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Abstract

We consider undirected graphs that arise as deterministic functions of stationary point processes such that each point has degree bounded by two. For a large class of point processes and edge-drawing rules, we show that the arising graph has no infinite connected component, almost surely. In particular, this extends our previous result for SINR graphs based on stabilizing Cox point processes and verifies the conjecture of Balister and Bollobás that the bidirectional k-nearest neighbor graph of a two-dimensional homogeneous Poisson point process does not percolate for $k = 2$.

1 Introduction

Continuum percolation was introduced by Gilbert [G61] in order to model connectivity in large telecommunication networks. In his graph model, the vertices form a homogeneous Poisson point process (PPP) in $\mathbb{R}^2$, and two points are connected whenever their distance is less than a fixed connection radius $r > 0$. He showed that this model undergoes a phase transition: if the spatial intensity $\lambda > 0$ of the PPP is sufficiently small, then the graph consists of finite components only, almost surely, whereas for large enough λ, the graph *percolates*, i.e., it has an unbounded connected component, also almost surely.

This model has been widely extended, for instance to the case of random connection radii and for various point processes, see [MR96, BY13, CD14, GKP16, HJC19, J16, S13, JTC20]. A drawback of Gilbert's model is that it allows for an arbitrarily large degree of the vertices, whereas for many applications, it is a reasonable assumption that the vertices should have bounded degree. Incorporating this property, Häggström and Meester [HM96] studied percolation in the so-called *undirected k-nearest neighbor* ($\mathbf{U}$-kNN) *graph*, based on a stationary PPP in $\mathbb{R}^d$, $d \geq 1$, see top line of Figure 1. Here, all points of the point process are connected to their k-nearest neighbors, for some fixed $k \in \mathbb{N}$. This results in a graph that is the undirected variant of a directed graph with out-degrees bounded by k, which itself also has degrees larger than k. Let us write $k_{\mathbf{U},d}$ for the minimum of all $k \in \mathbb{N}$ such that the $\mathbf{U}$-kNN-graph of the stationary PPP in $\mathbb{R}^d$ percolates with positive probability. It was shown in [HM96] that $k_{\mathbf{U},d} > 1$ for all $d \in \mathbb{N}$, however, $k_{\mathbf{U},d} = 2$ for all sufficiently large d. This was complemented in [TY07] by the assertion that $k_{\mathbf{U},d} < \infty$ for all $d \geq 2$.

Balister and Bollobás [BB13] studied the case $d = 2$. They also introduced another undirected graph, which is contained in the $\mathbf{U}$-kNN graph, called the *bidirectional k-nearest neighbor* ($\mathbf{B}$-kNN) *graph*, see bottom line of Figure 1. Here, one connects two points of the point process if and only if they are mutually among the k-nearest neighbors of each other. This graph has in fact degrees bounded by k, which immediately implies that there is no percolation for $k = 1$, whatever the vertex set is (note that in the PPP case this also follows from the results of [HM96]). Define the critical out-degree $k_{\mathbf{B},d}$ analogously to $k_{\mathbf{U},d}$ but with $\mathbf{U}$ replaced by $\mathbf{B}$. It was shown in [BB13] that $k_{\mathbf{U},2} \leq 11$ and $k_{\mathbf{B},2} \leq 15$.

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Further, high-confidence results of [BB13] indicate $k_{\mathbf{U},2} = 3$ and $k_{\mathbf{B},2} = 5$. These results follow once one shows that a certain deterministic integral exceeds a certain deterministic value, however, the integrals were only evaluated via Monte–Carlo methods so far.

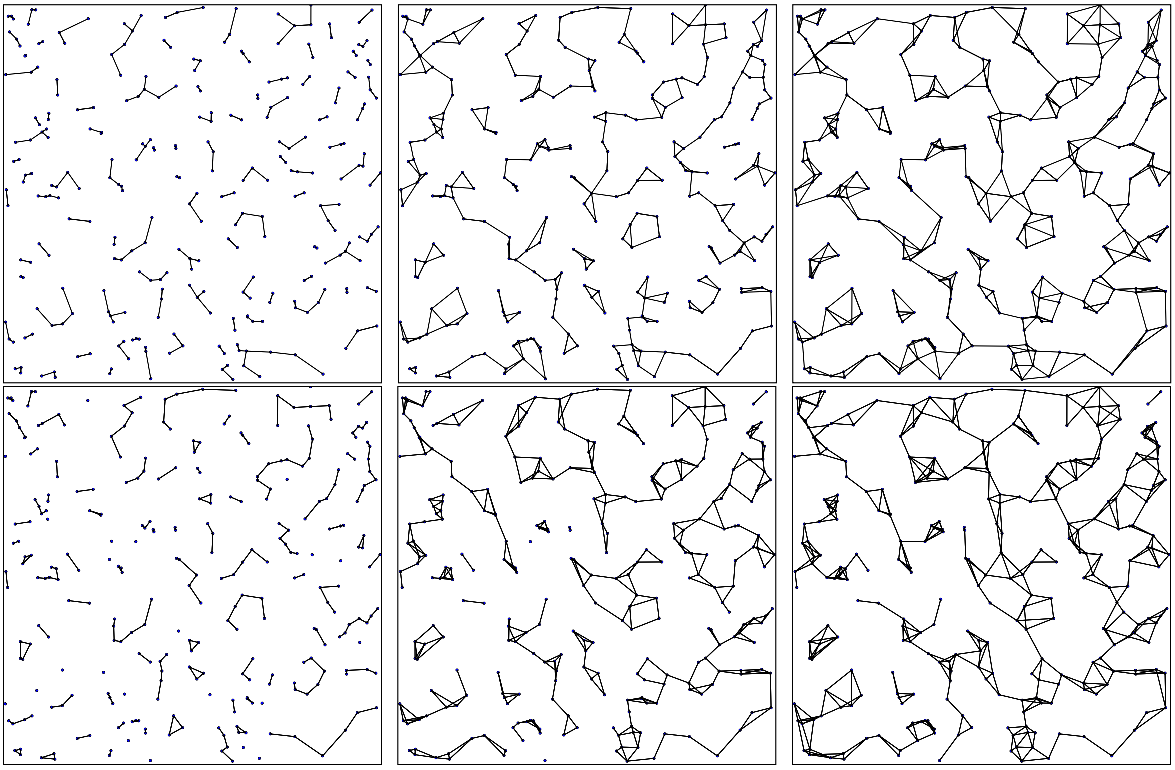

Figure 1: Top: Realizations of $\mathbf{U}$-kNN graphs based on a PPP with $k = 1$ (left), $k = 2$ (center) and $k = 3$ (right). Bottom: Realizations of $\mathbf{B}$-kNN graphs based on a PPP with $k = 2$ (left), $k = 4$ (center) and $k = 5$ (right).

Another line of research on percolation of bounded-degree spatial graphs with unbounded-range dependences, which is also close to applications in wireless networks, is *signal-to-interference ratio (SINR)* percolation, introduced in [DBT05, DFM$^+$06]. Here, a transmission in the network is considered successful if and only if, measured at the receiver, the incoming signal power of the transmitter is larger than a given threshold times the interference (sum of signal powers) coming from all other users plus some external noise. Then, the SINR graph is constructed by drawing an edge between two vertices whenever the transmission between them is successful in both directions, see Section 4.2 for more details. This graph has bounded degrees (see [DBT05, Theorem 1]), where the smallest degree bound k depends on the model parameters. If the transmitted signal powers are all equal, then the SINR graph is contained in the corresponding $\mathbf{B}$-kNN graph (see [T19, Lemma 4.1.13]) and hence also of the $\mathbf{U}$-kNN graph.

In general, if in an undirected graph all degrees are bounded by $k = 2$, all infinite connected components must be path graphs (no cycles, no branchings), infinite in one or two directions, which makes the graph similar to a one-dimensional continuum percolation model, indicating that under rather general conditions, there should be no infinite connected component. Certainly, there are deterministic point processes where percolation is possible, but a little bit of randomness can be expected to suffice for non-percolation. In our recent paper [JT19], we showed that in SINR graphs based on general stationary Cox point processes (CPPs) in any dimension, under rather general choices of the parameters resulting in degrees bounded by 2, there is no percolation.

In the present paper, moving away from the particular setting of SINR graphs, we present analogous results in a general framework, extending the methods of the proof of [JT19, Theorem 2.2]. We consider a generalization of the $\mathbf{B}$-kNN graph, called the f-kNN graph. Here, points of the underlying marked point process are connected by an edge whenever they are mutually among the k nearest neighbors of each other with respect to an ordering that may also depend on some marks of the points, apart from the (not necessarily Euclidean) distance of the points. The ordering is expressed in terms of a function f, hence the name f-kNN graph. We show that under suitable conditions on the underlying stationary marked point process, the f-kNN graph does not percolate for $k = 2$. This in particular implies non-percolation of subgraphs of the f-kNN graph depending on additional randomness. Our results extend to many stationary point processes, including all CPPs satisfying a basic nondegeneracy condition, and all Gibbs point processes satisfying a pointwise monotonicity property in the Hamiltonian.

