and to the monotonicity of H. Thus, there is a constant Λ_{24}, having the same dependences as Λ_2 does, such that

$$\begin{aligned} &\frac{\varphi_0}{2}\|\Theta(\cdot,t)\|^2 + \frac{k(0)}{4}\|(1*\nabla\Theta)(\cdot,t)\|^2 + \frac{\alpha}{2}\|\Theta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_t)} \\ &\leq \Lambda_{24}\left\{\|\Theta_{\mathrm{e}}\|^2_{L^2(\Sigma_t)} + \int_0^t \left(\|\Theta(\cdot,s)\|^2 + \|(1*\nabla\Theta)(\cdot,s)\|^2 + \|\Theta_{\mathbf{\Gamma}}\|^2_{L^2(\Sigma_s)}\right) ds\right\} \end{aligned}$$

for any $t \in [0,T]$. Finally, we just notice that it is not difficult to recover (2.6) taking advantage of the Gronwall lemma. $\square$

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