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Abstract

This contribution deals with the analysis of models for phase-field fracture in visco-elastic materials with dynamic effects. The evolution of damage is handled in two different ways: As a viscous evolution with a quadratic dissipation potential and as a rate-independent law with a positively 1-homogeneous dissipation potential. Both evolution laws encode a non-smooth constraint that ensures the unidirectionality of damage, so that the material cannot heal. Suitable notions of solutions are introduced in both settings. Existence of solutions is obtained using a discrete approximation scheme both in space and time. Based on the convexity properties of the energy functional and on the regularity of the displacements thanks to their viscous evolution, also improved regularity results with respect to time are obtained for the internal variable: It is shown that the damage variable is continuous in time with values in the state space that guarantees finite values of the energy functional.

1 Introduction

This work is concerned with the evolution of dynamic fracture in a visco-elastically deformable solid body occupying a domain $\Omega \subset \mathbb{R}^d$, $1 < d \in \mathbb{N}$. The process is monitored within a time interval $[0, \mathsf{T}]$. It is assumed that only sufficiently small external loadings are applied such that the setting of small strains is admissible. Here the displacement field $u : [0, \mathsf{T}] \times \Omega \to \mathbb{R}^d$ characterizes the elastic deformation of the fracturing solid and the linearized strain tensor $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ is a feasible measure of strain. To enable the model to capture complicated crack geometries the approach of phase-field fracture is applied [FM98, BFM00, MHW10, HW14, AGDL15, KM10], in which the $(d-1)$-dimensional crack surface is approximated by a d-dimensional volume where damage of the material occurs. In the spirit of generalized standard materials [HN75] the volume damage of the material is modelled with the aid of an internal variable

$$z : [0, \mathsf{T}] \times \Omega \to [0, 1],$$

called here phase-field or damage variable, which accounts for the state of material degradation in each point of the domain $\Omega \subset \mathbb{R}^d$. By taking values in $[0, 1]$, z represents in our notation the volume fraction of undamaged material, i.e., $z(t, x) = 1$ if the material is completely sound and $z(t, x) = 0$ in case of maximal damage in a material point $x \in \Omega$ at time $t \in [0, \mathsf{T}]$. As it is the case for metals or rubber we assume that healing of the material cannot occur, so that damage increases over time and hence in our notation z has to decrease in time. This unidirectional evolution is realized in the model by a non-smooth constraint, enforced by the characteristic function $\chi_{(-\infty,0]}$ of the interval $(-\infty, 0]$, i.e.,

$$\chi_{(-\infty,0]}(v) := \begin{cases} 0 & \text{if } v \in (-\infty, 0], \\ \infty & \text{otherwise,} \end{cases} \tag{1}$$

and the occurrence of this non-smooth function in the model turns the evolution law into a subdifferential inclusion, resp. variational inequality. In this work, we will study two different evolution laws for z: A viscous law and a rate-independent law. On a formal level, the Cauchy problem for phase-field fracture in visco-elastic materials at small strains with a viscous evolution of damage is given as follows:

$$\rho \ddot{u} - \mathrm{div}\big(\mathbb{D}(z)e(\dot{u}) + \mathbb{C}(z)e(u)\big) = f_V \;\text{ in } (0, \mathsf{T}) \times \Omega\,, \tag{2a}$$

$$M\dot{z} + \partial\chi_{(-\infty,0]}(\dot{z}) + \tfrac{1}{2}\mathbb{C}'(z)e(u){:}e(u) - \tfrac{1}{\ell}(1-z) - \ell\mathrm{div}\nabla z \ni 0 \;\text{ in } (0, \mathsf{T}) \times \Omega\,. \tag{2b}$$

In (2a), $\rho > 0$ is the mass density and $f_V : [0, \mathsf{T}] \times \Omega \to \mathbb{R}^d$ denotes an external volume force. Moreover, in (2b) the parameter $M > 0$ is the viscosity parameter and $\ell > 0$ controls the width of the diffusive crack zone. The evolution laws

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(2a) and (2b) are complemented by the boundary and initial conditions

$$u(t) = 0 \qquad \text{in } [0,\mathsf{T}] \times \partial_D\Omega \tag{2c}$$

$$\big(\mathbb{D}(z)e(\dot u) + \mathbb{C}(z)e(u)\big)\mathrm{n} = f_S \qquad \text{in } (0,\mathsf{T}) \times \partial_N\Omega, \tag{2d}$$

$$\ell\nabla z \cdot \mathrm{n} = 0 \qquad \text{in } (0,\mathsf{T}) \times \partial\Omega, \tag{2e}$$

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{2f}$$

$$\dot u(0) = \dot u_0 \qquad \text{in } \Omega, \tag{2g}$$

$$z(0) = z_0 \qquad \text{in } \Omega, \tag{2h}$$

where u_0, $\dot u_0$, z_0 are given initial data. The boundary of Ω is denoted by $\partial\Omega$ with outer unit normal n. On the Dirichlet boundary $\partial_D\Omega$ there are imposed homogeneous Dirichlet conditions at all times $t \in [0,\mathsf{T}]$, i.e., it is assumed that also $u_0 = \dot u_0 = 0$ on $\partial_D\Omega$. On the Neumann boundary $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ there acts an external surface force $f_S\colon [0,\mathsf{T}] \times \partial_N\Omega \to \mathbb{R}^d$.

In addition to the viscous evolution of z with $M > 0$ in (2b), we will also consider the case of a rate-independent evolution $M = 0$ in (2b). In particular, we will use a vanishing viscosity limit $M \to 0$ to prove the existence of solutions for the rate-independent setting. In order to better explain our methods and results we now define the function spaces

$$\mathbf{Z} := L^1(\Omega)\,, \quad \mathbf{Z}_M := L^2(\Omega)\,, \quad \mathbf{X} := H^1(\Omega)\,, \quad \mathbf{Y} := H^1(\Omega) \cap L^\infty(\Omega)\,, \tag{3a}$$

$$\mathbf{U} := \{v \in H^1(\Omega, \mathbb{R}^d),\ v = 0 \text{ on } \partial_D\Omega\}, \quad \mathbf{W} := L^2(\Omega; \mathbb{R}^d)\,, \tag{3b}$$

and introduce the functionals that lead to the evolution law (2). In particular, we define the viscous dissipation potential for the damage variable $\mathcal{R}_M : \mathbf{Z}_M \to [0,\infty]$,

$$\mathcal{R}_M(v) = \int_\Omega \mathrm{R}_M(v)\,\mathrm{d}x \ \text{ with } \ \mathrm{R}_M(v) := \frac{M}{2}|v|^2 + \chi_{(-\infty,0]}(v)\,. \tag{4}$$

The vanishing-viscosity limit $M \to 0$ results in the non-smooth, rate-independent potential $\mathcal{R} : \mathbf{Z} \to [0,\infty]$, which here only consists of the unidirectionality constraint,

$$\mathcal{R}(v) := \int_\Omega \chi_{(-\infty,0]}(v)\,\mathrm{d}x\,. \tag{5}$$

At this point we observe that $\mathcal{R}$ indeed is positively homogeneous of degree 1, since $\mathcal{R}(0) = 0$ and $\mathcal{R}(\lambda v) = \lambda\mathcal{R}(v)$ is trivially satisfied for all $\lambda > 0$ and $v \in \mathbf{Z}$.

In view of (2a) we also introduce the viscous dissipation potential of quadratic growth for the displacements $\mathcal{V} : \mathbf{X} \times \mathbf{U} \to [0,\infty)$,

$$\mathcal{V}(z; \dot u) := \int_\Omega \frac{1}{2}\mathbb{D}(z)e(\dot u) : e(\dot u)\,\mathrm{d}x\,, \tag{6}$$

and the kinetic energy $\mathcal{K} : \mathbf{W} \to [0,\infty)$,

$$\mathcal{K}(\dot u) := \int_\Omega \frac{\rho}{2}|\dot u|^2\,\mathrm{d}x\,. \tag{7}$$

Moreover, the energy functional $\mathcal{E} : [0,T] \times \mathbf{U} \times \mathbf{X} \to \mathbb{R}$ associated with system (2) is given by

$$\mathcal{E}(t,u,z) := \int_\Omega \Big(\tfrac{1}{2}\mathbb{C}(z)e(u) : e(u) + \Big(\tfrac{1}{2\ell}(1-z)^2 + \tfrac{\ell}{2}|\nabla z|^2\Big)\Big)\,\mathrm{d}x - \langle f(t), u\rangle_{\mathbf{U}^*,\mathbf{U}}\,, \tag{8}$$

where we have gathered the volume load f_V from (2a) and the surface load f_S from (2d) in the term

$$\langle f(t), u\rangle_{\mathbf{U}^*,\mathbf{U}} := \int_\Omega f_V(t)\cdot u\,\mathrm{d}x + \int_{\partial_N\Omega} f_S(t)\cdot u\,\mathrm{d}S;$$

the detailed assumptions on the external loadings are specified in (17). Note that $\mathcal{E}$ is a slight modification of the Ambrosio-Tortorelli functional for phase-field fracture as we will allow $\mathbb{C}$ to depend on z in a monotone, but non-convex way to keep $\mathbb{C}$ bounded, cf. assumptions (13) & (14) lateron. Moreover, we will assume both tensors $\mathbb{C}$ and $\mathbb{D}$ in (6) to be uniformly positive definite for all $z \in \mathbb{R}$, so that the material can bear loads and still shows a visco-elastic response even in the state of maximal damage $z = 0$. In this way, model (2) captures partial damage of the body, only. In the purely rate-independent case of quasistatic evolutions, i.e., in the setting of energetic solutions for rate-independent processes the systems given by $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{R})$ from (3), (5), and with an energy of the type (8), were shown in [Gia05] to approximate the Francfort-Marigo model for brittle fracture [FM98] as $\ell \to 0$. This model is a variational formulation of Griffith' energetic approach [Gri21] to the description of brittle crack growth in terms of competing elastic bulk and dissipative surface energies. Following Griffith' ideas for brittle solids, such as glass and certain metals, fracture is often modelled as a rate-independent process. This modelling approach captures the observation that cracks can form and evolve abruptly, much faster than the changes of the external loadings. In fact, solutions of purely rate-independent damage and fracture models do feature jumps with respect to time, cf. e.g., [KS12, RTP15]. More recently, research focus in both engineering applications [BVS+12, SWKM14, SKM+17] and in applied analysis [DMLT16, DMLT19, DMLT20, LRTT18, Rou19, RT17, SS19] is put on the investigation of *dynamic fracture*.

As an immediate approach based on well-established models for rate-independent phase-field fracture, the rate-independent evolution of the damage variable is coupled with a (visco-) dynamic evolution of the displacements as also done in (2). In order to achieve better stability in numerical simulations, often a viscosity for the damage variable is added to the model, as we also allow for in (2) if $M > 0$. It is the aim of this contribution to better investigate the interplay of the rate-independent evolution of the damage variable with the visco-elastodynamic evolution of the displacements.

For this, we will now give a suitable weak formulation for system (2). In this setting, we will show the existence of solutions and study their temporal regularity for both cases $M > 0$ and $M = 0$.

Definition 1.1. *In the spirit of [RT17] we call a system that combines the conservative process of elastodynamics with further dissipative processes a* damped inertial system*. We denote the damped inertial system with viscous regularization $M > 0$ for the damage variable from* (2) *by the tuple $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$. The damped inertial system obtained in the rate-independent limit $M \to 0$ is denoted by $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$.*

In the viscous case $M > 0$ a suitable weak formulation for the damped inertial system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ is introduced as follows:

Definition 1.2 (Solutions of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$, viscous case $M > 0$)**.** *A pair $(u_M, z_M)\colon [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{X}$ is a solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ if it satisfies the following four conditions:*

- *one-sided variational inequality for z_M for almost all $t \in [0, \mathsf{T})$:*

$$\int_\Omega \big[\tfrac12 \mathbb{C}'(z_M(t))e\big(u_M(t)\big) : e\big(u_M(t)\big) - \tfrac{1}{\ell}\big(1 - z_M(t)\big)\big) + M\dot z_M(t)\big]\eta \,\mathrm{d}x + \int_\Omega \ell\, \nabla z_M(t)\cdot\nabla\eta \,dx \ge 0 \tag{9a}$$

  for all $\eta \in \mathbf{Y}$ such that $\eta \le 0$ a.e. in Ω;
- *unidirectionality: for all $t_1 < t_2 \in [0, \mathsf{T}]$ it is $z_M(t_2) \le z_M(t_1)$ a.e. in Ω ;* (9b)
- *weak formulation of the momentum balance for all $t \in [0, \mathsf{T}]$:*

$$\begin{aligned}\rho\int_\Omega \dot u_M(t)\cdot v(t)\,\mathrm{d}x - \rho\int_0^t\int_\Omega \dot u_M(r)\cdot \dot v(r)\,\mathrm{d}x\,\mathrm{d}r \\ + \int_0^t\int_\Omega \big[\mathbb{D}(z_M)e(\dot u_M) + \mathbb{C}(z_M)e(u_M)\big] : e(v)\,\mathrm{d}x\,\mathrm{d}r \\ = \rho\int_\Omega \dot u_M(0)\cdot v(0)\,\mathrm{d}x + \int_0^t \langle f(r), v(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\end{aligned} \tag{9c}$$

  for all $v \in L^2(0, \mathsf{T}; \mathbf{U}) \cap H^1(0, \mathsf{T}; L^2(\Omega, \mathbb{R}^d))$;
- *energy-dissipation balance for almost all $t \in [0, \mathsf{T})$:*

$$\begin{aligned}\mathcal{K}(\dot u_M(t)) + \mathcal{E}(t, u_M(t), z_M(t)) + \int_0^t 2\big(\mathcal{V}(z_M; \dot u_M) + \mathcal{R}_M(\dot z_M)\big)\,\mathrm{d}r \\ = \mathcal{K}(\dot u_0) + \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_t\mathcal{E}\big(r, u(r), z(r)\big)\,\mathrm{d}r\,.\end{aligned} \tag{9d}$$

Above, in (9d) the term $\partial_t\mathcal{E}(r,u(r),z(r)) = -\langle \dot f(r),u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}$ stands for the partial time-derivative of the energy functional. We point out that the formulation of the viscous damage evolution (9a) in terms of a one-sided variational inequality was already used in e.g. [HK11] at small strains and e.g. in [TBW20, TBW18] at finite strains. We also refer to the works [HK11, BB08, RR15, HKRR17], where viscous damage models have been studied also in combination with dynamics and further dissipative effects such as heat transport and phase separation. Moreover, [Rou19] gives a comprehensive overview on different time-discretization schemes for damage models with viscous evolution and dynamics.

In analogy to the above viscous case, a suitable notion of weak solution for the damped inertial system $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ in the rate-independent case $M=0$ is given by:

Definition 1.3 (Solutions of $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$, rate-independent case $M=0$). *A pair $(u,z)\colon [0,\mathsf{T}]\to\mathbf{U}\times\mathbf{X}$ is a solution of $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ if it satisfies the the following four conditions:*

- *one-sided variational inequality for z for almost all $t\in[0,\mathsf{T})$:*

$$\int_\Omega \big[\tfrac{1}{2}\mathbb{C}'(z(t))e\big(u(t)\big):e\big(u(t)\big)-\tfrac{1}{\ell}\big(1-z(t)\big)\big]\eta+\ell\nabla z(t)\cdot\nabla\eta\,dx\geq 0 \tag{10a}$$

for all $\eta\in\mathbf{Y}$ such that $\eta\leq 0$ a.e. in Ω;

- *unidirectionality: for all $t_1<t_2\in[0,\mathsf{T}]$ it is $z(t_2)\leq z(t_1)$ a.e. in Ω ;* (10b)
- *weak formulation of the momentum balance for all $t\in[0,\mathsf{T}]$:*

$$\begin{aligned}&\rho\int_\Omega \dot u(t)\cdot v(t)\,\mathrm{d}x-\rho\int_0^t\int_\Omega \dot u(r)\cdot\dot v(r)\,\mathrm{d}x\,\mathrm{d}r\\&\qquad+\int_0^t\int_\Omega\big[\mathbb{D}(z)e(\dot u)+\mathbb{C}(z)e(u)\big]:e(v)\,\mathrm{d}x\,\mathrm{d}r\\&=\rho\int_\Omega\dot u(0)\cdot v(0)\,\mathrm{d}x+\int_0^t\langle f(r),v(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\end{aligned}\tag{10c}$$

for all $v\in L^2(0,\mathsf{T};\mathbf{U})\cap H^1(0,\mathsf{T};L^2(\Omega,\mathbb{R}^d))$;

- *energy-dissipation balance for almost all $t\in[0,\mathsf{T})$:*

$$\begin{aligned}&\mathcal{K}(\dot u(t))+\mathcal{E}(t,u(t),z(t))+\int_0^t 2\mathcal{V}(z(r);\dot u(r))\,\mathrm{d}r+\mathcal{R}(z(t)-z(0))\\&\quad=\frac{\rho}{2}\int_\Omega|\dot u(0)|^2\,\mathrm{d}x+\mathcal{E}(0,u(0),s(0))+\int_0^t\partial_t\mathcal{E}\big(r,u(r),z(r)\big)\,\mathrm{d}r\,.\end{aligned}\tag{10d}$$

Remark 1.4 (Semistable energetic solution of $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$). *The tensorial map $z\mapsto\mathbb{C}(z)$ is assumed to be non-convex, but with a convexity regime $(-\infty,z_*)$ with $z_*>1$, such that $\mathbb{C}$ is convex in particular on the interval $[0,1]$, see assumptions* (13) *&* (14) *for more details. Hence, the map $z\mapsto\mathcal{E}(t,u,z)$ is non-convex in general, but convex for functions $z\in\mathbf{X}$ that take values in $[0,1]$ a.e. in Ω. In fact, for solutions (u,z) in the sense of Def. 1.3 it will be shown in Theorem 5.1, and for the time-discrete version in Theorem 4.1, that $z:[0,\mathsf{T}]\to\mathbf{X}$ takes its values in the interval $[0,1]$ a.e. in Ω. Hence, convexity of $\mathcal{E}(t,u(t),\cdot)$ can be exploited along solutions. This is the reason why solutions of $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ in the sense of Definition 1.3 also fulfill the following semistability inequality for almost all $t\in[0,\mathsf{T})$:*

$$\mathcal{E}(t,u(t),z(t))\leq\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z-z(t))\quad\textit{for all } \tilde z\in\mathbf{X}\tag{11}$$

with $\mathcal{E}$ from (8) *and $\mathcal{R}$ from* (5)*. Hence, solutions in the sense of Definition 1.3 are also* semistable energetic solutions *in the spirit of [RT17].*

Remark 1.5 (Improved temporal regularity and (9d), (10), (11) for all $t\in[0,\mathsf{T})$). *Let $\mathsf{D}_c:=\{z\in\mathbf{Y},\,0\leq z(x)\leq z_*$ a.e. in $\Omega\}$ denote the convexity regime of $\mathbb{C}$. Thanks to the observations for $\mathbb{C}$ discussed above in Remark 1.4 one finds for $\mathcal{E}$ from* (8) *that $\mathcal{E}(t,u(t),\cdot):\mathsf{D}_c\to\mathbb{R}$ is even uniformly convex in the following sense: There is a constant $C_\star>0$ such that for all $z_0,z_1\in\mathsf{D}_c$ and for all $\lambda\in[0,1]$:*

$$\mathcal{E}(t,u(t),z_\lambda)+C_\star\lambda(1-\lambda)\|z_1-z_0\|^2_{\mathbf{X}}\leq\lambda\mathcal{E}(t,u(t),z_1)+(1-\lambda)\mathcal{E}(t,u(t),z_0)\,,$$

where we set $z_\lambda:=\lambda z_1+(1-\lambda)z_0$. This allows us to deduce improved regularity statements for the solution z by suitably adapting a general regularity result from [RT17, Thm. 3.8] for coupled rate-independent/rate-dependent systems. In the *rate-independent case $M = 0$ we prove in Theorem 5.2 an abstract result providing a modulus of continuity to control the expression $\|z(t) - z(s)\|_{\mathbf{X}}$ at any times $s, t \in [0, \mathsf{T}]$ in which the variational inequality* (10a) *and the energy-dissipation balance* (10d) *are valid. In analogy, for the viscous case $M > 0$ we deduce in Theorem 6.2 a modulus of continuity to control a kind of α-variation $\sum_{k\in\mathbb{N}} \|z(t_k) - z(t_{k-1})\|^{\alpha}_{\mathbf{X}}$ for partitions $(t_k)_{k\in\mathbb{N}}$ of any time interval $[s, t] \subset [0, \mathsf{T}]$ with s, t such that* (9a) *and the energy-dissipation balance* (9d) *are valid. In both cases, $M > 0$ and $M = 0$, the modulus of continuity emerges from terms related to the displacements and to their smoothness in time provided by the viscosity $\mathcal{V}$ from* (6)*. Further exploiting the unidirectionality* (10b)*, resp.* (9b)*, of the damage evolution the modulus of continuity can be extended to any time $t \in [0, \mathsf{T})$ in Corollary 5.5 for the rate-independent case $M = 0$ and in Corollary 6.4 for the viscous case $M > 0$. In this way, one ultimately finds that the map $z : [0, \mathsf{T}) \to \mathbf{X}$ is continuous if $M > 0$, cf. Theorem 6.1, and even Hölder-continuous if $M = 0$, cf. Theorem 5.1. Thus, in contrast to the purely rate-independent case, here in the coupled rate-independent/rate-dependent setting, the uniform convexity of $\mathcal{E}(t, u(t), \cdot) : \mathsf{D}_c \to \mathbb{R}$ rules out that solutions z have jumps in time, because the regularity of the displacements enhanced by the viscosity $\mathcal{V}$ also improves the temporal regularity of the internal variable z to a continuous evolution in time with values in the state space $\mathbf{X}$.*

Based on these continuity results, also properties (9d)*,* (10) *&* (11) *can be concluded to hold for all $t \in [0, \mathsf{T})$, cf. Corollaries 5.5 & 6.4 for more details.*

Outline of the paper. The purpose of this work is two-fold: On the one hand, as described in Remark 1.5, we investigate the influence of the coupling of the state variables on their temporal regularity. On the other hand we aim to bring the analytical approach closer to numerical methods. This is why we carry out the analysis for the existence of solutions in the sense of Definitions 1.2 and 1.3 for both systems $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ and $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ based on a full discretization both in space and time. After specifying the basic assumptions on the domain and given data in Section 2, we introduce in Section 3 the discrete scheme based on a staggered time-discrete method in combination with a Galerkin approach in space, cf. (25), and we establish the existence of discrete solutions in Proposition 3.1. In particular, as done for numerical simulations we understand on the discrete level the discretized version of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ as an approximation of the system. On the discrete level we also regularize the non-smooth unidirectionality constraint (1) with the aid of a Yosida approximation. While the fully discrete counterpart to (2a) reduces to solving a linear system of equations, it is more involved to find solutions for the discrete version of the damage evolution (2b) due to the nonlinearities stemming from the nonlinear z-dependence of the elastic tensor $\mathbb{C}$ and the Yosida term. The existence proof thus relies on arguments for nonlinear systems of equations based on Brouwer's fixed point theorem. We subsequently show that the discrete solutions obtained by the fully discrete scheme (25) approximate solutions of the systems $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ and $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in several steps: Based on uniform a-priori bounds in Prop. 3.2 being independent of the space-discretization we first pass to a space-continuous but time-discrete problem. In this setting it is possible to show that solutions satisfy the constraint $z \in [0, 1]$ a.e. in Ω and hence lie in the convexity regime D_c of the energy functional. In this way one can obtain further uniform a priori bounds for the time-discrete solutions based on energy-dissipation estimates. Section 5 treats the limit passage from time-discrete to continuous in the case $M \to 0$ and thus provides the existence of solutions to system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 1.3, cf. Theorem 5.1. Subsequently, Section 6 is devoted to the viscous analogon with $M > 0$ fixed and the existence of solutions to system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ in the sense of Def. 1.2 is obtained in Theorem 6.1. The abstract results on the temporal regularity of the solutions addressed in Remark 1.5 are provided in Theorem 5.2 for the case $M = 0$ and in Theorem 6.2 for the case $M > 0$. We also point out that we obtain strong convergence of the discrete solutions thanks to the validity of the energy-dissipation balance (10d) & (9d), cf. Theorems 5.1 & 6.1.

Comparison with other approaches in literature. For the limit passage from time-discrete to time-continuous in Sections 5 and 6 we adapt arguments from [LRTT18], where the existence of semistable energetic solutions has been shown for a system coupling rate-independent damage processes in thermo-viscoelastic materials with dynamic effects. This concerns in particular the proofs of the weak balance of momentum and the energy-dissipation estimates, whereas the limit passage in the variational inequality for the damage evolution is different here due to the viscous regularization $M > 0$. For simplicity, the present work only considers homogeneous Dirichlet conditions (2c) and postulates C^1-regularity in time for the external load f, cf. (17). We refer to [LRTT18] for a relaxation to H^1-regularity in time and to [LRTT16] for the treatment of inhomogeneous, time-dependent Dirichlet conditions. We further point to the recent work [KZ18] which extends the existence theory for the purely rate-independent setting to discontinuous loads using Kurzweil integrals. We emphasize that our approach on the discrete level regularizes the unidirectionality constraint in terms of the Yosida approximation. There are other techniques to provide monotonicity of the damage evolution. In many applications the problem is solved as an unconstrained minimization and imposed a posteriori by a truncation with the solution from the previous time-step. In a quasistatic $2d$-setting with a viscous regularization for the damage variable it is shown in [ABN18] that discrete solutions obtained with this method by unconstrained minimization in an alternate minimization scheme and a posteriori truncation converge to solutions of a unilateral L^2-gradient flow.

We apply a vanishing viscosity method on the discrete level, but we do not develop balanced viscosity solutions in the sense of [MRS12, MRS09, EM06] for general rate-independent systems, or like in [KRZ13, KRZ15, KRZ19] in the context of quasistatic, rate-independent damage models. The main difficulty to apply this approach lies in the stored elastic energy $\int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x$ that nonlinearly couples the damage variable with the strains. Solutions u for the displacement field, naturally found in the space $\mathbf{U} \subset H^1(\Omega;\mathbb{R}^d)$, are not regular enough to make the variational derivative $\mathrm{D}_z\mathcal{E}(t,z,u)$ a well-defined object in the dual space $\mathbf{X}^*$ or in $\mathbf{Z}_M$ in general space dimension $d > 2$, even if one finds z being bounded with values in $[0,1]$ a.e. in Ω. Due to this lack of regularity there is no chain rule available to calculate the time-derivative of the energy and hence, solutions cannot be a priori characterized in terms of an energy dissipation balance. In [KRZ13] or in [ABN18] in $2d$ this issue is solved with the aid of an elliptic regularity result [HMW11, Theorem 1.1, p. 803] which provides sufficiently improved regularity for the displacements to find a chain rule. However, because of the rate-dependence of the displacements in problem (2) due to viscosity and inertia such improved spatial regularity results for the displacements are not available here.

2 Notation and basic assumptions

We denote by $\mathcal{L}^m$ the m-dimensional Lebesgue measure for any $m \in \mathbb{N}$.

Assumptions on the domain: For the domain Ω we make the assumptions

$$\begin{aligned}&\Omega \subset \mathbb{R}^d \text{ is a bounded domain with Lipschitz-boundary } \partial\Omega, \text{ such that}\\ &\partial_D\Omega \subset \partial\Omega \text{ is non-empty and relatively open and } \partial_N\Omega := \partial\Omega \setminus \partial_D\Omega.\end{aligned} \tag{12}$$

Assumptions on the tensors $\mathbb{C}, \mathbb{D}$: The dependence of the material tensors $\mathbb{C}, \mathbb{D} : \mathbb{R} \to \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ on the phase-field parameter z is realized by functions $w_\mathbb{C}, w_\mathbb{D} : \mathbb{R} \to [w_0, w^*]$ being prefactors to constant tensors $\tilde{\mathbb{C}}, \tilde{\mathbb{D}}$, i.e.,

$$\mathbb{C}(z) := w_\mathbb{C}(z)\tilde{\mathbb{C}} \quad \text{and} \quad \mathbb{D}(z) := w_\mathbb{D}(z)\tilde{\mathbb{D}} \text{ for all } z \in \mathbb{R}, \tag{13a}$$

$$\text{with constant, symmetric, and positively definite tensors } \tilde{\mathbb{C}}, \tilde{\mathbb{D}}. \tag{13b}$$

For $w_\mathbb{C}, w_\mathbb{D}$ it is further assumed:

$$\bullet \text{ Differentiability \& boundedness: } w_\mathbb{C}, w_\mathbb{D} \in C^1(\mathbb{R}, [w_0, w^*]) \quad \text{with constants } 0 < w_0 < w^*, \tag{14a}$$

$$\bullet \text{ Monotonicity: } w'_\mathbb{C}(z) \geq 0 \text{ and } w'_\mathbb{D}(z) \geq 0 \text{ for all } z \in \mathbb{R}, \tag{14b}$$

$$\bullet \text{ Locally constant growth: } w'_\mathbb{C}(z) = 0 \text{ and } w'_\mathbb{D}(z) = 0. \quad \text{for all } z \in (-\infty, 0] \cup [z^*, \infty), \tag{14c}$$

$$\bullet \text{ Local convexity: There are } z_* \in (1, z^*) \text{ and } w_* \in (w_0, w^*) \text{ s.th. } w_\mathbb{C} : [0, z_*] \to [w_0, w_*] \text{ is convex.} \tag{14d}$$

Remark 2.1 (Properties of $w_\mathbb{C}, w_\mathbb{D}$ and consequences)**.** *Properties* (14) *imply the existence of constants* $0 < c^0_\mathbb{D} < c^*_\mathbb{D}$ *and* $0 < c^0_\mathbb{C} < c^*_\mathbb{C}$ *such that for all* $(z, A) \in \mathbb{R} \times \mathbb{R}^{d\times d}$ *we have*

$$c^0_\mathbb{D}\,|A|^2 \leq \mathbb{D}(z)A : A \leq c^*_\mathbb{D}\,|A|^2 \text{ and} \tag{15a}$$

$$c^0_\mathbb{C}\,|A|^2 \leq \mathbb{C}(z)A : A \leq c^*_\mathbb{C}\,|A|^2\ . \tag{15b}$$

Moreover, (14) *implies that* $w_\mathbb{C}$ *qualitatively is of the form indicated in Fig. 1.*

The non-convexity of $w_\mathbb{C}$ *on the interval* $[z_*, z^*]$ *entails that an upper energy-dissipation estimate alike* (9d) *is not available for fully discrete solutions* $(u^k_{\tau n}, z^k_{\tau n})_n$. *It will be only obtained in the limit* $n \to \infty$ *for the time-discrete, space-continuous solutions* (u^k_τ, z^k_τ), *since it will be shown in Theorem 4.1, Formula* (41) *that* z^k_τ *takes values in* $[0,1] \subset [z_*, z^*]$ *a.e. in* Ω.

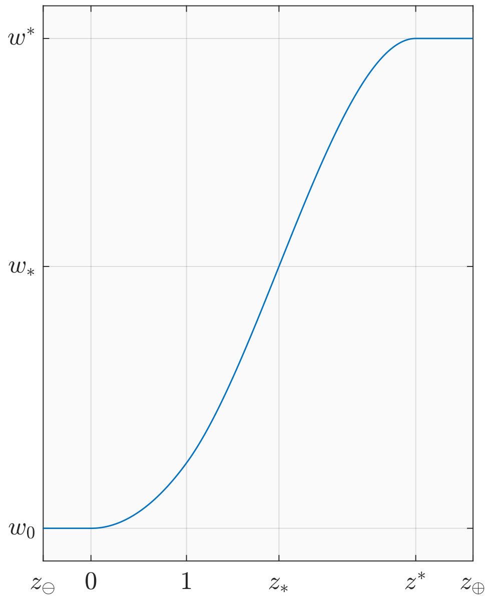

Figure 1: Qualitative shape of $w_{\mathbb{C}} : \mathbb{R} \to [w_0, w^*]$: The function is constant on the intervals $(-\infty, 0] \cup [z^*, \infty)$, monotonously increasing on $\mathbb{R}$, and convex on the interval $(-\infty, z_*)$ with $z_* > 1$ but non-convex on $[z_*, z^*)$. The points $z_\ominus \ll 0$ and $z_\oplus \gg z^*$ will play a role later in the proof of Theorem 4.1, Formula (41), when showing that solutions z_τ^k of the space-continuous problem (2b) are bounded in $[0, 1]$.

Assumptions on the given data: We assume for the external volume force f_V in (2a) and the surface load f_S in (2d) that $f_V \in C^1(0, \mathrm{T}; \mathbf{U}^*)$ and $f_S \in C^1(0, \mathrm{T}; L^2(\partial_N\Omega, \mathbb{R}^d))$. The combination of both forces

$$\langle f(t), v\rangle_{\mathbf{U}^*,\mathbf{U}} := \langle f_V(t), v\rangle_{\mathbf{U}^*,\mathbf{U}} + \int_{\partial_N\Omega} f_S(t) \cdot v \, d\mathcal{H}^{d-1} \text{ for all } v \in \mathbf{U} \tag{16}$$

has the following properties:

- Regularity: $f \in C^1\big(0, T; \mathbf{U}^*\big)\,,$ (17a)
- Bounded time derivative: $\sup_{t\in[0,\mathrm{T}]} \Big\|\dot{f}(t)\Big\|_{\mathbf{U}^*} < \infty\,.$ (17b)

In addition, from the set of initial data in (2f)-(2h) it is demanded that:

$$\begin{aligned} &u_0 \in \mathbf{U}\,,\\ &\dot{u}_0 \in \mathbf{U},\\ &z_0 \in \mathbf{X},\ z_0(x) \in [0,1] \text{ for almost all } x \in \Omega\,. \end{aligned} \tag{18}$$

Yosida-regularization: In the discrete setting, the non-smoothness in the dissipation potential $\mathcal{R}_M$ in (4) will be substituted by a smooth approximation in terms of the Yosida-regularization. For this, the characteristic function $\chi_{(-\infty,0]}$ in (4) is replaced by

$$r \mapsto \frac{N_\tau}{2} m_+(r)^2 \tag{19a}$$

with $m_+ \colon \mathbb{R} \to [0,\infty)$ the maximum function $m_+(r) := \max\{r, 0\}$ and $N_\tau \to \infty$ as time-step size $\tau \to 0$. Accordingly, R_M in (4) will be replaced in the discrete scheme by

$$\mathrm{R}_{M\tau}(v) := \frac{M}{2}\,|v|^2 + \frac{N_\tau}{2} m_+(v)^2 \tag{19b}$$

and we write $\mathcal{R}_{M\tau}$ for the corresponding integral functional. For shorter notation in the proofs lateron, we will also write $m_+^2(r)$ for $m_+(r)^2$ in (19a). We point out that (19a) indeed is a regularization of the non-smooth unidirectionality constraint since

$$\frac{\mathrm{d}}{\mathrm{d}r} m_+^2(r) = \begin{cases} 2r & \text{if } r > 0,\\ 0 & \text{if } r \le 0\,. \end{cases} \tag{20}$$

3 Existence of fully discrete solutions

The strategy to find solutions for the systems $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$ and $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ is to consider a fully discrete scheme at first. The spatial discretization follows a Galerkin approach:

Space discretization: For $V\in\{\mathbf{X},\mathbf{Y},\mathbf{U}\}$ let $V_n\subset V$, $n\in\mathbb{N}$, be finite-dimensional subspaces such that these spaces form ascending chains, i.e. $V_{n_1}\subset V_{n_2}$, if $n_1\le n_2$, and such that $\bigcup_{n>0}V_n\subset V$ densely. For $V_n=\mathbf{X}_n$ and $V_n=\mathbf{Y}_n$ the index $n\in\mathbb{N}$ coincides with the space dimension, while for $V_n=\mathbf{U}_n$ the space dimension is supposed to be dn, since elements $u\in\mathbf{U}_n$ are vector-valued functions of dimension d. Moreover, $P_n^V: V\to V_n$ denotes the projection onto V_n defined by

$$\left\|P_n^V(v)-v\right\|_V=\min_{w\in V_n}\|w-v\|_V\quad\text{for all } v\in V\,. \tag{21}$$

Let $(\varphi_j)_{j=1}^n$, resp. $(\boldsymbol{\varphi}_j)_{j=1}^{dn}$, be a basis for $\mathbf{X}_n$, resp. $\mathbf{U}_n$. Then $z\in\mathbf{X}_n$ and $u\in\mathbf{U}_n$ are represented by $z=\sum_{j=1}^n z_j\varphi_j$, $u=\sum_{j=1}^{dn}u_j\boldsymbol{\varphi}_j$ and we write $\mathbf{z}=(z_j)_{j=1}^n\in\mathbb{R}^n$, $\mathbf{u}=(u_j)_{j=1}^{dn}\in\mathbb{R}^{dn}$ for the vectors of coefficients.

