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Spatial decay of the vorticity field of time-periodic viscous flow past a body

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Abstract

We study the asymptotic spatial behavior of the vorticity field associated to a time-periodic Navier-Stokes flow past a body in the class of weak solutions satisfying a Serrin-like condition. We show that outside the wake region the vorticity field decays pointwise at an exponential rate, uniformly in time. Moreover, decomposing it into its time-average over a period and a so-called purely periodic part, we prove that inside the wake region, the time-average has the same algebraic decay as that known for the associated steady-state problem, whereas the purely periodic part decays even faster, uniformly in time. This implies, in particular, that “sufficiently far” from the body, the time-periodic vorticity field behaves like the vorticity field of the corresponding steady-state problem.

1 Introduction

Consider a (rigid) body, $\mathscr{B}$, translating with constant nonzero velocity, v_∞, in a viscous (Navier–Stokes) liquid, $\mathscr{L}$, that occupies the whole space outside $\mathscr{B}$. Without loss of generality, we assume that v_∞ is directed along the positive x_1-axis, namely, $v_\infty = \lambda\, \mathrm{e}_1$ with $\lambda > 0$. We also assume that $\mathscr{L}$ is subject to a body force and a distribution of boundary velocity, both being time-periodic of period $\mathcal{T}$. Then, the time-periodic dynamics of the liquid around the body are governed by the following set of equations

$$\begin{cases} \partial_t u - \Delta u - \lambda \partial_1 u + u \cdot \nabla u + \nabla \mathfrak{p} = f & \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \Omega, \\ u = u_* & \text{on } \mathbb{T} \times \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(t,x) = 0 & \text{for } t \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\Omega := \mathbb{R}^3 \backslash \mathscr{B}$ is the domain occupied by the liquid. Moreover, $u\colon \mathbb{T}\times\Omega \to \mathbb{R}^3$ and $\mathfrak{p}\colon \mathbb{T}\times\Omega \to \mathbb{R}$ are velocity and pressure fields of the liquid, $f\colon \mathbb{T} \times \Omega \to \mathbb{R}^3$ is the external body force, and $u_*\colon \mathbb{T} \times \partial\Omega \to \mathbb{R}^3$ the velocity field at the boundary. The time-axis is given by the torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$, which ensures that all functions appearing in (1.1) are time-periodic with a prescribed period $\mathcal{T} > 0$. Note that for a body at rest, that is, for $\lambda = 0$, the mathematical and physical characteristics of the flow are very different from those for $\lambda \neq 0$. For this issue, we refer the reader to the recent papers [13, 14].

Existence, uniqueness and spatial asymptotic behavior of solutions to (1.1) have been the object of several recent researches [15, 16, 7]. In particular, under suitable assumptions on the data, these results provide sharp pointwise *algebraic* decays for the velocity field and its first spatial derivatives; see [7] and Theorem 6.1 below. However, as suggested by physical grounds, the vorticity field $\omega := \operatorname{curl} u$ is expected to decay at an *exponential* rate, at least outside the “wake region"behind $\mathscr{B}$. It is just to this question that the present paper is devoted.

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More precisely, we shall study the asymptotic behavior of the vorticity field $\operatorname{curl} u(x,t)$ for $|x| \to \infty$, uniformly in time. In these regards, we recall that in the case of a steady-state flow, that is, when (v,p) is a time-independent solution to (1.1), a famous result of CLARK [3] and BABENKO and VASIL'EV [1] shows that for $|x|$ sufficiently large one has

$$|\operatorname{curl} v(x)| \leq C|x|^{-3/2} \mathrm{e}^{-\alpha s(x)} \tag{1.2}$$

for some constants $C,\ \alpha > 0$, where

$$s(x) := |x| + x_1.$$

In particular, this reflects the anisotropic behavior of the fluid flow and translates, in mathematical terms, the presence of a “wake region"behind $\mathscr{B}$. Estimate (1.2) implies that the vorticity, $\operatorname{curl} v$, decays exponentially fast on rays $\{x \in \mathbb{R}^3 \mid x_1 = \theta|x|\}$ for $\theta \in (-1,1]$, while inside parabolic regions $\{x \in \mathbb{R}^3 \mid s(x) \leq \beta\}$, $\beta > 0$, estimate (1.2) merely yields an algebraic decay rate. Since time-independent solutions are trivially also time-periodic, one would expect a similar behavior in the time-periodic case. As a matter of fact, we show that this is indeed true and that the vorticity field associated to a time-periodic flow is subject to an analogus estimate.

Actually, as proved in [7], if we split u into its time average v and a purely periodic part $w := u - v$, then the decay rates of v and ∇v are much slower than those of w and ∇w. Thus, also in the problem at hand, it seems reasonable to derive separate pointwise estimates for the two parts $\operatorname{curl} v$ and $\operatorname{curl} w$ of the vorticity $\operatorname{curl} u$. In doing so, we are indeed able to show that the time-independent part v satisfies (1.2) whereas the other part obeys the estimate

$$|\operatorname{curl} w(t,x)| \leq C|x|^{-9/2} \mathrm{e}^{-\alpha s(x)} \tag{1.3}$$

for all sufficiently large $|x|$, and therefore decays faster. It is worth emphasizing that we establish this result for *any* weak solution to (1.1) (see Definition 3.1), whose purely periodic part *only* satisfies the Serrin-like condition (3.3), provided the data are sufficiently smooth with f of bounded spatial support; see Theorem 3.2.

A main tool in our approach is the introduction of a time-periodic fundamental solution associated to the vorticity field $\operatorname{curl} u$. The concept of time-periodic fundamental solutions in the field of fluid dynamics is new and was recently introduced by KYED [20] and GALDI and KYED [16] in the case of a three-dimensional Navier–Stokes flow, and further extended by EITER and KYED [9] to the general n-dimensional case. The fundamental solution Γ^λ introduced there consists of the fundamental solution Γ_0^λ to the steady-state problem and a second so-called purely periodic part $\Gamma_\perp^\lambda$. Analogously, we define the time-periodic vorticity fundamental solution ϕ^λ as the sum of the corresponding steady-state fundamental solution ϕ_0^λ and a purely periodic part $\phi_\perp^\lambda$.

After introducing these time-periodic fundamental solutions, our procedure parallels that of [4], where DEURING and GALDI studied the vorticity field associated to the steady-state flow past a rotating body. Note that this problem is directly related to the one investigated here since a time-independent solution in the frame attached to the rotating body corresponds to a time-periodic solution in the inertial frame. By means of the above time-periodic fundamental solutions we deduce representation formulas for u and $\operatorname{curl} u$, which enable us to express u as a fixed point of a nonlinear map F_S of convolution type; see eq. (6.12). We then establish the existence of a fixed point $z = F_S(z)$ of this map in a class of functions such that $\operatorname{curl} z$ decays in the expected way; see Corollary 8.1 . Successively, we show that this fixed point is, in fact, unique in the *larger* class of functions that merely satisfy the pointwise estimates of u and ∇u established in [7]; see Theorem 8.3. Since u is a fixed point of F_S by construction, we thus conclude $u = z$ and that $u = v + w$ satisfies (1.2) and (1.3). Observe that, in order to employ the contraction mapping principle, the existence of the fixed point z is established in a class of functions that satisfy a slightly weaker estimate than that given in (1.3). However, by another application of the representation formulas via the vorticity fundamental solution, we finally obtain the asserted decay rates (1.2) and (1.3). The result just described is proved in the case where Ω is the whole space $\mathbb{R}^3$. However, we show that it can be readily transferred to the case of an exterior domain by a classical cut-off argument, provided u_* and f are sufficiently smooth, with u_* having zero total net flux at $\partial\Omega$. We leave it as an open question whether this condition can indeed be removed.

Finally, we observe that some of the intermediate results are contained in the first author's PhD thesis [6]. However, they were derived under the stringent assumption that both external force f and solution u are of class C^∞. In contrast, here we merely require summability assumptions on f and u (see (3.1) and (3.3)) which represents a rather significant improvement

The paper is structured as follows. After introducing the basic notation in Section 2, we present our main result on the decay of the vorticity field in Section 3. In Section 4 we recall the notion of a time-periodic fundamental solution to the Navier–Stokes equations and introduce the concept of a time-periodic vorticity fundamental solution. Section 5 is dedicated to the study of regularity of weak solutions to the time-periodic Navier–Stokes problem. The introduced fundamental solutions are employed in Section 6 in order to conclude a suitable fixed-point equation. After the derivation of appropriate estimates for the terms in this equation in Section 7, we finish the proof of the main result in Section 8.

2 Notation

Points in $\mathbb{T}\times\Omega$ for $\Omega\subset\mathbb{R}^3$ are usually denoted by (t,x) and consist of a time variable $t\in\mathbb{T}$ and a spatial variable $x\in\Omega$. For a sufficiently regular function $u\colon \mathbb{T}\times\Omega\to\mathbb{R}^3$ we write $\partial_j u := \partial_{x_j} u$, and we set $\Delta u := \partial_j\partial_j u$ and $\operatorname{div} u := \partial_j u_j$. Here we employ Einstein's summation convention, which we do frequently in the following. By δ_{jk} and $\varepsilon_{jk\ell}$ we denote the Kronecker delta and the Levi-Civita symbol, respectively.

For $R > 0$ and $x \in \mathbb{R}^3$ we set $\mathrm{B}_R(x) := \{y \in \mathbb{R}^3 \mid |x-y| < R\}$ and $\mathrm{B}^R(x) := \{y \in \mathbb{R}^3 \mid |x-y| > R\}$, and in the case $x = 0$ we write $\mathrm{B}_R := \mathrm{B}_R(0)$ and $\mathrm{B}^R := \mathrm{B}^R(0)$. Moreover, for $R > r > 0$ we set $\mathrm{B}_{r,R} := \mathrm{B}_R \cap \mathrm{B}^r$. For vectors $a, b \in \mathbb{R}^3$ their vector product $a\wedge b$ and their tensor product $a\otimes b$ are given by $(a\wedge b)_j = \varepsilon_{jk\ell}a_k b_\ell$ and $(a\otimes b)_{jk} = a_j b_k$, respectively. Moreover, we call a subset $\Omega\subset\mathbb{R}^3$ an exterior domain, if it is the complement of a non-empty compact subset of $\mathbb{R}^3$. Without loss of generality, we always assume that 0 is contained in the interior of $\mathbb{R}^3\setminus\Omega$.

In order to include the time periodicity in the formulation of the Navier–Stokes equations (1.1), we formulated the system on $\mathbb{T}\times\Omega$. In the case $\Omega=\mathbb{R}^3$, which plays a prominent role in our approach, the time-space domain is given by the locally compact Abelian group $G := \mathbb{T}\times\mathbb{R}^3$. The dual group of G can be identified with $\widehat{G} = \mathbb{Z}\times\mathbb{R}^3$, the elements of which we denote by $(k,\xi)\in\mathbb{Z}\times\mathbb{R}^3$. We equip the group $\mathbb{T}$ with the normalized Haar measure given by

$$\forall f \in \mathrm{C}(\mathbb{T}): \qquad \int_{\mathbb{T}} f(t)\,\mathrm{d}t = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} f(t)\,\mathrm{d}t,$$

the group $\mathbb{Z}$ with the counting measure, and G and $\widehat{G}$ with the corresponding product measures. The

Fourier transform $\mathscr{F}_G$ on G and its inverse $\mathscr{F}_G^{-1}$ are formally given by

$$\mathscr{F}_G[f](k,\xi) := \int_{\mathbb{T}}\int_{\mathbb{R}^3} f(t,x)\,\mathrm{e}^{-i\frac{2\pi}{\mathcal{T}}kt-ix\cdot\xi}\,\mathrm{d}x\mathrm{d}t,$$
$$\mathscr{F}_G^{-1}[f](t,x) := \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^3} f(k,\xi)\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt+ix\cdot\xi}\,\mathrm{d}\xi.$$

This defines an isomorphism $\mathscr{F}_G\colon \mathscr{S}(G)\to\mathscr{S}(\widehat{G})$ with inverse $\mathscr{F}_G^{-1}$, provided that the Lebesgue measure $\mathrm{d}\xi$ is normalized appropriately. Here $\mathscr{S}(G)$ is the so-called Schwartz–Bruhat space, which is a generalization of the classical Schwartz space in the Euclidean setting; see [2, 8]. By duality, this yields an isomorphism $\mathscr{F}_G\colon \mathscr{S}'(G)\to\mathscr{S}'(\widehat{G})$ between the corresponding dual spaces $\mathscr{S}'(G)$ and $\mathscr{S}'(\widehat{G})$, the spaces of tempered distributions.

