The invariance of asymptotic laws of stochastic systems under discretization

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Abstract

The stochastic trapezoidal rule provides the only discretization scheme from the family of implicit Euler methods (see [11]) which possesses the same asymptotic (stationary) law as underlying linear continuous time stochastic systems with white or coloured noise. This identity is shown for systems with multiplicative (parametric) and additive noise using fixed point principles and the theory of positive operators. The key result is useful for adequate implementation of stochastic algorithms applied to numerical solution of autonomous stochastic differential equations. In particular it has practical importance when accurate long time integration is required such as in the process of estimation of Lyapunov exponents or stationary measures for oscillators in Mechanical Engineering.

Contents

1 Introduction

Numerous applications in Natural Sciences and Econometrics lead to models which correspond to linear differential systems perturbed additively or parametrically by random noise. This noise could be modelled as real, coloured or white one. All our results hold for this general case too. For simplicity of this contribution, we confine ourselves to the white noise case where the basic ideas are clearly visible. The occuring systems can often be interpreted as systems of linear differential equations in $\mathbb{R}^d$. Consider systems

$$dX_t \;=\; A_0\,X_t\,dt \,+\, \sum_{j=1}^{m}[A_j\,X_t \,+\, a_j]\,dW_t^j \tag{1}$$

where A_j are $d \times d$ matrices, a_j d–dimensional vectors and W_t^j independent standard Wiener processes. Without loss of generality, we may suppose that system (1) is given in Itô interpretation. (Otherwise one transforms given stochastic calculus to Itô one.) For following analysis we decompose system (1) into related systems

$$dX_t \;=\; A_0\,X_t\,dt \,+\, \sum_{j=1}^{m} a_j\,dW_t^j \tag{2}$$

with purely additive noise and

$$dX_t \;=\; A_0\,X_t\,dt \,+\, \sum_{j=1}^{m} A_j\,X_t\,dW_t^j \tag{3}$$

with purely multiplicative noise. For basic facts on theory of stochastic differential equations (SDEs), see [1],[10] or more recent monographs [6],[8],[9]. For some applications in Mechanical Engineering, have a look in [20],[26].

Under discretization these systems have a large variety of analoga. For examples, see [2], [11], [13], [14], [17], [19], [24] or [25]. We are especially interested in qualitative behaviour of the simplest and most used numerical methods as integration time tends to infinity. Such methods are performed by the **family of implicit Euler methods**. Their scheme applied to system (1) with equidistant step size $h > 0$ can explicitly be written to

$$X_{n+1} \;=\; B_0\, X_n \;+\; \sqrt{h}\,\sum_{j=1}^{m} [B_j\, X_n \;+\; b_j]\, \xi_n^j \tag{4}$$

where

$$B_0 \;=\; \Big(I - \alpha h A_0\Big)^{-1}\Big(I + (1-\alpha)hA_0\Big),\; B_j \;=\; \Big(I - \alpha h A_0\Big)^{-1} A_j,\; b_j \;=\; \Big(I - \alpha h A_0\Big)^{-1} a_j,$$

$$\xi_n^j \;=\; \big(W^j(t_{n+1}) - W^j(t_n)\big)/\sqrt{h}$$

along time–discretization $0 < t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots$. Here $\alpha \in [0,1]$ (or $\alpha \geq 0$) represents the parameter of implicitness. Throughout the paper I denotes $d \times d$ unit matrix in $\mathbb{R}^{d\times d}$. Well–known members of this family are performed by (explicit) Euler (i.e. $\alpha = 0$), trapezoidal (i.e. $\alpha = 0.5$, sometimes called improved Euler method which is also identical with midpoint method in linear autonomous case) and implicit Euler method (i.e. $\alpha = 1$). They belong to the more general classes of stochastic θ–methods (for introduction, see [15]) and stochastic Runge–Kutta methods. Note that this class only takes into account an incorporation of implicitness carried by drift part of underlying continuous time dynamics. An appropriate incorporation of stochastic–implicit terms is fairly complicated without changing stochastic calculus in the presence of multiplicative noise. Besides stochastic implicitness shall not be necessary for the purpose of our considerations. Thereby we do not consider such representatives in this paper.