Discretization in time: Consider a partition $\Pi_\tau=\{0=t_\tau^0<t_\tau^1\ldots<t_\tau^{N_\tau}=\mathsf{T}\}$ of the time interval $[0,\mathsf{T}]$ with step size $\tau=t_\tau^k-t_\tau^{k-1}=\frac{\mathsf{T}}{N_\tau}$. For a sufficiently smooth function $v\colon[0,\mathsf{T}]\to V$ we set $v_\tau^k=v(t_\tau^k)$ for $t_\tau^k\in\Pi_\tau$ and we introduce the discrete approximations of the time derivatives by

$$\mathsf{D}_\tau v_\tau^k:=\frac{v_\tau^k-v_\tau^{k-1}}{\tau}\,, \tag{22a}$$

$$\mathsf{D}_\tau^2 v_\tau^k:=\frac{1}{\tau}(\mathsf{D}_\tau v_\tau^k-\mathsf{D}_\tau v_\tau^{k-1})=\frac{v_\tau^k-2v_\tau^{k-1}+v_\tau^{k-2}}{\tau^2}\,. \tag{22b}$$

For the discretization of the external loadings we use an approximation

$$f_\tau^k:=f(t_\tau^k) \tag{23}$$

and denote by $f_{\tau n}^k$ the restriction of $f_\tau^k\in\mathbf{U}^*$ to $\mathbf{U}_n$, where naturally

$$f_{\tau n}^k\to f_\tau^k\text{ strongly in }\mathbf{U}^*\text{ as }n\to\infty\text{ for all }k\in\{1,\ldots,N_\tau\}\text{ and }\tau>0\text{ fixed}\,. \tag{24}$$

Discrete approximation of $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$: Keep the time step-size $\tau>0$ fixed. For the initial data $(z_0,u_0,\dot u_0)$ from (18) set $z_\tau^0:=z_0$, $u_\tau^0:=u_0$, and $u_\tau^{-1}:=u_0-\tau\dot u_0$. For all $n\in\mathbb{N}$ let $(z_{\tau n}^0)_n$, $(u_{\tau n}^0)_n$, $(u_{\tau n}^{-1})_n$ with $z_{\tau n}^0\in\mathbf{X}_n$, $u_{\tau n}^0$, $u_{\tau n}^{-1}\in\mathbf{U}_n$ be approximations of the inital data such that $z_{\tau n}^0\to z_\tau^0$ in $\mathbf{X}$, $u_{\tau n}^0\to u_\tau^0$ in $\mathbf{U}$, and $u_{\tau n}^{-1}\to u_\tau^{-1}$ in $\mathbf{U}$ as $n\to\infty$. For each $\tau,n>0$ fixed, using the discrete intial data $(z_{\tau n}^0,u_{\tau n}^0,u_{\tau n}^{-1})$ our aim is to find for every time step $t_\tau^k\in\Pi_\tau$ solutions $z_{\tau n}^k\in\mathbf{X}_n$, $u_{\tau n}^k\in\mathbf{U}_n$ of the following staggered discrete Galerkin scheme:

$$0=\langle\mathrm{D}_z\mathcal{E}(t_\tau^k,u_{\tau n}^{k-1},z_{\tau n}^k)+\mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_{\tau n}^k),\eta_n\rangle_{\mathbf{X}^*,\mathbf{X}}\quad\text{for all }\eta_n\in\mathbf{Y}_n\,, \tag{25a}$$

$$\begin{aligned}0=&\int_\Omega\big(\mathsf{D}_\tau^2u_{\tau n}^k\cdot v_n+\big[\mathbb{D}(z_{\tau n}^k)e(\mathsf{D}_\tau u_{\tau n}^k)+\mathbb{C}(z_{\tau n}^k)e(u_{\tau n}^k)\big]:e(v_n)\big)\,\mathrm{d}x\\&-\big\langle f_{\tau n}^k,v_n\big\rangle_{\mathbf{U}^*,\mathbf{U}}\quad\text{for all }v_n\in\mathbf{U}_n\,.\end{aligned} \tag{25b}$$

We state the two results of this section, the existence of solutions $(u_{\tau n}^k,z_{\tau n}^k)$ for the Galerkin scheme (25) and their uniform boundedness with respect to the index $n\in\mathbb{N}$, cf. Propositions 3.1 and 3.2; the proofs will be carried out subsequently in Subsections 3.1 and 3.2.

Proposition 3.1 (Existence of fully discrete solutions)**.** *Let the assumptions* (12)–(19) *be satisfied. Keep $\tau>0$, $k\in\{1,\ldots,N_\tau\}$, and $n\in\mathbb{N}$ fixed. Then there exists a solution $(u_{\tau n}^k,z_{\tau n}^k)$ of the Galerkin scheme* (25) *corresponding to system $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$.*

Note that, due to assumptions (13)–(14), the stored elastic energy is non-convex in z on the subinterval $[z_*,z^*]$. Thus, one cannot expect to obtain an energy-dissipation estimate alike (9d) via convexity arguments. Nevertheless, thanks to assumptions (13b) and (14b), the following uniform a-priori bounds can be obtained for fully discrete solutions $(u_{\tau n}^k,z_{\tau n}^k)_n$ for all k.

Proposition 3.2 (Uniform a-priori bounds for fully discrete solutions)**.** *Let the assumptions of Theorem 3.1 be fulfilled. Further assume that the discrete initial data $(u^0_{\tau n})_n$, $(u^{-1}_{\tau n})_n$, and $(z^0_{\tau n})_n$, are uniformly bounded. Then, the fully discrete solutions $(u^k_{\tau n}, z^k_{\tau n})$ of problem* (25) *satisfy the following uniform a-priori bounds*

$$\|u^k_{\tau n}\|_{\mathbf{U}} \le \tilde{C}\,, \tag{26a}$$
$$\|z^k_{\tau n}\|_{\mathbf{Z}} \le \tilde{C}\,. \tag{26b}$$

with a constant $\tilde{C} = \tilde{C}(f, u_0, \dot{u}_0, z_0, \tau, M, \ell)$ depending on $f, u_0, \dot{u}_0, z_0, \tau, M, \ell$, but independent of $n \in \mathbb{N}$.

3.1 Proof of Proposition 3.1

In the following, $\tau > 0$ and $k \in \{1, \ldots, N_\tau\}$ are kept fixed. Using the notation introduced at the beginning of Section 3, the Galerkin scheme (25) can be rewritten as a system of (non-) linear equations for the coefficient vectors $\mathbf{z}^k_{\tau n} = (z^k_{\tau n i})^n_{i=1} \in \mathbb{R}^n$, $\mathbf{u}^k_{\tau n} \in \mathbb{R}^{dn}$:

Testing in (25b) with basis elements $\boldsymbol{\varphi}_j$ for $\mathbf{U}_n$, $j = 1, \ldots, dn$, and multiplying with τ^2 implies for all $j \in \{1, \ldots, dn\}$

$$\begin{aligned} 0 = \sum_{i=1}^{dn} u^k_{\tau n i}\Big(\int_\Omega \rho\,\boldsymbol{\varphi}_i\cdot\boldsymbol{\varphi}_j\,\mathrm{d}x + \int_\Omega \big(\tau\mathbb{D}(z^k_{\tau n}) + \tau^2\mathbb{C}(z^k_{\tau n})\big)e(\boldsymbol{\varphi}_i) : e(\boldsymbol{\varphi}_j)\,\mathrm{d}x\Big) \\ + \int_\Omega \rho\,(-2u^{k-1}_{\tau n} + u^{k-2}_{\tau n})\cdot\boldsymbol{\varphi}_j - \tau\mathbb{D}(z^k_{\tau n})e(u^{k-1}_{\tau n}) : e(\boldsymbol{\varphi}_j)\,\mathrm{d}x - \tau^2\langle f^k_{\tau n}, \boldsymbol{\varphi}_j\rangle_{\mathbf{U}^*,\mathbf{U}}\,. \end{aligned}$$

This is rewritten as matrix-vector multiplication using the coefficient vector $\mathbf{u}^k_{\tau n}$:

$$\begin{aligned} &\Big[\int_\Omega \rho\,\boldsymbol{\varphi}_i\cdot\boldsymbol{\varphi}_j\,\mathrm{d}x\Big]^{dn}_{i,j=1}\mathbf{u}^k_{\tau n} + \Big[\int_\Omega\Big(\tau\mathbb{D}(z^k_{\tau n}) + \tau^2\mathbb{C}(z^k_{\tau n})\Big)e(\boldsymbol{\varphi}_i) : e(\boldsymbol{\varphi}_j)\,\mathrm{d}x\Big]^{dn}_{i,j=1}\mathbf{u}^k_{\tau n} \\ &= \Big[\int_\Omega \rho\,(2u^{k-1}_{\tau n} - u^{k-2}_{\tau n})\cdot\boldsymbol{\varphi}_j + \tau\mathbb{D}(z^k_{\tau n})e(u^{k-1}_{\tau n}) : e(\boldsymbol{\varphi}_j)\,\mathrm{d}x + \tau^2\langle f^k_{\tau n}, \boldsymbol{\varphi}_j\rangle_{\mathbf{U}^*,\mathbf{U}}\Big]^{dn}_{j=1} \end{aligned} \tag{27a}$$

which is a linear system of equations $(\mathbb{M}_1 + \mathbb{M}_2)\mathbf{u}^k_{\tau n} = \mathbf{b}$. It is solvable since the mass matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ are positively definite by the linear independence of the basis elements and thanks to the assumptions (13) on $\mathbb{C}, \mathbb{D}$. Thus, finding a solution $\mathbf{u}_n$ amounts to solving the linear system of equations (27a) by directly inverting the mass matrices.

Testing (25a) with the basis elements φ_j of $\mathbf{Z}_n$, $j = 1, \ldots, n$, and using the notation $E := (\varphi_j)^n_{j=1}$, leads to

$$\begin{aligned} \mathbf{0} = &\int_\Omega\Big(\frac{1}{2}\mathbb{C}'(z^k_{\tau n})e(u^{k-1}_{\tau n}) : e(u^{k-1}_{\tau n}) + \tfrac{N_\tau}{2}\tfrac{\mathrm{d}}{\mathrm{d}z}m^2_+(\mathsf{D}_\tau z^k_{\tau n})\Big)E\,\mathrm{d}x \\ &+ \Big[\int_\Omega\Big(\frac{M}{\tau} + \frac{1}{\ell}\Big)\varphi_i\varphi_j\,\mathrm{d}x\Big]^n_{i,j=1}\mathbf{z}^k_{\tau n} + \Big[\int_\Omega \ell\nabla\varphi_i\cdot\nabla\varphi_j\,\mathrm{d}x\Big]^n_{i,j=1}\mathbf{z}^k_{\tau n} \\ &- \int_\Omega\Big(\frac{M}{\tau}z^{k-1}_{\tau n} + \frac{1}{\ell}\Big)E\,\mathrm{d}x \end{aligned}$$

which is a nonlinear system of equations

$$\mathbf{g}(\mathbf{z}^k_{\tau n}) := \mathbf{f}(\mathbf{z}^k_{\tau n}) + \mathbb{M}_3\mathbf{z}^k_{\tau n} + \mathbb{M}_4\mathbf{z}^k_{\tau n} - \mathbf{p} = \mathbf{0}\,. \tag{27b}$$

We show now that it posesses a solution for every fixed k, τ, n. To do so, we will make use of the following result:

Proposition 3.3 ([Zei86, Prop. 2.8, p. 53])**.** *Consider the system of equations*

$$\mathbf{g}(\mathbf{z}) = (g_i(\mathbf{z}))^n_{i=1} = \mathbf{0} \ \ \textit{where } \mathbf{z} \in \mathbb{R}^n. \tag{28}$$

Let $\overline{B}_R(0) := \{\mathbf{z} \in \mathbb{R}^n,\ \|\mathbf{z}\| \le R\}$ for fixed $R > 0$ and $\|\cdot\|$ a norm on $\mathbb{R}^n$. Let $g_i : \overline{B}_R(0) \to \mathbb{R}$ be continuous for $i = 1, \ldots, n$. Further assume that

$$\mathbf{g}(\mathbf{z})\cdot\mathbf{z} \ge 0 \quad \textit{for all } \mathbf{z} \textit{ with } \|\mathbf{z}\| = R. \tag{29}$$

Then (28) *has a solution $\mathbf{z}$ with $\|\mathbf{z}\| \le R$.*

In the following we thus verify that the nonlinear system (27b) satisfies the assumptions of Prop. 3.3. Here, we write $z = \sum_{i=1}^{n} z_i \varphi_i$ and $\mathbf{z} = (z_i)_{i=1}^{n}$. The continuity of $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ follows by the assumptions of Section 2. It remains to check condition (29). For that, exploiting the positive definiteness of $\mathbb{M}_3$ and $\mathbb{M}_4$ one directly estimates

$$\mathbf{g}(\mathbf{z}) \cdot \mathbf{z} = \mathbf{f}(\mathbf{z}) \cdot \mathbf{z} + \mathbb{M}_3 \mathbf{z} \cdot \mathbf{z} + \mathbb{M}_4 \mathbf{z} \cdot \mathbf{z} - \mathbf{p} \cdot \mathbf{z} \geq \mathbf{f}(\mathbf{z}) \cdot \mathbf{z} + c_1 |\mathbf{z}|^2 - c_2 |\mathbf{z}| \,, \tag{30}$$

where the constant $c_1 = c_1(\frac{M}{\tau}, \ell)$ is given by the smallest eigenvalue of $(\mathbb{M}_3 + \mathbb{M}_4)$ and $c_2 = c_2(\frac{M}{\tau}, \frac{1}{\ell}, z_{\tau n}^{k-1})$ originates from the term $\mathbf{p} = \int_\Omega \Big(\frac{M}{\tau} z_{\tau n}^{k-1} + \frac{1}{\ell} \Big) E \, \mathrm{d}x$. We now estimate in detail the nonlinear term $\mathbf{f}(\mathbf{z}) \cdot \mathbf{z}$ that involves the nonlinear functions $\mathbb{C}'$ and $\frac{\mathrm{d}}{\mathrm{d}z} m_+^2$. For these terms we use that $\mathbb{C}'$ takes its maximum value at z_* by (14) and that in view of (20)

$$\tfrac{\mathrm{d}}{\mathrm{d}z} m_+^2 \big(\tfrac{1}{\tau}(z - z_{\tau n}^{k-1}) \big) E \cdot \mathbf{z} \geq -\frac{2}{\tau} \big((1 - \frac{1}{2\varepsilon}) \, |\mathbf{z}|^2 + \frac{\varepsilon}{2} \left| z_{\tau n}^{k-1} \right|^2 \big) \tag{31}$$

with $\varepsilon > 0$ fixed but arbitrary such that $(1 - \frac{1}{2\varepsilon}) > 0$. In this way we find

$$\begin{aligned} \mathbf{f}(\mathbf{z}) \cdot \mathbf{z} &= \int_\Omega \frac{1}{2} \mathbb{C}'(z) e(u_{\tau n}^{k-1}) : e(u_{\tau n}^{k-1}) E \cdot \mathbf{z} \, \mathrm{d}x + \int_\Omega \tfrac{N_\tau}{2} \tfrac{\mathrm{d}}{\mathrm{d}z} m_+^2 \big(\tfrac{1}{\tau}(z - z_{\tau n}^{k-1}) \big) E \cdot \mathbf{z} \, \mathrm{d}x \\ &\geq -\int_\Omega \frac{1}{2} \left| \mathbb{C}'(z_*) | |e(u_{\tau n}^{k-1})|^2 |\mathbf{z}| \right| \mathrm{d}x - \int_\Omega \frac{N_\tau}{\tau} \Big((1 - \frac{1}{2\varepsilon}) \, |\mathbf{z}|^2 + \varepsilon \frac{\left| z_{\tau n}^{k-1} \right|^2}{2} \Big) \, \mathrm{d}x \\ &\geq -c_3 \, |\mathbf{z}| - \frac{N_\tau}{\tau} (1 - \frac{1}{2\varepsilon}) \, |\mathbf{z}|^2 \, \mathcal{L}^d(\Omega) - \int_\Omega \varepsilon \frac{N_\tau}{2\tau} \left| z_{\tau n}^{k-1} \right|^2 \, \mathrm{d}x \end{aligned}$$

with $c_3 = c_3(u_{\tau n}^{k-1})$. Now, choose $\varepsilon > 0$ such that with c_1 from (30) $c_4 := c_1 - \frac{N_\tau}{\tau}(1 - \frac{1}{2\varepsilon}) \mathcal{L}^d(\Omega) > 0$. Then, with Young's inequality it follows that

$$c_4 \, |\mathbf{z}|^2 - (c_2 + c_3) \, |\mathbf{z}| \geq c_4 \frac{|\mathbf{z}|^2}{2} - \frac{(c_2 + c_3)^2}{2 c_4} \, .$$

Putting everything together and inserting it into (30) results in

$$\mathbf{g}(\mathbf{z}) \cdot \mathbf{z} \geq \frac{c_4}{2} \, |\mathbf{z}|^2 - \frac{(c_2 + c_3)^2}{2 c_4} - c_5 \tag{32}$$

with $c_5 = c_5(\tau, z_{\tau n}^{k-1})$, more precisely

$$c_5 = \varepsilon \frac{N_\tau}{2\tau} \int_\Omega \left| z_{\tau n}^{k-1} \right|^2 \, \mathrm{d}x$$

and ε with the specific choice from above. From this we see that (29) is satisfied for $R \geq \sqrt{\frac{2c_5}{c_4} + \frac{2(c_2+c_3)^2}{2c_4^2}}$. □

3.2 Proof of Proposition 3.2

We proceed by induction and see that the assertion is satisfied for the initial step $k = 0$ thanks to the assumptions made on the initial data. For any step $k \in \mathbb{N}$, suppose that $(u_{\tau n}^{k-1})_n$, $(u_{\tau n}^{k-2})_n$ and $(z_{\tau n}^{k-1})_n$ are uniformly bounded in their respective state spaces. Testing (25a), (25b) with the solutions $z^k_{\tau n}$ and $u^k_{\tau n}$ respectively, we estimate

$$
\begin{aligned}
0 =& \big\langle \mathrm{D}_z\mathcal{E}(t^k_\tau, u^{k-1}_{\tau n}, z^k_{\tau n}) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_{\tau n}), z^k_{\tau n}\big\rangle_{\mathbf{X}^*,\mathbf{X}}\\
&+ \int_\Omega \rho \mathsf{D}^2_\tau u^k_{\tau n}\cdot u^k_{\tau n} + \big[\mathbb{D}(z^k_{\tau n})e(\mathsf{D}_\tau u^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})e(u^k_{\tau n})\big] : e(u^k_{\tau n})\,\mathrm{d}x\\
&- \big\langle f^k_{\tau n}, u^k_{\tau n}\big\rangle_{\mathbf{U}^*,\mathbf{U}}\\
\geq& \int_\Omega \frac{1}{2} z^k_{\tau n}\mathbb{C}'(z^k_{\tau n})e(u^{k-1}_{\tau n}) : e(u^{k-1}_{\tau n}) - \frac{1}{\ell}(1-z^k_{\tau n})z^k_{\tau n} + \ell|\nabla z^k_{\tau n}|^2\,\mathrm{d}x\\
&+ \int_\Omega M(\mathsf{D}_\tau z^k_{\tau n})z^k_{\tau n} + \tfrac{N_\tau}{2}\tfrac{\mathrm{d}}{\mathrm{d}z} m^2_+(\mathsf{D}_\tau z^k_{\tau n})z^k_{\tau n}\,\mathrm{d}x\\
&+ \int_\Omega \frac{\rho}{\tau^2}\big(|u^k_{\tau n}|^2 - \frac{1}{2}|u^k_{\tau n}|^2 - 2|u^{k-1}_{\tau n}|^2 - \frac{1}{2}|u^k_{\tau n}|^2 - \frac{1}{2}|u^{k-2}_{\tau n}|^2\big)\,\mathrm{d}x\\
&+ \int_\Omega \frac{1}{\tau}\big(c^0_{\mathbb{D}}|e(u^k_{\tau n})|^2 - \frac{c^0_{\mathbb{D}}}{2}|e(u^k_{\tau n})|^2 - \frac{{c^*_{\mathbb{D}}}^2}{2c^0_{\mathbb{D}}}\left|e(u^{k-1}_{\tau n})\right|^2\big)\,\mathrm{d}x\\
&+ \int_\Omega c^0_{\mathbb{C}}|e(u^k_{\tau n})|^2\,\mathrm{d}x - \|f^k_{\tau n}\|_{\mathbf{U}^*}\|u^k_{\tau n}\|_{\mathbf{U}}\,,
\end{aligned}
\tag{33}
$$

where Hölder's and Young's inequalities where used to estimate the momentum term. Observe that the first term on the right-hand side is non-negative since $z^k_{\tau n}\mathbb{C}'(z^k_{\tau n}) \geq 0$ by assumptions (14b) and (14c); it thus can be omitted to further estimate from below. For the phase-field term we estimate

$$
-\int_\Omega \tfrac{1}{\ell}(1-z^k_{\tau n})z^k_{\tau n}\,\mathrm{d}x \geq \tfrac{1}{2\ell}\|z^k_{\tau n}\|^2_{L^2(\Omega)} - \tfrac{1}{2\ell}\mathcal{L}^d(\Omega)
\tag{34}
$$

and for the viscous dissipation we find the lower bound

$$
\begin{aligned}
&\int_\Omega M(\mathsf{D}_\tau z^k_{\tau n})z^k_{\tau n} + \tfrac{N_\tau}{2}\tfrac{\mathrm{d}}{\mathrm{d}z} m^2_+(\mathsf{D}_\tau z^k_{\tau n})z^k_{\tau n}\,\mathrm{d}x\\
&\geq \big(\tfrac{M}{2\tau} - \tfrac{N_\tau}{\tau}(1-\tfrac{1}{2\varepsilon})\big)\|z^k_{\tau n}\|^2_{L^2(\Omega)} - \tfrac{M}{2\tau}\|z^{k-1}_{\tau n}\|^2_{L^2(\Omega)} - \tfrac{\varepsilon N_\tau}{2\tau}\|z^{k-1}_{\tau n}\|^2_{L^2(\Omega)}\,,
\end{aligned}
\tag{35}
$$

where again the lower bound on the Yosida-term in (31) was used and $\varepsilon > 0$ was chosen such that $c_6 = c_6(M,\tau) := \big(\frac{M}{2\tau} - \frac{N_\tau}{\tau}(1-\frac{1}{2\varepsilon})\big) > 0$. We set $c_7 = c_7(\frac{1}{\tau^2}) := \frac{\varepsilon N_\tau}{2\tau}$ with ε as above. The terms in the last line of (33) are estimated by Korn's inequality with constant c_K and by Young's inequality

$$
\int_\Omega c^0_{\mathbb{C}}\left|e(u^k_{\tau n})\right|^2\,\mathrm{d}x - \left\|f^k_{\tau n}\right\|_{\mathbf{U}^*}\left\|u^k_{\tau n}\right\|_{\mathbf{U}} \geq \big(\tfrac{c^0_{\mathbb{C}}}{c^2_K} - \tfrac{\delta}{2}\big)\left\|u^k_{\tau n}\right\|^2_{\mathbf{U}} - \tfrac{1}{2\delta}\left\|f^k_{\tau n}\right\|^2_{\mathbf{U}^*}\,,
\tag{36}
$$

where $\delta := \frac{c^0_{\mathbb{C}}}{c^2_K}$ is chosen such that $\big(\frac{c^0_{\mathbb{C}}}{c^2_K} - \frac{\delta}{2}\big) = \frac{c^0_{\mathbb{C}}}{2c^2_K}$. Using estimates (34)–(36) in (33) and putting all negative terms to the left-hand side results in

$$
\begin{aligned}
&\frac{1}{2\ell}\mathcal{L}^d(\Omega) + \frac{c^2_K}{2c^0_{\mathbb{C}}}\left\|f^k_{\tau n}\right\|^2_{\mathbf{U}^*} + \frac{\rho}{\tau^2}\Big(2\left\|u^{k-1}_{\tau n}\right\|^2_{L^2} + \frac{1}{2}\left\|u^{k-2}_{\tau n}\right\|^2_{L^2}\Big)\\
&\quad + \Big(\frac{M}{2\tau} + c_7\Big)\left\|z^{k-1}_{\tau n}\right\|^2_{L^2(\Omega)} + \frac{{c^*_{\mathbb{D}}}^2}{2\tau c^0_{\mathbb{D}}}\left\|e(u^{k-1}_{\tau n})\right\|^2_{L^2(\Omega)}\\
&\geq \Big(c_6 + \frac{1}{2\ell}\Big)\|z^k_{\tau n}\|^2_{L^2(\Omega)} + \ell\|\nabla z^k_{\tau n}\|^2_{L^2(\Omega)} + \frac{c^0_{\mathbb{C}}}{2c^2_K}\left\|u^k_{\tau n}\right\|^2_{\mathbf{U}}\,.
\end{aligned}
$$

Since $\left\|f^k_{\tau n}\right\|_{\mathbf{U}^*} \leq C$ uniformly for all k,τ,n the above estimate gives a bound on $(z^k_{\tau n})_n$ and $(u^k_{\tau n})_n$ in $\mathbf{Z}$ and $\mathbf{U}$ uniformly for all $n\in\mathbb{N}$ and fixed $\tau, k\in\mathbb{N}$, i.e., with a constant $\tilde{C} = \tilde{C}(f, u_0, \dot{u}_0, z_0, \tau, M, \ell)$ as indicated in (26). ☐

4 Limit passage from the space-discrete to the space-continuous setting

In this section we keep the parameters $M,\tau > 0$ fixed and pass to the limit $n\to\infty$ with the space discretization. In particular, we obtain the following result:

Theorem 4.1 (Existence of solutions in the space-continuous setting). *Let the assumptions of Proposition 3.1 and 3.2 be satisfied. For all $\tau > 0$, $k \in \{0, 1, \dots, N_\tau\}$, $n \in \mathbb{N}$ let $(u^k_{\tau n}, z^k_{\tau n})$ be a solution of* (25). *Keep $\tau > 0$ fixed. Then the following statements hold true:*

1 *For each $k \in \{1, \dots, N_\tau\}$ there is a (not relabelled) subsequence $(u^k_{\tau n}, z^k_{\tau n})_n$ and a limit pair $(u^k_\tau, z^k_\tau) \in \mathbf{U} \times \mathbf{X}$ such that the following convergence results hold true:*

$$u^k_{\tau n} \rightharpoonup u^k_\tau \quad \text{weakly in } \mathbf{U}, \tag{37a}$$

$$z^k_{\tau n} \rightharpoonup z^k_\tau \quad \text{weakly in } \mathbf{X}. \tag{37b}$$

2 *For each $k \in \{1, \dots, N_\tau\}$ the limit pair $(u^k_\tau, z^k_\tau) \in \mathbf{U} \times \mathbf{X}$ is a solution of the time-discrete problem*

$$0 = \langle \mathrm{D}_z \mathcal{E}(t^k_\tau, u^{k-1}_\tau, z^k_\tau) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_\tau), \eta \rangle_{\mathbf{X}^*, \mathbf{X}} \qquad \text{for all } \eta \in \mathbf{Y}, \tag{38a}$$

$$0 = \int_\Omega \big(\mathsf{D}^2_\tau u^k_\tau \cdot v + \big[\mathbb{D}(z^k_\tau) e(\mathsf{D}_\tau u^k_\tau) + \mathbb{C}(z^k_\tau) e(u^k_\tau) \big] : e(v) \big) \, \mathrm{d}x - \big\langle f^k_\tau, v \big\rangle_{\mathbf{U}^*, \mathbf{U}} \qquad \text{for all } v \in \mathbf{U}\,. \tag{38b}$$

3 *Assume that the discrete initial data satisfy*

$$u^0_{\tau n} \to u^0_\tau \ \text{ in } \mathbf{U} \quad \text{and} \quad u^{-1}_{\tau n} \to u^{-1}_\tau \ \text{ in } \mathbf{U}, \tag{39a}$$

$$z^0_{\tau n} \to z^0_\tau \ \text{ in } \mathbf{X}. \tag{39b}$$

Then, in addition to (37) *for each $k \in \{1, \dots, N_\tau\}$ also the following improved convergence results hold true:*

$$u^k_{\tau n} \to u^k_\tau \quad \text{strongly in } \mathbf{U}, \tag{40a}$$

$$z^k_{\tau n} \to z^k_\tau \quad \text{strongly in } \mathbf{X}. \tag{40b}$$

4 *Suppose that $z^0_{\tau n} \in [0, 1]$. Then, for each $k \in \{1, \dots, N_\tau\}$ the limit function z^k_τ satisfies*

$$z^k_\tau \in \mathbf{Y}, \quad \text{in particular } \ 0 \le z^k_\tau \le 1 \text{ a.e. in } \Omega. \tag{41}$$

5 *Let $L \in \{1, \dots, N_\tau\}$. The time-discrete solutions $(u^k_\tau, z^k_\tau)^{N_\tau}_{k=0}$ of* (38) *satisfy the following upper energy-dissipation estimate:*

$$\begin{aligned} &\mathcal{K}(\mathsf{D}_\tau u^L_\tau) + \mathcal{E}(t^L_\tau, u^L_\tau, z^L_\tau) + \sum_{k=1}^{L} \tau 2\mathcal{V}(z^k_\tau; \mathsf{D}_\tau u^k_\tau) + \sum_{k=1}^{L} \tau 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_\tau) \\ &\le \mathcal{K}(\mathsf{D}_\tau u^0_\tau) + \mathcal{E}(t^0_\tau, u^0_\tau, z^0_\tau) - \tau \sum_{k=1}^{L} \big\langle \mathsf{D}_\tau f^k_\tau, u^{k-1}_\tau \big\rangle_{\mathbf{U}^*, \mathbf{U}} \end{aligned} \tag{42}$$

Proof of Theorem 4.1. The weak convergence results (37) are direct consequences of the uniform a-priori bounds (26). The proofs of the remaining statements (38)–(42) will be carried out subsequently in Subsections 4.1–4.5. □

For solutions $(u^k_\tau, z^k_\tau)^{N_\tau}_{k=1}$ obtained by solving (38), piecewise constant interpolants $\bar{v}_\tau$, $\underline{v}_\tau$, and affine-linear approximations v_τ for $v \in \{u, z\}$ are introduced, defined for $t \in (t^{k-1}_\tau, t^k_\tau]$, $k = 1, \dots N_\tau$, by

$$\bar{v}_\tau(t) = v^k_\tau, \ \underline{v}_\tau(t) = v^{k-1}_\tau, \ v_\tau(t) = \frac{t - t^{k-1}_\tau}{\tau} v^k_\tau + \frac{t^k_\tau - t}{\tau} v^{k-1}_\tau\,. \tag{43}$$

In addition, we set for any $t \in (t^{k-1}_\tau, t^k_\tau]$

$$\bar{t}_\tau(t) := t^k_\tau, \quad \underline{t}_\tau(t) := t^{k-1}_\tau, \tag{44}$$

and for the stored energy

$$\hat{\mathcal{E}}(t,u,z) := \int_\Omega \left(\tfrac{1}{2}\mathbb{C}(z)e(u):e(u) + \left(\tfrac{1}{2\ell}(1-z)^2 + \tfrac{\ell}{2}|\nabla z|^2\right)\right)\mathrm{d}x - \left\langle \hat{f}(t),u\right\rangle_{\mathbf{U}^*,\mathbf{U}} \tag{45}$$

with $\hat{\mathcal{E}} \in \{\mathcal{E}_\tau, \bar{\mathcal{E}}_\tau, \underline{\mathcal{E}}_\tau\}$ depending on the choice of the interpolant for the external force $\hat{f} \in \{f_\tau, \bar{f}_\tau, \underline{f}_\tau\}$. In this way, the time-discrete problem (38) as well as the upper energy-dissipation estimate (42) can be reformulated also for the interpolants. Here, also discrete integration by parts is used

$$\tau\sum_{k=1}^{L}\int_\Omega \tfrac{\dot{u}_\tau^k - \dot{u}_\tau^{k-1}}{\tau}\cdot v_\tau^k\,\mathrm{d}x = \int_\Omega\left(\dot{u}_\tau^L\cdot v_\tau^L - \dot{u}_\tau^0\cdot v_\tau^0\right)\mathrm{d}x - \tau\sum_{k=1}^{L}\int_\Omega \dot{u}_\tau^{k-1}\cdot\tfrac{{v_\tau}^k - v_\tau^{k-1}}{\tau}\,\mathrm{d}x \tag{46}$$

for any tuple $(v_\tau^k)_{k=0}^L \subset L^2(0,\mathsf{T};L^2(\Omega))$, to state the weak balance of momentum for the interpolants. Then, we have

$$0 = \langle \mathrm{D}_z\bar{\mathcal{E}}_\tau(t,\underline{u}_\tau(t),\bar{z}_\tau(t)) + \mathrm{D}\mathcal{R}_{M\tau}(\dot{z}_\tau(t)),\eta\rangle_{\mathbf{X}^*,\mathbf{X}} \qquad \text{for all } \eta\in\mathbf{Y}, \tag{47a}$$

$$0 = \rho\int_\Omega \dot{u}_\tau(t)\cdot\bar{v}_\tau(t) - \dot{u}_\tau(0)\cdot\bar{v}_\tau(0)\,dx - \rho\int_0^{\bar{t}_\tau(t)}\int_\Omega \dot{u}_\tau(r-\tau)\dot{v}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \tag{47b}$$

$$+\int_0^{\bar{t}_\tau(t)}\int_\Omega\left[\mathbb{D}\big(\bar{z}_\tau(r)\big)e\big(\dot{u}_\tau(r)\big) + \mathbb{C}\big(\bar{z}_\tau(r)\big)e\big(\bar{u}_\tau(r)\big)\right] : e\big(\bar{v}_\tau(r)\big)\,\mathrm{d}x\,\mathrm{d}r$$

$$-\int_0^{\bar{t}_\tau(t)}\left\langle \bar{f}_\tau(r),\bar{v}_\tau(r)\right\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r \tag{47c}$$

for all tuples $(v_\tau^k)_{k=0}^{N_\tau} \subset \mathbf{U}$ setting $\bar{v}(s) := v_\tau^k$ and $v_\tau(s) := \frac{t-t_\tau^{k-1}}{\tau}v_\tau^k + \frac{t_\tau^k - t}{\tau}v_\tau^{k-1}$ for $s\in(t_\tau^{k-1},t_\tau^k]$, and

$$\begin{aligned}&\mathcal{K}(\dot{u}_\tau(t)) + \bar{\mathcal{E}}_\tau(t,\bar{u}_\tau(t),\bar{z}_\tau(t)) + \int_0^{\bar{t}_\tau(t)} 2\big(\mathcal{V}(\bar{z}_\tau(r);\dot{u}_\tau(r)) + \mathcal{R}_{M\tau}(\dot{z}_\tau(r))\big)\,\mathrm{d}r\\ &\quad\le \mathcal{K}(\dot{u}_\tau(0)) + \mathcal{E}(0,\bar{u}_\tau(0),\bar{z}_\tau(0)) - \int_0^{\bar{t}_\tau(t)}\left\langle \dot{f}_\tau(r),\underline{u}_\tau(r)\right\rangle_{\mathbf{U}^*,\mathbf{U}}\mathrm{d}r.\end{aligned} \tag{47d}$$

Estimate (47d) leads to the following uniform a priori estimates for the time-discrete interpolated solutions:

Proposition 4.2 (Uniform a-priori bounds for time-discrete solutions)**.** *Let the assumptions of Theorem 4.1 be satisfied. In addition, suppose that we have $z_\tau^0 = z_0$, $u_\tau^0 = u_0$ and $u_\tau^{-1} = u_0 - \tau\dot{u}_0$ for all $\tau > 0$. For the interpolants constructed by* (43) *with the time-discrete limit pairs $(u_\tau^k, z_\tau^k)_{k=1}^{N_\tau}$ found in* (37)*, the following a priori estimates hold true with a constant $C>0$ independent of τ and M:*

$$\|\underline{u}_\tau\|_{L^\infty(0,\mathsf{T};\mathbf{U})} + \|\bar{u}_\tau\|_{L^\infty(0,\mathsf{T};\mathbf{U})} \le C, \tag{48a}$$

$$\|\dot{u}_\tau\|_{B(0,\mathsf{T};L^2(\Omega,\mathbb{R}^d))} \le C, \tag{48b}$$

$$\|u_\tau\|_{H^1(0,\mathsf{T};\mathbf{U})} \le C, \tag{48c}$$

$$\|\mathsf{D}_\tau\dot{u}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U}^*)} \le C, \tag{48d}$$

$$\|\underline{z}_\tau\|_{L^\infty(0,\mathsf{T};\mathbf{X})} + \|\bar{z}_\tau\|_{L^\infty(0,\mathsf{T};\mathbf{X})} \le C, \tag{48e}$$

$$\|\dot{z}_\tau\|_{L^2(0,\mathsf{T};L^2(\Omega))} \le \frac{C}{\sqrt{M}}, \tag{48f}$$

$$\|z_\tau\|_{H^1(0,\mathsf{T};L^2(\Omega))} \le \frac{C}{\sqrt{M}}, \tag{48g}$$

$$\|\underline{z}_\tau\|_{BV(0,\mathsf{T};L^1(\Omega))} + \|\bar{z}_\tau\|_{BV(0,\mathsf{T};L^1(\Omega))} \le C, \tag{48h}$$

$$\|\dot{z}_\tau\|_{L^1(0,\mathsf{T};L^1(\Omega))} \le C. \tag{48i}$$

The proof of Proposition 4.2 is carried out in detail in Section 4.6.

