Persistence of a two-dimensional super-Brownian motion in a catalytic medium

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Abstract

The super-Brownian motion X^ϱ in a super-Brownian medium ϱ constructed in [DF96a] is known to be persistent (no loss of expected mass in the longtime behaviour) in dimensions one ([DF96a]) and three ([DF96b]). Here we fill the gap in showing that persistence holds also in the critical dimension two. The key to this result is that in any dimension $(d \leq 3)$, given the catalyst, the variance of the process is finite 'uniformly in time'. This is in contrast to the 'classical' super-Brownian motion where this holds only in high dimensions $(d \geq 3)$, whereas in low dimensions the variances grow without bound, and the process clusters leading to local extinction.

Contents

1 Introduction and result

We are concerned with the longtime behaviour of *'super-Brownian motion X^ϱ in a super-Brownian medium ϱ'* in dimension two. Both the process X^ϱ and the catalytic medium ϱ will be started from Lebesgue measure ℓ at time zero. This process was constructed in [DF96a] where the study of its longtime behaviour was also initiated. This was supplemented in [DF96b]. From these papers it is well known that the process is *persistent* in dimensions one and three. (In three dimensions the catalyst was actually started from its stationary distribution rather than from Lebesgue measure at time zero; of course, this simplification is not possible in lower dimensions where ϱ clusters in the longtime limit.)

In this note we are mainly interested in the critical dimension two. Here matters are more delicate, but we show that, in contrast to what one might expect at first sight, the process is *persistent* (Theorem 1 at p.4). In particular, it does not suffer local extinction.

We begin with a *heuristic description* of the catalytic super-Brownian motion (SBM). (A formal characterisation will be given in Section 2.) Recall that 'classical' SBM motion in R^d arises as a diffusion approximation to critical binary branching Brownian motion (infinite particle model) and so can loosely be thought of as a large number of small particles, each moving around according to an independent Brownian motion for a short (independent) random lifetime at the end of which it dies and is replaced (at the location where it died) by zero or two offspring with equal probability. Offspring continue to evolve in the same way as their parent. In this classical setting, the rate at which a particle dies is proportional to some constant $\gamma > 0$ called the *branching rate.*

In the catalytic setting, the heuristic picture is the same except that particles only die when they are in contact with a *catalyst.* This is a natural model if one is thinking of *chemical reaction diffusion systems.* Of course the catalyst itself may vary in time and space and may only be present in some localised regions such as networks of filaments. We are interested in the case where the catalyst is itself a SBM with constant branching rate $\gamma > 0$. We will denote the catalyst process by ϱ_t and the corresponding catalytic SBM by X_t^ϱ at time t. As a rule, we will take both ϱ_0 and X_0^ϱ to be Lebesgue measure ℓ.

Somewhat more formally then, if one thinks in terms of Dynkin's additive functional approach to superprocesses ([Dyn91]), given the medium ϱ, an intrinsic X-particle following a Brownian path W branches according to the clock given by the *collision local time,* $L_{[W,\varrho]}(\mathrm{d}s)$, of W with ρ ([BEP91]),

$$L_{[W,\varrho]}(\mathrm{d}s) \;:=\; \Big(\int \delta_y(W_s)\,\varrho_s(\mathrm{d}y)\Big)\,\mathrm{d}s. \tag{1}$$

Although the measures $\varrho_s(\mathrm{d}y)$ are singular in dimensions $d \geq 2$ ([DH79]), these collision local times $L_{[W,\varrho]}$ make sense non-trivially for $d \leq 3$ ([EP94]). For this reason, in these dimensions the *catalytic SBM* X^ϱ could successfully be constructed in [DF96a] as a continuous measure-valued (time-inhomogeneous) Markov process $(X^\varrho, P^\varrho_{r,\mu})$, given the *catalyst process* ϱ *(quenched approach).* In particular, $P^\varrho_{0,\ell}$ denotes the law of the process X^ϱ (for fixed ϱ)

started at time zero from Lebesgue measure ℓ. The law of the catalyst ϱ is denoted by $\mathbb{P}_\ell$.

In [DF96a] the longtime behaviour of X_t^ϱ was studied in one dimension. In contrast to the classical SBM, the one-dimensional process X^ϱ is *persistent.* Indeed

$$X_t^\varrho \underset{t\uparrow\infty}{\longrightarrow} \ell, \quad \text{in } P_{0,\ell}^\varrho\text{–probability}, \quad \text{for } \mathbb{P}_\ell\text{–almost all } \varrho, \tag{2}$$

([DF96a, Theorem 51]). That is, we have convergence *without* loss of expected mass. The intuitive explanation for this behaviour of X^ϱ is that at large times the one-dimensional catalyst process ϱ forms huge clusters that move out to infinity ([DF88]). In particular, any finite window in R eventually becomes empty, and the surviving (recurrent) Brownian X–particles moving in that window will not die (branch) since they do not meet any catalyst. An averaging effect then leads to the expected mass ℓ in the limit.