The class of implicit Euler methods provides numerically mean square converging solutions to SDEs (1) with convergence order $\gamma = 1.0$ for subsystems (2) and order $\gamma = 0.5$ for subsystems (3). Related discrete time subsystems are given by

$$X_{n+1} \;=\; B_0\, X_n \;+\; \sqrt{h}\,\sum_{j=1}^{m} b_j\, \xi_n^j \tag{5}$$

and

$$X_{n+1} \;=\; B_0\, X_n \;+\; \sqrt{h}\,\sum_{j=1}^{m} B_j\, X_n\, \xi_n^j \tag{6}$$

with matrices B_j and vectors b_j as above. As in deterministic analysis, these methods and implicit techniques at all are introduced to stabilize the behaviour of numerical solutions. The parameters $\alpha \geq 0$ and $h > 0$ turn out to be corresponding control parameters in the process of numerical stabilization. There the general concept of numerical stability (A–stability) has established to classify and decide the question of goodness and preference of corresponding approximations (cf. contributions of DAHLQUIST [3], [4]). For further aspects on deterministic numerical analysis, see [5], [7], [22] or [23]. In stochastic analysis, a possible counterpart to concept of numerical stability could be the concept of stationarity. Moreover, one is aiming to obtain an 'iff'–relation between discrete and underlying continuous time systems in view of asymptotic (stationary) probabilistic behaviour. First investigations in this respect can be found for simple systems in [16] and [18].

An obvious necessary condition for existence of asymptotic (stationary) laws of continuous time systems (1) is that all real parts of eigenvalues of drift matrix A_0 are exclusively negative. In another words, it holds

$$\forall\, \lambda\, \Big(\exists\, e \in \mathbb{R}^d\; A\, e \;=\; \lambda\, e\Big) \;:\; Re(\lambda(A_0)) \;<\; 0 \tag{7}$$

where $Re(\lambda(A_0))$ represents the real part of inscribed eigenvalue of matrix A_0. Requirement (7) also guarantees the existence of the inverse of matrices $I \,-\, \alpha\, h\, A_0$ for all parameters $\alpha\, h \geq 0$, which one needs for 'unconditioned' construction of methods (4). Let us assume assumption (7) throughout the remaining exposition of this paper. However, in case of more general discrete time systems of form (4) where B_0 is an arbitrary matrix, it nessitates (and suffices for subsystem (5)) to require

$$r(B_0) \; < \; 1$$

for existence of asymptotic laws. $r(B)$ denotes the spectral radius of inscribed matrix (or operator). In passing we note that from physical arguments it is also reasonable to require the existence of asymptotic (stationary) probabilistic laws. For example, this can be motivated by dissipation of energy of damped harmonic oscillations. Besides, for simplicity, we assume that matrices A_j, B_j and vectors a_j, b_j are deterministic throughout this paper.

The remaining part of this contribution is organized as follows. Section 2 states necessary and sufficient conditions for existence and form of asymptotic (stationary) laws of continuous and discrete time stochastic systems. First, we compile basic facts on asymptotic behaviour of systems (1) – (3), taken from Arnold [1]. Second, we shall prove a new theorem concerning asymptotic behaviour for discrete time stochastic systems (4). For this purpose we make use of standard fixed point principles and theory of linear, positive operators, as known from Krasnosel'skij, Lifshitz and Sobolev [12]. The invariance of asymptotic laws of linear SDEs under appropriate discretization is noticed for family of implicit Euler methods in section 3. After it we shall show results of some numerical experiments for damped harmonic oscillators excited by random noise. The paper is finished with a summary and remarks in section 5, supplemented with an appendix containing an auxiliary lemma on linear positive operators in section 6.

2 Asymptotic (stationary) laws of linear stochastic systems

Before coming to key result, we add some of basic results on stationary laws of linear stochastic systems. In stating assertions below, let $\mathcal{N}(\mu, \sigma^2)$ denote Gaussian distribution with mean μ and covariance matrix σ^2, $\mathcal{O}$ is $d \times d$ null matrix and $(.)^T$ the transpose of inscribed vector or matrix. Let X_∞ denote the random variable of asymptotic (stationary) solution.

Theorem 1. *Assume that* $\mathbb{E}\,||X_0||^2 < +\infty$. *Let* X_0 *be independent of* $\sigma\{W_t^j : j = 1, 2, ..., m; t \geq 0\}$. *Then system (2) has stationary law* $X_\infty \in \mathcal{N}(0, M)$ *if and only if*

(*i*) *condition* (7) *holds and*

(*ii*) $M \in \mathbb{R}^{d\times d}$ *satisfies*

$$A_0\,M \; + \; M\,A_0^T \; + \sum_{j=1}^{m} a_j\,a_j^T \; = \; \mathcal{O}\,. \tag{8}$$

Furthermore, system (3) has exponentially mean square stable null solution $X_\infty = 0$ *if and only if*

(*i*) *condition* (7) *holds and*

(*ii*) $\exists$ *positive–definite solution* $M \in \mathbb{R}^{d\times d}$ *of matrix equation*

$$A_0\,M \; + \; M\,A_0^T \; + \sum_{j=1}^{m} A_j\,M\,A_j^T \; = \; -\,C \tag{9}$$

for any positive–definite matrix $C \in \mathbb{R}^{d\times d}$.