4.1 Proof of (38b): Limit passage in the discrete momentum balance

We pass to the limit $n \to \infty$ in the fully discrete momentum balance (25b). For this, let $v \in \mathbf{U}$ be a test function of the space-continuous limit problem (38b) and $(v_n)_n \subset \mathbf{U}$ such that $v_n \in \mathbf{U}_n$ for all $n \in \mathbb{N}$ are test functions for the finite-dimensional problems (25b) with the property $v_n \to v$ strongly in $\mathbf{U}$. A sequence $(v_n)_n$ with these properties does exist, since $\cup_{n\in\mathbb{N}}\mathbf{U}_n$ is dense in $\mathbf{U}$ by assumption. Now, for the limit passage in (25b), i.e., in

$$0 = \int_\Omega \tfrac{u_{\tau n}^k - 2u_{\tau n}^{k-1} + u_{\tau n}^{k-2}}{\tau^2} \cdot v_n + \Big[\mathbb{D}(z_{\tau n}^k)e\Big(\tfrac{u_{\tau n}^k - u_{\tau n}^{k-1}}{\tau}\Big) + \mathbb{C}(z_{\tau n}^k)e(u_\tau^k)\Big] : e(v_n)\,\mathrm{d}x \\ - \langle f_{\tau n}^k, v_n\rangle_{\mathbf{U}^*,\mathbf{U}}\,,$$

we see that convergence of the first summand is ensured by the weak convergence of the displacements in $\mathbf{U}$ from (37a) and the strong convergence $v_n \to v$ in $\mathbf{U}$. For the second and third summand, (37b) implies by compactness that $z_{\tau n}^k \to z_\tau^k$ strongly in $L^1(\Omega)$, thus almost everywhere in Ω along a subsequence. Then, by continuity of $\mathbb{C}, \mathbb{D}$, cf. assumption (14a), there follows

$$\mathbb{D}(z_{\tau n}^k)e(v_n) \to \mathbb{D}(z_\tau^k)e(v) \ \text{ and } \mathbb{C}(z_{\tau n}^k)e(v_n) \to \mathbb{C}(z_\tau^k)e(v) \text{ pointwise a.e. in } \Omega\,.$$

Exploiting the uniform bounds on $\mathbb{D}$ and $\mathbb{C}$ in (15a) and (15b), we conclude the convergence of the integrals using the dominated convergence theorem in a version with n-dependent majorants, cf. [RF17, Sec. 4.4, Thm. 19, p. 89]. Convergence of the external loading term follows from the strong convergence of the test functions together with strong convergence (24). This results in (38b). □

4.2 Proof of (38a): Limit passage in the discrete phase-field equation

We consider the limit passage $n \to \infty$ in the discrete problem (25a). Let $\eta \in \mathbf{Y}$ be a test function for the space-continuous phase-field equation (38a). We obtain test functions for the discrete equation (25a) by projections $\eta_n = P_n^X(\eta) \in \mathbf{X}_n$, $n \in \mathbb{N}$ onto $\mathbf{X}_n \subset \mathbf{X}$ with the property $\eta_n \to \eta$ strongly in $\mathbf{X}$ and $\|\eta_n\|_{L^\infty(\Omega)} \le c_\eta$ uniformly for all $n \in \mathbb{N}$. Using these test functions we now pass to the limit in (25a), i.e., in

$$\begin{aligned} 0 &= \langle \mathrm{D}_z\bar{\mathcal{E}}_\tau(t_\tau^k, u_{\tau n}^{k-1}, z_{\tau n}^k) + \mathrm{D}\mathcal{R}_{M\tau}\Big(\tfrac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\Big), \eta_n\rangle_{\mathbf{X}^*,\mathbf{X}} \\ &= \int_\Omega \frac{1}{2}\mathbb{C}'(z_{\tau n}^k)e(u_{\tau n}^{k-1}) : e(u_{\tau n}^{k-1})\eta_n\,\mathrm{d}x + \int_\Omega \big(\ell\nabla z_{\tau n}^k \cdot \nabla\eta_n - \frac{1}{\ell}(1 - z_{\tau n}^k)\eta_n\big)\,\mathrm{d}x \\ &\quad + \int_\Omega M\tfrac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\eta_n\,\mathrm{d}x + \int_\Omega \frac{N_\tau}{2}\frac{\mathrm{d}}{\mathrm{d}z}m_+^2\Big(\tfrac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\Big)\eta_n\,\mathrm{d}x\,. \end{aligned}$$

For the second and the third integral term on the right-hand side, convergence follows by weak-strong convergence arguments using (37b) together with the strong convergence of $(\eta_n)_n$. For the fourth integral on the right-hand side, that is the Yosida-regularization of the unidirectionality constraint, we find with (20) that $\frac{N_\tau}{2}\frac{\mathrm{d}}{\mathrm{d}z}m_+^2\big(\frac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\big)\eta_n$ convergences pointwise almost everywhere in Ω. In addition,

$$\left|\frac{N_\tau}{2}\frac{\mathrm{d}}{\mathrm{d}z}m_+^2\big(\frac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\big)\eta_n\right| \le \left|N_\tau\big(\frac{z_{\tau n}^k - z_{\tau n}^{k-1}}{\tau}\big)\eta_n\right|\,,$$

which provides an admissible summable majorant. Based on this, one can pass to the limit using the dominated convergence theorem [RF17, Sec. 4.4, Thm. 19, p. 89]. It remains to discuss the limit passage in the first integral on the right-hand side. For this, observe that the assumptions (14) on $w_\mathbb{C}$ imply together with the uniform bound on η_n that

$$|\eta_n|w'_\mathbb{C}(z_{\tau n}^k) \le c_\eta w'_\mathbb{C}(z_*)$$

for all $z_{\tau n}^k$, and thus

$$|\eta_n\mathbb{C}'(z_{\tau n}^k)e(u_{\tau n}^{k-1}) : e(u_{\tau n}^{k-1})| \le c_\eta\mathbb{C}'(z^*)e(u_{\tau n}^{k-1}) : e(u_{\tau n}^{k-1})\,.$$

Arguing by the dominated convergence theorem with n-dependent majorants provides the convergence of the corresponding integral term. Here we explicitly use the *strong* convergence $u_{\tau n}^{k-1} \to u_\tau^{k-1}$ in $\mathbf{U}$, cf. (40a), which is proved by induction in Lemma 4.3 right below. All in all we obtain (38a). □

4.3 Proof of (40): Improved convergence

In the following we verify the strong convergence (40) with the aid of two separate lemmata:

Lemma 4.3 (Strong convergence of $(u^k_{\tau n})_n$)**.** *Keep $k \in \mathbb{N}$ fixed. Assume that*

$$u^{k-1}_{\tau n} \to u^{k-1}_{\tau} \;\text{ in } \mathbf{U} \quad \text{and} \quad u^{k-2}_{\tau n} \to u^{k-2}_{\tau} \;\text{ in } \mathbf{U}, \tag{49a}$$

$$z^{k}_{\tau n} \to z^{k}_{\tau} \;\text{ in } L^2(\Omega). \tag{49b}$$

Then, the fully discrete solutions $(u^k_{\tau n})_n$ satisfy the strong convergence result (40a)*.*

Proof. In a first step we show that

$$\int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})\big)e(u^k_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x \to \int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau}) + \mathbb{C}(z^k_{\tau})\big)e(u^k_{\tau}) : e(u^k_{\tau})\,\mathrm{d}x\,. \tag{50}$$

For this, we test (25b) with $u^k_{\tau n} \in \mathbf{U}_n$ and rearrange the terms as follows

$$\begin{aligned}
&\int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})\big)e(u^k_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x \\
&= \tfrac{1}{\tau^2}\int_\Omega -|u^k_{\tau n}|^2 + 2u^{k-1}_{\tau n}\cdot u^k_{\tau n} - u^{k-2}_{\tau n}\cdot u^k_{\tau n}\,\mathrm{d}x + \int_\Omega \tfrac{1}{\tau}\mathbb{D}(z^k_{\tau n})e(u^{k-1}_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x \\
&\quad + \langle f^k_{\tau n}, u^k_{\tau n}\rangle_{\mathbf{U}^*,\mathbf{U}}
\end{aligned} \tag{51}$$

As $n \to \infty$ we obtain convergence of all three integrals on the right-hand side by the following arguments: For the first integral we have convergence due to $u^k_{\tau n} \to u^k_\tau$ strongly in $L^2(\Omega)$ by (37a) and the compact embedding of $\mathbf{U}$ in $L^2(\Omega)$ together with convergence assumption (49a) on $(u^{k-1}_{\tau n})_n$ and $(u^{k-2}_{\tau n})_n$. Moreover, the convergence of the external loading-term is a consequence of the strong convergence of the external forces in (96c) and again (37a). The limit passage in the dissipation term on the right-hand side is guaranteed by (49a) together with the uniform bound on $\mathbb{D}$, providing that $\frac{1}{\tau}\mathbb{D}(z^k_{\tau n})e(u^{k-1}_{\tau n}) \to \frac{1}{\tau}\mathbb{D}(z^k_{\tau})e(u^{k-1}_{\tau})$ strongly in $L^2(\Omega)$. With the above arguments and using weak lower semicontinuity on the left-hand side of (51), we obtain the following chain of inequalities

$$\begin{aligned}
&\int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau}) + \mathbb{C}(z^k_{\tau})\big)e(u^k_{\tau}) : e(u^k_{\tau})\,\mathrm{d}x \\
&\le \liminf_{n\to\infty} \int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})\big)e(u^k_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x \\
&\le \limsup_{n\to\infty} \int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})\big)e(u^k_{\tau n}) : e(u^k_{\tau n})\,\mathrm{d}x \\
&= \frac{1}{\tau^2}\int_\Omega -|u^k_{\tau}|^2 + 2u^{k-1}_{\tau}\cdot u^k_{\tau} - u^{k-2}_{\tau}\cdot u^k_{\tau}\,\mathrm{d}x + \int_\Omega \frac{1}{\tau}\mathbb{D}(z^k_{\tau})e(u^{k-1}_{\tau}) : e(u^k_{\tau})\,\mathrm{d}x \\
&\quad + \langle f^k_{\tau}, u^k_{\tau}\rangle_{\mathbf{U}^*,\mathbf{U}} \\
&= \int_\Omega \big(\frac{1}{\tau}\mathbb{D}(z^k_{\tau}) + \mathbb{C}(z^k_{\tau})\big)e(u^k_{\tau}) : e(u^k_{\tau})\,\mathrm{d}x\,,
\end{aligned} \tag{52}$$

where the last equality in (52) is due to the fact that solutions (u^k_τ, z^k_τ) satisfy the weak balance of momentum (38b) with the test function $u^k_\tau \in \mathbf{U}$. Hence, (50) is proved.

Now, (50) can be used to conclude the desired strong convergence (40a). Making use of the projection operator $P^{\mathbf{U}}_n : \mathbf{U} \to \mathbf{U}_n$, Korn's inequality, and the positive definiteness of the tensors $\mathbb{C}$ and $\mathbb{D}$, we estimate

$$\begin{aligned}
&c_K^2\left\|u^k_{\tau n} - u^k_\tau\right\|^2_{\mathbf{U}} \le \left\|e(u^k_{\tau n}) - e(u^k_\tau)\right\|^2_{L^2(\Omega)} \\
&\le 2\left\|e(u^k_{\tau n}) - e\big(P^{\mathbf{U}}_n(u^k_\tau)\big)\right\|^2_{L^2(\Omega)} + 2\left\|\big(P^{\mathbf{U}}_n(u^k_\tau)\big) - e(u^k_\tau)\right\|^2_{L^2(\Omega)} \\
&\le 2(c^0_{\mathbb{D}} + c^0_{\mathbb{C}})^{-1}\int_\Omega \big(\mathbb{D}(z^k_{\tau n}) + \mathbb{C}(z^k_{\tau n})\big)\big[e(u^k_{\tau n}) - e\big(P^{\mathbf{U}}_n(u^k_\tau)\big)\big] : \big[e(u^k_{\tau n}) - e\big(P^{\mathbf{U}}_n(u^k_\tau)\big)\big]\,\mathrm{d}x \\
&\quad + \left\|e\big(P^{\mathbf{U}}_n(u^k_\tau)\big) - e(u^k_\tau)\right\|^2_{L^2} \quad \to 0\,.
\end{aligned}$$

The latter summand converges to 0 as an intrinsic property of the projection operator. The first summand on the rightmost side converges to 0 as a consequence of (50) and further weak-strong convergence arguments. Thus, the assertion follows. $\square$

Lemma 4.4 (Strong convergence of $(z^k_{\tau n})_n$). *Keep $k \in \mathbb{N}$ fixed. Assume that the first of* (49a) *holds true and in addition also*

$$z^{k-1}_{\tau n} \to z^{k-1}_\tau \text{ in } \mathbf{Z}. \tag{53}$$

Then, the fully discrete solutions $(z^k_{\tau n})_n$ satisfy the strong convergence result (40b).

Proof. To find the desired strong convergence (40b) we will show that

$$\|\nabla z^k_\tau\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|\nabla z^k_{\tau n}\|^2_{L^2(\Omega)} \le \limsup_{n\to\infty} \|\nabla z^k_{\tau n}\|^2_{L^2(\Omega)} \le \|\nabla z^k_\tau\|^2_{L^2(\Omega)} . \tag{54}$$

Here, the first estimate in (54) follows by weak lower semicontinuity and weak convergence (37b) and the second estimate is immediate. To verify the third estimate in (54) we will make use of the discrete equation (38a). More precisely, we test (38a) with $z^k_{\tau n} \in \mathbf{Z}_n$ and rearrange the terms as follows

$$\begin{aligned} &\int_\Omega \ell |\nabla z^k_{\tau n}|^2 \,\mathrm{d}x \\ &= -\int_\Omega \tfrac{1}{2} z^k_{\tau n} \mathbb{C}'(z^k_{\tau n}) e(u^{k-1}_{\tau n}) : e(u^{k-1}_{\tau n}) \,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\tfrac{1}{\ell}(1 - z^k_{\tau n}) z^k_{\tau n} - M \tfrac{z^k_{\tau n} - z^{k-1}_{\tau n}}{\tau} z^k_{\tau n} - \tfrac{N_\tau}{2} \tfrac{\mathrm{d}}{\mathrm{d}z} m^2_+ \Big(\tfrac{z^k_{\tau n} - z^{k-1}_{\tau n}}{\tau} \Big) z^k_{\tau n} \Big) \,\mathrm{d}x \,. \end{aligned} \tag{55}$$

We discuss the limit $n \to \infty$ for the terms on the right-hand side of (55). Thanks to the convergence $z^k_{\tau n} \rightharpoonup z^k_\tau$ in $\mathbf{X}$ by (37b) and by the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ we have $z^k_{\tau n} \to z^k_\tau$ in $L^2(\Omega)$. A similar result also holds true for $(z^{k-1}_{\tau n})_n$. Note that, by (20), $\frac{\mathrm{d}}{\mathrm{d}z} m^2_+ \Big(\frac{z^k_{\tau n} - z^{k-1}_{\tau n}}{\tau} \Big) = 2 \big(\frac{z^k_{\tau n} - z^{k-1}_{\tau n}}{\tau} \big)$ for $(z^k_{\tau n} - z^{k-1}_{\tau n}) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}z} m^2_+ \Big(\frac{z^k_{\tau n} - z^{k-1}_{\tau n}}{\tau} \Big) = 0$ for $(z^k_{\tau n} - z^{k-1}_{\tau n}) \le 0$, hence L^2-convergence supplemented by dominated convergence is sufficient to pass to the limit also in this term. With these arguments the convergence of the second integral on the right-hand side of (55) is ensured. Instead, the first integral on the right-hand side (55) requires further investigation. Since there is no uniform L^∞-bound available for $z^k_{\tau n}$, we instead exploit the properties of the degradation function $w_\mathbb{C}$. More precisely, properties (14) imply the estimate

$$0 \le z^k_{\tau n} w'_\mathbb{C}(z^k_{\tau n}) \le z^* w'_\mathbb{C}(z_*) \quad \text{for all } z^k_{\tau n} \in \mathbb{R} .$$

This further implies that

$$0 \le z^k_{\tau n} \mathbb{C}'(z^k_{\tau n}) e(u^{k-1}_{\tau n}) : e(u^{k-1}_{\tau n}) \le z^* \mathbb{C}'(z_*) e(u^{k-1}_{\tau n}) : e(u^{k-1}_{\tau n}) \,, \tag{56}$$

and the right-hand side of (56) provides a convergent, integrable majorant thanks to (49a). Hence, we can pass to the limit also in the first integral term on the right-hand side of (55) with the aid of the dominated convergence theorem [RF17, Sec. 4.4, Thm. 19, p. 89]. Since above arguments ensure the convergence of all the integral terms on the right-hand side of (55), we are entitled to conclude that

$$\begin{aligned} &\limsup_{n\to\infty} \int_\Omega \ell |\nabla z^k_{\tau n}|^2 \,\mathrm{d}x \\ &= -\int_\Omega \frac{1}{2} z^k_\tau \mathbb{C}'(z^k_\tau) e(u^{k-1}_\tau) : e(u^{k-1}_\tau) \,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\frac{1}{\ell}(1 - z^k_\tau) z^k_\tau - M \tfrac{z^k_\tau - z^{k-1}_\tau}{\tau} z^k_{\tau n} - \tfrac{N_\tau}{2} \tfrac{\mathrm{d}}{\mathrm{d}z} m^2_+ \big(\tfrac{z^k_\tau - z^{k-1}_\tau}{\tau} \big) z^k_\tau \Big) \,\mathrm{d}x. \\ &= \int_\Omega \ell |\nabla z^k_\tau|^2 \,\mathrm{d}x \,. \end{aligned}$$

Here, the last equality stems from the fact that z^k_τ satisfies the time-discrete evolution equation (38a) with the specific test function z^k_τ. In view of (54) the assertion is verified. $\square$

Conclusion of (40) We argue with the aid of Lemmata 4.3, 4.4 by induction. For this, we note that prerequisites (49a) and (53) are fulfilled by the initial data thanks to assumption (39) of Theorem 4.1. Moreover, for each $k \in \{1, \ldots, N_\tau\}$ prerequisite (49b) directly follows from weak convergence result (37b) by the compact embedding of $\mathbf{Z} = H^1(\Omega)$ into $L^2(\Omega)$. Hence, for $k = 1$ Lemmata 4.3, 4.4 provide the strong convergence of the fully discrete solutions $(u^1_{\tau n}, z^1_{\tau n})_n$. Now (40) follows by induction. □

4.4 Proof of (41): Boundedness of solutions z^k_τ in $[0, 1]$

We argue with a recursion argument by contradiction. For that, we will assume that $z^{k-1}_\tau \in [0, 1]$ a.e. in Ω, but that $z^k_\tau \notin [0, 1]$ on a set $B \subset \Omega$ of strictly positive measure. To simplify the argument we will assume that $z^k_\tau(x)$ for a.a. $x \in B$ takes its values in one of the three intervals $[z_\ominus, 0)$, $(1, z^*]$ and $(z^*, z_\oplus]$, see Fig. 1, and deduce a contradiction separately in each of the three intervals. For this, we will test the time-discrete phase-field equation (38a) by a suitable cut-off of a solution z^k_τ. More precisely, this will involve the composition of the Lipschitz-continuous functions $\max\{\cdot, \cdot\}$ and $\min\{\cdot, \cdot\}$ with Sobolev functions $z, g \in \mathbf{X} = H^1(\Omega)$. We remark that, indeed $\max\{z, g\}, \min\{z, g\} \in \mathbf{X}$ for $z, g \in \mathbf{X}$ thanks to [MM79].

Case $[z_\ominus, 0)$**:** Let $z_\ominus \ll 0$ as in Fig. 1, p. 7. Suppose that there is a set $B_1 \subset \Omega$ with $\mathcal{L}^d(B_1) > 0$ such that $z_\ominus \le z^k_\tau < 0$ a.e. in B_1. We define an admissible testfunction for (38a) by $\tilde\eta = -P_{[z_\ominus,0]}(z^k_\tau) = -\min\{0, \max\{z_\ominus, z^k_\tau\}\}$, which is the projection onto $[z_\ominus, 0] \subset \mathbb{R}$. Then

$$
\begin{aligned}
0 &= \langle \mathrm{D}_z\bar{\mathcal{E}}_\tau(t^k_\tau, u^{k-1}_\tau, z^k_\tau) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_\tau), \tilde\eta\rangle_{\mathbf{X}^*,\mathbf{X}} \\
&= \int_{\{z^k_\tau < z_\ominus\}} -\tfrac{1}{\ell}(1 - z^k_\tau)(-z_\ominus) + \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z^k_\tau)(-z_\ominus)\,\mathrm{d}x \\
&\quad + \int_{\{z_\ominus \le z^k_\tau < 0\}} -\tfrac{1}{\ell}(1 - z^k_\tau)(-z^k_\tau) - \ell|\nabla z^k_\tau|^2 + \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z^k_\tau)(-z^k_\tau)\,\mathrm{d}x \\
&\le \int_{B_1} -\tfrac{1}{\ell}(1 - z^k_\tau)(-z^k_\tau) < 0\,.
\end{aligned}
$$

The last inequality is strict and thus by contradiction it follows that $\mathcal{L}^d(B_1) = 0$. Here and in the following, we also use the notation $\{z < g\} := \{x \in \Omega,\ z(x) < g(x)\}$.

Case $(z^*, z_\oplus]$**:** Let $1 < z_* < z^* < z_\oplus$ as in Fig. 1. Assume that there is a set $B_2 \subset \Omega$ with $\mathcal{L}^d(B_2) > 0$ such that $z^* < z^k_\tau \le z_\oplus$ a.e. in B_2. As an admissible test function for (38a) we set $\tilde\eta = P_{[z^*,z_\oplus]}(z^k_\tau) - z^* = \min\{z_\oplus, \max\{z^*, z^k_\tau\}\} - z^*$. Then

$$
\begin{aligned}
0 &= \int_{\{z^* < z^k_\tau \le z_\oplus\}} -\tfrac{1}{\ell}(1 - z^k_\tau)(z^k_\tau - z^*) + \ell|\nabla z^k_\tau|^2 + \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z^k_\tau)(z^k_\tau - z^*)\,\mathrm{d}x \\
&\quad + \int_{\{z_\oplus < z^k_\tau\}} -\tfrac{1}{\ell}(1 - z^k_\tau)(z_\oplus - z^*) + \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z^k_\tau)(z_\oplus - z^*)\,\mathrm{d}x \\
&\ge \int_{B_2} -\tfrac{1}{\ell}(1 - z^k_\tau)(z_\oplus - z^*)\,\mathrm{d}x > 0\,,
\end{aligned}
$$

where the last inequality is strict by $1 < z^* < z_\oplus$ and the assumption on B_2. We obtain by contradiction that $\mathcal{L}^d(B_2) = 0$.

Case $(1,z^*]$**:** Suppose that there exists a set $B_3 \subset \Omega$ such that $\mathcal{L}^d(B_3) \neq 0$ and $1 < z_\tau^k \leq z^*$ a.e. in B_3. Let $\tilde{\eta} = -(P_{[1,z^*]}(z_\tau^k) - 1) = -(\min\{z^*, \max\{1, z_\tau^k\}\} - 1)$ be the test function for (38a), thus

$$
\begin{aligned}
0 = &\int_{\{1<z_\tau^k\leq z^*\}} \big(\tfrac{1}{2}\mathbb{C}'(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1})(1-z_\tau^k) - \tfrac{1}{\ell}(1-z_\tau^k)(1-z_\tau^k) - \ell|\nabla z_\tau^k|^2\big)\,\mathrm{d}x \\
&+ \int_{\{1<z_\tau^k\leq z^*\}} \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z_\tau^k)(1-z_\tau^k)\,\mathrm{d}x + \int_{\{z^*<z_\tau^k\}} \mathrm{DR}_{M\tau}(\mathsf{D}_\tau z_\tau^k)(1-z^*)\,\mathrm{d}x \\
&+ \int_{\{z^*<z_\tau^k\}} \big(\tfrac{1}{2}\mathbb{C}'(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1})(1-z^*) - \tfrac{1}{\ell}(1-z_\tau^k)(1-z^*)\big)\,\mathrm{d}x \\
\leq &\int_{B_3} -\tfrac{1}{\ell}(1-z_\tau^k)^2\,\mathrm{d}x < 0\,,
\end{aligned}
$$

which leads us to conclude that $\mathcal{L}^d(B_3) = 0$.

Since we require in (18) for the initial datum that $z_0(x) \in [0,1]$ for a.e. $x \in \Omega$, it follows that the time-discrete, space-continuous solutions for the phase-field variable are bounded with values in $[0,1]$ almost everywhere in Ω. □

4.5 Proof of (42): Upper energy dissipation estimate for $(u_\tau^k, z_\tau^k)_{k=0}^{N_\tau}$

To deduce the upper energy-dissipation estimate (42), we first test the discrete momentum balance (38b) at time-step $k \in \{1,\ldots,N_\tau\}$ with $\mathsf{D}_\tau u_\tau^k$, i.e.,

$$
0 = \langle \rho\,\mathsf{D}_\tau^2 u_\tau^k + \mathrm{D}_u\bar{\mathcal{E}}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \mathrm{D}\mathcal{V}(z_\tau^k; \mathsf{D}_\tau u_\tau^k), \mathsf{D}_\tau u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}}\,. \tag{57}
$$

Here, all the terms involved in (57) are derivatives of convex functionals and we will thus further estimate each of the terms separately by convexity arguments. We start with the elastic contribution contained in the energy given by the map $u \mapsto \int_\Omega \frac{1}{2}\mathbb{C}(z_\tau^k)e(u) : e(u)\,\mathrm{d}x$. By convexity we estimate

$$
\begin{aligned}
&\int_\Omega \mathbb{C}(z_\tau^k)e(u_\tau^k) : e(\mathsf{D}_\tau u_\tau^k)\,\mathrm{d}x = \frac{1}{\tau}\int_\Omega \mathbb{C}(z_\tau^k)e(u_\tau^k) : e(u_\tau^k - u_\tau^{k-1})\,\mathrm{d}x \\
&\geq \frac{1}{\tau}\int_\Omega \Big(\frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^k) : e(u_\tau^k) - \frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1})\Big)\,\mathrm{d}x
\end{aligned}
\tag{58}
$$

Since also the map $u \mapsto \int_\Omega \frac{\rho}{2}\frac{|u|^2}{\tau^2}\,\mathrm{d}x$ is convex, we can estimate the inertial term in (57) as follows

$$
\begin{aligned}
&\langle \rho\,\mathsf{D}_\tau^2 u_\tau^k, \mathsf{D}_\tau u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}} = \int_\Omega \rho\,\mathsf{D}_\tau^2 u_\tau^k \cdot \mathsf{D}_\tau u_\tau^k\,\mathrm{d}x = \int_\Omega \frac{\rho}{\tau}\,\mathsf{D}_\tau u_\tau^k \cdot (\mathsf{D}_\tau u_\tau^k - \mathsf{D}_\tau u_\tau^{k-1})\,\mathrm{d}x \\
&\geq \frac{1}{\tau}\int_\Omega \Big(\frac{\rho}{2}|\mathsf{D}_\tau u_\tau^k|^2 - \frac{\rho}{2}|\mathsf{D}_\tau u_\tau^{k-1}|^2\Big)\,\mathrm{d}x
\end{aligned}
\tag{59}
$$

Moreover, the term involving the external loading can be reformulated as

$$
\begin{aligned}
&-\langle f_\tau^k, \mathsf{D}_\tau u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}} = -\frac{1}{\tau}\langle f_\tau^k, u_\tau^k - u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}} \\
&= -\frac{1}{\tau}\langle f_\tau^k, u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}} + \frac{1}{\tau}\langle f_\tau^{k-1}, u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}} + \frac{1}{\tau}\langle f_\tau^k - f_\tau^{k-1}, u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}}
\end{aligned}
\tag{60}
$$

Using relations (58)–(60) in (57) we arrive at

$$
\begin{aligned}
0 &= \langle \rho\,\mathsf{D}_\tau^2 u_\tau^k + \mathrm{D}_u\bar{\mathcal{E}}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \mathrm{D}\mathcal{V}(z_\tau^k; \mathsf{D}_\tau u_\tau^k), \mathsf{D}_\tau u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}} \\
&\geq \frac{1}{\tau}\int_\Omega \Big(\frac{\rho}{2}|\mathsf{D}_\tau u_\tau^k|^2 - \frac{\rho}{2}|\mathsf{D}_\tau u_\tau^{k-1}|^2\Big)\,\mathrm{d}x + \int_\Omega \mathbb{D}(z_\tau^k)e(\mathsf{D}_\tau u_\tau^k) : e(\mathsf{D}_\tau u_\tau^k)\,\mathrm{d}x \\
&\quad + \frac{1}{\tau}\int_\Omega \Big(\frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^k) : e(u_\tau^k) - \frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1})\Big)\,\mathrm{d}x \\
&\quad - \frac{1}{\tau}\langle f_\tau^k, u_\tau^k\rangle_{\mathbf{U}^*,\mathbf{U}} + \frac{1}{\tau}\langle f_\tau^{k-1}, u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}} + \frac{1}{\tau}\langle f_\tau^k - f_\tau^{k-1}, u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}}\,.
\end{aligned}
\tag{61}
$$

Secondly, we test the time-discrete phase-field equation (38a) at time-step $k \in \{1, \ldots, N_\tau\}$ with $\mathsf{D}_\tau z_\tau^k$, i.e.,

$$0 = \langle \mathrm{D}_z \bar{\mathcal{E}}_\tau(t_\tau^k, u_\tau^{k-1}, z_\tau^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k), \mathsf{D}_\tau z_\tau^k \rangle_{\mathbf{X}^*,\mathbf{X}} \,. \tag{62}$$

We observe that $\mathrm{D}_z \mathcal{E}(t_\tau^k, u_\tau^{k-1}, z_\tau^k)$ stems from the following energy contributions: a convex map

$$z \mapsto \int_\Omega \big(\frac{\ell}{2}|\nabla z|^2 + \frac{1}{2\ell}(z^2+1)\big)\,\mathrm{d}x,$$

the linear contribution $z \mapsto \int_\Omega -\frac{1}{\ell} z \,\mathrm{d}x$ and the contribution $z \mapsto \int_\Omega \frac{1}{2}\mathbb{C}(z)e(u_\tau^{k-1}) : e(u_\tau^{k-1})\,\mathrm{d}x$. For this third contribution we observe that it is convex as well, if $z \in [0, z_*]$ by (14d). Since even $z_\tau^k \in [0,1]$ a.e. in Ω thanks to (41), this convexity relation is available for estimates in (62). In this way, we may estimate the energy terms in (62) from below by convexity and linearity as follows

$$\begin{aligned}
0 &= \langle \mathrm{D}_z \bar{\mathcal{E}}_\tau(t_\tau^k, u_\tau^{k-1}, z_\tau^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k), \mathsf{D}_\tau z_\tau^k \rangle_{\mathbf{X}^*,\mathbf{X}} \\
&\geq \frac{1}{\tau}\int_\Omega \frac{1}{2}\Big(\mathbb{C}(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1}) - \frac{1}{2}\mathbb{C}(z_\tau^{k-1})e(u_\tau^{k-1}) : e(u_\tau^{k-1})\Big)\,\mathrm{d}x \\
&\quad + \frac{1}{\tau}\int_\Omega \Big(\frac{1}{2\ell}(1-z_\tau^k)^2 + \frac{\ell}{2}|\nabla z_\tau^k|^2 - \frac{1}{2\ell}(1-z_\tau^{k-1})^2 - \frac{\ell}{2}|\nabla z_\tau^{k-1}|^2\Big)\,\mathrm{d}x \\
&\quad + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k)\,,
\end{aligned} \tag{63}$$

where we used that $\langle \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k), \mathsf{D}_\tau z_\tau^k \rangle_{\mathbf{X}^*,\mathbf{X}} = 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k)$ due to the quadratic growth of the terms involved.

