ENERGETIC SYSTEMS AND GLOBAL ATTRACTORS FOR THE 3D NAVIER–STOKES EQUATIONS

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1 INTRODUCTION

The problem of uniqueness and global regularity of solutions of the 3–dimensional Navier–Stokes equations (3D NSE)

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \ge 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega \subset R^3, \tag{1.4}$$

is one of the central open issues in Fluid Mechanics [1]–[3] and Partial Differential Equations [4], [5]. The fundamental unsolved question is whether or not solutions of 3D NSE develop singularities and non-uniqueness. Recall that the classical Leray–Hopf small–data–regularity theorem [6]–[8], [9] states that, for a given smooth domain $\Omega \subset R^3$ and a given viscosity $\nu > 0$, there exists a *small bounded* set of data $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating unique solutions $u \in C([0,\infty);V) \cap L^\infty(0,\infty;V) \cap L^2_{loc}(0,\infty;V^2)$ of 3D NSE,— such solutions are called *globally regular Leray–Hopf* solutions. (Here and below, H denotes the space of divergence–free $L^2(\Omega)$–vector fields satisfying prescribed boundary conditions, while V and V^2 stand for the subspaces of H consisting of $H^1(\Omega)$–vector fields and $H^2(\Omega)$–vector fields, respectively.) The importance of this theorem is that it is the first result which yields an example of a linear space $X \times Y$ such that an explicit construction of data $(u_0, f) \in X \times Y$ generating solutions $u = u(x,t)$, uniquely defined and bounded in the norm of X for all times $t \in [0,\infty)$, can be given without any further *a priori* assumptions on the solutions and/or the domain $\Omega \subset R^3$. In particular, it follows from the Leray–Hopf small–data–regularity theorem that, for a fixed small force $f \in H$, 3D NSE generate a dynamical system[1] on a *small bounded* subset of the space V. Later on,

[1]Recall [10]–[14] that a *dynamical system* S, or *semigroup*, or *semiflow*, on a subset M of a complete metric space W is a mapping $\sigma : [0,\infty) \times M \to M$, $(t,w) \mapsto S(t)w \in M$, such that the folowing three properties hold: 1) $S(0)w = w$, for all $w \in M$; 2) $S(t)S(s)w = S(t+s)w$, for all $w \in M$, $s, t \in [0,\infty)$; 3) the mapping $\sigma : (0,\infty) \times M \to M$ is continuous.

numerous results parallel to the Leray–Hopf small–data–regularity theorem were obtained for other spaces $X \times Y$ and/or for other classes of domains $\Omega \subset R^3$ (see, e.g., [15]–[27]). A general feature of all these results is that they yield *bounded* sets of data $(u_0, f) \in X \times Y$ generating solutions $u = u(x,t)$ uniquely defined and bounded in the norm of the space X for all times $t \in [0,\infty)$, — such solutions are called *globally X–regular* (in particular, globally regular Leray–Hopf solutions are globally V–regular). As a result, 3D NSE were known to generate a dynamical system only on some *bounded* subsets of the corresponding spaces X. Recently, under precisely the same hypotheses that are used in the Leray–Hopf small–data–regularity theorem, a large *unbounded open star–shaped* set of data $(u_0, f) \in V \times L^\infty(0,\infty;H)$ generating unique globally regular Leray–Hopf solutions has been constructed in [28]–[29] by using a new general method of finding large sets of data generating global solutions to nonlinear evolutionary equations (in the case of 3D NSE, the method can also be adapted to other spaces $X \times Y$ and/or other classes of domains $\Omega \subset R^3$). The construction of this unbounded open star–shaped set allows to conclude that, for a fixed small force $f \in H$, 3D NSE generate a dynamical system on a *large unbounded open star–shaped* subset of V, and the existence of the compact global attractor $\mathcal{A} \subset V$ of this dynamical system has been proven in [29]–[30]. Since a motivation for the introduction of the notion of a global attractor in the dynamical systems theory is to describe long–time behaviour of solutions of evolutionary equations (see, e.g., [10]–[14]), one can say that the long–time behaviour of solutions generated by the large unbounded open star–shaped set of initial functions $u_0 \in V$ has been shown to be described by a small, compact, set $\mathcal{A} \subset V$.

An important for applications question is whether or not there exists a *linear space* (not just a subset of a linear space!) X of initial functions u_0 generating global in time solutions $u = u(t,x)$, $t \in [0,\infty)$, long–time behaviour of which can be described by a compact set $\mathcal{A} \subset X$. This question was left open. The reason for the difficulty in resolving this question is that the uniqueness and global regularity problem is not solved yet. Because no linear space X of initial functions $u_0 = u_0(x)$, $x \in \Omega \subset R^3$, generating unique globally X–regular solutions $u = u(x,t)$, $t \in [0,\infty)$, is known to exist even for small forces, it is not known whether there exists a linear space X of initial functions for which 3D NSE define a dynamical system on X. As a result, methods and concepts of the dynamical systems theory cannot be directly applied to 3D NSE in an effective way. In particular, the notion of a global attractor, used for description of long–time behaviour of solutions of evolutionary equations generating dynamical systems, is not applicable to viscous incompressible fluids and all established results about global attractors are conditional in the sense that they assume *a priori*, without justification, that 3D NSE do generate a dynamical system on a hypothetical linear space X of initial functions (usually, the space V is taken for the hypothetical space X); see Sections 9.1 and 9.3 of [3], also [4], [14].