For an open set $\Omega\subset\mathbb{R}^3$ or $\Omega\subset\mathbb{T}\times\mathbb{R}^3$ and $q\in[1,\infty]$, $m\in\mathbb{N}$, we denote the classical Lebesgue and Sobolev spaces by $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{m,q}(\Omega)$, respectively. Moreover, $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is the set of all locally integrable functions, and $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ is the subset of $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with locally integrable weak derivatives. For an open subset $\Omega\subset\mathbb{R}^3$, homogeneous Sobolev spaces are denoted by

$$\mathrm{D}^{m,q}(\Omega) := \big\{u\in\mathrm{L}^1_{\mathrm{loc}}(\Omega)\;\big|\;\nabla^m u\in\mathrm{L}^q(\Omega)\big\},$$

where $\nabla^m u$ denotes the collection of all m-th weak derivatives of u. We further set

$$\mathrm{C}^\infty_{0,\sigma}(\Omega) := \{\varphi\in\mathrm{C}^\infty_0(\Omega)^3\mid\operatorname{div}\varphi=0\},$$

where $\mathrm{C}^\infty_0(\Omega)$ is the class of compactly supported smooth functions on Ω. For $q\in[1,\infty]$ and a (semi-)normed vector space X, $\mathrm{L}^q(\mathbb{T};X)$ denotes the corresponding Bochner–Lebesgue space on $\mathbb{T}$, and

$$\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega) := \big\{u\in\mathrm{L}^q(\mathbb{T};\mathrm{W}^{2,q}(\Omega))\;\big|\;\partial_t u\in\mathrm{L}^q(\mathbb{T}\times\Omega)\big\}.$$

We further define the projections

$$\mathcal{P}f(x) := \int_{\mathbb{T}} f(t,x)\,\mathrm{d}t, \qquad \mathcal{P}_\perp f := f-\mathcal{P}f,$$

which decompose $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)$ into a time-independent *steady-state* part $\mathcal{P}f$ and a remainder *purely periodic* part $\mathcal{P}_\perp f$. One readily sees that $\mathcal{P}$ and $\mathcal{P}_\perp$ are bounded operators on $\mathrm{L}^q(\mathbb{T}\times\Omega)$ for all $q\in[1,\infty]$ and that

$$\mathcal{P}f = \mathscr{F}_G^{-1}\big[\delta_{\mathbb{Z}}(k)\mathscr{F}_G[f]\big], \qquad \mathcal{P}_\perp f = \mathscr{F}_G^{-1}\big[(1-\delta_{\mathbb{Z}}(k))\mathscr{F}_G[f]\big],$$

where $\delta_{\mathbb{Z}}$ is the delta distribution on $\mathbb{Z}$.

The letter C always denotes a generic positive constant, the value of which may change from line to line. When we want to specify the dependence of the constant C on quantities $a,b,\ldots$, we write $C(a,b,\ldots)$.

3 Main result

As emphasized earlier on, our focus is the pointwise estimates of the vorticity field $\operatorname{curl} u$ associated to a solution $(u,\mathfrak{p})$ of (1.1). More precisely, we study the vorticity field of weak solutions to (1.1) defined as follows.

Definition 3.1. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$. A function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\Omega)^3$ is called* weak solution *to* (1.1) *if*

i. $\nabla u \in \mathrm{L}^2(\mathbb{T}\times\Omega)^{3\times 3}$, $u \in \mathrm{L}^2(\mathbb{T};\mathrm{L}^6(\Omega))^3$, $\operatorname{div} u = 0$ *in* $\mathbb{T}\times\Omega$, $u = u_*$ *on* $\mathbb{T}\times\partial\Omega$,

ii. $\mathcal{P}_\perp u \in \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\Omega))^3$,

iii. the identity

$$\int_{\mathbb{T}\times\Omega} \big[-u\cdot\partial_t\varphi + \nabla u : \nabla\varphi - \lambda\partial_1 u\cdot\varphi + (u\cdot\nabla u)\cdot\varphi \big]\,\mathrm{d}(t,x) = \int_{\mathbb{T}\times\Omega} f\cdot\varphi\,\mathrm{d}(t,x)$$

holds for all test functions $\varphi \in \mathrm{C}^\infty_{0,\sigma}(\mathbb{T}\times\Omega)$.

Let us explain the choice of the functional class for weak solutions. When $\Omega = \mathbb{R}^3$, condition i. is equivalent to $u \in \mathrm{L}^2(\mathbb{T};\mathrm{D}^{1,2}_{0,\sigma}(\mathbb{R}^3))$, where $\mathrm{D}^{1,2}_{0,\sigma}(\mathbb{R}^3)$ is the closure of $\mathrm{C}^\infty_{0,\sigma}(\mathbb{R}^3)$ with respect to the homogeneous norm $\|\nabla\cdot\|_2$. In this case, the class of solutions from Definition 3.1 is the same as considered in [16] and [7], where the asymptotic behavior of the velocity field u and its gradient ∇u was investigated. Moreover, for any $f \in \mathrm{L}^2(\mathbb{T};\mathrm{D}^{-1,2}_0(\mathbb{R}^3))^3$ the existence of a weak solution in the above sense was shown by KYED [19]. Therefore, this class of solutions is a natural candidate for further investigation of the associated vorticity field $\operatorname{curl} u$.

The goal of the present article is to prove the following result.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class C^2, and let $\lambda > 0$. Let f and u_* be such that*

$$\forall q \in (1,\infty):\ f \in \mathrm{L}^q(\mathbb{T}\times\Omega)^3, \qquad \operatorname{supp} f \textit{ bounded}, \tag{3.1}$$

$$u_* \in \mathrm{C}(\mathbb{T};\mathrm{C}^2(\partial\Omega))^3 \cap \mathrm{C}^1(\mathbb{T};(\partial\Omega))^3, \qquad \int_{\partial\Omega} u_*\cdot\mathrm{n}\,\mathrm{d}S = 0, \tag{3.2}$$

where n *denotes the unit outer normal at* $\partial\Omega$. *Let u be a weak time-periodic solution to* (1.1) *in the sense of Definition 3.1, which satisfies*

$$\exists r \in (5,\infty):\quad \mathcal{P}_\perp u \in \mathrm{L}^r(\mathbb{T}\times\Omega)^3. \tag{3.3}$$

Then there exist constants $C_1 > 0$ and $\alpha = \alpha(\lambda,\mathcal{T}) > 0$ such that

$$|\operatorname{curl}\mathcal{P}u(x)| \le C_1|x|^{-3/2}\,\mathrm{e}^{-\alpha s(x)}, \tag{3.4}$$

$$|\operatorname{curl}\mathcal{P}_\perp u(t,x)| \le C_1|x|^{-9/2}\,\mathrm{e}^{-\alpha s(x)} \tag{3.5}$$

for all $t \in \mathbb{T}$ and $x \in \Omega$.

Remark 3.3. The constant C_1 depends on Ω, λ and on norms of the solution u which, in turn, can be estimated in terms of the body force f. So, ultimately, C_1 depends on Ω, λ and f. If not specified otherwise, this may always be the case for all other constants C, C_i that we will introduce throughout the paper.

Remark 3.4. In our proof, we need the zero-flux condition $(3.2)_4$ on the boundary velocity u_*, which, instead, is not needed in the particular case of steady-state solutions [3, 1]. Though it is probable that our result continues to hold if the flux is only “sufficiently small,ïť is not clear whether the same conclusion may be drawn for flux of arbitrary magnitude.

Remark 3.5. Condition (3.3) is merely a technical assumption. As pointed out in [16] for the case $\Omega = \mathbb{R}^3$, it leads to additional local regularity of the solution but does not improve its spatial decay properties.

Remark 3.6. If f is time-independent, then $u \equiv \mathcal{P}u$, and our result reduces to that of CLARK [3] and BABENKO and VASIL'EV [1]. Actually –as it becomes clear from our proof– in such a case, we do not need the assumptions (3.2), and (3.3).

4 Time-periodic fundamental solutions

In this section, we consider the so-called Oseen linearization of (1.1) in the whole space given by

$$\begin{cases} \partial_t u - \Delta u - \lambda\partial_1 u + \nabla\mathfrak{p} = f & \text{in } \mathbb{T}\times\mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T}\times\mathbb{R}^3 \end{cases} \tag{4.1}$$

for $\lambda > 0$. In [16, 9], a velocity fundamental solution $\varGamma^\lambda$ to the time-periodic problem (4.1) was introduced such that

$$u = \varGamma^\lambda * f$$

with convolution taken with respect to the locally compact abelian group $G = \mathbb{T}\times\mathbb{R}^3$. It is given by

$$\varGamma^\lambda := \varGamma_0^\lambda \otimes 1_{\mathbb{T}} + \varGamma_\perp^\lambda, \tag{4.2}$$

where

$$\varGamma_0^\lambda\colon \mathbb{R}^3\setminus\{0\}\to\mathbb{R}^{3\times3}, \quad \varGamma_{0,j\ell}^\lambda(x) := \frac{1}{4\pi\lambda}\big[\delta_{j\ell}\Delta - \partial_j\partial_\ell\big]\int_0^{s(\lambda x)/2}\frac{1-\mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau, \tag{4.3}$$

$$\varGamma_\perp^\lambda := \mathscr{F}_G^{-1}\Big[\frac{1-\delta_{\mathbb{Z}}(k)}{|\xi|^2 + i(\frac{2\pi}{\mathcal{T}}k - \lambda\xi_1)}\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\Big], \tag{4.4}$$

the symbol $1_{\mathbb{T}}$ denotes the constant 1 distribution, and $s(x) = |x| + x_1$ as above. In particular, the fundamental solution $\varGamma^\lambda$ decomposes into a *steady-state* part $\varGamma_0^\lambda$ and a *purely periodic* part $\varGamma_\perp^\lambda$. The steady-state part $\varGamma_0^\lambda$ is the fundamental solution to the steady-state Oseen problem

$$\begin{cases} -\Delta v - \lambda\partial_1 v + \nabla p = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3; \end{cases} \tag{4.5}$$

see [12, Section VII.3]. This function shows strongly anisotropic behavior, which is reflected in the pointwise estimates

$$\forall\alpha\in\mathbb{N}_0^3\ \forall\varepsilon>0\ \exists C>0\ \forall|x|\geq\varepsilon: \quad |\mathrm{D}_x^\alpha\varGamma_0^\lambda(x)| \leq C\big[|x|(1+s(\lambda x))\big]^{-1-\frac{|\alpha|}{2}}; \tag{4.6}$$

see [10, Lemma 3.2]. For the purely periodic part $\varGamma_\perp^\lambda$ one can show the estimates

$$\forall\alpha\in\mathbb{N}_0^3\ \forall r\in[1,\infty)\ \forall\varepsilon>0\ \exists C>0\ \forall|x|\geq\varepsilon: \ \|\mathrm{D}_x^\alpha\varGamma_\perp^\lambda(\cdot,x)\|_{\mathrm{L}^r(\mathbb{T})} \leq C|x|^{-3-|\alpha|}; \tag{4.7}$$

see [9]. Observe that estimate (4.7) does not have an anisotropic character and that the purely periodic part $\varGamma_\perp^\lambda$ decays faster than the steady-state part $\varGamma_0^\lambda$.

In order to derive estimates of the solution u from those of the fundamental solution $\varGamma^\lambda$, one thus has to study convolutions of functions that satisfy pointwise estimates similar to those in (4.6) and (4.7). Convolutions of the first type were examined by Farwig [10, 11] in dimension $n = 3$, and later by Kračmar, Novotný and Pokorný [18] in the general n-dimensional case. We collect some of their results in the following theorem, which gives estimates of convolutions with $\varGamma_0^\lambda$ and $\nabla\varGamma_0^\lambda$.

Theorem 4.1. *Let $A\in[2,\infty)$ and $B\in[0,\infty)$, and let $g\in\mathrm{L}^\infty(\mathbb{R}^3)$ such that $|g(x)|\leq M(1+|x|)^{-A}(1+s(x))^{-B}$. Then there exists a constant $C = C(A,B,\lambda)>0$ with the following properties:*

1 If $A+\min\{1,B\}>3$*, then*

$$\big||\Gamma_0^\lambda|*g(x)\big|\leq CM\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-1}. \tag{4.8}$$

2 If $A+\min\{1,B\}>3$ *and* $A+B\geq 7/2$*, then*

$$\big||\nabla\Gamma_0^\lambda|*g(x)\big|\leq CM\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-3/2}. \tag{4.9}$$

Proof. These are special cases of [18, Theorems 3.1 and 3.2]. □

An analogous result for convolutions with $\Gamma_\perp^\lambda$ and $\nabla\Gamma_\perp^\lambda$ was derived in [7].

Theorem 4.2. *Let* $A\in\mathbb{R}$ *and* $g\in\mathrm{L}^\infty(\mathbb{T}\times\mathbb{R}^3)$ *such that* $|g(t,x)|\leq M(1+|x|)^{-A}$*. Then for any* $\varepsilon>0$ *there exists a constant* $C=C(A,\lambda,\mathcal{T},\varepsilon)>0$ *with the following properties:*

1 If $A>3$*, then*

$$\forall|x|\geq\varepsilon:\qquad \big||\Gamma_\perp^\lambda|*_G g(t,x)\big|\leq CM(1+|x|)^{-3}. \tag{4.10}$$

2 If $A>4$*, then*

$$\forall|x|\geq\varepsilon:\qquad \big||\nabla\Gamma_\perp^\lambda|*_G g(t,x)\big|\leq CM(1+|x|)^{-4}. \tag{4.11}$$

Proof. We refer to [7, Theorem 3.3]. □

Next we derive a fundamental solution for the vorticity field $\operatorname{curl}u$. For $u=\Gamma^\lambda*f$ a direct computation yields

$$(\operatorname{curl}u)_m=\varepsilon_{mhj}\partial_h\Gamma_{0,j\ell}^\lambda*\mathcal{P}f_\ell+\varepsilon_{mhj}\partial_h\Gamma_{\perp,j\ell}^\lambda*f_\ell=\varepsilon_{mh\ell}\partial_h\phi_0^\lambda*\mathcal{P}f_\ell+\varepsilon_{mh\ell}\partial_h\phi_\perp^\lambda*f_\ell$$

with

$$\phi_0^\lambda(x):=\frac{1}{4\pi|x|}\,\mathrm{e}^{-s(\lambda x)/2}, \tag{4.12}$$

$$\phi_\perp^\lambda:=\mathscr{F}_G^{-1}\Big[\frac{1-\delta_{\mathbb{Z}}(k)}{|\xi|^2-i\lambda\xi_1+i\frac{2\pi}{\mathcal{T}}k}\Big]. \tag{4.13}$$

In conclusion, we obtain

$$\operatorname{curl}u(t,x)=\int_G\nabla\phi^\lambda(t-s,x-y)\wedge f(s,y)\,\mathrm{d}(s,y), \tag{4.14}$$

where

$$\phi^\lambda:=\phi_0^\lambda\otimes 1_{\mathbb{T}}+\phi_\perp^\lambda. \tag{4.15}$$

We have thus found an integral formula for the vorticity $\operatorname{curl}u$. We call ϕ^λ the *vorticity fundamental solution*. As for the velocity fundamental solution Γ^λ, the vorticity fundamental solution ϕ^λ decomposes into a steady-state and a purely periodic part, which can be analyzed separately. A direct computation leads to the the following estimate of $\nabla\phi_0^\lambda$.