Remark. The proof of Theorem 1 can be omitted since it follows from Arnold [1]. More precisely, first part of Theorem 1 is a modified version of Theorem 8.2.12 from [1], whereas the second part is an immediate consequence of Theorem 11.4.11 from [1].

Theorem 2. *Assume that* $\mathbb{E}\,||X_0||^2 < +\infty$. *Let* X_0 *be independent of* $\sigma\{W_t^j : j = 1, 2, ..., m; t \geq 0\}$. *Fix arbitrary step size* $h > 0$. *Then system (5) has stationary law* $X_\infty \in \mathcal{N}(0, M)$ *if and only if*

(*i*) $r(B_0) \; < \; 1$ *holds and*

(*ii*) $M \in \mathbb{R}^{d\times d}$ *satisfies*

$$M \; = \; B_0\,M\,B_0^T \; + \; h \sum_{j=1}^{m} b_j\,b_j^T\,. \tag{10}$$

Furthermore, system (6) has exponentially mean square stable null solution $X_\infty = 0$ *if and only if*

(*i*) $r(B_0) < 1$ *holds and*

(*ii*) $\exists$ *positive–definite solution* $M \in \mathbb{R}^{d\times d}$ *of matrix equation*

$$- M + B_0 M B_0^T + h \sum_{j=1}^{m} B_j M B_j^T = - C \tag{11}$$

for any positive–definite matrix $C \in \mathbb{R}^{d\times d}$.

Proof. For mathematical convenience, consider separated systems with purely additive and multiplicative noise. Fix arbitrary step size $h > 0$. Consider system (5) at first. Suppose (*i*) and (*ii*) are valid. If asymptotic law exists then it must be Gaussian. This fact directly follows from explicit expansion of numerical solution which is possible under (*i*), cf. [18]. Therefore it suffices to show the existence of stationary first and second moments. That stationary solution X_∞ exists can be seen from the following argumentation. Define operators

$$\mathcal{H}_1(S) := B_0 S B_0^T, \; \mathcal{H}_2(S) := B_0 S B_0^T + h \sum_{j=1}^{m} b_j b_j^T = \mathcal{H}_1(S) + h \sum_{j=1}^{m} b_j b_j^T$$

mapping from the space of symmetric, real–valued, $d \times d$ matrices $\mathcal{S}_d$ into itself. Note that operator $\mathcal{H}_2$ describes the evolution of second moments of corresponding discrete time system (5). Namely, it holds

$$\mathbb{E}\,[X_{n+1} X_{n+1}^T] = \mathcal{H}_2(\mathbb{E}\,[X_n X_n^T]) = [\mathcal{H}_2]^{n+1}(\mathbb{E}\,[X_0 X_0^T]).$$

Introduce scalar product $< S_1, S_2 >_+ := trace(S_1 S_2)$ on the space $\mathcal{S}_d$, which renders $\mathcal{S}_d$ to a Hilbert space. Define $\mathcal{K}$ as the subspace of positive–semidefinite matrices which is a positive, reproducing, normal cone in $\mathcal{S}_d$. Obviously, operators $\mathcal{H}_1$ and $\mathcal{H}_2$ leave positive cone $\mathcal{K}$ invariant. Now, let us formulate the asymptotic behaviour of discrete time stochastic systems in terms of positive operators. The asymptotic law X_∞ with bounded second moments exists for system (5) if and only if operator $\mathcal{H}_2$ is contractive with respect to metric induced by scalar product $< .,. >_+$. This is equivalent with $r(\mathcal{H}_1) < 1$ where $r()$ is the spectral radius of linear, positive operator $\mathcal{H}_1$. This fact can be seen from a series of theorems from Krasnosel'skij, Lifshitz and Sobolev [12] as a consequence of well–known Krejn–Rutman Theorem (1948), see appendix. (Note that one operates on finite–dimensional spaces. In the infinite–dimensional setup this 'iff'–relation is not always true for linear operators, see [12] for an example.) Hence corresponding successive approximations $S_{n+1} = \mathcal{H}_2(S_n), S_0 \in \mathcal{K}$ must converge if and only if $r(\mathcal{H}_1) < 1$. This is equivalent with existence of unique fixed point M of (10). Moreover, as standard fixed point principle says, the limit $\lim_{n\to+\infty}[\mathcal{H}_2]^n(S_0)$ converges towards the unique fixed point M of (10) for any initial matrix $S_0 \in \mathcal{K}$ (with respect to metric induced by scalar product $< .,. >_+$). Then stationary law exists and is Gaussian with matrix M of second moments. Besides, operator $\mathcal{H}_1$ must have the unique fixed point $\mathcal{O}$ when fixed point of $\mathcal{H}_2$ exists. This also implies that the related deterministic system has asymptotically stable null solution. Therefore stationary solution X_∞ satisfies $\mathbb{E}\, X_\infty = 0$. An analogous argumentation one can make for systems (6). Define linear, positive operator

