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Abstract

We derive two different effective models from a heterogeneous Cosserat continuum taking into account the Cosserat intrinsic length of the constituents. We pass to the limit using homogenization via periodic unfolding and in doing so we provide rigorous proof to the results introduced by Forest, Pradel, and Sab (Int. J. Solids Structures 38 (26-27): 4585-4608 '01). Depending on how different characteristic lengths of the domain scale with respect to the Cosserat intrinsic length, we obtain either an effective classical Cauchy continuum or an effective Cosserat continuum. Moreover, we provide some corrector type results for each case.

1 Introduction

In recent recent years it has been widely observed that mechanical properties of composite materials that are used in a variety of applications depend on different characteristic lengths that are determined by the structure itself or the characteristics of an underlying microstructure or a combination of both [Lak83], [PL86], [Lak93], [Lak95], [ZGBG14], [RL17]. This results in corresponding macroscopic properties of the composite that may be vastly different from the underlying material properties. Moreover, in cases where these characteristic lengths of the problem become comparable with the characteristic length of the microstructure, classical theory of continuum mechanics loses its accuracy in describing the mechanical behavior of such materials. These type of phenomena described above are often referred to as *size effects* and one way of accounting for size effects in composites is to model them using generalized continuum theories. One of the earliest generalized continuum theories was that of the Cosserat brothers [CC09] where they introduced the notion of the couple stress. Their original development of the theory was largely underappreciated during their time only to be revisited again in the early sixties onwards [Tou62], [MT62], [Tou64], [Min64], [Min65], [ME68], [Now72]. Cosserat continuum mechanics incorporates size effects naturally through an intrinsic length scale parameter ℓ_c which, loosely speaking, can be considered as a measure of the absolute size of the constituents in the unit cell. Generalized continuum theories are thought to have applications in the modeling of materials with microstructure, such as granular or fibrous materials, or materials with a lattice structure [Min64], [Min65], [FS98], [TB96], [FPS01]. One of the methods specifically designed for analysis of highly heterogeneous and microstructured materials is the theory of homogenization [BLP78], [SP80], [BP89], [MV10]. In this theory, the effective material properties of periodic structures are defined on the analysis on a periodicity cell, and in turn these properties depend on the mechanics of constituents and the geometry of the periodic structure but are independent of the external boundary conditions and applied forces. They are normally determined in the limit as the size of the microstructure $\varepsilon \to 0$. Moreover, homogenization seems like a natural fit to explore connections between generalized continuum theories and classical theory for heterogenous structures.

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Generalized continuum theories fall into two categories: Higher grade theory that introduces higher gradients of the displacement field to the usual strain tensor and higher order theory that includes additional degrees of freedom. Regarding the former there is a vast literature of deriving second grade (and even third grade [ASd03]) models through homogenization either through two-scale asymptotic expansions or through variational convergence methods e.g. Γ-convergence [TB96], [PS97], [ASd03], [SAd11], [dCG17]. We point out that the authors in [dCG17] provide a historical perspective and theoretical overview of higher grade continua. Regarding the latter, where one allows for additional degrees of freedom, as in a Cosserat continuum, a series of works appeared in the late nineties addressing estimation of effective properties of heterogenous Cosserat materials taking into account size effects [FS98], [FS99], [FPS01]. In particular, the authors in [FPS01] consider periodic heterogenous Cosserat material taking into account a hierarchy of three characteristic lengths when obtaining the homogenous equivalent medium: the characteristic size of inhomogeneities, the Cosserat intrinsic length of the constituents, and the typical size of the considered structure. Heuristically, using two-scale expansions, the authors derived different homogenized models based on how the three characteristic lengths scale with respect to one another and, moreover, validated their results using finite element calculations.

The aim of this work is to provide the mathematical underpinnings that make the work in [FPS01] mathematically rigorous. Specifically, we consider a periodic Cosserat body Ω with body forces and body-couples acting on it,

$$\begin{aligned} -\partial_{x_j}\sigma_{ji} - f_i = 0 \quad & \text{in } \Omega, \\ -\partial_{x_j}\mu_{ji} - \epsilon_{ijk}\sigma_{jk} - g_i = 0 \quad & \text{in } \Omega, \end{aligned} \tag{1.1}$$

with σ_{ji} the non-symmetric strain, μ_{ji} the couple-stress, and ϵ_{ijk} the Levi-Civita tensor. Moreover, the constitutive relations are given by

$$\sigma_{ji} = C_{jik\ell}\gamma_{k\ell} + B_{jik\ell}\kappa_{k\ell}, \quad \mu_{ji} = B_{k\ell ji}\gamma_{k\ell} + L_{jik\ell}\kappa_{k\ell}. \tag{1.2}$$

where $\gamma_{ji} := \partial_{x_j}u_i - \epsilon_{kji}\varphi_k$ is the non-symmetric strain tensor, $\kappa_{ji} := \partial_{x_j}\varphi_i$ is the torsion tensor or curvature-twist tensor or curvature tensor or curvature, $\boldsymbol{u}$ is the displacement, and $\boldsymbol{\varphi}$ is the rotation. In this work we will assume we deal with centro-symmetric bodies and hence the fourth order tensor $B_{ijk\ell} \equiv 0$ [Now72]. Using the dimensional analysis done in [FPS01], to obtain the hierarchy of models based on the scaling of the Cosserat intrinsic length ℓ_c with respect to the overall length of the domain $\mathfrak{L}$ or the length of the periodic cell ℓ, and the periodic unfolding method we pass to the limit in each case. We obtain two different effective models: If ℓ_c/ℓ remains constant when $\ell/\mathfrak{L}$ goes to zero we obtain an effective Cauchy continuum where the effective moduli tensor depends on a standard set of local problems as in classical homogenization and on a set of local problems that contain the contribution of the rotations. If $\ell_c/\mathfrak{L}$ remains constant when $\ell/\mathfrak{L}$ goes to zero then we obtain an effective Cosserat continuum. In both cases we verify the results in [FPS01]. Additionally, we prove certain corrector type results using the adjoint of the unfolding operator (the averaging operator).

The paper is organized as follows: In Section 2 we reproduce the dimensional analysis in [FPS01], provide some background, and set up the model. In Section 3 we recall the definition of the unfolding and averaging operators and prove the main results. Section 4 is devoted to proving certain corrector type results using the averaging operator. We need to remark that we refer to the above as corrector type results as they involve both the displacement and the rotations unlike in classical elasticity where only the displacement is involved. To the author's knowledge, these corrector type results are new in their entirety. Finaly, in Section 5 we provide some conclusions and remarks.

2 Background and set up of the problem

2.1 Cosserat intrinsic length of the constituents

Let $\mathfrak{L}$ be the characteristic length of the domain Ω and ℓ the characteristic length of the periodic cell. We define the dimensionless coordinates, displacement, and rotation as in [FPS01],

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{\mathfrak{L}}, \quad \boldsymbol{u}^*(\boldsymbol{x}^*) = \frac{\boldsymbol{u}(\boldsymbol{x})}{\mathfrak{L}}, \quad \boldsymbol{\varphi}^*(\boldsymbol{x}^*) = \boldsymbol{\varphi}(\boldsymbol{x}). \tag{2.1}$$

In Cosserat media there is another length scale parameter that is of importance, namely, the Cosserat intrinsic length ℓ_c of the constituents [FPS01], [FS98]. The following nondimensionalization was done in [FPS01], [FS98] and we include it here for completion of the presentation. Hence, in accordance with [FPS01], the Cosserat intrinsic length is defined as follows,

$$\mathcal{C} = \mathcal{L}\,\ell_c^2, \tag{2.2}$$

where $\mathcal{L} = \max_{\boldsymbol{z}\in Y_\ell} |L_{jikl}(\boldsymbol{z})|$, $\mathcal{C} = \max_{\boldsymbol{z}\in Y_\ell} |C_{jikl}(\boldsymbol{z})|$, and $Y_\ell = (-\ell/2, \ell/2]^d$ is the periodic cell characterizing the body Ω. Addtionally, the non-symmetric strain and curvature non-dimensionalize respectively as,

$$\gamma^*_{ji} = \gamma_{ji} \text{ and } \kappa^*_{ji} = \mathfrak{L}\,\kappa_{ji}. \tag{2.3}$$

Moreover, we define the nondimensional stress, couple-stress, and fourth order material tensors as follows,

$$\begin{aligned} \sigma^*_{ji} &= \mathcal{L}^{-1}\sigma_{ji}, & L^*_{jik\ell}(\boldsymbol{x}^*) &= \mathcal{L}^{-1}L_{jik\ell}(\boldsymbol{x}), \\ \mu^*_{ji} &= (\mathcal{L}\mathfrak{L})^{-1}\mu_{ji}, & C^*_{jik\ell}(\boldsymbol{x}^*) &= \mathcal{C}^{-1}C_{jik\ell}(\boldsymbol{x}) \end{aligned} \tag{2.4}$$

We remark that the fourth order tensors $L^*_{jik\ell}(\boldsymbol{x}^*)$ and $\mathcal{C}^*_{jik\ell}(\boldsymbol{x}^*)$ are Y^* periodic where,

$$Y^* = \frac{\ell}{\mathfrak{L}}Y, \quad Y := \left(-\frac{1}{2}, \frac{1}{2}\right]^d. \tag{2.5}$$

Hence, the system of equations in (1.1) scales as,

$$\begin{aligned} -\partial_{x^*_j}\sigma^*_{ji} - f^*_i &= 0 && \text{in } \Omega, \\ -\partial_{x^*_j}\mu_{ji} - \epsilon_{ijk}\sigma^*_{jk} - g^*_i &= 0 && \text{in } \Omega, \end{aligned} \tag{2.6}$$

where f_i^*, and g_i^* are the appropriately scaled body forces and body couples (see [FPS01, Eq. (14), pg. 4589]) and with constitutive laws,

$$\sigma_{ji}^* = C_{jik\ell}^* \gamma_{k\ell}, \quad \mu_{ji}^* = \left(\frac{\ell_c}{\mathfrak{L}}\right)^2 L_{jik\ell}^* \kappa_{k\ell}^*. \tag{2.7}$$

Thus, one can generate an ε periodic problem by defining the nondimensional number ε as the ratio of $\ell/\mathfrak{L}$ and let $\varepsilon \to 0$ to obtain an effective medium. However, different cases ought to be considered depending on how ℓ_c scales with ℓ and $\mathfrak{L}$, respectively, as $\varepsilon \to 0$ [FPS01]. Here we consider the cases

$$\ell_c/\ell \sim 1, \tag{2.8}$$

$$\ell_c/\mathfrak{L} \sim 1. \tag{2.9}$$

If $\ell_c/\ell \sim 1$ then $\mu_{ji}^* = \left(\frac{\ell_c}{\mathfrak{L}}\right)^2 L_{jik\ell}^* \kappa_{k\ell}^*$, using the definition $\varepsilon = \ell/\mathfrak{L}$ and omitting the $*$ notation, becomes,

$$\mu_{ji}^\varepsilon = \varepsilon^2 L_{jik\ell}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\kappa_{k\ell}^\varepsilon. \tag{HS 1}$$

If $\ell_c/\mathfrak{L} \sim 1$ then $\mu_{ji}^* = \left(\frac{\ell_c}{\mathfrak{L}}\right)^2 L_{jik\ell}^* \kappa_{k\ell}^*$ becomes,

$$\mu_{ji}^\varepsilon = L_{jik\ell}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)\kappa_{k\ell}^\varepsilon. \tag{HS 2}$$

The former allows one to pass from a Cosserat continuum in the microscale to a Cauchy continuum in the macroscale, as $\varepsilon \to 0$, where all the relevant information are now captured in a new homogenized tensor which can be computed explicitly with the aid of an additional set of local problems. The latter allows one to obtain a Cosserat effective medium as $\varepsilon \to 0$.

Notation

In what follows $\alpha, \beta \in \mathbb{R}$ are generic constants such that $0 < \alpha$ and $0 < \beta$.