A persistence result holds also in dimension $d = 3$ ([DF96b]). Let $\mathbb{P}$ denote the law of the *time-stationary* catalyst process ϱ started at time 0 from its ergodic stationary distribution of uniform intensity. (That is the initial law is taken to be that of $\varrho_\infty := \lim_{t\uparrow\infty} \varrho_t$ with $\varrho_0 = \ell$.) Then again $(X^\varrho, P_{r,\mu}^\varrho)$ exists for $\mathbb{P}$–almost all ϱ, and we may consider $X = X^\varrho$ with respect to the *annealed* law $\mathcal{P} := \mathbb{P}\left[P_{0,\ell}^\varrho\right] = \int P_{0,\ell}^\varrho\{\cdot\}\mathbb{P}(\mathrm{d}\varrho)$. Then ([DF96b, Theorem 18 (a)])

$$X_t \underset{t\uparrow\infty}{\longrightarrow} X_\infty \quad \text{in } \mathcal{P}\text{–law, where } X_\infty \text{ has full intensity } \ell. \tag{3}$$

Here the intuitive explanation is that since the 'branching rate' ϱ is space-time stationary, it controls the transient Brownian X–particles in a similar way to a constant branching rate and so, as in the classical three-dimensional setting, we should expect persistence. (However, as conjectured in [DF96b], X_∞ should differ from the classical steady state.)

There was some dispute at the meeting on branching processes in Oberwolfach in December 1995, as to whether in dimension $d = 2$ the catalytic SBM X^ϱ dies out locally or not. (See also the open problem in [DF96b, Remark 14].) The situation is delicate in that although the catalyst ϱ dies out locally in the longtime limit, in contrast to $d = 1$ it does so only in probability. Indeed the time averaged ϱ has a nondegenerate limit (see e.g. [FG86]).

In particular, there is *not* a finite time after which a given ball becomes and remains empty. Consequently, at late times T, huge clusters of the catalyst (whose height is of order $\log T$, see [Fle78]) come back to our finite window in R^2 and (since critical binary branching with infinite rate degenerates to pure killing) could kill all the (recurrent) X–particles there.

The *main result* of the present paper, Theorem 1 below, contradicts this intuitive picture. (A somehow detailed heuristic argument for persistence will be given in § 3.1 below.)

For a precise formulation we introduce a number of laws of random measures on R^d, $d \leq 3$. Note that we reserve the letter Q for laws of states at a fixed time (including $t = \infty$), whereas P refers to process laws. For $t \geq 0$ fixed, the *quenched* laws

$$Q_t^\varrho \;:=\; P_{0,\ell}^\varrho \{X_t^\varrho \in (\cdot)\} \tag{4}$$

make sense for $\mathbb{P}_\ell$–almost all ϱ. So Q_t^ϱ is a *random* law whose distribution we denote by

$$\mathbf{Q}_t \;:=\; \mathbb{P}_\ell \{Q_t^\varrho \in (\cdot)\}\,. \tag{5}$$

On the other hand, integrating (4) with respect to $\mathbb{P}_\ell$, we obtain the *annealed* law

$$\mathcal{Q}_t \;:=\; \mathbb{P}_\ell\,[Q_t^\varrho] \;=\; \int Q_t^\varrho\{\cdot\}\,\mathbb{P}_\ell(\mathrm{d}\varrho). \tag{6}$$

In words, Q_t^ϱ is the law of the state of the catalytic SBM at time t, given the medium ϱ, whereas $\mathbf{Q}_t$ describes the random Q_t^ϱ, and finally $\mathcal{Q}_t$ results by mixing the Q_t^ϱ. Note that (5) and (6) imply that $\mathcal{Q}_t$ is the expectation of $\mathbf{Q}_t$:

$$\int V(\cdot)\,\mathbf{Q}_t(\mathrm{d}V) \;=\; \mathcal{Q}_t(\cdot). \tag{7}$$

Theorem 1 (persistence of catalytic SBM) *Suppose* $d \leq 3$.

(a) (annealed model) *Each limit point* $\mathcal{Q}$ *of the annealed laws* $\mathcal{Q}_t$ *as* $t \uparrow \infty$ *has full intensity:*

$$\int \langle f, \chi \rangle\, \mathcal{Q}(\mathrm{d}\chi) \;=\; \langle f, \ell \rangle\,, \qquad f \in \mathcal{B}_+^p, \tag{8}$$

that is, $\int \chi(\cdot)\,\mathcal{Q}(\mathrm{d}\chi) = \ell(\cdot)$.

