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Stochastic systems of particles with weights and approximation of the Boltzmann equation. The Markov process in the spatially homogeneous case

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ABSTRACT. A class of stochastic systems of particles with variable weights is studied. The corresponding empirical measures are shown to converge to the solution of the spatially homogeneous Boltzmann equation. In a certain sense, this class of stochastic processes generalizes the "Kac master process" ([4]).

CONTENTS

1. Introduction

We are concerned with stochastic systems of the form

$$(g_{i,n}(t), v_{i,n}(t)), \quad i = 1, ..., m_n(t), \quad t \in T. \tag{1.1}$$

The limiting behavior of the system (1.1) is studied as the parameter n tends to infinity. The number of elements in the system is denoted by $m_n(t)$, where $t \in T$, $T \subset [0, \infty)$, is the time parameter. The elements $v_{i,n}(t) \in \mathbb{R}^3$ are considered as "particles" (or "velocities"), and the elements $g_{i,n}(t) \in [0,1]$ – as "weights". The empirical measures corresponding to the system (1.1) are defined as

$$\mu_n(t, dv) = \sum_{i=1}^{m_n(t)} g_{i,n}(t) \delta_{v_{i,n}(t)}(dv), \tag{1.2}$$

where δ denotes the Dirac measure.

A simple model of the type (1.1) was studied in Illner and Wagner [3] and called "random discrete velocity model". This model was discrete in time, i.e. $T = \{k\Delta t;\ k = 0, 1, 2, ...\}$, $\Delta t > 0$. Also, a time–independent set of "discrete velocities" $(v_{i,n},\ i = 1, ..., n)$ was used. The random evolution of the weigths $(g_{i,n}(t),\ i = 1, ..., n)$, $t \in T$, was defined on the basis of a Broadwell–type equation. It has been proved that the empirical measures (1.2) approximate (as the number of discrete velocities tends to infinity) the solution of the time–discretized and spatially homogeneous Boltzmann equation

$$\begin{aligned} f(t + \Delta t, v) &= f(t, v) + \Delta t \int_{\mathbb{R}^3} dw \int_{\mathbb{S}^2} de\, q(v, w, e) \times \\ &\qquad \times \Big[f(t, v^*(v, w, e)) f(t, v^*(w, v, e)) - f(t, v) f(t, w) \Big], \end{aligned} \tag{1.3}$$

$$f(0, v) = f_0(v), \tag{1.4}$$

where $t \in T$ and $v \in \mathbb{R}^3$. The symbol $\mathbb{S}^2$ denotes the unit sphere in the Euclidean space $\mathbb{R}^3$, the positive measurable function q is called the collision kernel, and v^* denotes the collision transformation

$$v^*(v, w, e) = v + e(e, w - v), \quad v, w \in \mathbb{R}^3, \quad e \in \mathbb{S}^2, \tag{1.5}$$

where $(.,.)$ on the right-hand side of (1.5) is the scalar product in $\mathbb{R}^3$.

The precise formulation of what "approximation" means is the following. Let ρ denote the bounded Lipschitz distance between two finite measures ν_1 and ν_2, which is defined as

$$\rho(\nu_1, \nu_2) = \sup_{\varphi \in D} | \int_{\mathbb{R}^3} \varphi(v) \nu_1(dv) - \int_{\mathbb{R}^3} \varphi(v) \nu_2(dv) |, \tag{1.6}$$

where $D = \{\varphi : \mathbb{R}^3 \to [0,1]; \quad |\varphi(x) - \varphi(y)| \leq \|x - y\|\}$. Consider the measures

$$\lambda(t, dv) = f(t, v)dv, \quad t \in T, \tag{1.7}$$

where f is the solution of Eq.(1.3), (1.4). Under some assumptions concerning the collision kernel q it has been proved that

$$\lim_{n\to\infty} E^{(n)} \rho(\mu_n(t), \lambda(t)) = 0\,, \quad \forall t \in T\,,$$

while a certain stability condition holds. The symbol $E^{(n)}$ denotes the mathematical expectation.

The main result of this paper is the construction of a stochastic system of the form (1.1) such that the empirical measures (1.2) converge (in the sense described above) to the exact (i.e. without a time–discretization error) solution of the spatially homogeneous Boltzmann equation

$$\begin{aligned} \frac{\partial}{\partial t} f(t,v) &= \int_{\mathbb{R}^3} dw \int_{\mathbb{S}^2} de\, q(v,w,e) \times \\ &\quad \times \Big[f(t, v^*(v,w,e)) f(t, v^*(w,v,e)) - f(t,v) f(t,w) \Big]\,, \end{aligned} \tag{1.8}$$

$$f(0,v) = f_0(v), \tag{1.9}$$

where $t > 0$ and $v \in \mathbb{R}^3$.

The stochastic system contains certain free parameters that do not influence the convergence result. For a special choice of these parameters, the system (1.1) is closely related to the "master process" introduced by Kac in his famous paper [4] on the mathematical foundation of kinetic theory.

The paper is organized as follows. Section 2 contains the definition of the stochastic model. The precise formulation of the convergence result is given in Section 3. After some technical preparations collected in Section 4, the convergence result is proved in Section 5. In the proof, ideas from Skorokhod [5], Arsen'yev [2], Smirnov [6], and Wagner [7] are used. Some special cases are considered in Section 6. In particular, the relationship between the stochastic model introduced in this paper and the Kac model is discussed. Comments concerning some open problems conclude the paper.

2. The Model

We introduce a stochastic system of the form (1.1) as a sequence of Markov jump processes

$$\Big(z^{(n)}(t)\,, \quad t \geq 0 \Big)_{n=1,2,\ldots} \quad . \tag{2.1}$$

For a fixed n, the state space of the process $z^{(n)}$ is

$$Z^{(n)} = \bigcup_{1 \leq m \leq N(n)} \Big([0, \gamma(n)] \times \mathbb{R}^3 \Big)^m . \tag{2.2}$$

The positive numbers $N(n)$ and $\gamma(n)$ are bounds for the number of particles in the system and for the weights of the particles, respectively.

The process $z^{(n)}$ is defined by its infinitesimal generator

$$(A^{(n)}\Phi)(z) \;=\; \sum_{1 \leq i < j \leq m} \int_{\mathbb{S}^2} de D^{(n)}(z,i,j,e) \left[\Phi(J^{(n)}(z,i,j,e)) - \Phi(z) \right] , \tag{2.3}$$

where Φ is a bounded measurable function on the state space, and the state is of the form $z = ((g_1, v_1), \ldots, (g_m, v_m)) \in Z^{(n)}$.