- Throughout the article we employ the Einstein summation notation of repeated indices unless otherwise stated.
- $\mathcal{M}_d^4(\alpha, \beta, \Omega)$ = $\big\{$all fourth order tensors in $L^\infty(\Omega; \mathbb{R}^{d\times d\times d\times d})$ acting on matrices such that for any matrix $\zeta \in \mathbb{R}^{d\times d}$, $L(\boldsymbol{x})\,\zeta : \zeta \geq \alpha|\zeta|^2$ and $\beta\,|\zeta|^2 \leq L^{-1}(\boldsymbol{x})\,\zeta : \zeta$ for a.e. $\boldsymbol{x} \in \Omega\big\}$
- Any general fourth order tensor of the form $T_{ijk\ell}^\varepsilon(\boldsymbol{x})$ is defined, as usual, by $T_{ijk\ell}^\varepsilon(\boldsymbol{x}) := T_{ijk\ell}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big)$
- In addition to the standard Sobolev space $H^1(\Omega) := W^{1,2}(\Omega)$ we define the following spaces:

$$H^1_{\Gamma_0}(\Omega) = \big\{ w \in H^1(\Omega) \mid \operatorname{Tr}(w) = 0 \text{ on } \Gamma_0 \big\}$$

$$H(\text{curl}; \Omega) = \Big\{ w \in L^2(\Omega; \mathbb{R}^d) \mid \text{curl}(w) \in L^2(\Omega; R^d) \Big\}$$

- The third order tensor ϵ_{ijk} is the Levi-Civita symbol that is equal to 1 if (i,j,k) is an even permutation of $(1,2,3)$, -1 if it is an odd permuation, and zero if any index is repeated.
- We set $\widetilde{L}^\varepsilon(\boldsymbol{x})$ to be a general place holder under the schemes (HS 1) and (HS 2) as follows,

$$\widetilde{L}^\varepsilon(\boldsymbol{x}) := \varepsilon^2\, L^\varepsilon(\boldsymbol{x}) \text{ under the scheme (HS 1)}, \tag{2.10}$$

$$\widetilde{L}^\varepsilon(\boldsymbol{x}) := L^\varepsilon(\boldsymbol{x}) \text{ under the scheme (HS 2)}. \tag{2.11}$$

2.2 The model

We consider an elastic composite with periodic microstructure of period ε occupying a region $\Omega \subset \mathbb{R}^d,\ d = 2,3$. The region Ω that the composite occupies, is assumed to be bounded, open, and multiply connected. $Y = (-1/2, 1/2]^d$ is the unit cube in $\mathbb{R}^d$ and $\mathbb{Z}^d$ is the set of all d–dimensional vectors with integer components.

For every positive ε, let N_ε be the set of all points $m \in \mathbb{Z}^d$ such that $\varepsilon(m+Y)$ is strictly included in Ω and denote by $|N_\varepsilon|$ their total number. Let T be the closure of an open connected set with sufficiently smooth boundary, compactly included in Y. We define, for every $\varepsilon > 0$ and $m \in N_\varepsilon$, $T_m^\varepsilon := \varepsilon(m+T)$ as the region containing the distribution of space charges and by $S_m^\varepsilon = \partial T_m^\varepsilon$ denote the interphase boundary separating the region from the ambient surrounding material (see Fig. 3.1). We now define the following subsets of Ω:

$$\Omega_{1\varepsilon} := \bigcup_{m\in N_\varepsilon} T_m^\varepsilon\,, \quad \Omega_{2\varepsilon} := \Omega\backslash\overline{\Omega}_{1\varepsilon}, \quad \Omega := \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon} \cup (\cup_{m\in N_\varepsilon} S_m^\varepsilon).$$

Moreover, we denote by $\partial\Omega$ the boundary of Ω. The exterior will be denoted by Γ_0 and by S_m^ε, $m \in N_\varepsilon$ the remaining components of $\partial\Omega$. The vector $\boldsymbol{n}$ will be unit normal on Γ_0 pointing in the outward direction.

The heterogeneous Cosserat continuum is characterized by the following coupled system,

$$\begin{aligned}
-\partial_{x_j}\sigma_{ji}^\varepsilon - f_i = 0 \qquad & \text{in } \Omega,\\
-\partial_{x_j}\mu_{ji}^\varepsilon - \epsilon_{ijk}\sigma_{jk}^\varepsilon - g_i = 0 \qquad & \text{in } \Omega,\\
\boldsymbol{u}^\varepsilon = \boldsymbol{0} \qquad & \text{on } \Gamma_0,\\
\boldsymbol{\varphi}^\varepsilon = \boldsymbol{0} \qquad & \text{on } \Gamma_0.
\end{aligned} \tag{2.12}$$

Here σ_{ji}^ε is the stress while μ_{ji}^ε is the couple stress. Moreover, $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{\varphi}^\varepsilon$ are the displacement and rotation vector fields, respectively. The system of equations (2.12) characterizes the mechanical deformation that the body undergoes. The equations are fully coupled and the system is closed with homogeneous Dirichlet boundary conditions on Γ_0. In addition to computing the displacement we must also compute the rigid rotations which makes for a fully coupled system of partial differential equations.

Strain and torsion tensors. Define $\gamma_{ji}^\varepsilon := \partial_{x_j}u_i^\varepsilon - \epsilon_{kji}\varphi_k^\varepsilon$ and $\kappa_{ji}^\varepsilon := \partial_{x_j}\varphi_i^\varepsilon$. The term γ_{ji}^ε is the (non-symmetric) strain tensor and κ_{ji}^ε is referred to as the torsion tensor (or curvature-twist tensor or curvature tensor or curvature).

Constitutive relations. The stress is related to the strain though the fourth order material tensor $C_{jik\ell}^\varepsilon(\boldsymbol{x})$ by the relation,

$$\sigma_{ji}^\varepsilon = C_{jik\ell}^\varepsilon\gamma_{k\ell}^\varepsilon, \tag{2.13}$$

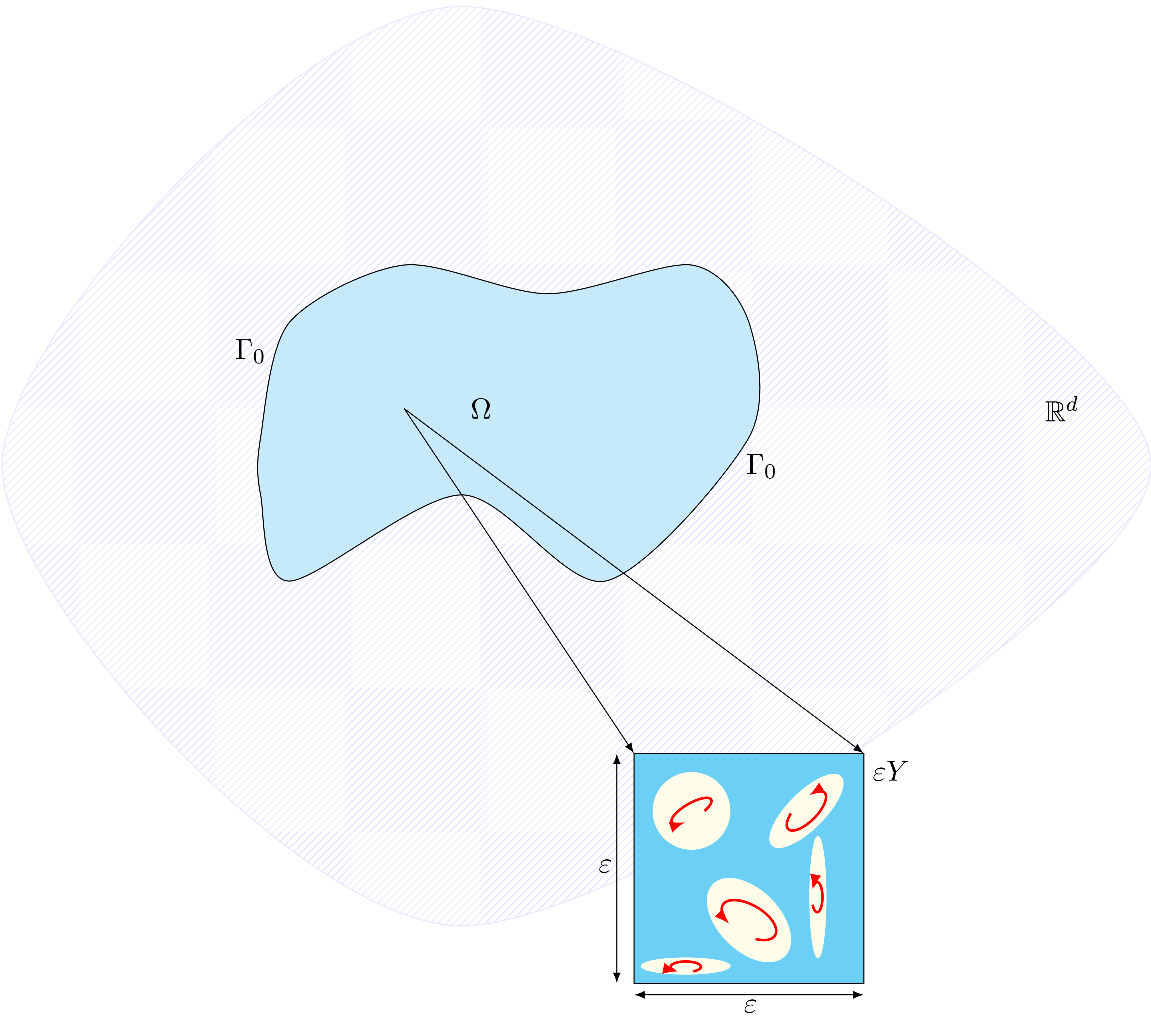

Figure 2.1: Schematic of the heterogeneous Cosserat medium in vaccum

while the couple stress is related to the curvature through the fourth order material tensor $\widetilde{L}^{\varepsilon}_{jik\ell}(\boldsymbol{x})$ by the relation

$$\mu^{\varepsilon}_{ji} = \widetilde{L}^{\varepsilon}_{jik\ell}\kappa^{\varepsilon}_{k\ell}. \tag{2.14}$$

The tensors $C^{\varepsilon}_{jik\ell}$ and $\widetilde{L}^{\varepsilon}_{jik\ell}$ are linear isotropic tensors in each phase and have the following form [Now72],

$$C^{\varepsilon}_{jik\ell}(\boldsymbol{x}) = (\mu_{\varepsilon}(\boldsymbol{x}) + \alpha_{\varepsilon}(\boldsymbol{x}))\,\delta_{jk}\,\delta_{i\ell} + (\mu_{\varepsilon}(\boldsymbol{x}) - \alpha_{\varepsilon}(\boldsymbol{x}))\,\delta_{j\ell}\,\delta_{ik} + \lambda_{\varepsilon}(\boldsymbol{x})\,\delta_{ij}\,\delta_{k\ell}, \tag{2.15}$$

$$\widetilde{L}^{\varepsilon}_{jik\ell}(\boldsymbol{x}) = (\theta_{\varepsilon}(\boldsymbol{x}) + \varrho_{\varepsilon}(\boldsymbol{x}))\,\delta_{jk}\,\delta_{i\ell} + (\theta_{\varepsilon}(\boldsymbol{x}) - \varrho_{\varepsilon}(\boldsymbol{x}))\,\delta_{j\ell}\,\delta_{ik} + \beta_{\varepsilon}(\boldsymbol{x})\,\delta_{ij}\,\delta_{k\ell}, \tag{2.16}$$

respectively. The coefficients $\mu_{\varepsilon}(\boldsymbol{x}) := \mu_{\varepsilon}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$, $\alpha_{\varepsilon}(\boldsymbol{x}) := \alpha\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$, $\lambda_{\varepsilon}(\boldsymbol{x}) := \lambda\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$, $\beta_{\varepsilon}(\boldsymbol{x}) := \beta\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$,

$\theta_\varepsilon(\boldsymbol{x}) := \theta\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$, $\varrho_\varepsilon(\boldsymbol{x}) := \varrho\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$ are material parameters that are piecewise constant in each phase and periodic.

The thermodynamic stability relations immediately yield that $C^\varepsilon_{jik\ell} = C^\varepsilon_{k\ell ji}$ and $\widetilde{L}^\varepsilon_{jik\ell} = \widetilde{L}^\varepsilon_{k\ell ji}$. Moreover, from the constitutive laws we have assumed above, (2.13) and (2.14), we content ourselves in the case of "centrosymmetric medium" (see [Now72]).