Theorem 4.3. *There exists* $C=C(\lambda)>0$ *such that for all* $x\in\mathbb{R}^3\setminus\{0\}$ *it holds*

$$|\nabla\phi_0^\lambda(x)|\leq C\big(|x|^{-2}+|x|^{-3/2}s(\lambda x)^{1/2}\big)\,\mathrm{e}^{-s(\lambda x)/2}\,. \tag{4.16}$$

Proof. The estimate follows directly by taking derivatives in (4.12) and using the identity $|\nabla[s(\lambda x)]|^2 = 2\lambda^2 s(x)/|x|$. □

The remainder of this section is dedicated to the derivation of an analogous estimate of $\nabla\phi_\perp^\lambda$. More precisely, we show the following result.

Theorem 4.4. *There exist constants $C = C(\lambda, \mathcal{T}, q, \gamma) > 0$ and $C_3 = C_3(\lambda, \mathcal{T}) > 0$ such that for all $\gamma \in (0,1)$, $q \in [1, \frac{1}{1-\gamma})$ and $x \in \mathbb{R}^3 \setminus \{0\}$ it holds*

$$\|\phi_\perp^\lambda(\cdot, x)\|_{\mathrm{L}^q(\mathbb{T})} \leq C|x|^{-(1+2\gamma)} \,\mathrm{e}^{-C_3|x|}, \tag{4.17}$$

$$\|\nabla\phi_\perp^\lambda(\cdot, x)\|_{\mathrm{L}^q(\mathbb{T})} \leq C|x|^{-(2+2\gamma)} \,\mathrm{e}^{-C_3|x|}. \tag{4.18}$$

For the proof of Theorem 4.4 we represent $\phi_\perp^\lambda$ in a different way. From $\mathscr{F}_G^{-1} = \mathscr{F}_\mathbb{T}^{-1} \otimes \mathscr{F}_{\mathbb{R}^3}^{-1}$ we conclude the identity

$$\phi_\perp^\lambda(t, x) = \mathscr{F}_\mathbb{T}^{-1}\big[k \mapsto \big(1 - \delta_\mathbb{Z}(k)\big)\varGamma_\mathrm{H}^{\frac{2\pi}{\mathcal{T}}k,\lambda}(x)\big](t), \tag{4.19}$$

where

$$\varGamma_\mathrm{H}^{\eta,\lambda} := \mathscr{F}_{\mathbb{R}^3}^{-1}\left[\frac{1}{|\xi|^2 - i\lambda\xi_1 + i\eta}\right]$$

is the fundamental solution to the equation

$$i\eta\, v - \Delta v - \lambda\partial_1 v = f \qquad \text{in } \mathbb{R}^3. \tag{4.20}$$

This function is explicitly given by

$$\varGamma_\mathrm{H}^{\eta,\lambda} \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}, \qquad \varGamma_\mathrm{H}^{\eta,\lambda}(x) = \frac{1}{4\pi|x|}\,\mathrm{e}^{i\sqrt{-\mu}|x| - \frac{\lambda}{2}x_1} \tag{4.21}$$

for $\eta \neq 0$ and $\mu := \mu(\eta, \lambda) := (\lambda/2)^2 + i\eta \in \mathbb{C} \setminus \mathbb{R}$; see [9, Lemma 3.3]. Here $\sqrt{z}$ is the square root of z with nonnegative imaginary part. We first derive pointwise estimates of $\varGamma_\mathrm{H}^{\eta,\lambda}$.

Lemma 4.5. *Let $\eta_0 > 0$. Then there exists $C_4 = C_4(\lambda, \eta_0) > 0$ such that*

$$|\varGamma_\mathrm{H}^{\eta,\lambda}(x)| \leq C|x|^{-1}\,\mathrm{e}^{-C_4|\eta|^{\frac12}|x|}, \tag{4.22}$$

$$|\nabla\varGamma_\mathrm{H}^{\eta,\lambda}(x)| \leq C\big(|x|^{-2} + |\eta|^{\frac12}|x|^{-1}\big)\,\mathrm{e}^{-C_4|\eta|^{\frac12}|x|}, \tag{4.23}$$

for all $\eta \in \mathbb{R}$ with $|\eta| > \eta_0$ and $x \in \mathbb{R}^3 \setminus \{0\}$.

Proof. As in [9, Lemma 3.2], we show the existence of a constant $C_4 = C_4(\lambda, \eta_0) > 0$ such that

$$\mathrm{Im}(\sqrt{-\mu}) - \frac{|\lambda|}{2} \geq C_4|\eta|^{\frac12}$$

for $|\eta| \geq \eta_0$ and $\mu = (\lambda/2)^2 + i\eta$. We thus have

$$\left|\mathrm{e}^{i\sqrt{-\mu}|x| - \frac{\lambda}{2}x_1}\right| \leq \mathrm{e}^{-\mathrm{Im}(\sqrt{-\mu})|x| + \frac{|\lambda|}{2}|x|} \leq \mathrm{e}^{-C_4|\eta|^{\frac12}|x|}.$$

This directly implies (4.22). Computing derivatives and employing this estimate again, we further deduce

$$\big|\nabla\varGamma_\mathrm{H}^{\eta,\lambda}(x)\big| \leq C\big(|x|^{-2} + |x|^{-1}(|\sqrt{-\mu}| + |\lambda|)\big)\,\mathrm{e}^{-C_4|\eta|^{\frac12}|x|},$$

which implies (4.23) by using $|\lambda| \leq 2|\sqrt{-\mu}| \leq C|\eta|^{\frac12}$ for $|\eta| \geq \eta_0$. □

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Now let $\chi \in \mathrm{C}^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi(\eta) = 0$ for $|\eta| \leq \frac{1}{2}$ and $\chi(\eta) = 1$ for $|\eta| \geq 1$. For $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $\gamma \in (0,1)$ and $x \in \mathbb{R}^3 \setminus \{0\}$ define the function

$$m_{\alpha,x}\colon \mathbb{R} \to \mathbb{R}, \qquad m_{\alpha,x}(\eta) := \chi(\eta)|\eta|^\gamma \mathrm{D}^\alpha \varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x). \tag{4.24}$$

We show that $m_{\alpha,x}$ is an $\mathrm{L}^q(\mathbb{R})$ multiplier and give an estimate of the multiplier norm by means of the Marcinkiewicz Multiplier Theorem.

Lemma 4.6. *Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $\gamma \in (0,1)$ and $x \in \mathbb{R}^3 \setminus \{0\}$. Then $m_{\alpha,x}$ is an $\mathrm{L}^q(\mathbb{R})$ multiplier for any $q \in (1,\infty)$, and there exist constants $C = C(\lambda, \mathcal{T}, q, \alpha, \gamma) > 0$ and $C_5 = C_5(\lambda, \mathcal{T}) > 0$ such that*

$$\big\| \mathscr{F}_{\mathbb{R}}^{-1}\big[m_{\alpha,x}\mathscr{F}_{\mathbb{R}}[f]\big]\big\|_{\mathrm{L}^q(\mathbb{R})} \leq C|x|^{-1-|\alpha|-2\gamma}\,\mathrm{e}^{-C_5|x|}\|f\|_{\mathrm{L}^q(\mathbb{R})}.$$

Proof. At first, let $\alpha = 0$. From (4.22) we conclude

$$|m_{0,x}(\eta)| \leq C\chi(\eta)|\eta|^\gamma|x|^{-1}\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|} \leq C|x|^{-1-2\gamma}\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|/2}$$

for $|\eta| \geq \frac{1}{2}$. Moreover, differentiating $\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}$ with respect to η, we obtain

$$\big|\partial_\eta \varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \leq C|\partial_\eta\sqrt{-\mu}|\,|x|\,\big|\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \leq C|\eta|^{-\frac12}|x|\,\big|\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big|,$$

so that (4.22) yields

$$\begin{aligned}
|\eta\partial_\eta m_{0,x}(\eta)| &\leq \big|\chi'(\eta)|\eta|^{\gamma+1}\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| + \big|\chi(\eta)\gamma|\eta|^\gamma\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| + \big|\chi(\eta)|\eta|^{\gamma+1}\partial_\eta\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \\
&\leq C\big(|\eta|^\gamma|x|^{-1} + |\eta|^{\gamma+\frac12}\big)\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|} \leq C|x|^{-1-2\gamma}\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|/2}
\end{aligned}$$

for $|\eta| \geq \frac{1}{2}$. Collecting these estimates and utilizing $m_{0,x}(\eta) = 0$ for $|\eta| \leq \frac{1}{2}$, we have

$$|m_{0,x}(\eta)| + |\eta\partial_\eta m_{0,x}(\eta)| \leq C|x|^{-1-2\gamma}\,\mathrm{e}^{-C_5|x|} \tag{4.25}$$

with $C_5 = \sqrt{\pi/\mathcal{T}}C_4/2$ for all $\eta \in \mathbb{R}$.

Next consider the case $\alpha = \mathrm{e}_j$ for some $j \in \{1,2,3\}$. Then (4.23) leads to

$$|m_{\alpha,x}(\eta)| \leq C\chi(\eta)|\eta|^\gamma\big(|x|^{-2} + |\eta|^{\frac12}|x|^{-1}\big)\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|} \leq C|x|^{-2-2\gamma}\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|/2}$$

for $|\eta| \geq \frac{1}{2}$. Moreover, a straightforward calculation yields

$$\big|\partial_\eta\partial_j\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \leq C\big(|\mu|^{-\frac12} + |x|\big)\big|\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \leq C\big(|\eta|^{-\frac12} + |x|\big)\big|\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big|,$$

so that we can employ Lemma 4.5 to estimate

$$\begin{aligned}
|\eta\partial_\eta\partial_j m_{\alpha,x}(\eta)| &\leq \big|\chi'(\eta)|\eta|^{\gamma+1}\partial_j\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| + \big|\chi(\eta)\gamma|\eta|^\gamma\partial_j\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \\
&\qquad + \big|\chi(\eta)|\eta|^{\gamma+1}\partial_\eta\partial_j\varGamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}\eta,\lambda}(x)\big| \\
&\leq C\big(|\eta|^\gamma|x|^{-2} + |\eta|^{\gamma+\frac12}|x|^{-1} + |\eta|^{\gamma+1}\big)\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|} \\
&\leq C|x|^{-2-2\gamma}\,\mathrm{e}^{-C_4|\frac{2\pi}{\mathcal{T}}\eta|^{\frac12}|x|/2}
\end{aligned}$$

for $|\eta| \geq \frac{1}{2}$. Collecting these estimates and utilizing $m_{\alpha,x}(\eta) = 0$ for $|\eta| \leq \frac{1}{2}$, we have

$$|m_{\alpha,x}(\eta)| + |\eta\partial_\eta m_{\alpha,x}(\eta)| \leq C|x|^{-2-2\gamma}\,\mathrm{e}^{-C_5|x|} \tag{4.26}$$

with $C_5 = \sqrt{\pi/\mathcal{T}}C_4/2$ as above.

By the Marcinkiewicz Multiplier Theorem (see [17, Corollary 5.2.5]), the assertion is now a direct consequence of (4.25) and (4.26). □

Using this result, we establish the pointwise estimates of $\phi_\perp^\lambda$ asserted in Theorem 4.4 by means of the so-called transference principle for Fourier multipliers.