Next, we add (61) and (63) and multiply by τ. Hereby, we also exploit the cancellation of the terms

$$\pm\frac{1}{\tau}\int_\Omega \frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^{k-1}) : e(u_\tau^{k-1})\,\mathrm{d}x,$$

which appear in (61) and in (63) with opposite signs. This procedure results in

$$\begin{aligned}
0 &\geq \int_\Omega \frac{\rho}{2}|\mathsf{D}_\tau u_\tau^k|^2 - \frac{\rho}{2}|\mathsf{D}_\tau u_\tau^{k-1}|^2\,\mathrm{d}x \\
&\quad + \tau\int_\Omega \mathbb{D}(z_\tau^k)e(\mathsf{D}_\tau u_\tau^k) : e(\mathsf{D}_\tau u_\tau^k)\,\mathrm{d}x + \tau 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau u_\tau^k) \\
&\quad + \int_\Omega \frac{1}{2}\mathbb{C}(z_\tau^k)e(u_\tau^k) : e(u_\tau^k) - \frac{1}{2}\mathbb{C}(z_\tau^{k-1})e(u_\tau^{k-1}) : e(u_\tau^{k-1})\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2\ell}(1-z_\tau^k)^2 + \frac{\ell}{2}|\nabla z_\tau^k|^2\,\mathrm{d}x - \int_\Omega \frac{1}{2\ell}(1-z_\tau^{k-1})^2 + \frac{\ell}{2}|\nabla z_\tau^{k-1}|^2\,\mathrm{d}x \\
&\quad - \langle f_\tau^k, u_\tau^k \rangle_{\mathbf{U}^*,\mathbf{U}} + \langle f_\tau^{k-1}, u_\tau^{k-1} \rangle_{\mathbf{U}^*,\mathbf{U}} + \tau\langle \mathsf{D}_\tau f_\tau^k, u_\tau^{k-1} \rangle_{\mathbf{U}^*,\mathbf{U}} \\
&= \mathcal{K}(\mathsf{D}_\tau u_\tau^k) + \mathcal{E}(t_\tau^k, u_\tau^k, z_\tau^k) + \tau\big(2\mathcal{V}(z_\tau^k; \mathsf{D}_\tau u_\tau^k) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k)\big) \\
&\quad - \mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) - \mathcal{E}(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) + \tau\langle \mathsf{D}_\tau f_\tau^k, u_\tau^{k-1} \rangle_{\mathbf{U}^*,\mathbf{U}}\,,
\end{aligned} \tag{64}$$

Now we sum (64) over $k = 1, \ldots, L$ for some index $L \in \{1, \ldots, N_\tau\}$. Exploiting further cancellations in the resulting telescopic sum ultimately leads to

$$\begin{aligned}
0 &\geq \mathcal{K}(\mathsf{D}_\tau u_\tau^L) + \mathcal{E}(t_\tau^L, u_\tau^L, z_\tau^L) + \sum_{k=1}^L \tau\big(2\mathcal{V}(z_\tau^k; \mathsf{D}_\tau u_\tau^k) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k)\big) \\
&\quad - \mathcal{K}(\mathsf{D}_\tau u_\tau^0) + \mathcal{E}(t_\tau^0, u_\tau^0, z_\tau^0) + \sum_{k=1}^L \tau\langle \mathsf{D}_\tau f_\tau^k, u_\tau^{k-1} \rangle_{\mathbf{U}^*,\mathbf{U}}\,,
\end{aligned}$$

which is the time-discrete upper energy-dissipation estimate (42). $\square$

4.6 Proof of Proposition 4.2

The proof of the a-priori bounds (48) is based on the upper energy-dissipation estimate (47d). Note that on the left-hand side in (47d) it appears the piecewise constant interpolant $\bar{\mathcal{E}}_\tau$ of the stored energy while on the right-hand side (the time-derivative of) the piecewise affine-linear interpolant (see definitions in (45)) is used. Both interpolants coincide on nodes t_τ^k of a partition $\Pi_\tau = \{0 = t_\tau^0 < t_\tau^1 \ldots < t_\tau^{N_\tau} = \mathsf{T}\}$ of the time interval.

To find a uniform bound for the right-hand side of (47d) requires an energetic control of the power of the time-discrete energy functional $\mathcal{E}_\tau$ from (45),

$$\begin{gathered}\text{There are constants } \tilde{c}, \hat{c} \text{ such that for all } (u,z) \text{ with } \mathcal{E}(0,u,z) < \infty \text{ it is}\\ \mathcal{E}_\tau(\cdot,u,z) \in W^{1,1}(0,\mathsf{T}), \partial_t\mathcal{E}_\tau(t,u,z) \text{ exists for a.a. } t \in (0,\mathsf{T}), \text{ and satisfies}\\ |\partial_t \mathcal{E}_\tau(t,u,z)| \le \tilde{c}\big(\mathcal{E}_\tau(t,u,z) + \hat{c}\big)\end{gathered} \tag{65}$$

cf. also [MR15b, Sec. 2] and [RT17]. The control of the power (65) allows for the application of Gronwall's inequality and thus implies the estimates

$$\mathcal{E}_\tau(t_2,u,z) \;\le\; (\mathcal{E}_\tau(t_1,u,z) + \hat{c}) \exp(\tilde{c}(t_2 - t_1)) - \hat{c}\,, \tag{66a}$$

$$|\partial_t \mathcal{E}_\tau(t_2,u,z)| \;\le\; \tilde{c}(\mathcal{E}_\tau(t_1,u,z) + \hat{c}) \exp(\tilde{c}(t_2 - t_1)) \tag{66b}$$

for all $t_1 < t_2 \in [0,\mathsf{T}]$ and $(u,z) \in \mathbf{U} \times \mathbf{X}$ with $\mathcal{E}(0,u,z) < \infty$. This also provides the absolute continuity of the map $t \mapsto \mathcal{E}(t,u,z)$.

Indeed, it can be checked that assumptions (17) on f allow it to prove for the linear interpolant f_τ constructed by (43) that the control of the power (65) is satisfied, analogously to e.g. [Rou06, (8.72), (8.73), pp. 219–220].

Uniform bound on the energy based on (47d): Based on the above ideas we now deduce the uniform bound on the energy following the lines of [MR15b, Sec. 2]. For this we observe that estimate (64) together with (66b) provides

$$\begin{aligned}
&\mathcal{K}(\mathsf{D}_\tau u_\tau^k) + \mathcal{E}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \tau\big(2\mathcal{V}(z_\tau^k; \mathsf{D}_\tau u_\tau^k) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k)\big)\\
&\le \mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) - \tau\langle \mathsf{D}_\tau f_\tau^k, u_\tau^{k-1}\rangle_{\mathbf{U}^*,\mathbf{U}}\,,\\
&\le \mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) + \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t \mathcal{E}_\tau(s, u_\tau^{k-1}, z_\tau^{k-1})\,\mathrm{d}s\\
&\le \mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1})\\
&\qquad + \int_{t_\tau^{k-1}}^{t_\tau^k} \tilde{c}(\mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) + \hat{c}) \exp(\tilde{c}(s - t_\tau^{k-1}))\,\mathrm{d}s\\
&\le \mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1})\\
&\qquad + \Big(\mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) + \hat{c}\Big)\big(\exp(\tilde{c}(t_\tau^k - t_\tau^{k-1})) - 1\big)\\
&= \Big(\mathcal{K}(\mathsf{D}_\tau u_\tau^{k-1}) + \mathcal{E}_\tau(t_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}) + \hat{c}\Big) \exp(\tilde{c}(t_\tau^k - t_\tau^{k-1})) - \hat{c}
\end{aligned} \tag{67}$$

By recursion we thus conclude for all $k \in \{1, \ldots, N_\tau\}$

$$\begin{aligned}
&\mathcal{K}(\mathsf{D}_\tau u_\tau^k) + \mathcal{E}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \hat{c}\\
&\le \Big(\mathcal{K}(\mathsf{D}_\tau u_\tau^0) + \mathcal{E}_\tau(t_\tau^0, u_\tau^0, z_\tau^0) + \hat{c}\Big)\Pi_{j=1}^k \exp(\tilde{c}(t_\tau^k - t_\tau^{k-1}))\\
&\le \Big(\mathcal{K}(\mathsf{D}_\tau u_\tau^0) + \mathcal{E}_\tau(t_\tau^0, u_\tau^0, z_\tau^0) + \hat{c}\Big) \exp(\tilde{c}T)
\end{aligned} \tag{68}$$

Exploiting cancellations we also find for all $k \in \{1, \ldots, N_\tau\}$

$$\begin{aligned}
&\mathcal{K}(\mathsf{D}_\tau u_\tau^k) + \mathcal{E}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \hat{c} + \sum_{j=1}^k \tau\big(2\mathcal{V}(z_\tau^j; \mathsf{D}_\tau u_\tau^j) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^j)\big)\\
&\le \Big(\mathcal{K}(\mathsf{D}_\tau u_\tau^0) + \mathcal{E}_\tau(t_\tau^0, u_\tau^0, z_\tau^0) + \hat{c}\Big) \exp(\tilde{c}T) \le \tilde{C}
\end{aligned} \tag{69}$$

with some positive constant $\tilde{C} > 0$ independent of τ, M thanks to the assumptions on the external loading (17) and on the initial data required in Prop. 4.2.

A priori estimates (48)**:** The uniform bound (69) puts us in the position to verify the a priori estimates (48). To this end, note that

$$0 \leq \frac{\ell}{2}\|\nabla z_\tau^k\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{2\ell}(1-z_\tau^k)^2\,\mathrm{d}x \leq \frac{\ell}{2}\|\nabla z_\tau^k\|^2_{L^2(\Omega)} + \frac{1}{2\ell}\mathcal{L}^d(\Omega) \tag{70}$$

for all $k \in \{1,\ldots,N_\tau\}$ thanks to (41). Being non-negative, these terms can be neglected on the left-hand side of (69) for the derivation of the uniform bounds related to the displacements. For this, coercivity estimate (15b) and the application of Korn's and Young's inequality together with the boundedness of f from (17a) allows us to find constants $c > 0, \varepsilon \in (0,1)$, $\hat{C} > 0$ such that

$$\begin{aligned} &c(1-\varepsilon)\left\|u_\tau^k\right\|^2_{\mathbf{U}} \\ &\leq \mathcal{K}(\mathsf{D}_\tau u_\tau^k) + \mathcal{E}_\tau(t_\tau^k, u_\tau^k, z_\tau^k) + \hat{c} \\ &\quad + \sum_{j=1}^k \tau\big(2\mathcal{V}(z_\tau^j;\mathsf{D}_\tau u_\tau^j) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^j)\big) + \frac{1}{c\varepsilon}\|f_\tau\|^2_{C([0,T],\mathbf{U}^*)} \\ &\leq \hat{C}(1 + \|f\|^2_{C([0,T],\mathbf{U}^*)}) \leq C \end{aligned} \tag{71}$$

for all $k \in \{0,\ldots,N_\tau\}$. This yields the uniform bound (48a) on $\bar{u}_\tau$, $\underline{u}_\tau$. Thanks to this we also read from (71) the bound on the kinetic energy, which implies (48b) because of $\rho > 0$ and the definition of the interpolants. Again by the definition of the interpolants estimate (71) also provides that

$$c^0_{\mathbb{D}} \int_0^{\mathsf{T}} \int_\Omega |e(\dot{u}_\tau)|^2\,\mathrm{d}x\,\mathrm{d}r \leq \sum_{j=1}^{N_\tau} \tau\big(2\mathcal{V}(z_\tau^j;\mathsf{D}_\tau u_\tau^j) + 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^j)\big) \leq C\,, \tag{72}$$

where we used that $\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^j) \geq 0$ and the positive definiteness (15a) of $\mathbb{D}$. Noting that (48a) implies that $\|u_\tau(t)\|_{\mathbf{U}} \leq C$ for all $t \in [0,\mathsf{T}]$ by the definition of the interpolants, estimate (72) leads to (48c).

We now verify the bound (48d) by a comparison argument. For this, we test the discrete momentum balance (38b) by functions $v \in C^0([0,\mathsf{T}],\mathbf{U})$. We estimate for $\mathsf{D}_\tau \dot{u}_\tau(t) = \mathrm{D}^2_\tau u_\tau^k$ for $t \in (t_\tau^{k-1}, t_\tau^k]$ that

$$\begin{aligned} &\|\rho \mathsf{D}_\tau \dot{u}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U}^*)} = \sup_{\substack{v\in C^0([0,\mathsf{T}];\mathbf{U})\\ \|v\|_{L^2(0,\mathsf{T};\mathbf{U})}=1}} \int_0^{\mathsf{T}}\int_\Omega \rho\,\mathsf{D}_\tau \dot{u}_\tau(t)\cdot v(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \sup_{\substack{v\in C^0([0,\mathsf{T}];\mathbf{U})\\ \|v\|_{L^2(0,\mathsf{T};\mathbf{U})}=1}} \int_0^{\mathsf{T}} \left|\langle \mathrm{D}_u\bar{\mathcal{E}}_\tau(t,\bar{u}_\tau(t),\bar{z}_\tau(t)) + \mathrm{D}\mathcal{V}(\bar{z}_\tau;\dot{u}_\tau(t)), v(t)\rangle_{\mathbf{U}^*,\mathbf{U}}\right|\mathrm{d}t \\ &\leq \|\mathbb{C}(\bar{z}_\tau)e(\bar{u}_\tau)\|_{L^2(0,\mathsf{T};L^2(\Omega))} + \|\mathbb{D}(\bar{z}_\tau)e(\dot{u}_\tau)\|_{L^2(0,\mathsf{T};L^2(\Omega))} + \|\bar{f}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U}^*)} \\ &\leq c^*_{\mathbb{C}}\|\bar{u}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U})} + c^*_{\mathbb{D}}\|\dot{u}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U})} + \|\bar{f}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U}^*)} \leq \hat{C}\,, \end{aligned} \tag{73}$$

where we used the growth property (15) of $\mathbb{C},\mathbb{D}$, the assumptions (17) on the loading, and the already deduced estimates (48a) and (48c). This proves the bound (48d) thanks to the density of $C^0([0,\mathsf{T}];\mathbf{U})$ in $L^2(0,\mathsf{T};\mathbf{U})$.

We also observe that the bound (48e) on $\bar{z}_\tau$ and $\underline{z}_\tau$ now directly follows from (70) and (71). The bound (48f) on the time derivative $\dot{z}_\tau$ follows from the bound on the viscous dissipation potential $\int_0^{\mathsf{T}} M\|\dot{z}_\tau(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq \int_0^{\mathsf{T}} 2\mathcal{R}_{M\tau}(\dot{z}_\tau(t))\,\mathrm{d}t \leq C$ provided by (72) when taking into account the definition of the interpolants; we point out the dependence on the viscous parameter M. The bound (48f) together with (48e) also yields (48g).

We now turn to the last two bounds (48h) and (48i), which remain active even if $M \to 0$ and thus allow us to deduce a rate-independent evolution for the phase-field variable in the limit. We start with (48h): The uniform bound on the viscous dissipation given by (72) implies

$$C \geq \frac{N_\tau}{2}\|\big(\mathsf{D}_\tau z_\tau\big)_+\|^2_{L^2([0,\mathsf{T}]\times\Omega)} \geq \frac{\mathsf{T}}{2\tau\mathcal{L}^{d+1}([0,\mathsf{T}]\times\Omega)^2}\|\big(\mathsf{D}_\tau z_\tau\big)_+\|^2_{L^1([0,\mathsf{T}]\times\Omega)}\,, \tag{74}$$

where we applied Hölder's inequality and used that $N_\tau = \frac{\mathsf{T}}{\tau}$. Taking the square root and making use of the definition of $\mathsf{D}_\tau z_\tau$ we deduce that

$$\mathcal{L}^{d+1}([0,\mathsf{T}]\times\Omega)\sqrt{\frac{2C\tau}{\mathsf{T}}} \geq \|\big(\mathsf{D}_\tau z_\tau\big)_+\|_{L^1([0,\mathsf{T}]\times\Omega)} = \sum_{k=1}^{N_\tau} \|(z_\tau^k - z_\tau^{k-1})_+\|_{L^1(\Omega)}\,. \tag{75}$$

Hence, we have a control on $\bar{z}_\tau$ where the damage evolves in the "wrong" direction, i.e., where it increases.

Next, we expand the quadratic lower order term $\frac{1}{2\ell}(1-z)^2 = \frac{1}{2\ell}(z^2+1) - \frac{1}{\ell}z$ and use the linear contribution to deduce an L^1-estimate that depends on the parameter ℓ but not on M. In this way we obtain

$$\begin{aligned} C \geq \int_\Omega z_\tau^0 - z_\tau^{\mathsf{T}}\,dx &= \sum_{k=1}^{N_\tau}\int_\Omega z_\tau^{k-1} - z_\tau^k\,\mathrm{d}x \\ &= \sum_{k=1}^{N_\tau}\Big(\int\limits_{\{z_\tau^{k-1}\geq z_\tau^k\}} \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x - \int\limits_{\{z_\tau^{k-1}< z_\tau^k\}} \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x\Big)\,. \end{aligned}$$

Together with (75) this implies

$$\begin{aligned} &\sum_{k=1}^{N_\tau}\int_\Omega \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x \\ &= \sum_{k=1}^{N_\tau}\Big(\int_{\{z_\tau^{k-1}\geq z_\tau^k\}} \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x + \int_{\{z_\tau^{k-1}< z_\tau^k\}} \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x\Big) \\ &\leq C + 2\sum_{k=1}^{N_\tau}\int_{\{z_\tau^{k-1}< z_\tau^k\}} \big|z_\tau^{k-1} - z_\tau^k\big|\,\mathrm{d}x \leq C + 2\mathcal{L}^{d+1}([0,\mathsf{T}]\times\Omega)\sqrt{\frac{2C\tau}{\mathsf{T}}}\,. \end{aligned}$$

Hence, the pointwise variation of $\bar{z}_\tau$ in time with values in $L^1(\Omega)$ is uniformly bounded and thus (48h) follows as well as (48i) by definition of the affine linear interpolants z_τ. □

5 Limit passage from the time-discrete to the time-continuous setting

In this section we discuss the limit passage $\tau \to 0$ starting out from tuples of interpolated time-discrete solutions $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ of problem (47).
In the case that also $M \to 0$ we obtain a solution of system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_1, \mathcal{E})$, more precisely we deduce the following

Theorem 5.1 (Existence of solutions in the rate-independent limit)**.** *Let the assumptions of Theorem 4.1 and Proposition 4.2 be satisfied, and assume that the one-sided variational inequality* (10a) *holds true at time $t = 0$ for the initial data $(u_0, z_0) \in \mathbf{U}\times\mathbf{X}$. Consider the viscosity parameter $M = M(\tau) > 0$ in* (4) *to depend on τ such that $M(\tau) \to 0$ as $\tau \to 0$. For all $\tau > 0$ let $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau)$ be a tuple of interpolated solutions of problem* (47) *corresponding to system*

$$(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})\,.$$

Then the following results hold true:

1 *Then, there exist functions* $u\colon [0,\mathsf{T}]\to\mathbf{U}$, $z\colon [0,\mathsf{T}]\to\mathbf{X}$ *such that following convergence statements are valid:*

$$\begin{aligned}
\bar u_\tau,\ \underline{u}_\tau &\overset{*}{\rightharpoonup} u && \text{weakly-}* \text{ in } L^\infty(0,\mathsf{T};\mathbf{U})\,, && \text{(76a)}\\
u_\tau &\rightharpoonup u && \text{weakly in } H^1(0,\mathsf{T};\mathbf{U})\,, && \text{(76b)}\\
\dot u_\tau &\overset{*}{\rightharpoonup} \dot u && \text{weakly-}* \text{ in } L^\infty\big(0,\mathsf{T};L^2(\Omega,\mathbb{R}^d)\big)\,, && \text{(76c)}\\
\bar u_\tau(t),\ \underline{u}_\tau(t) &\rightharpoonup u(t) && \text{weakly in } \mathbf{U} \text{ for all } t\in[0,\mathsf{T}]\,, && \text{(76d)}\\
\dot u_\tau(t) &\rightharpoonup \dot u(t) && \text{weakly in } L^2(\Omega,\mathbb{R}^d) \text{ for all } t\in[0,\mathsf{T}]\,, && \text{(76e)}\\
\bar z_\tau(t), \underline{z}_\tau(t) &\rightharpoonup z(t) && \text{weakly in } \mathbf{X} \text{ for all } t\in[0,\mathsf{T}], && \text{(76f)}\\
\bar z_\tau(t), \underline{z}_\tau(t) &\to z(t) && \text{strongly in } L^2(\Omega) \text{ for all } t\in[0,\mathsf{T}]\,, && \text{(76g)}\\
\bar z_\tau,\ \underline{z}_\tau &\overset{*}{\rightharpoonup} z && \text{weakly-}* \text{ in } L^\infty\big(0,\mathsf{T};\mathbf{X}\big)\,. && \text{(76h)}
\end{aligned}$$

2 *The limit pair* (u,z) *is a solution of* $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ *in the sense of Definition 1.3 and it is* $0\le z(t,x)\le 1$ *for a.a.* $x\in\Omega$ *and for all* $t\in[0,\mathsf{T}]$. *In addition, the limit* (u,z) *also satisfies semistablility inequality* (11) *for a.e.* $t\in(0,\mathsf{T})$.

3 *The limit function* u *has the following regularity:*

$$\begin{aligned}
&u\in H^1(0,\mathsf{T};\mathbf{U})\cap L^\infty(0,\mathsf{T};\mathbf{U})\cap W^{1,\infty}(0,\mathsf{T};L^2(\Omega))\cap C^0([0,\mathsf{T}];\mathbf{U})\,, && \text{(77a)}\\
&\ddot u\in L^2(0,\mathsf{T};\mathbf{U}^*)\,,\quad \text{and} && \text{(77b)}\\
&\textstyle\int_s^t\langle\ddot u(r),\dot u(r)\rangle\mathrm{d}r=\frac12\|\dot u(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}-\frac12\|\dot u(s)\|^2_{L^2(\Omega;\mathbb{R}^d)}\quad \text{for all } s,t\in[0,\mathsf{T}]\,, && \text{(77c)}
\end{aligned}$$

and, in addition to the regularity $z\in BV(0,\mathsf{T};L^1(\Omega))\cap L^\infty(0,\mathsf{T};\mathbf{X})$ *the limit function* z *even satisfies:*

$$z\in C^{0,1/4}([0,\mathsf{T});\mathbf{X})\,, \tag{78}$$

i.e., $z:[0,\mathsf{T})\to\mathbf{X}$ *is Hölder-continuous with Hölder-exponent* $h=1/4$. *Hence,* (u,z) *satisfies the one-sided variational inequality* (10a)*, semistability inequality* (11)*, and the energy-dissipation balance* (10d) *even for all* $t\in[0,\mathsf{T})$.

4 *In addition to convergence results* (76) *also the following improved convergences hold true:*

$$\begin{aligned}
e(\dot u_\tau) &\to e(\dot u) && \text{strongly in } H^1(0,T;\mathbf{U})\,, && \text{(79a)}\\
e(\bar u_\tau(t)) &\to e(u(t)) && \text{strongly in } \mathbf{U} \text{ for all } t\in[0,\mathsf{T})\,, && \text{(79b)}\\
\bar z_\tau(t) &\to z(t) && \text{strongly in } \mathbf{X} \text{ for all } t\in[0,\mathsf{T})\,. && \text{(79c)}
\end{aligned}$$

Proof. The proof of the convergence results (76) will be developed in Section 5.1. Subsequently the limit passage in the defining properties of the solutions, cf. Def. 1.3, properties (10), is carried out in Section 5.2. The regularity (77) of u will be discussed in Sec. 5.2.2 when passing to the limit in the weak momentum balance. The Hölder-continuity of $z:[0,\mathsf{T})\to\mathbf{X}$ is developed in Sec. 5.3 and it relies on a general regularity result stated here below in Theorem 5.2. The continuity of (u,z) in time allows it to conclude that the defining properties (10a), (11), and (10d) are valid even for all $t\in[0,\mathsf{T})$. Based on this, the improved convergences (79) are concluded in Section 5.4. □

The proof of the temporal Hölder-continuity of z relies on an adaption of a general regularity result for coupled rate-dependent/rate-independent systems obtained in [RT17, Thm. 3.8]. Let us point out that for purely rate-independent systems temporal (Hölder-) continuity stems from enhanced convexity properties of the energy functional for the pair (u,z), cf. [MT04, TM10] for more details. For damage models as in the current situation the energy functional is separately convex, only, so that improved temporal regularity cannot be expected in a purely rate-independent setting. As can be seen here in Theorem 5.2, in the coupled rate-dependent/rate-independent setting it is sufficient to have uniform convexity with respect to the rate-independent variable z, because the good regularity of the rate-dependent variable u partially carries over to z through estimates (90) or (91). In case of a unidirectional evolution of z as for damage it is even sufficient to have such estimates available for a.e. $t\in(0,\mathsf{T})$, only, because the information missing on a null-set $N\subset[0,\mathsf{T})$ is filled by unidirectionality to ultimately conclude regularity statement (78); see Sec. 5.3 for more details.

Theorem 5.2 (Adaption of [RT17, Thm. 3.8]). *Let $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ be a damped inertial system characterized by Banach spaces $\mathbf{U}, \mathbf{Z}$, and a Hilbert space $\mathbf{W}$, the kinetic energy $\mathcal{K} : \mathbf{W} \to [0, \infty)$, a dissipation potential $\mathcal{V} : \mathbf{Z} \times \mathbf{U} \to [0, \infty)$, a positively 1-homogeneous dissipation potential $\mathcal{R} : \mathbf{Z} \to [0, \infty]$, and an energy functional $\mathcal{E} : [0, \mathsf{T}] \times \mathbf{U} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$ such that for all $t \in [0, \mathsf{T}]$ the functional $\mathcal{E}(t, \cdot, \cdot)$ takes finite values on (a closed, convex subset $\mathsf{D}_u \times \mathsf{D}_z$ of) $\mathbf{V} \times \mathbf{X}$ with $\mathbf{X}$ a Banach space such that $\mathbf{X} \subset \mathbf{Z}$ compactly and $\mathbf{V}$ a Banach space such that $\mathbf{V} \subset \mathbf{U}$ continuously and densely. Further consider the following list of assumptions:*

A1) The pair $(u, z) : [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{Z}$ satisfies a semistability inequality for a.a. $t \in [0, \mathsf{T}]$:

$$\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) \quad \text{for all } \tilde{z} \in \mathbf{Z}\,. \tag{80}$$

Accordingly, define the $\mathcal{L}^1$-null set

$$N := \big\{\hat{t} \in [0, \mathsf{T}],\ (u(\hat{t}), z(\hat{t})) \text{ does not satisfy semistability (80)}\big\}\,. \tag{81}$$

A2) The pair $(u, z) : [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{Z}$ satisfies the following upper energy-dissipation estimate

$$\begin{aligned} \mathcal{K}(\dot{u}(t)) + \mathcal{E}(t, u(t), z(t)) + \mathcal{R}(z(t) - z(s)) + \int_s^t 2\mathcal{V}(z(r); \dot{u}(r))\,\mathrm{d}r \\ \leq \mathcal{K}(\dot{u}(s)) + \mathcal{E}(t, u(s), z(s)) + \int_s^t \partial_r \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r \end{aligned} \tag{82}$$

for all subintervals $[s, t] \subset [0, \mathsf{T}]$ with $s, t \in [0, \mathsf{T}] \backslash N$.

A3) $u \in W^{2,2}(0, \mathsf{T}; \mathbf{U}^*) \cap H^1(0, \mathsf{T}; \mathbf{U})$ and $t \mapsto |\langle \ddot{u}(t), \dot{u}(t)\rangle_{\mathbf{U}}| \in L^1(0, \mathsf{T})$.

A4) The energy functional $\mathcal{E}$ complies with the following power control: There are constants $\tilde{c}, \hat{c}$ such that for all $(u, z) \in \mathbf{U} \times \mathbf{Z}$ with $\mathcal{E}(0, u, z) < \infty$ it is $\mathcal{E}(\cdot, u, z) \in W^{1,1}(0, \mathsf{T})$, $\partial_t \mathcal{E}(t, u, z)$ exists for a.a. $t \in (0, \mathsf{T})$, and satisfies

$$|\partial_t \mathcal{E}(t, u, z)| \leq \tilde{c}\big(\mathcal{E}(t, u, z) + \hat{c}\big)\,. \tag{83}$$

A5) The functional $\mathcal{E}(t, u, \cdot) : \mathsf{D}_z \to \mathbb{R}$ is Gâteaux-differentiable and uniformly convex, i.e.,

$$\begin{aligned} &\exists\, \alpha \geq 2\ \exists\, C_\star > 0\ \forall\, t \in [0, \mathsf{T}],\ \forall\, (u, z_0), (u, z_1) \in \mathsf{D}_u \times \mathsf{D}_z, \forall\, \lambda \in [0, 1], \\ &\text{setting } z_\lambda := \lambda z_1 + (1 - \lambda) z_0 : \\ &\quad \mathcal{E}(t, u, z_\lambda) + C_\star \lambda(1 - \lambda)\|z_1 - z_0\|_{\mathbf{S}}^\alpha \leq \lambda \mathcal{E}(t, u, z_1) + (1 - \lambda)\mathcal{E}(t, u, z_0)\,, \end{aligned} \tag{84}$$

with $\mathbf{S}$ a Banach space such that $\mathbf{X} \subseteq \mathbf{S}$ continuously, that may or may not coincide with $\mathbf{X}$ or $\mathbf{Z}$.

A6) The functional $\mathcal{E}(t, \cdot, z) : \mathbf{U} \to \mathbb{R} \cup \{\infty\}$ is Hölder-continuous, i.e., there are constants $c_\star > 0, \beta_u \in (0, 1]$ such that for all $s, t \in [0, \mathsf{T}]$ and for all $(u_0, z_1), (u_1, z_1)$ with $\sup_{t \in [0, \mathsf{T}]} \mathcal{E}(t, u_i, z_1) \leq E$, $i \in \{0, 1\}$ we have

$$\big|\mathcal{E}(t, u_1, z_1) - \mathcal{E}(t, u_0, z_1)\big| \leq c_\star \|u_1 - u_0\|_{\mathbf{U}}^{\beta_u}\,. \tag{85}$$

A7) The functional $\mathcal{E}(t, \cdot, z) : \mathsf{D}_u \to \mathbb{R}$ is Gâteaux-differentiable for all $(t, z) \in [0, \mathsf{T}] \times \mathsf{D}_z$.

A8) The functional $\mathcal{E}(t, \cdot, z) : \mathsf{D}_u \to \mathbb{R}$ complies with the following gradient estimate: There exist constants $\hat{C}_1, \hat{C}_2, \hat{C}_3 > 0$ and $\sigma \in [1, \infty)$ such that

$$\|\mathrm{D}_u \mathcal{E}(t, u, z)\|_{\mathbf{U}^*}^\sigma \leq \hat{C}_1 \mathcal{E}(t, u, z) + \hat{C}_2 \|u\|_{\mathbf{U}} + \hat{C}_3 \tag{86}$$

for all $(t, u, z) \in [0, \mathsf{T}] \times \mathbf{U} \times \mathbf{X}$ with $\mathcal{E}(t, u, z) < \infty$.

A9) The pair $(u, z) : [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{Z}$ satisfies the weak momentum balance for all $t \in [0, \mathsf{T}]$:

$$\rho \ddot{u}(t) + \mathrm{D}_u \mathcal{E}(t, u(t), z(t)) + \mathrm{D}_{\dot{u}} \mathcal{V}(z(t); \dot{u}(t)) = 0 \quad \text{in } \mathbf{U}^*\,. \tag{87}$$

A10) The dissipation potential $\mathcal{V} : \mathbf{Z} \times \mathbf{U} \to [0, \infty)$ has quadratic growth, i.e., there are constants $\tilde{C}_1, \tilde{C}_2 > 0$ such that for all $(z, v) \in \mathbf{Z} \times \mathbf{U}$

$$\mathcal{V}(z; v) \geq \tilde{C}_1 \|v\|_{\mathbf{U}}^2 - \tilde{C}_2\,. \tag{88}$$

The following statements hold true:

1 *Let assumptions A1) and A5) be valid. Then* (u, z) *satisfies the following improved semistability inequality*

$$\mathcal{E}(s, u(s), z(s)) + C_\star \|z(t) - z(s)\|_{\mathbf{S}}^\alpha \leq \mathcal{E}(s, u(s), z(t)) + \mathcal{R}(z(t) - z(s)) \tag{89}$$

for all $s \in [0, \mathsf{T}] \backslash N$ *and for all* $t \in [0, \mathsf{T}]$.

2 *Let assumptions A1)– A6) be valid. Then,* z *complies with the estimate*

$$C_\star \|z(t) - z(s)\|_{\mathbf{S}}^\alpha \leq C|t-s| + \int_s^t |\langle \rho \ddot{u}(r), \dot{u}(r) \rangle_{\mathbf{U}}| \,\mathrm{d}r + c_\star \Big(\int_s^t \|\dot{u}(r)\|_{\mathbf{U}} \,\mathrm{d}r \Big)^{\beta_u} \tag{90}$$

for all subintervals $[s, t] \subset [0, \mathsf{T}]$ *with* $s, t \in [0, \mathsf{T}] \backslash N$.

3 *Let assumptions A1)–A9) be valid. Then,* z *complies with the estimate*

$$C_\star \|z(t) - z(s)\|_{\mathbf{S}}^\alpha \leq C|t-s| + c_\star \Big(\int_s^t \|\dot{u}(r)\|_{\mathbf{U}} \,\mathrm{d}r \Big)^{\beta_u} \tag{91}$$

for all subintervals $[s, t] \subset [0, \mathsf{T}]$ *with* $s, t \in [0, \mathsf{T}] \backslash N$.

4 *Let assumptions A1)– A6) be valid and let A1) and A2) be satisfied for all subintervals* $[s, t] \subset [s_*, t_*] \subset [0, \mathsf{T}]$, *even for all* $s, t \in [s_*, t_*]$. *Then also estimate* (90) *is valid even for all* $s, t \in [s_*, t_*]$ *and hence it implies that* $z \in C^0([s_*, t_*]; \mathbf{S})$.