$$\mathcal{H}(S) := B_0 S B_0^T + h \sum_{j=1}^{m} B_j S B_j^T$$

mapping from the space $\mathcal{S}_d$ into itself. Thus $\mathcal{H}$ is the sum of linear, positive operators. This implies the necessity of requirement $r(B_0) < 1$. As above, one notices that operator $\mathcal{H}$ describes the evolution of second moments of corresponding discrete time system (6). Then system (6) has exponentially stable null solution if and only if operator $\mathcal{H}$ possesses an unique fixpoint $\mathcal{O} \in \mathcal{K}$. Thanks to auxiliary lemma in appendix, it can be shown that this requirement is equivalent with $r(\mathcal{H}) < 1$. Under latter condition, inverse $(I - \mathcal{H})^{-1}$ exists, is linear and positive. Moreover, for all positive–definite matrices $S \in \mathcal{K}$, then it holds $\mathcal{H}(S) < S$. That is $\mathcal{H}(S) - S =: -C$ is negative–definite, and for all $C \in \mathcal{K}$, there exists matrix $M \in \mathcal{K}$ such that $-(I - \mathcal{H})^{-1}(C) = M$. Thus requirement (*ii*) of Theorem 2 is rather obvious. Thus this completes the proof. $\diamond$

Remark. Theorem 2 represents a natural discrete counterpart to Theorem 1. Its validity is not connected with the specific choice of matrices B_0, B_j and vectors b_0, b_j as in the family of implicit Euler methods (4). For example, its assertion is also true for discrete time stochastic systems (4) with arbitrary $d \times d$ matrices B_0. However, then we can not expect any relation to numerical solution of SDEs (1).

3 Coincidence of asymptotic laws of systems (1) with (4)

The asymptotic laws of both numerical solution and underlying continuous time stochastic dynamics can be identical. This observation has firstly been noted for one–dimensional case with multiplicative noise in [16], and for systems with diagonalizable drift matrix and additive noise in [18]. By evaluation of corresponding matrix–valued (Lyapunov) equations occuring in Theorems 1 and 2 we are able to generalize this observation to the following assertion.

Theorem 3. *Assume that*

(*i*) $\mathbb{E}\,\|X_0\|^2 < +\infty$,

(*ii*) X_0 *is independent of* $\sigma\{W_t^j : j = 1, 2, ..., m; t \geq 0\}$ *and*

(*iii*) (7) *holds.*

Then trapezoidal rule (i.e. $\alpha = 0.5$*) provides the only method with equidistant step size* $h > 0$ *from the family of implicit Euler methods (4) which gives the same asymptotic law as the class of stochastic processes (1). More precisely, system (2) has stationary Gaussian law* $\mathcal{N}(0, M)$ *with second moments* M *if and only if system (5) has stationary Gaussian law* $\mathcal{N}(0, M)$ *with second moments* M*. System (3) has exponentially mean square stable null solution if and only if system (6) has exponentially mean square stable null solution. Moreover, the choice of step size* $h > 0$ *as well as the coincidence of initial values of discrete and continuous dynamics plays no role for this identity.*

Proof. Suppose (7) holds. For mathematical convenience, once again we separate our considerations for systems with purely additive and multiplicative noise. Of course, the equivalence of asymptotic laws is valid for full original system (1). Elementary calculations show the equivalence of condition (7) with requirement $r(B_0) < 1$ for trapezoidal method (note $\alpha = 0.5$) and all step sizes $h > 0$. Consider systems (2) and (5) with $\alpha = 0.5$ at first. Then both systems possess a stationary Gaussian solution with first moment 0 and bounded second moments (use fixed point arguments as before) if and only if condition (7) holds. The coincidence of stationary second moments becomes clear after the following equivalent rearrangements. Suppose matrix M of stationary second moments satisfies (10). Then the simultanuous multiplication of matrices $I - \alpha h A_0$ from left and $(I - \alpha h A_0)^T$ from right does not change the unique solvability of stationary equation (10), since matrix $I - \alpha h A_0$ is invertible. Thus M is unique solution of (10) if and only if M uniquely solves

$$(I - \frac{1}{2}hA_0)M(I - \frac{1}{2}hA_0)^T = (I + \frac{1}{2}hA_0)M(I + \frac{1}{2}hA_0)^T + h\sum_{j=1}^{m} a_j a_j^T .$$