As a special case, our results imply that for general stationary CPPs, the $\mathbf{B}$-2NN graph does not percolate. This in particular implies that $k_{\mathbf{B},2} \geq 3$, which provides a partial verification of the high-confidence results of [BB13]. Note that this result does not follow from [JT19, Theorem 2.2] because in general, if the SINR graph is contained in a $\mathbf{B}$-2NN graph, it is a proper subgraph of it with substantially less edges. After stating and proving our main results, we also present examples of graphs with degrees bounded by 2 that are not contained in an f-2NN graph but where our proof techniques are also applicable, and also ones where they are not applicable.

Our setting is also related to the line of research on *outdegree-one* graphs, which were introduced in [CDS20]. However, our results do not follow from the results of that paper, and also not the other way around. We will comment on the similarities and differences of the two models in Section 4.4.

The rest of the paper is organized as follows. In Section 2 we present our setting and main result. In Section 3 we provide the proofs for our main result. Section 4 is devoted to examples, extensions of our methods, and discussions.

2 Model definition and main result

In this section we present our model definition and main results. Our setting is as follows. Let $d \in \mathbb{N}$, and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^d$. Further, let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel-σ-algebra of $\mathbb{R}^d$ (clearly, $\|\cdot\|$ generates $\mathcal{B}(\mathbb{R}^d)$). Moreover, consider the measurable space $(M, \mathcal{M})$, which serves as a mark space.

Next, let $\mathbf{X} = \{(X_i, P_i)\}_{i\in I}$ be a marked point process in $\mathbb{R}^d \times M$, so that $X = \{X_i\}_{i\in I}$ is a stationary point process in $\mathbb{R}^d$ with finite intensity $\lambda = \mathbb{E}[X([0,1]^d)]$, that is *nonequidistant*. This means that for all $i,j,k,l \in I$, $\|X_i - X_j\| = \|X_k - X_l\| > 0$ implies $\{i,j\} = \{k,l\}$ and $\|X_i\| = \|X_j\|$ implies $i \neq j$, almost surely. Clearly, this property implies that the point process X is *simple*, i.e., $\mathbb{P}(X_i \neq X_j, \forall i,j \in I \text{ with } i \neq j) = 1$. For illustration, note that the randomly shifted lattice $\mathbb{Z}^d + U$, where U is a uniform random variable in $[0,1]^d$, is a simple stationary but not nonequidistant point process on $\mathbb{R}^d$.

Next, we introduce a total ordering of the points. For this, let $f\colon [0,\infty) \times M \to [0,\infty)$ be a measurable function such that $a \mapsto f(a,p)$ is monotone decreasing for all $p \in M$. We call such a function an *ordering function*.

Definition 2.1. Let f be an ordering function and $(x,p),(y,q),(z,r) \in \mathbb{R}^d \times M$. We say that y *is f-closer to x than to z* if one of the following conditions is satisfied:

1 $f(\|x-y\|, p) < f(\|z-y\|, r)$, or

2 $f(\|x-y\|, p) = f(\|z-y\|, r)$ and $\|x-y\| < \|z-y\|$.

Then, it is elementary to verify the following lemma.

Lemma 2.2. *Let f be an ordering function. For $\mathbf{X}$ defined as above and X nonequidistant, almost surely, the following holds. For all $i \in I$, the relation "X_i is f-closer to X_j than to X_l" is a total ordering (i.e., irreflexive, antisymmetric and transitive, with any two elements being comparable) on the set $\{(j,l) \in I \times I : j \neq i \text{ and } l \neq i\}$, which we call the f-ordering.*

Thus, if $\boldsymbol{\omega} = \{(x_i, p_i)\}_{i \in I}$ is a deterministic, locally finite, infinite, and nonequidistant set of points in $\mathbb{R}^d \times M$ (for some countable index set I) and $x \in \omega := \{x_i\}_{i \in I}$, we can represent ω as $\omega = \{\mathrm{V}_n^f(x, \boldsymbol{\omega})\}_{n \in \mathbb{N}_0}$, where $\mathrm{V}_0^f(x, \boldsymbol{\omega}) = x$, and $\mathrm{V}_n^f(x, \boldsymbol{\omega})$ is the n-th nearest neighbor of x in ω with respect to the f-ordering for any $n \in \mathbb{N}_0$. Next, we build a graph based on the f-ordering.

Definition 2.3. Let f be an ordering function, $k \in \mathbb{N}$ and $\mathbf{X}$ defined as above with X nonequidistant, almost surely. The *f-k-nearest neighbor* (f-kNN) *graph* $g_{k,f}(\mathbf{X})$ is the random graph having vertex set X and for all $i \in I$ and $n \in \{1, \ldots, k\}$ an edge between X_i and $\mathrm{V}_n^f(X_i, \mathbf{X})$ whenever $X_i \in \{\mathrm{V}_1^f(\mathrm{V}_n^f(X_i, \mathbf{X}), \mathbf{X}), \ldots, \mathrm{V}_k^f(\mathrm{V}_n^f(X_i, \mathbf{X}), \mathbf{X})\}$.

As the next example shows, the $\mathbf{B}$-kNN graph is an f-kNN graph for a point process with trivial marks. Let us write $\{\star\}$ for the one-point measurable space (with $\mathcal{M} = \{\varnothing, \{\star\}\}$).

Example 2.4. Consider a nonequidistant point process X in $\mathbb{R}^d$, $d \geq 1$, and equip X with trivial marks in $M = \{\star\}$. Then, $f(x,p) = f(x) = x$ yields the $\mathbf{B}$-kNN graph based on X.

We will explain the relations between f-kNN graphs and SINR graphs in Section 4.2.

Apart from the basic requirement of being nonequidistant, the property of *stability under local thinning* introduced in the next definition is the most important requirement on the underlying stationarity point process. For $K > 0$, let $B_K(o)$ denote the open ℓ^2-ball of radius K around o. Let $\mathbf{X}^{K,p}$ be given as the union of $\mathbf{X} \setminus (B_K(o) \times M)$ and the independent thinning of $\mathbf{X} \cap (B_K(o) \times M)$ with survival probability $p \in [0,1]$. That is, conditional on $\mathbf{X}$, $\mathbf{X}^{K,p} \cap (B_K(o) \times M)$ contains each point of $\mathbf{X} \cap (B_K(o) \times M)$ with probability p independent of the other points of this point process, and it contains no other points.

Definition 2.5. The marked point process $\mathbf{X}$ is *stable under local thinning* if the law of $\mathbf{X}^{K,p}$ is absolutely continuous with respect to the one of $\mathbf{X}$ for any $K > 0$ and $p \in [0,1]$.

To be more precise, the absolute continuity is meant in this definition in the following way. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be any probability space on which $\mathbf{X}^{K,p}$ and $\mathbf{X}$ are jointly defined for all $K > 0$ and $p \in [0,1]$, in particular, $\mathbb{P}'(\mathbf{X} \in \cdot) = \mathbb{P}(\mathbf{X} \in \cdot)$. Let $G \in \mathcal{F}'$ be any event such that $\mathbb{P}'(\mathbf{X}^{K,p} \in G) > 0$, then we have $\mathbb{P}'(\mathbf{X} \in G) > 0$. We will present examples of marked point process that are stable under local thinning below. Equipped with the above definitions, we are now able to state our main result.

Theorem 2.6. *Let f be an ordering function and let the marked point process $\mathbf{X}$ be stable under local thinning and such that the underlying point process X is stationary and nonequidistant. Then, we have*

$$\mathbb{P}(g_{2,f}(\mathbf{X}) \textit{ percolates}) = 0.$$

The proof of this theorem is carried out in Section 3. In Section 4.2 we discuss the relation between this proof and the one of [JT19, Theorem 2.2]. In Section 4.3 we will explain how it extends to other graphs that are defined similarly, have degrees bounded by 2, but are not subgraphs of f-kNN graphs.

Note that stability under local thinning is satisfied by many point processes, as is shown by the following proposition.