2.3 Assumptions

We frame the heterogeneous Cosserat continuum model (2.12), (2.13), (2.14) under the following general assumptions:

- Ω is a bounded, multiply connected domain such that $\mathrm{mes}(\Gamma_0) > 0$, $\mathrm{mes}(S^\varepsilon_\ell) > 0$, and $S^\varepsilon_\ell \cap S^\varepsilon_p = \emptyset$ for $\ell \neq p$.
- Γ_0 and S^ε_ℓ are surfaces of class C^2, $S^\varepsilon_p \cap S^\varepsilon_q = \emptyset$ for $p, q \in N_\varepsilon$ with $p \neq q$, and $\Gamma_0 \cap S^\varepsilon_\ell = \emptyset$ for every $\ell \in N_\varepsilon$.
- The functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are such that $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ and $\boldsymbol{g} \in H(\mathrm{curl},\Omega)$.
- The fourth order material characterization tensors are such that $C^\varepsilon_{jik\ell}(\boldsymbol{x}) \in \mathcal{M}^4_d(\alpha,\beta,\Omega)$ and $\widetilde{L}^\varepsilon_{jik\ell}(\boldsymbol{x}) \in \mathcal{M}^4_d(\alpha,\beta,\Omega)$.

Existence and uniqueness. The Cosserat brothers [CC09] developed their theory to be derived from the principle of least action of Hamilton. Starting from total energy of the system described in (2.12),

$$\mathcal{E}_\varepsilon[\boldsymbol{v},\boldsymbol{\psi}] = \frac{1}{2}\int_\Omega C^\varepsilon\,\gamma : \gamma\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega \widetilde{L}^\varepsilon\,\kappa : \kappa\, d\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x} - \int_\Omega \boldsymbol{g}\cdot\boldsymbol{\psi}\, d\boldsymbol{x}, \tag{2.17}$$

where $C^\varepsilon\,\gamma : \gamma = C^\varepsilon_{jik\ell}\,\gamma_{\ell k}\,\gamma_{ji}$ and $\widetilde{L}^\varepsilon\,\kappa : \kappa = \widetilde{L}^\varepsilon_{jik\ell}\,\kappa_{\ell k}\,\kappa_{ji}$.

One can readily observe that the above energy is sequentially weakly lower semicontinuous and coersive by [HH69, Thm. 3.1] (see also [Nec67], [HN70]) in $H^1_{\Gamma_0}(\Omega,\mathbb{R}^d) \times H^1_{\Gamma_0}(\Omega,\mathbb{R}^d)$ and, moreover, the following estimates hold under (HS 1) and (HS 2), respectively:

$$\begin{aligned}\left(\|\boldsymbol{u}^\varepsilon\|^2_{H^1_0(\Omega,\mathbb{R}^d)} + \left(\|\boldsymbol{\varphi}^\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^d)} + \varepsilon\, \|\nabla\boldsymbol{\varphi}^\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\right)\right)^{1/2} \\ \leq c\left(\|\boldsymbol{f}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{g}\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)^{1/2},\end{aligned} \tag{2.18}$$

$$\left(\|\boldsymbol{u}^\varepsilon\|^2_{H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\varphi}^\varepsilon\|^2_{H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)}\right)^{1/2} \leq c\left(\|\boldsymbol{f}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{g}\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)^{1/2}, \tag{2.19}$$

for some generic constant c independent of ε. Additionally, it is convex in the arguments $\nabla\boldsymbol{u}$ and $\nabla\boldsymbol{\varphi}$. Hence, using the compact embedding of Rellich-Kondrachov we can apply the direct method to obtain existence and uniqueness. Hence, we can characterize the solution to (2.12) as the unique minimizer of,

$$(\boldsymbol{u}^\varepsilon,\boldsymbol{\varphi}^\varepsilon) = \mathrm{argmin}_{(\boldsymbol{v},\boldsymbol{\psi})\in H^1_{\Gamma_0}(\Omega,\mathbb{R}^d)\times H^1_{\Gamma_0}(\Omega,\mathbb{R}^d)}\,\mathcal{E}_\varepsilon[\boldsymbol{v},\boldsymbol{\psi}]. \tag{2.20}$$

By computing the first variation of $\mathcal{E}_\varepsilon$ we obtain the Euler-Lagrange equations,

$$\int_\Omega \sigma^\varepsilon_{jk} \left(\frac{\partial v_k}{\partial x_j} - \epsilon_{ijk}\psi_i \right) d\boldsymbol{x} - \int_\Omega \mu^\varepsilon_{ji} \frac{\partial \psi_i}{\partial x_j} d\boldsymbol{x} - \int_\Omega (f_i\, v_i + g_i\, \psi_i)\, d\boldsymbol{x} = 0. \tag{2.21}$$

If we group terms in the Euler-Lagrange equation above we obtain the weak form of (2.12) which reads as follows: Find $(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^d) \times H^1_{\Gamma_0}(\Omega, \mathbb{R}^d)$ such that,

$$\int_\Omega \sigma^\varepsilon_{ji} \frac{\partial v_i}{\partial x_j} d\boldsymbol{x} - \int_\Omega f_i\, v_i\, d\boldsymbol{x} = 0 \text{ for all } \boldsymbol{v} \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^d), \tag{2.22}$$

$$\int \mu^\varepsilon_{ji} \frac{\partial \psi_i}{\partial x_j} d\boldsymbol{x} - \int_\Omega \epsilon_{ijk}\sigma^\varepsilon_{jk}\, \psi_i\, d\boldsymbol{x} - \int_\Omega g_i\, \psi_i\, d\boldsymbol{x} = 0 \text{ for all } \boldsymbol{\psi} \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^d). \tag{2.23}$$

Using (2.22)–(2.23) we can recover the strong form of the equations in (2.12) in the sense of distributions as usual.

3 Homogenization of the Cosserat continuum

In the next two subsections we recall the of definitions and properties of the *periodic unfolding* and *averaging* operators [CDG02, Dam05, CDG08, CDG18] and present our main results. We will list some of their properties, leaving the interested reader to consult [CDG02, Dam05, CDG08] for further details regarding proofs.

3.1 The periodic unfolding and averaging operators

We define the following domain decompositions:

$$\Omega^-_\varepsilon := \operatorname{int}\left(\cup_{\ell \in K^-_\varepsilon} \varepsilon(\ell + Y) \right) \text{ with } K^-_\varepsilon := \left\{ \ell \in \mathbb{Z}^d \mid \varepsilon(\ell + Y) \subset \overline{\Omega} \right\}, \tag{3.1}$$

$$\Omega^+_\varepsilon := \operatorname{int}\left(\cup_{\ell \in K^+_\varepsilon} \varepsilon(\ell + Y) \right) \text{ with } K^+_\varepsilon := \left\{ \ell \in \mathbb{Z}^d \mid \varepsilon(\ell + Y) \cap \Omega \neq \emptyset \right\}. \tag{3.2}$$

Additionally, we define $\Lambda^-_\varepsilon := \Omega \backslash \Omega^-_\varepsilon$. It is now evident that $\Omega^-_\varepsilon \subset \Omega \subset \Omega^+_\varepsilon$ (see Fig. 3.1 (left)) and moreover, one can show that $\mathrm{mes}(\Omega^+_\varepsilon \backslash \Omega^-_\varepsilon) \to 0$ [Han11].

Let $[\boldsymbol{z}]_Y = (\lfloor z_1 \rfloor, \ldots, \lfloor z_d \rfloor)$ denote the integer part of $\boldsymbol{z} \in \mathbb{R}^d$ and denote by $\{\boldsymbol{z}\}_Y$ the difference $\boldsymbol{z} - [\boldsymbol{z}]_Y$ which belongs to Y. Regarding our multiscale problem that depends on a small length parameter $\varepsilon > 0$, we can decompose any $\boldsymbol{x} \in \mathbb{R}^d$ using the maps $[\cdot]_Y : \mathbb{R}^d \mapsto \mathbb{Z}^d$ and $\{\cdot\}_Y : \mathbb{R}^d \mapsto Y$ the following way (see Fig. 3.1 (right)),

$$\boldsymbol{x} = \varepsilon \left(\left[\frac{\boldsymbol{x}}{\varepsilon} \right]_Y + \left\{ \frac{\boldsymbol{x}}{\varepsilon} \right\}_Y \right). \tag{3.3}$$

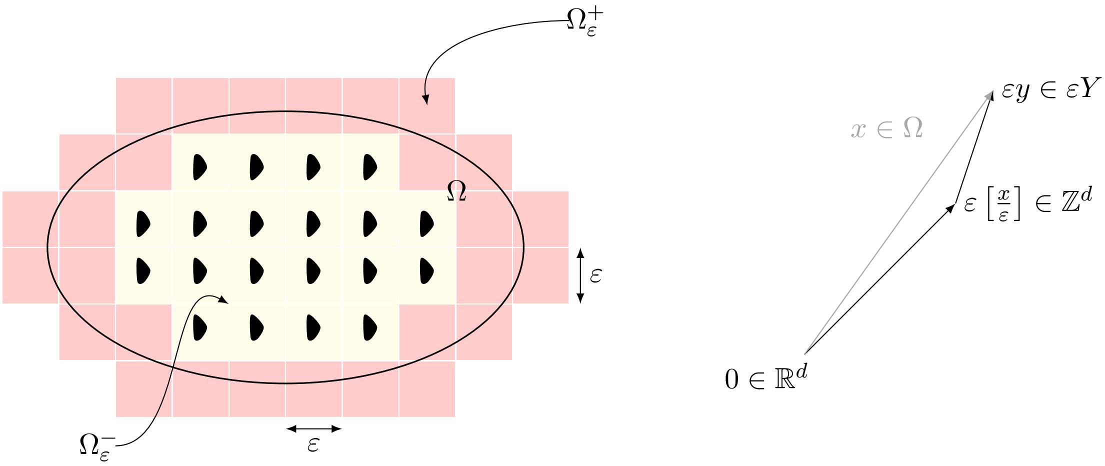

Figure 3.1: Unfolding operator on a periodic grid

For any Lebesgue measurable function φ on Ω we define the periodic unfolding operator by,

$$\mathcal{T}_\varepsilon(\varphi)(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \varphi\left(\varepsilon\left[\frac{\boldsymbol{x}}{\varepsilon}\right]_Y + \varepsilon\boldsymbol{y}\right) & \text{for a.e. } (\boldsymbol{x},\boldsymbol{y}) \in \Omega_\varepsilon^- \times Y \\ 0 & \text{for a.e. } (\boldsymbol{x},\boldsymbol{y}) \in \Lambda_\varepsilon^- \times Y. \end{cases} \tag{3.4}$$

Proposition 3.1. *For any $p \in [1,+\infty)$ the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \mapsto L^p(\Omega \times Y)$ is linear, continuous, and has the following properties:*

i. $\mathcal{T}_\varepsilon(\varphi\,\psi) = \mathcal{T}_\varepsilon(\varphi)\,\mathcal{T}_\varepsilon(\psi)$ *for every pair of Lebesgue measurable functions* φ, ψ *on* Ω

ii. *For every* $\varphi \in L^1(\Omega)$ *we have,*

$$\frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{x}\,d\boldsymbol{y} = \int_{\Omega_\varepsilon^-} \varphi(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\Omega_\varepsilon} \varphi(\boldsymbol{x})\,d\boldsymbol{x} - \int_{\Lambda_\varepsilon^-} \varphi(\boldsymbol{x})\,d\boldsymbol{x} \tag{3.5}$$

iii. $\|\mathcal{T}_\varepsilon(\varphi)\|_{L^p(\Omega\times Y)} \leq |Y|^{1/p}\,\|\varphi\|_{L^p(\Omega)}$ *for every* $\varphi \in L^p(\Omega)$

iv. $\mathcal{T}_\varepsilon(\varphi) \to \varphi$ *strongly in* $L^p(\Omega \times Y)$ *for* $\varphi \in L^p(\Omega)$ *as* $\varepsilon \to 0$

v. *If* $\{\varphi_\varepsilon\}_\varepsilon$ *is a sequence in* $L^p(\Omega)$ *such that* $\varphi_\varepsilon \to \varphi$ *strongly in* $L^p(\Omega)$*, then* $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi$ *strongly in* $L^p(\Omega \times Y)$

vi. *If* $\varphi \in L^p(Y)$ *is Y-periodic and* $\varphi_\varepsilon(\boldsymbol{x}) = \varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$ *then* $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi$ *strongly in* $L^p(\Omega \times Y)$ *as* $\varepsilon \to 0$

vii. *If* $\phi_\varepsilon \rightharpoonup \phi$ *in* $H^1(\Omega)$ *then there exists an non-relabelled subsequence and a* $\hat{\phi} \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ *such that*

   a. $\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \phi$ *in* $L^2(\Omega; H^1(Y))$
   b. $\mathcal{T}_\varepsilon(\nabla\phi_\varepsilon) \rightharpoonup \nabla_x\phi + \nabla_y\hat{\phi}$ *in* $L^2(\Omega \times Y)$

viii. *Let $\{\phi_\varepsilon\}_\varepsilon \in H^1(\Omega)$ and assume that $\{\phi_\varepsilon\}_\varepsilon$ is a bounded sequence in $L^2(\Omega)$ satisfying $\varepsilon \left\|\nabla \phi_\varepsilon\right\|_{L^2(\Omega;\mathbb{R}^d)} \leq c$ (c is a constant independent of ε) then there exists an non-relabelled subsequence and a $\hat{\phi} \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ such that*

   a. $\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \hat{\phi}$ *in* $L^2(\Omega; H^1(Y))$
   b. $\varepsilon\, \mathcal{T}_\varepsilon(\nabla \phi_\varepsilon) \rightharpoonup \nabla_y \hat{\phi}$ *in* $L^2(\Omega \times Y)$