After algebraic rearrangements and devision by h this equality is identical with equation (8). Consequently stationary second moments of systems (2) coincide with that of (5). Now, consider systems (3) and (6) with $\alpha = 0.5$. Then both systems have asymptotically vanishing first moments for all step sizes $h > 0$ if and only if (7). Suppose system (6) is exponentially mean square stable, i.e. second moments asymptotically vanish too. Then, thanks to Theorem 2, there exists unique solution M of matrix equation (11) for any given positive–definite matrix $C \in \mathbb{R}^{d \times d}$. By equivalent transformation of (11) with $I - \alpha h A_0$ from left and $(I - \alpha h A_0)^T$ from right one encounters with

$$-(I - \frac{1}{2}hA_0)M(I - \frac{1}{2}hA_0)^T + (I + \frac{1}{2}hA_0)M(I + \frac{1}{2}hA_0)^T + h\sum_{j=1}^{m} A_j M A_j^T = -(I - \frac{1}{2}hA_0)C(I - \frac{1}{2}hA_0)^T$$

which is equivalent to

$$A_0 M + M A_0^T + \sum_{j=1}^{m} A_j M A_j^T = -\frac{1}{h}(I - \frac{1}{2}hA_0)C(I - \frac{1}{2}hA_0)^T =: \hat{C} .$$

That is, for any positive–definite $C \in \mathcal{K}$, we find a positive–definite $\hat{C} \in \mathcal{K}$ such that M uniquely solves (9). Hence, thanks to Theorem 1, the null solution is exponentially mean square stable for system (3) too.

An analogous conclusion holds for 'vice versa' direction. Consequently, the equivalence of asymptotic laws (i.q.m.) is obvious when $\alpha = 0.5$. The fact that trapezoidal method is the only method with this equivalence property within equidistant integration of class of systems (1) can easily be seen in the one–dimensional linear case (see [16] and [18]). Thus this completes the proof. $\diamond$

R e m a r k . Theorem 3 has important practical meaning. For example, in all applications where exactness of asymptotic laws under discretization is required or even asymptotic characteristics of underlying continuous time dynamics (like Lyapunov exponents or invariant measures) are to be estimated. There one should make use of those numerical techniques which preserve the asymptotic law under discretization. It is worth noting that trapezoidal method has this invariance–property which turns out to be independent of step size $h > 0$ used in numerical integration. Note also we have not proven that under specific constellation of multi–dimensional systems (1) and (4) one can not find a choice of step size h, implicitness α and initial values X_0 such that asymptotic laws coincide.

Theorem 2 gives rise to introduce a new definition concerning the invariance of asymptotic (probabilistic) laws under discretization. For the sake of classification, let us complete this section with the notion of *asymptotic equivalence of stochastic systems.*

D e f i n i t i o n . Stochastic systems (1) and (4) are called **asymptotically equivalent** iff equivalence of conditions (i) and (ii) of Theorems 1 and 2, respectively, hold.

R e m a r k . The only case when the property of asymptotic equivalence could be proven so far is the case $\alpha = 0.5$ with autonomous and linear systems. It would be interesting to carry over the search for asymptotically equivalent systems to the case of nonlinear or nonautonomous coefficients. For example, for periodically excited autonomous oscillators or for oscillators with hysteretic forces as often met in Mechanical Engineering, see SOBCZYK [20].

4 Numerical experiments for randomly excited harmonic oscillator

The following numerical illustration supplements the presented theory. Consider a randomly excited linear oscillator with one degree of freedom. Let x be its displacement and $v = \dot{x}$ its velocity. After elimination of its mass the equation of motion reduces to

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \sigma_0\xi_0 + \sigma_1 x\xi_1 + \sigma_2\dot{x}\xi_2 \tag{12}$$

where $\omega \in \mathbb{R}^+$ is its eigenfrequency, $\zeta \in \mathbb{R}^+$ damping coefficient, $\sigma_i \in \mathbb{R}$ noise intensities, and ξ_i formal derivatives of independent standard Wiener processes. It is clear that system (12) can be rewritten to a system of form (1). For example, take $x_1 = x, x_2 = \dot{x}$ and