A purpose of the present paper is to address the above question and to establish unconditional results about global attractors for 3D NSE by bypassing the problem of uniqueness and global regularity. In particular, we show that, for a large set of "good" forces, the problem of uniqueness and global regularity is irrelevant for the long–time description of viscous incompressible fluids. In fact, we establish that, for any fixed force $f \in H$ belonging to some large unbounded open star–shaped set, 3D NSE possess the compact global attractor $\mathcal{A} \subset H$ of finite Hausdorff dimension regardless of whether or not solutions of 3D NSE develop singularities and loose uniqueness. However, since the uniqueness and global regularity problem is not solved yet, we cannot talk about a dynamical system generated by 3D NSE on $X = H$; hence, in this situation, we cannot talk about global attractors in the framework of the dynamical systems theory. Therefore, the first issue we need to resolve is to find an appropriate substitution for the notion of a dynamical system so that to incorporate 3D NSE into the new notion. Second, we need to ensure that the concept of a (global) attractor makes sense for the new notion. Only after having properly generalized the basic concepts of the dynamical systems theory and developed the theory of the new notions, are we eligible to apply them to 3D NSE. The criterion according to which we have to choose the substitution for the concept of a dynamical system is that the new notion should satisfy two opposite requirements:

1) it must be general enough; in particular, it must incorporate 3D NSE with initial functions $u_0 = u_0(x)$ forming a linear space (not just a subset of a linear space);

2) it must not be too general; in particular, it should not loose essential properties of 3D NSE and the theory based on the new notion must be rich enough to allow existence of the compact global attractor of finite Hausdorff dimension for 3D NSE with initial functions $u_0 = u_0(x)$ forming a linear space.

Because of these two opposite requirements, many candidates for the desirable generalization of the notion of a dynamical system do not work. For example, the notion of a multi–valued semigroup[2] (set–valued, or multi–valued, mapping) [31]–[33] certainly could satisfy the requirement 1), but it does not satisfy 2) because it looses some essential properties of 3D NSE (see Remark 3.1 below).

We propose the notion of an *energetic system*, defined in Section 2, as a suitable candidate for the generalization. This notion is capable of both incorporating 3D NSE and adequate modelling of properties of 3D NSE. The idea behind the notion of an energetic system is to define and study such a set–valued mapping σ on a metric space X that, while σ is not a dynamical system, its restriction on any bounded set $K \subset X$ "becomes"a dynamical system in a finite time t_K depending only on K. It is this property of becoming a dynamical system in finite time on any bounded set that is lost by the theory of multi–valued semigroups when applied to 3D NSE. We develop the theory of energetic systems in Section 2. Then, in Section 3, we apply this theory to 3D NSE. In particular, we show that, for a large set of "good"forces $f \in H$, 3D NSE do generate an energetic system on the space H. This allows us to ask whether 3D NSE possess the global attractor on H. Theorem 3.2 below answers this question by stating that, for "good"forces $f \in H$, the energetic system generated by 3D NSE possesses the compact global attractor $\mathcal{A} \subset H$ of finite Hausdorff dimension. We emphasize that, for any value of viscosity $\nu > 0$ and for any smooth bounded domain $\Omega \subset R^3$, the set of "good"forces is open and star–shaped in H and it includes forces of arbitrarily large magnitudes $||f||$. (Here and throughout the paper, $||\cdot||$ denotes the norm in H.) In Section 4, we consider the problem of finding a maximal dynamical subsystem (see definition below) of the energetic system generated by 3D NSE. In the case of periodic boundary conditions on a *thin* domain $\Omega_\epsilon = (0, l_1) \times (0, l_2) \times (0, \epsilon) \subset R^3$, $0 < \epsilon \leq l_2 \leq l_1 < \infty$, we strengthen a result of Section 3, which describes an unbounded open star–shaped dynamical subsystem of the energetic system generated by 3D NSE on a general bounded smooth domain $\Omega \subset R^3$ under the assumption that $f \in H$ is small. Namely, a corollary from Theorem 4.1 below states that if the domain $\Omega_\epsilon \subset R^3$ is thin then even for large forces $f \in H$ belonging to some open convex neighborhood of zero in H, there exists a dynamical subsystem S_ϵ defined on a set

[2] Recall (see, e.g., [31], [32]) that a *multi–valued semigroup* on a Banach space E is a family $\{S_t\}_{t\geq 0}$ of multi–valued operators $S_t : E \to E$ such that $S_{t+s}x \subset S_t S_s x$, or even $S_{t+s}x = S_t S_s x$, for all $x \in E$ and all t, $s \in [0, \infty)$. Here $S_t K = \cup_{x \in K} S_t x$, for all $K \subset E$.

$M_\epsilon \subset H$ containing an unbounded open star–shaped neighborhood $\mathcal{U}_\epsilon \subset V$ of the origin. Theorem 4.2 yields an explicit construction of $\mathcal{U}_\epsilon$. Then, in the case of a bounded smooth domain $\Omega \subset R^3$, we establish a necessary condition for a weak Leray–Hopf solution to be a globally regular one. Because of the space limitation, we only state main results; detailed proofs will appear elsewhere.

2 ELEMENTS OF THE ENERGETIC SYSTEMS THEORY

In this section, we introduce basic definitions and outline elements of the theory of energetic systems.