5 *Let assumptions A1)–A9) be valid and let A1) and A2) be satisfied for all subintervals* $[s, t] \subset [s_*, t_*] \subset [0, \mathsf{T}]$, *even for all* $s, t \in [s_*, t_*]$. *Then also estimate* (91) *holds true even for all* $s, t \in [s_*, t_*]$ *and thus it implies that* $z \in C^0([s_*, t_*]; \mathbf{S})$. *Additionally assume that A10) is valid. Then*

$$z \in C^{0,h}([s_*, t_*]; \mathbf{S}) \text{ with the Hölder-exponent } h = \frac{\beta_u}{(2\alpha)} < 1/2. \tag{92}$$

Proof. In [RT17, Thm. 3.8] the assumptions A1) and A2) are strengthened to hold for all $s, t \in [0, \mathsf{T}]$ and consequently it only ensures statements 4 and 5 of above Thm. 5.2 with $[s_*, t_*] = [0, \mathsf{T}]$. Moreover, in [RT17, Thm. 3.8] also assumption A10) on the rate-dependent dissipation is different: There, $\tilde{\mathcal{V}} : \mathbf{U} \to [0, \infty)$ is *independent of the rate-independent variable* z but allows for a general p-growth with $p > 1$ instead of $p = 2$ in (88). In this spirit, the viscous dissipation function $\int_s^t 2\mathcal{V}(z(r); \dot{u}(r)) \,\mathrm{d}r$ appearing in (82) is replaced in [RT17] by De Giorgi's expression $\int_s^t \tilde{\mathcal{V}}(\dot{u}(r)) + \tilde{\mathcal{V}}^*(-(\ddot{u}(r) + \mathrm{D}_u \mathcal{E}(r, u(r), z(r)))) \,\mathrm{d}r$ which involves the convex conjugate of the convex potential $\tilde{\mathcal{V}} : \mathbf{U} \to [0, \infty)$. We do not use this expression in (82) due to the quadratic, but z-dependent nature of $\mathcal{V}$. A close perusal of the proof of [RT17, Thm. 3.8] reveals that above estimates (89), (90), and (91) can be deduced to hold for all $s, t \in [0, \mathsf{T}] \backslash N$ under the relaxed assumptions that semistability (11) and the upper energy-dissipation estimate (82) are valid for exactly these $s, t \in [0, \mathsf{T}] \backslash N$. In this way, also statements 1 and 2 become valid. Moreover, since we here work in the setting of a quadratic, z-dependent dissipation $\mathcal{V} : \mathbf{X} \times \mathbf{U} \to [0, \infty)$ and use the dissipative term $\int_s^t 2\mathcal{V}(z(r); \dot{u}(r)) \,\mathrm{d}r$ in (82), certain estimates related to (91) can be carried out differently circumventing $\mathcal{V}^*$. In this way, also statement 3 of above Thm. 5.2 can be shown to hold for all $s, t \in [0, \mathsf{T}] \backslash N$. We also refer to Theorem 6.2 and in particular to estimates (143)–(158) in its proof, where analogous arguments are carried out in the setting of a viscous dissipation potential $\mathcal{R}_M$ for z. □

Remark 5.3 (Simultaneous limit and its connection to FE-approximations). *It is possible to formulate the defining properties* (10)*, resp.* (9) *of solutions in the sense of Def. 1.3, resp. Def. 1.2, already on the fully discrete level. In this context the Yosida regularization cannot have its full effect such that the discrete damage variable may take values outside of the interval* $[0, 1]$ *on sets of strictly positive measure. This entails that the discrete version of the functional* $z \mapsto \int_\Omega \frac{1}{2} \mathbb{C}(z) e(u) : e(u) \,\mathrm{d}x$ *is non-convex even for fixed displacements* u. *Hence, an upper energy-dissipation estimate holds true only up to an error generated by the non-convexity. In order to find compactness nevertheless, regions of non-convexity have to be controlled. In [BMT+20, Section 4] we show that this is indeed possible. Therein, the space discretization is realized with a finite element approximation in terms of P1 finite elements. Here, apart from the error due to the non-convexity also an error caused by only* approximately *solving the nonlinear phase-field evolution equation* (25a) *becomes relevant. The control of these error terms leads to additional conditions which can be regarded as stopping criteria for an algorithm solving* *the discrete problems. Moreover, the control of the non-convexity errors in the upper energy-dissipation estimate leads to a coupling relation between time-step size and mesh-size of the FE-space. We mention that it is also possible to show with an a-posteri argument the existence of a diagonal sequence converging to a solution in the sense of Def. 1.3, see [BMT+20, Sec. 4].*

5.1 Proof of Theorem 5.1, Item 1: Convergence statements (76).

Convergence statements (76a)–(76e) *for the displacements: The convergence statements* (76a)–(76c) *for* $\bar{u}_\tau, \underline{u}_\tau, u_\tau$ *and* $\dot{u}_\tau$ *follow by standard compactness arguments from the uniform bounds* (48a), (48b), *and* (48c), *at first each of them with a different limit function and it has to be shown that the limits coincide. For this, note that* (48c) *implies* $(\dot{u}_\tau)_\tau$ *to be uniformly bounded in* $L^2(0,\mathsf{T};\mathbf{U})$. *Then, the identities*

$$u_\tau(t) - \bar{u}_\tau(t) = (t - t_\tau^k)\dot{u}_\tau(t) \quad \textit{and} \quad u_\tau(t) - \underline{u}_\tau(t) = (t - t_\tau^{k-1})\dot{u}_\tau(t) \tag{93}$$

allow us to conclude that the limit functions of (76a) *and* (76b) *coincide. They also coincide with the limit obtained by convergence* (76c) *as can be deduced from the uniqueness of weak limits when taking into account* (76b).

For convergences (76d), *which hold pointwise for all* $t \in [0,\mathsf{T}]$, *we realize that* $(u_\tau)_\tau$ *is uniformly bounded in* $BV(0,\mathsf{T};\mathbf{U})$ *thanks to the* $H^1(0,\mathsf{T};\mathbf{U})$*-bound from* (48c) *and the continuous embedding of* $H^1(0,\mathsf{T};\mathbf{U})$ *in* $BV(0,\mathsf{T};\mathbf{U})$. *By the definition of the interpolants we also find that* $(\bar{u}_\tau)_\tau$ *and* $(\underline{u}_\tau)_\tau$ *are uniformly bounded in* $BV(0,\mathsf{T};\mathbf{U})$. *More precisely, we have the following estimate:*

$$\begin{aligned}\sum_{k=1}^{N_\tau} \big\|u_\tau^k - u_\tau^{k-1}\big\|_{\mathbf{U}} &= \sum_{k=1}^{N_\tau} \tau \left\|\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\right\|_{\mathbf{U}} = \int_0^{\mathsf{T}} \|\dot{u}_\tau(t)\|_{\mathbf{U}} \,\mathrm{d}t \\ &= \|\dot{u}_\tau\|_{L^1(0,\mathsf{T};\mathbf{U})} \le \sqrt{\mathsf{T}}\, \|\dot{u}_\tau\|_{L^2(0,\mathsf{T};\mathbf{U})} \le C\,,\end{aligned} \tag{94}$$

where the left-hand side of (94) *gives the total variation of the interpolants* $\bar{u}_\tau$ *and* $\underline{u}_\tau$. *Then, an application of Helly's theorem for Banach spaces [MT04, Thm. 6.1] allows us to conclude the pointwise convergences in* (76d) *upon extraction of a further subsequence. To conclude that* $(\bar{u}_\tau)_\tau$ *and* $(\underline{u}_\tau)_\tau$ *have the same limit pointwise in time that coincides with* u *we once more exploit the identities* (93) *together with the uniqueness of the weak limit already obtained in* (76a).

For (76e), *observe that after possibly extracting a subsequence,* (48d) *implies by [MR15a, Theorem B.5.10]:*

$$\dot{u}_\tau(t) \overset{*}{\rightharpoonup} \hat{v}(t) \textit{ weakly-}* \textit{ in } U^* \textit{ for all } t \in [0,\mathsf{T}]$$

with $\hat{v} \in BV(0,\mathsf{T};U^*)$. *Then, [DJ12, Thm. 1, p. 3073], a version of an Aubin-Lions compactness argument suited for a time-discretization and piecewise constant sequences in time* $(\dot{u}_\tau)_\tau$ *can be used to find for this subsequence*

$$\begin{aligned}&\dot{u}_\tau \to \dot{u} \textit{ strongly in } L^2\big(0,\mathsf{T};L^2(\Omega,\mathbb{R}^d)\_igr) \\ &\dot{u}_\tau \to \tilde{v} \textit{ strongly in } L^p(0,\mathsf{T};U^*) \textit{ for all } p \in [1,\,\infty)\end{aligned}$$

with $\tilde{v} \in C^0([0,\mathsf{T}],U^*)$. *This shows that*

$$\dot{u} = \tilde{v} = \hat{v} \;\textit{ in } L^2(0,T;U^*) \tag{95}$$

and we set $\dot{u}(t) = \hat{v}(t)$ *for all* $t \in [0,\mathsf{T}]$. *From this we infer* (76e) *with the following argument: For all* $t \in [0,\mathsf{T}]$, *every subsequence of* $(\dot{u}_\tau(t))_\tau$ *is bounded in* $L^2(\Omega)$ *by* (48b) *and admits a further subsequence weakly converging in* $L^2(\Omega)$ *to some limit* v_t. *In view of* (95) *we have* $v_t = \dot{u}(t)$ *identified in* $\mathbf{U}^*$ *for all* $t \in [0,\mathsf{T}]$. *Since the limit does not depend on the extracted subsequence, we conclude* (76e).

Convergence statements (76f)–(76h) *for the damage variable: To verify convergence statements* (76f)–(76h) *we observe that the uniform* BV*-bound* (48h) *justifies the use of a variant of Helly's Theorem, cf. [MR15b, Thm. 2.1.24, p. 72], since* $\|\cdot\|_{L^1(\Omega)}$ *defines a dissipation distance in the sense of [MR15b, (D1) and (D2), p. 46]. This provides the existence of an element* $z\colon [0,\mathsf{T}] \to \mathbf{X}$ *with* $z \in BV\big(0,\mathsf{T};L^1(\Omega)\big)$ *such that, along a (not relabelled) subsequence,* $\bar{z}_\tau(t) \rightharpoonup z(t) \in \mathbf{X}$ *weakly even in* $\mathbf{X}$ *for all* $t \in [0,\mathsf{T}]$, *which is the first of* (76f). *By the compact embedding* $\mathbf{X} \subset L^2(\Omega)$ *we thus obtain* *the first of* (76g)*. Thanks to the uniform bound* (48e) *we now also conclude that the first of* (76h) *holds true. Hence, the convergence statements* (76f)–(76h) *are verified for the sequence* $(\bar{z}_\tau)_\tau$.

Repeating above arguments for the sequence $(\underline{z}_\tau)_\tau$ *starting from the uniform* BV*-bound* (48h) *we also find that* $(\underline{z}_\tau)_\tau$ *converges to a limit function* $\underline{z}\colon [0,\mathsf{T}] \to \mathbf{X}$ *in the topologies of* (76f)–(76h) *and now it has to be shown that* $\underline{z}$ *indeed coincides with* z*. For this, we may follow the lines of [BMTW20]. We denote by* J_z *and* $J_{\underline{z}}$ *the countable jump sets of the two limit functions* $z, \underline{z} \in BV(0,\mathsf{T};L^1(\Omega))$*. Let* $t \in [0,\mathsf{T}] \setminus (J_z \cup J_{\underline{z}})$*. By the definition of the interpolants we have* $\bar{z}_\tau(t-\tau) = \underline{z}_\tau(t)$ *for all* $\tau > 0$ *and thus as* $\tau \to 0$ *if follows* $z(t) = \underline{z}(t)$ *for all* $t \in [0,\mathsf{T}] \setminus (J_z \cup J_{\underline{z}})$*. Now, let* $t \in J_z \cup J_{\underline{z}}$ *and w.l.o.g. assume* $t \in J_z$*. Then there are sequences* $(t_j^+)_j,\ (t_j^-)_j \subset [0,\mathsf{T}] \setminus (J_z \cup J_{\underline{z}})$ *such that* $t_j^+ \searrow t$*,* $t_j^- \nearrow t$*. But since* $z(t_j^\pm) = \underline{z}(t_j^\pm)$ *for* $t_j^+, t_j^- \in [0,\mathsf{T}] \setminus (J_z \cup J_{\underline{z}})$*, we find for the left limit that* $z^-(t) = \lim_{j\to\infty} z(t_j^-) = \lim_{j\to\infty} \underline{z}(t_j^-) = \underline{z}^-$ *and for the right limit that* $z^+(t) = \lim_{j\to\infty} z(t_j^+) = \lim_{j\to\infty} \underline{z}(t_j^+) = \underline{z}^+$*. This implies that* $J_z = J_{\underline{z}}$*. Thus,* $z = \underline{z}$ *on the whole interval* $[0,\mathsf{T}]$ *and hence convergence results* (76f)–(76h) *are verified.* □

5.2 Proof of Theorem 5.1, Item 2: Defining properties of the solutions and boundedness $z \in [0,1]$

In this section we show that the limit pair (u,z) *obtained through convergences* (76) *indeed is a solution of system* $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_1,\mathcal{E})$ *in the sense of Definition 1.3. For this we will pass to the limit* $\tau \to 0$ *in problem* (47) *for the interpolants* $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ *using the convergence results* (76) *and thus conclude properties* (10)*. For the limit passage we will also make use of the following convergence results for the interpolants of the external forces:*

$$\bar{f}_\tau \to f \qquad \text{strongly in } L^p\big(0,\mathsf{T};\mathbf{U}^*\big) \text{ for all } 1 \le p < \infty. \tag{96a}$$

$$\bar{f}_\tau \overset{*}{\rightharpoonup} f \qquad \text{weakly-* in } L^\infty\big(0,\mathsf{T};\mathbf{U}^*\big)\,. \tag{96b}$$

$$\bar{f}_\tau(t) \to f(t) \qquad \text{strongly in } \mathbf{U}^* \text{ for all } t \in [0,\mathsf{T}]\,. \tag{96c}$$

$$f_\tau \to f \qquad \text{strongly in } H^1\big(0,\mathsf{T};\mathbf{U}^*\big) \tag{96d}$$

by assumption (17) *on the regularity of the external load.*

First, it is shown in Section 5.2.1 that z *takes values bounded in* $[0,1]$ *and that it satisfies the unidirectionality property* (10b)*. Subsequently, Section 5.2.2 is devoted to the limit passage in the weak momentum balance* (10c)*. We will further verify in Section 5.2.3 that the one-sided variational inequality* (10a) *is valid for the limit pair* (u,z)*. There, we also show that solutions satisfy the semistability inequality* (11)*. Moreover, Section 5.2.4 establishes the energy dissipation balance* (10d)*.*

5.2.1 Proof of the boundedness of z and of the unidirectionality of the damage evolution (10b)

We first show the boundedness of z*, i.e., that*

$$z(t,x) \in [0,1] \text{ for a.e. } x \in \Omega \text{ and all } t \in [0,\mathsf{T}]. \tag{97}$$

Indeed, this can be concluded with the knowledge that the time-discrete approximants $(\bar{z}_\tau)_\tau$ *satisfy* $\bar{z}_\tau(t,x) \in [0,1]$ *for a.e.* $x \in \Omega$ *and all* $t \in [0,\mathsf{T}]$ *by* (41) *in Theorem 4.1. The strong* $L^2(\Omega)$*-convergence* (76g) *provides convergence in measure. Hence, assuming that* $z(t) \notin [0,1]$ *on a set* $B \subset \Omega$ *of strictly positive measure leads to a contradiction; thus* (97) *is verified.*

Unidirectionality (10b)*: We verify now that in the time-continuous limit the damage variable has a unidirectional evolution, i.e., for all* $t_1 < t_2 \in [0,\mathsf{T}]$ *it is* $z(t_1,x) \ge z(t_2,x)$ *for a.a.* $x \in \Omega$*, cf.* (10b)*. For this, assume the contrary, i.e., suppose that* $z(t_1,x) < z(t_2,x)$ *for a.a.* $x \in E$*, with* $\mathcal{L}^d(E) > 0$*. Hence* $\int_E z(t_2) - z(t_1)\,dx =: \alpha > 0$*. But thanks to the strong* $L^2(\Omega)$*-convergence pointwise in time, cf.* (76g)*, and with the aid of the control* (75) *on the Yosida-regularization we* *deduce*

$$\begin{aligned}0 < \alpha &= \int_E (z(t_2) - z(t_1))_+ \,\mathrm{d}x = \lim_{\tau\to 0}\int_E (\bar z_\tau(t_2) - \bar z_\tau(t_1))_+ \,\mathrm{d}x\\ &\le \lim_{\tau\to 0} \|(\bar z_\tau(t_2) - \bar z_\tau(t_1))_+\|_{L^1(\Omega)} \le \lim_{\tau\to 0}\sum_{k=1}^{N_\tau} \|(z_\tau^k - z_\tau^{k-1})_+\|_{L^1(\Omega)}\\ &\le \lim_{\tau\to 0}\mathcal{L}^{d+1}([0,\mathsf{T}]\times\Omega)\sqrt{\frac{2C\tau}{\mathsf{T}}} = 0\,,\end{aligned}$$

which states a contradiction. Hence the assertion is proven. □

5.2.2 Proof of the weak momentum balance (10c) for all $t \in [0,\mathsf{T}]$ and regularity (77) of the limit function u

Let

$$\tilde v \in L^2(0,\mathsf{T};\mathbf{U}) \cap H^1(0,T;L^2(\Omega,\mathbb{R}^d)) \tag{98}$$

be a test function for the weak momentum equation in (10c)*. We define*

$$v_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} v(r)\,\mathrm{d}r \tag{99}$$

and set the interpolants $\overline{v}_\tau$ and v_τ as in (43). With this definition there holds

$$\overline{v}_\tau \to v \text{ strongly in } L^2(0,\mathsf{T};\mathbf{U}) \tag{100}$$

and

$$v_\tau \to v \text{ strongly in } H^1(0,\mathsf{T};L^2(\Omega,\mathbb{R}^d)). \tag{101}$$

The latter implies $v_\tau(t) \to v(t)$ strongly in $L^2(\Omega,\mathbb{R}^d)$ everywhere in $[0,\mathsf{T}]$. Since (98) *implies that $v \in C([0,\mathsf{T}],L^2(\Omega))$ we conclude that also*

$$\overline{v}_\tau(t) \to v(t) \text{ strongly in } L^2(\Omega,\mathbb{R}^d). \tag{102}$$

Limit passage in the weak balance of momentum: In the time-discrete balance of momentum (38b) *the acceleration term is now rewritten using the discrete integration-by-parts formula* (46)

$$\tau\sum_{k=1}^L\int_\Omega \tfrac{\dot u_\tau^k - \dot u_\tau^{k-1}}{\tau}\cdot v_\tau^k\,\mathrm{d}x = \int_\Omega \left(\dot u_\tau^L\cdot v_\tau^L - \dot u_\tau^0\cdot v_\tau^0\right)\mathrm{d}x - \tau\sum_{k=1}^L\int_\Omega \dot u_\tau^{k-1}\cdot\tfrac{{v_\tau}^k - v_\tau^{k-1}}{\tau}\,\mathrm{d}x$$

to obtain

$$\begin{aligned}&\rho\int_\Omega \dot u_\tau(t)\cdot\overline{v}_\tau(t) - \dot u_\tau(0)\cdot v_\tau(0)\,\mathrm{d}x - \rho\int_0^{\bar t_\tau(t)}\int_\Omega \dot u_\tau(r-\tau)\dot v_\tau(r)\,\mathrm{d}x\,\mathrm{d}r\\ &+\int_0^{\bar t_\tau(t)}\int_\Omega\Big[\mathbb{D}(\bar z_\tau)e(\dot u_\tau) + \mathbb{C}(\bar z_\tau)e(\bar u_\tau)\Big] : e(\overline{v}_\tau)\,\mathrm{d}x\,\mathrm{d}r = \int_0^{\bar t_\tau(t)}\left\langle \bar f_\tau, \overline{v}_\tau\right\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\,.\end{aligned}\tag{103}$$

Then, passing to the limit we conclude by weak-strong convergence arguments with the aid of convergences (76e) *and* (100) *that*

$$\rho\int_\Omega \dot u_\tau(t)\cdot\overline{v}_\tau(t) - \dot u_\tau(0)\cdot v_\tau(0)\,\mathrm{d}x \to \rho\int_\Omega \dot u(t)\cdot v(t) - \dot u(0)\cdot v(0)\,\mathrm{d}x \text{ for all } t\in[0,\mathsf{T}] \tag{104}$$

Moreover, convergences (76b) *and* (101) *lead to*

$$\rho\int_0^{\bar t_\tau(t)}\int_\Omega \dot u_\tau(r-\tau)\cdot\dot v_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \to \rho\int_0^t\int_\Omega \dot u(r)\cdot\dot v(r)\,\mathrm{d}x\,\mathrm{d}r\,, \tag{105}$$

where the convergence of the translated functions $\dot u_\tau$ follows by an $\frac{\varepsilon}{3}$-argument using the density of smooth and compactly supported functions in $L^2(0,\mathsf{T};\mathbf{U})$. In addition, by (96a) *we also have $\bar f_\tau \to f$ in $L^p(0,\mathsf{T};\mathbf{U}^*)$ for $1\le p<\infty$, and thus*

$$\int_0^{\bar t_\tau(t)} \left\langle \bar f_\tau(r), \bar v_\tau(r)\right\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r \to \int_0^t \left\langle f(r), v(r)\right\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r$$

follows as well. For the convergence of the quadratic terms, we first realize that convergence (76g) *implies by the dominated convergence theorem that*

$$\bar z_\tau \to z \text{ strongly in } L^2\big(0,\mathsf{T};L^2(\Omega)\big)\,.$$

From this together with (100)*, and with the isometric isomorphism*

$$L^2\big(0,\mathsf{T};L^2(\Omega,\mathbb{R}^m)\big) \cong \{\tilde u\colon [0,T]\times\Omega\to\mathbb{R}^m,\ \textstyle\int_0^{\mathsf{T}}\big(\int_\Omega |\tilde u(t,x)|^2\,\mathrm{d}x\big)\,\mathrm{d}t<\infty\}\,,$$

where $m=d^2+1$ it follows that, up to a subsequence,

$$(\bar z_\tau(t,x), e(\bar v_\tau(t,x))) \to (z(t,x), e(v(t,x)))$$

pointwise for almost all $(t,x)\in[0,\mathsf{T}]\times\Omega$. Then, by continuity of $|\cdot|$, $\mathbb{D}$ and $\mathbb{C}$, cf. (14a)*, we obtain for a.a. $t\in[0,\mathsf{T}]$*

$$\big|[\mathbb{D}\big(\bar z_\tau(t)\big)+\mathbb{C}\big(\bar z_\tau(t)\big)]e\big(\bar v_\tau(t)\big)\big| \to \big|[\mathbb{D}\big(z(t)\big)+\mathbb{C}\big(z(t)\big)]e\big(v(t)\big)\big|$$

pointwise almost everywhere in Ω. In view of (15a) *and* (15b) *a summable L^2-majorant is given by $(c_{\mathbb{D}}^*+c_{\mathbb{D}}^*)\,|e(\bar v_\tau(t))|$ and we conclude by a version of the dominated convergence theorem with τ-dependent majorants, cf. [RF17, Sec. 4.4, Thm. 19, p. 89], that*

$$[\mathbb{D}\big(\bar z_\tau(t)\big)+\mathbb{C}\big(\bar z_\tau(t)\big)]e\big(\bar v_\tau(t)\big) \to [\mathbb{D}\big(z(t)\big)+\mathbb{C}\big(z(t)\big)]e\big(v(t)\big)$$

strongly in $L^2((0,\mathsf{T})\times\Omega;\mathbb{R}^{d\times d})$. In view of (76a) and (76b), which imply that $e(\bar u_\tau)\rightharpoonup e(u)$ as well as $e(\dot u_\tau)\rightharpoonup e(\dot u)$ weakly in $L^2(0,\mathsf{T};L^2(\Omega,\mathbb{R}^{d\times d}))$, it can be concluded by symmetry of $\mathbb{D}$ and $\mathbb{C}$, and again by weak-strong convergence arguments, that

$$\int_0^{\bar t_\tau}\!\!\int_\Omega \Big[\mathbb{D}(\bar z_\tau)e(\dot u_\tau)+\mathbb{C}(\bar z_\tau)e(\bar u_\tau)\Big]{:}e(\bar v_\tau)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega \Big[\mathbb{D}(z)e(\dot u)+\mathbb{C}(z)e(u)\Big]{:}e(v)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Altogether we conclude that (10c) *is satisfied for all $t\in[0,\mathsf{T}]$.* □

Regularity (77) *of the limit function u: So far, convergences* (76a)–(76c) *provide*

$$u\in H^1(0,\mathsf{T};\mathbf{U})\cap L^\infty(0,\mathsf{T};\mathbf{U})\cap W^{1,\infty}(0,\mathsf{T};L^2(\Omega))\,.$$

In view of [Bre73, Appendix, p. 140] this implies that $u:[0,\mathsf{T}]\to\mathbf{U}$ is absolutely continuous and hence we also have

$$u\in C^0([0,\mathsf{T}];\mathbf{U})\,.$$

This provides regularity statement (77a)*. We now turn to regularity statement* (77b) *for $\ddot u$: From the a priori bound* (48d) *we infer the existence of a subsequence $(\dot u_\tau)_\tau$ and of an element $\xi\in L^2(0,\mathsf{T};\mathbf{U}^*)$ such that $\mathrm{D}_\tau\dot u_\tau\rightharpoonup\xi$ in $L^2(0,\mathsf{T};\mathbf{U}^*)$. In view of the strong convergences* (101) *&* (102) *of the approximating test functions, the discrete integration-by-parts formula* (46) *and of the already deduced limits* (104) *and* (105)*, we see that*

$$\begin{aligned}
&\int_0^{\bar t_\tau}\langle\rho\,\mathrm{D}_\tau\dot u_\tau(r),\bar v_\tau(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\\
&=\rho\int_\Omega\big(\dot u_\tau(t)\cdot\bar v_\tau(t)-\dot u_\tau(0)\cdot v_\tau(0)\big)\,\mathrm{d}x-\rho\int_0^{\bar t_\tau(t)}\int_\Omega\dot u_\tau(r-\tau)\dot v_\tau(r)\,\mathrm{d}x\,\mathrm{d}r\\
&\qquad\qquad\downarrow\\
&\int_0^t\langle\rho\,\xi(r),v(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\\
&=\rho\int_\Omega\big(\dot u(t)\cdot v(t)-\dot u(0)\cdot v(0)\big)\,\mathrm{d}x-\rho\int_0^t\int_\Omega\dot u(r)\cdot\dot v(r)\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}\tag{106}$$

for all test functions $v \in H^1(0,\mathsf{T};L^2(\Omega)) \cap L^2(0,\mathsf{T};\mathbf{U})$*. This shows that*

$$\xi = \ddot{u} \in L^2(0,\mathsf{T};\mathbf{U}^*)\,,$$

and (106) *states an integration by parts formula for the limit function* u*. Moreover, since the spaces* $\mathbf{U} \subset L^2(\Omega;\mathbb{R}^d) \subset \mathbf{U}^*$ *form an evolution triple, in view of e.g. [Rou06, Lemma 7.3, p. 191] we also have an integration-by-parts formula for* $\dot{u}$

$$\begin{aligned}\int_s^t \langle \ddot{u}(r), \dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}} &= \frac{1}{2}\langle \dot{u}(t), \dot{u}(t)\rangle_{\mathbf{U}^*,\mathbf{U}} - \frac{1}{2}\langle \dot{u}(s), \dot{u}(s)\rangle_{\mathbf{U}^*,\mathbf{U}}\\ &= \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} - \frac{1}{2}\|\dot{u}(s)\|^2_{L^2(\Omega;\mathbb{R}^d)}\,.\end{aligned}$$

This concludes the proof of statements (77). □

5.2.3 Rate-independent evolution of the phase-field variable (10a) & (11)

We first show that the limit pair (u,z) *satisfies the one-sided variational inequality* (10a) *for a.a.* $t \in (0,\mathsf{T})$*, which provides a rate-independent evolution law for* z*. From this, we will deduce by convexity arguments that also semistability inequality* (11) *is valid.*

Proof of the one-sided variational inequality (10a)**.** *We test the time-discrete evolution equation* (47a) *with functions* $\eta \in \mathbf{Y}$ *with the property* $\eta \le 0$ *a.e. in* Ω*. Then, omitting the negative term* $\int_\Omega N_\tau(\dot{z}_\tau(t))_+\eta\,\mathrm{d}x$*, one obtains after rearranging*

$$\begin{aligned}&\int_\Omega \Big[-\frac{1}{\ell}\big(1-\bar{z}_\tau(t)\big) + M\dot{z}_\tau(t)\Big]\eta + \ell\nabla\bar{z}_\tau(t)\cdot\nabla\eta\,\mathrm{d}x\\ &\quad\ge \int_\Omega \big[\frac{1}{2}\mathbb{C}'\big(\bar{z}_\tau(t)\big)e\big(\underline{u}_\tau(t)\big) : e\big(\underline{u}_\tau(t)\big)\big](-\eta)\,\mathrm{d}x \ge 0\,.\end{aligned} \tag{107}$$

To pass to the limit in this inequality we want to make use of lower semicontinuity arguments on the right-hand side and upper semicontinuity on the left-hand side. For this we note that the term $\int_\Omega M\dot{z}_\tau(t)\eta\,\mathrm{d}x$ *cannot be handled pointwise in time. Hence, in the following we consider an arbitrary measureable set* $I \subset [0,T]$*. We integrate* (107) *over* I

$$\begin{aligned}&\int_I\int_\Omega \Big[-\frac{1}{\ell}\big(1-\bar{z}_\tau(t)\big) + M\dot{z}_\tau(t)\Big]\eta + \ell\nabla\bar{z}_\tau(t)\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\ge \int_I\int_\Omega \big[\frac{1}{2}\mathbb{C}'\big(\bar{z}_\tau(t)\big)e\big(\underline{u}_\tau(t)\big) : e\big(\underline{u}_\tau(t)\big)\big](-\eta)\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned} \tag{108}$$

and aim to pass to the limit in (108) *using lower and upper semicontinuity arguments.*

We first discuss the limit passage on the left-hand side by upper semicontinuity. In fact, the limes superior of the left-hand side of (108) *is further estimated by*

$$\begin{aligned}&\limsup_{\tau\to0}\int_I\int_\Omega \Big[-\frac{1}{\ell}\big(1-\bar{z}_\tau(t)\big) + M\dot{z}_\tau(t)\Big]\eta + \ell\nabla\bar{z}_\tau(t)\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\le \limsup_{\tau\to0}\int_I\int_\Omega M\dot{z}_\tau(t)\eta\,\mathrm{d}x\,\mathrm{d}t + \limsup_{\tau\to0}\int_I\int_\Omega \big[-\frac{1}{\ell}\big(1-\bar{z}_\tau(t)\big)\eta + \ell\nabla\bar{z}_\tau(t)\cdot\nabla\eta\big]\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{109}$$

For the first term on the right-hand side of (109) *we exploit the bound* (48f) *that provides*

$$\sqrt{M}\|\dot{z}_\tau\|_{L^2(0,T;L^2(\Omega))} \le C,$$

and also use that $M(\tau) \to 0$ *as* $\tau \to 0$*. In this way we obtain*

$$\begin{aligned}\limsup_{\tau\downarrow0}\left|\int_I\int_\Omega M\dot{z}_\tau(t)\eta\,\mathrm{d}x\,\mathrm{d}t\right| &\le \limsup_{\tau\downarrow0}\, M\,\|\dot{z}_\tau\|_{L^2(I,L^2(\Omega))}\,\|\eta\|_{L^2(\Omega)}\,\mathcal{L}^1(I)^{\frac{1}{2}}\\ &\le \limsup_{\tau\downarrow0}\,\sqrt{M}\,C\,\|\eta\|_{L^2(\Omega)}\,\mathcal{L}^1(I)^{\frac{1}{2}} = 0\,.\end{aligned} \tag{110}$$

For the second term on the right-hand side of (109) *we find with convergence* (76h)

$$\begin{aligned}\lim_{\tau\downarrow 0}\int_I\int_\Omega\Big[-\frac{1}{\ell}\big(1-\bar z_\tau(t)\big)\Big]\eta+\ell\nabla\bar z_\tau(t)\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ =\int_I\int_\Omega\Big[-\frac{1}{\ell}\big(1-z(t)\big)\Big]\eta+\ell\nabla z(t)\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\tag{111}$$

Thus (110) *and* (111) *provide an estimate for the limit superior of the left-hand side of* (108).