Proposition 2.7. *The i.i.d. marked point process $\mathbf{X}$ where X is stationary and nonequidistant is stable under local thinning if X is a*

1. *Cox point process, or*
2. *a well-defined Gibbs point process based on a monotone Hamiltonian H, where H is* monotone *if for all locally-finite configurations $\boldsymbol{\omega}$, all $K > 0$ and points $\mathbf{x} \in B_K(o) \times M$, we have that*

$$H_{B_K(o)}(\boldsymbol{\omega}) \leq H_{B_K(o)}(\boldsymbol{\omega} \cup \mathbf{x}).$$

The monotonicity condition on the Hamiltonian is for example satisfied if H is defined via a non-negative pair interaction. The proof of this proposition is presented in Section 4.1. The part regarding CPPs has already been verified before, cf. [JT19, Lemma 5.10], but we present the proof also in this paper for the reader's convenience. Note that the class of stationarity and nonequidistant CPPs includes the homogeneous PPPs. Moreover, there are also well-known point processes that are not stable under local thinning. Let us discuss a class of such examples. As introduced in [GP12], we say that the point process X is *rigid* if for any $K > 0$, there exists a deterministic measurable function h_K such that,

$$\#(X \cap B_K(o)) = h_K(X \setminus B_K(o)),$$

almost surely, i.e., X outside $B_K(o)$ determines the number of points of X in $B_K(o)$. The following proposition states that rigid point processes fail to be stable under local thinning and will be proven in Section 4.1.

Proposition 2.8. *If the non-marked version X of the marked point process $\mathbf{X}$ is stationary and rigid with positive intensity, then $\mathbf{X}$ is not stable under local thinning.*

According to [GP12], the Ginibre ensemble and the Gaussian zero process are rigid point processes in $\mathbb{R}^2$, which are also stationary, nonequidistant, and of positive intensity. Hence, the proof of Theorem 2.6 is not applicable for them. We nevertheless believe that they satisfy the assertion of the theorem, but the proof would require additional arguments.

3 Proof of Theorem 2.6

The proof of the Theorem 2.6 proceeds along the following line of arguments. We first show that up to $\mathbb{P}$-null sets, clusters, i.e., maximally connected components, are either finite or infinite in both directions, i.e., they contain no vertex of degree 1 in case it is infinite, see Lemma 3.1 below. Next, we assume for a contradiction that there exists an infinite cluster with positive probability. Then, we introduce a procedure that removes points from the infinite cluster that is closest to the origin in a certain sense. In the resulting configuration, the infinite cluster still remains infinite, but it contains a vertex of degree 1. Hence, the probability that the process takes values in the set of the resulting configurations is zero. Then it remains to show that also the probability that the process takes place in the set of original configurations is zero, which leads to the desired contradiction. At this point it will be useful to compare the resulting configuration with an independent thinning of the original configuration in a certain ball, and this is where we make use of the stability under local thinning.

Note that for the proof of Theorem 2.6, we can assume that the intensity λ of the underlying stationary point process is positive, since otherwise Theorem 2.6 is trivially true. We start with the following, previously proven lemma, which excludes existence of infinite clusters that have a degree-one point in the case of general random graphs based on stationary marked point processes.

Lemma 3.1. *[JT19, Lemma 5.6] Let $g(\mathbf{X})$ be a random graph based on a marked point process $\mathbf{X} = \{(X_i, P_i)\}_{i\in I}$ with values in $\mathbb{R}^d \times M$, with vertex set $X = \{X_i\}_{i\in I}$ such that the degree of all $X_i \in X$, $\deg(X_i)$, is bounded by 2, almost surely. Let X be stationary and have a finite intensity, and consider the point process of degree-one points in infinite clusters*

$$\mathcal{X}_0 = \sum_{i\in I} \delta_{X_i} \mathbb{1}\{\deg(X_i) = 1,\ X_i \text{ is part of an infinite cluster in } g(\mathbf{X})\}.$$

Then, $\mathbb{P}(\mathcal{X}_0(\mathbb{R}^d) = 0) = 1$.

The proof is based on a certain variant of the *mass-transport principle* (see [CDS20, Section 4.2] for instance). Informally speaking, the proof goes as follows: if there was an infinite cluster having a point of degree one, then by stationarity, the point process of degree-1 points of infinite clusters $\mathcal{X}_0$ would have to have a positive density. This however leads to a contradiction because any infinite cluster can only contain at most one degree-1 point and must contain infinitely many degree-2 points, which implies that the aforementioned density must be equal to zero. We refer the reader to [JT19, Section 5.2] for further details.

Let us denote by $(\mathcal{C}_i)_{0\le i\le L}$ the L-many infinite clusters in $g_{2,f}(\mathbf{X})$, where $L \in \mathbb{N}_0 \cup \{0, \infty\}$. For the proof of Theorem 2.6, it then suffices to show that

$$\mathbb{P}(L \ge 1) = 0. \tag{3.1}$$

We view $\mathbf{X}$ as the canonical process $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{\omega}$ on the set $\mathbf{N}$ of marked point configurations $\boldsymbol{\omega}$ in $\mathbb{R}^d \times M$ such that $\omega = \{x_i \colon (x_i, p_i) \in \boldsymbol{\omega}\}$ is an infinite, locally-finite, nonequidistant point configuration on $\mathbb{R}^d$. The set of such point configurations ω will be denoted by N. Note that $\mathbf{N}$ and N are equipped with the corresponding evaluation σ-fields.

Let us, for the remainder of this section, fix an ordering function f. Then, for $\boldsymbol{\omega} \in \mathbf{N}$ and $x_o \in \omega$, we can consider the vector $\mathbf{V}(x_o, \boldsymbol{\omega}) = (\mathbf{V}_n(x_o, \boldsymbol{\omega}))_{n\in\mathbb{N}_0}$ of the marked points of $\boldsymbol{\omega}$ ordered in increasing f-order of $\boldsymbol{\omega}$, measured from x_o. To lighten notation, we suppress the reference to f in $\mathbf{V}$ here and in the remaining document. Then, $\mathrm{V}_i(x_o, \boldsymbol{\omega})$ defined in Section 2 is the first component of $\mathbf{V}_i(x_o, \boldsymbol{\omega})$, which we call the *$i$-th f-nearest neighbor of x_o*. In particular, $\mathrm{V}_0(x_o, \boldsymbol{\omega}) = x_o$.

Now, if x_o has degree two in $g_{2,f}(\boldsymbol{\omega})$, then x_o must be connected by an edge to both $\mathrm{V}_1(x_o, \boldsymbol{\omega})$ and $\mathrm{V}_2(x_o, \boldsymbol{\omega})$ since the degree bound applies already for the edges towards x_o. Moreover, both $\mathrm{V}_1(x_o, \boldsymbol{\omega})$ and $\mathrm{V}_2(x_o, \boldsymbol{\omega})$ must also have x_o as one of their first two f-nearest neighbors, that is,

$$x_o \in \big\{\mathrm{V}_1(\mathrm{V}_i(x_o, \boldsymbol{\omega}), \boldsymbol{\omega}), \mathrm{V}_2(\mathrm{V}_i(x_o, \boldsymbol{\omega}), \boldsymbol{\omega})\big\},$$

for all $i \in \{1, 2\}$. These f-nearest neighbor relations hold almost surely, in particular for every nonequidistant configuration $\boldsymbol{\omega}$. The goal of using the configuration space $\mathbf{N}$ is to entirely exclude configurations that offend the degree bound or the f-nearest neighbor relations or are not nonequidistant.

In the event $\{L \geq 1\}$, let $\mathbf{Z} = (Z, R)$ denote the closest point to the origin that has degree two and is contained in an infinite cluster. Without loss of generality, we will assume that this cluster is always equal to $\mathcal{C}_1$. Now, Theorem 2.6 immediately follows once we have verified the following proposition.

Proposition 3.2. *Consider the event* $\{L \geq 1\}$ *and define the random variable*

$$J = \inf\{i \geq 3\colon \mathrm{V}_i(Z, \mathbf{X}) \in \mathcal{C}_1\}.$$

Then, under the assumptions of Theorem 2.6, for any $i \geq 3$*, we have*

$$\mathbb{P}(\{L \geq 1\} \cap \{J = i\}) = 0. \tag{3.2}$$

Proof of Theorem 2.6. Using a union bound and noting that $\{L \geq 1\} \subset \{J < \infty\}$, Proposition 3.2 implies $\mathbb{P}(L \geq 1) = 0$, which is (3.1), and thus finishes the proof of Theorem 2.6. □

Proof of Proposition 3.2. For $\boldsymbol{\omega} \in \{L \geq 1\}$, by definition, we have that $Z(\boldsymbol{\omega})$ is connected by an edge both to $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ in $g_{2,f}(\boldsymbol{\omega})$. Further, thanks to the degree bound of two, in the event $\{L \geq 1\}$, $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ have no further joint neighbor in $g_{2,f}(\boldsymbol{\omega})$ since otherwise $\mathcal{C}_1(\boldsymbol{\omega})$ has a loop and can not be infinite by the degree bound. This way, for any $i \geq 3$, there exists $l \in \{1, 2\}$ such that $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_l(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ are not connected by an edge in $g_{2,f}(\boldsymbol{\omega})$. Let us denote the corresponding $\mathrm{V}_l(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ by $M_i(\boldsymbol{\omega})$, and define $M_i(\boldsymbol{\omega}) = \mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ if neither $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ nor $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ is connected to $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ by an edge. The element of $\{\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega}), \mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})\}$ not being equal to $M_i(\boldsymbol{\omega})$ is denoted by $N_i(\boldsymbol{\omega})$. We will write Q for the mark of $M_i(\boldsymbol{\omega})$.

