OPTIMAL BOUNDARY CONTROL PROBLEMS FOR SHAPE MEMORY ALLOYS UNDER STATE CONSTRAINTS FOR STRESS AND TEMPERATURE

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Abstract

We consider two optimal control problems for first order martensitic phase transitions in a deformation–driven experiment on shape memory alloys including state constraints for the total stress and the temperature. We control by the elongation of a thin rod and by the outside temperature. The control problems are stated, and the necessary conditions of optimality are derived.

1 Introduction

In this paper, we consider optimal control problems for a deformation–driven experiment on shape memory alloys (SMA) with state constraints for the total stress and the temperature. SMA exhibit a non–monotone temperature–dependent hysteretic behaviour in their load–deformation cycles leading to interesting industrial applications. In a series of papers (cf. [6],[7],[8], for example), Falk introduced a one–dimensional model that is based on the Landau–Ginzburg theory of phase transitions and uses the linearized shear strain $\varepsilon = u_x$,

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where u denotes the displacement, as order parameter. The corresponding (Helmholtz–) free energy $F = F(u, \theta)$, where θ denotes the absolute temperature, is given by

$$F(\varepsilon, \theta) = F_0(\theta) + \theta F_1(\varepsilon) + F_2(\varepsilon), \tag{1.1}$$

where

$$F_0(\theta) = -c_e \theta \, \log\left(\frac{\theta}{\tilde{\theta}}\right) + c_e\theta + C, \tag{1.2}$$

and

$$F_1(\varepsilon) = \frac{1}{2}\gamma\varepsilon^2, \quad F_2(\varepsilon) = -\frac{1}{2}\gamma\theta_1\varepsilon^2 - \frac{1}{4}\beta\varepsilon^4 + \frac{1}{6}\alpha\varepsilon^6, \tag{1.3}$$

with positive constant heat capacity c_e, a critical temperature θ_1, and positive material constants $\tilde{\theta}, C, \alpha, \beta$, and γ, which have to be determined for each specimen. For thermodynamical reasons, i.e. in order to comply with the second principle, the constitutive equations yield for the total stress:

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon, \theta) = -\gamma\,(\theta - \theta_1)\,\varepsilon - \beta\,\varepsilon^3 + \alpha\,\varepsilon^5. \tag{1.4}$$

In a deformation–driven experiment, a thin rod of a SMA is fixed on one side and pushed and pulled on the other side in the course of time by an elongation m. In such experiments, the order parameter is taken to be $\varepsilon = u_x$, u denoting the displacement *in* the direction of the rod. For a detailed description of the physical background, we refer the reader to [2],[3]. Summarizing, we have the following system ($\Omega := (0, l)$, $Q := \Omega \times (0, T)$):

$$\rho\, u_{tt} - (\gamma\,(\theta - \theta_1)\,u_x - \beta\,u_x^3 + \alpha\,u_x^5)_x + \delta\,u_{xxxx} = 0, \quad \text{in} \quad Q, \tag{1.5a}$$

$$c_e\,\theta_t - \kappa\,\theta_{xx} - \gamma\,\theta\,u_x\,u_{xt} = g(x,t), \quad \text{in} \quad Q, \tag{1.5b}$$

$$\begin{aligned} & u(0,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad u(l,t) = m(t), \quad \forall\, t \in [0,T], \\ & \theta_x(0,t) = 0, \quad -\kappa\,\theta_x(l,t) = \bar{\kappa}\,(\theta(l,t) - \theta_\Gamma(t)), \quad \forall\, t \in [0,T], \\ & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \forall\, x \in \overline{\Omega}, \end{aligned} \tag{1.5c}$$

The equations (1.5a) and (1.5b) represent the balance laws of momentum and energy, respectively. The physical meanings of the involved quantities are: ρ – constant mass density, κ – positive constant heat conductivity, g – density of heat sources or sinks, l – length of the rod (which is normalized to unity: $l := 1$), $\bar{\kappa}$ – positive constant heat exchange coefficient, θ_Γ – temperature of the surrounding medium. The couple stress leads to the Ginzburg–term $\delta \cdot u_{xxxx}$, δ being another positive material constant. The boundary condition for u at $x = 1$ reflects the pulling and pushing of the rod in the course of time by a prescribed elongation m. The other boundary condition for the momentum balance has been taken in analogy to [11]. The boundary condition for the energy balance models a heat exchange with the surrounding temperature at $x = 1$ using Newton's law. We normalize all physical constants to 1, except for θ_1 which is set to 0. In order to deal with homogeneous boundary conditions, we transform the system (1.5) by $\tilde{u}(x,t) := u(x,t) - x \cdot m(t)$. An additional term $\rho \cdot x \cdot \ddot{m}(t)$ appears only on the left hand side of the momentum balance. We now have $\varepsilon = u_x + m(t)$ instead of $\varepsilon = u_x$. For simplicity, the tilde for u and u_x, repectively, is omitted. We denote by $\tilde{\sigma}$ the polynom (1.4) where $\varepsilon = u_x$ is replaced by $\varepsilon = u_x + m(t)$.