In a similar fashion we define the averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega \times Y) \to L^p(\Omega)$ for $p \in [1, +\infty)$, which acts as a pseudo-inverse of the unfolding operator, by:

$$\mathcal{U}_\varepsilon(\Phi)(\boldsymbol{x}) = \begin{cases} \int_Y \Phi\left(\varepsilon\left[\frac{\boldsymbol{x}}{\varepsilon}\right] + \varepsilon \boldsymbol{z}, \left\{\frac{\boldsymbol{x}}{\varepsilon}\right\}\right) d\boldsymbol{z} & \text{for a.e. } \boldsymbol{x} \in \Omega_\varepsilon^- \\ 0 & \text{for a.e. } \boldsymbol{x} \in \Lambda_\varepsilon^- . \end{cases} \tag{3.6}$$

Proposition 3.2. *For any $p \in [1, +\infty)$ the averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega \times Y) \mapsto L^p(\Omega)$ has the following properties:*

i. *If $\{w_\varepsilon\}_\varepsilon \in L^p(\Omega \times Y)$ is a bounded sequence such that $w_\varepsilon \rightharpoonup w$ in $L^p(\Omega \times Y)$ as $\varepsilon \to 0$ then*

$$\mathcal{U}_\varepsilon(w_\varepsilon) \rightharpoonup \int_Y w(\cdot, \boldsymbol{y})\, d\boldsymbol{y} \text{ in } L^p(\Omega). \tag{3.7}$$

   If w is independent of $\boldsymbol{y}$ then the convergence above is strong (see [CDG08, Cor. 2.26, pg. 1599]).

ii. *If $\{w_\varepsilon\}_\varepsilon$ is a sequence in $L^p(\Omega)$ then the following are equivalent:*

   a. $\mathcal{T}_\varepsilon(w_\varepsilon) \to \hat{w}$ *in* $L^p(\Omega \times Y)$
   b. $w_\varepsilon\, \mathbb{I}_{\Omega_\varepsilon^-} - \mathcal{U}_\varepsilon(\hat{w}) \to 0$ *in* $L^p(\Omega)$

iii. *If $\{w_\varepsilon\}_\varepsilon$ is a sequence in $L^p(\Omega)$ then the following are equivalent:*

   a. $\mathcal{T}_\varepsilon(w_\varepsilon) \to \hat{w}$ *in* $L^p(\Omega \times Y)$ *and* $\int_{\Lambda_\varepsilon} |w_\varepsilon|^p\, d\boldsymbol{x} \to 0$
   b. $w_\varepsilon - \mathcal{U}_\varepsilon(\hat{w}) \to 0$ *in* $L^p(\Omega)$

3.2 Main results

3.2.1 Homogenization under the $\mathrm{HS}\ 1$ scheme

Theorem 3.1. *If $(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}^\varepsilon)$ is the solution set to (2.12) then, under the $\mathrm{HS}\ 1$ scheme, there exist $\boldsymbol{u}^0 \in H^1(\Omega; \mathbb{R}^d)$, $\boldsymbol{u}^1 \in L^2(\Omega; H^1_{per}(Y; \mathbb{R}^d))$, $\boldsymbol{\varphi}^0 \in L^2(\Omega; H^1_{\mathrm{per}}(Y; \mathbb{R}^d))$ such that*

$$\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) \rightharpoonup \boldsymbol{u}^0 \text{ in } L^2(\Omega; H^1(Y; \mathbb{R}^d)) \tag{3.8}$$
$$\mathcal{T}_\varepsilon(\nabla \boldsymbol{u}^\varepsilon) \rightharpoonup \nabla_x \boldsymbol{u}^0 + \nabla_y \boldsymbol{u}^1 \text{ in } L^2(\Omega \times Y; \mathbb{R}^{d\times d}) \tag{3.9}$$
$$\mathcal{T}_\varepsilon(\boldsymbol{\varphi}^\varepsilon) \rightharpoonup \boldsymbol{\varphi}^0 \text{ in } L^2(\Omega; H^1(Y; \mathbb{R}^d)) \tag{3.10}$$

and $(\boldsymbol{u}^0, \boldsymbol{\varphi}^0, \boldsymbol{u}^1)$ is the unique solution set of

$$\begin{aligned}\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell} u_k^0 + \partial_{x_\ell} u_k^1 - \epsilon_{\nu k\ell}\varphi_\nu^0 - \frac{1}{2}\epsilon_{\nu ij} g_\nu\Big)\,(\partial_{x_i} V_j + \partial_{y_i}\overline{W}_j)\,d\boldsymbol{y}\,d\boldsymbol{x} \\ - \frac{1}{2}\int_{\Omega\times Y} \epsilon_{\nu ij}\,g_\nu\,\,(\partial_{x_i} V_j + \partial_{y_i}\overline{W}_j)\,d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega f_i\,V_i\,d\boldsymbol{x} = 0,\end{aligned} \tag{3.11}$$

for all $\boldsymbol{V} \in H_0^1(\Omega;\mathbb{R}^d)$ and $\overline{\boldsymbol{W}} \in L^2(\Omega; H^1(Y;\mathbb{R}^d))$. If in addition $\boldsymbol{u}^1$ and $\boldsymbol{\varphi}^0$ have the following form,

$$u_i^1(\boldsymbol{x},\boldsymbol{y}) = \zeta_i^{k\ell}(\boldsymbol{y})\,\partial_{x_\ell} u_k^0(\boldsymbol{x}) + c_i(\boldsymbol{x}) \tag{3.12}$$

$$\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y}) = \xi_\nu^{k\ell}(\boldsymbol{y})\,\partial_{x_\ell} u_k^0(\boldsymbol{x}) \tag{3.13}$$

and we select $\overline{\boldsymbol{W}} \equiv \mathbf{0}$ then equation (3.25) *takes the following more familiar form:*

$$\int_\Omega \sigma_{ij}^{\text{eff}}\,\partial_{x_j} V_i\,d\boldsymbol{x} = \frac{1}{2}\int_\Omega \epsilon_{\nu ij}\,g_\nu\,\,\partial_{x_i} V_j\,d\boldsymbol{x} + \int_\Omega f_i\,V_i\,d\boldsymbol{x}, \tag{3.14}$$

where $\sigma_{ij}^{\text{eff}} := \Big(C_{ijpq}^{\text{eff}}\,\partial_{x_q} u_p^0 - \frac{1}{2}\epsilon_{\nu ij}\,g_\nu\Big)$ is the Cauchy stress in classical elasticity with symmetry $\sigma_{ij}^{\text{eff}} = \sigma_{ji}^{\text{eff}}$ and

$$C_{jipq}^{\text{eff}} = \int_Y C_{jik\ell}(\boldsymbol{y})\left(\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\zeta_k^{pq} - \epsilon_{\nu k\ell}\xi_\nu^{pq}\right)\,d\boldsymbol{y}. \tag{3.15}$$

Furthermore, $\boldsymbol{\zeta}^{k\ell}$ and $\boldsymbol{\xi}^{k\ell}$ are the local solutions satisfying the following variational problems,

$$\int_Y C_{jik\ell}\,\left(\frac{1}{2}\,\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\zeta_k^{pq}\right)\partial_{y_i} v_j\,d\boldsymbol{y} = 0 \text{ for all } \boldsymbol{v} \in H_{\text{per}}^1(Y;\mathbb{R}^d), \tag{3.16}$$

$$\int_Y C_{jik\ell}\,\left(\frac{1}{2}\,\delta_{kp}\,\delta_{q\ell} + \epsilon_{\nu k\ell}\,\xi_\nu^{pq}\right)\,\partial_{y_i} v_j\,d\boldsymbol{y} = 0 \text{ for all } \boldsymbol{v} \in H_{\text{per}}^1(Y;\mathbb{R}^d). \tag{3.17}$$

Proof. The weak form (2.21) of the Cosserat continuum reads: Find $(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in H_{\Gamma_0}^1(\Omega_0;\mathbb{R}^d) \times H_{\Gamma_0}^1(\Omega;\mathbb{R}^d)$ such that,

$$\begin{aligned}\int_\Omega C_{jik\ell}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)(\partial_{x_\ell} u_k^\varepsilon - \epsilon_{\nu k\ell}\varphi_\nu^\varepsilon)\,(\partial_{x_j} v_i - \epsilon_{\nu ij}\psi_\nu)\,d\boldsymbol{x} \\ - \int_\Omega \varepsilon^2\,L_{jik\ell}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\,\partial_{x_\ell}\varphi_k^\varepsilon\,\partial_{x_i}\psi_j\,d\boldsymbol{x} - \int_\Omega (f_i\,v_i + g_i\,\psi_i)\,d\boldsymbol{x} = 0.\end{aligned} \tag{3.18}$$

Unfold the above expression using Proposition 3.1 properties $i., ii.,$ and $iv)$ and obtain,

$$
\begin{aligned}
&\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\left(\mathcal{T}_\varepsilon(\partial_{x_\ell}u_k^\varepsilon)-\epsilon_{\nu k\ell}\mathcal{T}_\varepsilon(\varphi_\nu^\varepsilon)\right)\left(\mathcal{T}_\varepsilon\left(\partial_{x_j}v_i\right)-\epsilon_{\nu ij}\mathcal{T}_\varepsilon(\psi_\nu)\right)\,d\boldsymbol{y}\,d\boldsymbol{x}\\
&-\int_{\Omega\times Y}\varepsilon^2 L_{jik\ell}(\boldsymbol{y})\,\mathcal{T}_\varepsilon\left(\partial_{x_\ell}\varphi_k^\varepsilon\right)\mathcal{T}_\varepsilon\left(\partial_{x_i}\psi_j\right)\,d\boldsymbol{y}\,d\boldsymbol{x}\\
&-\int_{\Omega\times Y}\left(\mathcal{T}_\varepsilon(f_i)\,\mathcal{T}_\varepsilon(v_i)+\mathcal{T}_\varepsilon(g_i)\mathcal{T}_\varepsilon(\psi_i)\right)d\boldsymbol{y}\,d\boldsymbol{x}=0.
\end{aligned}
\tag{3.19}
$$

Set $\boldsymbol{v}:=\boldsymbol{V}(\boldsymbol{x})$ and $\boldsymbol{\psi}:=\boldsymbol{\Psi}(\boldsymbol{x})$ for any test functions $\boldsymbol{V}\in C_0^\infty(\Omega;\mathbb{R}^d)$ and $\boldsymbol{\Psi}\in C_0^\infty(\Omega;\mathbb{R}^d)$ in (3.19) and using (2.18) and Proposition 3.1 $vii.$ and $viii.$ we obtain, as ε tends to 0,

$$
\begin{aligned}
\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}u_k^0+\partial_{y_\ell}u_k^1-\epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})\right)\left(\partial_{x_j}V_i-\epsilon_{\nu ij}\,\Psi_\nu\right)d\boldsymbol{y}\,d\boldsymbol{x}&\\
-\int_\Omega\left(f_i\,V_i+g_i\,\Psi_i\right)d\boldsymbol{x}=0&
\end{aligned}
\tag{3.20}
$$