Proof of Theorem 4.4. It suffices to consider $q \in (1, \infty)$. Due to (4.19), we have

$$\mathrm{D}_x^\alpha \phi_\perp^\lambda(\cdot, x) = \mathscr{F}_{\mathbb{T}}^{-1}\big[M_{\alpha,x}\mathscr{F}_{\mathbb{T}}[\varphi_\gamma]\big] \tag{4.27}$$

with

$$M_{\alpha,x}(k) := \big(1 - \delta_{\mathbb{Z}}(k)\big)|k|^\gamma \mathrm{D}_x^\alpha \Gamma_{\mathrm{H}}^{\frac{2\pi}{\mathcal{T}}k,\lambda}(x), \qquad \varphi_\gamma := \mathscr{F}_{\mathbb{T}}^{-1}\big[k \mapsto \big(1 - \delta_{\mathbb{Z}}(k)\big)|k|^{-\gamma}\big].$$

First, note that $M_{\alpha,x} = m_{\alpha,x}|_{\mathbb{Z}}$. Since $m_{\alpha,x}$ is a continuous $\mathrm{L}^q(\mathbb{R})$ multiplier by Lemma 4.6, the transference principle (see [5, Theorem B.2.1] or [8, Theorem 2.15]) implies that $M_{\alpha,x}$ is an $\mathrm{L}^q(\mathbb{T})$ multiplier for any $q \in (1, \infty)$ and that

$$\|\mathscr{F}_{\mathbb{T}}^{-1}\big[M_{\alpha,x}\mathscr{F}_{\mathbb{T}}[f]\big]\|_{\mathrm{L}^q(\mathbb{T})} \leq C|x|^{-1-|\alpha|-2\gamma}\,\mathrm{e}^{-C_3|x|}\|f\|_{\mathrm{L}^q(\mathbb{T})}.$$

Moreover, we have $\varphi_\gamma \in \mathrm{L}^q(\mathbb{T})$ provided $q < 1/(1-\gamma)$, which is a direct consequence of [17, Example 3.1.19] for example. Finally, the assertion follows from (4.27). □

5 Regularity results

Here we collect some results concerning the regularity of weak solutions to (1.1) and its linearization, which is given by

$$\begin{cases} \partial_t u - \Delta u - \lambda\partial_1 u + \nabla\mathfrak{p} = f & \text{in } \mathbb{T}\times\Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T}\times\Omega, \\ u = u_* & \text{on } \mathbb{T}\times\partial\Omega. \end{cases} \tag{5.1}$$

First of all, we derive the following regularity theorem for solutions to (5.1) in the case $\mathcal{P}f = 0$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class C^2, let u_* be as in* (3.2)*, and let $f \in \mathrm{L}^q(\mathbb{T}\times\Omega)$ for some $q \in (1,\infty)$. Assume $\mathcal{P}f = 0$ and that u is a weak solution to* (5.1)*, that is, $u = u_*$ on $\mathbb{T}\times\partial\Omega$, $\operatorname{div} u = 0$ and*

$$\int_{\mathbb{T}\times\Omega} \big[-u\cdot\partial_t\varphi + \nabla u : \nabla\varphi - \lambda\partial_1 u\cdot\varphi\big]\,\mathrm{d}(t,x) = \int_{\mathbb{T}\times\Omega} f\cdot\varphi\,\mathrm{d}(t,x) \tag{5.2}$$

for all $\varphi \in \mathrm{C}_{0,\sigma}^\infty(\mathbb{T}\times\Omega)$. Assume that $u \in \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\Omega))^3$ and $\nabla u \in \mathrm{L}^2(\mathbb{T}\times\Omega)^{3\times3}$. Then $u \in \mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)^3$, and there exists $\mathfrak{p} \in \mathrm{L}^q(\mathbb{T};\mathrm{D}^{1,q}(\Omega))$ such that $(u,\mathfrak{p})$ is a strong solution to (5.1)*.*

Proof. First of all, using classical arguments (see [12, Section III.3] for example) one can show the existence of a function $U \in \mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)^3$ such that $U = u_*$ on $\mathbb{T}\times\partial\Omega$ and $\operatorname{div} U = 0$. Moreover, since $\mathcal{P}f = 0$, by [15, Theorem 5.1] there exist $z \in \mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)^3$ and $\mathfrak{p} \in \mathrm{L}^q(\mathbb{T};\mathrm{D}^{1,q}(\Omega))$ such that

$$\begin{cases} \partial_t z - \Delta z - \lambda\partial_1 z + \nabla\mathfrak{p} = f - \partial_t U - \Delta U - \lambda\partial_1 U & \text{in } \mathbb{T}\times\Omega, \\ \operatorname{div} z = 0 & \text{in } \mathbb{T}\times\Omega, \\ z = 0 & \text{on } \mathbb{T}\times\partial\Omega. \end{cases} \tag{5.3}$$

Then $(\tilde{u}, \mathfrak{p}) := (z + U, \mathfrak{p})$ solves (5.1), and for the completion of the proof it remains to show $u = \tilde{u}$. For this purpose, we employ a duality argument. Let $\psi \in \mathrm{C}_0^\infty(\mathbb{T}\times\Omega)^3$. By [15, Theorem 5.1] there exist functions $w \in \mathrm{W}^{1,2,2}(\mathbb{T}\times\Omega)^3 \cap \mathrm{W}^{1,2,q'}(\mathbb{T}\times\Omega)^3$ and $\mathfrak{q} \in \mathrm{L}^2(\mathbb{T};\mathrm{D}^{1,2}(\Omega)) \cap \mathrm{L}^{q'}(\mathbb{T};\mathrm{D}^{1,q'}(\Omega))$, where $q' = q/(q-1)$, which satisfy

$$\begin{cases} \partial_t w - \Delta w + \lambda\partial_1 w + \nabla\mathfrak{q} = \mathcal{P}_\perp\psi & \text{in } \mathbb{T}\times\Omega, \\ \operatorname{div} w = 0 & \text{in } \mathbb{T}\times\Omega, \\ w = 0 & \text{on } \mathbb{T}\times\partial\Omega. \end{cases} \tag{5.4}$$

By a standard density argument one shows that we can let $\varphi = w$ in the weak formulation (5.2). Then, by an integration by parts, we get

$$\begin{aligned} \int_{\mathbb{T}\times\Omega} (u-\tilde{u})\cdot\mathcal{P}_\perp\psi\,\mathrm{d}(t,x) &= \int_{\mathbb{T}\times\Omega} (u-\tilde{u})\cdot\big(\partial_t w - \Delta w + \lambda\partial_1 w + \nabla\mathfrak{q}\big)\,\mathrm{d}(t,x) \\ &= \int_{\mathbb{T}\times\Omega} \big[u\cdot\partial_t w + \nabla u : \nabla w - \lambda\partial_1 u\cdot w\big]\,\mathrm{d}(t,x) \\ &\qquad - \int_{\mathbb{T}\times\Omega} \big(\partial_t\tilde{u} - \Delta\tilde{u} - \lambda\partial_1\tilde{u} + \nabla\mathfrak{p}\big)\cdot w\,\mathrm{d}(t,x) \\ &= \int_{\mathbb{T}\times\Omega} f\cdot w - \int_{\mathbb{T}\times\Omega} f\cdot w = 0. \end{aligned}$$

Since $\mathcal{P}u = \mathcal{P}\tilde{u} = 0$, we thus conclude

$$\int_{\mathbb{T}\times\Omega} (u-\tilde{u})\cdot\psi\,\mathrm{d}(t,x) = \int_{\mathbb{T}\times\Omega} (u-\tilde{u})\cdot\mathcal{P}\psi\,\mathrm{d}(t,x) + \int_{\mathbb{T}\times\Omega} (u-\tilde{u})\cdot\mathcal{P}_\perp\psi\,\mathrm{d}(t,x) = 0$$

for arbitrary $\psi \in \mathrm{C}_0^\infty(\mathbb{T}\times\Omega)^3$, which implies $u = \tilde{u}$ and completes the proof. □

Based on this result for the linearized problem (5.1), we can now show that the additional integrability condition assumed in (3.3) leads to higher regularity of the weak solution.

Lemma 5.2. *In the hypotheses of Theorem 3.2 we have* $u = v + w$*, with* $v = \mathcal{P}u$, $w = \mathcal{P}_\perp u$ *such that*

$$\forall q \in (1,\infty):\ v \in \mathrm{D}^{2,q}(\Omega),\ \ \forall r \in (\tfrac{4}{3},\infty]:\ v \in \mathrm{D}^{1,r}(\Omega),\ \ \forall s \in (2,\infty]:\ v \in \mathrm{L}^s(\Omega), \tag{5.5}$$

$$\forall q \in (1,\infty):\ w \in \mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega). \tag{5.6}$$

Moreover, there exists a pressure field $\mathfrak{p}$ *with*

$$\forall q \in (1,\infty):\ \mathfrak{p} \in \mathrm{L}^q(\mathbb{T};\mathrm{D}^{1,q}(\Omega)) \tag{5.7}$$

such that (1.1) *is satisfied in the strong sense.*

Proof. At first, observe that v is a weak solution to the steady-state Navier–Stokes problem

$$\begin{cases} -\Delta v - \lambda\partial_1 v + v\cdot\nabla v + \nabla p = \mathcal{P}f - \mathcal{P}[w\cdot\nabla w] & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = \mathcal{P}u_* & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

Hölder's inequality and Definition 3.1 yield $w\cdot\nabla w\in \mathrm{L}^1(\mathbb{T};\mathrm{L}^{3/2}(\Omega))\cap\mathrm{L}^2(\mathbb{T};\mathrm{L}^1(\Omega))$. Therefore, we have $\mathcal{P}f-\mathcal{P}(w\cdot\nabla w)\in\mathrm{L}^1(\Omega)\cap\mathrm{L}^{3/2}(\Omega)$, and [12, Lemma X.6.1] implies

$$\forall q\in(1,\frac{3}{2}]:\, v\in\mathrm{D}^{2,q}(\Omega),\ \ \forall r\in(\frac{4}{3},3]:\, v\in\mathrm{D}^{1,r}(\Omega),\ \ \forall s\in(2,\infty):\, v\in\mathrm{L}^s(\Omega) \tag{5.9}$$

and the existence of $p\in\mathrm{D}^{1,q}(\Omega)$ for all $q\in(1,\frac{3}{2}]$ such that (5.8) is satisfied in the strong sense. Moreover, w is a weak solution to

$$\begin{cases}\partial_t w-\Delta w-\lambda\partial_1 w+\nabla\mathfrak{q}=\mathcal{P}_\perp f-v\cdot\nabla w-w\cdot\nabla v-\mathcal{P}_\perp(w\cdot\nabla w) & \text{in } \mathbb{T}\times\Omega,\\ \operatorname{div} w=0 & \text{in } \mathbb{T}\times\Omega,\\ w=\mathcal{P}_\perp u_* & \text{on } \mathbb{T}\times\partial\Omega.\end{cases} \tag{5.10}$$

By a standard interpolation argument, the assumptions from Definition 3.1 imply $w\in\mathrm{L}^{10/3}(\mathbb{T}\times\Omega)$. Since $v\in\mathrm{L}^{10/3}(\Omega)$ by (5.9) and $\nabla u\in\mathrm{L}^2(\mathbb{T}\times\Omega)$ by assumption, this implies

$$\mathcal{P}_\perp f-v\cdot\nabla w-w\cdot\nabla v-\mathcal{P}_\perp(w\cdot\nabla w)\in\mathrm{L}^s(\mathbb{T}\times\Omega) \tag{5.11}$$

for $s=5/4$. Now Lemma 5.1 shows that there exists $\mathfrak{q}$ such that

$$w\in\mathrm{W}^{1,2,5/4}(\mathbb{T}\times\Omega),\ \mathfrak{q}\in\mathrm{L}^{5/4}(\mathbb{T};\mathrm{D}^{1,5/4}(\Omega)), \tag{5.12}$$

and (5.10) holds in a strong sense. Starting from (5.12), we now employ a boot-strap argument to conclude the proof.

If $w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)$ for some $q\in(1,15/8)$, then the embedding theorem from [15, Theorem 4.1] implies $\nabla w\in\mathrm{L}^{5q/(5-q)}(\mathbb{T}\times\Omega)$. In virtue of (3.3) and (5.9), this implies $w\cdot\nabla w,\ v\cdot\nabla w\in\mathrm{L}^s(\mathbb{T}\times\Omega)$ for $\frac{1}{s}=\frac{1}{q}+\frac{1}{r}-\frac{1}{5}$. Moreover, [15, Theorem 4.1] yields $w\in\mathrm{L}^{5q/(5-q)}(\mathbb{T};\mathrm{L}^{15q/(15-8q)}(\Omega))$, so that $w\cdot\nabla v\in\mathrm{L}^s$ by (5.9). In total, we thus obtain (5.11) for $\frac{1}{s}=\frac{1}{q}+\frac{1}{r}-\frac{1}{5}$, and Lemma 5.1 leads to the implication

$$\exists\, q\in\big(1,\frac{15}{8}\big):\, w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)\implies\forall\frac{1}{s}\in\big[\frac{1}{q}+\frac{1}{r}-\frac{1}{5},\frac{1}{q}\big]:\, w\in\mathrm{W}^{1,2,s}(\mathbb{T}\times\Omega). \tag{5.13}$$

If $w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)$ for some $q\in[5/3,5/2)$, then [15, Theorem 4.1] yields $w\in\mathrm{L}^{5q/(5-2q)}(\mathbb{T}\times\Omega)$ and $\nabla w\in\mathrm{L}^{5q/(5-q)}(\mathbb{T}\times\Omega)$, which implies $w\cdot\nabla w\in\mathrm{L}^{s_1}(\mathbb{T}\times\Omega)$ for all $\frac{1}{s_1}\in[\frac{2}{q}-\frac{3}{5},\frac{2}{q}]$. Hence we have $\mathcal{P}(w\cdot\nabla w)\in\mathrm{L}^{s_1}(\Omega)$, and another application of [12, Lemma X.6.1] in view of (5.9) yields $\nabla v\in\mathrm{L}^t(\Omega)$ for all $t\in[4/3,15/4]$ and $v\in\mathrm{L}^t(\Omega)$ for all $t\in(2,\infty]$. We thus conclude $w\cdot\nabla v\in\mathrm{L}^{s_2}(\mathbb{T}\times\Omega)$ for $\frac{1}{s_2}\in[\frac{1}{q}-\frac{2}{15},\frac{1}{q}+\frac{3}{4}]$ and $v\cdot\nabla w\in\mathrm{L}^{s_3}(\mathbb{T}\times\Omega)$ for $\frac{1}{s_3}\in[\frac{1}{q}-\frac{1}{5},\frac{1}{q}+\frac{1}{2}]$. In particular, we obtain (5.11) for $\frac{1}{s}=\frac{2}{q}-\frac{3}{5}$ if $q\leq 15/7$, and $\frac{1}{s}=\frac{1}{q}-\frac{2}{15}$ if $q\geq 15/7$. By Lemma 5.1, this implies

$$\exists\, q\in(\frac{5}{3},\frac{15}{7}]:\, w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)\implies\forall\frac{1}{s}\in[\frac{2}{q}-\frac{3}{5},\frac{1}{q}]:\, w\in\mathrm{W}^{1,2,s}(\mathbb{T}\times\Omega), \tag{5.14}$$

$$\exists\, q\in[\frac{15}{7},\frac{5}{2}):\, w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)\implies\forall\frac{1}{s}\in[\frac{1}{q}-\frac{2}{15},\frac{1}{q}]:\, w\in\mathrm{W}^{1,2,s}(\mathbb{T}\times\Omega). \tag{5.15}$$