DEFINITION 2.1. We will say that a set–valued mapping $\sigma : [0,\infty) \times X \to 2^X$, $(t,x) \mapsto S^*(t)x \in 2^X$, on a complete metric space X is an *energetic system S^* on X* if the following properties are satisfied:

1) $S^*(0)x = x$, for all $x \in X$;

2) for any bounded set $K \subset X$, there exists a time $t_K \in [0,\infty)$ such that, for all $x \in S^*(t_K)K$ and all s, $t \in [0,\infty)$, one has

$$S^*(t)x \in X \quad \text{and } S^*(s)S^*(t)x = S^*(s+t)x \quad \text{and } S^*(t+t_K)K = S^*(t)S^*(t_K)K$$

and the restricted mapping $\sigma : (0,\infty) \times S^*(t_K)K \to X$ is continuous. Here and below, 2^X is the set of non–empty subsets of X and $S^*(t)M := \cup_{x\in M} S^*(t)x$, for any $M \subset X$, $t \in [0,\infty)$.

The time t_K will be called the *transition time for the set K*. Clearly, if t_K is a transition time for K then any $t > t_K$ is so. From now on, we will always assume that S^* denotes an energetic system with a fixed choice of t_K, for any bounded set $K \subset X$.

If $N \subset X$ then we will denote the restricted mapping $\sigma : [0,\infty) \times N \mapsto 2^X$ by S^*_N and call it the *restriction of S^* onto N*. It can happen that, for some set $M \subset X$, the restriction $S = S^*_M$ of S^* onto M is a dynamical system; in this case we call this restriction the *dynamical subsystem of S^* defined on the set M* and we write $S < S^*$.

Clearly, every dynamical system is an energetic system with $t_K = 0$, for any bounded set $K \subset X$.

DEFINITION 2.2. A set $L \subset X$ is a *gate* for S^* if any bounded set $K \subset X$ *passes through* L, i.e. there exists a time $t^* = t^*(K) \geq t_K$ such that $S^*(t^*)K \subset L$.

Note that, in contrast to the dynamical systems theory concept of an absorbing set (which has a natural generalization to the case of energetic systems), Definition 2.2 does not require that $S^*(t)K \subset L$ for $t > t^*(K)$.

A useful property of the notion of an energetic system is that many fundamental concepts of the dynamical systems theory have straightforward generalizations in the framework of the energetic systems theory. For example, we have the following generalizations of the notions of an attractor and global attractor.

DEFINITION 2.3. A non-empty set $\mathcal{A} \subset X$ is an *attractor* of S^* if:

1) $\mathcal{A}$ is an invariant set, i.e. $S^*(t)\mathcal{A} = \mathcal{A}, \quad t \geq 0$, and

2) there exists an open neighborhood $\mathcal{N} \subset X$ of $\mathcal{A}$ such that, for every $x \in \mathcal{N}$, one has

$$dist(S^*(t)x, \mathcal{A}) \stackrel{\text{def}}{=} \sup_{y \in S^*(t)x} \inf_{z \in \mathcal{A}} d_X(y,z) \to 0 \qquad \text{as} \qquad t \to \infty,$$

where d_X is the distance function in X. The largest such neighborhood $\mathcal{N}$ is called the *basin* of attraction of $\mathcal{A}$ and is denoted by $\mathcal{B}(\mathcal{A})$.

DEFINITION 2.4. An attractor $\mathcal{A} \subset X$ of S^* is *global* if $\mathcal{A}$ uniformly attracts all bounded sets of X, i.e. for any bounded set $K \subset X$ one has $dist(S^*(t)K, \mathcal{A}) \to 0$ as $t \to \infty$.

EXAMPLE 2.1. Let $X = R$. Define a set–valued mapping S^* as follows.

$$S^*(t)x_0 := \begin{cases} e^{-2t}x_0, & \text{if } x_0 \in [-1,1], \quad t \in [0,\infty), \\ e^{-2t}x_0, & \text{if } x_0 \in (1,\infty), \quad t \in [0,1), \\ e^{-2(t-1)}\{\chi_1,\chi_2\}, & \text{if } x_0 \in (1,\infty), \quad t \in [1,\infty), \\ x_0 + t, & \text{if } x_0 \in (-\infty,-1), \quad t \in [0,-1-x_0], \\ -e^{-2(t+1+x_0)}, & \text{if } x_0 \in (-\infty,-1), \quad t \in [-1-x_0,\infty), \end{cases}$$

where $\chi_1 = e^{-2}x_0 + \frac{1}{2e^2}$, $\chi_2 = e^{-2}x_0 - \frac{1}{2e^2}$. Then S^* is an energetic system on $X = R$. Note that S^* is not a dynamical system on R. For any $a \in (0,1)$, the restriction $S^*_{(-a,a)}$ is a dynamical subsystem of S^*. For any bounded set $K \subset (1,\infty)$, the minimal transition time is $t_K = 1 + \frac{1}{2}\ln(e^{-2}x^* + \frac{1}{2e^2})$, where $x^* = \sup_{x \in K} x$. For any bounded set $K \subset (-\infty,-1)$, any time $t > -1 - x_*$ is a transition time for K, where $x_* = \inf_{x \in K} x$. Despite the fact that S^* is not a dynamical system on R, the global attractor for S^* exists. In fact, the set $\{0\}$ is the global attractor.