Select now test functions of the form $\boldsymbol{v}=\boldsymbol{v}^\varepsilon:=\varepsilon\,U(\boldsymbol{x})\,\boldsymbol{W}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$ where $U\in C_0^\infty(\Omega)$ and $\boldsymbol{W}\in H^1_{\rm per}(Y;\mathbb{R}^d)$. It is cleat that $\boldsymbol{v}^\varepsilon\to\boldsymbol{0}$ in $L^2(\Omega;\mathbb{R}^d)$. Moreover, we have $\partial_{x_j}v_i^\varepsilon(\boldsymbol{x})=\varepsilon\partial_{x_j}U(\boldsymbol{x})W_i(\boldsymbol{x}/\varepsilon)+U(\boldsymbol{x})\partial_{y_j}W_i(\boldsymbol{x}/\varepsilon)$ which implies $\mathcal{T}_\varepsilon(\partial_{x_j}v_i^\varepsilon)\to\partial_{y_j}\overline{W}_i(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$ as $\varepsilon\to 0$ where $\overline{W}_i(\boldsymbol{x},\boldsymbol{y}):=U(\boldsymbol{x})\,W_i(\boldsymbol{y})$. Likewise, we select as test function for the rotations $\boldsymbol{\psi}=\boldsymbol{\psi}^\varepsilon:=\varepsilon\,\Phi(\boldsymbol{x})\,\boldsymbol{\Xi}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$ where $\Phi\in C_0^\infty(\Omega)$ and $\boldsymbol{\Xi}\in H^1_{\rm per}(Y;\mathbb{R}^d)$ with $\boldsymbol{\psi}^\varepsilon\to\boldsymbol{0}$ in $L^2(\Omega;\mathbb{R}^d)$ and $\mathcal{T}_\varepsilon(\partial_{x_j}\psi_i^\varepsilon)\to\partial_{y_j}\overline{\Xi}_i(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$ as $\varepsilon\to 0$ where $\overline{\Xi}_i(\boldsymbol{x},\boldsymbol{y}):=\Psi(\boldsymbol{x})\,\Xi_i(\boldsymbol{y})$. Hence, unfolding (3.19) with the above test functions we obtain,

$$
\begin{aligned}
&\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\left(\mathcal{T}_\varepsilon(\partial_{x_\ell}u_k^\varepsilon)-\epsilon_{\nu k\ell}\mathcal{T}_\varepsilon(\varphi_\nu^\varepsilon)\right)\left(\mathcal{T}_\varepsilon\left(\partial_{x_j}v_i^\varepsilon\right)-\epsilon_{\nu ij}\mathcal{T}_\varepsilon(\psi_\nu^\varepsilon)\right)\,d\boldsymbol{y}\,d\boldsymbol{x}\\
&-\int_{\Omega\times Y}\varepsilon^2 L_{jik\ell}(\boldsymbol{y})\,\mathcal{T}_\varepsilon\left(\partial_{x_\ell}\varphi_k^\varepsilon\right)\mathcal{T}_\varepsilon\left(\partial_{x_i}\psi_j^\varepsilon\right)\,d\boldsymbol{y}\,d\boldsymbol{x}\\
&-\int_{\Omega\times Y}\left(\mathcal{T}_\varepsilon(f_i)\,\mathcal{T}_\varepsilon(v_i^\varepsilon)+\mathcal{T}_\varepsilon(g_i)\mathcal{T}_\varepsilon(\psi_i^\varepsilon)\right)d\boldsymbol{y}\,d\boldsymbol{x}=0.
\end{aligned}
\tag{3.21}
$$

Letting ε tend to zero in (3.21) we obtain,

$$
\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}u_k^0+\partial_{y_\ell}u_k^1-\epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})\right)U(\boldsymbol{x})\,\partial_{y_j}\,W_i(\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}=0.
\tag{3.22}
$$

Defining $\overline{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{y}):=U(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{y})$ and adding (3.20) and (3.22) we obtain,

$$
\begin{aligned}
\int_{\Omega\times Y}\left\{C_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}u_k^0+\partial_{y_\ell}u_k^1-\epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})\right)\right\}\left(\partial_{x_j}V_i+\partial_{y_j}\overline{W}_i-\epsilon_{\nu ij}\Psi_\nu\right)d\boldsymbol{y}\,d\boldsymbol{x}&\\
-\int_\Omega\left\{f_i\,V_i+g_i\,\Psi_i)\right\}d\boldsymbol{y}\,d\boldsymbol{x}=0.&
\end{aligned}
\tag{3.23}
$$

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By the density of $C_0^\infty(\Omega) \otimes H^1_{\mathrm{per}}(Y;\mathbb{R}^d)$ in $L^2(\Omega; H^1_{\mathrm{per}}(Y;\mathbb{R}^d))$ the result holds for all $\overline{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega; H^1_{\mathrm{per}}(Y;\mathbb{R}^d))$.

The above equation is not immediately in a form that we recognize. However, using the properties of the Levi-Civita tensor, we can re-write $g_i = \frac{1}{2}\epsilon_{ipq}\,\epsilon_{jpq}\,g_j$. Grouping the terms in (3.23) by test function we obtain,

$$\begin{aligned}\int_{\Omega\times Y} \Big\{C_{jik\ell}(\boldsymbol{y})(\partial_{x_\ell}u_k^0 + \partial_{y_\ell}u_k^1 - \epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{2}\,\epsilon_{sij}g_s\Big\}\Big(\partial_{x_j}V_i + \partial_{y_j}\overline{W}_i\Big)\,d\boldsymbol{y}\,d\boldsymbol{x}\\ - \int_{\Omega\times Y}\Big\{C_{jik\ell}(\boldsymbol{y})\,(\partial_{x_\ell}u_k^0 + \partial_{y_\ell}u_k^1 - \epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{2}\,\epsilon_{sij}\,g_s\Big\}\,\epsilon_{\nu ij}\,\Psi_\nu)\,d\boldsymbol{y}\,d\boldsymbol{x}\\ - \int_\Omega \Big\{f_i\,V_i + \frac{1}{2}\epsilon_{sij}\,g_s\,\partial_{x_j}V_i\Big\}\,d\boldsymbol{y}\,d\boldsymbol{x} = 0. \end{aligned} \tag{3.24}$$

From here we can obtain two sets of equations. By considering, first, that $(\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{\Psi}) = (\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{0})$ we have,

$$\begin{aligned}\int_{\Omega\times Y}\Big\{C_{jik\ell}(\boldsymbol{y})\,(\partial_{x_\ell}u_k^0 + \partial_{y_\ell}u_k^1 - \epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{2}\,\epsilon_{sij}\,g_s\Big\}\,(\partial_{x_j}V_i + \partial_{y_j}\overline{W}_i)\,d\boldsymbol{y}\,d\boldsymbol{x}\\ - \int_\Omega\Big\{f_i\,V_i + \frac{1}{2}\epsilon_{sij}\,g_s\,(\partial_{x_j}V_i + \partial_{y_j}\overline{W}_i)\Big\}\,d\boldsymbol{y}\,d\boldsymbol{x} = 0.\end{aligned} \tag{3.25}$$

By considering $(\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{\Psi}) = (\boldsymbol{0},\boldsymbol{0},\boldsymbol{\Psi})$ we have,

$$-\int_{\Omega\times Y}\Big\{C_{jik\ell}(\boldsymbol{y})\,(\partial_{x_\ell}u_k^0 + \partial_{y_\ell}u_k^1 - \epsilon_{\nu k\ell}\,\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{2}\,\epsilon_{sij}\,g_s\Big\}\,\epsilon_{\nu ij}\,\Psi_\nu\,d\boldsymbol{y}\,d\boldsymbol{x} = 0. \tag{3.26}$$

If in (3.25) we select $\boldsymbol{V} \equiv \boldsymbol{0}$ we can see that both $\boldsymbol{u}^1$ and $\boldsymbol{\varphi}^0$ depend on $\nabla_x\boldsymbol{u}^0$ linearly. In some sense, this could be interpreted that the macroscopic displacement has contributions from microscopic displacements and rotations, independently. Hence, the form of $\boldsymbol{u}^1$ and $\boldsymbol{\varphi}^0$ look as follows,

$$u_i^1(\boldsymbol{x},\boldsymbol{y}) = \zeta_i^{k\ell}(\boldsymbol{y})\,\partial_{x_\ell}u_k^0(\boldsymbol{x}) + c_i(\boldsymbol{x}) \tag{3.27}$$

$$\varphi_\nu^0(\boldsymbol{x},\boldsymbol{y}) = \xi_\nu^{k\ell}(\boldsymbol{y})\,\partial_{x_\ell}u_k^0(\boldsymbol{x}) \tag{3.28}$$

where the correctors $\boldsymbol{\zeta}^{k\ell}$ and $\boldsymbol{\xi}^{k\ell}$ are the local solutions satisfying the following variational problems,

$$\begin{aligned}&\boldsymbol{\zeta}^{pq} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d), \quad \int_Y \boldsymbol{\zeta}^{pq}\,d\boldsymbol{y} = 0,\\ &\int_Y C_{jik\ell}\,\left(\frac{1}{2}\,\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\zeta_k^{pq}\right)\partial_{y_i}v_j\,d\boldsymbol{y} = 0 \text{ for all } \boldsymbol{v} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d),\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&\boldsymbol{\xi}^{pq} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d), \quad \int_Y \boldsymbol{\xi}^{pq}\, d\boldsymbol{y} = 0,\\
&\int_Y C_{jik\ell}\left(\frac{1}{2}\,\delta_{kp}\,\delta_{q\ell} + \epsilon_{\nu k\ell}\,\xi^{pq}_{\nu}\right)\,\partial_{y_i} v_j\, d\boldsymbol{y} = 0 \text{ for all } \boldsymbol{v} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d).
\end{aligned} \tag{3.30}$$

Existence and uniqueness for (3.29) follows from classical theory of variational inequalities [Lio69]. While for problem (3.30) one can show existence (up to an additive constant) as in [GR11, Thm. 3.4, pg. 45].

Remark 3.1. *Equation* (3.30) *is a new local problem that does not appear in the classical homogenization approach. Its appearance is solely a contribution of the non-symmetric part of the strain tensor and upon closer examination, the local problem is one that involves rotations (or curls) which implies that certain curvature-twist effects are present in the microscale. Moreover, these curvature-twist effects manifest themselves macroscopically as part of the effective material tensor of a linear elastic material and not independently.*

Returning to (3.25) and substituting $\overline{\boldsymbol{W}} = \boldsymbol{0}$, $\boldsymbol{u}^1$ and $\boldsymbol{\varphi}^0$ from (3.27) and (3.28) respectively, we obtain,

$$\int_\Omega \sigma^{\mathrm{eff}}_{ij}\,\partial_{x_j} V_i\, d\boldsymbol{x} = \frac{1}{2}\int_\Omega \epsilon_{\nu ij}\, g_\nu\;\partial_{x_i} V_j\, d\boldsymbol{x} + \int_\Omega f_i\, V_i\, d\boldsymbol{x}, \tag{3.31}$$

where

$$\sigma^{\mathrm{eff}}_{ij} := \Big(C^{\mathrm{eff}}_{jipq}\,\partial_{x_q} u^0_p - \frac{1}{2}\epsilon_{\nu ij}\, g_\nu\Big), \tag{3.32}$$

and

$$C^{\mathrm{eff}}_{jipq} = \int_Y C_{jik\ell}(\boldsymbol{y})\,\left(\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\zeta^{pq}_k - \epsilon_{\nu k\ell}\xi^{pq}_\nu\right)\; d\boldsymbol{y}. \tag{3.33}$$

Using equation (3.26) we see that $\sigma^{\mathrm{eff}}_{ij} = \sigma^{\mathrm{eff}}_{ji}$ exactly like the Cauchy stress in classical linear elasticity. These are precisely the homogenized equations obtained by [FPS01] using two-scale expansions under their scheme $\mathrm{HS1}$. □