The function $D^{(n)}$ is supposed to be nonnegative, measurable, and such that the jump intensity

$$\pi^{(n)}(z) = \sum_{1 \leq i < j \leq m} \int_{\mathbb{S}^2} de \, D^{(n)}(z,i,j,e) \tag{2.4}$$

is bounded, i.e.

$$\pi^{(n)}(z) \leq \pi^{(n)}_{max} , \qquad \forall z \in Z^{(n)} , \quad \text{for some constant } \pi^{(n)}_{max} . \tag{2.5}$$

The jump transformation $J^{(n)}$ is defined as follows,

$$J^{(n)}(z,i,j,e) = \begin{cases} ((\tilde{g}_1, \tilde{v}_1), \ldots, (\tilde{g}_{m+2}, \tilde{v}_{m+2})) & , \text{if} \quad m+2 \leq N(n) , \\ z & , \text{otherwise} , \end{cases} \tag{2.6}$$

$$\tilde{v}_k = \tilde{v}_k(z,i,j,e) = \begin{cases} v_k & , \text{if} \quad k \leq m , \\ v^*(v_i, v_j, e) & , \text{if} \quad k = m+1 , \\ v^*(v_j, v_i, e) & , \text{if} \quad k = m+2 , \end{cases} \tag{2.7}$$

where v^* is the collision transformation given in (1.5), and

$$\tilde{g}_k = \tilde{g}_k(z,i,j,e) = \begin{cases} g_k & , \text{if} \quad k \leq m , \, k \neq i,j , \\ g_k - G^{(n)}(z,i,j,e) & , \text{if} \quad k = i,j , \\ G^{(n)}(z,i,j,e) & , \text{if} \quad k > m . \end{cases} \tag{2.8}$$

The function $G^{(n)}$ is supposed to be nonnegative, measurable, and such that

$$G^{(n)}(z,i,j,e) \leq \gamma(n)^{-1} \, g_i \, g_j , \tag{2.9}$$

for all $z = ((g_1, v_1), \ldots, (g_m, v_m)) \in Z^{(n)}$, $1 \leq i,j \leq m$, $e \in \mathbb{S}^2$.

The pathwise behavior of the process (2.1) is the following.

At a given state $z=((g_1,v_1),...,(g_m,v_m))$, the process waits a random time exponentially distributed with the parameter $\pi^{(n)}(z)$.

Then, it performs a jump into a state $\tilde{z}$ distributed according to the probability distribution

$$P^{(n)}(z,d\tilde{z})=\frac{1}{\pi^{(n)}(z)}\sum_{1\le i<j\le m}\int_{\mathbb{S}^2} deD^{(n)}(z,i,j,e)\delta_{J^{(n)}(z,i,j,e)}(d\tilde{z})\,, \tag{2.10}$$

where $\pi^{(n)}$ is defined in (2.4).

If $m>N(n)-2$, then $P^{(n)}(z,d\tilde{z})=\delta_z(d\tilde{z})$, i.e. the jump is fictitious. Otherwise, the jump looks as follows. A random pair of indices (i,j) is chosen according to the probabilities

$$\frac{1}{\pi^{(n)}(z)}\int_{\mathbb{S}^2} deD^{(n)}(z,i,j,e)\,. \tag{2.11}$$

A random direction vector e is generated according to the probability density

$$\frac{D^{(n)}(z,i,j,e)\,de}{\int_{\mathbb{S}^2} de\,D^{(n)}(z,i,j,e)}\,. \tag{2.12}$$

A collision is performed with the particles v_i and v_j, and the vector e. The post–collisional velocities $v^*(v_i,v_j,e)$ and $v^*(v_j,v_i,e)$ are added to the system. A part $G^{(n)}(z,i,j,e)$ of the weights of the pre–collisional velocities is given to the post–collisional velocities. Condition (2.9) assures that the weights remain in the interval $[0,\gamma(n)]$.

3. The Convergence Result

The following properties of the collision kernel q of the Boltzmann equation (1.8) are assumed, for arbitrary $v,w,v_1,v_2\in\mathbb{R}^3$ and $e\in\mathbb{S}^2$,

$$q(v,w,e)=q(w,v,e)\,, \tag{3.1}$$

$$q(v,w,e)=q(v^*(v,w,e),v^*(w,v,e),e)\,, \tag{3.2}$$

where v^* is defined in (1.5),

$$\int_{\mathbb{S}^2} de\,q(v,w,e)\le Q_{max}\,, \tag{3.3}$$

$$\int_{\mathbb{S}^2} de\,|q(v_1,w,e)-q(v_2,w,e)|\le Q_L\,\|v_1-v_2\|\,. \tag{3.4}$$

The function f_0 appearing in the initial condition (1.9) is supposed to be such that

$$\int_{\mathbb{R}^3} \|v\|^2 f_0(v) dv < \infty . \tag{3.5}$$

We refer to Arkeryd [1] concerning the following results. Under the assumptions (3.1)-(3.3) there exists a unique solution $f(t,v) \in L^1(\mathbb{R}^3)$, $t > 0$, of Eq.(1.8), (1.9) for every nonnegative function $f_0 \in L^1(\mathbb{R}^3)$. Moreover, if the function f_0 satisfies condition (3.5), then certain conservation properties hold, namely

$$\int_{\mathbb{R}^3} \bar{\psi}_i(v) f(t,v) dv = \int_{\mathbb{R}^3} \bar{\psi}_i(v) f_0(v) dv , \quad \forall t \geq 0 , \quad i = 0,1,2 , \tag{3.6}$$

where

$$\bar{\psi}_0(v) = 1, \quad \bar{\psi}_1(v) = v, \quad \bar{\psi}_2(v) = \|v\|^2, \quad \forall v \in \mathbb{R}^3 . \tag{3.7}$$

The following assumptions concerning the parameters of the stochastic model (2.1)–(2.9) are made;

– number of particles at the beginning:

$$m_n(0) = n , \quad \forall n = 1,2,... ; \tag{3.8}$$

– bound for the number of particles:

$$N(n) = c_N\, n , \quad \forall n = 1,2,... , \quad \text{for some constant } c_N > 1 ; \tag{3.9}$$

– bound for the weights of the particles:

$$\gamma(n) = c_\gamma\, n^{-1} , \quad \forall n = 1,2,... , \quad \text{for some constant } c_\gamma . \tag{3.10}$$

Concerning the function $D^{(n)}$, it is assumed that

$$\int_{\mathbb{S}^2} de\, D^{(n)}(z,i,j,e) \leq c_D\, n^{-1} , \quad \forall n = 1,2,... , \quad \text{for some constant } c_D , \tag{3.11}$$

and

$$D^{(n)}(z,i,j,e) G^{(n)}(z,i,j,e) = q(v_i, v_j, e)\, g_i\, g_j , \tag{3.12}$$

for all $z = ((g_1,v_1),...,(g_m,v_m)) \in Z^{(n)}$, $1 \leq i,j \leq m$, $e \in \mathbb{S}^2$. Notice that assumption (3.11) implies (2.5), with

$$\pi^{(n)}_{max} = \frac{c_D\, c_N}{2}\, N(n) . \tag{3.13}$$

Theorem 3.1. *Suppose the assumptions* (3.1)–(3.5) *concerning the parameters of the Boltzmann equation and the assumptions* (3.8)–(3.12) *concerning the parameters of the stochastic model to be fulfilled.*

Let $\lambda(t)$, $t \geq 0$, *be the measures related to the solution of Eq.* (1.8), (1.9) *via the formula* (1.7), *and* $\mu_n(t)$, $t \geq 0$, *be the empirical measures* (1.2) *associated with the stochastic system* (1.1).