In this paper, we consider the optimal control of the phase transitions governed by the following weak formuation of (1.5):

$$\int_0^T < u_{tt}(s), \phi(s) >_{H^{-1}\times H_0^1} \,\mathrm{d}s \;+ \int_0^T \int_\Omega x\, \ddot{m}(s)\, \phi \,\mathrm{d}x\,\mathrm{d}s \;+ \int_0^T \int_\Omega \tilde{\sigma}\, \phi_x \,\mathrm{d}x\,\mathrm{d}s$$
$$- \int_0^T \int_\Omega u_{xxx}\, \phi_x \,\mathrm{d}x\,\mathrm{d}s \;= 0, \quad \forall\, \phi \in L^2(0,T;H_0^1(\Omega)), \tag{1.6a}$$
$$\theta_t - \theta\,(u_x + m(t))\,(u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad \text{a.e.} \quad \text{in} \quad Q, \tag{1.6b}$$
$$u(0,t) = u(1,t) = 0, \quad \forall\, t \in [0,T], \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad \text{a.e.} \quad \text{in} \quad (0,T),$$
$$\theta_x(0,t) = 0, \quad -\theta_x(1,t) = \theta(1,t) - \theta_\Gamma(t), \quad \text{a.e.} \quad \text{in} \quad (0,T),$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \forall\, x \in \overline{\Omega}, \tag{1.6c}$$

where we want to admit state constraints for the total stress σ defined by (1.4) and the temperature θ. Under the following assumptions

$$\begin{aligned} (H1) \qquad & m \in H^3(0,T); \quad g \in L^2(0,T;L^2(\Omega)); \quad g(x,t) \geq 0 \quad \text{on} \quad \overline{Q}; \\ & \theta_\Gamma \in H^1(0,T); \quad \theta_\Gamma(t) > 0 \quad \text{on} \quad [0,T]; \end{aligned} \tag{1.7}$$
$$\begin{aligned} (H2) \qquad & u_0 \in H_E^3(\Omega) := \{u \in H^3(\Omega) \;\big|\; u(0) = u''(0) = u(1) = u''(1) = 0\}; \\ & u_1 \in H_0^1(\Omega); \quad \theta_0 \in H^1(\Omega); \quad \theta_0(x) > 0 \quad \text{on} \quad \overline{\Omega}, \end{aligned} \tag{1.8}$$

the existence and uniqueness of a weak solution has been proved in [4].

Theorem 1.1 *Suppose that (H1) and (H2) are satisfied. Then the system (1.6) has a solution* (u,θ) *on* $\overline{Q}$ *satisfying*

$$u \;\in\; X_{1,T} := W^{2,\infty}(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H_E^3(\Omega)) \quad \textit{and}$$
$$\theta \;\in\; X_{2,T} := H^{2,1}(Q) \cap L^\infty(0,T;H^1(\Omega)), \tag{1.9}$$

for any $T > 0$.

Lemma 3.5 in [4] states uniqueness. We recall that, with stronger assumptions for the data, the existence and uniqueness of a classical solution can be proved (see [11],[2],[4]).

Related optimal control problems have been studied so far in [1] concerning load–driven experiments, state constraints for those problems have been imposed in [9] and [10]. Therein, boundary control problems with state constraints for the transversal displacement and on the shear strain, respectively, were introduced. It has been left out as an open problem whether state constraints for total stress and for the temperature are possible.

Now, in [4] we have shown the differentiability of the observation operator as mapping into the solution space $X_{1,T} \times X_{2,T}$, while in [1] only the differentiability into the Banach space

$$
\begin{aligned}
B \;=\; & W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \\
\times \;\; & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))
\end{aligned} \tag{1.10}
$$

has been proved. Since $X_{2,T}$ is continuously imbedded in $C(\overline{\Omega_T})$, this means that also pointwise constraints on the temperature θ and therefore on the stress σ, too, can now be included in the control problem. This was not possible in [9] and [10] where only pointwise constraints on the displacement u and the strain ε, respectively, could be admitted. Note that pointwise constraints for θ are very realistic for the particular experimental setup discussed here, where θ is kept close to a prescribed (constant) temperature $\overline{\theta}$ (see also remark 4.1 in [4]). Since we do not want to differentiate with respect to the distributed heat sources and sinks, g, we even have Fréchet differentiability with the result given in [4].

We define

$$
\mathcal{M} := M_m \times M_{\theta_\Gamma}, \tag{1.11}
$$

where

$$
\begin{aligned}
M_m \;&:=\; \left\{ \; m \in H^3(0,T) \;\middle|\; m(0)=0,\;\; \dot{m}(0)=0,\;\; \ddot{m}(0)=0 \; \right\}, \\
M_{\theta_\Gamma} \;&:=\; \left\{ \; \theta_\Gamma \in H^1(0,T) \;\middle|\; \theta_\Gamma(t)>0 \;\text{ on }\; [0,T] \; \right\},
\end{aligned} \tag{1.12}
$$

and the control space

$$
\mathcal{Z} := H^3(0,T) \times H^1(0,T), \tag{1.13}
$$

therefore $\mathcal{M} \subset \mathcal{Z}$. The solution operator is denoted by

$$
\mathcal{G}(\cdot,\cdot) \;:\; \mathcal{M} \ni (m,\theta_\Gamma) \mapsto (u,\theta) \in X_{1,T} \times X_{2,T} \subset C(\overline{Q}) \times C(\overline{Q}). \tag{1.14}
$$

Note that $u \in X_{1,T}$ implies $u_x \in C(\overline{Q})$ and therefore, $\sigma \in C(\overline{Q})$, too. From [4] we have the following properties of the solution operator.