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \; A_3 = \mathcal{O},$$

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix}.$$

Obviously, key assumption (7) for existence of asymptotic laws is satisfied when $\omega, \zeta > 0$. Hence we may apply theorems 1 – 3. Consequently, trapezoidal method (remember that $\alpha = 0.5$ in (4)) is a favourite method for numerical integration of system (12). The system components B_j, b_j for trapezoidal method of form (4) applied to (12) can easily be found. For example, one gets

$$B_0 = \frac{1}{det}\begin{pmatrix} 1 + h\zeta\omega - 0.25h^2\omega^2 & h \\ -h\omega^2 & 1 - h\zeta\omega - 0.25h^2\omega^2 \end{pmatrix}, \; B_1 = \frac{\sigma_1}{det}\begin{pmatrix} 0.5h & 0 \\ 1 & 0 \end{pmatrix},$$

$$B_2 = \frac{\sigma_2}{det}\begin{pmatrix} 0 & 0.5h \\ 0 & 1 \end{pmatrix}, \; b_3 = \frac{\sigma_0}{det}\begin{pmatrix} 0.5h \\ 1 \end{pmatrix},$$

where $h > 0$ represents any equidistant step size, $det = det(I - 0.5hA_0) = 1 + h\zeta\omega + 0.25h^2\omega^2$, and other elements B_3, b_1, b_2 vanish. For numerical illustration, we shall separate both continuous and corresponding discrete time systems with purely additive and multiplicative noise. In passing we give a short remark on practical application of methods (4). The explicit form of (4) used here is possible because of its relatively simple specific structure. In very high–dimensional situation one would prefer to solve algebraic implicit equations by numerical procedures instead of explicit inversion of factors $I - \alpha h A_0$. Nevertheless, this fact does not hinder us to advice to prefer such numerical methods which coincide with trapezoidal method (4) under linearization.

First, the case of purely additive noise, i.e. $\sigma_1 = \sigma_2 = 0$. For abbreviation, we shall only refer to 'dimensionless' parameters σ_i, ζ, ω. Let us estimate the mean square evolution of displacement and velocity of corresponding linear oscillator with $\omega = 5$, $\zeta = 0.04$ and $\sigma_0 = 2.0$. This is done by means of trapezoidal method using equidistant step size $h = 0.001$. From Theorem 3 we know about coincidence of stationary law of continuous time oscillator (12) under the absence of multiplicative noise. This stationary law is Gaussian with mean zero and second moments

$$\mathbb{E}\,[x^2] \;=\; \frac{\sigma_0^2}{4\,\zeta\,\omega^3}\,(=\;0.2), \qquad \mathbb{E}\,[\dot{x}^2] \;=\; \frac{\sigma_0^2}{4\,\zeta\,\omega}\,(=\;5.0), \qquad \mathbb{E}\,[x\,\dot{x}] \;=\; 0\,.$$

The exact replication of this law can numerically be checked. For this purpose we plot the numerical mean square evolution of displacement x and velocity $v = \dot{x}$ in figure 1. Figure 1 confirms our theoretical results. One obviously recognizes that both numerical mean square evolutions converge towards stationary values of exact solution as integration time tends to infinity. There we used sample size $N = 50000$ for statistical estimation. Besides, numerical system has started in deterministic initial value $(x, \dot{x}) = (1, 0)$ at time $t = 0$.

Now, the case of purely multiplicative noise, i.e. $\sigma_1 = 0$. Consider mean square evolution of displacement and velocity of system (12) under the absence of additive noise. Suppose $\sigma = \sigma_1 = \sigma_2$ and parameters ω, ζ are chosen as above. The statistical estimation of functionals of components of oscillators attracts special interest. In particular, one desire is to estimate functionals

$$f(t) \;:=\; \mathbb{E}\,[\omega^2 x^2(t) + \dot{x}^2(t)]/2 \;=\; \mathbb{E}\,[25 x^2(t) + \dot{x}^2(t)]/2$$

which could be carried out using trapezoidal method with step size $h = 0.001$, noise intensity $\sigma = 0.1$ and sample size $N = 50000$. Functional f characterizes the mean total energy of damped harmonic oscillations (as sum of kinetic and strain energy). Thus, from physical point of view, we are interested in estimation of dissipation of energy. Note that the energy for continuous time system (12) decreases as time tends to infinity until $\dot{x}$ coordinate reaches zero, and the term $2\,\zeta\,\omega$ accounts for the dissipation of energy in the absence of random perturbations. Numerical results for estimation of evolution of total energy of randomly perturbed system (12) are visualized in figure 2. It is clearly visible that the mean energy of stochastic oscillator dissipates for given parameter choice. In another words, system (12) has asymptotically mean square stable null solution for sufficiently small intensity σ. In another words, the mean energy of stochastic oscillator dissipates. For sufficiently large noise intensities, the system is mean square instable and statistical estimation would be more and more problematic since higher moments diverge. Then a more accurate estimation needs an appropriate refinement of step sizes or more robust statistical procedures. So the plot of estimates for very large spread of intensities requires very laborious work. Besides, energy dissipation cannot be observed in this case.

