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Abstract

Generalized Nash equilibrium problems in function spaces involving PDEs are considered. One of the central issues arising in this context is the question of existence, which requires the topological characterization of the set of minimizers for each player of the associated Nash game. In this paper, we propose conditions on the operator and the functional that guarantee the reduced formulation to be a convex minimization problem. Subsequently, we generalize results of convex analysis to derive optimality systems also for non-smooth operators. Our theoretical findings are illustrated by examples.

1 Introduction

Nash Equilibrium Problems (NEPs) received a considerable amount of attention in the recent past; see, e.g., [FK07], [HSK15], [PF05], [KKSW19], [FFP09], [HS13], [BK13] and the references therein. Concerning problems posed in function space, however, the topic is significantly less researched. In the latter context, NEPs may arise in connection with optimal control problems involving partial differential equations (PDEs) along with a condition restricting the control. One of the key tasks in this context is the derivation of existence of equilibria. This question is closely related to the existence of a fixed point for a set-valued operator. Results regarding the latter significantly rely on a topological description of the operator's values [Gli52, EM46]. When addressing optimal control problems with non-linear PDE constraints a topological characterization of the solution set may not be (immediately) available.
Taking the latter as a starting point, the present work discusses a class of problems governed by solution operators fulfilling a preorder based convexity concept to guarantee the convexity of the optimization problem. Subsequently, a subdifferential concept related to coderivatives (cf. [Mor06]) is discussed and used to derive first-order conditions. A characterization of the subderivatives is given for a selection of practically relevant examples. The work is concluded with a study of a Nash equilibrium problem involving a global variational inequality constraint. The latter poses a challenge due to the non-smoothness of the solution operator and has been subject of various investigations (cf. [Mig76], [HK09], [HMS14] or [Wac16b] and the references therein).
In order to address some of the analytical difficulties, we consider the following class of optimal control problems, in which the state of the system is defined as the solution of a generalized equation (GE). In fact, we study

$$\begin{aligned} &\text{minimize } \ J^1(y) + J^2(u) \text{ over } u \in U_{\mathrm{ad}} \\ &\text{subject to } f + Bu \in A(y), \end{aligned} \tag{1}$$

where $B \in \mathcal{L}(U, W)$ is bounded, linear and the multifunction $A : Y \rightrightarrows W$ is a set-valued operator. The objective is assumed to be separated into the functional J^1 explicitly depending on the state only as well as a part solely and directly influenced by the control. Moreover, the control is assumed to be constrained via a set U_{ad}. A restriction of the state is not considered. In case, A is single-valued the

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(GE) becomes a classical operator equation, which might be a partial differential equation in specific cases. In general the solution mapping will be non-linear and oftentimes non-smooth. Due to the presence of the state in the objective the convexity cannot always be guaranteed.
In the course of this paper we will therefore address conditions on the solution mapping (control-to-state-mapping) as well as the objective to guarantee the convexity of problems of the structure presented in (1). In order to achieve this, a generalized convexity concept based on preorder relations related to [CLV13, Chapter 19] and [BS00, Section 2.3.5] is utilized. Along with a suitable condition on objective convexity can be guaranteed. Based on this, calculus for this class of operators is developed and hence first-order conditions for optimization problems like (1) are derived. The results are applied to a selection of meaningful applications.
This paper is organized as follows. In section 2 we introduce notation and preliminaries used in the rest of the work, In section 3 we introduce the notion of K-convex operators and investigate their properties as well as conditions guaranteeing this property. In section 4 we draw our attention to optimization problems and develop a subdifferential concept to derive first order optimality systems. In section 5 our abstract findings are applied to a class of semilinear elliptic partial differential equations, VIs as well as to a Nash equilibrium problem.