Suppose that

$$\sup_n E^{(n)} \int_{\mathbb{R}^3} \|v\|^2 \mu_n(0, dv) < \infty \tag{3.14}$$

and

$$\lim_{n\to\infty} E^{(n)} \rho(\mu_n(0), \lambda(0)) = 0 \,. \tag{3.15}$$

Then,

$$\lim_{n\to\infty} E^{(n)} \sup_{t\in[0,\Delta t]} \rho(\mu_n(t), \lambda(t)) = 0 \,, \tag{3.16}$$

for all

$$\Delta t < \frac{1}{c_D} \left[\frac{1}{c_N} \left(1 - \frac{1}{c_N} \right) \right] \,. \tag{3.17}$$

4. Technical Preparations

Let $\mathcal{B}(\mathbb{R}^3)$, $\mathcal{C}_b(\mathbb{R}^3)$, and $\mathcal{C}_L(\mathbb{R}^3)$, respectively, denote the spaces of bounded measurable, bounded continuous, and bounded Lipschitz continuous functions on $\mathbb{R}^3$. The norms $\|\varphi\|_\infty = \sup_{v\in\mathbb{R}^3} |\varphi(v)|$ and

$$\|\varphi\|_L = max \left\{ \|\varphi\|_\infty ; \sup_{v,w\in\mathbb{R}^3} \frac{|\varphi(v) - \varphi(w)|}{\|v - w\|} \right\}$$

are used. Let $\mathcal{M}^+(\mathbb{R}^3)$ denote the space of finite positive measures on $\mathbb{R}^3$.

We will use the following abbreviations,

$$\langle \varphi, \nu \rangle = \int_{\mathbb{R}^3} \varphi(v)\, \nu(dv) \,, \tag{4.1}$$

$$\beta^*(\varphi)(v, w, e) = \varphi(v^*(v, w, e)) + \varphi(v^*(w, v, e)) - \varphi(v) - \varphi(w) \,, \tag{4.2}$$

$$\beta(\varphi)(v, w) = \int_{\mathbb{S}^2} de\, q(v, w, e) \frac{1}{2} \beta^*(\varphi)(v, w, e) \,, \tag{4.3}$$

$$B(\varphi, \nu) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \beta(\varphi)(v, w) \nu(dv) \nu(dw) \,, \tag{4.4}$$

where $v, w \in \mathbb{R}^3$, $e \in \mathbb{S}^2$, $\varphi \in \mathcal{B}(\mathbb{R}^3)$, $\nu \in \mathcal{M}^+(\mathbb{R}^3)$, and v^* is defined in (1.5).

Lemma 4.1. *Suppose assumptions* (3.1)–(3.3) *to be fulfilled. Let* $\lambda(t)$, $t \geq 0$, *be the measures related to the solution of Eq.* (1.8), (1.9) *via the formula* (1.7).

Then, the following equation is satisfied,

$$\langle \varphi, \lambda(t) \rangle = \langle \varphi, \lambda(0) \rangle + \int_0^t ds\, B(\varphi, \lambda(s)), \quad t \geq 0, \quad \forall \varphi \in \mathcal{B}(\mathbb{R}^3). \tag{4.5}$$

Proof. From (1.8), (1.9) one obtains the equation

$$\begin{aligned} \langle \varphi, \lambda(t) \rangle &= \langle \varphi, \lambda(0) \rangle + \int_0^t ds \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dw \int_{\mathbb{S}^2} de\, \varphi(v)\, q(v, w, e) \times \\ &\times \left[f(s, v^*(v, w, e)) f(s, v^*(w, v, e)) - f(s, v) f(s, w) \right] . \end{aligned}$$

A substitution of the integration variables

$$(v, w) \longrightarrow \mathcal{T}(v, w) = (v^*(v, w, e), v^*(w, v, e)) ,$$

which has the property that $\mathcal{T}^2$ is the identity, yields the assertion. □

Lemma 4.2. *The function* β^* *defined in* (4.2) *has the following properties,*

$$\beta^*(\varphi)(v, v, e) = 0 , \tag{4.6}$$

$$\beta^*(\varphi)(v, w, e) = \beta^*(\varphi)(w, v, e) , \tag{4.7}$$

$$|\beta^*(\varphi)(v, w, e)| \leq 4\, \|\varphi\|_\infty , \quad \varphi \in \mathcal{B}(\mathbb{R}^3) , \tag{4.8}$$

$$|\beta^*(\varphi)(v_1, w, e) - \beta^*(\varphi)(v_2, w, e)| \leq 4\, \|\varphi\|_L\, \|v_1 - v_2\| , \quad \varphi \in \mathcal{C}_L(\mathbb{R}^3) , \tag{4.9}$$

for all $v, w, \in \mathbb{R}^3$, $e \in \mathbb{S}^2$.

Proof. Properties (4.6)–(4.8) are obvious consequences of the definitions (4.2) and (1.5). Property (4.9) is shown as follows,

$$\begin{aligned} &|\beta^*(\varphi)(v_1, w, e) - \beta^*(\varphi)(v_2, w, e)| \leq \\ &\quad \|\varphi\|_L \Big[\|v^*(v_1, w, e) - v^*(v_2, w, e)\| + \|v^*(w, v_1, e) - v^*(w, v_2, e)\| + \|v_1 - v_2\| \Big] \leq \\ &\quad \leq 4\, \|\varphi\|_L\, \|v_1 - v_2\| , \end{aligned}$$

where (1.5) has been used. □

Lemma 4.3. *Suppose assumptions* (3.1) *and* (3.3) *to be fulfilled.*
Then, the function β *defined in* (4.3) *has the following properties,*

$$\beta(\varphi)(v,w) = \beta(\varphi)(w,v)\,, \tag{4.10}$$

$$|\beta(\varphi)(v,w)| \leq 2\,\|\varphi\|_\infty\, Q_{max}\,, \tag{4.11}$$

for all $v, w, \in \mathbb{R}^3$, $\varphi \in \mathcal{B}(\mathbb{R}^3)$.
If, in addition, assumption (3.4) *is fulfilled, then the following inequality holds,*

$$|\beta(\varphi)(v_1,w) - \beta(\varphi)(v_2,w)| \leq 2\,\|\varphi\|_L(Q_{max} + Q_L)\,\|v_1 - v_2\|\,, \tag{4.12}$$

for all $v_1, v_2, w \in \mathbb{R}^3$, $\varphi \in \mathcal{C}_L(\mathbb{R}^3)$.