Theorem 1.2 *$\mathcal{G}(\cdot,\cdot)$ is Fréchet differentiable as mapping between the open set $\mathcal{M}$ and $X_{1,T} \times X_{2,T}$, and the Fréchet derivative $\mathcal{G}'(m,\theta_\Gamma)\cdot(h,l) =: (\phi,\psi)$ of $\mathcal{G}$ at (m,θ_Γ) applied to $(h,l) \in \mathcal{Z}$ is given as the unique solution to the system*

$$\int_0^T < \phi_{tt}(s), \xi(s) >_{H^{-1}\times H_0^1} \, \mathrm{d}s \; - \int_0^T \int_\Omega \phi_{xxx} \, \xi_x \, \mathrm{d}x \, \mathrm{d}s \; = - \int_0^T \int_\Omega x \, \ddot{h}(s) \, \xi \, \mathrm{d}x \, \mathrm{d}s$$

$$- \int_0^T \int_\Omega \Big(\varepsilon \, \psi + (\theta + F_2''(\varepsilon)) \, (\phi_x + h(s)) \Big) \, \xi_x \, \mathrm{d}x \, \mathrm{d}s \, , \quad \forall \, \xi \in L^2(0,T; H_0^1(\Omega)), \quad (1.15a)$$

$$\psi_t - \psi_{xx} = \theta \, \varepsilon_t \, (\phi_x + h(t)) + \varepsilon \, \varepsilon_t \, \psi + \theta \, \varepsilon \, (\phi_{xt} + \dot{h}(t)), \quad a.e. \quad in \quad Q, \quad (1.15b)$$

$$\phi(x,0) = \phi_t(x,0) = 0 = \psi(x,0), \qquad \forall \, x \in \overline{\Omega},$$

$$\phi(0,t) = \phi(1,t) = 0, \quad \forall \, t \in [0,T], \quad \phi_{xx}(0,t) = \phi_{xx}(1,t) = 0, \quad a.e. \quad in \quad (0,T),$$

$$\psi_x(0,t) = 0, \;\; -\psi_x(1,t) = \psi(1,t) - l(t), \qquad a.e. \quad in \quad (0,T), \quad (1.15c)$$

where $\mathcal{G}(m, \theta_\Gamma) = (u, \theta)$ *and* $\varepsilon = u_x + m$.

Clearly, we have $(\phi, \psi) \in X_{1,T} \times X_{2,T} \subset C(\overline{Q}) \times C(\overline{Q})$, and again, $\phi_x \in C(\overline{Q})$, too.

Since the strain ε plays the role of the order parameter, it is quite natural to consider cost functionals involving ε. On the other hand, the natural control variables are m and θ_Γ; in fact, these variables are used to control the processes in actual industrial applications of SMA.

We are going to consider two problems. First, we take the elongation m as the control variable, and, to simplify, we consider θ_Γ as given data. We impose state constraints for both the stress and the temperature. Second, we take θ_Γ as control variable, m as given data and prescribe constraints for the total stress.

2 Control by Elongation

We study the following problem.

(CP1) Minimize $J(m)$, subject to (1.6), $(\theta, \sigma) \in \mathcal{C}$ and $m \in \mathcal{U}_{ad}$.

Here, $\mathcal{U}_{ad}$ denotes the set of admissible controls, and is some nonempty, convex, bounded, and closed subset of M_m. $\mathcal{C}$ is given by

$$\mathcal{C} := \Big\{ \;\; (\theta, \sigma) \in C(\overline{Q}) \times C(\overline{Q}) \;\; \Big| \;\; c_1 \le \theta(x,t) \le c_2, \;\; c_3 \le \sigma(x,t) \le c_4, \;\; \forall \, (x,t) \in \overline{Q} \;\; \Big\}. \quad (2.1)$$

The cost functional is assumed in the form

$$J(m) = \int_0^T \int_\Omega \Phi_1(u_x(x,t), \theta(x,t)) \; \mathrm{d}x \; \mathrm{d}t \; + \int_0^T \Phi_2(\dddot{m} \, (t)) \; \mathrm{d}t \, , \quad (2.2)$$

where $\Phi_1 \in C^2(\mathbb{R}^2), \Phi_2 \in C^1(\mathbb{R})$, and where Φ_2 is convex in its argument. A particular form could be

$$J(m) = \alpha_1 \left(\| \, \sigma - \overline{\sigma} \, \|^2_{L^2(Q)} + \| \, \theta - \overline{\theta} \, \|^2_{L^2(Q)} \right) + \alpha_2 \| \, \dddot{m} \, \|^2_{L^2(0,T)}, \quad (2.3)$$

where α_1 and α_2 are non–negative constants, and where $\overline{\theta}$ and $\overline{\sigma}$ denote the desired temperature and stress distributions during the evolution of the process, respectively. Of course, also other cost functionals are conceivable in actual applications.