It is worthwhile to notice that, in general, if S is a dynamical system possessing a (global) attractor $\mathcal{A}$ and, for some energetic system S^*, one has $S < S^*$ then $\mathcal{A}$ may not be an attractor of S^* (see Example 2.2 below, which shows that this can occur even if S^* is a dynamical system). On the other hand, if an energetic system S^* has the global attractor $\mathcal{A}$ and S^* happens to be a dynamical system then $\mathcal{A}$ is the global attractor of S^* considered as a dynamical system.

EXAMPLE 2.2. Let $X = R^2$. Let $S^*(t)x^0 = x(t;x_0)$, $t \in [0,\infty)$, $x^0 \in R^2$, where $x(\cdot;x^0) = (x_1(\cdot), x_2(\cdot))$ is the solution of the system of ordinary differential equations

$$\dot{x}_1 = -x_1 \tag{2.1}$$

$$\dot{x_2} = x_2 \tag{2.2}$$

with initial condition $x(0) = x^0$. Then S^* is an energetic system on R^2 which is a dynamical system. The restriction $S = S^*_M$, where $M = \{x \in R^2 : x_2 = 0\}$, is a dynamical subsystem of S^* possessing the global attractor $\mathcal{A} = \{0\}$. However, $\mathcal{A}$ is not an attractor of S^*.

The following general result gives a sufficient condition for an energetic system to possess the global attractor. In the next section, it will be used in a specific situation.

THEOREM 2.1. Let S^* be an energetic system on X. Assume there exist a gate $L \subset X$ and a dynamical subsystem $S < S^*$ defined on a set $N \subset X$ such that $L \subset N$. If the dynamical subsystem $S < S^*$ has an attractor $\mathcal{A} \subset N$ uniformly attracting L then $\mathcal{A}$ is the global attractor of the energetic system S^*.

We note that the set–theoretic relation of inclusion induces a partial order in the set of dynamical subsystems of a given energetic system. An interesting intrinsic problem of the theory of energetic systems is the following one.

PROBLEM. Given an energetic system S^* on X, find its *maximal* dynamical subsystem $S < S^*$, that is a set $N \subset X$ such that S^*_N is a dynamical system, but no other set containing N possesses this property.

EXAMPLE 2.3. In the example 2.1 above, $S^*_{[-1,1]}$ is the unique maximal dynamical subsystem of S^*.

3 APPLICATIONS OF THE ENERGETIC SYSTEMS THEORY: GLOBAL ATTRACTOR OF FINITE HAUSDORFF DIMENSION FOR 3D NSE

In this section, we apply the theory of energetic systems to 3D NSE and we establish the existence of the compact global attractor $\mathcal{A} \subset H$. We remark that the existence of a (local) attractor $\mathcal{A} \subset V$ with the *small bounded* basin $\mathcal{B}(\mathcal{A}) \subset V$ follows from classical works of Leray [6]–[8] and Hopf [9]. By restricting the class of domains $\Omega \subset R^3$ to the special case of *thin* domains $\Omega_\epsilon = (0, l_1) \times (0, l_2) \times (0, \epsilon) \subset R^3$, $0 < \epsilon \le l_2 \le l_1 < \infty$, this result was improved by Raugel and Sell in [22]–[24]; however, their method cannot be generalized to non–thin domains $\Omega \subset R^3$ because it is based on a perturbation argument with respect to the small parameter $\epsilon > 0$. Without the thinness assumption, the existence of a compact attractor $\mathcal{A} \subset V$ with the *large unbounded* basin $\mathcal{B}(\mathcal{A}) \subset V$ was established in [29]–[30]. Because the uniqueness and global regularity problem is not solved in the space V (as well as in any linear space of initial functions u_0!), there is no linear space X of initial functions for which 3D NSE are known to generate a dynamical system on X. Therefore, there is no linear space X of initial functions for which we are eligible to talk about global attractors of 3D NSE in the framework of the dynamical systems theory (note that the spaces W_{LH} and W constructed by Sell in [13] cannot be taken as linear spaces of initial functions $u_0 = u_0(x)$ because both W and W_{LH} contain functions depending on time t and, moreover, no linear space of functions $u = u_0(x)$ independent of time is contained in W or W_{LH}). We emphasize that for the applications (see, e.g., [34], [35]) it is important to know if there exists a linear space X of initial functions $u_0 = u_0(x)$ such that the long–time behaviour of solutions of 3D NSE can be described by a compact set in X of finite Hausdorff dimension. Because of the above difficulties of the dynamical systems theory, neither such space X nor the desired compact set in X are known to exist even in the case of small forces f. Theorem 3.2 below gives the answer to this question in the case of arbitrarily large forces f belonging to some special unbounded open star–shaped set $\mathcal{F}$ sitting in the space $Y = H$. Before we formulate Theorem 3.2, we state Theorem 3.1 which makes us eligible to talk about global attractors for 3D NSE in the framework of the energetic systems theory. More precisely, Theorem 3.1 below describes an energetic system on H generated by 3D NSE.