(b) (quenched model) *Each limit point* $\mathbf{Q}$ *of* $\mathbf{Q}_t$ *as* $t \uparrow \infty$ *is concentrated on laws* V *of random measures* χ *on* R^d *with full intensity:*

$$\mathbf{Q}\left\{V : \int \chi(\cdot)V(\mathrm{d}\chi) = \ell(\cdot)\right\} = 1. \tag{9}$$

As already mentioned, in view of the results of [DF96a] and [DF96b] the principal interest of this theorem is the case $d = 2$. Note also that in contrast to classical SBM in dimensions $d \leq 2$,

$$X_t^{\varrho}(B) \underset{t\uparrow\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{Q}_t\text{-law is } \textit{violated}, \tag{10}$$

for any ball B in R^d of (strictly) positive radius (*non-extinction*).

Remark 2 (open problems)
(i) (convergence) Of course, the existence of limit points follows from the *relative compactness* of the $\mathcal{Q}_t$ and $\mathbf{Q}_t$ in the corresponding weak topologies (see step 1° of the proof in § 3.3 below). It would be interesting to know whether the limit points are unique also in dimension two, that is whether we have *convergence* as $t \uparrow \infty$. Since the process will encounter catalyst at arbitrarily large times (see the discussion before Theorem 1), we cannot expect the analogue of the one-dimensional result to hold. Thus it seems reasonable to use the weak notions of convergence indicated in the Theorem.
(ii) (absolutely continuous states) Recall also that by self-similarity in dimension two (see (24) below), a *persistent convergence* $\mathcal{Q}_t \to \mathcal{Q}$ as $t \uparrow \infty$ is related to an *absolute continuity* of the measures X_t^{ϱ} with annealed law $\mathcal{Q}_t$, for each fixed t ([DF96b, Remark 14]). ◇

Recall that for *classical* SBM we see the following *dichotomy* of behaviour. In dimensions one and two, the *variance* of the amount of mass in a ball B of positive radius grows unboundedly with time, reflecting the *clumping* of the process, and the process suffers *local extinction.* On the other hand in dimensions three (and above), the variance of the amount of mass in B remains bounded and the process survives, and is actually *persistent.*

In the *catalytic* SBM the situation changes drastically. Here even in low dimensions the variances, given the medium (catalyst), *remain finite* as the process evolves (see Theorem 3 at p.11 below). This is our *key result,* valid in any dimension $d \leq 3$, from which it is easy to conclude persistence.

The rest of the note is laid out as follows. In Section 2 we recall the formal characterisation of the catalytic SBM X^{ϱ}. In Section 3 we provide a rigorous (if intuitively unhelpful) proof, which we preface in § 3.1 by a heuristic argument in the critical dimension two which perhaps better explains why persistence in the case $d = 2$ is true (even if the formal proof is simpler in a sense). § 3.2 contains the finiteness of variances result with a surprisingly straightforward proof. In the final § 3.3 the persistence proof is completed.

For a comprehensive reference on SBM we recommend [Daw93].

2 Notation and background

In this section we recall the formal characterisation of the catalytic SBM in terms of its Laplace transition functional and also recall expressions for the mean and variance of the integral of the process against a test function.

Fix a number $p > d$, and introduce the reference function

$$\phi_p(x) := \frac{1}{(1+|x|^2)^{p/2}}, \qquad x \in \mathsf{R}^d. \tag{11}$$

Write $\mathcal{B}_+^p$ for the set of all functions f on R^d such that $0 \le f \le c_f \phi_p$ for some (finite) constant c_f. Let $\mathcal{M}_p$ denote the set of all (non-negative) measures μ defined on R^d such that $\langle \phi_p, \mu \rangle := \int \phi_p(x)\,\mu(\mathrm{d}x) < \infty$ (*p–tempered measures*). $\mathcal{M}_p$ is endowed with the coarsest topology such that the map $\mu \mapsto \langle f, \mu \rangle$ is continuous for $f = \phi_p$ and for each continuous $f \ge 0$ on R^d with compact support.

Fix a constant $\gamma > 0$. The *catalyst process ϱ with rate γ* is a continuous $\mathcal{M}_p$–valued time-homogeneous Markov process $(\varrho, \mathbb{P}_\mu)$ with Laplace transition functional

$$\mathbb{P}_\mu \exp \langle -f, \varrho_t \rangle = \exp \langle -u(t), \mu \rangle, \qquad t \ge 0, \quad \mu \in \mathcal{M}_p, \quad f \in \mathcal{B}_+^p. \tag{12}$$

Here[1] $u = \{u(t) : t \ge 0\} = \left\{u(t,x) : t \ge 0, \ x \in \mathsf{R}^d\right\}$ is the unique non-negative solution to the *basic cumulant equation*

$$\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u - \gamma u^2 \quad \text{on } (0,\infty) \times \mathsf{R}^d \tag{13}$$

[1] Where the meaning is clear, we suppress dependence on the space variable and write simply $u(t)$.

with *initial* condition $u(0,x) = f(x),\ x \in \mathsf{R}^d$. (Where necessary, 'solution' has to be understood in a *mild* sense.) In other words, ϱ is a continuous (critical) SBM with constant branching rate γ. It will serve as our *random medium (catalyst).*