We now aim to pass to the limit on the right-hand side of (108) *by weak lower semicontinuity. For this we observe that property* (14b) *for the degradation function implies that $\mathbb{C}'(z)$ is positive definite. Hence, $-\eta\mathbb{C}'(z)$ is positive semidefinite thanks to $-\eta\geq 0$ a.e. in Ω. Invoking the lower semicontinuity result [Dac12, Thm. 3.4, p. 74] we conclude that the functional $(z,\xi)\mapsto\int_\Omega(-\eta)\mathbb{C}'(z)e(\xi):e(\xi)\,\mathrm{d}x$ is lower semi-continuous with respect to convergences* (76d) *and* (76g)*. Hence,*

$$\begin{aligned}\liminf_{\tau\downarrow 0}\int_\Omega\big[\frac{1}{2}\mathbb{C}'\big(\bar z_\tau(t)\big)e\big(\underline{u}_\tau(t)\big):e\big(\underline{u}_\tau(t)\big)\big](-\eta)\,\mathrm{d}x\\ \geq\int_\Omega\big[\frac{1}{2}\mathbb{C}'\big(z(t)\big)e\big(u(t)\big):e\big(u(t)\big)\big](-\eta)\,\mathrm{d}x\geq 0\end{aligned}$$

for all $t\in[0,\mathsf{T}]$. Then Fatou's lemma yields

$$\begin{aligned}\liminf_{\tau\downarrow 0}\int_I\int_\Omega\big[\frac{1}{2}\mathbb{C}'\big(\bar z_\tau(t)\big)e\big(\underline{u}_\tau(t)\big):e\big(\underline{u}_\tau(t)\big)\big](-\eta)\,\mathrm{d}x\,\mathrm{d}t\\ \geq\int_I\int_\Omega\big[\frac{1}{2}\mathbb{C}'\big(z(t)\big)e\big(u(t)\big):e\big(u(t)\big)\big](-\eta)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\tag{112}$$

Putting together (108)–(112) *it follows for the limit that*

$$\int_I\int_\Omega\big[\frac{1}{2}\mathbb{C}'\big(z(t)\big)e\big(u(t)\big):e\big(u(t)\big)-\frac{1}{\ell}\big(1-z(t)\big)\big]\eta+\ell\nabla z(t)\cdot\nabla\eta\,dx\,\mathrm{d}t\geq 0$$

holds for every measurable set $I\subset[0,\mathsf{T}]$. This implies that

$$\int_\Omega\Big(\big[\frac{1}{2}\mathbb{C}'\big(z(t)\big)e\big(u(t)\big):e\big(u(t)\big)-\frac{1}{\ell}\big(1-z(t)\big)\big]\eta+\ell\nabla z(t)\cdot\nabla\eta\Big)\,\mathrm{d}x\geq 0\tag{113}$$

for almost every $t\in(0,\mathsf{T})$ *and for all test functions $\eta\in\mathbf{Y}$ with $\eta\leq 0$ a.e. in Ω, that is* (10a). ☐

Proof of the semistability inequality (11)**.** *The one-sided variational inequality* (10a) *is now used to show semistability* (11). *Thanks to* (97) *we have $0\leq z(t)\leq 1$ a.e. in Ω for all $t\in[0,\mathsf{T}]$. By assumptions* (14d) *the interval $[0,1]$ is contained in the convexity regime of the degradation function, so that the functional $\mathcal{E}(t,\cdot,u(t))$ is convex. Hence, for any test function $\tilde z$ with $0\leq\tilde z\leq z(t)$ a.e. in Ω it follows from* (10a) *by convexity*

$$0\geq\langle-\mathrm{D}_z\mathcal{E}(t,u(t),z(t)),\tilde z-z(t)\rangle_{\mathbf{X}^*,\mathbf{X}}\geq\mathcal{E}\big(t,u(t),z(t)\big)-\mathcal{E}(t,u(t),\tilde z)\tag{114}$$

for a.e. $t\in(0,\mathsf{T})$. In view of the definition of $\mathcal{R}$ in (5) *this implies*

$$\mathcal{E}\big(t,u(t),z(t)\big)\leq\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}\big(\tilde z-z(t)\big)\tag{115}$$

for all $\tilde z\in\mathbf{X}$ with $0\leq\tilde z\leq 1$ a.e. in Ω for a.e. $t\in(0,\mathsf{T})$. ☐

5.2.4 Proof of the energy-dissipation balance (10d) for a.e. $t\in(0,\mathsf{T})$.

We first pass to the limit in the time-discrete upper energy dissipation estimate (47d) *by exploiting weak lower semicontinuity arguments on its left-hand side and the well-preparedness of the given data on its right-hand side. Secondly, the energy-dissipation balance* (10d) *will be concluded by exploiting the already deduced weak momentum balance* (10c) *and* *semistability* (11) *for the limit pair* (u, z) *in a Riemann-sum argument as commonly used for rate-independent systems, cf. e.g., [DMFT05, MR06, MR15b]. Note that our proofs provide the weak momentum balance to hold for all* $t \in [0, \mathsf{T}]$, *cf. Sec. 5.2.2, whereas the semistability inequality so far has been deduced in Sec. 5.2.3 to hold for a.e.* $t \in (0, \mathsf{T})$, *only. This is why we here as a first step find the energy-dissipation balance* (10d) *to be valid for a.e.* $t \in (0, \mathsf{T})$, *only. Yet, this gives the basis to apply the regularity result stated in Theorem 5.2 to obtain the temporal continuity of* z *and thus to conclude that* (10d) *holds true for all* $t \in [0, \mathsf{T}]$; *we refer to the subsequent Sec. 5.3 for this proof.*

Proof of an upper energy-dissipation estimate for all $t \in [0, \mathsf{T}]$**:** *We pass to the limit in* (47d) *by adapting the arguments of [LRTT18, Lemma 4.4]. We first discuss the limit passage on the left-hand side of* (47d) *exploiting the weak lower semicontinuity and positivity of the functionals involved. In difference to [LRTT18] in* (47d) *there also appears the viscous contribution of the damage evolution. For all* $\tau > 0$ *this term is non-negative, so that we estimate it from below by* $\int_0^t 2\mathcal{R}_{M\tau}(\dot{z}_\tau(r))\,\mathrm{d}r \geq 0$. *For the viscous dissipation of the displacements we argue by weak lower semicontinuity. For this, we realize that the map* $(z, \xi) \mapsto \mathbb{D}(z)e(\xi) : e(\xi)$ *is continuous and that the map* $\xi \mapsto \mathbb{D}(z)e(\xi) : e(\xi)$ *is convex for all* $(z, \xi) \in \mathbb{R} \times \mathbb{R}^{d\times d}$ *by the assumptions on regularity and positive definiteness of* $\mathbb{D}$ *in* (13) *and* (14). *Thus, [Dac12, Theorem 3.4, p. 74] provides the lower semicontinuity of the functional* $\mathcal{V}$ *with respect to the topologies given by* (76e) *and 76g), so that we find* $\liminf_{\tau\to 0} \mathcal{V}(\bar{z}_\tau(r); \dot{u}_\tau(r)) \geq \mathcal{V}(z(r); \dot{u}(r)) \geq 0$ *for all* $r \in [0, \mathsf{T}]$, *also thanks to the positive definiteness of* $\mathbb{D}$. *This justifies the application of Fatou's lemma, so that we conclude*

$$\begin{aligned}\liminf_{\tau\to 0} \int_0^{\bar{t}_\tau(t)} 2\mathcal{V}(\bar{z}_\tau(r); \dot{u}_\tau(r))\,\mathrm{d}r &\geq \int_0^t \liminf_{\tau\to 0} 2\mathcal{V}(\bar{z}_\tau(r); \dot{u}_\tau(r))\,\mathrm{d}r \\ &\geq \int_0^t 2\mathcal{V}(z(r); u(r))\,\mathrm{d}r\,,\end{aligned}$$

where we also used that $\bar{t}_\tau(t) \geq t$ *for all* $t \in [0, \mathsf{T}]$ *by construction* (44).

For the kinetic energy we also have $\liminf_{\tau\to 0} \mathcal{K}(u_\tau(t)) \geq \mathcal{K}(u(t))$ *for all* $t \in [0, \mathsf{T}]$ *by the weak convergence* (76e) *and thanks to the weak lower semicontinuity of the* $L^2(\Omega, \mathbb{R}^d)$*-norm.*

We now comment on the weak lower semi-continuity of $\bar{\mathcal{E}}_\tau$*: With the same arguments as for* $\mathcal{V}$, *making use of [Dac12, Theorem 3.4, p. 74], we deduce that the stored elastic energy* $(z, u) \mapsto \int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\mathrm{d}x$ *is lower semicontinuous with respect to the topologies given by* (76d) *and* (76g) *and also that the phase-field energy* $(z, \xi) \mapsto \int_\Omega \frac{\ell}{2}|\xi|^2 + \frac{1}{2\ell}(1-z)^2\,\mathrm{d}x$ *is lower semicontinuous with respect to the topologies* (76f) *and* (76g). *Additionally, the convergence of the external loading term follows from the strong convergence* (96c) *together with the weak convergence* (76d). *In this way, we pass to the limit on the left-hand side of* (47d).

As for the right-hand side of (47d), *we realize that* $\mathcal{K}(u_\tau) + \bar{\mathcal{E}}_\tau(0, \bar{u}_\tau(0), \bar{z}_\tau(0)) = \mathcal{K}(\dot{u}_0) + \bar{\mathcal{E}}_\tau(0, u_0, z_0)$ *is constant for all* $\tau > 0$. *In the power of the external loadings we pass to the limit using the uniform bounds on* $\dot{f}_\tau$ *and* $\underline{u}_\tau$ *and having* $\bar{t}_\tau(t) \geq t$ *for all* $t \in [0, \mathsf{T}]$, *supplemented by strong* $H^1(0, \mathsf{T}; \mathbf{U}^*)$*-convergence* (96d) *of* $(f_\tau)_\tau$ *guaranteed by the regularity assumption* (17), *and weak* $L^\infty(0, \mathsf{T}; \mathbf{U})$*-convergence of* $(\underline{u}_\tau)_\tau$. *In this way we conclude the upper energy dissipation estimate for the limit system* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_1, \mathcal{E})$

$$\begin{aligned}&\mathcal{K}(\dot{u}(t)) + \mathcal{E}(t, u(t), s(t)) + \int_0^t 2\mathcal{V}(z(r); \dot{u}(r))\,\mathrm{d}r \\ &\leq \mathcal{K}(\dot{u}_0) + \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_t\mathcal{E}\big(r, u(r), z(r)\big)\,\mathrm{d}r\,.\end{aligned} \tag{116}$$

for all $t \in [0, \mathsf{T}]$. □

Proof of the energy-dissipation balance (10d) **for a.e.** $t \in (0, \mathsf{T})$**.** *We now discuss that* (116) *even holds as an* equality *for a.e.* $t \in (0, \mathsf{T})$. *For this, we follow standard arguments for rate-independent systems, cf. e.g. [DMFT05, MR06, MR15b] and also [RT17] for abstract results on coupled rate-independent/rate-dependent systems, which deduce a lower energy-dissipation estimate opposite to* (116) *by exploiting a Riemann-sum argument using the momentum balance* (10c) *and the semistability inequality* (11) *of the* limit system. *We only point out here the main ingredients and refer to [LRTT18, Sec. 4.3] for the details of the calculation.*

So far, semistability inequality (11) *is valid a.e. in* $(0, \mathsf{T})$, *only. Hence, let* $t \in (0, \mathsf{T})$ *be such that* (11) *holds true. Moreover, it is possible to choose a sequence of partitions* $(\Pi_\theta)_\theta$ *with* $\Pi_\theta = \{0 = t_\theta^0 < t_\theta^1 < \ldots < t_\theta^{N_\theta} = t\}$ *of the interval* $[0, t]$

such that (11) *also holds true for the collection of nodes and such that also*

$$\lim_{\theta\downarrow 0}\sum_{k=1}^{N_\theta}\int_{t_\theta^{k-1}}^{t_\theta^k}\int_\Omega \mathbb{C}\big(z(t_\theta^k)\big)e\big(u(r)\big):e\big(\dot u(r)\big)\,\mathrm{d}x\,\mathrm{d}r \\ =\int_0^t\int_\Omega \mathbb{C}\big(z(r)\big)e\big(u(r)\big):e\big(\dot u(r)\big)\,\mathrm{d}x\,\mathrm{d}r\,; \tag{117}$$

for this, see also Remark 6.3. Semistability inequality (11) *for the limit pair* (u,z) *at time* t_θ^{k-1} *is now tested with* z_θ^k*, which is a bounded test function by* (41) *and ensures that* $\mathcal{R}(z_\theta^k-z_\theta^{k-1})=0$ *by unidirectionality property* (10b)*. Summing up over* $k\in\{1,\ldots,N_\theta\}$ *and taking the limit as* $\theta\downarrow 0$ *results in*

$$\mathcal{E}\big(0,u(0),s(0)\big)\le\mathcal{E}\big(t,u(t),z(t)\big)-\int_0^t\int_\Omega \mathbb{C}\big(z(r)\big)e\big(u(r)\big):e\big(\dot u(r)\big)\,\mathrm{d}x\,\mathrm{d}r \\ +\int_0^t\langle f(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r-\int_0^t\partial_t\mathcal{E}(r,u(r),s(r))\,\mathrm{d}r\,. \tag{118}$$

where one also uses (117) *and that* $-\Big\langle \dot f(r),u(r)\Big\rangle_{\mathbf{U}^*,\mathbf{U}}=\partial_t\mathcal{E}(r,u(r),s(r))$.

Secondly, the weak momentum balance (10c) *at time* t *is tested by* $\dot u$*. This is admissible thanks to the regularity statements* (77) *already deduced in Sec. 5.2.2. Applying the integration-by-parts formula* (77c) *to the kinetic term then results in*

$$\frac{\rho}{2}\|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^d)}+\int_0^t\int_\Omega\big[\mathbb{D}(s)e(\dot u)+\mathbb{C}(s)e(u)\big]:e(\dot u)\,\mathrm{d}x\,\mathrm{d}r \\ =\frac{\rho}{2}\|\dot u(0)\|^2_{L^2(\Omega,\mathbb{R}^d)}+\int_0^t\langle f(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r \tag{119}$$

Summing up (118) *and* (119) *ultimately yields*

$$\mathcal{E}\big(0,u(0),z(0)\big)\le\mathcal{E}\big(t),u(t),z(t)\big)+\frac{\rho}{2}\|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^d)}-\frac{\rho}{2}\|\dot u(0)\|^2_{L^2(\Omega,\mathbb{R}^d)} \\ +\int_0^t\int_\Omega\mathbb{D}(z(r))e(\dot u(r)):e(\dot u(r))\,\mathrm{d}x\,\mathrm{d}r-\int_0^t\partial_r\mathcal{E}(r,u(r),s(r))\,\mathrm{d}r\,,$$

which is the estimate opposite to (116)*. In this way, the energy-dissipation balance* (10d) *is deduced to hold for a.e.* $t\in(0,\mathsf{T})$. ☐

5.3 Proof of the temporal Hölder-continuity $z\in C^{0,1/4}([0,\mathsf{T});\mathbf{X})$ and validity of properties (10a), (11) & (10d) for all $t\in[0,\mathsf{T})$

To deduce that the limit z *has the temporal Hölder-regularity* (78) *assumptions A1)-A8) of Theorem 5.2 have to be checked. To this end, we collect the corresponding properties of* $\mathcal{E}$ *from* (8) *and* $\mathcal{V}$ *from* (6) *in the following*

Lemma 5.4. *Let* $\mathcal{E}$ *and* $\mathcal{V}$ *be given as in* (8) *and* (6) *such that assumptions* (12)–(18) *hold true. The following statements are valid for the energy functional* $\mathcal{E}$*:*

1. *Let* $\mathsf{D}_c:=\{\tilde z\in\mathbf{X},\,0\le\tilde z\le z_*$ *a.e. in* $\Omega\}$ *denote the convexity regime of* $\mathcal{E}(t,u,\cdot)$ *in accordance with* (14)*. Then for all* $t\in[0,\mathsf{T}]$ *and* $u:[0,\mathsf{T}]\to\mathbf{U}$ *a solution of* (10c) *the energy functional* $\mathcal{E}(t,u(t),\cdot):\mathsf{D}_c\to\mathbb{R}$ *is uniformly convex. More precisely, it satisfies inequality* (84) *with the constants* $\alpha=2$, $C_\star=\min\big\{\frac{\ell}{2},\frac{1}{2\ell}\big\}$*, and the Banach space* $\mathbf{S}=\mathbf{X}$*.*
2. *The functional* $\mathcal{E}(t,\cdot,z):\mathbf{U}\to\mathbb{R}$ *satisfies Hölder estimate* (85) *with* $\beta_u=1$ *and a constant* $c_\star=c_\star(E,f)>0$.
3. *The functional* $\mathcal{E}(t,\cdot,z):\mathbf{U}\to\mathbb{R}$ *is Gâteaux-differentiable and it satisfies the gradient estimate* (86) *with the exponent* $\sigma=2$.

Moreover, the dissipation potential $\mathcal{V}$ has p-growth for $p = 2$, i.e.,

$$\mathcal{V}(z; v) \geq \frac{c_{\mathbb{D}}^0}{2C_{\mathrm{K}}^2}\|v\|_{\mathbf{U}}^2 \quad \textit{for all } (z,v) \in \mathbf{Z} \times \mathbf{U}\,, \tag{120}$$

with Korn's constant $C_{\mathrm{K}} > 0$ and $c_{\mathbb{D}}^0 > 0$ from coercivity assumption (15a).

Proof. To Item 1., uniform convexity: Recall that the stored elastic energy functional $z \mapsto \int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x$ is convex for all $z \in \mathrm{D}_c$ by assumption (14). Moreover, for the phase-field functional $z \mapsto \int_\Omega \frac{\ell}{2}|\nabla z|^2 + \frac{1}{2\ell}|z|^2 \mathrm{d}x$ we see that the quadratic map $a \mapsto c|a|^2$ satisfies for all $a_1, a_2 \in \mathbb{K} \in \{\mathbb{R}, \mathbb{R}^d\}$ and for all $\lambda \in [0,1]$

$$c|\lambda a_1 + (1-\lambda)a_2|^2 = \lambda c|a_1|^2 + (1-\lambda)c|a_2|^2 - \lambda(1-\lambda)c|a_1 - a_2|^2$$

From that we conclude the statement when setting $a_i = \nabla z_i$ with $c = \frac{\ell}{2}$ and $a_i = (1 - z_i)$ with $c = \frac{1}{2\ell}$, for $i = 1, 2,$ and by adding the two results.

To Item 2., Hölder-continuity of $\mathcal{E}$: Let (u_i, z_1) so that $\sup_{t\in[0,\mathsf{T}]} \mathcal{E}(t, u_i, z_1) \leq E$ for $i = 0, 1$. In view of assumptions (15b) on $\mathbb{C}$ and (17) on f we find for all $t \in [0, \mathsf{T}]$

$$\begin{aligned}
&\big|\mathcal{E}(t,u_1,z_1) - \mathcal{E}(t,u_0,z_1)\big| \\
&= \Big|\int_\Omega \frac{1}{2}\mathbb{C}(z_1)\big(e(u_1)) : e(u_1)) - e(u_0)) : e(u_0))\big)\,\mathrm{d}x - \langle f(t), u_1 - u_0\rangle_{\mathbf{U}^*,\mathbf{U}}\Big| \\
&\leq \frac{c_{\mathbb{C}}^*}{2}\big(\|u_1\|_{\mathbf{U}} + \|u_0\|_{\mathbf{U}}\big)\|u_1 - u_0\|_{\mathbf{U}} + \sup_{t\in[0,\mathsf{T}]}\|f(t)\|_{\mathbf{U}^*}\|u_1 - u_0\|_{\mathbf{U}} \leq c_\star\|u_1 - u_0\|_{\mathbf{U}}\,,
\end{aligned}$$

Here we also checked that $\|u_1\|_{\mathbf{U}} + \|u_0\|_{\mathbf{U}} \leq 2\big(E + \frac{c_{\mathrm{K}}^2}{2c_{\mathbb{C}}^0}\sup_{t\in[0,\mathsf{T}]}\|f(t)\|_{\mathbf{U}^*}^2\big)^{1/2}$ by repeating the calculations for a priori bound (71).

To Item 3., Gâteaux-differentiability of $\mathcal{E}$ and gradient estimate (86)*:* Gâteaux-differentiability of the functional $u \mapsto \mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x - \langle f(t), u\rangle_{\mathbf{U}}$ is clear and we now deduce gradient estimate (86). For this, we calculate

$$\begin{aligned}
\|\mathrm{D}_u\mathcal{E}(t,u,z)\|_{\mathbf{U}^*} &= \sup_{\substack{v\in\mathbf{U}\\ \|v\|_{\mathbf{U}}=1}} \langle \mathrm{D}_u\mathcal{E}(t,u,z), v\rangle_{\mathbf{U}^*,\mathbf{U}} \\
&\leq \sup_{\substack{v\in\mathbf{U}\\ \|v\|_{\mathbf{U}}=1}} \Big(\frac{c_{\mathbb{C}}^*}{2}\|e(u)\|_{L^2(\Omega)}\|e(v)\|_{L^2(\Omega)} + \sup_{t\in[0,\mathsf{T}]}\|f(t)\|_{\mathbf{U}^*}\|v\|_{\mathbf{U}}\Big) \\
&\leq \frac{c_{\mathbb{C}}^*}{2}\Big(\frac{4C_{\mathrm{K}}}{c_{\mathbb{C}}^0}\Big(\int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x - \langle f(t), u\rangle_{\mathbf{U}}\Big)\Big)^{1/2} + \sup_{t\in[0,\mathsf{T}]}\|f(t)\|_{\mathbf{U}^*} \\
&\leq \max\Big\{\frac{c_{\mathbb{C}}^*}{2}, \sup_{t\in[0,\mathsf{T}]}\|f(t)\|_{\mathbf{U}^*}\Big\}\Big(\frac{4C_{\mathrm{K}}}{c_{\mathbb{C}}^0}\Big(\int_\Omega \frac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x - \langle f(t), u\rangle_{\mathbf{U}}\Big) + 1\Big)^{1/2},
\end{aligned}$$

which shows that $\|\mathrm{D}_u\mathcal{E}(t,u,z)\|_{\mathbf{U}^*}^2 \leq \tilde{c}(\hat{c}\,\mathcal{E}(t,u,z) + 1)$ and thus establishes (86) with the exponent $\sigma = 2$. □

Consider now the pair $(u,z) : [0,\mathsf{T}] \to \mathbf{U} \times \mathbf{Z}$ obtained by convergences (76)*. Recall that the results of Sec. 5.2.3 and 5.2.4 already provide the semistability inequality* (11) *and the energy-dissipation balance* (10d) *to hold for a.e. $t \in [0,\mathsf{T}]$, i.e., for all $t \in [0,\mathsf{T}]\backslash N$ with the $\mathcal{L}^1$-null set N as in* (81)*. Balance* (10d) *also directly implies the upper energy-dissipation estimate* (82) *to be valid for all subintervals $[s,t] \subset [0,\mathsf{T}]$ with $s,t \in [0,\mathsf{T}]\backslash N$. Thus, assumptions A1) and A2) of Theorem 5.2 are satisfied. Moreover, regularity assumption A3) for u is clearly ensured by regularity statements* (77a) *&* (77b)*. In Section 5.2.2 we already verified that the weak momentum balance* (10c) *holds true for all $t \in [0,\mathsf{T}]$, which gives A9). We further note that above Lemma 5.4 also provides the validity of assumptions A5)–A8), and A10) while the power control A4) can be proven using coercivity of the system energy and the uniform bound on $\dot{f}$. Consequently, we are now in the position to conclude the temporal Hölder-continuity $z \in C^{0,1/4}([0,\mathsf{T});\mathbf{X})$ and the validity of properties* (10a)*,* (11)*, and* (10d) *on all of $[0,\mathsf{T})$ as a corollary:*

Corollary 5.5. *Let the assumptions of Lemma 5.4 be satisfied and let the variational inequality* (10a) *hold true for the initial datum* (u_0, z_0)*. Then the functionals* $\mathcal{E}$ *and* $\mathcal{V}$ *comply with the assumptions A1)–A10) of Theorem 5.2 and thus, for the pair* (u, z) *obtained by convergences* (76)*, inequalities* (90) *and* (91) *are valid for all subintervals* $[s, t] \subset [0, \mathsf{T}]$ *with* $s, t \in [0, \mathsf{T}]\backslash N$.

1 *For all* $\hat{t} \in N \cap (0, \mathsf{T})$ *there are sequences* $(t_n^{\pm})_n \subset [0, \mathsf{T}]\backslash N$ *such that* $t_n^- \nearrow \hat{t}$, $t_n^+ \searrow \hat{t}$ *as* $n \to \infty$ *and,* $z^- = \lim_{n\to\infty} z(t_n^-) = \lim_{n\to\infty} z(t_n^+) = z^+$ *in* $\mathbf{X}$ *thanks to the validity of inequalities* (90) *and* (91) *for* $[t_n^-, t_n^+]$, $n \in \mathbb{N}$.
2 *Further let* $\mathcal{R}$ *as in* (5) *encode a unidirectional evolution of the rate-independent variable. Then* $z(t_n^+) \leq z(\hat{t}) \leq z(t_n^-)$ *for all* $n \in \mathbb{N}$ *and thus*

$$z^- = z(\hat{t}) = z^+ \text{ in } \mathbf{X}, \tag{121}$$

i.e. $z \in C^0([0, \mathsf{T}); \mathbf{X})$.
3 *In addition, let* $u \in C^0([0, \mathsf{T}], \mathbf{U})$*, as guaranteed by* (77a)*. Then, the one-sided variational inequality* (10a)*, semistability inequality* (11)*, and the energy-dissipation balance* (10d) *are valid even for all* $t \in [0, \mathsf{T})$*. Consequently, also estimate* (91) *is valid for all* $t \in [0, \mathsf{T})$ *and thus ensures the temporal Hölder-continuity* $z \in C^{0,h}([0, \mathsf{T}); \mathbf{X})$ *with the Hölder-exponent* $h = \frac{\beta_u}{2\alpha} = \frac{1}{4}$ *for* $\beta_u = 1$ *and* $\alpha = 2$ *obtained in Lemma 5.4.*

To see the continuity of z *in Item 2 of Corollary 5.5, consider* $\hat{t}_1,\ \hat{t}_2 \in [0, \mathsf{T}],\ \hat{t}_1 < \hat{t}_2$ *and sequences* $(t_n^1)_n$, $(t_n^2)_n \subset [0, \mathsf{T}] \setminus N$ *such that* $t_n^1 \nearrow \hat{t}_1$ *and* $t_n^2 \searrow \hat{t}_2$*. Then, one may estimate*

$$\|z(\hat{t}_1) - z(\hat{t}_2)\|_{\mathbf{X}} \leq \|z(\hat{t}_1) - z(t_n^1)\|_{\mathbf{X}} + \|z(t_n^1) - z(t_n^2)\|_{\mathbf{X}} + \|z(t_n^2) - z(\hat{t}_2)\|_{\mathbf{X}}$$

where the first and the third term on the right-hand side can be made arbitrarily small by Item 1 of Corollary 5.5 complemented by (121)*, and the second is estimated by* (90) *or* (91)*.*
We point out that the initial time $t = 0$ *is a (Hölder-) continuity point of* z*, since the variational inequality* (10a) *and thus semistability* (11) *are satisfied by assumption. Hence,* $0 \in [0, \mathsf{T}]\backslash N$ *and one obtains the validity of the inequalities* (90) *and* (91) *for intervals* $[0, t]$ *with* $t \in [0, \mathsf{T}]\backslash N$*. With the arguments of Cor. 5.5, Item 1., one can consider the limit* $t \nearrow 0$ *and thus conclude that* $z_0 = \lim_{t\to 0} z(t)$ *in* $\mathbf{X}$ *thanks to* (90) *and* (91)*. Instead, for the final time* T *it may happen that* $\mathsf{T} \in N$*, so that* (90) *and* (91) *are not guaranteed. Since one can only consider the limit from the left for sequences* $t \searrow \mathsf{T}$*, but not from the right, it is thus possible for* $z \in BV([0, \mathsf{T}], L^1(\Omega))$ *that* $z_{\mathsf{T}}^- := \lim_{t\to\mathsf{T}} z(t) > z(\mathsf{T})$ *with* $z(\mathsf{T})$ *the value extracted by convergences* (76)*. A solution of* (10) *that is (Hölder-) continuous on all of* $[0, \mathsf{T}]$ *can be rendered by replacing* $z(\mathsf{T})$ *with* z_{T}^-*.* □

5.4 Proof of Theorem 5.1, Item 4: Improved convergence (79c)

For the proof of the strong convergence (79a) *for the sequence* $(\dot{u}_\tau)_\tau$ *we refer to [LRTT18, L. 4.8].*

To conclude the strong convergences (79b) *and* (79c) *we shall exploit the validity of the energy-dissipation balance* (10d) *at all* $t \in [0, \mathsf{T})$ *for the limit pair* (u, z)*. More precisely, in view of the weak convergence results* $\bar{u}_\tau(t) \rightharpoonup u(t)$ *in* $\mathbf{U}$ *and* $\bar{z}_\tau(t) \rightharpoonup z(t)$ *in* $\mathbf{X}$ *by* (76d) *and* (76f) *in the separable, reflexive Banach spaces* $\mathbf{U}, \mathbf{X}$ *the strong convergence of the sequences can be concluded if also their norms can be shown to converge, i.e., if it can be shown that*

$$\|u(t)\|_{\mathbf{U}}^2 \leq \liminf_{\tau\to 0} \|\bar{u}_\tau(t)\|_{\mathbf{U}}^2 \quad \leq \quad \limsup_{\tau\to 0} \|\bar{u}_\tau(t)\|_{\mathbf{U}}^2 \leq \|z(t)\|_{\mathbf{U}}^2\,, \tag{122a}$$

$$\|z(t)\|_{\mathbf{X}}^2 \leq \liminf_{\tau\to 0} \|\bar{z}_\tau(t)\|_{\mathbf{X}}^2 \quad \leq \quad \limsup_{\tau\to 0} \|\bar{z}_\tau(t)\|_{\mathbf{X}}^2 \leq \|z(t)\|_{\mathbf{X}}^2\,. \tag{122b}$$

While the first set of inequalities in (122) *is due to weak convergence and the weak lower semicontinuity of the norms, the last set of inequalities in* (122) *will now be concluded with the aid of the energy-dissipation balance of the limit system.*

We first carry out the argument for $(\bar{z}_\tau)_\tau$ *to deduce the last inequality in* (122b)*. For this, at any time* $t \in [0, \mathsf{T})$*, we* *rearrange the discrete energy-dissipation inequality* (42) *as follows*

$$
\begin{aligned}
&\int_\Omega \Big(\frac{1}{2\ell}\big(1-\bar z_\tau(t)\big)^2+\frac{\ell}{2}\,|\nabla \bar z_\tau(t)|^2\Big)\,\mathrm{d}x\\
&\leq \mathcal{K}(\dot u_0)+\mathcal{E}(0,u_0,z_0)-\int_0^{\bar t_\tau(t)}\langle \dot f_\tau(r),\underline{u}_\tau(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\\
&\quad-\mathcal{K}(\dot u_\tau(t))-\int_\Omega\frac12\mathbb{C}(\bar z_\tau(t))e\big(\bar u_\tau(t)\big):e(\bar u_\tau(t))\,\mathrm{d}x+\big\langle \bar f_\tau(t),\bar u_\tau(t)\big\rangle_{\mathbf{U}^*,\mathbf{U}}\\
&\quad-\int_0^{\bar t_\tau(t)}2\mathcal{R}_M(\dot z_\tau(r))\,\mathrm{d}r-\int_0^{\bar t_\tau(t)}2\mathcal{V}(\bar z_\tau(r);\dot u_\tau(r))\,\mathrm{d}r\,,
\end{aligned}
\tag{123}
$$

and we take the limit superior as $\tau\to 0$ *on both sides of* (123)*. By making use of convergences* (76)*, we can pass to the limit on the right-hand side by weak lower semicontinuity and weak-strong convergence arguments, essentially by repeating the argumentation of Sec. 5.2.4 for the upper energy-dissipation estimate, and we also use the estimate* $-\int_0^{\bar t_\tau(t)}2\mathcal{R}_M(\dot z_\tau(r))\,\mathrm{d}r\leq 0$*. In this way we find*

$$
\begin{aligned}
&\limsup_{\tau\downarrow 0}\int_\Omega \Big(\frac{1}{2\ell}\big(1-\bar z_\tau(t)\big)^2+\frac{\ell}{2}\,|\nabla \bar z_\tau(t)|^2\Big)\,\mathrm{d}x\\
&\leq \mathcal{K}(\dot u_0)+\mathcal{E}(0,u_0,z_0)-\int_0^t\langle \dot f(r),u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r-\mathcal{K}(\dot u(t))\\
&\quad-\int_\Omega\frac12\mathbb{C}(z(t))e(u(t)):e(u(t))\,\mathrm{d}x+\langle f(t),u(t)\rangle_{\mathbf{U}^*,\mathbf{U}}-\int_0^t 2\mathcal{V}(z(r);\dot u(r))\,\mathrm{d}r\\
&=\int_\Omega \Big(\frac{1}{2\ell}\big(1-z(t)\big)^2+\frac{\ell}{2}\,|\nabla z(t)|^2\Big)\,\mathrm{d}x
\end{aligned}
$$

for all $t\in[0,\mathsf{T})$*, where the last equality follows from the validity of the energy-dissipation balance* (10d) *of the limit. This provides* (122b)*.*

To deduce (122a) *we repeat the above line of arguments. Accordingly, in the analogon of* (123) *we keep the stored-elastic-energy term on the left-hand side and move the phase-field term to the right-hand side. In this term we can also pass to the limit via convergences* (76) *and weak lower semicontinuity, as already argued in Sec. 5.2.4. Thus, we obtain*

$$
\begin{aligned}
&\limsup_{\tau\downarrow 0}\int_\Omega\frac12\mathbb{C}(\bar z_\tau(t))e(\bar u_\tau(t)):e(\bar u_\tau(t))\,\mathrm{d}x\\
&\leq \mathcal{K}(\dot u_0)+\mathcal{E}(0,u_0,z_0)-\int_0^t\langle \dot f(r),u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r-\int_0^t 2\mathcal{V}(z(r);\dot u(r))\,\mathrm{d}r\\
&\quad-\mathcal{K}(\dot u(t))-\int_\Omega \Big(\frac{1}{2\ell}\big(1-\bar z_\tau(t)\big)^2+\frac{\ell}{2}\,|\nabla \bar z_\tau(t)|^2\Big)\,\mathrm{d}x+\langle f(t),u(t)\rangle_{\mathbf{U}^*,\mathbf{U}}\\
&=\int_\Omega\frac12\mathbb{C}(z(t))e(u(t)):e(u(t))\,\mathrm{d}x\,.
\end{aligned}
\tag{124}
$$

From this, (122a) *is concluded with the aid of the following lemma:*

Lemma 5.6 (Adaption of [LRTT18, L. 4.7])**.** *Given two constants* C_1,C_2 *with* $0<C_1\leq C_2$*, let* $\mathcal{T}_{C_1,C_2}$ *denote the class of tensors* $\mathbb{C}\in\mathbb{R}^{d\times d\times d\times d}$ *that are symmetric, i.e.,*

$$
\mathbb{C}_{ijkl}=\mathbb{C}_{jikl}=\mathbb{C}_{ijlk}=\mathbb{C}_{klij}\,,
$$

positive definite and bounded:

$$
C_1|A|^2\leq \mathbb{C}\,A:A\leq C_2|A|^2\quad\text{for every } A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\,.
\tag{125}
$$

Let $\mathcal{I}_n$ *be the functional defined by*

$$
\mathcal{I}_n(e):=\int_\Omega\mathbb{C}_n(x)e(x):e(x)\,\mathrm{d}x\quad\text{for every } e\in L^2(\Omega;\mathbb{R}^{d\times d})\,,
$$

where $\mathbb{C}_n \in L^\infty(\Omega;\mathcal{T}_{C_1,C_2})$ are such that

$$\mathbb{C}_n(x) \to \mathbb{C}_\infty(x) \qquad \text{for a.e. } x \in \Omega\,, \tag{126a}$$
$$e_n \rightharpoonup e_\infty \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d})\,, \tag{126b}$$
$$\limsup_{n\to\infty} \mathcal{I}_n(e_n) \le \mathcal{I}_\infty(e_\infty)\,, \tag{126c}$$

and $\mathcal{I}_\infty$ is defined by

$$\mathcal{I}_\infty(e) := \int_\Omega \mathbb{C}_\infty(x)e(x) : e(x)\,\mathrm{d}x \quad \text{for every } e \in L^2(\Omega;\mathbb{R}^{d\times d})\,.$$

Then, $\lim_{n\to\infty} \mathcal{I}_n(e_n) = \mathcal{I}_\infty(e_\infty)$ and

$$e_n \to e_\infty \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{d\times d})\,. \tag{127}$$

Note that [LRTT18, L. 4.7] states the result for tensors $\mathbb{C} \in L^\infty((0,\mathsf{T})\times\Omega;\mathcal{T}_{C_1,C_2})$ that additionally depend on time and for functions $e \in L^2((0,\mathsf{T})\times\Omega;\mathbb{R}^{d\times d})$, so that the functionals $\mathcal{I}_n, \mathcal{I}$ are defined by additionally integrating over $(0,\mathsf{T})$. Accordingly, [LRTT18, L. 4.7] provides strong convergence of $(e_n)_n$ in $L^2((0,\mathsf{T})\times\Omega;\mathbb{R}^{d\times d})$. But the arguments of the proof remain valid, if we drop the time-dependence as here in Lemma 5.6. □