Arguing in a similar fashion, one shows the further implications

$$\exists\, q\in[\frac{5}{2},5):\, w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)\implies\forall\frac{1}{s}\in(\frac{1}{q}-\frac{1}{5},\frac{1}{q}]:\, w\in\mathrm{W}^{1,2,s}(\mathbb{T}\times\Omega), \tag{5.16}$$

$$\exists\, q\in[5,\infty):\, w\in\mathrm{W}^{1,2,q}(\mathbb{T}\times\Omega)\implies\forall\, s\in[q,\infty):\, w\in\mathrm{W}^{1,2,s}(\mathbb{T}\times\Omega). \tag{5.17}$$

Using now (5.12) as starting point, we can iteratively employ (5.13)–(5.17) to obtain $w \in \mathrm{W}^{1,2,s}(\mathbb{T} \times \Omega)$ for all $s \in [5/4, \infty)$. Firstly, this yields $w \cdot \nabla w \in \mathrm{L}^\infty(\mathbb{T} \times \Omega)$, so that $\mathcal{P}f - \mathcal{P}(w \cdot \nabla w) \in \mathrm{L}^q(\Omega)$ for all $q \in [1, \infty)$. Now (5.5) is a direct consequence of [12, Theorem X.6.4]. Secondly, this shows that (5.11) holds for all $s \in [1, \infty)$, whence Lemma 5.1 implies (5.6). Finally, the claimed regularity (5.7) of $\mathfrak{p} = p + \mathfrak{q}$ is a direct consequence. This completes the proof. □

6 The fixed-point problem

In this section we derive a suitable fixed-point equation satisfied by weak solutions in the whole space, and we introduce the necessary functional framework. More precisely, the main focus of the subsequent analysis lies on the study of problem (1.1) when $\Omega = \mathbb{R}^3$, namely,

$$\begin{cases} \partial_t u - \Delta u - \lambda \partial_1 u + u \cdot \nabla u + \nabla \mathfrak{p} = f & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \lim_{|x| \to \infty} u(t,x) = 0 & \text{for } t \in \mathbb{T}. \end{cases} \tag{6.1}$$

The case of an exterior domain will be treated at the end of the last section.

We begin to observe that asymptotic properties of weak solutions to (6.1) were studied in [16] and [7], where the following decay estimates of u and ∇u were derived.

Theorem 6.1. *Let $\lambda > 0$ and $f \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)^3$ for all $q \in (1, \infty)$ and let $\operatorname{supp} f$ be compact. Let u be a weak solution to* (6.1) *that satisfies* (3.3) *for $\Omega = \mathbb{R}^3$. Then there is $C_2 > 0$ such that for all $(t,x) \in \mathbb{T} \times \mathbb{R}^3$ the function u satisfies*

$$|\mathcal{P}u(x)| \leq C_2 \big[\big(1 + |x|\big)\big(1 + s(\lambda x)\big) \big]^{-1}, \tag{6.2}$$
$$|\nabla \mathcal{P}u(x)| \leq C_2 \big[\big(1 + |x|\big)\big(1 + s(\lambda x)\big) \big]^{-\frac{3}{2}}, \tag{6.3}$$
$$|\mathcal{P}_\perp u(t,x)| \leq C_2 \big(1 + |x|\big)^{-3}, \tag{6.4}$$
$$|\nabla \mathcal{P}_\perp u(t,x)| \leq C_2 \big(1 + |x|\big)^{-4}. \tag{6.5}$$

Proof. Under the assumption $f \in \mathrm{C}_0^\infty(\mathbb{T} \times \mathbb{R}^3)^3$, this result was shown in [7, Theorem 4.5] based on estimates of the velocity field u derived in [16]. However, a careful study of the proofs shows that these results continue to be valid under the stated weaker assumption on f. □

To derive a suitable fixed-point equation, we exploit the following representation formulas that result from the time-periodic fundamental solutions introduced in the previous section.

Proposition 6.2. *Let u be a weak solution as in Theorem 6.1. Then*

$$D_x^\alpha u = D_x^\alpha \mathit{\Gamma}^\lambda * [f - \operatorname{curl} u \wedge u] \tag{6.6}$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. In particular, the steady-state part $v := \mathcal{P}u$ and the purely periodic part $w := \mathcal{P}_\perp u$ satisfy

$$D_x^\alpha v = D_x^\alpha \mathit{\Gamma}_0^\lambda * \big[\mathcal{P}f - \operatorname{curl} v \wedge v - \mathcal{P}(\operatorname{curl} w \wedge w) \big], \tag{6.7}$$
$$D_x^\alpha w = D_x^\alpha \mathit{\Gamma}_\perp^\lambda * \big[\mathcal{P}_\perp f - \operatorname{curl} v \wedge w - \operatorname{curl} w \wedge v + \mathcal{P}_\perp(\operatorname{curl} w \wedge w) \big]. \tag{6.8}$$

Moreover, we have

$$\operatorname{curl} u(t,x) = \int_G \nabla\phi^\lambda(t-s,x-y)\wedge\big[f-\operatorname{curl} u\wedge u\big](s,y)\,\mathrm{d}(s,y), \tag{6.9}$$

and

$$\operatorname{curl} v(x) = \int_{\mathbb{R}^3} \nabla\phi_0^\lambda(x-y)\wedge\big[\mathcal{P}f-\operatorname{curl} v\wedge v-\mathcal{P}(\operatorname{curl} w\wedge w)\big](y)\,\mathrm{d}y, \tag{6.10}$$

as well as

$$\begin{aligned}\operatorname{curl} w(t,x) = \int_G \nabla\phi_\perp^\lambda(t-s,x-y)\wedge\big[\mathcal{P}_\perp f-\operatorname{curl} v\wedge w \\ -\operatorname{curl} w\wedge v-\mathcal{P}_\perp(\operatorname{curl} w\wedge w)\big](s,y)\,\mathrm{d}(s,y).\end{aligned} \tag{6.11}$$

Proof. Since $u\cdot\nabla u = \frac{1}{2}\nabla\big(|u|^2\big)+\operatorname{curl} u\wedge u$ and $\varGamma^\lambda*\nabla\big(|u|^2\big)=\operatorname{div}\big(\varGamma^\lambda*|u|^2\big)=0$, the equations (6.6), (6.7) and (6.8) are direct consequences of [7, Proposition 4.8]. The remaining identities follow by applying the curl operator to both sides of these formulas and repeating the computations from Section 4. □

Remark 6.3. In view of Proposition 6.2 and the pointwise estimates of ϕ^λ from Theorem 4.3 and Theorem 4.4, we can explain, at this point, the origin of the pointwise estimates stated in Theorem 3.2. Comparing (3.4) and (4.16), we see that the asserted decay rates of the steady-state parts $\operatorname{curl} v$ and $\nabla\phi_0^\lambda$ coincide, which is the optimal result one can expect to derive from equation (6.10) for general $f\in\mathrm{C}_0^\infty(G)^3$. In contrast, the asserted decay rates of the purely periodic parts $\operatorname{curl} w$ and $\varGamma_\perp^\lambda$ given in (3.5) and (4.18), respectively, do not coincide. The reason is due to the presence of the term $\operatorname{curl} v\wedge w$ in (6.11). By assuming the—to some extent—optimal decay rate (3.4) for $\operatorname{curl} v$, the pointwise estimate of w from (6.4) implies

$$|\operatorname{curl} v\wedge w|(t,x)\leq C|x|^{-9/2}\,\mathrm{e}^{-\alpha s(\lambda x)}.$$

In the end, this term dominates the decay of the right-hand side of (6.11) and thus the pointwise estimates of $\operatorname{curl} w$. As a result, the decay of $\operatorname{curl} w$ is slower than that of $\nabla\phi_\perp^\lambda$ but, however, still faster than the decay rate of the steady-state vorticity field $\operatorname{curl} v$.

Proposition 6.2 yields fixed-point equations for u and $\operatorname{curl} u$ and the respective steady-state and purely periodic parts, which we now decompose in an appropriate way. Let $\chi\in\mathrm{C}_0^\infty(\mathbb{R};[0,1])$ with $\chi(s)=1$ for $|s|\leq 5/4$ and $\chi(s)=0$ for $|s|\geq 7/4$. For $S>0$ define $\chi_S\in\mathrm{C}_0^\infty(\mathbb{R}^3;[0,1])$ by $\chi_S(x):=\chi(S^{-1}|x|)$, and fix $S_0>0$ such that $\operatorname{supp} f\subset\mathbb{T}\times\mathrm{B}_{S_0}$. For $S\in[2S_0,\infty)$ we express (6.6) as the sum of two terms, namely

$$u=\varGamma^\lambda*\big[-(1-\chi_S)\operatorname{curl} u\wedge u\big]+\varGamma^\lambda*\big[f-\chi_S\operatorname{curl} u\wedge u\big].$$

Due to $\operatorname{supp}(1-\chi_S)\subset\mathrm{B}^S$, this yields

$$u|_{\mathbb{T}\times\mathrm{B}^S}=\mathcal{F}_S(u|_{\mathbb{T}\times\mathrm{B}^S})+\mathcal{H}_S, \tag{6.12}$$

where

$$\begin{aligned}\mathcal{F}_S(z) &:= \big(\varGamma^\lambda*\big[-(1-\chi_S)\operatorname{curl} z\wedge z\big]\big)|_{\mathbb{T}\times\mathrm{B}^S},\\ \mathcal{H}_S &:= \big(\varGamma^\lambda*\big[f-\chi_S\operatorname{curl} u\wedge u\big]\big)|_{\mathbb{T}\times\mathrm{B}^S}.\end{aligned}$$

We set $\mathcal{A}(z) := -\operatorname{curl} z \wedge z$ and

$$\mathcal{A}_0(z) := \mathcal{P}\mathcal{A}(z) = -\operatorname{curl} z_0 \wedge z_0 - \mathcal{P}(\operatorname{curl} z_\perp \wedge z_\perp), \tag{6.13}$$
$$\mathcal{A}_\perp(z) := \mathcal{P}_\perp\mathcal{A}(z) = -\operatorname{curl} z_0 \wedge z_\perp - \operatorname{curl} z_\perp \wedge z_0 - \mathcal{P}_\perp(\operatorname{curl} z_\perp \wedge z_\perp), \tag{6.14}$$

with $z_0 := \mathcal{P}z$ and $z_\perp := \mathcal{P}_\perp z$. For $(t,x) \in \mathbb{T} \times \mathrm{B}^S$ from Proposition 6.2 we then obtain

$$D_x^\alpha \mathcal{P}\mathcal{F}_S(z)(x) = D_x^\alpha \varGamma_0^\lambda * \big[(1-\chi_S)\mathcal{A}_0(z)\big](x), \tag{6.15}$$
$$D_x^\alpha \mathcal{P}_\perp\mathcal{F}_S(z)(t,x) = D_x^\alpha \varGamma_\perp^\lambda * \big[(1-\chi_S)\mathcal{A}_\perp(z)\big](t,x), \tag{6.16}$$
$$\operatorname{curl} \mathcal{P}\mathcal{F}_S(z)(x) = \int_{\mathbb{R}^3} \nabla\phi_0^\lambda(x-y) \wedge \big[(1-\chi_S)\mathcal{A}_0(z)\big](y)\,\mathrm{d}y, \tag{6.17}$$
$$\operatorname{curl} \mathcal{P}_\perp\mathcal{F}_S(z)(t,x) = \int_{\mathbb{T}\times\mathbb{R}^3} \nabla\phi_\perp^\lambda(t-s,x-y) \wedge \big[(1-\chi_S)\mathcal{A}_\perp(z)\big](s,y)\,\mathrm{d}(s,y), \tag{6.18}$$

and

$$D_x^\alpha \mathcal{P}\mathcal{H}_S(x) = D_x^\alpha \varGamma_0^\lambda * \big[\mathcal{P}f + \chi_S\mathcal{A}_0(u)\big](x), \tag{6.19}$$
$$D_x^\alpha \mathcal{P}_\perp\mathcal{H}_S(t,x) = D_x^\alpha \varGamma_\perp^\lambda * \big[\mathcal{P}_\perp f + \chi_S\mathcal{A}_\perp(u)\big](t,x), \tag{6.20}$$
$$\operatorname{curl} \mathcal{P}\mathcal{H}_S(x) = \int_{\mathbb{R}^3} \nabla\phi_0^\lambda(x-y) \wedge \big[\mathcal{P}f + \chi_S\mathcal{A}_0(u)\big](y)\,\mathrm{d}y, \tag{6.21}$$
$$\operatorname{curl} \mathcal{P}_\perp\mathcal{H}_S(t,x) = \int_{\mathbb{T}\times\mathbb{R}^3} \nabla\phi_\perp^\lambda(t-s,x-y) \wedge \big[\mathcal{P}_\perp f + \chi_S\mathcal{A}_\perp(u)\big](s,y)\,\mathrm{d}(s,y). \tag{6.22}$$

In the next step we introduce the functional framework for the analysis of the fixed-point equation (6.12). Let $\varepsilon \in (0, \frac{1}{4})$ and fix a radius $S > S_0$. We define the following (semi-)norms, which take into account different decay rates of the steady-state and the purely periodic parts:

$$\begin{aligned}
\mathrm{M}_S(z) :=& \operatorname*{ess\,sup}_{x\in\mathrm{B}^S} \Big[|x|(1+s(x))|\mathcal{P}z(x)| + \big[|x|(1+s(x))\big]^{3/2}|\nabla\mathcal{P}z(x)|\Big] \\
&\qquad + \operatorname*{ess\,sup}_{(t,x)\in\mathbb{T}\times\mathrm{B}^S} \Big[|x|^3|\mathcal{P}_\perp z(t,x)| + |x|^4|\nabla\mathcal{P}_\perp z(t,x)|\Big], \\
\mathrm{N}_S^\varepsilon(z) :=& \operatorname*{ess\,sup}_{x\in\mathrm{B}^S} |x|^{3/2}\,\mathrm{e}^{\frac{s(Kx)}{1+S}}\,|\operatorname{curl}\mathcal{P}z(x)| \\
&\qquad + \operatorname*{ess\,sup}_{(t,x)\in\mathbb{T}\times\mathrm{B}^S} |x|^{9/2-\varepsilon}\,\mathrm{e}^{\frac{s(Kx)}{1+S}}\,|\operatorname{curl}\mathcal{P}_\perp z(t,x)|,
\end{aligned}$$

where $K := \frac{1}{4}\min\{\lambda, C_3\}$ with C_3 from Theorem 4.4. The function spaces associated to these (semi-)norms are given by

$$\begin{aligned}
\mathcal{M}_S &:= \big\{ z \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{T}\times\mathrm{B}^S) \;\big|\; \mathrm{M}_S(z) < \infty \big\}, \\
\mathcal{N}_S^\varepsilon &:= \big\{ z \in \mathcal{M}_S \;\big|\; \mathrm{N}_S^\varepsilon(z) < \infty \big\},
\end{aligned}$$

which are Banach spaces with respect to the norms

$$\|z\|_{\mathcal{M}_S} := \mathrm{M}_S(z), \qquad \|z\|_{\mathcal{N}_S^\varepsilon} := \mathrm{M}_S(z) + \mathrm{N}_S^\varepsilon(z),$$

respectively.