We need some notation. Given a metric space Y, we let $B_Y(0, a)$ denote the open ball $\{y \in Y : d_Y(0, y) < a\}$ of radius $a > 0$. Given a self-adjoint operator $\mathbf{A} = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$ in a

Hilbert space and a number $r > 0$, we let $L_r(\mathbf{A})$ denote the projection defined by

$$L_r(\mathbf{A}) \stackrel{\text{def}}{=} \int_{\delta_r} dE_\lambda, \qquad \delta_r \stackrel{\text{def}}{=} [-r, r].$$

If $\mathbf{A}$ is the Stokes operator $A = -P\Delta$, where $P : L^2(\Omega) \to H$ is the orthogonal projection, then we write L_r for $L_r(A)$ and we denote the first eigenvalue of A by λ_1. Recall [4]–[5] that, in the case of the Dirichlet non–slip boundary conditions (1.3) on a smooth bounded domain $\Omega \subset R^3$, one has $H = Cl_{L^2(\Omega)}\{u \in C_0^\infty(\Omega) : \nabla \cdot u = 0\}$, $V = Cl_{H^1(\Omega)}\{u \in C_0^\infty(\Omega) : \nabla \cdot u = 0\}$, $V^2 = Cl_{H^2(\Omega)}\{u \in C_0^\infty(\Omega) : \nabla \cdot u = 0\}$, and 3D NSE (1.1)–(1.4) can be rewritten in the equivalent form of the evolutionary equation

$$\begin{aligned} u' + \nu A u + B(u,u) &= f, && (3.1)\\ u(0) &= u_0 && (3.2) \end{aligned}$$

on the space H, where $B(u,v) \stackrel{\text{def}}{=} P(u \cdot \nabla)v$.

Given $f \in H$, we let $S^*(t)u_0 \subset 2^H$ denote the set of values of all weak Leray–Hopf solutions at time $t \geq 0$, which are generated by the initial function $u_0 \in H$ (these values are well–defined since weak Leray–Hopf solutions belong to the space $C_w(0,\infty; H)$; *cf.* remark after Definition 8.5 on p. 71 of [4] or remark after the statement of Problem 2.1 in Section 2 of [5]). Let p and s be arbitrary positive numbers such that $p < 3/16$ and $s < 3/4$.

THEOREM 3.1. For any 3-dimensional bounded smooth domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, let $\alpha > 0$, $\beta > 0$, and $r_0(\nu, \Omega, \alpha, \beta, p, s) > 0$ be as in Theorem 1 of [29] and satisfy $\nu^{-2}(\lambda_1^{-1} + 1)\beta < \alpha/2$. Let $r \geq r_0(\nu, \Omega, \alpha, \beta, p, s)$ be so large that $r^{s-1} < \beta$. Let $f \in \mathcal{F}_r$, where

$$\mathcal{F}_r \stackrel{\text{def}}{=} \{f \in H : ||L_r f||^2 < \beta \text{ and } ||(I - L_r)f||^2 < r^s\}.$$

Then S^* is an energetic system on H possessing a dynamical subsystem $S < S^*$ defined on a set $M_r \subset H$ containing an open convex set $\mathcal{U}_r \subset V$ of radius $r^{p/2}$. If $||f||^2 < \beta$ then S is defined on a set $M \subset H$ containing the large, unbounded open star–shaped, set $\mathcal{U} \subset V$ given by $\mathcal{U} = \cup_{r \geq r_0} \mathcal{U}_r$. In either case, the ball $B_V(0, \sqrt{\alpha}) \subset \mathcal{U}_r \subset \mathcal{U}$ is a gate for S^*, and any bounded set $K \subset H$ passes through $B_V(0, \sqrt{\alpha})$ no later than the moment $t = 2(\nu\alpha)^{-1}a^2$, where $a > 0$ is such that $K \subset B_H(0, a)$; in particular, $t_K \leq 2(\nu\alpha)^{-1} \sup\{||u_0||^2 : u_0 \in K\}$. Moreover, for any $u_0 \in H$, there exists $t_{u_0} < 2(\nu\alpha)^{-1}||u_0||^2$ such that $S^*(t_{u_0})u_0 \subset B_V(0, \sqrt{\alpha})$. Furthermore, the dynamical subsystem $S^*_{M \cap V} < S^*$ is generated by strong solutions of 3D NSE, that is, for any $u_0 \in M \cap V$, $S^*_{M \cap V}(t)u_0 \in V$ is the value of the unique globally regular Leray–Hopf solution at time $t \geq 0$, generated by u_0.

COROLLARY. For any force f belonging to the unbounded open star–shaped set $\mathcal{F} \subset H$ given by

$$\mathcal{F} \stackrel{\text{def}}{=} \bigcup_{r \geq r_0} \mathcal{F}_r,$$

3D NSE generate an energetic system on H.

REMARK 3.1. The fact that 3D NSE generate an energetic system on the linear space H of initial functions $u_0 = u_0(x)$ refers, in particular, to the property 2) of Definition 2.1.

Note that this property would be lost by saying that 3D NSE generate a multi–valued semigroup (set–valued mapping).