2 Notation and Preliminaries

In the following let X denote a topological vector space X with X^* its *topological dual space* and associated dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{X^*,X} : X^* \times X \to \mathbb{R}$ defined by $\langle x^*, x\rangle_{X^*,X} := x^*(x)$. Oftentimes we simply denote $\langle\,\cdot\,,\,\cdot\,\rangle$ if the corresponding spaces are clear from the context. Two elements $x^* \in X^*$ and $x \in X$ are called *orthogonal* if $\langle x^*, x\rangle = 0$ and we write $x^* \perp x$ or $x \perp x^*$. The *annihilator* of a subset $M \subseteq X$ is defined as

$$M^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in M\}\,,$$

and analogously for a set $M^* \subseteq X^*$ as

$$M^{*\perp} = \{x \in X : \langle x^*, x\rangle = 0 \text{ for all } x^* \in M^*\}\,.$$

For a single element we simply write $x^\perp := \{x\}^\perp$. A subset $C \subseteq X$ is called *convex* if for all $t \in (0,1)$ and $x_0, x_1 \in C$ it holds that $tx_1 + (1-t)x_0 \in C$. A set $K \subseteq X$ is called a *cone* if for all $t \in \mathbb{R}$, $t \geq 0$ and $x \in K$ also $tx \in K$ holds. The *radial cone* of C in $x \in C$ is defined by

$$\begin{aligned}\mathcal{R}_C(x) &:= \{d \in X : \text{ there exists } t > 0 : x + td \in C\}\\ &= \mathbb{R}_+(C - x) = \{\lambda(y - x) : y \in C,\, \lambda \in \mathbb{R}_+\}\,,\end{aligned}$$

with $\mathbb{R}_+ := \{\lambda \in \mathbb{R} : \lambda \geq 0\}$. The *tangential cone* of C in $x \in C$ is defined as

$$\begin{aligned}T_C(x) &:= \{d \in X : \text{ there exist } t_k \searrow 0, d_k \to d \text{ with } x + t_k d_k \in C\ \forall k \in N\}\\ &= \operatorname{cl}\left(\mathcal{R}_C(x)\right),\end{aligned}$$

together with the normal cone as

$$N_C(x) := \{x^* \in X^* : \langle x^*, x' - x\rangle_{X^*,X} \leq 0 \text{ fo all } x' \in C\}\,.$$

The *core* (or *algebraic interior*) of a set $M \subseteq X$ is defined by

$$\operatorname{core}(M) := \{x \in M : \forall d \in X\ \exists\,\bar{t} > 0 : x + td \in M \text{ for all } |t| < \bar{t}\}\,.$$

A subset $S \subseteq X$ is called *absorbing*, if for all $x \in X$ there exists $r > 0$ such that for all $|t| \leq r$ one has $tx \in S$.
A convex subset $C \subseteq X$ is called *cs-closed* (convex series closed) if for every sequence $(t_i)_{i\in\mathbb{N}}$ of non-negative numbers with $\sum_{i=1}^{\infty} t_i = 1$ and sequence $(x_i)_{i\in\mathbb{N}} \subseteq C$ such that $x := \sum_{i=1}^{\infty} t_i x_i$ exists, the inclusion $x \in C$ follows. Moreover, C is called *cs-compact* (convex series compact) if for all sequences $(t_i)_{i\in\mathbb{N}}$ of non-negative numbers with $\sum_{i=1}^{\infty} t_i = 1$ and an arbitrary sequence $(x_i)_{i\in\mathbb{N}} \subseteq C$ the limit $x := \sum_{i=1}^{\infty} t_i x_i$ exists and $x \in C$ holds.
For $(X, \|\cdot\|)$ being a normed vector space the closed unit ball of X is denoted as

$$\mathbb{B}_X := \{x \in X : \|x\| \leq 1\}.$$

The interior of a set $M \subseteq X$ is defined as

$$\operatorname{int}(M) := \{x \in M : \text{ there exists } \varepsilon > 0 : x + \varepsilon\mathbb{B}_X \subseteq M\}.$$

and its closure as

$$\operatorname{cl}(M) := \{x \in M : \text{ there exists } (x_n)_{n\in\mathbb{N}} \subseteq M \text{ with } x_n \to x\}.$$

Let Y be another topological vector space. The *coordinate projections* $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are defined by

$$\mathrm{pr}_X(x, y) := x \text{ and } \mathrm{pr}_Y(x, y) := y,$$

respectively.
A function $F : X \to \mathcal{P}(Y)$ is called a *set-valued operator* or *correspondence* and is denoted by $F : X \rightrightarrows Y$. Its *graph* is defined by

$$\mathrm{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

and its domain by

$$\mathcal{D}(F) := \{x \in X : F(x) \neq \emptyset\}.$$

In the scope of this work we make use of some terminology and aspects of order theory, which will be introduced next. For further references as well as details and additional information the interested reader is referred to the monographs [Sch74, Bec08]. A binary relation $\leq$ on a set X is called a *preorder relation* (or just *preorder*) if for all $x, x_0, x_1, x_2 \in X$ it holds that