The following existence result can be shown with standard compactness arguments.

Theorem 2.1 *Assume that there is at least one admissible control m such that the solution to (1.6) yields $(\theta, \sigma) \in \mathcal{C}$. Then there exists an optimal solution to the above control problem.*

The necessary optimality conditions for the control problem are given by the following theorem. Since here θ_Γ is given, we write $\mathcal{G}(m)$ instead of $\mathcal{G}(m, \theta_\Gamma)$.

Theorem 2.2 *Let $m \in \mathcal{U}_{ad}$ denote any solution to the optimal control problem* **(CP1)**, *and let $(u, \theta) = \mathcal{G}(m)$. Then there exist a real number $\lambda_1 \geq 0$ and Borel measures $(\mu_1, \mu_2) =: \mu \in (C(\overline{Q}) \times C(\overline{Q}))'$ with $\lambda_1 + \|\mu\|_{(C(\overline{Q}) \times C(\overline{Q}))'} > 0$ such that $\int (\hat{\theta} - \theta)\,\mathrm{d}\mu_1 + \int (\hat{\sigma} - \sigma)\,\mathrm{d}\mu_2 \leq 0$, $\forall\, (\hat{\sigma}, \hat{\theta}) \in \mathcal{C}$, as well as functions $(p, q) \in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; L^2(\Omega))$ satisfying the following optimality conditions.*

State equations:

$$\int_0^T < u_{tt}(s), \phi(s) >_{H^{-1} \times H_0^1}\, \mathrm{d}s + \int_0^T \int_\Omega x\, \ddot{m}(s)\, \phi \, \mathrm{d}x \, \mathrm{d}s + \int_0^T \int_\Omega \Big(\theta\, (u_x + m(s))$$
$$+ F_2'(u_x + m(s)) \Big)\, \phi_x \, \mathrm{d}x \, \mathrm{d}s - \int_0^T \int_\Omega u_{xxx}\, \phi_x \, \mathrm{d}x \, \mathrm{d}s = 0, \quad \forall\, \phi \in L^2(0, T; H_0^1(\Omega)), \tag{2.4a}$$
$$\theta_t - \theta\, (u_x + m(t))\, (u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad a.e. \quad in \quad Q, \tag{2.4b}$$
$$u(0,t) = u(1,t) = 0, \quad \forall\, t \in [0,T], \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad a.e. \quad in \quad (0,T),$$
$$\theta_x(0,t) = 0, \quad -\theta_x(1,t) = \theta(1,t) - \theta_\Gamma(t), \quad a.e. \quad in \quad (0,T), \tag{2.4c}$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \forall\, x \in \overline{\Omega}. \tag{2.4d}$$

Adjoint state equations:

$$\int_0^T < \xi_{tt}(s), p(s) >_{H^{-1} \times H_0^1}\, \mathrm{d}s - \int_0^T \int_\Omega \xi_{xxx}\, p_x \, \mathrm{d}x \, \mathrm{d}s + \int_0^T \int_\Omega \Big(\Big((\theta + F_2''(\varepsilon))\, p_x$$
$$- \theta\, \varepsilon_t\, q \Big)\, \xi_x - \theta\, \varepsilon\, q\, \xi_{xt} \Big)\, \mathrm{d}x \, \mathrm{d}s = \lambda_1 \int_0^T \int_\Omega D_1 \Phi_1(u_x, \theta)\, \xi_x \, \mathrm{d}x \, \mathrm{d}s + \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon}\, \xi_x \, \mathrm{d}\mu_2\,,$$
$$\forall\, \xi \in X_{1,T}, \tag{2.5a}$$
$$\int_0^T \int_\Omega \Big(q\, (\varphi_t - \varphi_{xx} - \varepsilon\, \varepsilon_t\, \varphi) + \varepsilon\, p_x\, \varphi \Big)\, \mathrm{d}x \, \mathrm{d}s = \lambda_1 \int_0^T \int_\Omega D_2\, \Phi_1(u_x, \theta)\, \varphi \, \mathrm{d}x \, \mathrm{d}s$$
$$+ \int \varepsilon\, \varphi \, \mathrm{d}\mu_2 + \int \varphi \, \mathrm{d}\mu_1\,, \quad \forall\, \varphi \in X_{2,T}. \tag{2.5b}$$

Optimality conditions:

$$\int_0^T \int_\Omega \Big\{ -\ddot{h}(s)\, p\, x + \dot{h}(s)\, q\, \theta\, \varepsilon - h(s) \left(p_x \left(\theta + F_2''(\varepsilon)\right) + q\, \theta\, \varepsilon_t \right) \Big\} \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \quad \lambda_1 \int_0^T \Big\{ \Phi_2'(\dddot{m}\,(s))\ \dddot{h}\,(s) \Big\}\,\mathrm{d}s + \int \frac{\partial \tilde{\sigma}}{\partial m}\, h \ \mathrm{d}\mu_2 \quad \geq \quad 0,$$
$$h = \hat{m} - m, \quad \forall\, \hat{m} \in \mathcal{U}_{ad}. \tag{2.6}$$