To characterise the *catalytic* SBM, roughly speaking, we have to replace the constant rate γ in (13) by the (randomly) varying rate $\varrho_t(x)$, where $\varrho_t(x)$ is the *generalised* derivative $\frac{\varrho_t(\mathrm{d}x)}{\mathrm{d}x}(x)$ of the measure $\varrho_t(\mathrm{d}x)$. Because of the time-inhomogeneity, it is convenient to write the related formal equation in a *backward setting:*

$$-\frac{\partial}{\partial r}\, v_t^\varrho(r,x) \;=\; \frac{1}{2}\,\Delta v_t^\varrho(r,x) - \varrho_r(x)\, v_t^\varrho(r,x)^2, \tag{14}$$

$0 \le r \le t,\ \ x \in \mathsf{R}^d$. The initial condition becomes a *terminal* condition: $v_t^\varrho(t) = f$. After a formal integration, we can rewrite (14) probabilistically as

$$v_t^\varrho(r,x) \;=\; \Pi_{r,x}\left[f(W_t) - \gamma \int_r^t v_t^\varrho(s,W_s)^2\, L_{[W,\varrho]}(\mathrm{d}s)\right], \tag{15}$$

$0 \le r \le t,\ \ x \in \mathsf{R}^d$, where $\Pi_{r,x}$ is the law of (standard) Brownian motion W starting at time r from x, and $L_{[W,\varrho]}$ is the *collision local time* of W with ϱ, formally introduced in (1). Based on the finite measure case [EP94], in [DF96a, Theorem 42] it was shown that this collision local time $L_{[W,\varrho]}$ makes sense non-trivially for $\mathbb{P}_\ell$–almost all ϱ, as a continuous additive functional of Brownian motion W, provided that $d \le 3$. *From now on we assume* $d \le 3$. Moreover ([DF96a, Proposition 6]), for t, f fixed, and $\mathbb{P}_\ell$–a.a. ϱ, there is a unique non-negative solution v_t^ϱ to (15). Finally ([DF96a, § 5.4]), for $\mathbb{P}_\ell$–almost all ϱ, there exists a continuous $\mathcal{M}_p$–valued time-inhomogeneous Markov process $(X^\varrho, P_{r,\mu}^\varrho)$ with Laplace transition functional

$$P_{r,\mu}^\varrho \exp\langle -f, X_t^\varrho\rangle \;=\; \exp\langle -v_t^\varrho(r), \mu\rangle\,, \tag{16}$$

$0 \le r \le t,\ \ \mu \in \mathcal{M}_p,\ \ f \in \mathcal{B}_+^p$, and v_t^ϱ the solution to (15). This is the desired *catalytic SBM,* with catalyst ϱ, which was intuitively introduced in Section 1.

By the criticality of the branching mechanism, X^ϱ has *expectation*

$$P_{r,\mu}^\varrho\,[\langle f, X_t^\varrho\rangle] \;=\; \langle S_{t-r} f, \mu\rangle \tag{17}$$

which is independent of the medium ϱ. Here $S = \{S_t : t \geq 0\}$ is the semigroup of Brownian motion. In particular,

$$P^{\varrho}_{0,\ell}\,[X^{\varrho}_t] \;\equiv\; \ell, \qquad \mathsf{IP}_{\ell}\text{--a.s.}, \tag{18}$$

is even independent of time. For the related *variances* (given ϱ), following [DF96a] we have

$$\mathrm{Var}^{\varrho}_{0,\ell}\,[\langle f, X^{\varrho}_t\rangle] \;=\; 2\int \Pi_{0,x}\left[\int_0^t \left(\left(S_{t-s}f\right)(W_s)\right)^2 L_{[W,\varrho]}(\mathrm{d}s)\right]\ell(\mathrm{d}x). \tag{19}$$

Using the (intuitive) definition (1) of the collision local time $L_{[W,\varrho]}$, and computing the $\Pi_{0,x}$–expectation, the r.h.s. of (19) becomes

$$2\int\int_0^t\int \mathrm{p}(s, y-x)\left(S_{t-s}f(y)\right)^2 \varrho_s(\mathrm{d}y)\,\mathrm{d}s\;\ell(\mathrm{d}x).$$

Here p denotes the Brownian transition density

$$\mathrm{p}(t,x) = (2\pi t)^{-d/2}\exp\left(\frac{-|x|^2}{2t}\right), \qquad t>0, \quad x\in \mathsf{R}^d. \tag{20}$$

Interchanging the order of integration gives, for IP_{ℓ}–almost all paths ϱ,

$$\mathrm{Var}^{\varrho}_{0,\ell}\,[\langle f, X^{\varrho}_t\rangle] \;=\; 2\int_0^t \left\langle \left(S_{t-s}f\right)^2, \varrho_s\right\rangle \mathrm{d}s, \tag{21}$$

$t\geq 0$, $f\in\mathcal{B}^p_+$ ([DF96a, formula (95)]).