Let us fix $i \geq 3$. Let $\boldsymbol{\omega} \in \{L \geq 1\}$ be such that $J(\boldsymbol{\omega}) = i$. Let us define a thinned configuration

$$\boldsymbol{\omega}^i = \boldsymbol{\omega} \setminus \{(M_i(\boldsymbol{\omega}), Q), \mathbf{V}_3(Z(\boldsymbol{\omega}), \boldsymbol{\omega}), \dots, \mathbf{V}_{i-1}(Z(\boldsymbol{\omega}), \boldsymbol{\omega})\}.$$

We claim for $\mathbb{P}$-almost all $\boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}$ also $\boldsymbol{\omega}^i \in \{L \geq 1\}$. For this, first note that the removal of finitely many points and their associated edges from an infinite cluster does not change the property of the cluster to be infinite. However, the removal of points can still change the edge structure of the remaining points. In order to exclude this, we can use the fundamental property of the f-kNN graph that, if we remove points from a configuration, then edges between remaining points are preserved.

Definition 3.3. Let $g\colon \mathbf{N} \to N \times (N \times N)$, $\boldsymbol{\omega} \mapsto g(\boldsymbol{\omega}) = (\omega, E_g(\boldsymbol{\omega}))$ be a function that maps a marked point configuration $\boldsymbol{\omega}$ to a graph with vertex set ω. We say that g is *edge-preserving* if for all $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbf{N}$ with $\boldsymbol{\omega} \subseteq \boldsymbol{\omega}'$, for all $(x, p), (y, q) \in \boldsymbol{\omega}$ such that $(x, y) \in E_g(\boldsymbol{\omega}')$, one has $(x, y) \in E_g(\boldsymbol{\omega})$.

It is easy to see that the f-kNN graph $g_{k,f}\colon \boldsymbol{\omega} \mapsto g_{k,f}(\boldsymbol{\omega})$ is edge-preserving for all $k \in \mathbb{N}$. In particular, for all $\boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}$, since $g_{2,f}$ is edge-preserving, also all edges between two points of ω^i in $g_{2,f}(\boldsymbol{\omega})$ exist in $g_{2,f}(\boldsymbol{\omega}^i)$. This implies that $L(\boldsymbol{\omega}^i) \geq 1$, hence the claim. Let us note that in Section 4.3 we will also present some examples that are not subgraphs of the f-kNN graph but constructed similarly and are still edge-preserving.

Then, the next claim is that for $\boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}$, we have that $\boldsymbol{\omega}^i$ is contained in

$$B = \{\boldsymbol{\eta}\colon L(\boldsymbol{\eta}) \geq 1 \text{ and } \mathcal{C}_1(\boldsymbol{\eta}) \text{ contains a point of degree one}\} \subset \{L \geq 1\}.$$

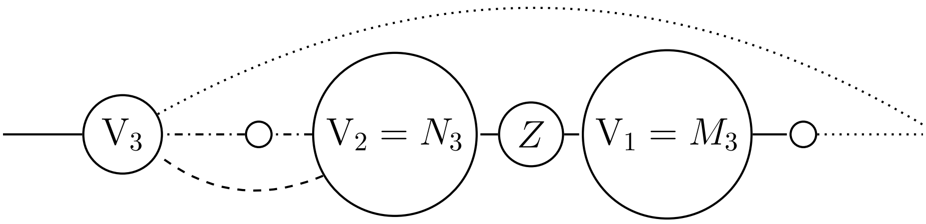

Figure 2: An illustration of the case $J(\boldsymbol{\omega}) = 3$ for some realization $\boldsymbol{\omega} \in \{L \geq 1\}$. $\mathrm{V}_3 = \mathrm{V}_3(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ is contained in the infinite cluster $\mathcal{C}_1 = \mathcal{C}_1(\boldsymbol{\omega})$ including $Z = Z(\boldsymbol{\omega})$, and it is not a neighbor of $M_3 = M_3(\boldsymbol{\omega})$, which in this example equals $\mathrm{V}_1 = \mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$, whereas $\mathrm{V}_2 = \mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) = N_3 = N_3(\boldsymbol{\omega})$. Hence, if V_3 has degree two in $\mathcal{C}_1$, then there are various possibilities respecting the degree bound of two to connect V_3 to $\mathcal{C}_1$ so that it is not connected to M_3 by an edge. V_3 can either be a direct neighbor of V_2 (see dashed line) or a later point of the path from Z to infinity starting with the edge from Z to V_2 (dash-dotted lines) or a non-direct neighbor of V_1 on the path from Z to infinity starting with the edge from Z to V_1 (dotted lines). Now, removing M_3 from the realization, both edges adjacent to V_3 are preserved. Also all edges from Z to infinity starting with the edge from Z to V_2 are preserved, hence Z is still contained in an infinite cluster, but the edge from Z to V_1 is removed. In the resulting new configuration, the second-nearest f-neighbor towards Z is V_3, and hence this is the only point of the configuration that could be connected to Z by an edge. But V_3 still cannot have degree 3 or more, hence it cannot be connected to Z, which implies that in the new configuration Z is in an infinite cluster containing a point of degree one.

The proof of this claim in the simplest case $i = 3$ is illustrated in Figure 2. For general $i \geq 3$, recall that Z cannot have degree higher than two in $g_{2,f}(\boldsymbol{\omega}^i)$, whereas it has degree at least one and its cluster $\mathcal{C}_1(\boldsymbol{\omega}^i)$ is infinite in $g_{2,f}(\boldsymbol{\omega}^i)$. Note also that the edge between $Z(\boldsymbol{\omega})$ and $N_i(\boldsymbol{\omega})$ still exists in $g_{2,f}(\boldsymbol{\omega}^i)$. Further, if $Z(\boldsymbol{\omega})$ has degree two in $g_{2,f}(\boldsymbol{\omega}^i)$, then it is connected to the second-nearest f-neighbor towards $Z(\boldsymbol{\omega})$ in $\boldsymbol{\omega}^i$, which is $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega}^i) = \mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$, whereas $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega}^i) = N_i(\boldsymbol{\omega})$. Now, since $\boldsymbol{\omega} \notin B$, $\boldsymbol{\omega} \in \{L \geq 1\}$ and $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in \mathcal{C}_1(\boldsymbol{\omega})$, it follows that $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ has degree equal to two in $g_{2,f}(\boldsymbol{\omega})$. Further, it is neither connected to $M_i(\boldsymbol{\omega})$ by an edge nor to $Z(\boldsymbol{\omega})$ in this graph. Hence, both edges adjacent to $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ also exist in $g_{2,f}(\boldsymbol{\omega}^i)$. But since $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ has degree at most two in $g_{2,f}(\boldsymbol{\omega}^i)$, it follows that $Z(\boldsymbol{\omega})$ and $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ are not connected by an edge in this graph. Hence, $\boldsymbol{\omega}^i \in B$, which implies the claim.

Note that by Lemma 3.1, the set B is a $\mathbb{P}$-null set, i.e.,

$$\mathbb{P}(\{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}\}) = 0. \tag{3.3}$$

This implies (3.2) and concludes the proof of Proposition 3.2 as soon as the following lemma is verified.

Lemma 3.4. *Under the assumptions of Theorem 2.6, for any $i \geq 3$, $\mathbb{P}(\{L \geq 1\} \cap \{J = i\}) > 0$ implies $\mathbb{P}(\{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}\}) > 0$.*

By Lemma 3.4, where we show that if the collection of thinned configurations is contained in a $\mathbb{P}$-null set, also the non-thinned configurations form a $\mathbb{P}$-null set, we see that (3.3) implies (3.2), which concludes the proof of Proposition 3.2. □

Proof of Lemma 3.4. The proof strongly relies on the stability under local thinning. Let us fix $i \geq 3$ and assume that $\mathbb{P}(\{L \geq 1\} \cap \{J = i\}) > 0$. Then, by continuity of measures, there exists $K > 0$ such that

$$\mathbb{P}\big(\{\boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\} \colon \mathrm{V}_j(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o),\ \forall j \in \{1, \dots, i\}\}\big) > 0,$$

where $B_K(o)$ denotes the open Euclidean ball of radius K in $\mathbb{R}^d$. Hence, there exists $n \geq i$ such that $\mathbb{P}(C_{i,K,n}) > 0$, where

$$C_{i,K,n} = \big\{\boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\} \colon \#\big(\omega \cap B_K(o)\big) = n+1 \\ \text{and } \mathrm{V}_j(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o), \forall j \in \{1, \dots, i\}\big\}.$$

Conditional on the event $C_{i,K,n}$, the marked point process $(\mathbf{X} \setminus \{\mathbf{Z}\}) \cap B_K(o)$ has precisely n points $\mathbf{X}_1, \dots \mathbf{X}_n$.