(reflexivity) $x \leq x$ for all $x \in X$;

(transivity) $x_0 \leq x_1$ and $x_1 \leq x_2$ imply $x_0 \leq x_2$;

A preorder relation $\leq$ is called *(partial) order* if moreover for all $x_0, x_1 \in X$ it holds;

(antisymmetry) $x_0 \leq x_1$ and $x_1 \leq x_0$ implies $x_0 = x_1$.

Let a subset $A \subseteq X$ be given. The infimum of A is an element $x \in X$ such that $x \leq a$ for all $a \in A$ and for every $y \in X$ with $y \leq a$ for $a \in A$ one infers $y \leq x$. The supremum is defined analogously.
A set X equipped with an order relation is called *ordered set*. It is called a *lattice*, if for two elements x_0, x_1 the *infimum* $\min(x_0, x_1) = x_0 \wedge x_1 := \inf\{x_0, x_1\}$ as well as the *supremum* $\max(x_0, x_1) = x_0 \vee x_1 := \sup\{x_0, x_1\}$ exist, respectively. For $x \in X$ we also abbreviate $x^+ := \max(x, 0)$.
A (real) vector space X equipped with a (pre)order relation is called an *(pre)ordered vector space*, if moreover for all $x_0, x_1, z \in X$ and $t \geq 0$ it holds that

- $x_0 \leq x_1$ implies $(x_0 + z) \leq (x_1 + z)$ and
- $x_0 \leq x_1$ implies $tx_0 \leq tx_1$.

An ordered vector space that is also a lattice is called a *vector lattice* and a Banach space that is also a (pre)ordered vector lattice is called a *(pre)ordered Banach space* (cf. [Bec08]). One can prove for vector lattices (see [Sch74, Proposition 1.4]) that $x = x^+ - (-x)^+$ and that there exists one and only one decomposition $x = x_1 - x_2$ with $x_1, x_2 \geq 0$ and x_1, x_2 being disjoint (i.e. $\min(x_1, x_2) = 0$).
Let $K := \{x \in X : x \geq 0\}$. If $(X, \leq)$ is a preordered vector space, then K is a non-empty, closed, convex cone. On the other hand, let a vector space X and a non-empty, closed convex cone $K \subseteq X$ be given. Then K induces a preorder relation $\leq_K$ for all $x_0, x_1 \in X$ by $x_0 \leq_K x_1$ if $x_1 - x_0 \in K$. By definition $(X, \leq_K)$ is a preordered vector space and $\leq_K$ induces an order if and only if $K \cap (-K) = \{0\}$. In this sense it is possible to characterize the order equivalently by the cone of non-negative elements. A subset $\underline{C} \subseteq X$ of some vector lattice is called a *set with lower bound*, if $\underline{C} + K \subseteq \underline{C}$ and for all $x_0, x_1 \in \underline{C}$ it holds that $\min(x_0, x_1) \in \underline{C}$ (see [Wac16a, Definition 5.4.9]).
Let $d \in \mathbb{N}\backslash\{0\}$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded, open domain. Associated to this domain we denote the *Borel algebra* $\mathcal{B}(\Omega)$ as the smallest σ-algebra generated by the system of open subsets of Ω. The Lebesgue measure on the Borel-algebra is denoted by $\lambda^d : \mathcal{B}(\Omega) \to [0, \infty]$. For a set $A \in \mathcal{B}(\Omega)$ we the *characteristic function* of A is given by

$$\mathbb{1}_A(x) := \begin{cases} 1, & x \in A, \\ 0, & \text{else.} \end{cases}$$

For $p \in [1, \infty)$ denote the Lebesgue space as

$$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u|^p \mathrm{d}x < +\infty \right\}$$

with its elements only identified up to null sets, i.e. sets of Lebesgue measure zero. This space equipped with the norm $\|u\|_{L^p} := \left(\int_\Omega |u|^p \mathrm{d}x\right)^{\frac{1}{p}}$ is a Banach space for all $p \in [1, \infty)$ and a reflexive Banach space for $p \in (1, \infty)$. The *Sobolev spaces* $W^{1,p}(\Omega)$ are defined as