We shall also require an expression for the *Laplace transform of this variance.* Using (21), an application of Iscoe's [Isc86] characterisation of the 'weighted occupation time' for classical SBM now gives

$$\mathsf{IP}_{\ell}\exp\left(-\mathrm{Var}^{\varrho}_{0,\ell}\,[\langle f, X^{\varrho}_t\rangle]\right) \;=\; \exp\left\langle -w_f(t), \ell\right\rangle, \qquad f\in\mathcal{B}^p_+, \tag{22}$$

where the log-Laplace functional w_f is the unique non-negative solution to[2]

$$\frac{\partial}{\partial t}w_f \;=\; \frac{1}{2}\,\Delta w_f - \gamma\, w_f^2 + 2\,[S_t f]^2 \quad \text{ on } (0,\infty)\times\mathsf{R}^d \tag{23}$$

[2]By an abuse of notation, we retain the t–dependence only where it requires emphasis.

with initial condition $w_f(0,x) \equiv 0$. Although this equation looks a bit complicated at first sight, in step 1° of proof of Theorem 3 below we will show that w_f has a very simple bound which implies all we need regarding the L^1–behaviour of solutions as $t \uparrow \infty$.

Finally, recall that in the critical dimension $d = 2$ the catalytic SBM $X = X^\varrho$ is *self-similar* with respect to the *annealed* law $\mathcal{P}_\ell := \mathbb{P}_\ell \left[P^\varrho_{0,\ell} \right]$:

$$K^{-1} X_{Kt}(K^{1/2}\cdot\,) \overset{\mathcal{L}}{=} X_t(\cdot), \qquad t \geq 0, \quad K > 0, \tag{24}$$

([DF96b, Proposition 13 (b)]).

3 Proof of the persistence theorem

Before presenting a rigorous proof of Theorem 1, we give a heuristic argument for persistence in the critical dimension $d = 2$.

3.1 Heuristic argument $(d = 2)$

Here we restrict our attention to the critical dimension two, where the persistence result is completely new.

Consider the case $f = \mathrm{p}(a)$ with $a := 2\mathrm{e}^4$. Then $S_r f = \mathrm{p}(a + r)$, and using the identity

$$\mathrm{p}(r)^2 = \frac{1}{4\pi r}\,\mathrm{p}\,(r/2)\,, \qquad r > 0, \tag{25}$$

the corresponding (random) variance (recall (21)) becomes

$$\mathcal{V}\mathrm{ar}^\varrho_{0,\ell}\left[\langle \mathrm{p}(a), X^\varrho_t\rangle\right] \;=\; \frac{1}{2\pi}\int_0^t \frac{1}{a+t-s}\left\langle \mathrm{p}\left(\frac{a+t-s}{2}\right),\, \varrho_s\right\rangle \mathrm{d}s. \tag{26}$$

We can view the r.h.s. of (26) as

$$\frac{1}{2\pi}\,\log(a+t)\int_0^t \left\langle \mathrm{p}\left(\frac{a+t-s}{2}\right), \varrho_s\right\rangle \sigma_t(\mathrm{d}s). \tag{27}$$

Here σ_t is the following (probability) law on $[0,t]$:

$$\frac{1}{\log(a+t)}\int_0^t \frac{1}{a+t-s}\,\delta_s(\cdot)\,\mathrm{d}s \;=\; \int_0^1 \frac{1}{\log(a+t)\left(\frac{a}{t}+1-s\right)}\,\delta_{ts}(\cdot)\,\mathrm{d}s \;\approx\; \delta_t$$

as $t \uparrow \infty$. Hence, (27) should be approximately

$$\approx \frac{1}{2\pi} \log(a+t) \langle \mathrm{p}(a/2), \varrho_t \rangle .$$

But $\langle \mathrm{p}(a/2), \varrho_t \rangle$ suffers extinction in $\mathbb{P}_\ell$–probability as $t \uparrow \infty$, and should in fact be of order $1/\log t$, compensating $\log(a+t)$. More precisely, from (12) we obtain

$$\mathbb{P}_\ell \exp \Big\langle -\varepsilon \log(a+t)\, \mathrm{p}\Big(\tfrac{a}{2}\Big), \varrho_t \Big\rangle \;=\; \exp \langle -\mathrm{u}^\varepsilon(t), \ell \rangle, \qquad \varepsilon \geq 0,$$

with u^ε solving (13) but this time with initial condition

$$u^\varepsilon(0) \;=\; \varepsilon \log(a+t) \mathrm{p}\Big(\tfrac{a}{2}\Big)$$

(depending on t). Using well-known super-solutions to (13) (see e.g. [BCG93, Lemma 1]), one easily verifies that

$$\limsup_{t\uparrow\infty} \langle u^\varepsilon(t), \ell \rangle \;=\; O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