Proof of Theorem 3.1 follows from the properties of the set $Q = Q_{(\alpha,\beta,p,s)}$ constructed in [29]. Here, we will show only that the ball $B_V(0, \sqrt{\alpha})$ is a gate for S^*. Indeed, consider the energy inequality

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 + \nu||A^{\frac{1}{2}}u(t)||^2 \leq |\langle f, u(t)\rangle|,$$

satisfied by all weak Leray–Hopf solutions (see, e.g., Remark 3.3 in [5]). An application of the Young inequality to the term $|\langle f, u(t)\rangle| = |\langle A^{-\frac{1}{2}}f, A^{\frac{1}{2}}u(t)\rangle|$ and subsequent integration of the obtained inequality yield

$$\nu\int_0^t ||A^{\frac{1}{2}}u||^2 ds \leq ||u_0||^2 + \frac{1}{\nu}||A^{-\frac{1}{2}}f||^2 t.$$

By applying the projection L_r and using the Poincaré inequality $||A^{-\frac{1}{2}}L_r f||^2 \leq \lambda_1^{-1}||L_r f||^2$ and the inequality $||A^{-\frac{1}{2}}(I - L_r)f||^2 \leq r^{-1}||(I - L_r)f||^2$, we obtain that

$$\nu\int_0^t ||A^{\frac{1}{2}}u||^2 ds \leq ||u_0||^2 + \frac{1}{\nu}\left(\lambda_1^{-1}||L_r f||^2 + r^{-1}||(I - L_r)f||^2\right)t,$$

which yields, according to our choice of α, β, r, and f, the following estimate

$$\nu\int_0^t ||A^{\frac{1}{2}}u||^2 ds < ||u_0||^2 + \frac{1}{\nu}\left(\lambda_1^{-1}\beta + r^{-1}r^s\right)t < ||u_0||^2 + \frac{1}{2}\nu\alpha t.$$

Therefore, for $T = \frac{2||u_0||^2}{\nu\alpha}$, we obtain

$$\frac{1}{T}\int_0^T ||A^{\frac{1}{2}}u||^2 ds < \alpha,$$

which implies the existence of $t_0 \in [0, T)$ such that $||A^{\frac{1}{2}}u(t_0)||^2 = ||u(t_0)||_V^2 < \alpha$. QED.

Now, having established that 3D NSE generate an energetic system on H, we are eligible to raise the issue about existence of the global attractor in H for 3D NSE. By using Theorem 2.1, we obtain the following result.

THEOREM 3.2. For any bounded smooth domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, let α, β satisfy assumptions of Theorem 3.1. Let $f \in \mathcal{F}$. Then the energetic system S^* constructed in Theorem 3.1 possesses the compact global attractor $\mathcal{A} \subset H$. Moreover, $\mathcal{A}$ is compact in V, bounded in $V^2 = D(A)$, and has finite Hausdorff dimension both in H and in V.

4 LOOKING FOR A MAXIMAL DYNAMICAL SUBSYSTEM

In this section, we address the problem of finding a maximal dynamical subsystem of the energetic system S^* generated by 3D NSE. From the previous section, we already know that, in the case of a small force $f \in H$, the energetic system generated by 3D NSE supplemented with the Dirichlet boundary conditions on an arbitrary smooth bounded domain $\Omega \subset R^3$ possesses a dynamical subsystem $S = S_M^*$ defined on a set $M \subset H$

containing the large unbounded open star–shaped neighborhood $\mathcal{U} \subset V$ of the origin. Here, in subsection 4.1, we show that if the 3–dimensional domain $\Omega = \Omega_\epsilon$ is *thin* (i.e. the height of the domain is a small number $\epsilon > 0$) and boundary conditions are periodic, then even for large forces $f \in H$ forming an open convex neighborhood of the zero in H , S^* possesses the dynamical subsystem defined on a set $M_\epsilon \subset H$ containing a large unbounded open star–shaped neighborhood $\mathcal{U}_\epsilon \subset V$ of the origin. This result follows from Theorem 4.1 below, while Theorem 4.2 provides information allowing one to construct the neighborhood $\mathcal{U}_\epsilon$ explicitly.

Since the problem of finding a maximal dynamical subsystem of the energetic system generated by 3D NSE is closely related to the global regularity problem for 3D NSE, we are interested in necessary conditions for a weak Leray–Hopf solution to be globally regular. One such condition is given in subsection 4.2 (Theorem 4.3).

4.1 3D NSE on thin domains

Given a set $K \subset V \times L^\infty(0,\infty;H)$, we will say that K *generates* globally regular Leray–Hopf solutions if, for any $(u_0, f) \in K$, 3D NSE (3.1)–(3.2) possess a (unique) globally regular Leray–Hopf solution.

In the case of a domain $\Omega_\epsilon = (0,l_1) \times (0,l_2) \times (0,\epsilon) \subset R^3$, $0 < \epsilon \le l_2 \le l_1 < \infty$, with a sufficiently small height $\epsilon > 0$, it was shown [22]–[24] by using a perturbation argument with respect to the parameter $\epsilon > 0$ that, for 3D NSE with periodic boundary conditions on Ω_ϵ, there exists a *bounded* set $\mathcal{R}(\epsilon) \times \mathcal{S}(\epsilon) \subset V \times L^\infty(0,\infty;H)$ generating globally regular Leray–Hopf solutions and such that the radius of $\mathcal{R}(\epsilon) \times \mathcal{S}(\epsilon)$ in $V \times L^\infty(0,\infty;H)$ is proportional to ϵ^{-s}, where $0 < s < \frac{1}{2}$ is some number. By combining the perturbation method of [22]–[24] with the method [29] of finding large sets of data generating global solutions to nonlinear evolutionary equations, one can show that, under the same hypotheses that are used in [22]–[24], there exists an *unbounded* open star–shaped set generating globally regular Leray–Hopf solutions and containing $\mathcal{R}(\epsilon) \times \mathcal{S}(\epsilon)$. In particular, the following theorem holds.