In addition, $\lambda_1 = 1$ if the Slater condition is satisfied, i.e. there exists some $\hat{m} \in \mathcal{U}_{ad}$ such that the unique solution (ϕ, ψ) of the linearized state equations (1.15) corresponding to $h = \hat{m} - m$ satisfies the condition

$$c_1 \ < \ \theta(x,t) + \psi(x,t) \ < \ c_2, \quad \text{and} \tag{2.7}$$
$$c_3 \ < \ \tilde{\sigma}(x,t) + \psi(x,t)\,\varepsilon(x,t) + (\phi_x(x,t) + h(t))\Big(\theta(x,t) + F_2''(\varepsilon(x,t))\Big) \ < \ c_4, \quad \forall\,(x,t) \in \overline{Q}.$$

PROOF. Now, a solution to **(CP1)** is denoted by m^*, and therefore $h = m - m^*$. Let us denote by $J'(m) \in (H^3(0,T))'$ the Fréchet derivative of the cost functional $J(m)$, by $\mathcal{F}(\mathcal{G}(m)) = \tilde{\sigma}$, and by $\Big[D_m\Big(\theta, \mathcal{F}(\mathcal{G}(m))\Big)\Big]^*$ the adjoint mapping of the differential. Moreover, let $< .,. >$ denote the dual pairing between the spaces $(H^3(0,T))'$ and $H^3(0,T)$. Applying theorem 5.2 of [5], we conclude that there exist Borel measures $(\mu_1, \mu_2) = \mu \in (C(\overline{Q}) \times C(\overline{Q}))'$ and some $\lambda_1 \geq 0$ satisfying

$$\lambda_1 + \|\mu\|_{(C(\overline{Q})\times C(\overline{Q}))'} \ > \ 0, \tag{2.8}$$
$$< \mu, z - \Big(\theta^*, \mathcal{F}(\mathcal{G}(m^*))\Big) > \ \leq \ 0, \quad \forall\, z \in \mathcal{C}, \tag{2.9}$$
$$< \lambda_1 J'(m^*) + \Big[D_m\Big(\theta^*, \mathcal{F}(\mathcal{G}(m^*))\Big)\Big]^* \mu, m - m^* > \ \geq \ 0, \quad \forall\, m \in \mathcal{K}. \tag{2.10}$$

Furthermore, we have $\lambda_1 = 1$ if the *Slater condition*

$$\exists\, \tilde{m} \in \mathcal{U}_{ad} \quad \text{such} \quad \text{that} \quad \mathcal{G}(m) + \mathcal{G}'(m)\cdot(\tilde{m} - m) \in \mathrm{int}(\mathcal{C}) \tag{2.11}$$

is satisfied. Recalling the definition of $\mathcal{C}$, we find that this condition is eqivalent to (2.7). Now, to continue in a simplified manner, we set $\lambda_1 = 1$.

We introduce the linear and bijective operators $\mathcal{L}_1 : X_{1,T} \times X_{2,T} \to L^2(0,T; H^{-1}(\Omega))$ and $\mathcal{L}_2 : X_{1,T} \times X_{2,T} \to L^2(0,T; L^2(\Omega))$ with

$$\int_0^T < \mathcal{L}_1(\phi,\psi)(s), \xi(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s \equiv \int_0^T < \phi_{tt}(s), \xi(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s$$
$$- \int_0^T \int_\Omega \phi_{xxx}\,\xi_x\,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_\Omega \Big(\varepsilon\,\psi + (\theta + F_2''(\varepsilon))\,\phi_x\Big)\xi_x\,\mathrm{d}x\,\mathrm{d}s\,,$$
$$\forall\,\xi \in L^2(0,T; H_0^1(\Omega)), \quad \text{and} \tag{2.12}$$
$$\int_0^T \int_\Omega \mathcal{L}_2(\phi,\psi)\,\varphi\,\mathrm{d}x\,\mathrm{d}s \equiv \int_0^T \int_\Omega \Big(\psi_t - \psi_{xx} - \theta\,\varepsilon_t\,\phi_x - \varepsilon\,\varepsilon_t\,\psi - \theta\,\varepsilon\,\phi_{xt}\Big)\,\varphi\,\mathrm{d}x\,\mathrm{d}s\,,$$
$$\forall\,\varphi \in L^2(0,T; L^2(\Omega)). \tag{2.13}$$