Hence, the $\mathbb{P}_\ell$–laws of the variances $P^\varrho_{0,\ell} \langle \mathrm{p}(a), X^\varrho_t \rangle$ should be *relatively compact,* which indicates that $\langle \mathrm{p}(a), X^\varrho_t \rangle$, with respect to the annealed law $\mathcal{P}_\ell$ (defined before (24)) should be *persistent.*

Theorem 3 below is a *rigorous version* of the above. Notice however that things are rather delicate. Indeed, $\mathbb{P}_\ell [\varrho_s] \equiv \ell$, and so for the $\mathbb{P}_\ell$–expectation of the $P^\varrho_{0,\ell}$–variance of $\langle \mathrm{p}(a), X^\varrho_t \rangle$, from (26) we obtain

$$\mathbb{P}_\ell \Big[\mathrm{Var}^\varrho_{0,\ell} [\langle \mathrm{p}(a), X^\varrho_t \rangle] \Big] \;=\; \frac{1}{2\pi} \int_0^t \frac{1}{a+t-s}\, \mathrm{d}s \;=\; \frac{1}{2\pi} \log(a+t) \underset{t\uparrow\infty}{\longrightarrow} \infty.$$

Consequently, since the expectation (18) is independent of ϱ, the $\mathcal{P}_\ell$–variance of $\langle \mathrm{p}(a), X^\varrho_t \rangle$ grows without bound as $t \uparrow \infty$, even though the random variance remains finite (in $\mathbb{P}_\ell$–law, according to Theorem 3 below).

3.2 Finiteness of variances ($d \leq 3$)

Here is our key result on the variances:

Theorem 3 (relative compactness of the law of variances) *In any dimension $d \le 3$, for $f \in \mathcal{B}_+^p$ fixed, the laws*

$$q_t(\cdot) \;:=\; \mathbb{P}_\ell \left\{ \mathrm{Var}^\varrho_{0,\ell}\left[\langle f, X_t^\varrho \rangle\right] \in (\cdot) \right\}, \qquad t \ge 0,$$

of the random variances $\mathrm{Var}^\varrho_{0,\ell}\left[\langle f, X_t^\varrho \rangle\right]$ of $\langle f, X_t^\varrho \rangle$ form a relatively compact family.

Proof Set $\zeta_t^\varrho(f) := \mathrm{Var}^\varrho_{0,\ell}\left[\langle f, X_t^\varrho \rangle\right]$. Recall that the Laplace functional of ζ_t^ϱ is given by (22) and is determined by the unique solution $w_f \ge 0$ of equation (23).

1° (*domination of the log-Laplace functional*) We claim that w_f is *dominated* by the heat flow

$$\overline{w}_f(t) \;:=\; \sqrt{2/\gamma}\, S_t f, \qquad t \ge 0.$$

In fact, $\overline{w}_f$ solves the heat equation

$$\frac{\partial}{\partial t} \overline{w}_f \;=\; \frac{1}{2} \Delta \overline{w}_f \quad \text{with} \quad \overline{w}_f(0) = \sqrt{2/\gamma}\, f.$$

Subtracting and adding again $\gamma\, \overline{w}_f^2$ on the r.h.s. of this equation shows that $\overline{w}_f$ solves (23). But since $\overline{w}_f(0) \ge 0 = w_f(0)$, by uniqueness of solutions and monotonicity in the initial data, the claim follows.

2° (*relative compactness of the laws of the variances*) Let $\varepsilon > 0$. Then

$$\mathbb{P}_\ell \left\{ \zeta_t^\varrho(f) \ge \frac{1}{\varepsilon^2} \right\} \;=\; \mathbb{P}_\ell \left\{ 1 - \exp\left(-\varepsilon^2\, \zeta_t^\varrho(f) \right) \ge 1 - \frac{1}{\mathrm{e}} \right\}.$$

Writing $c := \left(1 - \mathrm{e}^{-1}\right)^{-1}$, the r.h.s. can be estimated from above by

$$c\, \mathbb{P}_\ell \left[1 - \exp\left(-\varepsilon^2\, \zeta_t^\varrho(f) \right) \right] \;=\; c \left(1 - \exp\left\langle -\, w_{\varepsilon f}(t), \ell \right\rangle \right) \;\le\; c \left\langle w_{\varepsilon f}(t), \ell \right\rangle$$

where we used the Laplace functional representation (22) with f replaced by εf. But by step 1°, the latter term can be dominated by

$$c \left\langle \overline{w}_{\varepsilon f}(t), \ell \right\rangle \;=\; c \sqrt{2/\gamma} \left\langle S_t\, (\varepsilon f)\,, \ell \right\rangle \;=\; \varepsilon\, c \sqrt{2/\gamma}\; \langle f, \ell \rangle$$

which is independent of t and converges to 0 as $\varepsilon \downarrow 0$. This implies the relative compactness of the family $\{ q_t : t \ge 0 \}$, finishing the proof. ■

3.3 Completion of the persistence proof $(d \leq 3)$

Now we turn to the proof of our main result, Theorem 1 from p.4. Recall the notations (4)–(6).