Remark 6.4. Let us explain the terms appearing in these definitions. The definition of $\mathrm{M}_S(z)$ is chosen to capture the asymptotic behavior of u and ∇u described in Theorem 6.1. A justification for the denominator $1+S$ in the exponential term in the definition of $\mathrm{N}^\varepsilon_S(z)$ is given by Lemma 7.1 below. The choice of the constant K ensures the validity of the inequalities

$$\mathrm{e}^{2s(Kx)} \leq \mathrm{e}^{s(\lambda x)/2}, \qquad \mathrm{e}^{2s(Kx)} \leq \mathrm{e}^{C_3|x|}, \tag{6.23}$$

so that the exponential term can be related with the exponential terms in the decay rates of $\nabla\phi_0^\lambda$ and $\nabla\phi_\perp^\lambda$ from Theorem 4.3 and Theorem 4.4, respectively. Moreover, the second term in the definition of $\mathrm{N}^\varepsilon_S(z)$ contains the factor $|x|^{9/2-\varepsilon}$ instead of $|x|^{9/2}$, which one would expect, in view of the asserted estimate (3.5). Later on we shall see that this discrepancy is necessary to ensure that $\mathcal{F}_S$ is a contraction in the underlying function space.

7 Estimates

In this section, we collect estimates of $\mathcal{H}_S$ and $\mathcal{F}_S(z)$ with respect to the (semi-)norms introduced above, which ensure that $z \mapsto \mathcal{F}_S(z) + \mathcal{H}_S$ is a contractive self-mapping when we choose S sufficiently large. We begin with the following elementary lemma, which explains the term $1+S$ in the definition of $\mathrm{N}^\varepsilon_S(z)$.

Lemma 7.1. *Let $a,\ S>0$. If $x,y\in\mathbb{R}^3$ with $|y|\leq 2S$, then*

$$\mathrm{e}^{-s(a(x-y))} \leq \mathrm{e}^{4a}\,\mathrm{e}^{-\frac{s(ax)}{1+S}}, \tag{7.1}$$

$$\mathrm{e}^{-a|x-y|} \leq \mathrm{e}^{2a}\,\mathrm{e}^{-\frac{a|x|}{1+S}}. \tag{7.2}$$

Proof. For $|y|\leq 2S$ we have $s(ay)/(1+S) \leq 2a|y|/(1+S) \leq 4a$. Together with $s(a(x-y)) \geq s(ax)-s(ay)$, this implies

$$\mathrm{e}^{-s(a(x-y))} \leq \mathrm{e}^{-\frac{s(a(x-y))}{1+S}} \leq \mathrm{e}^{-\frac{s(ax)}{1+S}}\,\mathrm{e}^{\frac{s(ay)}{1+S}} \leq \mathrm{e}^{-\frac{s(ax)}{1+S}}\,\mathrm{e}^{4a}.$$

Similarly, we have $|y|/(1+S)\leq 2$, which implies

$$\mathrm{e}^{-a|x-y|} \leq \mathrm{e}^{-\frac{a|x-y|}{1+S}} \leq \mathrm{e}^{-\frac{a|x|}{1+S}}\,\mathrm{e}^{\frac{a|y|}{1+S}} \leq \mathrm{e}^{-\frac{a|x|}{1+S}}\,\mathrm{e}^{2a}.$$

This completes the proof. □

We further employ the following lemma in order to estimate convolutions of functions with anisotropic decay behavior.

Lemma 7.2. *Let $A\in(2,\infty)$, $B\in[0,\infty)$ with $A+\min\{1,B\}>3$. Then there exists $C=C(A,B)>0$ such that for all $x\in\mathbb{R}^3$ it holds*

$$\int_{\mathbb{R}^3} \big[(1+|x-y|)(1+s(x-y))\big]^{-3/2}(1+|y|)^{-A}(1+s(y))^{-B}\,\mathrm{d}y \leq C(1+|x|)^{-3/2}.$$

Proof. See [4, Theorem 5]. □

The next lemma treats convolutions of functions that are homogeneous in space.

Lemma 7.3. *Let $A \in (0,3)$, $B \in (0,\infty)$, $\alpha \in (0,\infty)$. Then there exists a constant $C = C(A,B,\alpha) > 0$ such that for all $x \in \mathbb{R}^3$ it holds*

$$\int_{\mathbb{R}^3} |x-y|^{-A}\,\mathrm{e}^{-\alpha|x-y|}(1+|y|)^{-B}\,\mathrm{d}y \leq C(1+|x|)^{-B}.$$

Proof. For $x=0$ the integral is finite, so that it remains to consider $x \neq 0$. We split the integral into two parts

$$I_1 := \int_{\mathrm{B}_{|x|/2}(x)} |x-y|^{-A}\,\mathrm{e}^{-\alpha|x-y|}(1+|y|)^{-B}\,\mathrm{d}y,$$
$$I_2 := \int_{\mathrm{B}^{|x|/2}(x)} |x-y|^{-A}\,\mathrm{e}^{-\alpha|x-y|}(1+|y|)^{-B}\,\mathrm{d}y,$$

which we estimate separately. On the one hand, since $|x-y| \leq |x|/2$ implies $|y| \geq |x| - |x-y| \geq |x|/2$, we have

$$I_1 \leq C(1+|x|)^{-B} \int_{\mathbb{R}^3} |x-y|^{-A}\,\mathrm{e}^{-\alpha|x-y|}\;\mathrm{d}y \leq C(1+|x|)^{-B},$$

where the integral is finite due to $A<3$. On the other hand, we obtain

$$I_2 \leq C\,\mathrm{e}^{-\alpha|x|/4} \int_{\mathbb{R}^3} \mathrm{e}^{-\alpha|x-y|/2}\;\mathrm{d}y \leq C\,\mathrm{e}^{-\alpha|x|/4} \leq C(1+|x|)^{-B}.$$

This completes the proof. □

Since our assumptions do not provide pointwise information on the body force f, we estimate the convolutions of the fundamental solutions with f in a different way, which leads to the following lemma.

Lemma 7.4. *There exists a constant $C>0$ such that for $\alpha \in \mathbb{N}_0$, $|\alpha| \leq 1$, we have*

$$\big|D_x^\alpha \varGamma_0^\lambda * \mathcal{P}f(x)\big| \leq C\big[|x|\big(1+s(\lambda x)\big)\big]^{-1-\frac{|\alpha|}{2}}, \tag{7.3}$$
$$\big|D_x^\alpha \varGamma_\perp^\lambda * \mathcal{P}_\perp f(t,x)\big| \leq C|x|^{-3-|\alpha|}, \tag{7.4}$$
$$\Big|\int_{\mathbb{R}^3} \nabla\phi_0^\lambda(x-y)\wedge\mathcal{P}f(y)\,\mathrm{d}y\Big| \leq C|x|^{-3/2}\,\mathrm{e}^{-\frac{s(\lambda x)}{4(1+S_0)}}, \tag{7.5}$$
$$\Big|\int_{\mathbb{T}\times\mathbb{R}^3} \nabla\phi_\perp^\lambda(t-s,x-y)\wedge\mathcal{P}_\perp f(s,y)\,\mathrm{d}(s,y)\Big| \leq C|x|^{-9/2}\,\mathrm{e}^{-\frac{C_3|x|}{2(1+S_0)}}\,. \tag{7.6}$$

for all $t \in \mathbb{T}$ and $|x| \geq 2S_0$.

Proof. For $|x| \geq 2S_0 \geq 2|y|$ we have

$$(1+2\lambda S_0)(1+\lambda s(x-y)) \geq 1+\lambda s(x) + 2\lambda S_0 - \lambda s(y) \geq 1+\lambda s(x)$$

and $|x-y| \geq |x|-|y| \geq |x|/2 \geq S_0$. Therefore, (4.6) and $\operatorname{supp} f \subset \mathbb{T}\times\mathrm{B}_{S_0}$ imply

$$\begin{aligned}\big|D_x^\alpha \varGamma_0^\lambda * \mathcal{P}f(x)\big| &\leq C\int_{\mathrm{B}_{S_0}} \big[|x-y|\big(1+\lambda s(x-y)\big)\big]^{-1-\frac{|\alpha|}{2}}|\mathcal{P}f(y)|\,\mathrm{d}y\\ &\leq C\big[|x|\big(1+\lambda s(x)\big)\big]^{-1-\frac{|\alpha|}{2}}\int_{\mathrm{B}_{S_0}} |\mathcal{P}f(y)|\,\mathrm{d}y,\end{aligned}$$

which yields (7.3). Using Hölder's inequality in time and (4.7), for any $q \in (1,\infty)$ we obtain in a similar way

$$\begin{aligned}\big|D_x^\alpha \varGamma_\perp^\lambda * \mathcal{P}_\perp f(t,x)\big| &\leq \int_{\mathrm{B}_{S_0}} \Big(\int_{\mathbb{T}} |D_x^\alpha \varGamma_\perp^\lambda(s,x-y)|^{\frac{q}{q-1}}\,\mathrm{d}s\Big)^{\frac{q-1}{q}} \Big(\int_{\mathbb{T}} |\mathcal{P}_\perp f(s,y)|^q\,\mathrm{d}s\Big)^{1/q}\mathrm{d}y \\ &\leq C\int_{\mathrm{B}_{S_0}} |x-y|^{-3-|\alpha|}\Big(\int_{\mathbb{T}} |\mathcal{P}_\perp f(s,y)|^q\,\mathrm{d}s\Big)^{1/q}\mathrm{d}y \\ &\leq C|x|^{-3-|\alpha|}\Big(\int_{\mathbb{T}}\int_{\mathrm{B}_{S_0}} |\mathcal{P}_\perp f(s,y)|^q\,\mathrm{d}s\mathrm{d}y\Big)^{1/q},\end{aligned}$$

which shows (7.4). In virtue of the estimates (4.16) and (4.18) and Lemma 7.1, for $|x| \geq 2S_0 \geq 2|y|$ we further derive

$$\begin{aligned}|\nabla\phi_0^\lambda(x-y)| &\leq C|x-y|^{-3/2}\,\mathrm{e}^{-s(\lambda(x-y))/4} \leq C|x|^{-3/2}\,\mathrm{e}^{-\frac{s(\lambda x)}{4(1+S_0)}}, \\ \|\nabla\phi_\perp^\lambda(\cdot,x-y)\|_{\mathrm{L}^q(\mathbb{T})} &\leq C|x-y|^{-9/2}\,\mathrm{e}^{-C_3|x-y|/2} \leq C|x|^{-9/2}\,\mathrm{e}^{-\frac{-C_3|x|}{2(1+S_0)}}.\end{aligned}$$

From these estimates we conclude (7.5) and (7.6) with the same argument as above. □

After these preparations, we show in the next two lemmas that the norm of $\mathcal{H}_S$ in both $\mathcal{M}_S$ and $\mathcal{N}_S^\varepsilon$ is bounded by a constant independent of $S \geq 2S_0$.