Furthermore, denoting

$$
\begin{aligned}
&\mathcal{X} := X_{1,T} \times X_{2,T}, \quad \mathcal{Y} := L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;L^2(\Omega)), && (2.14)\\
&z := (z_1, z_2) \in \mathcal{Y}, \quad \text{with}\\
&\int_0^T < z_1(s), \xi(s) >_{H^{-1}\times H_0^1} \, \mathrm{d}s \; := -\int_0^T \int_\Omega x \, \ddot{h}(s) \, \xi \, \mathrm{d}x \, \mathrm{d}s\\
&\qquad - \int_0^T \int_\Omega (\theta + F_2''(\varepsilon)) \, h(s) \, \xi_x \, \mathrm{d}x \, \mathrm{d}s \,, \quad \forall \, \xi \in L^2(0,T;H_0^1(\Omega)), \quad \text{and} && (2.15)\\
&\int_0^T \int_\Omega z_2 \, \varphi \, \mathrm{d}x \, \mathrm{d}s \; := \int_0^T \int_\Omega \Big(\theta \, \varepsilon_t \, h(s) + \theta \, \varepsilon \, \dot{h}(s)\Big) \, \varphi \, \mathrm{d}x \, \mathrm{d}s \,, \quad \forall \, \varphi \in L^2(0,T;L^2(\Omega)),\\
&\mathcal{L} : \mathcal{X} \to \mathcal{Y}, \quad \text{with} \quad \mathcal{L}[(\phi,\psi)] := (\mathcal{L}_1(\phi,\psi), \mathcal{L}_2(\phi,\psi)), && (2.16)
\end{aligned}
$$

the linearized state equations (1.15) take the form

$$
\begin{aligned}
&\text{Find} \quad (\phi,\psi) \quad \text{such} \quad \text{that} \quad \mathcal{L}[(\phi,\psi)] = z \in \mathcal{Y}, && (2.17\text{a})\\
&\phi(x,0) = \phi_t(x,0) = 0 = \psi(x,0), \quad \forall \, x \in \overline{\Omega},\\
&\phi(0,t) = \phi(1,t) = 0, \quad \forall \, t \in [0,T], \quad \phi_{xx}(0,t) = \phi_{xx}(1,t) = 0, \quad \text{a.e.} \quad \text{in} \quad (0,T),\\
&\psi_x(0,t) = 0, \;\; -\psi_x(1,t) = \psi(1,t), \quad \text{a.e.} \quad \text{in} \quad (0,T). && (2.17\text{b})
\end{aligned}
$$

For any continuous linear form $\Psi(.,.)$ on $C(\overline{Q}) \times C(\overline{Q})$ we have an unique element $\overline{v} \in \mathcal{Y}' = L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;L^2(\Omega))$ such that

$$
\Big(\mathcal{L}[(r,s)], \overline{v}\Big)_{\mathcal{Y}} = \Psi(r_x, s), \quad \forall \, (r,s) \in \mathcal{X}, \tag{2.18}
$$

because for an element $(r,s) \in \mathcal{X}$ we have $(r,s) = \mathcal{L}^{-1}[(z)]$ for the unique element $z \in \mathcal{Y}$ and $\|r_x\|_{C(\overline{Q})} + \|s\|_{C(\overline{Q})} \le C\|(r,s)\|_{\mathcal{X}}$, $\forall \, (r,s) \in \mathcal{X}$, $C > 0$. We select the following continuous linear form on $C(\overline{Q}) \times C(\overline{Q})$

$$
\begin{aligned}
\Psi(\phi_x, \psi) \equiv \lambda_1 \Big[\int_0^T \int_\Omega \Big(D_1\Phi_1(u_x,\theta) \, \phi_x + D_2\Phi_1(u_x,\theta) \, \psi \Big) \Big] \, \mathrm{d}x \, \mathrm{d}s \; + \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon} \, \phi_x \, \mathrm{d}\mu_2\\
+ \int \varepsilon \, \psi \, \mathrm{d}\mu_2 + \int \psi \, \mathrm{d}\mu_1 \,, \quad (\phi,\psi) \in \mathcal{X}.
\end{aligned} \tag{2.19}
$$

Then there exists a unique adjoint state $v^* = (p^*, q^*) \in \mathcal{Y}'$ such that the following adjoint state equation is satisfied

$$
\Big(\mathcal{L}[(r,s)], v^*\Big)_{\mathcal{Y}} = \Psi(r_x, s), \quad \forall \, (r,s) \in \mathcal{X}. \tag{2.20}
$$

This leads to the adjoint system (2.5), and for any solution (ϕ,ψ) of the linearized state equations we have $\forall \, m \in \mathcal{K}$, $h = m - m^*$,

$$
< J'(m^*) + \Big[D_m\Big(\theta^*, \mathcal{F}(\mathcal{G}(m^*))\Big)\Big]^* \mu, h > = \int_0^T \Phi_2'(\dddot{m}\,(s)) \, \dddot{h}\,(s) \, \mathrm{d}s
$$