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega; \mathbb{R}^d) \right\},$$

where ∇u denotes the distributional derivative of u. Equipped with the norm

$$\|u\|_{W^{1,p}} := \left(\|u\|_{L^p}^p + \sum_{i=1}^d \|\partial_i u\|_{L^p}^p \right)^{\frac{1}{p}},$$

the space $W^{1,p}(\Omega)$ is a Banach space, and for $p \in (1, \infty)$ a reflexive Banach space. For $p = 2$ one also denotes $H^1(\Omega) = W^{1,2}(\Omega)$.

3 K-Convex Operators

Targeting solution maps of variational inequalities and generalized equations, we introduce and briefly study K-convex operators in this section. Later, this class of operators will be of help to study convexity properties of non-linear and possibly non-smooth minimization problems.
We start with basic facts on the polar respectively dual cone associated with a subset $A \subseteq X$.

Definition 1. Let X be a topological vector space and $A \subseteq X$. The *polar cone of* A is defined as

$$A^\circ := \{x^* \in X^* : \langle x^*, x\rangle \le 0 \text{ for all } x \in A\}.$$

The *dual cone* A^+ is defined as $A^+ := -A^\circ = \{x^* \in X^* : \langle x^*, x\rangle \ge 0 \text{ for all } x \in A\}$.

Using the notation of Definition 1 above one observes $N_C(x) := (C - x)^\circ$ (cf. [Sch07, Definition 11.2.1 and Lemma 11.2.2]). Next we establish some calculus rules for the dual cone. For the statements as well as their proofs we refer to [RWW09, Corollary 11.25] (in finite dimensions) as well as to [BS00, Proposition 2.40]. However, we provide short proofs in the appendix.

Lemma 2. *Let X be a topological vector space and let the subsets $A, A_1, A_2 \subseteq X$ be given. Then the following assertions hold true.*

(i) If $A_1 \subseteq A_2$, then $A_2^+ \subseteq A_1^+$.

(ii) $A^+ = (\mathrm{cl}\,(A))^+$.

(iii) If A is a non-empty, closed, convex cone, then $A^{++} = A$.

(iv) If $0 \in A_1 \cap A_2$, then we have

$$(A_1 + A_2)^+ = A_1^+ \cap A_2^+.$$

(v) Let A_j be closed, convex cones, then it holds that

$$(A_1 \cap A_2)^+ = \mathrm{cl}\left(A_1^+ + A_2^+\right).$$

The notion of K-convex mappings is introduced next.

Definition 3. Let X, Y be topological vector spaces. Let a non-empty closed, convex cone $K \subseteq Y$ inducing a preorder relation $\ge_K$ on Y be given. A set-valued mapping $\Phi : X \rightrightarrows Y$ is called *K-convex*, if for all $t \in (0,1)$ and $x_0, x_1 \in X$ the relation

$$t\Phi(x_1) + (1-t)\Phi(x_0) \subseteq \Phi(tx_1 + (1-t)x_0) + K$$

holds true, and Φ is called K-concave if it is ($-K$)-convex, i.e., for all $x_0, x_1 \in X$ and $t \in (0,1)$

$$t\Phi(x_1) + (1-t)\Phi(x_0) \subseteq \Phi(tx_1 + (1-t)x_0) - K.$$

We note here that $\Phi : X \rightrightarrows Y$ is K-convex if and only if the epigraph $\mathrm{epi}_K\,(\Phi)$ of Φ with respect to K defined as

$$\mathrm{epi}_K\,(\Phi) := \{(x, y) : \Phi(x) \le_K y\}$$

is a convex subset of $X \times Y$. By defining the set-valued mapping $\Phi_K : X \rightrightarrows Y$ via $\Phi_K(x) := \Phi(x) + K$ one can rewrite

$$\mathrm{epi}_K\,(\Phi) = \mathrm{gph}(\Phi_K).$$

A special instance is the case of a single-valued operator $T : X \to Y$. Then the K-convexity reads