Proposition 3.3. *If we use the notation* $\gamma^\varepsilon_{ji} := \partial_{x_j} u^\varepsilon_i - \epsilon_{kji}\varphi^\varepsilon_k$, $\gamma^0_{ji} := \partial_{x_j} u^0_i$, *and* $\gamma^1_{ji} := \partial_{y_j} u^1_i - \epsilon_{kji}\varphi^0_k$ *then under the assumptions of Theorem 3.1 we have the following convergence results,*

$$\lim_{\varepsilon\to 0}\int_\Omega C^\varepsilon(\boldsymbol{x})\,\gamma^\varepsilon : \gamma^\varepsilon\, d\boldsymbol{x} = \int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\, d\boldsymbol{y}\, d\boldsymbol{x} \tag{3.34}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega^-_\varepsilon} C^\varepsilon(\boldsymbol{x})\,\gamma^\varepsilon : \gamma^\varepsilon\, d\boldsymbol{x} = 0. \tag{3.35}$$

Proof. Using the weak lower semicontinuity of the integrals, the fact that tensors C^ε and $\widetilde{L}^\varepsilon$ belong in $\mathcal{M}^4_d(\alpha,\beta,\Omega)$, and properties of the limit infimum we obtain,

$$\int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\, d\boldsymbol{y} d\boldsymbol{x} \le \liminf_{\varepsilon\to 0}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(C^\varepsilon)\,\mathcal{T}_\varepsilon(\gamma^\varepsilon) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y} d\boldsymbol{x}$$

$$
\begin{aligned}
&\leq \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(C^\varepsilon)\,\mathcal{T}_\varepsilon(\gamma^\varepsilon) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} + \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\widetilde{L}^\varepsilon)\,\mathcal{T}_\varepsilon(\kappa^\varepsilon) : \mathcal{T}_\varepsilon(\kappa^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x}\\
&\leq \liminf_{\varepsilon\to 0} \left\{ \int_{\Omega\times Y} \mathcal{T}_\varepsilon(C^\varepsilon)\,\mathcal{T}_\varepsilon(\gamma^\varepsilon) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} + \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\widetilde{L}^\varepsilon)\,\mathcal{T}_\varepsilon(\kappa^\varepsilon) : \mathcal{T}_\varepsilon(\kappa^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} \right\}\\
&\leq \liminf_{\varepsilon\to 0} \left\{ \int_{\Omega} C^\varepsilon\,\gamma^\varepsilon : \gamma^\varepsilon\, d\boldsymbol{x} + \int_{\Omega} \widetilde{L}^\varepsilon\,\kappa^\varepsilon : \kappa^\varepsilon\, d\boldsymbol{x} \right\}\\
&= \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \left(\boldsymbol{f}\cdot\boldsymbol{u}^\varepsilon + \boldsymbol{g}\cdot\boldsymbol{\varphi}^\varepsilon\right)\, d\boldsymbol{y}d\boldsymbol{x}\\
&= \int_{\Omega\times Y} \left(\boldsymbol{f}\cdot\boldsymbol{u}^0 + \boldsymbol{g}\cdot\boldsymbol{\varphi}^0\right)\, d\boldsymbol{y}d\boldsymbol{x}\\
&= \int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\, d\boldsymbol{y}d\boldsymbol{x},
\end{aligned} \tag{3.36}
$$

which is precisely (3.34). We remark that the last equality came from equation (3.23). Moreover, (3.34) implies (3.35). □

Remark 3.2. *Immediately one can observe from Proposition 3.3 that the following result holds,*

$$
\lim_{\varepsilon\to 0} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(C^\varepsilon)\,\mathcal{T}_\varepsilon(\gamma^\varepsilon) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} = \int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\, d\boldsymbol{y}d\boldsymbol{x} \tag{3.37}
$$

Corollary 3.1. *The Cosserat strain γ^ε converges strongly in $L^2(\Omega\times Y;\mathbb{R}^{d\times d})$,*

$$
\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon(\gamma^\varepsilon) \to \gamma^0+\gamma^1 \text{ in } L^2(\Omega\times Y;\mathbb{R}^{d\times d}) \tag{3.38}
$$

Proof. By expanding the square of the expression below we have,

$$
\begin{aligned}
&\int_{\Omega\times Y} C(\boldsymbol{y})(\mathcal{T}_\varepsilon(\gamma^\varepsilon)-\gamma^0-\gamma^1) : (\mathcal{T}_\varepsilon(\gamma^\varepsilon)-\gamma^0-\gamma^1)\, d\boldsymbol{y}d\boldsymbol{x}\\
&\quad= \int_{\Omega\times Y} C(\boldsymbol{y})\mathcal{T}_\varepsilon(\gamma^\varepsilon) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} - \int_{\Omega\times Y} C(\boldsymbol{y})\mathcal{T}_\varepsilon(\gamma^\varepsilon) : (\gamma^0-\gamma^1)\, d\boldsymbol{y}d\boldsymbol{x}\\
&\quad- \int_{\Omega\times Y} C(\boldsymbol{y})(\gamma^0-\gamma^1) : \mathcal{T}_\varepsilon(\gamma^\varepsilon)\, d\boldsymbol{y}d\boldsymbol{x} + \int_{\Omega\times Y} C(\boldsymbol{y})(\gamma^0-\gamma^1) : (\gamma^0-\gamma^1)\, d\boldsymbol{y}d\boldsymbol{x}.
\end{aligned} \tag{3.39}
$$

The first term converges from (3.37) while the rest of the terms converge by (3.9) and properties of the unfolding operator. Hence, all the terms on the right hand side above sum to zero in the limit and the result follows. □

3.2.2 Homogenization under the HS 2 scheme

Theorem 3.2. *If $(\boldsymbol{u}^\varepsilon,\boldsymbol{\varphi}^\varepsilon)$ is the solution set to* (2.12) *then, under the* HS 2 *scheme, there exist $\boldsymbol{u}^0 \in H^1(\Omega;\mathbb{R}^d)$, $\boldsymbol{u}^1 \in L^2(\Omega; H^1_{per}(Y;\mathbb{R}^d))$, $\boldsymbol{\varphi}^0 \in H^1(\Omega;\mathbb{R}^d)$, $\boldsymbol{\varphi}^1 \in L^2(\Omega; H^1_{per}(Y;\mathbb{R}^d))$ such that,*

$$
\mathcal{T}_\varepsilon(\boldsymbol{u}^\varepsilon) \rightharpoonup \boldsymbol{u}^0 \text{ in } L^2(\Omega; H^1(Y;\mathbb{R}^d)) \tag{3.40}
$$

$$\mathcal{T}_\varepsilon(\nabla \boldsymbol{u}^\varepsilon) \rightharpoonup \nabla_x \boldsymbol{u}^0 + \nabla_y \boldsymbol{u}^1 \text{ in } L^2(\Omega \times Y; \mathbb{R}^{d\times d}) \tag{3.41}$$

$$\mathcal{T}_\varepsilon(\boldsymbol{\varphi}^\varepsilon) \rightharpoonup \boldsymbol{\varphi}^0 \text{ in } L^2(\Omega; H^1(Y; \mathbb{R}^d)) \tag{3.42}$$

$$\mathcal{T}_\varepsilon(\nabla \boldsymbol{\varphi}^\varepsilon) \rightharpoonup \nabla_x \boldsymbol{\varphi}^0 + \nabla_y \boldsymbol{\varphi}^1 \text{ in } L^2(\Omega \times Y; \mathbb{R}^{d\times d}) \tag{3.43}$$

and $(\boldsymbol{u}^0, \boldsymbol{u}^1, \boldsymbol{\varphi}^0, \boldsymbol{\varphi}^1)$ *is the unique solution set of*

$$\begin{aligned} &\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell} u_k^0 + \partial_{x_\ell} u_k^1 - \epsilon_{\nu k\ell}\varphi_\nu^0\Big)\left(\partial_{x_i} V_j + \partial_{y_i} \overline{W}_j - \varepsilon_{\nu ij}\Psi_\nu\right) d\boldsymbol{y}\, d\boldsymbol{x} \\ &- \int_{\Omega\times Y} L_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}\varphi_k^0 + \partial_{y_\ell}\varphi_k^1\right)\left(\partial_{x_i}\Psi_j + \partial_{y_i}\overline{\Xi}_j\right) d\boldsymbol{y}\, d\boldsymbol{x} \\ &- \int_\Omega f_i\, V_i + g_i\, \Psi_i\, d\boldsymbol{x} = 0, \end{aligned} \tag{3.44}$$

for all $\boldsymbol{V} \in H_0^1(\Omega; \mathbb{R}^d)$ *and* $\overline{\boldsymbol{W}} \in L^2(\Omega; H^1(Y; \mathbb{R}^d))$. *If in addition* $\boldsymbol{u}^1$ *and* $\boldsymbol{\varphi}^1$ *have the following form,*

$$u_i^1(\boldsymbol{x}, \boldsymbol{y}) = \zeta_i^{pq}(\boldsymbol{y})\left(\partial_{x_p} u_q^0(\boldsymbol{x}) - \varepsilon_{\nu pq}\varphi_\nu^0(\boldsymbol{x})\right) + \kappa_i(\boldsymbol{x}) \tag{3.45}$$

$$\varphi_\nu^1(\boldsymbol{x}, \boldsymbol{y}) = \xi_\nu^{pq}(\boldsymbol{y})\, \partial_{x_p}\varphi_q^0(\boldsymbol{x}) + \kappa_i(\boldsymbol{x}) \tag{3.46}$$

and we select $\overline{\boldsymbol{W}} \equiv \boldsymbol{0}$ *and* $\overline{\boldsymbol{\Xi}} \equiv \boldsymbol{0}$ *then equation* (3.44) *takes the following form:*

$$\begin{aligned} \int_\Omega C^{\text{eff}}_{jipq}\Big(\partial_{x_p} u_q^0 - \epsilon_{\nu pq}\varphi_\nu^0\Big)\left(\partial_{x_i} V_j - \varepsilon_{\nu ij}\Psi_\nu\right) d\boldsymbol{x} - \int_\Omega L^{\text{eff}}_{jipq}\, \partial_{x_p}\varphi_q^0\, \partial_{x_i}\Psi_j\, d\boldsymbol{x} \\ - \int_\Omega \left(f_i\, V_i + g_i\, \Psi_i\right) d\boldsymbol{x} = 0, \end{aligned} \tag{3.47}$$

where

$$C^{\text{eff}}_{jipq} = \int_Y C_{jik\ell}(\boldsymbol{y})\left(\delta_{kp}\,\delta_{\ell q} + \partial_{y_\ell}\zeta_k^{pq}\right) d\boldsymbol{y}, \tag{3.48}$$

$$L^{\text{eff}}_{jipq} = \int_Y L_{jik\ell}(\boldsymbol{y})\left(\delta_{kp}\,\delta_{\ell q} + \partial_{y_\ell}\xi_k^{pq}\right) d\boldsymbol{y}, \tag{3.49}$$

with $\boldsymbol{\zeta}^{pq}$ *and* $\boldsymbol{\xi}^{pq}$ *being the local solutions on the unit cell satisfying the second order elliptic problems,*

$$\begin{aligned} &\boldsymbol{\zeta}^{pq} \in H^1_{\text{per}}(Y; \mathbb{R}^d), \quad \int_Y \boldsymbol{\zeta}^{pq}\, d\boldsymbol{y} = 0, \\ &\int_Y C_{jik\ell}(\boldsymbol{y})\, \left(\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\zeta_k^{pq}\right)\partial_{y_i} w_j\, d\boldsymbol{y} = 0 \text{ for all } \boldsymbol{w} \in H^1_{\text{per}}(Y; \mathbb{R}^d), \end{aligned} \tag{3.50}$$

$$
\begin{aligned}
&\boldsymbol{\xi}^{pq} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d), \quad \int_Y \boldsymbol{\xi}^{pq}\, d\boldsymbol{y} = 0, \\
&\int_Y L_{jik\ell}(\boldsymbol{y})\, \left(\delta_{kp}\,\delta_{q\ell} + \partial_{y_\ell}\xi_k^{pq}\right) \partial_{y_i} v_j\, d\boldsymbol{y} = 0 \textit{ for all } \boldsymbol{v} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^d),
\end{aligned}
\tag{3.51}
$$