Proof. The properties (4.10) and (4.11) are obvious consequences of (3.1), (4.7), (3.3), and (4.8). Property (4.12) follows from

$$\begin{aligned}
|\beta(\varphi)(v_1,w) - \beta(\varphi)(v_2,w)| \;\leq\; & \int_{\mathbb{S}^2} de\, |q(v_1,w,e) - q(v_2,w,e)|\frac{1}{2}|\beta^*(\varphi)(v_1,w,e)| + \\
& \int_{\mathbb{S}^2} de\, q(v_2,w,e)\frac{1}{2}|\beta^*(\varphi)(v_1,w,e) - \beta^*(\varphi)(v_2,w,e)| \\
\leq\; & 2\,\|\varphi\|_\infty\, Q_L\, \|v_1 - v_2\| + 2\,\|\varphi\|_L\, Q_{max}\, \|v_1 - v_2\|\,,
\end{aligned}$$

where (4.8), (4.9), and the assumptions (3.3), (3.4) have been used. □

Lemma 4.4. *Suppose assumptions* (3.1) *and* (3.3) *to be fulfilled.*
Then, the function B *defined in* (4.4) *satisfies the inequality,*

$$|B(\varphi,\nu)| \leq 2\,\|\varphi\|_\infty\, Q_{max}\,\nu(\mathbb{R}^3)^2\,, \quad \forall \varphi \in \mathcal{B}(\mathbb{R}^3)\,, \quad \nu \in \mathcal{M}^+(\mathbb{R}^3)\,. \tag{4.13}$$

If, in addition, assumption (3.4) *is fulfilled, then the following property holds,*

$$|B(\varphi,\nu_1) - B(\varphi,\nu_2)| \leq 2\,\|\varphi\|_L(Q_{max} + Q_L)\,\rho(\nu_1,\nu_2)\,[\nu_1(\mathbb{R}^3) + \nu_2(\mathbb{R}^3)]\,, \tag{4.14}$$

for all $\varphi \in \mathcal{C}_L(\mathbb{R}^3)$, $\nu_1, \nu_2 \in \mathcal{M}^+(\mathbb{R}^3)$.

Proof. Property (4.13) is an immediate consequence of (4.11). To prove (4.14), we consider the function

$$\beta_1(\varphi,\nu)(v) = \int_{\mathbb{R}^3} \beta(\varphi)(v,w)\nu(dw)\,,$$

where β is defined in (4.3) and $\nu \in \mathcal{M}^+(\mathbb{R}^3)$. One easily obtains from (4.11), (4.12) that

$$|\beta_1(\varphi,\nu)(v)| \leq 2\,\|\varphi\|_\infty\, Q_{max}\,\nu(\mathbb{R}^3)$$

and

$$|\beta_1(\varphi,\nu)(v_1) - \beta_1(\varphi,\nu)(v_2)| \leq 2\,\|\varphi\|_L(Q_{max} + Q_L)\,\|v_1 - v_2\|\,\nu(\mathbb{R}^3)\,.$$

Consequently,

$$\|\beta_1(\varphi,\nu)\|_L \leq 2\,\|\varphi\|_L(Q_{max}+Q_L)\,\nu(\mathbb{R}^3)\,. \tag{4.15}$$

Using (4.10), (4.15), and the definition (1.6), one obtains

$$\begin{aligned} &|B(\varphi,\nu_1)-B(\varphi,\nu_2)| \leq \\ &\quad \leq |\langle\beta_1(\varphi,\nu_1),\nu_1\rangle - \langle\beta_1(\varphi,\nu_1),\nu_2\rangle| + |\langle\beta_1(\varphi,\nu_2),\nu_1\rangle - \langle\beta_1(\varphi,\nu_2),\nu_2\rangle| \leq \\ &\quad \leq 2\,\|\varphi\|_L(Q_{max}+Q_L)\,\rho(\nu_1,\nu_2)\,[\nu_1(\mathbb{R}^3)+\nu_2(\mathbb{R}^3)]\,, \end{aligned}$$

where $\langle .,. \rangle$ is defined in (4.1). □

For any $r>0$, we consider the function χ_r on $\mathbb{R}^3$,

$$\chi_r(v)=\begin{cases} 1 & , \quad \|v\|\leq r\,, \\ r+1-\|v\| & , \quad \|v\|\in[r,r+1]\,, \\ 0 & , \quad \|v\|\geq r+1\,. \end{cases}$$

For $\varphi\in\mathcal{B}(\mathbb{R}^3)$, we denote

$$\varphi_r(v)=\varphi(v)\,\chi_r(v)\,,\quad v\in\mathbb{R}^3\,. \tag{4.16}$$

Lemma 4.5. *Let $\varphi\in\mathcal{C}_L(\mathbb{R}^3)$.*
Then,

$$\|\varphi_r\|_L \leq 2\,\|\varphi\|_L\,,\quad \forall r>0\,. \tag{4.17}$$

Proof. Obviously, $\|\varphi_r\|_\infty \leq \|\varphi\|_\infty$. The assertion follows from

$$\begin{aligned} &|\varphi_r(v)-\varphi_r(w)| \leq \\ &\quad\leq\ |\varphi(v)\chi_r(v)-\varphi(v)\chi_r(w)|+|\varphi(v)\chi_r(w)-\varphi(w)\chi_r(w)| \\ &\quad\leq\ \|\varphi\|_\infty\,\|v-w\|+\|\varphi\|_L\,\|v-w\|\,, \end{aligned}$$

where the Lipschitz property of the function χ_r has been used. □

Given $\varphi\in\mathcal{B}(\mathbb{R}^3)$, we introduce the function

$$F(\varphi)(z)=\sum_{i=1}^m g_i\,\varphi(v_i)\,,\quad z=((g_1,v_1),\ldots,(g_m,v_m))\in Z^{(n)}\,. \tag{4.18}$$

Notice that

$$F(\varphi)(z^{(n)}(t))=\langle\varphi,\mu_n(t)\rangle\,. \tag{4.19}$$

Let $\P_A$ denote the indicator function of a set A.

Lemma 4.6. *Conservation of mass, momentum, and energy holds for the empirical measures* (1.2) *associated with the stochastic system* (1.1), *i.e.*

$$\langle \bar{\psi}_i, \mu_n(t)\rangle = \langle \bar{\psi}_i, \mu_n(0)\rangle\,, \quad \forall t \geq 0\,, \quad i = 0,1,2\,, \tag{4.20}$$

where the functions $\bar{\psi}_0$, $\bar{\psi}_1$, *and* $\bar{\psi}_2$ *are defined in* (3.7).

Proof. Using the definitions (2.6)–(2.8) and (4.2), we find that

$$\begin{aligned} F(\varphi)(J^{(n)}(z,i,j,e)) &= F(\varphi)(z) + \\ &\quad + ¶_{\{m\leq N(n)-2\}}(m)\, G^{(n)}(z,i,j,e)\, \beta^*(\varphi)(v_i,v_j,e)\,, \end{aligned} \tag{4.21}$$

for all $z \in Z^{(n)}$, $1 \leq i < j \leq m$, $e \in \mathbb{S}^2$. The jumps of the process $z^{(n)}(t)$ are of the form $z \to J^{(n)}(z,i,j,e)$. Thus, one obtains from (4.19), (4.21) that

$$\langle \varphi, \mu_n(t)\rangle = \langle \varphi, \mu_n(0)\rangle\,, \quad \forall t \geq 0\,,$$

if

$$\beta^*(\varphi)(v,w,e) = 0\,, \quad \forall v,w \in \mathbb{R}^3\,, \quad e \in \mathbb{S}^2\,.$$