5 Conclusions and remarks

This contribution represents a continuation of papers [15] – [18]. Therein and here in a more general context, a remarkable coincidence between asymptotic (stationary) laws of discrete and continuous time systems could be noticed. For the purpose of classification, we have introduced the notion of asymptotic equivalence of stochastic systems. It has been shown that stochastic trapezoidal and midpoint rule are preferable within equidistant integration of linear autonomous stochastic systems. Their advantage particularly comes up when stationary probabilistic law of underlying continuous time system should exactly be preserved under discretization (i.e. the invariance of stationary law under discretization or, in another words, asymptotic equivalence of continuous and discrete systems). However, for nonautonomous systems (1), nonlinear systems or variable step size integration, we have not clarified this fact so far. This is due to a lack of knowledge on fixed points of sequences of nonidentical (nonlinear) positive operators. It would also be interesting to carry over the presented analysis to the more general class of stochastic θ–methods (for introduction, see [15]) or other stochastic Runge–Kutta techniques. Anyway, the presented results also are valid for other classes of discrete time stochastic systems. The range of validity within theory of numerical integration is established by all those (nonlinear) methods which possess the form (4) under linearization. Nontrivial examples differing

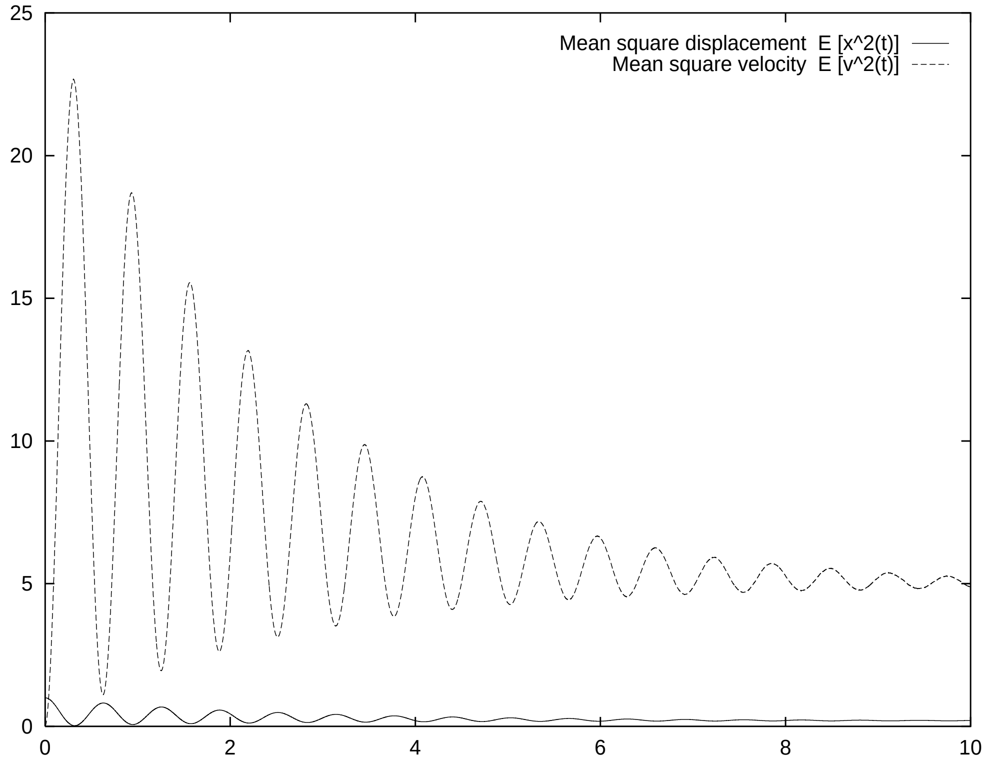

Figure 1. Temporal evolution of mean square displacement and velocity of damped harmonic oscillator (12) perturbed by additive noise.