Proof. Once again, we start with the weak form (2.21) of the Cosserat continuum (in this scheme the tensor L^ε scales with 1 instead of ε^2): Find $(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in H^1_\Gamma(\Omega_0;\mathbb{R}^d) \times H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$ such that,

$$
\begin{aligned}
&\int_\Omega C_{jik\ell}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)(\partial_{x_\ell}u_k^\varepsilon - \epsilon_{\nu k\ell}\varphi_\nu^\varepsilon)\,(\partial_{x_j}v_i - \epsilon_{\nu ij}\psi_\nu)\, d\boldsymbol{x} \\
&\qquad - \int_\Omega L_{jik\ell}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\, \partial_{x_\ell}\varphi_k^\varepsilon\, \partial_{x_i}\psi_j\, d\boldsymbol{x} - \int_\Omega (f_i\, v_i + g_i\, \psi_i)\, d\boldsymbol{x} = 0.
\end{aligned}
\tag{3.52}
$$

Unfold the above expression using Proposition 3.1 properties $i., ii.,$ and $iv)$ and obtain,

$$
\begin{aligned}
&\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\,(\mathcal{T}_\varepsilon(\partial_{x_\ell}u_k^\varepsilon) - \epsilon_{\nu k\ell}\mathcal{T}_\varepsilon(\varphi_\nu^\varepsilon))\,\left(\mathcal{T}_\varepsilon\left(\partial_{x_j}v_i\right) - \epsilon_{\nu ij}\mathcal{T}_\varepsilon(\psi_\nu)\right)\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&- \int_{\Omega\times Y} \varepsilon^2\, L_{jik\ell}(\boldsymbol{y})\, \mathcal{T}_\varepsilon\left(\partial_{x_\ell}\varphi_k^\varepsilon\right)\, \mathcal{T}_\varepsilon\left(\partial_{x_i}\psi_j\right)\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&- \int_{\Omega\times Y} (\mathcal{T}_\varepsilon(f_i)\, \mathcal{T}_\varepsilon(v_i) + \mathcal{T}_\varepsilon(g_i)\mathcal{T}_\varepsilon(\psi_i))\, d\boldsymbol{y}\, d\boldsymbol{x} = 0.
\end{aligned}
\tag{3.53}
$$

Select the same test functions as before. Namely, set $\boldsymbol{v} := \boldsymbol{V}(\boldsymbol{x})$ and $\boldsymbol{\psi} := \boldsymbol{\Psi}(\boldsymbol{x})$ for any test functions $\boldsymbol{V} \in C_0^\infty(\Omega;\mathbb{R}^d)$ and $\boldsymbol{\Psi} \in C_0^\infty(\Omega;\mathbb{R}^d)$ in (3.53) and using (2.19) and Proposition 3.1 $vii.$ we obtain, as ε tends to 0,

$$
\begin{aligned}
&\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell}u_k^0 + \partial_{x_\ell}u_k^1 - \epsilon_{\nu k\ell}\varphi_\nu^0\Big)\,(\partial_{x_i}V_j - \varepsilon_{\nu ij}\Psi_\nu)\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&\qquad\qquad - \int_{\Omega\times Y} L_{jik\ell}(\boldsymbol{y})\,(\partial_{x_\ell}\varphi_k^0 + \partial_{y_\ell}\varphi_k^1)\,\partial_{x_i}\Psi_j\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&\qquad\qquad - \int_\Omega (f_i\, V_i + g_i\, \Psi_i)\, d\boldsymbol{x} = 0,
\end{aligned}
\tag{3.54}
$$

Select now the test functions from before $\boldsymbol{v} = \boldsymbol{v}^\varepsilon := \varepsilon\, U(\boldsymbol{x})\, \boldsymbol{W}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$ and $\boldsymbol{\psi} = \boldsymbol{\psi}^\varepsilon := \varepsilon\, \Phi(\boldsymbol{x})\, \Xi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$. Recall that $\boldsymbol{v}^\varepsilon \to 0$, $\boldsymbol{\psi}^\varepsilon \to \boldsymbol{0}$ in $L^2(\Omega;\mathbb{R}^d)$ and $\mathcal{T}_\varepsilon(\partial_{x_j}\psi_i^\varepsilon) \to \partial_{y_j}\overline{\Xi}_i(\boldsymbol{x},\boldsymbol{y})$, $\mathcal{T}_\varepsilon(\partial_{x_j}v_i^\varepsilon) \to \partial_{y_j}\overline{W}_i(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$.

With the above test functions we obtain,

$$
\begin{aligned}
&\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell}u_k^0 + \partial_{x_\ell}u_k^1 - \epsilon_{\nu k\ell}\varphi_\nu^0\Big)\,\partial_{y_i}\overline{W}_j\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&\qquad\qquad - \int_{\Omega\times Y} L_{jik\ell}(\boldsymbol{y})\,(\partial_{x_\ell}\varphi_k^0 + \partial_{y_\ell}\varphi_k^1)\,\partial_{y_i}\overline{\Xi}_j\, d\boldsymbol{y}\, d\boldsymbol{x} = 0,
\end{aligned}
\tag{3.55}
$$

Adding (3.54) and (3.55) we obtain,

$$\begin{aligned}\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell}u_k^0+\partial_{x_\ell}u_k^1-\epsilon_{\nu k\ell}\varphi_\nu^0\Big)\left(\partial_{x_j}V_i+\partial_{y_i}\overline{W}_j-\epsilon_{\nu ij}\Psi_\nu\right)d\boldsymbol{y}\,d\boldsymbol{x}&\\ -\int_{\Omega\times Y} L_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}\varphi_k^0+\partial_{y_\ell}\varphi_k^1\right)\left(\partial_{x_j}\Psi_i+\partial_{y_i}\overline{\Xi}_j\right)d\boldsymbol{y}\,d\boldsymbol{x}&\\ -\int_{\Omega}\left(f_i\,V_i+g_i\,\Psi_i\right)d\boldsymbol{x}=0.&\end{aligned} \tag{3.56}$$

Using the density of $C_0^\infty(\Omega;\mathbb{R}^d)$ in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$ and $C_0^\infty(\Omega)\otimes H^1_{\mathrm{per}}(Y;\mathbb{R}^d)$ in $L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^d))$ the result holds true for all $\boldsymbol{V}\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$ and all $\overline{\boldsymbol{W}}\in L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^d))$.

Taking $(\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{\Psi},\overline{\boldsymbol{\Xi}})=(\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{0},\boldsymbol{0})$ we obtain,

$$\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell}u_k^0+\partial_{x_\ell}u_k^1-\epsilon_{\nu k\ell}\varphi_\nu^0\Big)\left(\partial_{x_j}V_i+\partial_{y_i}\overline{W}_j\right)d\boldsymbol{y}\,d\boldsymbol{x}-\int_{\Omega} f_i\,V_i\,d\boldsymbol{x}=0. \tag{3.57}$$

Taking $(\boldsymbol{V},\overline{\boldsymbol{W}},\boldsymbol{\Psi},\overline{\boldsymbol{\Xi}})=(\boldsymbol{0},\boldsymbol{0},\boldsymbol{\Psi},\overline{\boldsymbol{\Xi}})$ we obtain,

$$\begin{aligned}-\int_{\Omega\times Y} C_{jik\ell}(\boldsymbol{y})\Big(\partial_{x_\ell}u_k^0+\partial_{x_\ell}u_k^1-\epsilon_{\nu k\ell}\varphi_\nu^0\Big)\,\epsilon_{\nu ij}\Psi_\nu\,d\boldsymbol{y}\,d\boldsymbol{x}&\\ -\int_{\Omega\times Y} L_{jik\ell}(\boldsymbol{y})\left(\partial_{x_\ell}\varphi_k^0+\partial_{y_\ell}\varphi_k^1\right)\left(\partial_{x_j}\Psi_i+\partial_{y_i}\overline{\Xi}_j\right)d\boldsymbol{y}\,d\boldsymbol{x}&\\ -\int_{\Omega} g_i\,\Psi_i\,d\boldsymbol{x}=0.&\end{aligned} \tag{3.58}$$

Moreover, if we select $\boldsymbol{V}\equiv\boldsymbol{0}$ in (3.57) we obtain that $\boldsymbol{u}^1$ has the following form,

$$u_i^1(\boldsymbol{x},\boldsymbol{y})=\zeta_i^{pq}(\boldsymbol{y})\left(\partial_{x_p}u_q^0(\boldsymbol{x})-\varepsilon_{\nu pq}\varphi_\nu^0(\boldsymbol{x})\right)+\kappa_i(\boldsymbol{x}). \tag{3.59}$$

While, if we set $\boldsymbol{\Psi}\equiv\boldsymbol{0}$ in (3.58) we obtain $\boldsymbol{\varphi}^1$ the following form,

$$\varphi_\nu^1(\boldsymbol{x},\boldsymbol{y})=\xi_\nu^{pq}(\boldsymbol{y})\,\partial_{x_p}\varphi_q^0(\boldsymbol{x})+\kappa_i(\boldsymbol{x}), \tag{3.60}$$

where $\boldsymbol{\zeta}^{pq}$ and $\boldsymbol{\xi}^{pq}$ satisfy (3.50) and (3.51), respectively and $\kappa_i(\boldsymbol{x})$ is some generic constant function in $\boldsymbol{y}$. We remark that existence and uniqueness for $\boldsymbol{\zeta}^{pq}$ and $\boldsymbol{\xi}^{pq}$ follow from the theory of variational inequalities.

Rewriting, (3.56) and substituting in $\overline{\boldsymbol{W}}\equiv\boldsymbol{0}$, $\overline{\boldsymbol{\Xi}}\equiv\boldsymbol{0}$, $\boldsymbol{u}^1$, and $\boldsymbol{\varphi}^1$ from (3.57) and (3.58), respectively and factoring out common terms we obtain,

$$\begin{aligned}\int_{\Omega} C^{\mathrm{eff}}_{jipq}\Big(\partial_{x_p}u_q^0-\epsilon_{\nu pq}\varphi_\nu^0\Big)\left(\partial_{x_i}V_j-\varepsilon_{\nu ij}\Psi_\nu\right)d\boldsymbol{x}-\int_{\Omega} L^{\mathrm{eff}}_{jipq}\,\partial_{x_p}\varphi_q^0\,\partial_{x_i}\Psi_j\,d\boldsymbol{x}&\\ -\int_{\Omega}\left(f_i\,V_i+g_i\,\Psi_i\right)d\boldsymbol{x}=0,&\end{aligned} \tag{3.61}$$

where

$$C^{\mathrm{eff}}_{jipq} = \int_Y C_{jik\ell}(\boldsymbol{y})\,(\delta_{kp}\,\delta_{\ell q} + \partial_{y_\ell}\zeta^{pq}_k)\,d\boldsymbol{y}, \tag{3.62}$$

$$L^{\mathrm{eff}}_{jipq} = \int_Y L_{jik\ell}(\boldsymbol{y})\,(\delta_{kp}\,\delta_{\ell q} + \partial_{y_\ell}\xi^{pq}_k)\,d\boldsymbol{y}. \tag{3.63}$$

□

Remark 3.3. *Under the* HS 2 *scheme we obtain an effective Cosserat continuum in* (3.61)*. As a result we have two sets of effective coefficients in* (3.62) *and* (3.63)*. One that relates the non-symmetric strain tensor to the stress and one that relates the curvature-twist tensor to the couple stress. Unlike in the first homogenization scheme,* HS 1*, the curvature-twist effects manifest themselves as a separate equation.*

Proposition 3.4. *If we use the notation* $\gamma^{\varepsilon}_{ji} := \partial_{x_j}u^{\varepsilon}_i - \epsilon_{kji}\varphi^{\varepsilon}_k$, $\gamma^0_{ji} := \partial_{x_j}u^0_i - \epsilon_{kji}\varphi^0_k$, $\gamma^1_{ji} := \partial_{y_j}u^1_i - \epsilon_{kji}\varphi^1_k$, $\kappa^0 := \partial_{x_j}\varphi^0_i$, *and* $\kappa^1 := \partial_{y_j}\varphi^1_i$ *then under the assumptions of Theorem 3.2 we have the following convergence results,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega \Big(C^{\varepsilon}(\boldsymbol{x})\,\gamma^{\varepsilon} : \gamma^{\varepsilon} + \widetilde{L}^{\varepsilon}(\boldsymbol{x})\,\kappa^{\varepsilon} : \kappa^{\varepsilon}\Big)\,d\boldsymbol{x}\\ &= \int_{\Omega\times Y}\big(C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1) + L(\boldsymbol{y})\,(\kappa^0+\kappa^1) : (\kappa^0+\kappa^1)\big)\,d\boldsymbol{y}\,d\boldsymbol{x}\end{aligned} \tag{3.64}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega^-_{\varepsilon}} \Big(C^{\varepsilon}(\boldsymbol{x})\,\gamma^{\varepsilon} : \gamma^{\varepsilon} + \widetilde{L}^{\varepsilon}(\boldsymbol{x})\,\kappa^{\varepsilon} : \kappa^{\varepsilon}\Big)\,d\boldsymbol{x} = 0. \tag{3.65}$$