1° (*existence of weak limit points*) By the expectation formula (18),

$$\int \chi(\cdot)\, Q_t^{\varrho}(\mathrm{d}\chi) \;\equiv\; \ell(\cdot), \quad \text{for } \mathbb{P}_\ell\text{−almost all } \varrho, \tag{28}$$

which after integration with $\mathbb{P}_\ell$ gives $\int \chi(\cdot) \mathcal{Q}_t(\mathrm{d}\chi) \equiv \ell(\cdot)$. Therefore, the subset $\{\mathcal{Q}_t : t \geq 0\}$ of $\mathcal{M}_1(\mathcal{M}_p)$ (the set of probability laws on $\mathcal{M}_p$ endowed with the topology of weak convergence) is *relatively compact* (see, e.g. [Daw92, Lemma 3.2.8]). On the other hand, $\{\mathbf{Q}_t : t \geq 0\}$ belongs to the *compact* space $\mathcal{M}_1(\mathcal{M}_1(\mathcal{M}_p))$ (again with the weak topology). Hence, the sets of limit points $\mathcal{Q}$ and $\mathbf{Q}$ of the $\mathcal{Q}_t$ and $\mathbf{Q}_t$, respectively, are *not empty.* We have to show that they satisfy the claims in **(a)** and **(b)**, respectively, of the theorem.

2° ((**b**) *implies* (**a**)) Let $\mathcal{Q}$ be any (weak) limit point of the $\mathcal{Q}_t$. Then we may choose a sequence $t_n \uparrow \infty$ so that for some (weak) limit point $\mathbf{Q}$ of the $\mathbf{Q}_t$

$$\mathcal{Q}_{t_n} \to \mathcal{Q} \quad \text{and} \quad \mathbf{Q}_{t_n} \to \mathbf{Q}, \quad \text{as } n \uparrow \infty. \tag{29}$$

Combined with $\int\int F(\chi)\, V(\mathrm{d}\chi)\, \mathbf{Q}_t(\mathrm{d}V) = \int F(\chi)\, \mathcal{Q}_t(\mathrm{d}\chi)$, for each bounded and continuous $F : \mathcal{M}_p \to \mathsf{R}_+$, (which holds by (7)), the limit statements (29) imply the identity

$$\int V(\cdot)\, \mathbf{Q}(\mathrm{d}V) = \mathcal{Q}(\cdot). \tag{30}$$

Then by (30) and **(b)**,

$$\int \chi(\cdot)\, \mathcal{Q}(\mathrm{d}\chi) \;=\; \int\int \chi(\cdot)\, V(\mathrm{d}\chi)\, \mathbf{Q}(\mathrm{d}V) \;=\; \int \ell(\cdot)\, \mathbf{Q}(\mathrm{d}V) \;=\; \ell(\cdot),$$

that is, we get the claim **(a)** of the theorem. Thus *it suffices to prove* **(b)**.

3° (*upper bound* ℓ) Let $\mathbf{Q}$ be any limit point of the $\mathbf{Q}_t$. Choose again a sequence $t_n \uparrow \infty$ such that the second convergence statement in (29) holds. We begin by proving that $\mathbf{Q}$–a.s. all intensity measures are bounded by ℓ :

$$\mathbf{Q}\left\{V : \int \chi(\cdot)\, V(\mathrm{d}\chi) \;\leq\; \ell(\cdot)\right\} \;=\; 1. \tag{31}$$

Fix $\varepsilon > 0$ and let $\varphi \geq 0$ denote a continuous function on R^d with compact support. Assume for the moment that

$$\mathbf{Q}\left\{V: \int \langle\varphi,\chi\rangle\, V(\mathrm{d}\chi) \;>\; \langle\varphi,\ell\rangle + 2\varepsilon\right\} \;=:\; 2\delta \;>\; 0. \tag{32}$$

Then there exists a number $K > 0$ such that

$$\mathbf{Q}\left\{V: \int (\,\langle\varphi,\chi\rangle \wedge K)\, V(\mathrm{d}\chi) \;>\; \langle\varphi,\ell\rangle + \varepsilon\right\} \;\geq\; 2\delta \;>\; 0. \tag{33}$$