Now, for some fixed $p \in (0,1)$, we can represent $\mathbf{X}$ as $\mathbf{X}^{K,p} \cup \mathbf{X}'$, where $\mathbf{X}^{K,p}$ is the thinning of $\mathbf{X}$ corresponding to Definition 2.5, and $\mathbf{X}'$ is the complementary thinning. Then, $\mathbf{X}'$ and $\mathbf{X}^{K,p} \cap (B_K(o) \times M)$ are independent thinnings of $\mathbf{X} \cap (B_K(o) \times M)$ with survival probability $1-p$ respectively p, further, $\mathbf{X}^{K,p} = \mathbf{X} \setminus \mathbf{X}'$, $\mathbf{X}' \setminus (B_K(o) \times M) = \varnothing$, and $\mathbf{X}^{K,p} \setminus (B_K(o) \times M) = \mathbf{X} \setminus (B_K(o) \times M)$. In order to provide a precise construction of the thinned processes, we choose a sequence $(T_m)_{m \in \mathbb{N}}$ of i.i.d. Bernoulli random variables with parameter p that is independent of $\mathbf{X}$, and given the realization $\boldsymbol{\omega} = \mathbf{X}(\boldsymbol{\omega}) = (\mathbf{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega}))_{i \in \mathbb{N}_0}$, the realizations of $\mathbf{X}^{K,p}(\boldsymbol{\omega})$ and $\mathbf{X}'(\boldsymbol{\omega})$ are defined as follows, depending also on $(T_m)_{m \in \mathbb{N}}$:

$$\mathbf{X}^{K,p}(\boldsymbol{\omega}) = \mathbf{X}^{K,p}(\boldsymbol{\omega}, (T_m)_{m \in \mathbb{N}}) = \{\mathbf{Z}(\boldsymbol{\omega})\} \cup \{\mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \colon T_m = 1, \mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o)\} \\ \cup \{\mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \colon \mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in \mathbb{R}^d \setminus B_K(o)\}$$

and

$$\mathbf{X}'(\boldsymbol{\omega}) = \mathbf{X}'(\boldsymbol{\omega}, (T_m)_{m \in \mathbb{N}}) = \{\mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \colon T_m = 0, \mathbf{V}_m(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o)\}.$$

Clearly, the projections $X^{K,p}$ and X' of $\mathbf{X}^{K,p}$ respectively $\mathbf{X}'$ to $\mathbb{R}^d$ are nonequidistant, further, $\mathbf{X}^{K,p}$ can be represented as a random variable with values in $\mathbf{N}$, defined on an enlarged probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ governing both the point process $\mathbf{X}$ and the sequence $(T_m)_{m \in \mathbb{N}}$. In particular, $\mathbb{P}'(\mathbf{X} \in \cdot) = \mathbb{P}(\mathbf{X} \in \cdot)$.

Now, thanks to the assumption that $\mathbb{P}(C_{i,K,n}) > 0$ and using the definition of $\mathbf{X}^{K,p}$,

$$\begin{aligned} \mathbb{P}'\big(\mathbf{X}^{K,p} \in &\{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}\}\big) \geq \mathbb{P}'\big(\mathbf{X}^{K,p} \in \{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in C_{i,K,n}\}\big) \\ &\geq \mathbb{P}'\big(\mathbf{X}^{K,p} \in \{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in C_{i,K,n}\}, \mathbf{X} \in C_{i,K,n}\big) \\ &= \mathbb{P}(C_{i,K,n})\mathbb{P}'\big(\mathbf{X}^{K,p} \in \{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in C_{i,K,n}\} | \mathbf{X} \in C_{i,K,n}\big) \\ &= \mathbb{P}(C_{i,K,n}) p^{n-i+2} (1-p)^{i-2} > 0. \end{aligned} \tag{3.4}$$

Finally, since $\mathbf{X}$ is stable under local thinning, under $\mathbb{P}'$ the distribution of $\mathbf{X}^{K,p}$ is absolutely continuous with respect to the one of $\mathbf{X}$. Hence, it follows from (3.4) that

$$\mathbb{P}\big(\mathbf{X} \in \{\boldsymbol{\omega}^i \colon \boldsymbol{\omega} \in \{L \geq 1\} \cap \{J = i\}\}\big) > 0,$$

which implies the lemma. □

4 Examples, discussion and extensions

4.1 Stability under local thinning

In this section, we prove Propositions 2.7 and 2.8. In general, Proposition 2.7 claims that under $\mathbb{P}'$, the distribution of $\mathbf{X}^{K,p}$ is absolutely continuous with respect to the one of $\mathbf{X}$. Let F be an element of the evaluation σ-algebra of $\mathbf{N}$ such that $\mathbb{P}'(\mathbf{X}^{K,p} \in F) > 0$. We have to show that then also $\mathbb{P}(\mathbf{X} \in F) > 0$. Under the assumption that $\mathbb{P}'(\mathbf{X}^{K,p} \in F) > 0$, by continuity of measures, we can find $K, l \in \mathbb{N}$ such that

$$\varepsilon := \mathbb{P}'(\mathbf{X}^{K,p} \in F, \#(\mathbf{X}^{K,p} \cap (B_K(o) \times M)) = l) > 0. \tag{4.1}$$

In other words, we have $0 < \varepsilon = \mathbb{P}'(\mathbf{X}^{K,p} \in G)$ where $G = \{\boldsymbol{\omega} \in F\colon \#(\boldsymbol{\omega} \cap (B_K(o) \times M)) = l\}$. Thus,

$$\begin{aligned}\mathbb{P}(\mathbf{X} \in F) &\geq \mathbb{P}'(\mathbf{X} \in G, \mathbf{X}^{K,p} = \mathbf{X}) \geq \mathbb{P}'(\mathbf{X}^{K,p} \in G)\mathbb{P}'(\mathbf{X}^{K,p} = \mathbf{X} | \mathbf{X}^{K,p} \in G)\\ &= \varepsilon \mathbb{P}'(\mathbf{X}^{K,p} = \mathbf{X} | \mathbf{X}^{K,p} \in G),\end{aligned}$$

and further,

$$\mathbb{P}'(\mathbf{X}^{K,p} = \mathbf{X} | \mathbf{X}^{K,p} \in G) \geq \mathbb{P}'(\mathbf{X}^{K,p} = \mathbf{X}, \mathbf{X}^{K,p} \in G) = \mathbb{P}'(\mathbf{X}' = \varnothing, \mathbf{X}^{K,p} \in G). \tag{4.2}$$

Now we perform a case distinction depending of the type of the point process. We start with the case of CPPs.

Proof of Proposition 2.7 for Cox point processes. Let $\mathbf{X}$ be a stationary CPP based on the stationary random intensity measure Λ. Recall that a CPP is characterized by the property that conditional on its directing measure Λ, it is a PPP with intensity measure Λ, see e.g. [LP17, Section 13]. According to (4.1), we have

$$0 < \varepsilon = \mathbb{P}'(\mathbf{X}^{K,p} \in G) = \sum_{n=0}^{\infty} a_n,$$

where $a_n = \mathbb{E}'\big[\mathbb{P}'(\mathbf{X}^{K,p} \in G | \Lambda) \mathbb{1}\{\Lambda(B_K(o)) \in [n, n+1)\}\big]$, and thus there exists $m \in \mathbb{N}_0$ with $a_m > 0$. Now, conditional on Λ, $\mathbf{X}^{K,p}$ is an i.i.d. marked PPP, and hence a PPP on $\mathbb{R}^d \times M$, which also implies that the complementary thinnings $\mathbf{X}^{K,p}$ and $\mathbf{X}'$ are independent given Λ, see [K93, Colouring Theorem and Marking Theorem]. Hence, we obtain

$$\begin{aligned}\mathbb{P}'(\mathbf{X}' = \varnothing, \mathbf{X}^{K,p} \in G) &= \mathbb{E}'\big[\mathbb{P}'(\mathbf{X}' = \varnothing | \Lambda) \mathbb{P}'(\mathbf{X}^{K,p} \in G | \Lambda)\big]\\ &= \sum_{n=0}^{\infty} \mathbb{E}'\big[\mathrm{e}^{-(1-p)\Lambda(B_K(o))} \mathbb{P}'(\mathbf{X}^{K,p} \in G | \Lambda) \mathbb{1}\{\Lambda(B_K(o)) \in [n, n+1)\}\big]\\ &\geq \sum_{n=0}^{\infty} \mathrm{e}^{-(1-p)(n+1)} a_n \geq \mathrm{e}^{-(1-p)(m+1)} a_m > 0,\end{aligned}$$

which verifies the claim that the distribution of $\mathbf{X}^{K,p}$ is absolutely continuous with respect to the one of $\mathbf{X}$. □

Next, we handle the case of Gibbs point processes.