This is fulfilled for $\varphi = \bar{\psi}_i$, $i = 0,1,2$. □

Lemma 4.7. *Let* $\varphi \in \mathcal{B}(\mathbb{R}^3)$. *Suppose assumptions* (3.1), (3.3), *and* (3.12) *to be fulfilled.*

Then,

$$\begin{aligned} \langle \varphi, \mu_n(t)\rangle &= \langle \varphi, \mu_n(0)\rangle + \int_0^t ds\, B(\varphi, \mu_n(s)) - \\ &\quad - \int_0^t ds\, ¶_{\{m>N(n)-\}}\,(m_n(s))\, B(\varphi,\mu_n(s)) + M_n(5)(t)\,, \end{aligned} \tag{4.22}$$

where $M_n(\varphi)(t)$ *is a martingale such that*

$$|\mathrm{g}_n(\varphi)(t)| \leq 2\,\|\varphi\|_\infty \mu_n(0,\mathbb{R}^3)\,[1 + tQ_{max}\mu_n(0,\mathbb{R}^3)] \tag{4.23}$$

and

$$E^{(n)}\,[M_n(\varphi)(t)]^2 \leq 8\,\|\varphi\|_\infty^2\, \gamma(n)\, Q_{max}\, t\, E^{(n)}[\mathrm{y}_n(0,\mathbb{R}^3)]^2\,. \tag{4.24}$$

Proof. Since the function $F(\varphi)$ defined in (4.18) belongs to the domain of definition of the infinitesimal generator (2.3), the following representation holds,

$$F(\varphi)(\mathrm{r}^{(n)}(t)) = F(\varphi)(z^{(n)}(0)) + \int_0^t ds\, A^{(n)}(F(\varphi))(z^{(n)}(s)) + M_n(\varphi)(t)\,, \tag{4.25}$$

where $M_n(\varphi)(t)$ is a martingale, and

$$E^{(n)}\left[M_n(\varphi)(t)\right]^2 = E^{(n)}\int_0^t ds\,\left[A^{(n)}(F(\varphi)^2) - 2\,F(\varphi)\,A^{(n)}(F(\varphi))\right](z^{(n)}(s))\,. \quad (4.26)$$

We obtain from (2.3) and (4.21) that

$$\begin{aligned} A^{(n)}(F(\varphi))(z) &= \sum_{1\le i<j\le m}\int_{\mathbb{S}^2} de\,D^{(n)}(z,i,j,e)\times \\ &\qquad \times \P_{\{m\le N(n)-2\}}(m)\,G^{(n)}(z,i,j,e)\,\beta^*(\varphi)(v_i,v_j,e)\,. \end{aligned}$$

Now we use (3.1), (3.12), (4.7), and (4.6) to obtain

$$\begin{aligned} A^{(n)}(F(\varphi))(z) &= \P_{\{m\le N(n)-2\}}(m)\sum_{i,j=1}^m g_i\,g_j\int_{\mathbb{S}^2} de\,q(v_i,v_j,e)\,\frac{1}{2}\,\beta^*(\varphi)(v_i,v_j,e) \\ &= \P_{\{m\le N(n)-2\}}(m)\sum_{i,j=1}^m g_i\,g_j\,\beta(\varphi)(v_i,v_j)\,. \end{aligned}$$

From (4.19), (4.25) we obtain

$$\begin{aligned} \langle\varphi,\mu_n(t)\rangle &= \langle\varphi,\mu_n(0)\rangle + \\ &\quad + \int_0^t ds\,\P_{\{m\le N(n)-2\}}(m_n(s))\,B(\varphi,\mu_n(s)) + M_n(\varphi)(t)\,, \end{aligned} \quad (4.27)$$

and assertion (4.22) follows.

Moreover, it follows from (4.27) and (4.13) that

$$|M_n(\varphi)(t)| \;\le\; \|\varphi\|_\infty\,\mu_n(t,\mathbb{R}^3) + \|\varphi\|_\infty\,\mu_n(0,\mathbb{R}^3) + \int_0^t ds\,2\,Q_{max}\,\|\varphi\|_\infty\,[\mu_n(s,\mathbb{R}^3)]^2\,.$$

According to Lemma 4.6, we obtain assertion (4.23).

To show (4.24), we derive from (4.21) that

$$\begin{aligned} & F(\varphi)^2(J^{(n)}(z,i,j,e)) = F(\varphi)^2(z) + \\ & \quad +\;\; 2\,F(\varphi)(z)\,\P_{\{m\le N(n)-2\}}(m)\,G^{(n)}(z,i,j,e)\,\beta^*(\varphi)(v_i,v_j,e) + \\ & \quad +\;\; \P_{\{m\le N(n)-2\}}(m)\,\left[G^{(n)}(z,i,j,e)\,\beta^*(\varphi)(v_i,v_j,e)\right]^2\,. \end{aligned}$$

Consequently, we obtain from (2.3) that

$$\begin{aligned} & A^{(n)}(F(\varphi)^2)(z) = 2\,F(\varphi)(z)\,A^{(n)}(F(\varphi))(z) + \\ & \quad + \sum_{1\le i<j\le m}\int_{\mathbb{S}^2} de\,D^{(n)}(z,i,j,e)\,\P_{\{m\le N(n)-2\}}(m)\,\left[G^{(n)}(z,i,j,e)\,\beta^*(\varphi)(v_i,v_j,e)\right]^2\,. \end{aligned} \quad (4.28)$$

It follows from (2.9) that

$$G^{(n)}(z,i,j,e) \leq \gamma(n)\,. \tag{4.29}$$

Applying (3.12), (3.3), (4.8), and (4.29), we conclude from (4.28) that the function appearing on the right-hand side of (4.26) satisfies the inequalities

$$0 \leq \left[A^{(n)}(F(\varphi)^2) - 2\,F(\varphi)\,A^{(n)}(F(\varphi))\right](z) \leq 16\,\|\varphi\|_\infty^2\,Q_{max}\,\gamma(n)\,\frac{1}{2}\left(\sum_{i=1}^m g_i\right)^2\,.$$

Consequently, assertion (4.24) follows from (4.26), and Lemma 4.6. □

The following lemmas are related to the estimation of the probability that the number of particles in the system becomes close to the bound $N(n)$.

Lemma 4.8. *Consider a Markov jump process $z(t)$ given by the generator*

$$(A\phi)(z) = \pi(z)\int_Z P(z,d\tilde{z})\,[\phi(\tilde{z}) - \phi(z)]\,, \quad z \in Z\,, \tag{4.30}$$

and suppose that the jump intensity is bounded, i.e.

$$\pi(z) \leq \pi_{max} < \infty\,, \quad \forall z \in Z\,. \tag{4.31}$$

Let $\alpha(t)$ denote the number of jumps of the process $z(t)$ on the time interval $[0,t)$. Then,

$$Prob(\alpha(t) \geq c) \leq Prob(\xi \geq c)\,, \quad \forall c \in \mathbb{R}\,, \tag{4.32}$$

where the random variable ξ has a Poisson distribution with the parameter $t\,\pi_{max}$.