THEOREM 4.1. For any viscosity $\nu > 0$ and any sufficiently thin domain $\Omega_\epsilon = (0,l_1) \times (0,l_2) \times (0,\epsilon) \subset R^3$, $0 < \epsilon \le l_2 \le l_1 < \infty$, there exists a large unbounded star-shaped set $Q(\epsilon) \subset V \times L^\infty(0,\infty;H)$ generating unique globally regular Leray–Hopf solutions to 3D NSE supplemented with periodic boundary conditions on Ω_ϵ. Moreover, $\mathcal{R}(\epsilon) \times \mathcal{S}(\epsilon) \subset Q(\epsilon)$ and

$$Q(\epsilon) = \bigcup_{p_2<0} Q_{p_2}(\epsilon),$$

where, for each $p_2 < 0$, $Q_{p_2} \subset V \times L^\infty(0,\infty;H)$ is a convex set equal to the closure of its own interior. Furthermore, for any $f \in \mathcal{S}(\epsilon)$, the f–section $\mathcal{U}(\epsilon) = \{u_0 : (u_0, f) \in Q(\epsilon)\}$ of $Q(\epsilon)$ is a large unbounded star–shaped set in V which is independent of f, contains $\mathcal{R}(\epsilon)$, and is represented by

$$\mathcal{U}(\epsilon) = \bigcup_{p_2<0} \mathcal{U}_{p_2}(\epsilon),$$

where, for each $p_2 < 0$, $\mathcal{U}_{p_2}(\epsilon) \subset V$ is a convex neighborhood of zero, coinciding with the closure of its own interior.

COROLLARY. Let $\nu > 0$ and $\Omega_\epsilon = (0,l_1) \times (0,l_2) \times (0,\epsilon) \subset R^3$ be sufficiently thin. For any $f \in \mathcal{S}(\epsilon) \cap H$, the energetic system S^* generated by 3D NSE on Ω_ϵ possesses a dynamical subsystem $S^*_{M_\epsilon}$ defined on a set $M_\epsilon \subset H$ containing a large unbounded open star–shaped neighborhood $\mathcal{U}_\epsilon \subset V$ of the origin. The dynamical subsystem $S^*_{M_\epsilon \cap V}$ is generated by strong solutions of 3D NSE.

REMARK. The set $\mathcal{S}(\epsilon) \cap H$ is a convex bounded neighborhood of the origin in H coinciding with the closure of its own interior and the radius of the set $\mathcal{S}(\epsilon) \cap H$ in H increases unboundedly as $\epsilon \to 0$, *cf.* [22]–[24]. In particular, for small $\epsilon > 0$, $\mathcal{S}(\epsilon) \cap H$ contains forces f of large magnitudes $||f||$.

The construction of the sets $Q_{p_2}(\epsilon)$ and $\mathcal{U}_{p_2}(\epsilon)$, $p_2 < 0$, is given in Theorem 4.2 below. To formulate this theorem, we need more notation.

Let $Q_3 = (0, l_1) \times (0, l_2) \times (0, 1) \subset R^3$. The dilation $(x_1, x_2, x_3) \to (x_1, x_2, \epsilon x_3)$, $(x_1, x_2, x_3) \in Q_3$ induces the bijective linear map J_ϵ between the set of measurable functions on Q_3 and the set of measurable functions on Ω_ϵ, which is given by $J_\epsilon u(y_1, y_2, y_3) = u(y_1, y_2, \epsilon^{-1} y_3)$, $(y_1, y_2, y_3) \in \Omega_\epsilon$ [22]–[24]. Let P_ϵ be the orthogonal projection which maps $L^2(Q_3)$ onto the L^2–closure H_ϵ of smooth periodic functions u on Q_3 satisfying the conditions $\int_{Q_3} u\, dx = 0$ and $\nabla_\epsilon u \overset{\text{def}}{=} \partial_1 u_1 + \partial_2 u_2 + \epsilon^{-1} \partial_3 u_3 = 0$. Then, by composing P_ϵ with J_ϵ^{-1} and applying the composition $P_\epsilon J_\epsilon^{-1}$ to 3D NSE, one obtains the *dilated evolutionary NSE* on the space H_ϵ [22]–[24]

$$u_t + \nu A_\epsilon u + B_\epsilon(u, u) = P_\epsilon f, \tag{4.1.1}$$

$$u(0) = u_0, \tag{4.1.2}$$

where $A_\epsilon = -P_\epsilon \Delta_\epsilon = -P_\epsilon(\partial_1^2 + \partial_2^2 + \epsilon^{-2}\partial_3^2)$(with periodic boundary conditions) and $B_\epsilon(u, v) = P_\epsilon(u \cdot \nabla_\epsilon) v$. Recall also that [23]–[24] make use of bounded monotone functions $\eta_i(\epsilon)$ defined for $0 < \epsilon \le 1$, $i = 1, 2, 3, 4$, and negative constants p, q_1, q_2, r satisfying the following Hypothesis H*:

1) $r > -2$, $p > -\frac{29}{24}$, $q_1 > -\frac{5}{12}$, $q_2 > -\frac{5}{12}$,

2) there exist positive constants α_i, β_i such that, for small $\epsilon > 0$,

$$\beta_i \le \eta_i^{-1}(\epsilon) \le (-\ln \epsilon)^{\alpha_i}, \qquad i = 1, 2, 3, 4.$$