$$T(tx_1 + (1-t)x_0) \le_K tT(x_1) + (1-t)T(x_0)$$

for all $x_0, x_1 \in X$ and $t \in (0,1)$. It is noteworthy that for a convex set $\overline{C} \subseteq Y$ with $\overline{C} - K \subseteq \overline{C}$ its preimage under T is convex. In order to see this take $x_0, x_1 \in T^{-1}(\overline{C})$ and $t \in (0,1)$. By the K-convexity of T we obtain

$$T(tx_1 + (1-t)x_0) \in tT(x_1) + (1-t)T(x_0) - K \in \overline{C} - K \subseteq \overline{C},$$

and thus $tx_1 + (1-t)x_0 \in \overline{C}$. In this setting we can establish the following characterization.

Lemma 4. *Let $T : X \to Y$ be an operator with X, Y as in Definition 3 and let $DT : X \to \mathcal{L}(X, Y)$ denote its first-order Fréchet-derivative. Then the following statements are equivalent:*

(i) T is K-convex.

(ii) For all $y^* \in K^+$ the functional $x \mapsto \langle y^*, T(x)\rangle$ is convex.

(iii) If T is continuously Fréchet-differentiable, then for all $x_1, x_0 \in X$ it holds that

$$DT(x_0)(x_1 - x_0) + T(x_0) \leq_K T(x_1).$$

(iv) If T is continuously differentiable, then for all $x_1, x_0 \in X$ it holds that

$$(DT(x_1) - DT(x_0))(x_1 - x_0) \geq_K 0.$$

(v) If, moreover, T is twice continuously differentiable, then for all $x \in X$ and $d \in X$ it holds that

$$D^2T(x)(d, d) \geq_K 0.$$

Proof. Consider $K^+ = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in K\}$. Then we know by *(iii)* in Lemma 2 that $y \in K$ if and only if $y \in K^{++}$, which is equivalent to $\langle y^*, y\rangle \geq 0$ for all $y^* \in K^+$. By this we get that T is K-convex if and only if the functionals $x \mapsto \langle y^*, T(x)\rangle$ are convex for all $y^* \in K^+$, which proves the first assertion. For the C^1- and C^2-case we can hence utilize the characterization of convex functionals and obtain the equivalence of the remaining statements. □

3.1 K-Convexity for Solution Operators of Inverse Problems

We draw our attention to solution operators of equations and generalized equations involving set-valued operators $A : Y \rightrightarrows W$ of the form

$$w \in A(y).$$

Here $w \in W$ is given, whereas $y \in Y$ is the desired solution. In the following theorem we derive conditions on the operator A that guarantee the convexity of the solution mapping.

Theorem 5. *Let Y, W be Banach spaces both equipped with non-empty closed, convex cones $K \subseteq Y$ and $K_W \subseteq W$, respectively. Let $A : Y \rightrightarrows W$ be a set-valued operator fulfilling the following assumptions:*

(i) A is K_W-concave.

(ii) The mapping $A^{-1} : W \rightrightarrows Y$ is single-valued, its domain is W and it is K_W-K-isotone, i.e. for $w_1, w_0 \in W$ with $w_1 \geq_{K_W} w_0$ it holds that $A^{-1}(w_1) \geq_K A^{-1}(w_0)$.

Then the mapping $A^{-1} : W \rightrightarrows Y$ is K-convex.

Proof. Let $t \in (0, 1)$ and $w_0, w_1 \in W$. We denote by $y_j \in Y$ the unique solution of $w_j \in A(y_j)$ for $j = 0, 1$. Let $y \in Y$ be the solution of $tw_1 + (1 - t)w_0 \in A(y)$. Then we obtain by the assumed K_W-concavity

$$tw_1 + (1 - t)w_0 \in tA(y_1) + (1 - t)A(y_0) \subseteq A(ty_1 + (1 - t)y_0) - K_W.$$

Hence, there exists $k_W \in K_W$ with $tw_1 + (1-t)w_0 + k_W \in A(ty_1 + (1-t)y_0)$ and by the assumed isotonicity of the inverse A^{-1} we obtain

$$y = A^{-1}(tw_1 + (1-t)w_0) \leq_K A^{-1}(tw_1 + (1-t)w_0 + k_W) = ty_1 + (1-t)y_0,$$

which proves the K-convexity of A^{-1}. □

Next we prove the following important consequence of Theorem 5. It classifies the K-convexity of the solution map for a parametrized generalized equation of the form:
Given $u \in U, w \in W$, find $y \in Y$ such that

$$w \in A(u, y).$$

Corollary 6. *Consider the Banach spaces U, Y and W, the latter two equipped with non-empty, closed, convex cones $K \subseteq Y$ and $K_W \subseteq W$, respectively, and the set-valued operator $A : U \times Y \rightrightarrows W$. Let A fulfil the following assumptions:*

(i) The mapping A is K_W-concave.