Lemma 7.5. *There exists a constant $C_6 > 0$ such that for all $S \in [2S_0,\infty)$ we have*

$$\mathrm{M}_S(\mathcal{H}_S) \leq C_6.$$

Proof. From the decay estimates of u and ∇u from Theorem 6.1 we conclude

$$\big|\chi_S(x)\mathcal{A}_0(u)(x)\big| \leq C\big[(1+|x|)(1+s(x))\big]^{-5/2}, \tag{7.7}$$

$$\big|\chi_S(x)\mathcal{A}_\perp(u)(t,x)\big| \leq C(1+|x|)^{-9/2}. \tag{7.8}$$

By Theorem 4.1 and Theorem 4.2, these estimates and the formulas (6.19) and (6.20) together with Lemma 7.4 imply

$$\begin{aligned}|\mathcal{P}\mathcal{H}_S(x)| &\leq C\big[(1+|x|)(1+s(x))\big]^{-1}, \\ |\nabla\mathcal{P}\mathcal{H}_S(x)| &\leq C\big[(1+|x|)(1+s(x))\big]^{-3/2}, \\ |\mathcal{P}_\perp\mathcal{H}_S(t,x)| &\leq C(1+|x|)^{-3}, \\ |\nabla\mathcal{P}_\perp\mathcal{H}_S(t,x)| &\leq C(1+|x|)^{-4}\end{aligned}$$

for all $t \in \mathbb{T}$ and $|x| \geq S_0$. Collecting these, we arrive at the claimed estimate. □

Lemma 7.6. *There exists a constant $C_7 > 0$ such that for all $S \in [2S_0,\infty)$ we have*

$$\mathrm{N}_S^\varepsilon(\mathcal{H}_S) \leq C_7.$$

Proof. At first, let $x \in \mathbb{R}^3$ with $|x| \geq 2S$. For $|y| \leq 7S/4$ we have

$$|x-y| \geq |x|-|y| \geq |x|-7S/4 \geq |x|-7|x|/8 = |x|/8 \geq S/4 \geq S_0/2.$$

From (4.16) and Lemma 7.1, we then conclude

$$\begin{aligned}|\nabla\phi_0^\lambda(x-y)| &\leq C\big(|x-y|^{-2}+|x-y|^{-3/2}s(\lambda(x-y))^{1/2}\big)\,\mathrm{e}^{-s(\lambda(x-y))/2}\\ &\leq C\big(1+|x-y|^{-3/2}\big(1+s(\lambda(x-y))\big)^{-3/2}\big)\,\mathrm{e}^{-s(\lambda(x-y))/4}\\ &\leq C\big[(1+|x-y|)\big(1+s(\lambda(x-y))\big)\big]^{-3/2}\,\mathrm{e}^{-\frac{s(\lambda x)}{4(1+S)}}\,.\end{aligned}$$

In virtue of (6.21), (7.5) and (7.7) we thus obtain

$$\begin{aligned}&|\operatorname{curl}\mathcal{P}\mathcal{H}_S(x)| \leq C\int_{\mathrm{B}_{7S/4}}\big|\nabla\phi_0^\lambda(x-y)\big|\,\big|\mathcal{P}f+\chi_S\mathcal{A}_0(u)\big|(y)\,\mathrm{d}y\\ &\leq C|x|^{-3/2}\,\mathrm{e}^{-\frac{s(\lambda x)}{4(1+S_0)}}\\ &\quad + C\,\mathrm{e}^{-\frac{s(\lambda x)}{4(1+S)}}\int_{\mathbb{R}^3}\big[\big(1+|x-y|\big)\big(1+s(x-y)\big)\big]^{-3/2}\big[(1+|y|)(1+s(y))\big]^{-5/2}\,\mathrm{d}y\end{aligned}$$

for $|x|\geq 2S\geq 4S_0$. By estimating the remaining integral with the help of Lemma 7.2 and employing (6.23), we deduce

$$|\operatorname{curl}\mathcal{P}\mathcal{H}_S(x)| \leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-3/2} \tag{7.9}$$

for $|x|\geq 2S$. If $S\leq|x|\leq 2S$, then Lemma 7.5 yields

$$|\operatorname{curl}\mathcal{P}\mathcal{H}_S(x)| \leq C|\nabla\mathcal{P}\mathcal{H}_S(x)| \leq C\big[(1+|x|)(1+s(x))\big]^{-3/2} \leq C|x|^{-3/2}.$$

Since $|x|\leq 2S$ implies $s(Kx)/(1+S)\leq 2|Kx|/(1+S)\leq 4KS/(1+S)\leq 4K$, we have $1\leq \mathrm{e}^{4K}\,\mathrm{e}^{-s(Kx)/(1+S)}$, so that (7.9) also holds for $S\leq|x|\leq 2S$.

Now let us turn to $\operatorname{curl}\mathcal{P}_\perp\mathcal{H}_S$. From (4.18) and (7.2), for $|y|\leq 2S$ we conclude

$$\int_{\mathbb{T}}|\nabla\phi_\perp^\lambda(t-s,x-y)|\,\mathrm{d}s \leq C|x-y|^{-5/2}\,\mathrm{e}^{-\frac{C_3|x-y|}{2}}\,\mathrm{e}^{-\frac{C_3|x|}{2(1+S)}},$$

so that (6.22), (7.6) and (7.8) lead to

$$\begin{aligned}|\operatorname{curl}\mathcal{P}_\perp\mathcal{H}_S(t,x)| &\leq C\int_{\mathrm{B}_{7S/4}}\int_{\mathbb{T}}\big|\nabla\phi_\perp^\lambda(t-s,x-y)\big|\,\big|\mathcal{P}_\perp f+\chi_S\mathcal{A}_\perp(u)\big|(s,y)\,\mathrm{d}s\mathrm{d}y\\ &\leq C|x|^{-9/2}\,\mathrm{e}^{-\frac{C_3|x|}{2(1+S_0)}} + C\,\mathrm{e}^{-\frac{C_3|x|}{2(1+S)}}\int_{\mathbb{R}^3}|x-y|^{-5/2}\,\mathrm{e}^{-C_3|x-y|/2}(1+|y|)^{-9/2}\,\mathrm{d}y.\end{aligned}$$

The remaining integral can be estimated with Lemma 7.3. Further using (6.23), we end up with

$$|\operatorname{curl}\mathcal{P}_\perp\mathcal{H}_S(t,x)| \leq C\,\mathrm{e}^{-\frac{C_3|x|}{2(1+S)}}\,|x|^{-9/2} \leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-9/2+\varepsilon}$$

for $|x|\geq S\geq 2S_0$ and $t\in\mathbb{T}$. A combination of this estimate with (7.9) finishes the proof. □

In the next two lemmas we provide appropriate estimates of $\mathcal{F}_S(z)$. Observe that, in contrast to $\mathcal{H}_S$, this term depends on the (unknown) function z. In order to eventually obtain a contraction for large S, we factor out the term $S^{-\varepsilon}$ in the estimates.

Lemma 7.7. *There exists a constant $C_8>0$ such that for all $S\in[2S_0,\infty)$ and all $z_1,z_2\in\mathcal{M}_S$ we have*

$$\mathrm{M}_S(\mathcal{F}_S(z_1)) \leq C_8 S^{-\varepsilon}\mathrm{M}_S(z_1)^2, \tag{7.10}$$

$$\mathrm{M}_S(\mathcal{F}_S(z_1)-\mathcal{F}_S(z_2)) \leq C_8 S^{-\varepsilon}\big(\mathrm{M}_S(z_1)+\mathrm{M}_S(z_2)\big)\mathrm{M}_S(z_1-z_2). \tag{7.11}$$

Proof. For $z \in \mathcal{M}_S$ we immediately deduce

$$\begin{aligned}
\big|(1-\chi_S(x))\mathcal{A}_0(z)(x)\big| &\leq C\mathrm{M}_S(z)^2(1-\chi_S(x))\big[(1+|x|)(1+s(x))\big]^{-5/2}\\
&\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2(1+|x|)^{-5/2+\varepsilon}(1+s(x))^{-5/2},\\
\big|(1-\chi_S(x))\mathcal{A}_\perp(z)(t,x)\big| &\leq C\mathrm{M}_S(z)^2(1-\chi_S(x))(1+|x|)^{-9/2}\\
&\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2(1+|x|)^{-9/2+\varepsilon}
\end{aligned}$$

for $|x| \geq S$. By Theorem 4.1 and Theorem 4.2, from these estimates and the formulas (6.15) and (6.16) we conclude

$$\begin{aligned}
|\mathcal{P}\mathcal{F}_S(z)(x)| &\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2\big[(1+|x|)(1+s(x))\big]^{-1},\\
|\nabla\mathcal{P}\mathcal{F}_S(z)(x)| &\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2\big[(1+|x|)(1+s(x))\big]^{-3/2},\\
|\mathcal{P}_\perp\mathcal{F}_S(z)(t,x)| &\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2(1+|x|)^{-3},\\
|\nabla\mathcal{P}_\perp\mathcal{F}_S(z)(t,x)| &\leq CS^{-\varepsilon}\mathrm{M}_S(z)^2(1+|x|)^{-4}.
\end{aligned}$$

Collecting these estimates, we obtain (7.10). The inequality (7.11) is proved in the same fashion. □

Lemma 7.8. *There exists a constant $C_9 > 0$ such that for all $S \in [2S_0,\infty)$ and all $z_1, z_2 \in \mathcal{N}_S^\varepsilon$ we have*

$$\mathrm{N}_S^\varepsilon(\mathcal{F}_S(z_1)) \leq C_9 S^{-\varepsilon}\mathrm{M}_S(z_1)\mathrm{N}_S^\varepsilon(z_1), \tag{7.12}$$

$$\mathrm{N}_S^\varepsilon(\mathcal{F}_S(z_1)-\mathcal{F}_S(z_2)) \leq C_9 S^{-\varepsilon}\big(\|z_1\|_{\mathcal{N}_S^\varepsilon}+\|z_2\|_{\mathcal{N}_S^\varepsilon}\big)\|z_1-z_2\|_{\mathcal{N}_S^\varepsilon}. \tag{7.13}$$

Proof. For $z \in \mathcal{N}_S^\varepsilon$ we have

$$\begin{aligned}
\big|(1-\chi_S(x))\mathcal{A}_0(z)(x)\big| &\leq C\mathrm{M}_S(z)\mathrm{N}_S^\varepsilon(z)(1-\chi_S(x))|x|^{-5/2}(1+s(x))^{-1}\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\\
&\leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}_S^\varepsilon(z)|x|^{-5/2+\varepsilon}(1+s(x))^{-1}\,\mathrm{e}^{-\frac{s(Kx)}{1+S}},
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
\big|(1-\chi_S(x))\mathcal{A}_\perp(z)(t,x)\big| &\leq C\mathrm{M}_S(z)\mathrm{N}_S^\varepsilon(z)(1-\chi_S(x))|x|^{-9/2}\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\\
&\leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}_S^\varepsilon(z)|x|^{-9/2+\varepsilon}\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}
\end{aligned} \tag{7.15}$$

for $|x| \geq S$. Exploiting the representation formula (6.17), we can employ (4.16) and (7.14) to estimate

$$\begin{aligned}
|\mathrm{curl}\,\mathcal{P}\mathcal{F}_S(z)(x)| &\leq C\int_{\mathbb{R}^3}\big|\nabla\phi_0^\lambda(x-y)\big|\,\big|(1-\chi_S(y))\mathcal{A}_0(z)(y)\big|\,\mathrm{d}y\\
&\leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}_S^\varepsilon(z)\big(I_1+I_2\big),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_{\mathrm{B}^S\cap\mathrm{B}_{S_0}(x)}|x-y|^{-2}\,\mathrm{e}^{-\frac{s(\lambda(x-y))}{4}}\,|y|^{-5/2+\varepsilon}(1+s(y))^{-1}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}}\,\mathrm{d}y,\\
I_2 &:= \int_{\mathrm{B}^S\cap\mathrm{B}^{S_0}(x)}\big[|x-y|s(\lambda(x-y))\big]^{-3/2}\,\mathrm{e}^{-\frac{s(\lambda(x-y))}{4}}\,|y|^{-5/2+\varepsilon}(1+s(y))^{-1}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}}\,\mathrm{d}y.
\end{aligned}$$

To give estimates of these integrals, we first note that by $s(\lambda(x-y)) \geq s(\lambda x)-s(\lambda y)$ and (6.23), we have

$$\mathrm{e}^{-\frac{s(\lambda(x-y))}{4}}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}} \leq \mathrm{e}^{-\frac{s(\lambda x)}{4(1+S)}}\,\mathrm{e}^{\frac{s(\lambda y)}{4(1+S)}}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}} \leq \mathrm{e}^{-\frac{s(Kx)}{1+S}} \tag{7.16}$$

for all $x,\, y \in \mathbb{R}^3$. On the one hand, exploiting this estimate and that $|x-y| \leq S_0 \leq |x|/2$ implies $|y| \geq |x| - |x-y| \geq |x| - S_0 \geq |x|/2$, we conclude

$$I_1 \leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-5/2+\varepsilon}\int_{\mathrm{B}_{S_0}(x)} |x-y|^{-2}\,\mathrm{d}y \leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-3/2}$$

for $|x| \geq S \geq 2S_0$. On the other hand, due to (7.16) and the fact that $|y| \geq S \geq 2S_0$ implies $|y| \geq C(1+|y|)$, we obtain

$$\begin{aligned}I_2 &\leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\int_{\mathbb{R}^3}\big[(1+|x-y|)s(x-y)\big]^{-3/2}(1+|y|)^{-5/2+\varepsilon}(1+s(y))^{-1}\,\mathrm{d}y\\ &\leq C\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-3/2}\end{aligned}$$

by Lemma 7.2. From the estimates of I_1 and I_2 we deduce

$$|\mathrm{curl}\,\mathcal{P}\mathcal{F}_S(z)(x)| \leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}^{\varepsilon}_S(z)\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-3/2}.$$

Now let us turn to the purely periodic part $\mathcal{P}_\perp\mathcal{F}_S(z)$. From (4.18) (with $q=1$ and $\gamma = 1/4$) we conclude

$$\int_{\mathbb{T}} |\nabla\phi^{\lambda}_{\perp}(t-s,x-y)|\,\mathrm{d}s \leq C|x-y|^{-5/2}\,\mathrm{e}^{-C_3|x-y|}\,.$$