Proof. As before, using the weak lower semicontinuity of the integrals, the fact that tensors C^{ε} and $\widetilde{L}^{\varepsilon}$ belong in $\mathcal{M}^4_d(\alpha,\beta,\Omega)$, and properties of the limit infimum we obtain,

$$\begin{aligned}&\int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x} + \int_{\Omega\times Y} L(\boldsymbol{y})\,(\kappa^0+\kappa^1) : (\kappa^0+\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&\le \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(C^{\varepsilon})\,\mathcal{T}_{\varepsilon}(\gamma^{\varepsilon}) : \mathcal{T}_{\varepsilon}(\gamma^{\varepsilon})\,d\boldsymbol{y}d\boldsymbol{x} + \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(\widetilde{L}^{\varepsilon})\,\mathcal{T}_{\varepsilon}(\kappa^{\varepsilon}) : \mathcal{T}_{\varepsilon}(\kappa^{\varepsilon})\,d\boldsymbol{y}d\boldsymbol{x}\\
&\le \liminf_{\varepsilon\to 0}\left\{\int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(C^{\varepsilon})\,\mathcal{T}_{\varepsilon}(\gamma^{\varepsilon}) : \mathcal{T}_{\varepsilon}(\gamma^{\varepsilon})\,d\boldsymbol{y}d\boldsymbol{x} + \int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(\widetilde{L}^{\varepsilon})\,\mathcal{T}_{\varepsilon}(\kappa^{\varepsilon}) : \mathcal{T}_{\varepsilon}(\kappa^{\varepsilon})\,d\boldsymbol{y}d\boldsymbol{x}\right\}\\
&\le \liminf_{\varepsilon\to 0}\left\{\int_\Omega C^{\varepsilon}\,\gamma^{\varepsilon} : \gamma^{\varepsilon}\,d\boldsymbol{x} + \int_\Omega \widetilde{L}^{\varepsilon}\,\kappa^{\varepsilon} : \kappa^{\varepsilon}\,d\boldsymbol{x}\right\}\\
&= \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\left(\boldsymbol{f}\cdot\boldsymbol{u}^{\varepsilon} + \boldsymbol{g}\cdot\boldsymbol{\varphi}^{\varepsilon}\right)\,d\boldsymbol{y}d\boldsymbol{x}\\
&= \int_{\Omega\times Y}\left(\boldsymbol{f}\cdot\boldsymbol{u}^0 + \boldsymbol{g}\cdot\boldsymbol{\varphi}^0\right)\,d\boldsymbol{y}d\boldsymbol{x}\\
&= \int_{\Omega\times Y} C(\boldsymbol{y})\,(\gamma^0+\gamma^1) : (\gamma^0+\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x} + \int_{\Omega\times Y} L(\boldsymbol{y})\,(\kappa^0+\kappa^1) : (\kappa^0+\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x},\end{aligned} \tag{3.66}$$

which is precisely (3.64). We remark that the last equality came from equation (3.56). Moreover, (3.64) implies (3.65). ☐

Remark 3.4. *Immediately one can observe from Proposition 3.4 that the following result holds,*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega\times Y}\Big(\mathcal{T}_\varepsilon(C^\varepsilon)\,\mathcal{T}_\varepsilon(\gamma^\varepsilon):\mathcal{T}_\varepsilon(\gamma^\varepsilon)+\mathcal{T}_\varepsilon(\widetilde{L}^\varepsilon)\,\mathcal{T}_\varepsilon(\kappa^\varepsilon):\mathcal{T}_\varepsilon(\kappa^\varepsilon)\Big)\,d\boldsymbol{y}d\boldsymbol{x}\\
&=\int_{\Omega\times Y}C(\boldsymbol{y})\,(\gamma^0+\gamma^1):(\gamma^0+\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x}+\int_{\Omega\times Y}L(\boldsymbol{y})\,(\kappa^0+\kappa^1):(\kappa^0+\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x}
\end{aligned}\tag{3.67}$$

Corollary 3.2. *The following convergence results hold for the Cosserat strain and curvature-twist tensors,*

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon(\gamma^\varepsilon)\to\gamma^0+\gamma^1 \text{ in } L^2(\Omega\times Y;\mathbb{R}^{d\times d})\tag{3.68}$$

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon(\kappa^\varepsilon)\to\kappa^0+\kappa^1 \text{ in } L^2(\Omega\times Y;\mathbb{R}^{d\times d})\tag{3.69}$$

Proof. By expanding the square of the expressions below we have,

$$\begin{aligned}
&\int_{\Omega\times Y}C(\boldsymbol{y})(\mathcal{T}_\varepsilon(\gamma^\varepsilon)-\gamma^0-\gamma^1):(\mathcal{T}_\varepsilon(\gamma^\varepsilon)-\gamma^0-\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&+\int_{\Omega\times Y}L(\boldsymbol{y})(\mathcal{T}_\varepsilon(\kappa^\varepsilon)-\kappa^0-\kappa^1):(\mathcal{T}_\varepsilon(\kappa^\varepsilon)-\kappa^0-\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&=\int_{\Omega\times Y}C(\boldsymbol{y})\mathcal{T}_\varepsilon(\gamma^\varepsilon):\mathcal{T}_\varepsilon(\gamma^\varepsilon)\,d\boldsymbol{y}d\boldsymbol{x}-\int_{\Omega\times Y}C(\boldsymbol{y})\mathcal{T}_\varepsilon(\gamma^\varepsilon):(\gamma^0-\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&-\int_{\Omega\times Y}C(\boldsymbol{y})(\gamma^0-\gamma^1):\mathcal{T}_\varepsilon(\gamma^\varepsilon)\,d\boldsymbol{y}d\boldsymbol{x}+\int_{\Omega\times Y}C(\boldsymbol{y})(\gamma^0-\gamma^1):(\gamma^0-\gamma^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&+\int_{\Omega\times Y}L(\boldsymbol{y})\mathcal{T}_\varepsilon(\kappa^\varepsilon):\mathcal{T}_\varepsilon(\kappa^\varepsilon)\,d\boldsymbol{y}d\boldsymbol{x}-\int_{\Omega\times Y}L(\boldsymbol{y})\mathcal{T}_\varepsilon(\kappa^\varepsilon):(\kappa^0-\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x}\\
&-\int_{\Omega\times Y}L(\boldsymbol{y})(\kappa^0-\kappa^1):\mathcal{T}_\varepsilon(\kappa^\varepsilon)\,d\boldsymbol{y}d\boldsymbol{x}+\int_{\Omega\times Y}L(\boldsymbol{y})(\kappa^0-\kappa^1):(\kappa^0-\kappa^1)\,d\boldsymbol{y}d\boldsymbol{x}
\end{aligned}\tag{3.70}$$

Combining terms, using (3.37) and (3.40)–(3.43) we obtain that the right hand side is zero and the results follows. ☐

4 Some results regarding correctors

In this section, we provide some results that can be interpreted as "corrector" type results for a Cosserat continuum under each homogenization scheme. Classical correctors in the theory of homogenization for linear elasticity, transform the weak converge of the displacement $\boldsymbol{u}^\varepsilon\rightharpoonup\boldsymbol{u}^0$ to strong convergence by subtracting a term involving the gradient of $\boldsymbol{u}^0$ and the local solutions on the unit cell. Given that kinematics of a Cosserat continuum are more convoluted, it is not immediately clear what the form of a corrector should be.

The correctors are obtained using the averaging operator in (3.6) as is done in [CDG08]. Hence, we do not require any additional regularity assumptions of the local solutions (3.29), (3.30), (3.50), (3.51) as is done in standard homogenization problems (see e.g. [OV07]).

4.1 Correctors under $\mathrm{HS}\ 1$ scheme

Theorem 4.1. *Under the assumptions of Theorem 3.1 we have the following strong convergence,*

$$\gamma^{\varepsilon} - \gamma^{0} - \mathcal{U}_{\varepsilon}(\gamma^{1}) \to 0 \textit{ in } L^{2}(\Omega, \mathbb{R}^{d\times d}) \tag{4.1}$$

Proof. Using (3.34), (3.35), (3.38), and Proposition 3.2 iii. we have,

$$\gamma^{\varepsilon} - \mathcal{U}_{\varepsilon}(\gamma^{0}) - \mathcal{U}_{\varepsilon}(\gamma^{1}) \to 0 \text{ in } L^{2}(\Omega; R^{d\times d}). \tag{4.2}$$

Since γ^0 is independent of $\boldsymbol{y}$ we can use Proposition 3.2 i to obtain,

$$\mathcal{U}_{\varepsilon}(\gamma^{0}) \to \gamma^{0} \text{ in } L^{2}(\Omega; R^{d\times d}). \tag{4.3}$$

Hence, the results follows. □

4.2 Correctors under $\mathrm{HS}\ 2$ scheme

Theorem 4.2. *Under the assumptions of Theorem 3.2 we have the following strong convergence results,*

$$\gamma^{\varepsilon} - \gamma^{0} - \mathcal{U}_{\varepsilon}(\gamma^{1}) \to 0 \textit{ in } L^{2}(\Omega, \mathbb{R}^{d\times d}) \tag{4.4}$$

$$\kappa^{\varepsilon} - \kappa^{0} - \mathcal{U}_{\varepsilon}(\kappa^{1}) \to 0 \textit{ in } L^{2}(\Omega, \mathbb{R}^{d\times d}) \tag{4.5}$$

Proof. Using (3.64), (3.65), (3.68), (3.69) and Proposition 3.2 iii. we have,

$$\gamma^{\varepsilon} - \mathcal{U}_{\varepsilon}(\gamma^{0}) - \mathcal{U}_{\varepsilon}(\gamma^{1}) \to 0 \text{ in } L^{2}(\Omega; R^{d\times d}). \tag{4.6}$$

$$\kappa^{\varepsilon} - \mathcal{U}_{\varepsilon}(\kappa^{0}) - \mathcal{U}_{\varepsilon}(\kappa^{1}) \to 0 \text{ in } L^{2}(\Omega; R^{d\times d}). \tag{4.7}$$

Since both γ^0 and κ^0 are independent of $\boldsymbol{y}$ we can use Proposition 3.2 i to obtain,

$$\mathcal{U}_{\varepsilon}(\gamma^{0}) \to \gamma^{0} \text{ in } L^{2}(\Omega; R^{d\times d}), \tag{4.8}$$

and

$$\mathcal{U}_{\varepsilon}(\kappa^{0}) \to \kappa^{0} \text{ in } L^{2}(\Omega; R^{d\times d}). \tag{4.9}$$

Thus, completing the proof. □

5 Conclusions

We derived effective models for a heterogeneous Cosserat continuum taking into account the Cosserat intrinsic length of the constituents by the method of homogenization and periodic unfolding. In doing so, we provide rigorous proof to the results obtain in [FPS01] by two-scale expansion.

Depending on how the Cosserat intrinsic length scales with respect to the characteristic length of the domain or the chatacteristic length of the periodic cell, we are led to two different effective models. The first effective model is of a classical Cauchy continuum where all the information regarding displacements and rotations at the unit cell are contained in the fourth order stiffness tensor characterizing the material and can be computed by the help of two local problems one of which is related to the curvature-twist. The second effective model is of an Cosserat continuum with two fourth order effective tensors relating the non-symmetric strain to the non-symmetric stress and the curvature-twist to the couple stress, proving new constitutive laws for Cosserat media.

Additionally, we provide some corrector type results using the averaging operator for each of the effective models. By and large the results should hold true in the case where one of the materials is a void by adjusting the unfolding and averaging operators, respectively, as in [CDD+12].

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