Proof of Proposition 2.7 for Gibbs point processes. We consider a well-defined infinite-volume Gibbs point process $\mathbf{X}$ based on the stationary and monotone Hamiltonian H, i.e., the distribution of $\mathbf{X}$ is a solution for the DLR equations for the family of finite-volume Gibbs measures

$$Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int \mathcal{P}(\mathrm{d}\boldsymbol{\omega}_\Lambda) \mathrm{e}^{-H_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c})},$$

where Z_Λ is the partition function associated to the Hamiltonian H, $\boldsymbol{\omega}_{\Lambda^c}$ is a given boundary condition, and $\mathcal{P}_\Lambda$ is an i.i.d. marked homogeneous PPP with some intensity $\lambda > 0$ in Λ. We use the convention and write $\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}$ instead of $\boldsymbol{\omega}_\Lambda \cup \boldsymbol{\omega}_{\Lambda^c}$ For details on infinite-volume Gibbs point processes see for example [D19]. Again, by an application of the [K93, Colouring Theorem and Marking Theorem], via the DLR equation with $\Lambda = B_K(o)$, for the canonical process, we have that

$$\begin{aligned} 0 < \varepsilon &= \mathbb{P}'(\mathbf{X}^{K,p} \in G) \\ &= \int \mathbb{P}(\mathrm{d}\boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int \mathcal{P}^1_\Lambda(\mathrm{d}\boldsymbol{\omega}^1_\Lambda) \mathbb{1}\{\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}_{\Lambda^c} \in G\} \int \mathcal{P}\Lambda^2(\mathrm{d}\boldsymbol{\omega}^2_\Lambda) \mathrm{e}^{-H_\Lambda(\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}^2_\Lambda \boldsymbol{\omega}_{\Lambda^c})}, \end{aligned}$$

where $\mathcal{P}^1_\Lambda$ is an i.i.d. marked homogeneous PPP with intensity $p\lambda$ in Λ and $\mathcal{P}^2_\Lambda$ is an i.i.d. marked homogeneous PPP with intensity $(1-p)\lambda$, also in Λ. Hence, using the condition of monotonicity of the Hamiltonian, we obtain

$$\begin{aligned} &\mathbb{P}'(\mathbf{X}' = \varnothing, \mathbf{X}^{K,p} \in G) \\ &= \int \mathbb{P}(\mathrm{d}\boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int \mathcal{P}^1_\Lambda(\mathrm{d}\boldsymbol{\omega}^1_\Lambda) \mathbb{1}\{\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}_{\Lambda^c} \in G\} \int \mathcal{P}^2_\Lambda(\mathrm{d}\boldsymbol{\omega}^2_\Lambda) \mathbb{1}\{\boldsymbol{\omega}^2_\Lambda = \varnothing\} \mathrm{e}^{-H_\Lambda(\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}^2_\Lambda \boldsymbol{\omega}_{\Lambda^c})} \\ &= e^{-(1-p)\lambda|\Lambda|} \int \mathbb{P}(\mathrm{d}\boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int \mathcal{P}^1_\Lambda(\mathrm{d}\boldsymbol{\omega}^1_\Lambda) \mathbb{1}\{\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}_{\Lambda^c} \in G\} \mathrm{e}^{-H_\Lambda(\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}_{\Lambda^c})} \\ &\geq e^{-(1-p)\lambda|\Lambda|} \int \mathbb{P}(\mathrm{d}\boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int \mathcal{P}^1_\Lambda(\mathrm{d}\boldsymbol{\omega}^1_\Lambda) \mathbb{1}\{\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}_{\Lambda^c} \in G\} \int \mathcal{P}^2_\Lambda(\mathrm{d}\boldsymbol{\omega}^2_\Lambda) \mathrm{e}^{-H_\Lambda(\boldsymbol{\omega}^1_\Lambda \boldsymbol{\omega}^2_\Lambda \boldsymbol{\omega}_{\Lambda^c})} \\ &> e^{-(1-p)\lambda|\Lambda|} \varepsilon > 0, \end{aligned}$$

as desired. □

Finally, we prove Proposition 2.8.

Proof of Proposition 2.8. Assume that $\mathbf{X}$ is stationary with positive intensity and rigid, and fix $K > 0$. Let $\mathbf{X}^{K,p}$ be the thinning obtained from $\mathbf{X}$ via keeping all points of $\mathbf{X}$ in $\mathbb{R}^d \setminus B_K(o) \times M$ and the points of $\mathbf{X}$ within $B_K(o) \times M$ independently with survival probability $p \in (0,1)$. Let X and $X^{K,p}$ be the non-marked versions of $\mathbf{X}$ and $\mathbf{X}^{K,p}$ respectively. Since $\mathbf{X}$ is of positive intensity, we can fix $n \in \mathbb{N}$ such that $\mathbb{P}(X \cap B_K(o) = n) > 0$. Then, using the configuration spaces $\mathbf{N}$ and N introduced in the proof of Theorem 2.6, let us define the event

$$F = \big\{\boldsymbol{\omega} \in \mathbf{N} \colon \#(\omega \cap B_K(o)) = 0, \omega \cap (\mathbb{R}^d \setminus B_K(o)) \in h_K^{-1}(n)\big\}.$$

Then, we have that $\mathbb{P}(\mathbf{X} \in F) = 0$ since rigidity implies that $\{X \cap \mathbb{R}^d \setminus B_K(o) = g_K^{-1}(n)\} = \{\#(X \cap B_K(o)) = n\}$ apart from $\mathbb{P}$-nullsets. However, we have

$$\begin{aligned} \mathbb{P}\big(\mathbf{X}^{K,p} \in F\big) &\geq \mathbb{P}\big(\#(X \cap B_K(o)) = n\big)\mathbb{P}\big(\#(X^{K,p} \cap B_K(o)) = 0 | \#(X \cap B_K(o)) = n\big) \\ &= \mathbb{P}(\#(X \cap B_K(o)) = n)(1-p)^n > 0. \end{aligned}$$

Since $\mathbb{P}(\mathbf{X}^{K,p} \in F) > 0$ but $\mathbb{P}(\mathbf{X} \in F) = 0$, the distribution of $\mathbf{X}^{K,p}$ is not absolutely continuous with respect to the one of $\mathbf{X}$, which finishes the proof. □

4.2 SINR graphs as subgraphs of f-kNN graphs

Let us briefly summarize the relation between Theorem 2.6 and [JT19, Theorem 2.2]. Indeed, the two proofs are similar, in particular, two steps of the proof, Lemma 3.1 and the part of Proposition 2.7 regarding the Cox case already appeared in [JT19]. However, in [JT19] we focused on the particular case of SINR graphs based on stationary and nonequidistant CPPs, having concrete applications in telecommunications in mind, and we did not aim at checking whether our proof works also for a wider class of point processes or graphs. Thus, the main novelty in this paper is not that we exploit new proof techniques (although the proofs of Proposition 2.8 and the part of Proposition 2.7 regarding Gibbs point processes have no analogues in [JT19]). Instead, we highlight that apart from the general combinatorial condition of working with undirected and stationary random graphs with degrees bounded by two, two properties are crucial for a straightforward generalization of the proof in [JT19]: (1) the stability of the underlying point process under local thinning and (2) the edge-preserving property of the graph. The latter observation allows for the extensions of Theorem 2.6 presented in Section 4.3.

This puts the result into a general framework and allows for generalizations of the result both with respect to the type of graph and with respect to the kind of point process. Here, let us note that the SINR graph is not a special case of an f-kNN graph, but a proper subgraph of an f-2NN graph under particular choices of the parameters, which is itself edge-preserving. Non-percolation in f-2NN graphs was not even known before the present paper in the simplest case represented by the $\mathbf{B}$-2NN graph.

In order to make the relation between f-kNN graphs and SINR graphs explicit we recall the definition and interpretation of the latter graphs. Let $M = N = [0,\infty)$, $\|\cdot\| = \|\cdot\|_2$, P_o be a nonnegative random variable, and $\mathbf{X} = \{(X_i, P_i)\}_{i\in I}$ an i.i.d. marked point process in $\mathbb{R}^d \times [0,\infty)$ such that all P_i are distributed as P_o. Let $\ell\colon (0,\infty) \to [0,\infty)$, the so-called *path-loss function*, be a monotone decreasing function. Typical examples of path-loss functions correspond to *Hertzian propagation* (see [DBT05, DFM+06]), e.g., for $\alpha > d$, the unbounded function $\ell(r) = r^{-\alpha}$, its truncated variant $\ell(r) = \min\{1, r^{-\alpha}\}$, and its "shifted" variant $\ell(r) = (1+r)^{-\alpha}$. Now, define $f(x,p) = 1/(p\ell(\|x\|))$. In a telecommunication context, for $(X_i, P_i) \in \mathbf{X}$ and $x \in \mathbb{R}^d$, P_i expresses the signal power transmitted by a device at spatial position X_i, and ℓ describes propagation of signal strength over distance. Note that ℓ need not be strictly decreasing, which gives relevance to Part (2) of Definition 2.1 in order to make the f-ordering well-defined. We observe that in case P_o is almost surely equal to a fixed positive constant, then the arising f-kNN graph is the $\mathbf{B}$-kNN graph.