With formula (6.18) and estimate (7.15) we thus obtain

$$\begin{aligned}|\mathrm{curl}\,\mathcal{P}_\perp\mathcal{F}_S(z)(t,x)| &\leq C\int_{\mathbb{T}}\int_{\mathbb{R}^3}\big|\nabla\phi^{\lambda}_{\perp}(t-s,x-y)\big|\,\big|(1-\chi_S(y))\mathcal{A}_\perp(z)(s,y)\big|\,\mathrm{d}y\mathrm{d}s\\ &\leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}^{\varepsilon}_S(z)\int_{\mathrm{B}^S}|x-y|^{-5/2}\,\mathrm{e}^{-C_3|x-y|}\,|y|^{-9/2+\varepsilon}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}}\,\mathrm{d}y.\end{aligned}$$

By (6.23) we have

$$\mathrm{e}^{-\frac{C_3|x-y|}{2}}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}} \leq \mathrm{e}^{-s(K(x-y))}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}} \leq \mathrm{e}^{-\frac{s(K(x-y))}{1+S}}\,\mathrm{e}^{-\frac{s(Ky)}{1+S}} \leq \mathrm{e}^{-\frac{s(Kx)}{1+S}}\,.$$

This yields

$$\begin{aligned}&|\mathrm{curl}\,\mathcal{P}_\perp\mathcal{F}_S(z)(t,x)|\\ &\quad\leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}^{\varepsilon}_S(z)\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\int_{\mathbb{R}^3}|x-y|^{-5/2}\,\mathrm{e}^{-\frac{C_3|x-y|}{2}}(1+|y|)^{-9/2+\varepsilon}\,\mathrm{d}y.\end{aligned}$$

Employing Lemma 7.3 to estimate the remaining integral, we end up with

$$|\mathrm{curl}\,\mathcal{P}_\perp\mathcal{F}_S(z)(t,x)| \leq CS^{-\varepsilon}\mathrm{M}_S(z)\mathrm{N}^{\varepsilon}_S(z)\,\mathrm{e}^{-\frac{s(Kx)}{1+S}}\,|x|^{-9/2+\varepsilon}.$$

In total, we have thus shown (7.12). Estimate (7.13) is derived in the same way. □

8 Conclusion of the proof

After the preparatory results from the previous section, we now prove the existence of a function $z \in \mathcal{N}^{\varepsilon}_S$ satisfying the fixed-point equation

$$z = \mathcal{F}_S(z) + \mathcal{H}_S$$

provided $S \geq 2S_0$ is chosen sufficiently large. Afterwards, we show uniqueness of this fixed point in the function class $\mathcal{M}_S$. Since $u|_{\mathbb{T}\times\mathrm{B}^S}$ is another solution to this fixed-point equation and belongs to $\mathcal{M}_S$ by Theorem 6.1, we then conclude that z coincides with $u|_{\mathbb{T}\times\mathrm{B}^S}$. This yields the decay rate of the vorticity field asserted in Theorem 3.2 up to a factor $|x|^{-\varepsilon}$ for the purely periodic part. Returning to the representation formula (6.11), we finally omit this factor and complete the proof of Theorem 3.2.

To begin with, for $S \in [2S_0, \infty)$ consider the closed subset

$$\mathcal{B}_S := \big\{ z \in \mathcal{N}_S^\varepsilon \mid \|z\|_{\mathcal{N}_S^\varepsilon} \leq C_6 + C_7 + 1 \big\}$$

of the Banach space $\mathcal{N}_S^\varepsilon$. Choose $S_1 \in [2S_0, \infty)$ so large that for all $S \in [S_1, \infty)$ we have

$$\begin{aligned}(C_8 + C_9)(C_6 + C_7 + 1)^2 S^{-\varepsilon} &\leq 1,\\ (C_8 + C_9)(C_6 + C_7 + 1) S^{-\varepsilon} &\leq \frac{1}{4}.\end{aligned}$$

Thus, we obtain the existence of a fixed point of $z \mapsto \mathcal{F}_S(z) + \mathcal{H}_S$.

Corollary 8.1. *For any $S \in [S_1, \infty)$ there is a function $z_S \in \mathcal{B}_S$ with $z_S = \mathcal{F}_S(z_S) + \mathcal{H}_S$.*

Proof. By the Lemma 7.5, Lemma 7.6, Lemma 7.7 and Lemma 7.8 and the choice of S_1, the mapping

$$F_S \colon \mathcal{B}_S \to \mathcal{B}_S, \qquad F_S(z) := \mathcal{F}_S(z) + \mathcal{H}_S$$

is a well-defined contractive self-mapping for any $S \geq S_1$. The contraction mapping principle thus implies the existence of the asserted fixed point $z_S \in \mathcal{B}_S$ of F_S. □

Next we show that z_S coincides with $u|_{\mathbb{T}\times\mathrm{B}^S}$ for S sufficiently large. This yields pointwise estimates of u.

Lemma 8.2. *There exists $S_2 \in [S_1, \infty)$ such that for all $S \in [S_2, \infty)$ we have*

$$\begin{aligned}|\operatorname{curl} \mathcal{P} u(x)| &\leq (C_6 + C_7 + 1)|x|^{-3/2}\, \mathrm{e}^{-\frac{s(Kx)}{1+S}},\\ |\operatorname{curl} \mathcal{P}_\perp u(t,x)| &\leq (C_6 + C_7 + 1)|x|^{-9/2+\varepsilon}\, \mathrm{e}^{-\frac{s(Kx)}{1+S}}\end{aligned}$$

for all $t \in \mathbb{T}$ and $x \in \mathrm{B}^S$.

Proof. For $S \geq 2S_0$ we set $U_S := u|_{\mathbb{T}\times\mathrm{B}^S}$. By Theorem 6.1 we know $U_S \in \mathcal{M}_S$ with $\mathrm{M}_S(U) \leq C_2$, and by (6.12) we have $U_S = \mathcal{F}_S(U_S) + \mathcal{H}_S$ for any $S \geq 2S_0$. Now let $S \geq S_1$ and let $z_S \in \mathcal{B}_S$ be the function from Corollary 8.1. Then Lemma 7.7 implies

$$\begin{aligned}\mathrm{M}_S(z_S - U_S) = \mathrm{M}_S(\mathcal{F}_S(z_S) - \mathcal{F}_S(U_S)) &\leq C_8 S^{-\varepsilon}\big(\mathrm{M}_S(z_S) + \mathrm{M}_S(U_S)\big)\mathrm{M}_S(z_S - U_S)\\ &\leq C_8 S^{-\varepsilon}\big(C_6 + C_7 + 1 + C_2\big)\mathrm{M}_S(z_S - U_S).\end{aligned}$$

Choosing $S_2 \in [S_1, \infty)$ such that for all $S \in [S_2, \infty)$ we have

$$C_8\big(C_6 + C_7 + 1 + C_2\big)S^{-\varepsilon} \leq \frac{1}{2},$$

we conclude $\mathrm{M}_S(z_S - U_S) \leq \mathrm{M}_S(z_S - U_S)/2$ and hence $\mathrm{M}_S(z_S - U_S) = 0$ for all $S \in [S_2, \infty)$. This implies $z_S = U_S = u|_{\mathbb{T}\times\mathrm{B}^S}$. In particular, we have $\mathrm{N}_S^\varepsilon(u|_{\mathbb{T}\times\mathrm{B}^S}) = \mathrm{N}_S^\varepsilon(z_S) \leq C_6 + C_7 + 1$ for all $S \in [S_2, \infty)$. This completes the proof. □

Another application of the convolution formula (6.11) enables us to omit the term ε in the estimate of $\operatorname{curl} \mathcal{P}_\perp u$, which yields the estimates from Theorem 3.2 in the case $\Omega = \mathbb{R}^3$.

Theorem 8.3. *Let $\lambda > 0$ and let $f \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)^3$ for all $q \in (1, \infty)$ have bounded support. Let u be a weak time-periodic solution to* (6.1) *in the sense of Definition 3.1, which satisfies* (3.3)*. Then there exist constants $C_1 > 0$ and $\alpha = \alpha(\lambda, \mathcal{T}) > 0$ such that the estimates* (3.4) *and* (3.5) *hold for all $(t, x) \in \mathbb{T} \times \mathbb{R}^3$.*

Proof. We decompose $u = v+w$ into steady-state part $v := \mathcal{P}u$ and purely periodic part $w := \mathcal{P}_\perp u$. Since $\operatorname{curl} u$ is bounded by Theorem 6.1, Lemma 8.2 implies

$$\begin{aligned} |\operatorname{curl} v(x)| &\leq C(1+|x|)^{-3/2} \,\mathrm{e}^{-\alpha s(x)}, \\ |\operatorname{curl} w(t,x)| &\leq C(1+|x|)^{-9/2+\varepsilon} \,\mathrm{e}^{-\alpha s(x)} \end{aligned} \tag{8.1}$$

for all $(t, x) \in \mathbb{T} \times \mathbb{R}^3$, where $\alpha = (1 + S_2)^{-1} K$. In particular, this implies (3.4), and for (3.5) it remains to remove ε in the second inequality. Due to Theorem 6.1, the estimates (8.1) further yield

$$\big|\operatorname{curl} v(y) \wedge w(s,y) + \operatorname{curl} w(s,y) \wedge v(y) + \mathcal{P}_\perp[\operatorname{curl} w \wedge w](s,y)\big| \leq C(1+|y|)^{-9/2} \,\mathrm{e}^{-\alpha s(\lambda y)}$$

for all $(t, x) \in \mathbb{T} \times \mathbb{R}^3$. Moreover, by Theorem 4.4 we have

$$\int_{\mathbb{T}} |\nabla \phi_\perp^\lambda(t-s, x-y)| \,\mathrm{d}s \leq C|x-y|^{-5/2} \,\mathrm{e}^{-C_3|x-y|}.$$

Using these estimates and (7.6) in the representation formula (6.11), we conclude

$$|\operatorname{curl} w(t,x)| \leq C|x|^{-9/2} \,\mathrm{e}^{-\frac{C_3|x|}{2(1+S_0)}} + C \int_{\mathbb{R}^3} |x-y|^{-5/2} \,\mathrm{e}^{-C_3|x-y|} (1+|y|)^{-9/2} \,\mathrm{e}^{-\alpha s(\lambda y)} \,\mathrm{d}y.$$

Due to $2s(Kx) \leq C_3|x|$, we have

$$\frac{1}{2} C_3|x-y| + \alpha s(\lambda y) \geq s(K(x-y)) + \frac{s(Ky)}{1+S_2} \geq \frac{s(Kx)}{1+S_2} = \alpha s(x),$$

and we obtain

$$\begin{aligned} |\operatorname{curl} w(t,x)| \leq\ & C(1+|x|)^{-9/2} \,\mathrm{e}^{-\alpha s(x)} \\ & + C \,\mathrm{e}^{-\alpha s(x)} \int_{\mathbb{R}^3} |x-y|^{-5/2} \,\mathrm{e}^{-C_3|x-y|/2} (1+|y|)^{-9/2} \,\mathrm{d}y, \end{aligned}$$

where we used (6.23). We estimate the remaining integral with Lemma 7.3, which leads to (3.5) and completes the proof. □

Finally, we employ a classical cut-off argument to extend the result to an exterior domain and to finish the proof of Theorem 3.2.

Proof of Theorem 3.2. First of all, Lemma 5.2 implies the existence of a pressure field $\mathfrak{p}$ such that $(u, \mathfrak{p})$ is a strong solution to (1.1) satisfying (5.5)–(5.7). Fix radii $R > r > 0$ such that $\partial\Omega \subset \mathrm{B}_r$, and let $\chi \in \mathrm{C}^\infty(\mathbb{R}^3)$ be a cut-off function such that $\chi(x) = 0$ for $|x| \leq r$ and $\chi(x) = 1$ for $|x| \geq R$. By the divergence theorem, $\operatorname{div} u = 0$ and (3.2), we have

$$\int_{\mathrm{B}_{r,R}} u \cdot \nabla\chi \,\mathrm{d}x = \int_{\mathrm{B}_R} \operatorname{div}\big(u(\chi-1)\big) \,\mathrm{d}x = -\int_{\partial\Omega} u \cdot \mathrm{n} \,\mathrm{d}S = 0.$$

Therefore, there exists a function V with $V \in \mathrm{W}^{1,2,q}(\mathbb{T}\times\mathbb{R}^3)^3$ for all $q \in (1,\infty)$ and $\operatorname{supp} V \subset \mathbb{T}\times \mathrm{B}_{r,R}$ such that $\operatorname{div} V = u\cdot\nabla\chi$; see [12, Section III.3] for example. We define $U := \chi u - V$ and $\mathfrak{P} := \chi\mathfrak{p}$. Then $U \in \mathrm{L}^r(\mathbb{T}\times\mathbb{R}^3)$ for some $r\in(5,\infty)$ and U is a weak solution to

$$\begin{cases} \partial_t U - \Delta U - \lambda\partial_1 U + U\cdot\nabla U + \nabla\mathfrak{P} = F & \text{in } \mathbb{T}\times\mathbb{R}^3,\\ \operatorname{div} U = 0 & \text{in } \mathbb{T}\times\mathbb{R}^3,\\ \lim_{|x|\to\infty} U(t,x) = 0 & \text{for } t\in\mathbb{T}, \end{cases} \tag{8.2}$$

in the sense of Definition 3.1, where $F \in \mathrm{L}^q(\mathbb{T}\times\mathbb{R}^3)^3$ for all $q\in(1,\infty)$, and $\operatorname{supp} F$ is compact. Now the assertion follows from Theorem 8.3 and the identity $U(t,x) = u(t,x)$ for $|x| > R$. □

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