6 Limit passage in the viscous case

We now discuss the limit from time-discrete to time-continuous to obtain solutions for system

$$(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$$

with a viscous evolution of the phase-field variable, when the parameter $M>0$ is kept fixed in the limit passage:

Theorem 6.1 (Existence of solutions in the viscous limit)**.** *Let the assumptions of Theorem 4.1 and Proposition 4.2 be satisfied and assume that the one-sided variational inequality* (9a) *holds true at $t=0$ for the initial datum $(u_0,z_0)\in \mathbf{U}\times\mathbf{X}$. Let the viscosity parameter $M>0$ in* (4) *be fixed and let $\tau\to 0$. For all $\tau>0$ let $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau)$ be a tuple of interpolated solutions of problem* (47) *corresponding to system $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$. Then there holds:*

1. *There exists a pair $(u_M,z_M)\colon [0,\mathsf{T}]\to \mathbf{U}\times\mathbf{Z}$ such that, up to a (not relabeled) subsequence, the solutions $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ converge to (u_M,z_M) in the topologies of* (76) *and additionally also in the following sense:*

$$z_\tau \rightharpoonup z_M \quad \text{weakly in } H^1(0,T;\mathbf{Z}_M)\,. \tag{128}$$

2. *The limit pair (u_M,z_M) is a solution of $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$ in the sense of Definition 1.2 and it is $0\le z_M(t,x)\le 1$ for a.a. $x\in\Omega$ and for all $t\in[0,\mathsf{T}]$.*
3. *The limit function u_M complies with the regularity properties* (77)*. The limit function z_M the has regularity properties*

$$z_M \in H^1\big(0,\mathsf{T};L^2(\Omega)\big)\cap L^\infty(0,\mathsf{T};\mathbf{X})\cap C^0\big((0,\mathsf{T});\mathbf{X}\big)\,. \tag{129}$$

4. *In addition to the convergence results stated in Item 1, also the following improved convergence statements hold true:*

$$e(\dot{u}_\tau)\to e(\dot{u}_M) \quad \text{strongly in } H^1(0,\mathsf{T};\mathbf{U})\,, \tag{130a}$$
$$e(\bar{u}_\tau(t))\to e(u_M(t)) \quad \text{strongly in } \mathbf{U} \text{ for all } t\in[0,\mathsf{T})\,, \tag{130b}$$
$$\bar{z}_\tau(t)\to z_M(t) \quad \text{strongly in } \mathbf{X} \text{ for all } t\in[0,\mathsf{T})\,. \tag{130c}$$

Proof. The proof of Theorem 6.1 is discussed in Section 6.2 by mainly pointing out the differences to the rate-independent case given in Theorem 5.1. The proof of the continuity of z_M in the interval $[0,\mathsf{T})$ with values in $\mathbf{X}$ is based on a similar argumentation as the improved regularity the rate-independent setting, cf. Theorem 5.2. We state the abstract result for the viscous evolution below in Theorem 6.2 and verify regularity statement (129) in Section 6.4. □

The proof of the continuity-result stated in Theorem 6.2 below will be elaborated in detail in Section 6.1. Compared to the regularity result in [RT17, Thm. 3.8], the situation here is different due to a quadratic dissipation $\mathcal{R}_M$ instead of a 1-homogeneous rate-independent potential and due to the state-dependence of the viscous dissipation $\mathcal{V}$. The result is based on the one-sided variational inequality (9a)*, which is valid for a.e. $t \in (0,\mathsf{T})$, only, due to the appearance of $\dot z_M \in L^2(0,\mathsf{T};\mathbf{Z}_M)$. To estimate this expression we will make use of a Riemann-sum approach relying on a sequence of partitions $\Pi := (\{t^n_k,\, k=0,\dots n\})_n$ for which* (9a) *holds true in each of the nodes t^n_k, see Thm. 6.2, Item 4 and also Remark 6.3. An outcome of this will be the term*

$$\mathrm{Var}^{\alpha}_{\Pi,\mathbf{S}}(z;[s,t]) := \lim_{n\to\infty}\sum_{k=1}^{n}\|z(t^n_k)-z(t^n_{k-1})\|^{\alpha}_{\mathbf{S}}\,, \quad \alpha>1, \tag{131}$$

with the exponent $\alpha>1$ and the Banach space $\mathbf{S}$ given by the uniform convexity property (84)*. We remark that the expression $\mathrm{Var}^{\alpha}_{\Pi,\mathbf{S}}(z;[s,t])$ resembles a variation of power α, which appears in stochastics for $\alpha=2$, but differently to a true variation, in* (131) *it is not possible to consider the supremum over all the partitions of $[s,t]$.*

Theorem 6.2 (Improved temporal regularity)**.** *Let $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$ be a damped inertial system characterized by Banach spaces $\mathbf{U},\mathbf{Z}_M,$ and a Hilbert space $\mathbf{W},$ the kinetic energy $\mathcal{K}:\mathbf{W}\to[0,\infty),$ a dissipation potential $\mathcal{V}:\mathbf{Z}_M\times\mathbf{U}\to[0,\infty),$ a quadratic dissipation potential $\mathcal{R}_M:\mathbf{Z}_M\to[0,\infty],$ and an energy functional $\mathcal{E}:[0,\mathsf{T}]\times\mathbf{U}\times\mathbf{Z}_M\to\mathbb{R}\cup\{\infty\}$ such that for all $t\in[0,\mathsf{T}]$ the functional $\mathcal{E}(t,\cdot,\cdot)$ is coercive and takes finite values on (a closed, convex subset of) $\mathbf{V}\times\mathbf{X}$ with $\mathbf{X}$ a Banach space such that $\mathbf{X}\subset\mathbf{Z}_M$ compactly and $\mathbf{V}$ a Banach space such that $\mathbf{V}\subset\mathbf{U}$ continuously and densely. Moreover, let $\mathbf{S}$ be a Banach space such that $\mathbf{X}\subseteq\mathbf{S}\subseteq\mathbf{Z}_M$ continuously, which may or may not coincide with $\mathbf{X}$ or $\mathbf{Z}_M$. Further consider the list of assumptions A1)–A10) from Theorem 5.2, where A1) and A2) are now replaced by:*

$\widetilde{A1}$*) The pair $(u,z):[0,\mathsf{T}]\to\mathbf{U}\times\mathbf{X}$ satisfies the one-sided variational inequality* (132) *for a.a. $t\in[0,\mathsf{T}]$:*

$$\langle \mathrm{D}_z\mathcal{E}(t,u(t),z(t))+\mathrm{D}\mathcal{R}_M(\dot z(t)),\eta\rangle_{\mathbf{X}^*,\mathbf{X}}\geq 0 \quad \textit{for all } \eta\in\mathbf{K}(t) \tag{132}$$

with $\mathbf{K}(t)\subset\mathbf{X}$ a closed, convex subset of $\mathbf{X}$. Define the $\mathcal{L}^1$-null set

$$\widetilde{N} := \big\{\hat t\in[0,\mathsf{T}]\;|\;(u(\hat t),z(\hat t)) \textit{ does not satisfy } (132)\big\}\,. \tag{133}$$

$\widetilde{A2}$*) The pair $(u,z):[0,\mathsf{T}]\to\mathbf{U}\times\mathbf{X}$ satisfies the following upper energy-dissipation estimate*

$$\begin{aligned}\mathcal{K}(\dot u(t))+\mathcal{E}(t,u(t),z(t))&+\int_s^t 2(\mathcal{V}(z(r);\dot u(r))+\mathcal{R}_M(\dot z(r)))\,\mathrm{d}r\\ &\leq \mathcal{K}(\dot u(s))+\mathcal{E}(s,u(s),z(s))+\int_s^t \partial_r\mathcal{E}(r,u(r),z(r))\,\mathrm{d}r\end{aligned} \tag{134}$$

for all subintervals $[s,t]\subset[0,\mathsf{T}]$ with $s,t\in[0,\mathsf{T}]\backslash\widetilde{N}$.

The following statements hold true:

1 *Let assumptions $\widetilde{A1}$) and A5) be valid. Then (u,z) satisfies*

$$\begin{aligned}&\mathcal{E}(s,u(s),z(s))+C_\star\|z(t)-z(s)\|^{\alpha}_{\mathbf{S}}\\ &\leq \mathcal{E}(s,u(s),z(t))+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\end{aligned} \tag{135}$$

for all $s\in[0,\mathsf{T}]\backslash\widetilde{N}$ and for all $t\in[0,\mathsf{T}]$ such that $(z(t)-z(s))\in\mathbf{K}(s)$.

2 *Let assumptions $\widetilde{A1}$), $\widetilde{A2}$), and A3)–A6) be valid. Then, z complies with the following estimate*

$$\begin{aligned}C_\star\|z(t)-z(s)\|^{\alpha}_{\mathbf{S}}\leq c_\star\Big(\int_s^t\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\Big)^{\beta_u}+C(t-s)+\int_s^t\Big|\langle\rho\ddot u(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\Big|\,\mathrm{d}r\\ +\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_s^t 2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\end{aligned} \tag{136}$$

for all subintervals $[s,t]\subset[0,\mathsf{T}]$ with $s,t\in[0,\mathsf{T}]\backslash\widetilde{N}$ and $(z(t)-z(s))\in\mathbf{K}(s)$.

3 *Let assumptions $\widetilde{A1}$), $\widetilde{A2}$) and A3)–A9) be valid. Then, z complies with the estimate*

$$\begin{aligned} C_\star \|z(t)-z(s)\|_{\mathbf{S}}^\alpha \le \hat{C}\int_s^t \|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r + c_\star\Big(\int_s^t \|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\Big)^{\beta_u} + C(t-s) \\ + \langle \mathrm{D}\mathcal{R}_M(\dot z(s)), z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_s^t 2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r \end{aligned} \tag{137}$$

for all subintervals $[s,t]\subset[0,\mathsf{T}]$ with $s,t\in[0,\mathsf{T}]\backslash\widetilde{N}$ and $(z(t)-z(s))\in\mathbf{K}(s)$.

4 *Let $[s_*,t_*]\subset[0,\mathsf{T}]$ and consider a sequence of partitions $\Pi:=(\Pi_n)_{n\in\mathbb{N}}$ with $\Pi_n=\{s_*=t_0^n<t_1^n<\ldots<t_n^n=t_*\}$ such that*

$$\big(z(t_k^n)-z(t_{k-1}^n)\big)\in\mathbf{K}(t_{k-1}^n) \quad \text{for all } k\in\{1,\ldots,n\} \text{ and } n\in\mathbb{N}\,, \tag{138a}$$

$$\lim_{n\to\infty}\sum_{k=1}^n \big\langle \mathrm{D}\mathcal{R}_M(\dot z(t_{k-1}^n)), z(t_k^n)-z(t_{k-1}^n)\big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{s_*}^{t^*} 2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r = 0\,. \tag{138b}$$

Set

$$\mathrm{Var}^\alpha_{\Pi,\mathbf{S}}(z;[s_*,t_*]) := \lim_{n\to\infty}\sum_{k=1}^n \big\|z(t_k^n)-z(t_{k-1}^n)\big\|^\alpha_{\mathbf{S}}\,. \tag{139}$$

Further suppose for the nodes of Π that $t_k^n\in[0,\mathsf{T}]\setminus\widetilde{N}$ for all $k\in\{0,\ldots,n\}$, and $n\in\mathbb{N}$. Assume that estimate (136) *is valid with $\beta_u=1$ in all the nodes of Π. Then,*

$$C_\star \mathrm{Var}^\alpha_{\Pi,\mathbf{S}}(z;[s_*,t_*]) \le c_\star\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r + C(t_*-s_*) + \int_{s_*}^{t_*}\Big|\langle\rho\ddot u(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\Big|\,\mathrm{d}r\,. \tag{140a}$$

If estimate (137) *is valid with $\beta_u=1$ in all the nodes of Π, then,*

$$C_\star \mathrm{Var}^\alpha_{\Pi,\mathbf{S}}(z;[s_*,t_*]) \le (\hat C + c_\star)\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r + C(t_*-s_*)\,. \tag{140b}$$

If in addition also assumption A10) is valid, then

$$C_\star \mathrm{Var}^\alpha_{\Pi,\mathbf{S}}(z;[s_*,t_*]) \le (\hat C + c_\star)(t_*-s_*)^{\frac12}\,\|\dot u\|_{L^2(s_*,t_*;\mathbf{U})} + C(t_*-s_*)\,. \tag{140c}$$

5 *Let the conditions of Item 4 be valid and assume that one of* (140a), (140b) *holds true. For all $\hat t\in\widetilde{N}\cap(0,\mathsf{T})$ there are sequences $(\hat t_l^\pm)_{l\in\mathbb{N}}\subset(0,\mathsf{T})\backslash\widetilde{N}$ such that*

$$\begin{aligned} &\hat t_l^- \nearrow \hat t, \quad \hat t_l^+ \searrow \hat t, \quad \text{and } \|z(\hat t_l^+)-z(\hat t_l^-)\|_{\mathbf{S}}\to 0\,, \\ &z_{\hat t}^- = \lim_{l\to\infty} z(\hat t_l^-) = \lim_{l\to\infty} z(\hat t_l^+) = z_{\hat t}^+ \text{ in } \mathbf{S} \text{ as } l\to\infty\,. \end{aligned} \tag{141}$$

6 *Assume that $\mathcal{R}_M$, resp. the closed, convex subset $\mathbf{K}(t)$, $t\in[0,\mathsf{T}]$, encodes a unidirectionality constraint, i.e., $(z(\hat t)-z(t))\in\mathbf{K}(t)$ for all $\hat t,t\in[0,\mathsf{T}]$ with $\hat t\ge t$. Let the prerequisites of Item 5 be valid. Suppose that $\mathbf{Z}_M=L^p(\Omega)$ and that $\mathbf{S}\in\{L^{\tilde p}(\Omega),W^{m,\tilde p}(\Omega)\}$ with $p,\tilde p>1$ and $m\in\mathbb{N}$, and such that $\mathbf{X}\subseteq\mathbf{S}\subseteq\mathbf{Z}_M$ continuously. Then, it is $z_{\hat t}^-=z(\hat t)=z_{\hat t}^+$ in $\mathbf{S}$ for all $\hat t\in(0,\mathsf{T})$ and for the left- and right-continuous limits of the sequence* (141)*. Moreover, there even holds $z\in C^0((0,\mathsf{T});\mathbf{S})$.*

Proof. The proof is carried out in Section 6.1 below. □

Remark 6.3 (Approximation by Riemann sums)**.** *For a Banach space V, every $f\in L^1(0,T;V)$ can be approximated by Riemann sums, i.e. there exists a sequence of partitions $\Pi_n=\{0=t_0^n<t_1^n<\ldots<t_{N_n}^n\}$ such that*

$$\lim_{n\to\infty}\max_{1\le k\le N_n} t_k^n - t_{k-1}^n = 0 \tag{142}$$

and $\lim_{n\to\infty}\sum_{k=1}^{N_n}\int_{t_{k-1}^n}^{t_k^n}\|f(t_k^n)-f(r)\|\,\mathrm{d}r=0$ *[DMFT05, Lemma 4.12, p. 26]). There is even a freedom of choice in the selection of the partition because the approximation property also holds true if one takes into account* almost all *sequences of partitions with the property* (142)*, cf. [MR15b, footnote 35, p. 604]. This may be justified by applying for the* $L^1(0,\mathsf{T})$*-integrable functions* $\|f(\cdot)\|_V:[0,\mathsf{T}]\to[0,\infty)$ *the definition of gauge integrals, e.g. in the sense of Denjoy-Perron [Maw97, p. 349] or Henstock-Kurzweil [ebPM16, Ch. 4], and the fact that every Lebesgue-integrable function is gauge-integrable in the sense of Denjoy-Perron [Maw97, p. 385f] or Henstock-Kurzweil [Sch09]. In this way one may restrict the partitions to those with nodes in* $[0,\mathsf{T}]\backslash\widetilde{N}$ *and thus ensure the prerequisites of Thm. 6.2, Item 4.*

6.1 Proof of Theorem 6.2: Improved temporal regularity for the internal variable

To Item 1, estimate (135)***:*** *Based on* $\widetilde{A1}$*), we test the variational inequality* (132) *at time* $s\in[0,\mathsf{T}]\setminus\widetilde{N}$ *by* $\tilde z-z(s)\in\mathbf{K}(s)$ *with* $\tilde z\in\mathbf{X}$ *suitably. By the Gâteaux-differentiablility and convexity of* $\mathcal{E}(t,u,\cdot)$ *ensured by A5), one finds*

$$\begin{aligned}0&\le\langle \mathrm{D}_z\mathcal{E}(s,u(s),z(s))+\mathrm{D}\mathcal{R}_M(\dot z(s)),\tilde z-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\\&\le\mathcal{E}(s,u(s),\tilde z)-\mathcal{E}(s,u(s),z(s))+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),\tilde z-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\,.\end{aligned}\tag{143}$$

Let now $z_0,z_1\in\mathbf{X}$ *such that* $z_i-z(s)\in\mathbf{K}(s)$ *for* $i\in\{0,1\}$ *and* $\lambda\in(0,1)$*. Then, for* $\tilde z=\lambda z_1+(1-\lambda)z_0$ *in* (143)*, it is* $\tilde z-z(s)\in\mathbf{K}(s)$*. Exploiting the uniform convexity estimate* (84)*, it follows*

$$\begin{aligned}0\le&\,\mathcal{E}(s,u(s),\tilde z)-\mathcal{E}(s,u(s),z(s))+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),\tilde z-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\\\le&\,\lambda\mathcal{E}(s,u(s),z_1)+(1-\lambda)\mathcal{E}(s,u(s),z_0)-\lambda(1-\lambda)C_\star\|z_1-z_0\|_{\mathbf{S}}^\alpha-\mathcal{E}(s,u(s),z(s))\\&+\lambda\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z_1-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}+(1-\lambda)\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z_0-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\,.\end{aligned}\tag{144}$$

Now, the choice $z_0:=z(s)$ *in* (144)*, where clearly* $z(s)-z(s)=0\in\mathbf{K}(s)$*, leads to*

$$\begin{aligned}0\le&\,\lambda\big[\mathcal{E}(s,u(s),z_1)-\mathcal{E}(s,u(s),z(s))-(1-\lambda)C_\star\|z_1-z(s)\|_{\mathbf{S}}^\alpha\\&+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z_1-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\big]\,.\end{aligned}$$

Dividing by $\lambda>0$ *and letting* $\lambda\downarrow0$ *one arrives at*

$$C_\star\|z_1-z(s)\|_{\mathbf{S}}^\alpha\le\mathcal{E}(s,u(s),z_1)-\mathcal{E}(s,u(s),z(s))+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z_1-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\,.$$

The choice $z_1:=z(t)$ *for* $t\in[0,\mathsf{T}]$ *such that* $(z(t)-z(s))\in\mathbf{K}(s)$ *shows the validity of Theorem 6.2, Item 1, that is*

$$C_\star\|z(t)-z(s)\|_{\mathbf{S}}^\alpha\le\mathcal{E}(s,u(s),z(t))-\mathcal{E}(s,u(s),z(s))+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}\tag{145}$$

To Item 2, estimate (136)***:*** *Let now also* $t\in[0,\mathsf{T}]\backslash\widetilde{N}$*. In a first step, the right-hand side of* (145) *by adding and subtracting terms, can be rewritten as*

$$\begin{aligned}\text{(145)}=&\,\mathcal{E}(t,u(t),z(t))-\mathcal{E}(s,u(s),z(s))+\int_s^t2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\\&+\mathcal{E}(s,u(s),z(t))-\mathcal{E}(t,u(t),z(t))\\&+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_s^t2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\,.\end{aligned}\tag{146}$$

In view of the upper energy-dissipation estimate (134) *ensured in* $\widetilde{A2}$) *for* $s,t\in[0,\mathsf{T}]\backslash\widetilde{N}$ *one obtains*

$$\begin{aligned}\text{(146)}\le&\,\mathcal{K}(\dot u(s))-\mathcal{K}(\dot u(t))-\int_s^t2\mathcal{V}(z(r);\dot u(r))\,\mathrm{d}r+\int_s^t\partial_r\mathcal{E}(r,u(r),z(r))\,\mathrm{d}r\\&+\mathcal{E}(s,u(s),z(t))-\mathcal{E}(t,u(t),z(t))\\&+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_s^t2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\,.\end{aligned}\tag{147}$$

Now the terms on the right-hand side of (147) *will be further estimated from above individually. In view of assumption A3) on the regularity of* u*, the result [Rou06, Lemma 7.3, p. 191] together with the non-negativity of* $\mathcal{V}(z(r);\dot{u}(r))$*, provides that*

$$\mathcal{K}(\dot{u}(s)) - \mathcal{K}(\dot{u}(t)) - \int_s^t 2\mathcal{V}(z(r);\dot{u}(r))\,\mathrm{d}r \leq \int_s^t \big|\,\langle \rho\ddot{u}(r),\dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\big|\,\mathrm{d}r\,. \tag{148}$$

In addition, we make use of the absolute continuity of $r\mapsto\mathcal{E}(r,u,z)$ *and Hölder-estimate* (85) *for* $\mathcal{E}(t,\cdot,z(t))$ *provided by A6), to deduce that*

$$\begin{aligned}&\mathcal{E}(s,u(s),z(t)) - \mathcal{E}(t,u(t),z(t))\\&= \mathcal{E}(s,u(s),z(t)) - \mathcal{E}(t,u(s),z(t)) + \mathcal{E}(t,u(s),z(t)) - \mathcal{E}(t,u(t),z(t))\\&\leq -\int_s^t \partial_r\mathcal{E}(r,u(s),z(t))\,\mathrm{d}r + c_\star\,\|u(s)-u(t)\|_{\mathbf{U}}^{\beta_u}\,,\end{aligned} \tag{149}$$

and by the absolute continuity of u *we further note that*

$$c_\star\,\|u(s)-u(t)\|_{\mathbf{U}}^{\beta_u} \leq c_\star \left\|\int_s^t \dot{u}(r)\,\mathrm{d}r\right\|_{\mathbf{U}}^{\beta_u} \leq c_\star\Big(\int_s^t \|\dot{u}(r)\|_{\mathbf{U}}\,\mathrm{d}r\Big)^{\beta_u}\,. \tag{150}$$

In summary, we conclude for all $s,t\in[0,\mathsf{T}]\setminus\widetilde{N}$ *with* $(z(t)-z(s))\in\mathbf{K}(s)$ *that*

$$\begin{aligned}C_\star\,\|z(t)-z(s)\|_{\mathbf{S}}^{\alpha} \leq &\int_s^t \big|\,\langle\rho\ddot{u}(r),\dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\big|\,\mathrm{d}r + c_\star\Big(\int_s^t \|\dot{u}(r)\|_{\mathbf{U}}\,\mathrm{d}r\Big)^{\beta_u}\\&+\int_s^t \Big(\partial_r\mathcal{E}(r,u(r),z(r)) - \partial_r\mathcal{E}(r,u(s),z(t))\Big)\,\mathrm{d}r\\&+\langle \mathrm{D}\mathcal{R}_M(\dot{z}(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_s^t 2\mathcal{R}_M(\dot{z}(r))\,\mathrm{d}r\,,\end{aligned} \tag{151}$$

where the term involving the partial time derivatives of $\mathcal{E}$ *can be further estimated from above by the power control* (83) *provided in assumption A4) as follows*

$$\begin{aligned}&\left|\int_s^t \partial_r\mathcal{E}(r,u(r),z(r)) - \partial_r\mathcal{E}(r,u(s),z(t))\,\mathrm{d}r\right|\\&\leq \int_s^t \big(\,|\partial_r\mathcal{E}(r,u(r),z(r))| + |\partial_r\mathcal{E}(r,u(s),z(t))|\,\big)\,\mathrm{d}r\\&\leq \int_s^t \tilde{c}\big[\mathcal{E}(r,u(r),z(r)) + \mathcal{E}(r,u(s),z(t)) + 2\hat{c}\big]\,\mathrm{d}r \leq C(t-s)\,.\end{aligned} \tag{152}$$

Note here that the uniform bound on $\mathcal{E}(r,u(r),z(r)) + \mathcal{E}(r,u(s),z(t))$ *is guaranteed by the upper energy-dissipation estimate* (134) *and the coercivity of* $\mathcal{E}(t,\cdot,\cdot)$ *on (a closed, convex subset of)* $\mathbf{V}\times\mathbf{X}$ *as claimed in the general assumptions of Thm. 6.2. Inserting* (152) *into* (151) *proves the validity of estimate* (136) *for all* $s,t\in[0,\mathsf{T}]\setminus\widetilde{N}$ *such that* $(z(t)-z(s))\in\mathbf{K}(s)$*, that is Thm. 6.2, Item 2.*

To Item 3, estimate (137)***:*** *To deduce* (137) *we return to estimate* (147) *and, instead of using* (148)*, we argue as follows: Again, by assumption A3) on the regularity of* u *and [Rou06, Lemma 7.3, p. 191] we have* $\mathcal{K}(\dot{u}(s)) - \mathcal{K}(\dot{u}(t)) = -\int_s^t \langle\rho\ddot{u}(r),\dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r$*. Moreover,* $\mathcal{E}(r,\cdot,z(r))$ *is Gâteaux-differentiable by A7) and the weak momentum balance* (87) *holds true by assumption A9). We thus test* (87) *by* $\dot{u}$ *to obtain the identity*

$$\int_s^t \langle\rho\ddot{u}(r) + \mathrm{D}_u\mathcal{E}(r,u(r),z(r)),\dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r = -\int_s^t 2\mathcal{V}(z(t);\dot{u}(r))\,\mathrm{d}r\,.$$

Hence, the kinetic and the viscous terms on the right-hand side of (147) *amount to*

$$\begin{aligned}
&\mathcal{K}(\dot u(s))-\mathcal{K}(\dot u(t))-\int_s^t 2\mathcal{V}(z(r);\dot u(r))\,\mathrm{d}r\\
&=\int_s^t \langle -\rho\ddot u(r)+\rho\ddot u(r)+\mathrm{D}_u\mathcal{E}(r,u(r),z(r)),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\\
&=\int_s^t \langle \mathrm{D}_u\mathcal{E}(r,u(r),z(r)),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\,.
\end{aligned}\tag{153}$$

This term is now further estimated with the aid of the gradient estimate (86) *provided in A8) in the following way:*

$$\begin{aligned}
&\Big|\int_s^t \langle \mathrm{D}_u\mathcal{E}(r,u(r),z(r)),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r\Big|\le\int_s^t\|\mathrm{D}_u\mathcal{E}(r,u(r),z(r))\|_{\mathbf{U}^*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\\
&\le\int_s^t\big(\hat C_1\mathcal{E}(r,u(r),z(r))+\hat C_2\|u(r)\|_{\mathbf{U}}+\hat C_3\big)^{1/\sigma}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\le\hat C\int_s^t\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\,,
\end{aligned}\tag{154}$$

where the uniform boundedness of $u\in H^1(0,\mathsf{T};\mathbf{U})$ *claimed in A3) was used together with the uniform bound on the energy provided by the upper energy-dissipation estimate* (82). *Putting together estimates* (147), (149), (150), (152), *and* (154) *results in*

$$\begin{aligned}
C_\star\|z(t)-z(s)\|_{\mathbf{S}}^\alpha\le\hat C\int_s^t\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+c_\star\Big(\int_s^t\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r\Big)^{\beta_u}+C(t-s)\\
+\langle\mathrm{D}\mathcal{R}_M(\dot z(s)),z(t)-z(s)\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_s^t 2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\,,
\end{aligned}\tag{155}$$

which finishes the proof of estimate (137) *for all* $s,t\in[0,\mathsf{T}]\backslash\widetilde N$ *with* $(z(t)-z(s))\in\mathbf{K}(s)$, *i.e., Thm. 6.2, Item 3.*

To Item 4, estimates (140)***:*** *Consider now a sequence of partitions* $\Pi=(\Pi_n)_{n\in\mathbb{N}}$, $\Pi_n:=\{s_*=t_0^n<t_1^n<\ldots<t_n^n=t_*\}$ *with the properties* (138) *and such that* $t_k^n\in[0,\mathsf{T}]\backslash\widetilde N$ *for all the nodes of* Π. *First, assume that estimate* (136) *is valid with* $\beta_u=1$ *for all the nodes of* Π. *Hence, using* $s=t_{k-1}^n$, $t=t_k^n$ *and* $\beta_u=1$ *in* (136), *summing up from* $k=1$ *to* n, *and letting* $n\to\infty$, *gives*

$$\begin{aligned}
C_\star\mathrm{Var}_{\Pi,\mathbf{S}}^\alpha(z;[s_*,t_*])&\le c_\star\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+C(t_*-s_*)+\int_{s_*}^{t_*}\Big|\langle\rho\ddot u(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\Big|\,\mathrm{d}r\\
&+\lim_{n\to\infty}\sum_{k=1}^n\big\langle\mathrm{D}\mathcal{R}_M(\dot z(t_{k-1}^n)),z(t_k^n)-z(t_{k-1}^n)\big\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_{s_*}^{t_*}2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\\
&=c_\star\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+C(t_*-s_*)+\int_{s_*}^{t_*}\Big|\langle\rho\ddot u(r),\dot u(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\Big|\,\mathrm{d}r
\end{aligned}\tag{156}$$

by assumption on the convergence of the Riemann sum in (138b). *This shows* (140a). *Analogously one obtains from estimate* (137) *with* $\beta_u=1$ *that*

$$\begin{aligned}
C_\star\mathrm{Var}_{\Pi,\mathbf{S}}^\alpha(z;[s_*,t_*])&\le\hat C\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+c_\star\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+C(t_*-s_*)\\
&+\lim_{n\to\infty}\sum_{k=1}^n\big\langle\mathrm{D}\mathcal{R}_M(\dot z(t_{k-1}^n)),z(t_k^n)-z(t_{k-1}^n)\big\rangle_{\mathbf{X}^*,\mathbf{X}}-\int_{s_*}^{t_*}2\mathcal{R}_M(\dot z(r))\,\mathrm{d}r\\
&=(\hat C+c_\star)\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+C(t_*-s_*)\,,
\end{aligned}\tag{157}$$

that is (140b). *For estimate* (140c) *we observe from the quadratic growth* (88) *of* $\mathcal{V}$ *claimed in A10) that we may use Hölder's inequality with power* $p=2$ *for the first term on the right-hand side of* (140b), *resp. above in* (157). *Thus,*

$$\begin{aligned}
C_\star\mathrm{Var}_{\Pi,\mathbf{S}}^\alpha(z;[s_*,t_*])&\le(\hat C+c_\star)\int_{s_*}^{t_*}\|\dot u(r)\|_{\mathbf{U}}\,\mathrm{d}r+C(t_*-s_*)\\
&\le(\hat C+c_\star)(t_*-s_*)^{\frac12}\|\dot u\|_{L^2(s_*,t_*;\mathbf{U})}+C(t_*-s_*)\,,
\end{aligned}\tag{158}$$

which is (140c).

To Item 5, existence of* $\mathbf{S}$*-convergent sequences (141)***:*** *Let* $\hat{t} \in \widetilde{N} \cap (0,\mathsf{T})$ *and consider a sequence* $\varepsilon \searrow 0$ *with* $\varepsilon > 0$ *and such that* $\hat{t}-\varepsilon,\ \hat{t}+\varepsilon \in [0,\mathsf{T}] \setminus \widetilde{N}$*. This is possible in view of Remark 6.3. Assume that one of* (140a), (140b) *is valid. Without loss of generality we here carry out the proof under the assumption that* (140b) *is valid together with growth property* (88) *from A10); the proof based on* (140a) *or without* (88) *proceeds in an analogous way. Then, by assumption, there exists partitions* $(\Pi_n^\varepsilon)_{n\in\mathbb{N}} = \{\hat{t}-\varepsilon = t_0^{\varepsilon n} < t_1^{\varepsilon n} < \ldots < t_n^{\varepsilon n} = \hat{t}+\varepsilon\}$ *with nodes* $t_k^{\varepsilon n} \in [0,\mathsf{T}]\backslash\widetilde{N}$*. Hence,* (140b)*, resp.* (157) *above, together with* (88) *yields that*

$$
\begin{aligned}
&\mathrm{Var}^\alpha_{\Pi,\mathbf{S}}(z;[\hat{t}-\varepsilon,\hat{t}+\varepsilon]) = \lim_{n\to\infty}\sum_{k=1}^n \left\|z(t_k^{\varepsilon n}) - z(t_{k-1}^{\varepsilon n})\right\|_{\mathbf{S}}^\alpha\\
\leq&\frac{1}{C_\star}\Big((\hat{C}+c_\star)(2\varepsilon)^{\frac12}\,\|\dot{u}\|_{L^2(0,\mathsf{T};\mathbf{U})} + C2\varepsilon\\
&\qquad + \lim_{n\to\infty}\sum_{k=1}^n\big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^n)), z(t_k^n)-z(t_{k-1}^n)\big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{s_*}^{t_*} 2\mathcal{R}_M(\dot{z}(r))\,\mathrm{d}r\Big)\,,
\end{aligned}
$$

where we also divided by $C_\star$ *and used that* $[\hat{t}-\varepsilon,\hat{t}+\varepsilon] \subset [0,\mathsf{T}]$*. In view of* (138b)*, for all* $\nu > 0$ *and each* $\varepsilon > 0$ *there is an index* $n(\nu,\varepsilon) \in \mathbb{N}$ *such that*

$$
\sum_{k=1}^n\big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^n)), z(t_k^n)-z(t_{k-1}^n)\big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{s_*}^{t_*} 2\mathcal{R}_M(\dot{z}(r)) < \nu \tag{159}
$$

for all $n \geq n(\nu,\varepsilon)$ *such that* Π_n^ε *is an admissible partition of* $[\hat{t}-\varepsilon,\hat{t}+\varepsilon]$*. In particular, also the choice* $\nu = \varepsilon$ *is possible. In this way, for each* $\varepsilon > 0$ *we have found an index* $n(\varepsilon) = n(\varepsilon,\varepsilon) \in \mathbb{N}$ *marking the partition* $\Pi^\varepsilon_{n(\varepsilon)}$ *and constants* $C_1, C_2 > 0$ *such that*

$$
\begin{aligned}
\sum_{k=1}^{n(\varepsilon)} \left\|z(t_k^{\varepsilon n}) - z(t_{k-1}^{\varepsilon n})\right\|_{\mathbf{S}}^\alpha &\leq \frac{1}{C_\star}\Big((\hat{C}+c_\star)(2\varepsilon)^{\frac12}\,\|\dot{u}\|_{L^2(0,\mathsf{T};\mathbf{U})} + C2\varepsilon + \varepsilon\Big)\\
&\leq C_1(2\varepsilon)^{\frac12} + C_2\varepsilon\,.
\end{aligned} \tag{160}
$$

Moreover, there is an index $k(\varepsilon) \in \{1,\ldots,n(\varepsilon)\}$ *such that* $\hat{t} \in [t^{\varepsilon n}_{k(\varepsilon)-1}, t^{\varepsilon n}_{k(\varepsilon)}]$*. Hence,*

$$
\left\|z(t^{\varepsilon n}_{k(\varepsilon)}) - z(t^{\varepsilon n}_{k(\varepsilon)-1})\right\|_{\mathbf{S}}^\alpha \leq \sum_{k=1}^{n(\varepsilon)} \left\|z(t_k^{\varepsilon n}) - z(t_{k-1}^{\varepsilon n})\right\|_{\mathbf{S}}^\alpha \leq C_1(2\varepsilon)^{\frac12} + C_2\varepsilon\,.
$$

Choosing now a subsequence $(\varepsilon_l)_{l\in\mathbb{N}}$ *with* $\varepsilon_l \to 0$ *as* $l \to \infty$*, and* $\hat{t}_l^- := t^{\varepsilon_l n}_{k(\varepsilon_l)-1}$ *as well as* $\hat{t}_l^+ := t^{\varepsilon_l n}_{k(\varepsilon_l)}$ *proves the existence of sequences* $(\hat{t}_l^\pm)_{l\in\mathbb{N}} \subset [0,\mathsf{T}] \setminus \widetilde{N}$ *such that* $\hat{t}_l^- \nearrow \hat{t}$ *and* $\hat{t}_l^+ \searrow \hat{t}$*, and such that also* $\|z(t^{\varepsilon n}_{k(\varepsilon)}) - z(t^{\varepsilon n}_{k(\varepsilon)-1})\|_{\mathbf{S}} \to 0$*, and thus* $z_{\hat{t}}^- = \lim_{l\to\infty} z(\hat{t}_l^-) = \lim_{l\to\infty} z(\hat{t}_l^+) = z_{\hat{t}}^+$*, i.e., with properties* (141)*. This finishes the proof of Thm. 6.2, Item 5.*

To Item 6, continuity of the internal variable: *Given the prerequisites of Item 5, for every* $\hat{t} \in (0,T)$ *we find sequences* $(\hat{t}_l^-)_{l\in\mathbb{N}}$*,* $(\hat{t}_l^+)_{l\in\mathbb{N}}$ *such that* $\hat{t}_l^- \nearrow \hat{t}$ *and* $\hat{t}_l^+ \searrow \hat{t}$*, and such that* $z_{\hat{t}}^- = \lim_{l\to\infty} z(\hat{t}_l^-) = \lim_{l\to\infty} z(\hat{t}_l^+) = z_{\hat{t}}^+$ *in* $\mathbf{S}$*. Like in Corollary 5.5, we now exploit the unidirectionality of* $\mathcal{R}_M$ *to show that indeed*

$$
z_{\hat{t}}^- = \lim_{l\to\infty} z(\hat{t}_l^-) = z(\hat{t}) = \lim_{l\to\infty} z(\hat{t}_l^+) = z_{\hat{t}}^+ \quad \textit{in}\,\mathbf{S}\,. \tag{161}
$$

For this, we argue as follows: Since $\mathbf{S} \subset \mathbf{Z}_M$ *continuously, we also have* $z_{\hat{t}}^- = z_{\hat{t}}^+$ *in* $\mathbf{Z}_M$*. Moreover, by the unidirectionality constraint we have* $(z(\hat{t}) - z(\hat{t}_l^-)) \in \mathbf{K}(\hat{t}_l^-)$*,* $(z(\hat{t}_l^+) - z(\hat{t}_l^-)) \in \mathbf{K}(\hat{t}_l^-)$ *and* $(z(\hat{t}_l^+) - z(\hat{t})) \in \mathbf{K}(\hat{t})$*. Hence* $z(\hat{t}_l^-) \preceq z(\hat{t}) \preceq z(\hat{t}_l^+)$ *for all* $l \in \mathbb{N}$*, where* $\preceq$ *indicates the symbol for the unidirectionality relation. Thus* $z_{\hat{t}}^- \preceq z(\hat{t}) \preceq z_{\hat{t}}^+ = z_{\hat{t}}^-$*, which implies*

$$
z_{\hat{t}}^- = z(\hat{t}) = z_{\hat{t}}^+\,, \quad \textit{first in}\,\mathbf{Z}_M = L^p(\Omega)\,. \tag{162}
$$

By assumption, it is $\mathbf{S} = W^{m,\tilde{p}}(\Omega)$ *with* $\mathbf{X} \subseteq \mathbf{S} \subseteq Z_M$. *Hence, equality* (162) *also holds true in* $\mathbf{S}$ *if* $m = 0$. *Moreover, for* $m > 0$ *we also find equality* (162) *to hold true in* $\mathbf{S}$ *by the uniqueness of weak derivatives. This proves* (161).