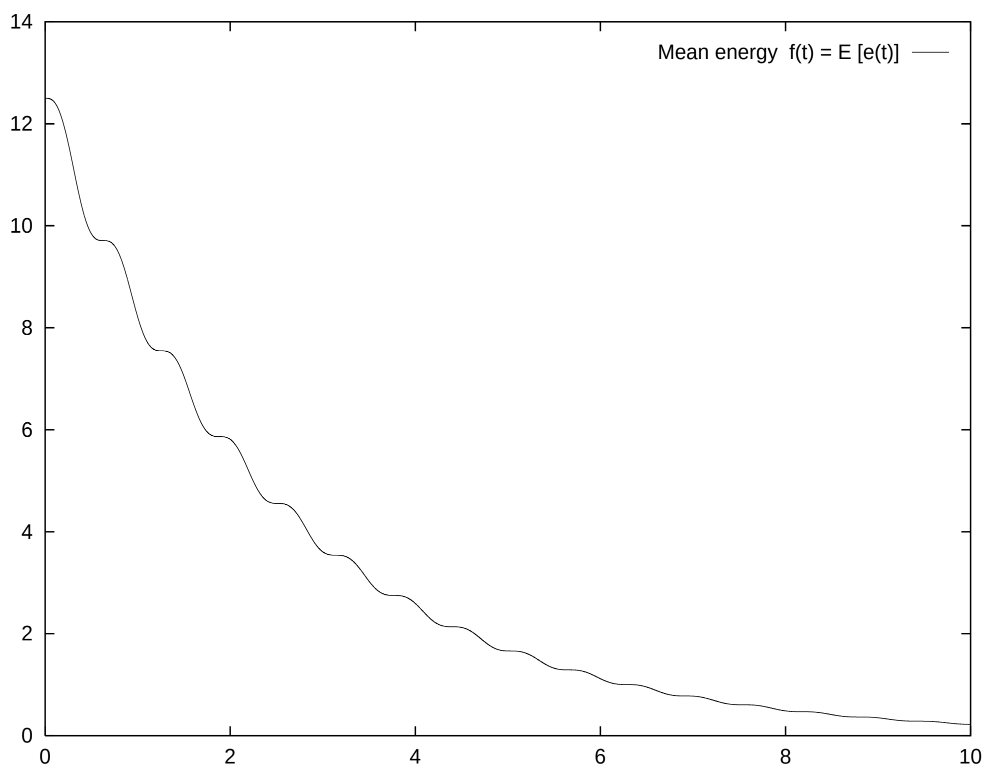

Figure 2. Temporal evolution of mean energy of damped harmonic oscillator (12) perturbed by multiplicative noise.

from family of implicit Euler methods in nonlinear situation are given by stochastic Rosenbrook (see [2]) or linear–implicit methods (see forthcoming paper [19]). Asymptotic moments of considered systems up to second order also coincide in a NonGaussian framework. For example, when W_t^j, ξ_n^j are only independently identically distributed with corresponding finite second moments.

The dynamics of deterministic (implicit) trapezoidal method and more general that of θ–methods is fairly well–understood nowadays. For example, the trapezoidal rule does not admit the existence of spurious solutions (see STEWART AND PEPLOW [21]) or this rule provides A–stable numerical solutions with highest possible accuracy (see DAHLQUIST [3],[4]) within the class of linear multi–step methods. Thus, together with main result of this contribution, one has received some advice to prefer implicit trapezoidal method (or midpoint rule) in numerical integration of autonomous stochastic differential equations too.

6 Appendix: An auxiliary lemma

Lemma. Let $\mathcal{H}$ be a linear, continuous, positive operator mapping from Banach space $(E, ||.||)$ into itself. Assume that $\mathcal{H}$ leaves reproducing cone $\mathcal{K} \subset E$ invariant, i.e. $\mathcal{H}(\mathcal{K}) \subset \mathcal{K}$. Furthermore E is finite–dimensional or $\mathcal{H}$ is completely continuous. Then the sequence of successive approximations $S_{n+1} = \mathcal{H}(S_n)$ converges to unique fixed point $\mathcal{O} \in \mathcal{K}$ for all initial values $S_0 \in \mathcal{K}$ if and only if $r(\mathcal{H}) < 1$ where $r(\mathcal{H}) = \lim_{n\to+\infty} ||\mathcal{H}^n||^{1/n}$.

Proof. The proof immediately follows from KRASNOSEL'SKIJ, LIFSHITZ AND SOBOLEV [12]. Theorem 15.1 from [12] yields sufficiency of $r(\mathcal{H}) < 1$ for convergence of successive approximations towards unique fixed point. This fixed point must be $\mathcal{O} \in \mathcal{K}$ since $\mathcal{H}$ is linear. From Theorem 15.2 in [12] we know that $r(\mathcal{H}) \leq 1$ is necessary for its convergence. The case $r(\mathcal{H}) = 1$ can be excluded under assumptions above. Suppose $r(\mathcal{H}) = 1$. Then, thanks to Theorem 9.1 in [12] (when E finite–dimensional) and Theorem 9.2 in [12] (when $\mathcal{H}$ completely continuous), there exists (nonvanishing) eigenelement $\hat{S} \in \mathcal{K}$ belonging to eigenvalue $r(\mathcal{H}) = 1$. Form $S_n = \mathcal{H}^n(\hat{S})$. Then $S_n = r^n \hat{S} = \hat{S}$. This would obviously imply convergence of S_n towards nonvanishing element $\hat{S} \in \mathcal{K}$ which contradicts to unique convergence of successive approximations. Hence, necessity of $r(\mathcal{H}) < 1$ is clear too. Consequently, the proof has been completed. $\diamond$

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