Proof. The assertion of the lemma is intuitively clear. To prove it formally, one considers the extended Markov process $(z(t),\alpha(t),\beta(t)), t \geq 0$, given by the generator

$$\begin{aligned}(\hat{A}\phi)(z,\alpha,\beta) \;=\; & \int_Z\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\pi(z)P(z,d\tilde{z})\,\delta_{\alpha+1}(d\tilde{\alpha})\,\delta_{\beta+1}(d\tilde{\beta})\,+ \\ & +[\pi_{max} - \pi(z)]\,\delta_z(d\tilde{z})\,\delta_\alpha(d\tilde{\alpha})\,\delta_{\beta+1}(d\tilde{\beta})\Big\}\left[\phi(\tilde{z},\tilde{\alpha},\tilde{\beta}) - \phi(z,\alpha,\beta)\right]\,.\end{aligned}$$

The jump intensity for the extended process is constant, $\hat{\pi}(z,\alpha,\beta) = \pi_{max}$. With probability $\frac{\pi(z)}{\pi_{max}}$, the process jumps into the state $(\tilde{z},\alpha+1,\beta+1)$, with probability $\left(1 - \frac{\pi(z)}{\pi_{max}}\right)$ — into the state $(z,\alpha,\beta+1)$. If $\alpha(0) = \beta(0) = 0$, then $\alpha(t) \leq \beta(t)$, $\forall t \geq 0$, and $\beta(t)$ has a Poisson distribution with the parameter $t\,\pi_{max}$. □

Lemma 4.9. *Let the random variable ξ have a Poisson distribution with the parameter $\bar{\pi}$.*

Then,

$$Prob(\xi \geq \bar{\pi} + C) \leq \frac{\bar{\pi}}{C^2}, \quad \forall C > 0\,. \tag{4.33}$$

Proof. The assertion follows from the properties of the Poisson distribution $E\,\xi = E\,(\xi - E\,\xi)^2 = \bar{\pi}$, and Chebyshev's inequality. □

Lemma 4.10. *Consider the process* (2.1). *Suppose that assumption* (2.5) *is fulfilled with*

$$\pi^{(n)}_{max} \leq c_\pi\, N(n)\,, \quad \text{for some constant}\, c_\pi\,. \tag{4.34}$$

Suppose that

$$\lim_{n\to\infty} N(n) = \infty\,, \tag{4.35}$$

and

$$\lim_{n\to\infty} Prob^{(n)}\Big(m_n(0) \geq \kappa\, N(n)\Big) = 0\,, \quad \text{for some}\, \kappa \in (0,1)\,. \tag{4.36}$$

Then,

$$\lim_{n\to\infty} Prob^{(n)}\Big(m_n(t) \geq N(n) - k\Big) = 0\,, \quad \forall k = 1,2,\dots\,,$$

for all $t < \frac{1-\kappa}{2\,c_\pi}$.

Proof. Let $\alpha_n(t)$ denote the number of jumps of the process $z^{(n)}(t)$ on the interval $[0,t)$. Then,

$$m_n(t) \leq m_n(0) + 2\,\alpha_n(t)\,. \tag{4.37}$$

Using (4.37), one estimates

$$\begin{aligned} & Prob^{(n)}\Big(m_n(t) \geq N(n) - k\Big) = \\ &= \; Prob^{(n)}\Big(m_n(t) \geq N(n) - k\,; \quad m_n(0) \geq \kappa\, N(n)\Big) + \\ &\quad + Prob^{(n)}\Big(m_n(t) \geq N(n) - k\,; \quad m_n(0) < \kappa\, N(n)\Big) \leq \\ &\leq \; Prob^{(n)}\Big(m_n(0) \geq \kappa\, N(n)\Big) + Prob^{(n)}\Big(\kappa\, N(n) + 2\,\alpha_n(t) \geq N(n) - k\Big)\,, \end{aligned} \tag{4.38}$$

for any $k = 1,2,\dots$. The first term on the right–hand side of (4.38) tends to zero as $n \to \infty$ because of assumption (4.36).

Lemma 4.8 is to be applied to estimate the second term on the right–hand side of (4.38). The generator (2.3) has the form (4.30) (cf. (2.10), (2.4)). Assumption

(4.31) is fulfilled with $\pi_{max} = c_\pi N(n)$ because of (4.34). Consequently, we obtain from (4.32),

$$Prob^{(n)}\Big(\kappa\, N(n) + 2\,\alpha_n(t) \geq N(n) - k\Big) \leq Prob\left(\xi \geq \frac{1}{2}\,[N(n)\,(1-\kappa) - k]\right), \tag{4.39}$$

where ξ is a random variable having a Poisson distribution with the parameter $t\,c_\pi\, N(n)$.

The probability on the right–hand side of (4.39) is estimated via Lemma 4.9. The positivity of the expression

$$\frac{1}{2}\,[N(n)\,(1-\kappa) - k] - t\,c_\pi\, N(n)$$

is assured as $n \to \infty$ by assumption (4.35), provided that $1 - \kappa - 2\,t\,c_\pi > 0$. Thus, we obtain from (4.33), with $\bar{\pi} = t\,c_\pi\, N(n)$,

$$Prob^{(n)}\Big(\kappa\, N(n) + 2\,\alpha_n(t) \geq N(n) - k\Big) \leq \frac{4\,t\,c_\pi\, N(n)}{[N(n)\,(1-\kappa-2\,t\,c_\pi) - k]^2}\,.$$

Thus, the second term on the right–hand side of (4.38) tends to zero as $n \to \infty$ because of assumption (4.35) provided that $t < \frac{1-\kappa}{2\,c_\pi}$. □

5. Proof of Theorem 3.1

Let $\mathbb{B}_r$, $r > 0$, denote the ball with the radius r in $\mathbb{R}^3$. Using (4.5), (4.22), and (4.16), we obtain the estimate

$$\begin{aligned}
&|\langle\varphi, \mu_n(t)\rangle - \langle\varphi, \lambda(t)\rangle| \leq \\
&\qquad |\langle\varphi_r, \mu_n(t)\rangle - \langle\varphi_r, \lambda(t)\rangle| + |\langle\varphi - \varphi_r, \mu_n(t)\rangle| + |\langle\varphi - \varphi_r, \lambda(t)\rangle| \leq \\
&\leq \quad |\langle\varphi_r, \mu_n(0)\rangle - \langle\varphi_r, \lambda(0)\rangle| + \int_0^t ds\, |B(\varphi_r, \mu_n(s)) - B(\varphi_r, \lambda(s))| + \\
&\qquad + \int_0^t ds\, \P_{\{m > N(n)-2\}}(m_n(s))\, |B(\varphi_r, \mu_n(s))| + |M_n(\varphi_r)(t)| + \\
&\qquad \Big[\mu_n(t, \mathbb{R}^3 \setminus \mathbb{B}_r) + \lambda(t, \mathbb{R}^3 \setminus \mathbb{B}_r)\Big]\, \|\varphi\|_\infty\,,
\end{aligned}$$

for any $\varphi \in \mathcal{B}(\mathbb{R}^3)$. Now we apply (4.13), (4.14), and (4.17) to obtain

$$\begin{aligned}\rho(\mu_n(t), \lambda(t)) \;\leq\; & 2\,\rho(\mu_n(0), \lambda(0)) + \\ & + \int_0^t ds\, 4\,(Q_{max} + Q_L)\, \rho(\mu_n(s), \lambda(s)) \left[\mu_n(s, \mathbb{R}^3) + \lambda(s, \mathbb{R}^3)\right] + \\ & + \int_0^t ds\, ¶_{\{m > N(n)-2\}}(m_n(s))\, 2\, Q_{max} \left[\mu_n(s, \mathbb{R}^3)\right]^2 + \\ & + \sup_{\|\varphi\|_L \leq 1} |M_n(\varphi_r)(t)| + \mu_n(t, \mathbb{R}^3 \setminus \mathbb{B}_r) + \lambda(t, \mathbb{R}^3 \setminus \mathbb{B}_r)\,.\end{aligned}$$