The function $\chi \mapsto \langle\varphi,\chi\rangle \wedge K$ is bounded and continuous, hence the map $V \mapsto \int (\,\langle\varphi,\chi\rangle \wedge K)\, V(\mathrm{d}\chi)$ is bounded and continuous too. Thus the set of V in (33) is an *open* subset of $\mathcal{M}_1(\mathcal{M}_p)$. Then from the assumed second weak convergence statement in (29) we conclude

$$\mathbf{Q}_{t_n}\left\{V: \int (\,\langle\varphi,\chi\rangle \wedge K)\, V(\mathrm{d}\chi) \;>\; \langle\varphi,\ell\rangle + \varepsilon\right\} \;\geq\; \delta \;>\; 0,$$

for some sufficiently large n. Hence

$$\mathbf{Q}_{t_n}\left\{V: \int \langle\varphi,\chi\rangle\, V(\mathrm{d}\chi) \;>\; \langle\varphi,\ell\rangle + \varepsilon\right\} \;\geq\; \delta \;>\; 0.$$

But this contradicts

$$\mathbf{Q}_{t_n}\left\{V: \int \langle\varphi,\chi\rangle\, V(\mathrm{d}\chi) \;\not\equiv\; \langle\varphi,\ell\rangle\right\} \;=\; 0$$

which holds by (28). Consequently, for all considered ε and φ, the l.h.s. of formula line (32) vanishes. Hence,

$$\mathbf{Q}\left\{V: \int \langle\varphi,\chi\rangle\, V(\mathrm{d}\chi) \;>\; \langle\varphi,\ell\rangle\right\} \;=\; 0,$$

and a separability argument yields (31).

It remains to show that

$$\int \chi(\cdot)\, V(\mathrm{d}\chi) \;\geq\; \ell(\cdot) \quad \text{for } \mathbf{Q}\text{−almost all } V.$$

For this purpose, we may fix any continuous function $\varphi \geq 0$ on R^d with compact support and $\langle\varphi,\ell\rangle > 0$. Then *it suffices to show*

$$\int \langle\varphi,\chi\rangle\, V(\mathrm{d}\chi) \;\geq\; \langle\varphi,\ell\rangle \quad \text{for } \mathbf{Q}\text{−almost all laws } V. \tag{34}$$

4° (*lower bound* ℓ) Fix $0 < \varepsilon < \langle\varphi, \ell\rangle$. Consider the probability

$$\mathbf{Q}\left\{V : \int \langle\varphi, \chi\rangle \, V(\mathrm{d}\chi) \, \geq \, \langle\varphi, \ell\rangle - \varepsilon\right\}. \tag{35}$$

For each $K > 0$, it is bounded below by

$$\mathbf{Q}\left\{V : \int (\langle\varphi, \chi\rangle \wedge K) \, V(\mathrm{d}\chi) \, \geq \, \langle\varphi, \ell\rangle - \varepsilon\right\}.$$

Since the set of these V is *closed,* again by the second weak convergence statement in (29) this expression is bounded below by

$$\limsup_{n\uparrow\infty} \mathbf{Q}_{t_n}\left\{V : \int (\langle\varphi, \chi\rangle \wedge K) \, V(\mathrm{d}\chi) \, \geq \, \langle\varphi, \ell\rangle - \varepsilon\right\}.$$

Now

$$\int (\langle\varphi, \chi\rangle \wedge K) \, V(\mathrm{d}\chi) = \int \langle\varphi, \chi\rangle \, V(\mathrm{d}\chi) - \int \Big(\langle\varphi, \chi\rangle - (\langle\varphi, \chi\rangle \wedge K)\Big) \, V(\mathrm{d}\chi).$$

The first term on the r.h.s. is $\mathbf{Q}_{t_n}$–a.s. $\langle\varphi, \ell\rangle$ (recall (18)), and so it suffices to estimate

$$\mathbf{Q}_{t_n}\left\{V : \int \Big(\langle\varphi, \chi\rangle - (\langle\varphi, \chi\rangle \wedge K)\Big) \, V(\mathrm{d}\chi) \leq \varepsilon\right\}$$

from below, uniformly in n. Observe first that

$$\begin{aligned} \int \Big(\langle\varphi, \chi\rangle - (\langle\varphi, \chi\rangle \wedge K)\Big) \, V(\mathrm{d}\chi) \;&\leq\; \int_{\langle\varphi,\chi\rangle \geq K} \langle\varphi, \chi\rangle \, V(\mathrm{d}\chi) \\ &\leq\; \frac{1}{K} \int \langle\varphi, \chi\rangle^2 \, V(\mathrm{d}\chi). \end{aligned}$$

Rewriting the second moment $\int \langle\varphi, \chi\rangle^2 \, V(\mathrm{d}\chi)$ as $\langle\varphi, \ell\rangle^2$ plus the related variance expression, we see from the relative compactness Theorem 3 that

$$\mathbf{Q}_{t_n}\left\{V : \int \langle\varphi, \chi\rangle^2 \, V(\mathrm{d}\chi) \leq \varepsilon K\right\} \longrightarrow 1 \quad \text{as} \quad K \uparrow \infty,$$

uniformly in n. Thus (35) is equal to one, and since ε was arbitrary, this implies (34), and the proof is complete. ■

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