The convergence (161) *along the special sequences* $(\hat{t}_l^{\pm})_{l\in\mathbb{N}} \subset [0,\mathsf{T}]\backslash\widetilde{N}$ *will be used now to show continuity of* z *in* $(0,\mathsf{T})$. *For that, consider a general sequence*

$$(\hat{s}^l)_{l\in\mathbb{N}} \subset (0,T) \text{ such that } \hat{s}^l \to \hat{t} \text{ as } l \to \infty\,, \tag{163a}$$

i.e., here in particular also $s^l \in \widetilde{N}$ *is allowed, and we aim to prove that also*

$$z(\hat{s}^l) \to z(\hat{t}) \quad \text{as } l \to \infty\,. \tag{163b}$$

Now, let $(\hat{t}_j)_{j\in\mathbb{N}} \subset (0,T) \setminus \widetilde{N}$ *denote the special sequence* $(\hat{t}_l^-)_{l\in\mathbb{N}}$ *with* (161) *obtained by the construction in Item 5, i.e., we have*

$$\hat{t}_j \to \hat{t} \text{ and } z(\hat{t}_j) \to z(\hat{t})\,. \tag{164}$$

By construction of Item 5, for each $j \in \mathbb{N}$ *there is a partition* $\Pi^{\varepsilon}_{n(j)}$ *such that* $\hat{t}_j = t_{k-1}^{\varepsilon n(j)}$ *and* $\hat{t} \in [t_{k-1}^{\varepsilon n(j)}, t_k^{\varepsilon n(j)}]$ *for some* $k \in \{1,\ldots,n(j)\}$ *and such that*

$$\sum_{k=1}^{n(j)} \big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t^n_{k-1})), z(t^n_k) - z(t^n_{k-1})\big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{\hat{t}+\varepsilon}^{\hat{t}-\varepsilon} 2\mathcal{R}_M(\dot{z}(r)) < \varepsilon\,, \tag{165}$$

since $n(j) \geq n(\varepsilon,\varepsilon)$ *in view of* (159). *Similarly, for each* $\hat{s}^l$, *for all* $l \in \mathbb{N}$ *there is also a special sequence* $(\hat{s}^l_i)_{i\in\mathbb{N}} \subset (0,T) \setminus \widetilde{N}$ *such that*

$$\hat{s}^l_i \to \hat{s}^l \text{ as } i \to \infty \text{ and } z(\hat{s}^l_i) \to z(\hat{s}^l)\,.$$

Let $\tilde{\varepsilon} \in (0,\varepsilon]$ *be general but fixed. Then, for all* $l \in \mathbb{N}$ *there is an index* $i(l,\tilde{\varepsilon}) \in \mathbb{N}$ *and there is an index* $j(\tilde{\varepsilon}) \in \mathbb{N}$ *such that*

$$\text{for all } i > i(l,\tilde{\varepsilon}) : \ \big\|z(\hat{s}^l_i) - z(\hat{s}^l)\big\|_{\mathbf{S}} < \tilde{\varepsilon}\,, \tag{166a}$$

$$\text{for all } j > j(\tilde{\varepsilon}) : \ \big\|z(\hat{t}_j) - z(\hat{t})\big\|_{\mathbf{S}} < \tilde{\varepsilon}\,. \tag{166b}$$

Now we also fix $j > j(\tilde{\varepsilon})$ *and we know that there is a partition* $\Pi^{\varepsilon}_{n(j)}$ *such that* $\hat{t}_j$ *coincides with one of its nodes, in particular* $\hat{t}_j = t_{k-1}^{\varepsilon n(j)}$ *by construction. Then, one finds* $l \in \mathbb{N}$ *large enough such that* $\hat{s}^l \in [t_{k-1}^{\varepsilon n(j)}, t_k^{\varepsilon n(j)}]$ *and also* $\hat{s}^l_i \in [t_{k-1}^{\varepsilon n(j)}, t_k^{\varepsilon n(j)}]$, *in addition to* (166a). *Thanks to this, we estimate*

$$\begin{aligned}
\big\|z(\hat{s}^l) - z(\hat{t})\big\|_{\mathbf{S}} &\leq \big\|z(\hat{s}^l) - z(\hat{s}^l_i)\big\|_{\mathbf{S}} + \big\|z(\hat{s}^l_i) - z(\hat{t})\big\|_{\mathbf{S}} \\
&\leq \tilde{\varepsilon} + \Big\|z(\hat{s}^l_i) - z(t_{k-1}^{\varepsilon n(j)})\Big\|_{\mathbf{S}} + \Big\|z(t_{k-1}^{\varepsilon n(j)}) - z(\hat{t})\Big\|_{\mathbf{S}} \\
&\leq 2\tilde{\varepsilon} + \Big\|z(\hat{s}^l_i) - z(t_{k-1}^{\varepsilon n(j)})\Big\|_{\mathbf{S}}\,,
\end{aligned}$$

where we again used (166). *From this, it follows*

$$\big\|z(\hat{s}^l) - z(\hat{t})\big\|^{\alpha}_{\mathbf{S}} \leq 2^{\alpha-1}\Big((2\tilde{\varepsilon})^{\alpha} + \Big\|z(\hat{s}^l_i) - z(t^{\varepsilon n(j)}{}_{k-1})\Big\|^{\alpha}_{\mathbf{S}}\Big)\,, \tag{167}$$

and it remains to deduce an estimate for the term $\|z(\hat{s}^l_i) - z(t^{n(j)}_{k-1})\|^{\alpha}_{\mathbf{S}}$. *Thanks to* $\hat{s}^l_i, t^{n(j)}_{k-1} \in (0,\mathsf{T}) \setminus \widetilde{N}$ *this can be achieved with the aid of* (137), *keeping in mind that here* $\beta_u = 1$. *Hence, it follows that*

$$\begin{aligned}
C_\star \Big\|z(\hat{s}^l_i) - z(t_{k-1}^{\varepsilon n(j)})\Big\|^{\alpha}_{\mathbf{S}} &\leq (\hat{C} + c_\star)\int_{t_{k-1}^{\varepsilon n(j)}}^{\hat{s}^l_i} \|\dot{u}(r)\|_{\mathbf{U}}\,\mathrm{d}r + C(\hat{s}^l_i - t_{k-1}^{\varepsilon n(j)}) \\
&+ \Big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^{\varepsilon n(j)})), z(\hat{s}^l_i) - z(t_{k-1}^{\varepsilon n(j)})\Big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{t_{k-1}^{\varepsilon n(j)}}^{\hat{s}^l_i} 2\mathcal{R}_M(\dot{z}(r))\,\mathrm{d}r\,.
\end{aligned} \tag{168}$$

In order to further estimate (168) *from above, we once more make use of the unidirectionality constraint. For this, we need to distinguish the following two cases: decay, i.e., for all* $t_1 < t_2 \in [0, \mathsf{T}]$ *it is* $z(t_1) \geq z(t_2)$ *a.e.* $\in \Omega$ *together with* $\dot{z} \leq 0$ *a.e. in* $(0, \mathsf{T}) \times \Omega$, *and growth, i.e., for all* $t_1 < t_2 \in [0, \mathsf{T}]$ *it is* $z(t_1) \leq z(t_2)$ *a.e.* $\in \Omega$ *together with* $\dot{z} \geq 0$ *a.e. in* $(0, \mathsf{T}) \times \Omega$. *We evaluate these two cases for the times* $t_{k-1}^{\varepsilon n(j)} \leq \hat{s}_i^l \leq t_k^{\varepsilon n(j)}$. *In case of decay we thus have* $z(t_{k-1}^{\varepsilon n(j)}) \geq z(\hat{s}_i^l) \geq z(t_k^{\varepsilon n(j)})$ *and hence* $0 \geq z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \geq z(t_k^{\varepsilon n(j)}) - z(t_{k-1}^{\varepsilon n(j)})$. *Together with* $\mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^{\varepsilon n(j)})) \leq 0$ *we see that*

$$
\begin{aligned}
0 &\leq \Big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(\varepsilon t_{k-1}^{\varepsilon n(j)})), z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \Big\rangle_{\mathbf{X}^*,\mathbf{X}} \\
&\leq \Big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^{\varepsilon n(j)})), z(t_k^{\varepsilon n(j)}) - z(t_{k-1}^{\varepsilon n(j)}) \Big\rangle_{\mathbf{X}^*,\mathbf{X}} .
\end{aligned}
\tag{169}
$$

Analogously, in case of growth we have $z(t_{k-1}^{\varepsilon n(j)}) \leq z(\hat{s}_i^l) \leq z(t_k^{\varepsilon n(j)})$ *and hence* $0 \leq z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \leq z(t_k^{\varepsilon n(j)}) - z(t_{k-1}^{\varepsilon n(j)})$. *Together with* $\mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^{\varepsilon n(j)})) \geq 0$ *we again observe* (169) *to hold true.*

Inserting (169) *into* (168) *and exploiting the additivity of the integral gives*

$$
\begin{aligned}
C_\star \Big\| z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \Big\|_{\mathbf{S}}^\alpha &\leq (\hat{C} + c_\star) \int_{t_{k-1}^{\varepsilon n(j)}}^{t_k^{\varepsilon n(j)}} \|\dot{u}(r)\|_{\mathbf{U}} \,\mathrm{d}r + C(t_k^{\varepsilon n(j)} - t_{k-1}^{\varepsilon n(j)}) \\
&+ \Big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{k-1}^{\varepsilon n(j)})), z(t_k^{\varepsilon n(j)}) - z(t_{k-1}^{\varepsilon n(j)}) \Big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{t_{k-1}^{\varepsilon n(j)}}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r \\
&+ \int_{\hat{s}_i^l}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r .
\end{aligned}
\tag{170}
$$

We add the analogous estimates (137) *for each of the nodes* $\tilde{k} \neq k \in \{1, \dots, n(j)\}$ *to* (170) *and divide the result by* $C_\star$. *In this way we obtain*

$$
\begin{aligned}
&\Big\| z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \Big\|_{\mathbf{S}}^\alpha \leq \Big\| z(\hat{s}_i^l) - z(t_{k-1}^{\varepsilon n(j)}) \Big\|_{\mathbf{S}}^\alpha + \sum_{\substack{\tilde{k}=1 \\ \tilde{k}\neq k}}^{n(j)} \Big\| z(t_k^{\varepsilon n(j)}) - z(t_{k-1}^{\varepsilon n(j)}) \Big\|_{\mathbf{S}}^\alpha \\
&\leq \frac{1}{C_\star} \Big((\hat{C} + c_\star) \sum_{\tilde{k}=1}^{n(j)} \int_{t_{\tilde{k}-1}^{\varepsilon n(j)}}^{t_{\tilde{k}}^{\varepsilon n(j)}} \|\dot{u}(r)\|_{\mathbf{U}} \,\mathrm{d}r + C \sum_{\tilde{k}=1}^{n(j)} (t_{\tilde{k}}^{\varepsilon n(j)} - t_{\tilde{k}-1}^{\varepsilon n(j)}) \\
&\quad + \sum_{\tilde{k}=1}^{n(j)} \Big(\Big\langle \mathrm{D}\mathcal{R}_M(\dot{z}(t_{\tilde{k}-1}^{\varepsilon n(j)})), z(t_{\tilde{k}}^{\varepsilon n(j)}) - z(t_{\tilde{k}-1}^{\varepsilon n(j)}) \Big\rangle_{\mathbf{X}^*,\mathbf{X}} - \int_{t_{\tilde{k}-1}^{\varepsilon n(j)}}^{t_{\tilde{k}}^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r \Big) \\
&\quad + \int_{\hat{s}_i^l}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r \Big) \\
&\leq \frac{1}{C_\star} \Big((\hat{C} + c_\star) \int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} \|\dot{u}(r)\|_{\mathbf{U}} \,\mathrm{d}r + C2\varepsilon + \varepsilon + \int_{\hat{s}_i^l}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r \Big) \\
&\leq C_1 (2\varepsilon)^{1/2} + C_2 \varepsilon + \frac{1}{C_\star} \int_{\hat{s}_i^l}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r .
\end{aligned}
$$

Here we used (165) *and growth estimate* (88), *similarly as for* (160). *Putting this together with* (167) *we find*

$$
\begin{aligned}
\big\| z(\hat{s}^l) - z(\hat{t}) \big\|_{\mathbf{S}}^\alpha &\leq 2^{\alpha-1} \Big((2\tilde{\varepsilon})^\alpha + \Big\| z(\hat{s}_i^l) - z(t_{k-1}^{n(j)}) \Big\|_{\mathbf{S}}^\alpha \Big) \\
&\leq 2^{\alpha-1} \Big((2\tilde{\varepsilon})^\alpha + C_1 (2\varepsilon)^{1/2} + C_2 \varepsilon + \frac{1}{C_\star} \int_{\hat{s}_i^l}^{t_k^{\varepsilon n(j)}} 2\mathcal{R}_M(\dot{z}(r)) \,\mathrm{d}r \Big) ,
\end{aligned}
\tag{171}
$$

where $\tilde\varepsilon \in (0,\varepsilon]$ and $[\hat s_i^l, t_k^{\varepsilon n(j)}] \subset [t_{k-1}^{\varepsilon n(j)}, t_k^{\varepsilon n(j)}] \subset [\hat t-\varepsilon, \hat t+\varepsilon]$. Hence, by the absolute continuity of the integral, the right-hand side of (171) *can be made arbitrarily small as $\varepsilon \to 0$. This shows that indeed* (163) *holds true for any sequence $(\hat s^l)_l \subset (0,\mathsf{T})$ with $\hat s^l \to \hat t \in (0,\mathsf{T})$ as $l\to\infty$. Thus we are now in the position to conclude that $z \in C^0((0,\mathsf{T});\mathbf{S})$.* □

6.2 Proof of Theorem 6.1: Viscous case

We carry out the proof of Thm. 6.1 following the lines of Sections 5.1–5.4, by pointing out the arguments which have to be done in a different way due to the presence of the quadratic dissipation potential $\mathcal{R}_M : \mathbf{Z}_M \to [0,\infty)$. In particular, in view of the uniform a priori bounds (48)*, the convergence of a subsequence of the interpolated solutions $(\bar u_\tau, \underline{u}_\tau, u_\tau, \bar z_\tau, \underline{z}_\tau, z_\tau)_\tau$ to a limit pair (u_M, z_M) in the topologies* (76) *is concluded in the same way as already done in Section 5.1. Also the* **boundedness** *$0 \le z_M(t,x) \le 1$ for a.e. $x\in\Omega$ and for all $t\in[0,\mathsf{T}]$ is concluded here like in Section 5.2.1 from the knowledge of this bound for the approximants $(\bar z_\tau(t))_\tau$ together with the strong $L^2(\Omega)$-convergence of this sequence ensured by* (76g) *for all $t\in[0,\mathsf{T}]$.*

Proof of Theorem 6.1, Item 1: Convergence statement (128)**.** *For fixed $M>0$ the uniform a priori bound*

$$\|z_\tau\|_{H^1(0,\mathsf{T};L^2(\Omega))} \le C/\sqrt{M}$$

provided in (48g) *implies the existence of $\tilde z \in H^1(0,\mathsf{T};\mathbf{Z}_M)$ such that, up to a subsequence, $z_\tau \rightharpoonup \tilde z$ weakly in $H^1(0,\mathsf{T};\mathbf{Z}_M)$. It has to be concluded that $\tilde z$ coincides in with z_M, the latter already obtained by convergences* (76f)–(76h)*. Indeed, by the definition of the interpolants* (43) *it is $z_\tau(t) - \bar z_\tau(t) = (t - t_\tau^k)\dot z_\tau(t)$ for any $t \in (t_\tau^{k-1}, t_\tau^k]$, and in view of the bound* (48g) *it thus follows*

$$\begin{aligned}
&\int_0^{\mathsf{T}}\int_\Omega \big(\tilde z - z_M\big) v \,\mathrm{d}x\,\mathrm{d}t = \lim_{\tau\to0}\int_0^{\mathsf{T}}\int_\Omega \big(z_\tau(t) - \bar z_\tau(t)\big)v(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \lim_{\tau\to0}\int_0^{\mathsf{T}}\int_\Omega \big(\tau \dot z_\tau\big)v(t)\,\mathrm{d}x\,\mathrm{d}t \le \lim_{\tau\to0}\tau\|\dot z_\tau\|_{L^2((0,\mathsf{T})\times\Omega)}\|v\|_{L^2((0,\mathsf{T})\times\Omega)} = 0
\end{aligned}$$

for all $v\in L^2((0,\mathsf{T})\times\Omega)$, which proves the assertion. □

Proof of Theorem 6.1, Item 2: Defining properties (9) **of the solution.** *As $\tau\to0$ the* ***weak balance of momentum*** (9c) *is obtained from its time-discrete version* (47b) *thanks to convergences* (76) *by repeating the lines of Section 5.2.2. Also an* ***upper energy-dissipation estimate for all*** *$t\in[0,\mathsf{T}]$ can be deduced following the arguments of Section 5.2.4 by exploiting the lower semicontinuity properties of the functionals $\mathcal{K}$, $\mathcal{E}(t,\cdot,\cdot)$, and $\int_0^t 2\mathcal{V}(\cdot;\cdot)\mathrm{d}r$ with respect to convergences* (76)*, the non-negativity of the Yosida-term $\int_0^t\int_\Omega \frac{N_\tau}{2}(\dot z_\tau)_+\,\mathrm{d}x\,\mathrm{d}r \ge 0$, together with the lower semicontinuity of the quadratic dissipation $\int_0^t\int_\Omega \frac{M}{2}(\cdot)\,\mathrm{d}x\,\mathrm{d}r$ with respect to the weak $L^2((0,\mathsf{T})\times\Omega)$-convergence obtained in* (128)*. For all $t\in[0,\mathsf{T}]$ this results in the upper energy-dissipation estimate*

$$\begin{aligned}
&\mathcal{K}(\dot u_M(t)) + \mathcal{E}(t,u_M(t),z_M(t)) + \int_0^t 2\big(\mathcal{V}(z_M;\dot u_M) + \mathcal{R}_M(\dot z_M)\big)\,\mathrm{d}r\\
&\quad\le \mathcal{K}(\dot u_0) + \mathcal{E}(0,u_0,z_0) + \int_0^t \partial_t\mathcal{E}\big(r,u(r),z(r)\big)\,\mathrm{d}r\,.
\end{aligned}\tag{172}$$

The opposite inequality will be deduced below in (174) *with the aid of a Riemann-sum argument once the one-sided variational inequality* (9a) *is verified.*

Unidirectionality (9b)**.** *The deduction of the a priori bounds* (48) *was carried out in Section 4.6 and also led to the estimates* (72) *and* (74)*. The latter yields $\int_0^{\mathsf{T}}\int_\Omega |(\dot z_\tau)_+|^2\,\mathrm{d}x\,\mathrm{d}r \le \frac{C}{N_\tau} = C\tau$ for all $\tau>0$. Hence, by lower semi-continuity of the map $z\mapsto \int_0^{\mathsf{T}}\int_\Omega |(z)_+|^2\,\mathrm{d}x\,\mathrm{d}r$ with respect to weak $L^2((0,\mathsf{T})\times\Omega)$-convergence, it follows by convergence* (128)

$$0 = \lim_{\tau\to0} C\tau = \liminf_{\tau\to0}\int_0^{\mathsf{T}}\int_\Omega |(\dot z_\tau(r))_+|^2\,\mathrm{d}x\,\mathrm{d}r \ge \int_0^{\mathsf{T}}\int_\Omega |(\dot z_M(r))_+|^2\,\mathrm{d}x\,\mathrm{d}r\,. \tag{173}$$

Since $\int_\Omega |(\dot z_M(r))_+|^2\,\mathrm{d}x \ge 0$ for all $r\in[0,\mathsf{T}]$, by the non-negativity of the integrand $|(\dot z_M(r))_+|^2$ we conclude from (173) *that there has to hold $\int_\Omega |(\dot z_M(r))_+|^2\,\mathrm{d}x = 0$ for a.a. $r\in[0,\mathsf{T}]$, and also $\int_\Omega (\dot z_M(r))_+\,\mathrm{d}x = 0$ for a.a. $r\in[0,\mathsf{T}]$*

by Hölder's inequality. Consider now any interval $[s,t] \subset [0,\mathsf{T}]$*. Then, by the convexity of the function* $(\cdot)_+$ *and Jensen's inequality we deduce*

$$\int_\Omega \big(z_M(t) - z_M(s)\big)_+ \,\mathrm{d}x = \int_\Omega \Big(\int_s^t \dot z_M(r)\,\mathrm{d}r\Big)_+ \,\mathrm{d}x \le \int_s^t \int_\Omega (\dot z_M(r))_+ \,\mathrm{d}x\,\mathrm{d}r = 0\,,$$

which proves that $z_M(t) \le z_M(s)$ *a.e. in* Ω *for all* $s < t \in [0,\mathsf{T}]$*.*

Viscous phase-field evolution (9a) ***for a.a.*** $t \in [0,\mathsf{T})$***.*** *Also the limit passage in the time-discrete damage evolution* (47a) *to the viscous evolution* (9a) *is proven similar to the rate-independent case. We thus proceed along the lines of Section 5.2: Testing* (47a) *with* $\eta \in \mathbf{Y}$ *such that* $\eta \le 0$ *a.e. in* Ω*, omitting the negative term* $\int_\Omega N_\tau(\dot z_\tau)_+ \eta\,\mathrm{d}x$*, and integrating over an arbitrary measurable set* $I \subset [0,\mathsf{T}]$ *one arrives at the inequality* (108)*, i.e.,*

$$\begin{aligned}\int_I\int_\Omega \Big[-\frac{1}{\ell}\big(1-\bar z_\tau(t)\big) + M\dot z_\tau(t)\Big]\eta + \ell\nabla \bar z_\tau(t)\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t&\\ \ge \int_I\int_\Omega \big[\frac{1}{2}\mathbb{C}'\big(\bar z_\tau(t)\big)e\big(\underline{u}_\tau(t)\big) : e\big(\underline{u}_\tau(t)\big)\big](-\eta)\,\mathrm{d}x\,\mathrm{d}t\,.&\end{aligned}$$

To pass to the limit one uses the lower and upper semicontinuity arguments from (109)–(112)*. Yet, for the limit passage in the viscous term the argument from* (110) *is replaced in view of weak convergence* (128) *by the following*

$$\int_I\int_\Omega M\dot z_\tau\,\eta\,\mathrm{d}x\,\mathrm{d}r \to \int_I\int_\Omega M\dot z_M\,\eta\,\mathrm{d}x\,\mathrm{d}r\,.$$

In this way one obtains the time-integrated one-sided variational inequality

$$\begin{aligned}\int_I\int_\Omega \big[\frac{1}{2}\mathbb{C}'\big(z_M(t)\big)e\big(u_M(t)\big) : e\big(u_M(t)\big) - \frac{1}{\ell}\big(1-z_M(t)\big) + M\dot z_M\big]\eta\,dx\,\mathrm{d}t&\\ + \int_I\int_\Omega \ell\nabla z_M(t)\cdot\nabla\eta\,dx\,\mathrm{d}t \ge 0&\end{aligned}$$

to hold for every measurable set $I \subset [0,\mathsf{T}]$*. From this, we conclude the assertion, i.e., that the one-sided variational inequality* (9a) *holds true for a.e.* $t \in [0,\mathsf{T})$*.*

Energy-dissipation balance (9d) ***for a.a.*** $t \in [0,\mathsf{T})$***.*** *In view of* (172) *it now remains to show the opposite estimate*

$$\begin{aligned}\mathcal{K}(\dot u_M(t)) + \mathcal{E}(t,u_M(t),z_M(t)) + \int_0^t 2\big(\mathcal{V}(z_M;\dot u_M) + \mathcal{R}_M(\dot z_M)\big)\,\mathrm{d}r&\\ \ge \mathcal{K}(\dot u_0) + \mathcal{E}(0,u_0,z_0) + \int_0^t \partial_t\mathcal{E}\big(r,u(r),z(r)\big)\,\mathrm{d}r\,.&\end{aligned}\tag{174}$$

Like for the rate-independent setting in Section 5.2.4 we will first obtain (174) *to hold for a.e.* $t \in [0,\mathsf{T})$*, only. In analogy to these arguments the proof for* (174) *also uses a Riemann-sum argument applied to the one-sided variational inequality* (9a) *that was shown above to be valid for a.e.* $t \in [0,\mathsf{T})$*, only. Let* $\widetilde N \subset [0,\mathsf{T}]$ *denote the* $\mathcal{L}^1$*-null set for which* (9a) *does not hold and consider any* $t \in (0,\mathsf{T}]\backslash\widetilde N$*. Then, thanks to Remark 6.3 we find a sequence of (not necessarily uniform) partitions* $\Pi_\theta = \{0 = t_\theta^0 < t_\theta^1 < \ldots < t_\theta^{N_\theta} = t\}$ *with (possibly variable) step-size* $\theta_k = t_\theta^k - t_\theta^{k-1}$*,* $\theta = \max_{k\in\{1,\ldots,N_\theta\}}\big|t_\theta^k - t_\theta^{k-1}\big|$ *and* $\theta \downarrow 0$ *as* $N_\theta \to \infty$ *such that* $t_\theta^k \in [0,\mathsf{T})\backslash\tilde N$ *and such that*

$$\begin{aligned}&\sum_{k=1}^{N_\theta}\theta_k\int_\Omega\Big(\tfrac{1}{2}\mathbb{C}(z_M(t_\theta^k))\tfrac{|e(u_M(t_\theta^{k-1}))|^2 - |e(u_M(t_\theta^k))|^2}{\theta_k} + M\dot z_M(t_\theta^{k-1})\tfrac{z_M(t_\theta^k) - z_M(t_\theta^{k-1})}{\theta_k}\Big)\,\mathrm{d}x\,\mathrm{d}r\\ &\qquad\qquad\qquad\qquad\downarrow\ \theta\to 0\\ &\int_0^{\mathsf{T}}\int_\Omega\Big(-\mathbb{C}\big(z_M(r)e(u_M(r)) : e(\dot u_M(r))\big) + M|\dot z_M(r)|^2\Big)\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}\tag{175}$$

Now we test the one-sided variational inequality (9a) *at time* t_θ^{k-1} *by* z_θ^k, *sum up over* $k \in \{1, \ldots, N_\theta\}$ *and take the limit* $\theta \to 0$. *Thanks to the convergence of the Riemann-sums* (175) *this results in*

$$\begin{aligned}\mathcal{E}\big(0, u(0), s(0)\big) \leq \mathcal{E}\big(t, u(t), z(t)\big) &+ \int_0^t 2\mathcal{R}_M(\dot{z}_M(r))\,\mathrm{d}r \\ &- \int_0^t \int_\Omega \mathbb{C}\big(z(r)\big)e\big(u(r)\big) : e\big(\dot{u}(r)\big)\,\mathrm{d}x\,\mathrm{d}r \\ &+ \int_0^t \langle f(r), \dot{u}(r)\rangle_{\mathbf{U}^*,\mathbf{U}}\,\mathrm{d}r - \int_0^t \partial_t \mathcal{E}(r, u(r), s(r))\,\mathrm{d}r\,,\end{aligned}$$

which is the viscous analogon of (118). *This is combined with* (119), *the latter obtained by testing the weak momentum balance* (9c) *by* $\dot{u}$. *This procedure yields* (174) *and thus proves the energy-dissipation balance to hold for a.e.* $t \in [0, \mathsf{T})$.

Proof of Theorem 6.1, Item 3: Regularity & energy-dissipation balance (9d) **for all** $t \in [0, \mathsf{T})$, **and Item 4: Improved convergence** (130). *For the regularity statements* (77) *for the displacements we point to Section 5.2.2, where assertion was obtained based on the convergence results* (76a)–(76c) *and a priori bound* (48d) *in* $L^2(0, \mathsf{T}; \mathbf{U}^*)$. *Similary, also the regularity* $z \in H^1(0, \mathsf{T}; L^2(\Omega)) \cap L^\infty(0, \mathsf{T}; \mathbf{X})$ *in* (129) *is a direct consequence of convergence results* (76f)–(76h) *together with the weak* $H^1(0, \mathsf{T}; \mathbf{Z}_M)$*-convergence* (128). *We now discuss the last statement of* (129), *i.e.,* $z \in C^0([0, \mathsf{T}); \mathbf{X})$. *For this, we consult Theorem 6.2. The above discussed regularity for* u *provides assumption A3). We further note that assumption* $\widetilde{A1}$*) is satisfied by* (9a) *with the closed, convex set*

$$\mathbf{K}(t) := \{\eta \in \mathbf{Y},\ -1 \leq \eta \leq 0 \text{ a.e. in } \Omega\} \text{ for all } t \in [0, \mathsf{T}]\backslash \widetilde{N}\,.$$

Similarly, also the upper energy dissipation estimate (134) *claimed in assumption* $\widetilde{A2}$*) is valid on* $[0, \mathsf{T}]\backslash\widetilde{N}$ *thanks to* (9d).

Moreover, the weak momentum balance (9c) *holds true for all* $t \in [0, \mathsf{T}]$ *and thus yields A9). As already checked in Section 5.3, in view of Lemma 5.4 also the properties claimed in assumptions A4)-A8) and A10) apply to system* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$. *Thus, in view of Lemma 5.4 all statements of Theorem 6.2 are valid for*

$$(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})\,.$$

In particular, estimates (135) *and* (140) *are valid with* $\alpha = 2$, $\mathbf{S} = \mathbf{X}$, *and* $\beta_u = 1$. *Since the dissipation potential* $\mathcal{R}_M$ *encodes a unidirectionality condition we conclude that* $z \in C^0((0, \mathsf{T}); \mathbf{X})$ *by Theorem 6.2. The continuity in* $t = 0$ *stems from the fact that* $0 \in [0, \mathsf{T}]\backslash\widetilde{N}$ *by assumption so that one can deduce continuity from the right following the lines of the proof of Items 5 and 6. Now, by the continuity properties of* $(u, z) \in C([0, \mathsf{T}]; \mathbf{U}) \times C([0, \mathsf{T}); \mathbf{X})$ *we see that the validity of the energy balance can be carried over from* $[0, \mathsf{T}]\backslash\widetilde{N}$ *to all of* $[0, \mathsf{T})$. *We summarize these results in the following*

Corollary 6.4. *Let the assumptions of Theorem 6.1 be satisfied and let the one-sided variational inequality* (9a) *hold true for the initial datum* (u_0, z_0). *Then system* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ *complies with the assumptions* $\widetilde{A1}$*),* $\widetilde{A2}$*), A3)–A10) of Theorem 6.2. Hence, a pair* (u, z) *obtained by convergences* (76) *is continuous with respect to time, in particular* $(u, z) \in C([0, \mathsf{T}]; \mathbf{U}) \times C([0, \mathsf{T}); \mathbf{X})$, *and it complies with the energy dissipation balance* (9d) *for all* $t \in [0, \mathsf{T})$.

Based on the energy dissipation balance (9d) *also the improved, strong convergence statements* (130) *can be concluded for all* $t \in [0, \mathsf{T})$ *by repeating the arguments of Section 5.4.* □

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