Now, if the Fourier series expansion of a function $u \in L^2(Q_3)$ is

$$u(x) = \sum_{k \in Z^3} c^k e^{2\pi i k a \cdot x},$$

where $a = (a_1, a_2, a_3) = (l_1^{-1}, l_2^{-1}, 1)$ and $ka = (k_1 a_1, k_2 a_2, k_3 a_3)$, $k = (k_1, k_2, k_3) \in Z^3$, then we set, for any integer $K \ge 0$,

$$M_K u(x) = \sum_{k \in Z^3, |k_3| \le K} c^k e^{2\pi i k a \cdot x}.$$

It can be shown that, for any integer $K \ge 0$, M_K is an orthogonal projection on $L^2(Q_3)$ commuting with ∇_ϵ and A_ϵ. We note that the projection M_K generalizes the projection M used in [22]–[24]; in fact, $M_0 = M$. Define $||f||_\infty := \text{ess sup}_{0<t<\infty} ||f(t)||$.

Let $\xi_2 = \xi_2(\epsilon)$ be a bounded monotone function defined for $0 < \epsilon \le 1$ such that there exist positive constants α, β satisfying, for small $\epsilon > 0$,

$$\beta \le \xi_2^{-1}(\epsilon) \le (-\ln \epsilon)^{\alpha}.$$

Then we have the following theorem.

THEOREM 4.2. Let η_i, $i = 1, 2, 3, 4$, p, q_1, q_2, and r satisfy the Hypothesis H* and let ξ_2 be as above. Then there exists a number $\epsilon_* > 0$ such that, for any $\epsilon \in (0, \epsilon_*)$ and any $p_2 < 0$, there exists a number $K_0 = K_0(\epsilon, p_2) > 0$ with the property that (4.1.1)–(4.1.2) have a unique globally regular (strong) solution

$$u \in C([0,\infty); D(A_\epsilon^{\frac{1}{2}})) \cap L^\infty(0,\infty; D(A_\epsilon^{\frac{1}{2}})) \cap L^2_{loc}(0,\infty; D(A_\epsilon))$$

whenever $K_0(\epsilon, p_2)$ satisfy

$$||A_\epsilon^{\frac{1}{2}} v_0||^2 \leq \epsilon^{q_1}\eta_1^{-2} \qquad ||MP_\epsilon f||^2_\infty \leq \epsilon^{q_2}\eta_2^{-2}, \tag{4.1.3}$$

$$||A_\epsilon^{\frac{1}{2}} w_{1,0}||^2 \leq \epsilon^{p}\eta_3^{-2} \qquad ||(M_K - M)P_\epsilon f||^2_\infty \leq \epsilon^{r}\eta_4^{-2}, \tag{4.1.4}$$

$$||A_\epsilon^{\frac{1}{2}} w_{2,0}||^2 \leq \epsilon^{p_2}\xi_2^{-2} \qquad ||(I - M_K)P_\epsilon f||^2_\infty \leq \epsilon^{r}\eta_4^{-2}, \tag{4.1.5}$$

where $v_0 = Mu_0,\ w_{1,0} = (M_K - M)u_0,\ w_{2,0} = (I - M_K)u_0$.

This theorem yields an explicit construction of the desired sets $Q_{p_2}(\epsilon)$ and $\mathcal{U}_{p_2}(\epsilon)$, for any $p_2 < 0$ and any $\epsilon \in (0, \epsilon^*)$. Indeed, $\mathcal{U}_{p_2}(\epsilon)$ is given by the image (under the map J_ϵ) of the set of initial functions described by the left inequalities in (4.1.3)–(4.1.5), while $Q_{p_2}(\epsilon)$ is the Cartesian product of $\mathcal{U}_{p_2}(\epsilon)$ with the image (under the map J_ϵ) of the set of forces described by the right inequalities in (4.1.3)–(4.1.5).

4.2 A necessary condition for global regularity of a weak Leray–Hopf solution

Necessary conditions for global regularity of weak solutions yield upper bound estimates for maximal dynamical subsystems of energetic systems generated by 3D NSE. The following theorem gives a necessary condition in terms of some *uniform* inequalities.

THEOREM 4.3. Let $\nu > 0$ be a positive number and $\Omega \subset R^3$ a bounded smooth domain. If, for some $\tau \geq 0$, a weak Leray–Hopf solution $u = u(x,t)$ of 3D NSE is regular on the interval $[\tau, \infty)$, (i.e. $u \in C([\tau,\infty); V) \cap L^\infty(\tau,\infty; V) \cap L^2_{loc}(\tau,\infty; V^2)$) then, for any $\gamma > 0$, there exists $r_* > 0$ such that

$$||(I - L_r)A^{\frac{1}{2}}u(t)||^2 \leq \gamma, \qquad t \in [\tau, \infty), \qquad r \geq r_*.$$

Moreover, for any $r \geq r_*$, there exists time $T_1 = T_1(r) \geq \tau$ such that

$$||(I - L_r)A^{\frac{1}{2}}u(t)||^2 \leq Dr^{-\frac{1}{2}}, \qquad t \in [T_1, \infty), \quad r \geq r_*,$$

where $D > 0$ is a constant independent of t and r.

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