It follows from (4.20), (3.6), and the monotonicity of $m_n(s)$ with respect to s that

$$\begin{aligned}\rho(\mu_n(t), \lambda(t)) \leq 4\,(Q_{max} + Q_L) \left[\mu_n(0, \mathbb{R}^3) + \lambda(0, \mathbb{R}^3)\right] \int_0^t ds\, \rho(\mu_n(s), \lambda(s)) + \\ +2\,\rho(\mu_n(0), \lambda(0)) + 2\, Q_{max} \left[\mu_n(0, \mathbb{R}^3)\right]^2 \Delta t ¶_{\{m > N(n)-2\}}(m_n(\Delta t)) + \\ + \sup_{t \in [0, \Delta t]} \sup_{\|\varphi\|_L \leq 1} |M_n(\varphi_r)(t)| + \sup_{t \in [0, \Delta t]} \mu_n(t, \mathbb{R}^3 \setminus \mathbb{B}_r) + \sup_{t \in [0, \Delta t]} \lambda(t, \mathbb{R}^3 \setminus \mathbb{B}_r)\,,\end{aligned}$$

for all $t \in [0, \Delta t]$, $\Delta t > 0$. Notice that

$$\mu_n(0, \mathbb{R}^3) \leq c_\gamma\,, \quad \forall n\,, \tag{5.1}$$

according to (3.8) and (3.10). Now we conclude from Gronwall's inequality that

$$\begin{aligned}\sup_{t \in [0, \Delta t]} \rho(\mu_n(t), \lambda(t)) \;\leq\; & C_1 \Big[\rho(\mu_n(0), \lambda(0)) + \\ & + ¶_{\{m > N(n)-2\}}(m_n(\Delta t)) + \sup_{t \in [0, \Delta t]} \sup_{\|\varphi\|_L \leq 1} |M_n(\varphi_r)(t)| + \\ & + \sup_{t \in [0, \Delta t]} \mu_n(t, \mathbb{R}^3 \setminus \mathbb{B}_r) + \sup_{t \in [0, \Delta t]} \lambda(t, \mathbb{R}^3 \setminus \mathbb{B}_r)\Big]\,,\end{aligned} \tag{5.2}$$

where the constant C_1 does not depend on r and n.

The next step is the estimation of the mathematical expectation of the terms on the right–hand side of (5.2).

Using Chebyshev's inequality, (4.20), and assumption (3.14), we estimate the term

$$\begin{aligned}E^{(n)} \sup_{t \in [0, \Delta t]} \mu_n(t, \mathbb{R}^3 \setminus \mathbb{B}_r) \;\leq\; & E^{(n)} \sup_{t \in [0, \Delta t]} \frac{1}{r^2} \int_{\mathbb{R}^3} \|v\|^2 \mu_n(t, dv) \\ =\; & \frac{1}{r^2} E^{(n)} \int_{\mathbb{R}^3} \|v\|^2 \mu_n(0, dv) \leq \frac{C_2}{r^2}\,,\end{aligned} \tag{5.3}$$

where the constant C_2 does not depend on n and r.

Analogously, using (3.6), we obtain

$$\sup_{t \in [0, \Delta t]} \lambda(t, \mathbb{R}^3 \setminus \mathbb{B}_r) \leq \frac{1}{r^2} \int_{\mathbb{R}^3} \|v\|^2 \lambda(0, dv)\,. \tag{5.4}$$

Lemma 4.10 is to be applied to estimate the term $E^{(n)}\P_{\{m>N(n)-2\}}(m_n(\Delta t))$. Condition (4.34) is fulfilled with $c_\pi = \frac{c_D c_N}{2}$ because of (3.13). Condition (4.36) is fulfilled for all $\kappa \in \left(\frac{1}{c_N}, 1\right)$ because of (3.8) and (3.9). Assumption (3.9) also assures condition (4.35). Consequently,

$$\lim_{n\to\infty} Prob^{(n)}\Big(m_n(t) \geq N(n) - 2\Big) = 0\,, \tag{5.5}$$

for all $t < \frac{1}{c_D}\frac{1}{c_N}\left(1 - \frac{1}{c_N}\right)$.

Finally, we estimate the term $E^{(n)} \sup_{t\in[0,\Delta t]} \sup_{\|\varphi\|_L\leq 1} |M_n(\varphi_r)(t)|$. Notice that the set $D_r = \{\varphi_r; \|\varphi\|_L \leq 1\}$ is compact in $\mathcal{C}(\mathbb{B}_{r+1})$. Consequently, for any $\varepsilon > 0$, there exists a finite set of functions (ψ_i) from $\mathcal{C}(\mathbb{B}_{r+1})$ such that, for any $\psi \in D_r$, $\min_i \|\psi - \psi_i\|_\infty \leq \varepsilon$. The functions (ψ_i) are continued by zero to the space $\mathbb{R}^3$. From the inequality

$$|M_n(\varphi_r)(t)| \leq |M_n(\varphi_r - \psi_i)(t)| + |M_n(\psi_i)(t)|\,, \quad \forall i\,,$$

we obtain the estimate

$$\begin{aligned} |M_n(\varphi_r)(t)| &\leq \min_i |M_n(\varphi_r - \psi_i)(t)| + \sum_i |M_n(\psi_i)(t)| \leq \\ &\leq \sup_{\|\psi\|_\infty\leq\varepsilon} |M_n(\psi)(t)| + \sum_i |M_n(\psi_i)(t)|\,. \end{aligned} \tag{5.6}$$

Using (4.23) and (5.1), we derive from (5.6) that

$$\sup_{t\in[0,\Delta t]} \sup_{\|\varphi\|_L\leq 1} |M_n(\varphi_r)(t)| \leq C_3\,\varepsilon + \sum_i \sup_{t\in[0,\Delta t]} |M_n(\psi_i)(t)|\,, \tag{5.7}$$

where the constant C_3 does not depend on n, r, and ε. Applying the martingale inequality, (4.24), and (5.1), we obtain from (5.7) that

$$E^{(n)} \sup_{t\in[0,\Delta t]} \sup_{\|\varphi\|_L\leq 1} |M_n(\varphi_r)(t)| \leq C_3\,\varepsilon + C_4 \sum_i \|\psi_i\|_\infty\, \gamma(n)^{1/2}\,, \tag{5.8}$$

where the constant C_4 does not depend on n, r, and ε.