(ii) For every fixed $u \in U$, the mapping $A(u, \cdot)^{-1} : W \rightrightarrows Y$ is single-valued and its domain is W. Moreover, the inverse is K_W-K-isotone, i.e., for all $w_0, w_1 \in W$ with $w_1 \geq_{K_W} w_0$ it holds that $A(u, \cdot)^{-1}(w_1) \geq_K A(u, \cdot)^{-1}(w_0)$.

Then the solution mapping $S : W \times U \to Y$ of the equation $w \in A(u, y)$ is K-convex.

Proof. In order to apply Theorem 5, we define $\bar{A} : U \times Y \rightrightarrows W \times U$ by

$$\bar{A}(v, y) := A(v, y) \times \{v\}$$

and equip the product spaces with the non-empty, closed, convex cones

$$\bar{K} := \{0\} \times K \subseteq U \times Y \text{ and } \bar{K}_W := K_W \times \{0\} \subseteq W \times U.$$

We check the conditions of the previous theorem:
The $\bar{K}_W$-concavity is immediately clear from the definition of $\bar{A}$. Considering the inverse, we see $\bar{A}^{-1}(w, u) = (A^{-1}(u, \cdot)(w), u)$ and obtain $(w_1, u_1) \geq_{\bar{K}_W} (w_2, u_2)$ if and only if $u_1 = u_2 =: u$ and $w_1 \geq_{K_W} w_2$. By our assumption it holds that $A^{-1}(u, \cdot)(w_1) \geq_K A^{-1}(u, \cdot)(w_2)$ and we deduce $\bar{A}^{-1}(w_1, u_1) \geq_{\bar{K}} \bar{A}^{-1}(w_2, u_2)$, which proves the isotonicity and by Theorem 5, the $\bar{K}$-convexity. Hence, we see that $(w, u) \mapsto (u, S(w, u)) = \bar{A}^{-1}(w, u)$ is $\bar{K}$-convex, which is equivalent to S being K-convex. □

The following corollary addresses a yet more specific form of the generalized equation.

Corollary 7. *Consider the Banach spaces U, Y and W and equip the latter two with the non-empty, closed, convex cones K and K_W, respectively. Let $A : Y \to W$ be invertible and K_W-concave and $B : U \to W$ be K_W-convex. Assume that A, B are Fréchet-differentiable and the operator $DA(y) \in \mathcal{L}(Y, W)$ has an isotone inverse, i.e., $w \geq_{K_W} 0 \Rightarrow DA(y)^{-1}w \geq_K 0$. Then the solution mapping $S : U \to Y$ of the equation*

$$A(y) = B(u)$$

is K-convex.

Proof. Consider the mapping $\widetilde{A}(u,y) := A(y) - B(u)$. Obviously the mapping is K_W-concave and $\widetilde{A}(u,\cdot\,)^{-1}$ is a singleton and defined on all of W. Moreover, it is K_W-K-isotone. Let therefore $w_1 \geq_{K_W} w_0$, and we see, writing $A(y_t) = tw_1 + (1-t)w_0$, that

$$\begin{aligned} A^{-1}(w_1) - A^{-1}(w_0) &= \int_0^1 D(A^{-1})(tw_1 + (1-t)w_0)(w_1 - w_0)\mathrm{d}t \\ &= \int_0^1 DA(y_t)^{-1}(w_1 - w_0)\mathrm{d}t \geq_K 0. \end{aligned}$$

The assertion follows by Corollary 6. □

We next illustrate these results by several practically relevant examples.