$$
\begin{aligned}
&\quad + \int_0^T \int_\Omega \Big(D_1\Phi_1(u_x,\theta)\,\phi_x + D_2\Phi_1(u_x,\theta)\,\psi \Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int \frac{\partial\tilde\sigma}{\partial\varepsilon}\,\phi_x\,\mathrm{d}\mu_2 + \int \varepsilon\,\psi\,\mathrm{d}\mu_2 + \int \psi\,\mathrm{d}\mu_1 + \int \frac{\partial\tilde\sigma}{\partial m}\,h\,\mathrm{d}\mu_2 \\
&= \int_0^T \Phi_2'(\dddot{m}\,(s))\,\dddot{h}\,(s)\,\mathrm{d}s + \Psi(\phi,\psi) + \int \frac{\partial\tilde\sigma}{\partial m}\,h\,\mathrm{d}\mu_2 \\
&= \int_0^T \Phi_2'(\dddot{m}\,(s))\,\dddot{h}\,(s)\,\mathrm{d}s + \int \frac{\partial\tilde\sigma}{\partial m}\,h\,\mathrm{d}\mu_2 \\
&\quad + \int_0^T < \mathcal{L}_1(\phi,\psi)(s), \xi(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s + \int_0^T \int_\Omega \mathcal{L}_2(\phi,\psi)\,\varphi\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T \Phi_2'(\dddot{m}\,(s))\,\dddot{h}\,(s)\,\mathrm{d}s + \int \frac{\partial\tilde\sigma}{\partial m}\,h\,\mathrm{d}\mu_2 \\
&\quad + \int_0^T < z_1(s), \xi(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s + \int_0^T \int_\Omega z_2\,\varphi\,\mathrm{d}x\,\mathrm{d}s\,, \qquad (2.21)
\end{aligned}
$$

whence (2.6) follows from (2.10). □

3 Control by Temperature

Now, we study the following problem.

(CP2) Minimize $J(\theta_\Gamma)$, subject to (1.6), $\sigma \in \mathcal{S}$ and $\theta_\Gamma \in \mathcal{U}_{ad}$.

Here, $\mathcal{U}_{ad} \subset M_{\theta_\Gamma}$. $\mathcal{S}$ is given by

$$
\mathcal{S} := \Big\{ \ \sigma \in C(\overline{Q}) \ \Big| \ c_5 \le \sigma(x,t) \le c_6, \ \ \forall\,(x,t) \in \overline{Q} \ \Big\}. \qquad (3.1)
$$

The cost functional is assumed in the form

$$
J(\theta_\Gamma) = \int_0^T \int_\Omega \Phi_1(u_x(x,t))\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \Phi_2(\theta_\Gamma(t))\ \mathrm{d}t\,, \qquad (3.2)
$$

where $\Phi_1 \in C^2(\mathbb{R}^2), \Phi_2 \in C^1(\mathbb{R})$, and where Φ_2 is convex in its argument. A particular form could be

$$
J(g,\theta_\Gamma) = \alpha_1 \,\|\, u_x - \overline{u_x} \,\|_{L^2(Q)}^2 + \alpha_2 \|\theta_\Gamma\|_{L^2(0,T)}^2, \qquad (3.3)
$$

where α_1, α_2, and α_3 are non–negative constants, and where $\overline{u_x}$ denotes the desired strain distribution during the evolution of the process. Again, also other cost functionals are conceivable.

The following existence result can be shown with standard compactness arguments as before.

Theorem 3.1 *Assume that there is at least one admissible control θ_Γ such that the solution to (1.6) yields $\sigma \in \mathcal{S}$. Then there exists an optimal solution to the above control problem.*

We give the necessary conditions of optimality in the following theorem. Since now m is given, we write $\mathcal{G}(\theta_\Gamma)$ instead of $\mathcal{G}(m,\theta_\Gamma)$.

Theorem 3.2 *Let $\theta_\Gamma \in \mathcal{U}_{ad}$ denote any solution to the optimal control problem* **(CP2)**, *and let $(u,\theta)=\mathcal{G}(\theta_\Gamma)$. Then there exist a real number $\lambda_2 \geq 0$ and a Borel measure $\mu_3 \in (C(\overline{Q}))'$ with $\lambda_2 + \|\mu_3\|_{(C(\overline{Q}))'} > 0$ such that $\int(\hat\sigma-\sigma)\,\mathrm{d}\mu_3 \leq 0,\ \forall\,\hat\sigma \in \mathcal{S}$, as well as functions $(p,q)\in L^2(0,T;H_0^1(\Omega))\times L^2(0,T;H^1(\Omega))$ satisfying the following optimality conditions.*

State equations:

$$\int_0^T < u_{tt}(s),\phi(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s + \int_0^T\int_\Omega x\,\ddot m(s)\,\phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_\Omega \Big(\theta\,(u_x+m(s))$$
$$+F_2'(u_x+m(s))\Big)\,\phi_x\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega u_{xxx}\,\phi_x\,\mathrm{d}x\,\mathrm{d}s = 0, \quad \forall\,\phi\in L^2(0,T;H_0^1(\Omega)), \tag{3.4a}$$
$$\theta_t - \theta\,(u_x+m(t))\,(u_{xt}+\dot m(t)) - \theta_{xx} = g, \quad a.e. \quad in \quad Q, \tag{3.4b}$$
$$u(0,t)=u(1,t)=0, \quad \forall\,t\in[0,T], \quad u_{xx}(0,t)=u_{xx}(1,t)=0, \quad a.e. \quad in \quad (0,T),$$
$$\theta_x(0,t)=0, \quad -\theta_x(1,t)=\theta(1,t)-\theta_\Gamma(t), \quad a.e. \quad in \quad (0,T), \tag{3.4c}$$
$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \theta(x,0)=\theta_0(x), \quad \forall\,x\in\overline{\Omega}. \tag{3.4d}$$