In this setting, the *SINR graph* [DBT05] is usually introduced in the following way. Let N_o be another nonnegative random variable independent of $\mathbf{X}$. Choose two further parameters $\gamma, \tau > 0$, the so-called *interference-cancellation factor* and the *SINR threshold*, respectively, and for $i, j \in I$, $i \neq j$, connect X_i and X_j by an edge whenever the *SINR constraint* is satisfied in both directions, i.e.,

$$P_i\ell(|X_i - X_j|) > \tau\Big(N_o + \gamma \sum_{k\in I\setminus\{i,j\}} P_k\ell(|X_k - X_j|)\Big), \tag{4.3}$$

and the same holds with the roles of i and j interchanged. Then, it is known from [DBT05, Theorem 1] that if $\mathbf{X}$ is a simple point process (even if not stationarity or not nonequidistant), all degrees in the SINR graph are less than $k = 1 + 1/(\tau\gamma)$. Using the elementary arguments of [T19, Lemma 4.1.13], one can easily verify that if $X = \{X_i\}_{i\in I}$ is also nonequidistant, then the SINR graph is a subgraph of the f-kNN graph of the present example. Further, if $N_o > 0$ is deterministic, then the SINR graph has bounded edge lengths and hence is a subgraph of the Gilbert graph introduced in [G61]. The same assertion holds also for $N_o = 0$ in case ℓ has bounded support. For positive assertions about percolation in SINR graphs based on various kinds of point processes, we refer the reader e.g. to [DBT05, DFM+06, BY13, T19, T20, L19, JT19].

Hence, for point processes satisfying the conditions of Theorem 2.6, there is no percolation in the SINR graph if its degrees are bounded by 2, which is always the case if $\gamma \geq 1/(2\tau)$. Thanks to Proposition 2.7, in particular, Gibbs point processes are covered by this result. To the best of our knowledge, there have been no results about SINR percolation for Gibbs point processes before (apart from the degree bounds themselves). Regarding non-percolation in case degrees are bounded by two, the case of CPPs was handled in [JT19, Theorem 2.2]. Here, based on the observation [JT19, Section 5.2, Proof of Proposition 5.8] that SINR graphs are edge-preserving on their own right in the sense of Definition 3.3, we carried out a certain variant of the proof of Theorem 2.6 for the SINR graph directly, with no direct reference to f-kNN (or even $\mathbf{B}$-kNN) graphs.

The aforementioned positive results on SINR percolation guarantee that for various kinds of point processes, the SINR graph percolates with positive probability for some positive γ given that λ is sufficiently large, while all the other parameters (depending on the type of point process) are fixed. In other words, we know that $k^* < \infty$ holds for the infimum k^* of all degree bounds k such that there exists an SINR graph with largest degree equal to k that percolates. There are multiple interesting open questions related to this. First, what is the smallest value of such k^*, and how does it depend on the type of the point process? The main results of the present paper imply that for stationary and nonequidistant point processes that are stable under local thinning, we have $k^* \geq 3$. Further, according to the high-confidence results of [BB13], $k^* \geq 5$ for the two-dimensional PPP. Second, is the smallest such degree bound the same for SINR graphs as for the underlying $\mathbf{B}$-kNN graph? While the relationship between Gilbert graphs and SINR graphs is clear (namely, Gilbert graphs are increasing limits of SINR graphs as $\gamma \downarrow 0$), we are not aware of results stating that the $\mathbf{B}$-kNN graph is an increasing limit of certain SINR graphs with degree bound k, and such a result may not be true in general. Namely, it may be the case that an SINR constraint of the form (4.3) with degree bound $k \in \mathbb{N}$ poses stronger restrictions on the edges of the graph than a $\mathbf{B}$-kNN constraint for the same k. We defer the investigation of such questions to future work, noting that numerical evidence indicates that the two critical degree bounds are not the same in general, see e.g. Figure 3.

4.3 Extensions and limitations of the proof of Theorem 2.6

We now present examples of graphs with degrees bounded by two that are not contained in an f-2NN graph but have rather similar properties to it, to the extent that the proof techniques of Theorem 2.6 are applicable to it.

Example 4.1 (Locally furthest neighbors)**.** The edge-preserving property of f-kNN graphs (see Definition 3.3) also holds if we replace the "k-nearest neighbors with respect to the f-ordering" in their definition by "k-furthest neighbors in a bounded (possibly random) set shifted to the point, w.r.t. f-ordering". For the sake of simplicity of notation, let us explain how this works in the case of $\mathbf{B}$-kNN graph. The case of general f-kNN graphs can be handled similarly, taking into account also the marks and using the f-ordering instead of the ordering of Euclidean norms. We assume throughout this discussion that the point process X is stationary, nonequidistant, and stable under local thinning.

Let us fix a deterministic measurable set $A \subseteq \mathbb{R}^d$ of finite Lebesgue measure and define a random graph with vertex set $\mathbf{X}$ via connecting two different points X_i, X_j of the point process X by an edge whenever X_j is one of the $k \in \mathbb{N}$ furthest neighbors in $(A + X_i) \setminus \{X_i\}$ and the same holds with the roles of i and j interchanged. It is easy to see that this graph is well-defined and edge-preserving. Clearly, for $k = 2$ it has degrees bounded by two. Hence, non-percolation of the graph can be verified along the lines of the proof of Theorem 2.6 also in the case of this graph. If A is bounded, then the graph has bounded edge lengths (unlike the $\mathbf{B}$-kNN graph).

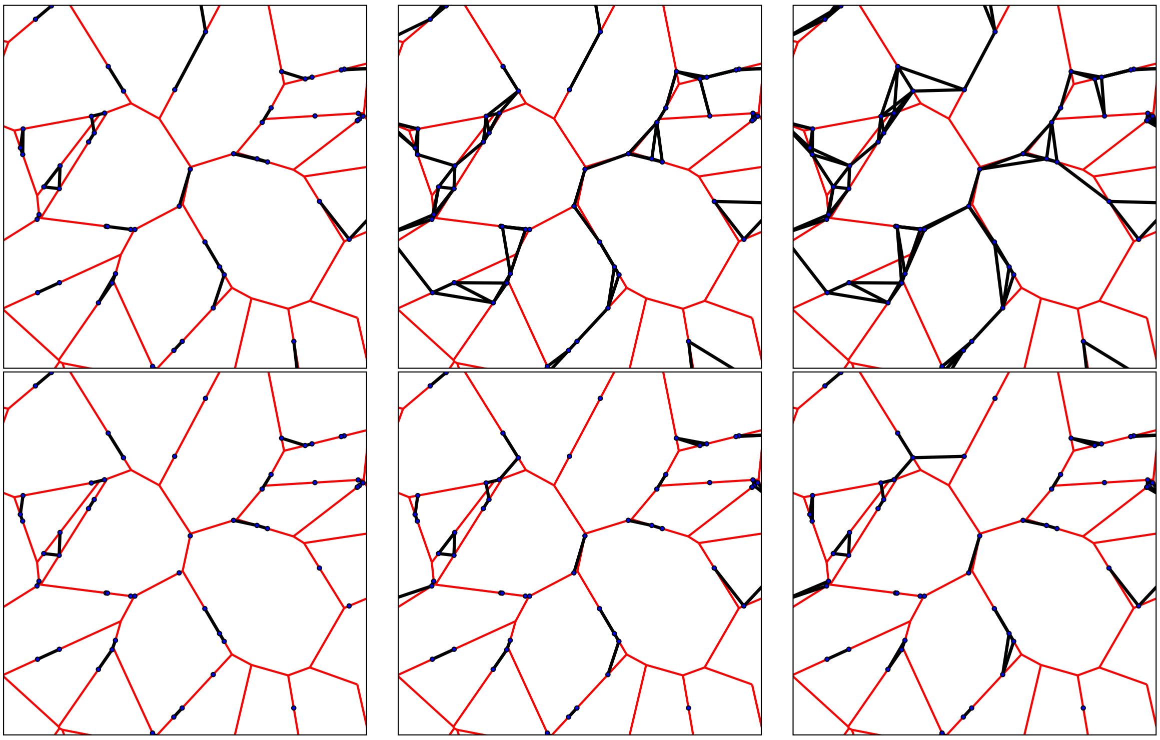

Figure 3: $\mathbf{B}$-kNN graphs (in the first line) and SINR graphs with degree bound k (in corresponding order in the second line) for $k = 2, 4, 5$, for X being a stationary CPP. The random intensity measure Λ is given as the edge-length measure (i.e., the one-dimensional Hausdorff measure) of a two-dimensional Poisson–Voronoi tessellation. The simulation leads to the conjecture that the smallest k such that the $\mathbf{B}$-kNN graph percolates is $k = 5$, which would mean that it equals the one of the two-dimensional PPP (which is 5 according to the high-confidence results of [BB13]). Further, it is known from [T20] that in this case, for large enough λ and accordingly chosen small $\gamma > 0$, there is also percolation in the SINR graph. However, it does not seem to be the case that this already happens when the degree bound equals 5, as the simulation suggests. Here, γ is just slightly bigger than $1/(5\tau)$, i.e., a small further increase of γ would increase the degree bound to 6, but the SINR graph is still much less connected than the corresponding $\mathbf{B}$-$5\mathrm{NN}$ graph.