Using (3.15), (5.5), (5.8), (3.10), (5.3), (5.4), and (3.5), we conclude from (5.2) that

$$\limsup_{n\to\infty} E^{(n)} \sup_{t\in[0,\Delta t]} \rho(\mu_n(t), \lambda(t)) \leq \frac{C_5}{r^2} + C_3\,\varepsilon\,,$$

where the constant C_5 does not depend on r and ε, provided that Δt satisfies (3.17). Since r and ε are arbitrary, the assertion of Theorem 3.1 follows.

6. Examples and Comments

The stochastic model described in Section 2 contains certain free parameters, namely the bound $N(n)$ for the number of particles in the system, the bound $\gamma(n)$ for the weights of the particles, the initial state $z^{(n)}(0)$, the function $D^{(n)}$ influencing the jump intensity (2.4), and the function $G^{(n)}$ determining the part of the weights transfered to the post–collisional velocities during a jump. Certain restrictions concerning these parameters have been introduced in Section 3 in connection with the convergence theorem. It will be illustrated now that there still remains considerable freedom in the choice of the parameters of the stochastic model.

First we consider the functions $G^{(n)}$ and $D^{(n)}$. Let $\tilde{q}$ be a positive measurable function of the same arguments as the collision kernel q. We introduce the function

$$G^{(n)}(z,i,j,e) = \gamma(n)^{-1} \left[\frac{q(v_i,v_j,e)}{\tilde{q}(v_i,v_j,e)}\right]^{\alpha} g_i\, g_j\,, \tag{6.1}$$

for some $\alpha \in [0,1]$. Condition (2.9) takes the form

$$q(v,w,e)^{\alpha} \le \tilde{q}(v,w,e)^{\alpha}\,, \quad \forall v,w \in \mathbb{R}^3\,,\ e \in \mathbb{S}^2\,. \tag{6.2}$$

It follows from (3.12) that

$$D^{(n)}(z,i,j,e) = \gamma(n)\,[q(v_i,v_j,e)]^{1-\alpha}\,[\tilde{q}(v_i,v_j,e)]^{\alpha}\,. \tag{6.3}$$

Condition (3.11) takes the form

$$c_\gamma \int_{\mathbb{S}^2} de\,[q(v,w,e)]^{1-\alpha}\,[\tilde{q}(v,w,e)]^{\alpha} \le c_D\,, \quad \forall v,w \in \mathbb{R}^3\,, \tag{6.4}$$

for some constant c_D .

Example 6.1. *In the case $\alpha = 0$, we obtain from (6.1), (6.3), and (3.10) that*

$$G^{(n)}(z,i,j,e) = c_\gamma^{-1}\, n\, g_i\, g_j$$

and

$$D^{(n)}(z,i,j,e) = c_\gamma\, n^{-1}\, q(v_i,v_j,e)\,.$$

Condition (6.2) is trivial. Condition (6.4) is fulfilled if

$$c_D \ge c_\gamma\, Q_{max}\,,$$

according to (3.3).

If, in addition,

$$g_{i,n}(0) = c_\gamma\, n^{-1}\,, \quad i = 1, \ldots, n\,,$$

then we obtain a model, in which the number of particles with non–zero weights remains constant. If two such particles collide, they give their weights to the post–-collisional particles.

Example 6.2. *In the case* $\alpha = 1\,,\quad \tilde{q}(v,w,e) = q_{max}\,,\quad \forall v,w \in \mathbb{R}^3\,,\ e \in \mathbb{S}^2\,,$ *we obtain from* (6.1), (6.3) *and* (3.10) *that*

$$G^{(n)}(z,i,j,e) = c_\gamma^{-1}\, n\, q_{max}^{-1}\, q(v_i,v_j,e)\, g_i\, g_j\,,$$

and

$$D^{(n)}(z,i,j,e) = c_\gamma\, n^{-1}\, q_{max}\,.$$

Condition (6.2) *reduces to*

$$q(v,w,e) \le q_{max}\,,\qquad \forall v,w \in \mathbb{R}^3\,,\ e \in \mathbb{S}^2\,.$$

Condition (6.4) *is fulfilled if*

$$c_D \ge 4\,\pi\, c_\gamma\, q_{max}\,,$$

where 4π *is the surface measure of the unit sphere.*

We obtain a model, for which the jump intensity (2.4) *depends only on the number of particles, i.e.*

$$\pi^{(n)}(z) = 4\,\pi\, c_\gamma\, q_{max}\, \frac{m(m-1)}{2n}\,,\qquad z = ((g_1,v_1),\ldots,(g_m,v_m)) \in Z^{(n)}\,.$$

The jump parameters $1 \le i < j \le m$ *and* $e \in \mathbb{S}^2$ *are distributed uniformly (cf.* (2.11), (2.12)*).*

Condition (3.14) concerning the initial state of the system follows from assumption (3.5), if one starts with independent samples $v_{i,n}(0)\,,\ i = 1,\ldots,n\,,$ of the appropriately normalized initial density f_0, and constant weights

$$g_{i,n}(0) = n^{-1} \int_{\mathbb{R}^3} dv\, f_0(v)\,,\qquad i = 1,\ldots,n\,.$$

However, it is also possible to start with a deterministic approximation of the initial measure $\lambda(0)$ such that condition (3.14) is fulfilled.

The convergence result (3.16) has been proved only for time intervals with a length satisfying condition (3.17). This condition suggests the choice $c_N = 2$. In this case, the restriction of the time interval takes the form $\Delta t < [4\, c_\gamma\, Q_{max}]^{-1}$ in Example 6.1, and $\Delta t < [16\,\pi\, c_\gamma\, q_{max}]^{-1}$ in Example 6.2, respectively.

Finishing the paper, we mention two problems, which are important for possible applications of the model in the context of stochastic particle methods for the numerical treatment of the Boltzmann equation.

In order to extend the convergence result to larger time intervals, one should introduce a certain mechanism to reduce the number of particles in the system. A specific example of such a mechanism has been given in [3], but this problem needs further investigations. In particular, the constant c_γ, which has no specific function in the present model, since the maximum of the weights decreases in time, might be helpful in this direction. Notice that after a reduction of the system to the particles with non–zero weights Example 6.1 reduces to the "master process" known from [4].

For numerical applications, it is necessary to replace the random waiting time with the parameter (2.4) by certain approximations. For instance, in Example 6.2, a simple approximation is obtained when replacing the random time by its expectation $\left[4\,\pi\,c_\gamma\,q_{max}\,\frac{m(m-1)}{2n}\right]^{-1}$. The uniformity in time of the convergence result (3.16) is useful (cf. [7], concerning the case of particle systems with constant weights) for proving convergence for modified processes with certain approximations of the time scale.

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21. E. Platen (ed.): Proceedings of the 1^{st} workshop on stochastic numerics.

22. S. Prößdorf (ed.): International Symposium "Operator Equations and Numerical Analysis" September 28 – October 2, 1992 Gosen (nearby Berlin).

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