Adjoint state equations:

$$\int_0^T < \xi_{tt}(s),p(s) >_{H^{-1}\times H_0^1}\ \mathrm{d}s - \int_0^T\int_\Omega \xi_{xxx}\,p_x\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_\Omega \Big(\big((\theta+F_2''(\varepsilon))\,p_x$$
$$-\theta\,\varepsilon_t\,q\big)\,\xi_x - \theta\,\varepsilon\,q\,\xi_{xt}\Big)\,\mathrm{d}x\,\mathrm{d}s = \lambda_1\int_0^T\int_\Omega D_1\Phi_1(u_x)\,\xi_x\,\mathrm{d}x\,\mathrm{d}s + \int\frac{\partial\tilde\sigma}{\partial\varepsilon}\,\xi_x\,\mathrm{d}\mu_3\,,$$
$$\forall\,\xi\in X_{1,T}, \tag{3.5a}$$
$$\int_0^T\int_\Omega \Big(q\,(\varphi_t-\varepsilon\,\varepsilon_t\,\varphi) + q_x\,\varphi_x + \varepsilon\,p_x\,\varphi\Big)\,\mathrm{d}x\,\mathrm{d}s + \int_0^T \varphi(1,s)\,q(1,s)\,\mathrm{d}s = \int\varepsilon\,\varphi\,\mathrm{d}\mu_3\,,$$
$$\forall\,\varphi\in X_{2,T}. \tag{3.5b}$$

Optimality conditions:

$$\int_0^T \Big\{\ \Phi_2'(\theta_\Gamma(s)) \ - \ q(1,s)\ \Big\}\ l(s)\ \ \mathrm{d}s \quad \geq \quad 0, \quad l=\hat\theta_\Gamma-\theta_\Gamma, \quad \forall\,\hat\theta_\Gamma\in\mathcal{U}_{ad}. \tag{3.6}$$

Again, $\lambda_2=1$ if the Slater condition is satisfied, i.e. there exists some $\hat\theta_\Gamma\in\mathcal{U}_{ad}$ such that the unique solution (ϕ,ψ) of the linearized state equations (1.15) corresponding to $l=\hat\theta_\Gamma-\theta_\Gamma$ satisfies the condition

$$c_5 \ < \ \tilde\sigma(x,t)+\psi(x,t)\,\varepsilon(x,t)+\phi_x(x,t)\Big(\theta(x,t)+F_2''(\varepsilon(x,t))\Big) \ < \ c_6, \quad \forall\,(x,t)\in\overline{Q}. \tag{3.7}$$

PROOF. The proof to this theorem is analogue to the last one with the difference that the adjoint variable $q \in L^2(0,T;H^1(\Omega))$. $\square$

References

[1] Brokate, M., Sprekels, J.: *Optimal Control of Thermomechanical Phase Transitions in Shape Memory Alloys: Necessary Conditions of Optimality,* Mathematical Methods in the Applied Sciences **14** (1991) 265–280.

[2] Bubner, N.: *Modellierung dehnungsgesteuerter Phasenübergänge in Formgedächtnislegierungen,* Doctoral Dissertation, Universität–GH Essen, Verlag Shaker, Aachen, 1995.

[3] Bubner, N.: *Landau–Ginzburg Model for a Deformation–Driven Experiment on Shape Memory Alloys,* Continuum Mechanics and Thermodynamics, to appear.

[4] Bubner, N., Sprekels, J.: *Optimal control of martensitic phase transitions in a deformation–driven experiment on shape memory alloys,* Advances in Mathematical Sciences and Applications, to appear.

[5] Casas, E.: *Boundary control of semilinear elliptic equations with pointwise state constraints,* SIAM Journal of Control and Optimization **31** (1993) 993–1006.

[6] Falk, F.: *Landau Theory and Martensitic Phase Transitions,* Journal de Physique **C4** (1982) 3–15.

[7] Falk, F.: *One–dimensional model of shape memory alloys,* Archives of Mechanics **35** (1983) 63–84.

[8] Falk, F.: *Elastic Phase Transitions and Nonconvex Energy Functions.* In: "Free Boundary Problems: Theory and Applications I" (K.–H. Hoffmann, J. Sprekels, eds.), 45–59, Longman, London, 1990.

[9] Sokołowski, J., Sprekels, J.: *Control Problems with State Contraints for Shape Memory Alloys,* Mathematical Methods in the Applied Sciences **17** (1994) 943–952.

[10] Sokołowski, J., Sprekels, J.: *Control problems for shape memory alloys with constraints on the shear strain.* In: "Control of Partial Differential Equations" (G. Da Prato, L. Tubaro, eds.), Lecture Notes in Pure and Applied Mathematics Vol. **165** 189–195, Marcel Dekker, New York, 1994.

[11] Sprekels, J., Zheng, S.: *Global Solutions to the Equations of a Ginzburg–Landau Theory for structural Phase Transitions in Shape Memory Alloys,* Physica D **39** (1989) 59–76.