This approach can be extended to the case when the deterministic set $A + X_i$ is replaced by a random set A_{X_i} in such a way that $\{A_{X_i}\}_{i\in I}$ are stationary, since the edge-preserving property and the degree bound of two are still preserved. E.g., the proof techniques of Theorem 2.6 are still applicable if $\{A_{X_i}\}_{i\in I}$ is a Boolean model with random radii based on $X = \{X_i\}_{i\in I}$ [MR96]. Instead of connecting X_i to the two furthest neighbors in $X \cap A_{X_i}$ by an edge, one can also connect it to the two nearest neighbors in $X \cap A_{X_i}$ and obtain the same result. We refrain from presenting further details here.

The next example shows that there are graphs defined very similarly to the f-2NN graph for which our methods are not applicable.

Example 4.2 (k_1-th and k_2-th nearest neighbors)**.** Let $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$. Similarly to Definition 2.3, the *f-k_1-th or k_2-th-nearest neighbor* (f-(k_1,k_2)*NN*) *graph* $g_{(k_1,k_2),f}(\mathbf{X})$ is defined as the random graph having vertex set X and for all $i \in I$ and $n \in \{1,2\}$ an edge between X_i and $\mathrm{V}^f_n(X_i,\mathbf{X})$ whenever $X_i \in \{\mathrm{V}^f_{k_1}(\mathrm{V}^f_n(X_i,\mathbf{X}),\mathbf{X}),\dots,\mathrm{V}^f_{k_2}(\mathrm{V}^f_n(X_i,\mathbf{X}),\mathbf{X})\}$. In the case $k_1 = 1$ and $k_2 = 2$, $g_{(1,2),f}(\mathbf{X})$ is equal to the f-2NN graph $g_{2,f}(\mathbf{X})$. However, it is easy to see that if $(k_1,k_2) \neq (1,2)$, then the f-(k_1,k_2)NN graph is in general not edge-preserving in the sense of Definition 2.3. This is a major obstacle for generalizing the proof of Theorem 2.6 to the case $(k_1,k_2) \neq (1,2)$, despite the fact that many of the proof ingredients of the theorem are still available in this case.

4.4 Relation of our model to outdegree-one graphs

In the setting of outdegree-one graphs [CDS20], one considers directed percolation in a directed graph arising as a deterministic and stationary function of a PPP in $\mathbb{R}^d$, where each vertex has precisely one out-degree. It was shown in [CDS20] that under certain stabilization and looping conditions of the edge-drawing mechanism, this model does not percolate, in the sense that the out-component (or the in-component) of any vertex is almost-surely finite, see also [H16]. In [S18], it was shown that this result is applicable for the example of the *k-th nearest neighbor graph*, where the outgoing edge of a vertex points to the k-th nearest neighbor of the vertex in the point process. This setting looks rather similar to the one that we are considering but is still different from it, for at least two reasons. First, although it is tempting to think that the $\mathbf{B}$-kNN graph can be obtained as a deterministic transformation of a (stationary) outdegree-one graph satisfying the conditions of [CDS20], we were not able to find such an outdegree-one graph. Second, the k-th nearest neighbor graph is only contained in the $\mathbf{U}$-kNN graph, not in the $\mathbf{B}$-kNN one; in particular, the results of [S18] cannot be derived from our ones.

References

[BB13] P. Balister and B. Bollobás, Percolation in the k-nearest neighbor graphs, In *Recent Results in Designs and Graphs: a Tribute to Lucia Gionfriddo, Quaderni di Matematica*, **28**, Editors: M. Buratti and C. Lindner and F. Mazzocca and N. Melone, 83–100 (2013)

[BY13] B. Blaszczyszyn and D. Yogeshwaran, Clustering and percolation of point processes, *Electron. J. Probab.*, **18**, 1–20 (2013)

[CDS20] D. Coupier, D. Dereudre and S. Le Stum, Absence of percolation for Poisson outdegree-one graphs, *Ann. Inst. Henri Poincaré Probab. Stat.*, **56:2**, 1179–1202 (2020)

[CD14] D. Coupier and D. Dereudre, Continuum percolation for quermass interaction model, *Electron. J. Probab.*, **19:35**, 1–19 (2014)

[D19] D. Dereudre, Introduction to the theory of Gibbs point processes, In *Stochastic Geometry*, Springer, 181–229 (2019)

[DBT05] O. Dousse, F. Baccelli and P. Thiran, Impact of interferences on connectivity in ad hoc networks, *IEEE/ACM Trans. Networking*, **1**, 425–436 (2005)

[DFM+06] O. Dousse, M. Franceschetti, N. Macris, R. Meester, and P. Thiran, Percolation in the signal to interference ratio graph, *J. Appl. Probab.*, **43**, 552–562 (2006)

[GKP16] S. Ghosh, M. Krishnapur and Y. Peres, Continuum percolation for Gaussian zeroes and Ginibre eigenvalues, *Ann. Probab.*, **44:5**, 3357–3384 (2016)

[GP12] S. Ghosh and Y. Peres, Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues, *Duke Math. J.*, **166:10**, 1789–1858 (2017)

[G61] E.N. Gilbert, Random plane networks, *J. Soc. Indust. Appl. Math.*, **9**, 533–543 (1961)

[H16] C. Hirsch, On the absence of percolation in a line-segment based lilypond model, *Ann. Inst. Henri Poincaré Probab. Stat.*, **52:1**, 127–145 (2016)

[HJC19] C. Hirsch, B. Jahnel and E. Cali, Continuum percolation for Cox point processes, *Stoch. Process. Their Appl.*, **129:10**, 3941–3966 (2019)

[HM96] O. Häggström and R. Meester, Nearest neighbor and hard sphere models in continuum percolation, *Random Structures and Algorithms*, **9**, 295-–315 (1996)

[JT19] B. Jahnel and A. Tóbiás, SINR percolation for Cox point processes with random powers, *arXiv:1912.07895* (2019)

[JTC20] B. Jahnel, A. Tóbiás and E. Cali, Phase transitions for the Boolean model of continuum percolation for Cox point processes, *arXiv:2003.06206* (2020)

[J16] S. Jansen, Continuum percolation for Gibbsian point processes with attractive interactions, *Electron. J. Probab.*, **21:47**, 1–22 (2016)

[K93] J.F.C. Kingman, *Poisson Processes*, Oxford University Press, New York (1993)

[L19] R. Löffler, Percolation phase transitions for the SIR model with random powers, Master's thesis. TU Berlin Institute of Mathematics, *arXiv:1908.07375* (2019)

[LP17] G. Last and M. Penrose, *Lectures on the Poisson Process*, Cambridge University Press (2017)

[MR96] R. Meester and R. Roy, *Continuum Percolation*, Cambridge University Press (1996)

[S13] K. Stucki, Continuum percolation for Gibbs point processes, *Electron. Commun. Probab.*, **18:67**, 1–10 (2013)

[S18] S. Le Stum, Deterministic walks on Poisson point process, *ESAIM: Proceedings and Surveys*, **60**, editors: J-F. Coeurjolly and A. Leclercq-Samson, 266–275 (2018)

[T19] A. TÓBIÁS, *Message Routeing and Percolation in Interference Limited Multi-hop Networks*. Doctoral thesis. TU Berlin Institute of Mathematics. Available at `https://depositonce.tu-berlin.de/handle/11303/9293` (2019)

[T20] A. TÓBIÁS, Signal to interference ratio percolation for Cox point processes, *Lat. Am. J. Probab. Math. Stat.*, **17**, 273–308 (2020)

[TY07] S.H. TENG and F. YAO, k-nearest-neighbor clustering and percolation theory, *Algorithmica*, **49**, 192–211 (2007)