Example 8 (Semilinear Elliptic PDE)*.* Take $Y := H_0^1(\Omega)$ for a bounded, open domain $\Omega \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}\backslash\{0\}$ and consider the following PDE:
Given $w \in H^{-1}(\Omega)$, find $y \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta y + \Phi(y) &= w \text{ in } \Omega, \\ y &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing and concave function inducing a continuous superposition operator $\Phi : L^2(\Omega) \to L^2(\Omega)$. Then the solution operator $S : H^{-1}(\Omega) \to H_0^1(\Omega)$ is K-convex with respect to $K := \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. on } \Omega\}$. We apply Theorem 5 to show this result. For this purpose, let $W = H^{-1}(\Omega)$ and

$$K_W := K^+ = \left\{\xi \in H^{-1}(\Omega) : \langle \xi, v\rangle \geq 0 \text{ for all } v \in H_0^1(\Omega)\right\}.$$

By the assumed concavity of $\Phi : \mathbb{R} \to \mathbb{R}$ we obtain for an arbitrary test function $v \in K$ that

$$\begin{aligned} \langle A(ty_1 + (1-t)y_0), v\rangle_{H^{-1},H_0^1} &= t(\nabla y_1, \nabla v)_{L^2} + (1-t)(\nabla y_0, \nabla v)_{L^2} \\ &+ (\Phi(ty_1 + (1-t)y_0), v) \geq t\left((\nabla y_1, \nabla v)_{L^2} + (\Phi(y_1), v)_{L^2}\right) \\ &+ (1-t)\left((\nabla y_0, \nabla v)_{L^2} + (\Phi(y_0), v)_{L^2}\right) \\ &= t\langle A(y_1), v\rangle_{H^{-1},H_0^1} + (1-t)\langle A(y_0), v\rangle_{H^{-1},H_0^1} \end{aligned}$$

and hence the K_W-concavity of A. By the monotonicity of Φ the operator A is strongly monotone and moreover, it is continuous by the assumed continuity of Φ and hence we obtain its invertibility by the Browder-Minty Theorem, see [Cia13, Theorem 9.14-1]. Using Theorem 5 yields the claimed K-convexity.

As a practically relevant case, choose $\Phi(y) := -(-y)^+$, which is equivalent to the setting of [CMWC18]. Alternatively one might be interested in the non-linearities of the type $\Phi(y) = y^3(+y)$, connected to Ginzburg–Landau-type equations (cf. [IK96, Equation (4.12)], as well as [KS20, Section 5.1]) or $\Phi(y) = \sinh(y)$ in a two-dimensional setting, associated to problems arising in semiconductor physics (cf. [KS20, Section 5.2] as well as [FI92], [FI94]). Provided that the right hand side is non-positive, the state is non-positive as well. Then one can substitute Φ by $\tilde{\Phi}$, with $\tilde{\Phi}(y) = \min(\Phi(y), 0)$ and reobtain the setting described in Example 8.
In the setting of [KS20, Section 5] this can be guaranteed via a sign condition incorporated in the constraint set Z_{ad} of the controls. Along with the isotonicity of the objective chosen therein, the convexity of the corresponding deterministic minimization problem can be achieved in this particular case.

Example 9 (Variational Inequality). Let Y be a reflexive vector lattice with order cone K and consider a K^+-concave, demicontinuous (i.e. for all sequences $(y_n)_{n\in\mathbb{N}}$ with $y_n \to y$ in Y it holds $A(y_n) \rightharpoonup A(y)$ in Y^*, cf. [Rou05, Definition 2.3]) and strongly monotone (cf. [Rou05, Definition 2.1 (iii)]) operator $A : Y \to Y^*$, that is strictly T-monotone (cf. [Rod87, Equation (5.7)]), i.e.,

$$\langle A(y+z) - A(y), (-z)^+\rangle < 0 \text{ for } z \text{ with } (-z)^+ \neq 0.$$

Furthermore, let $C : U \rightrightarrows Y$ be a set-valued operator with a convex graph and values *with lower bound*, i.e. for all $u \in U$ it holds that $C(u) + K \subseteq C(u)$ and $y_0, y_1 \in C(u)$ implies $\min(y_0, y_1) \in C(u)$. Moreover, let $w \in Y^*$ be given. We consider the following variational inequality problem (VI): Find $y \in C(u)$ such that

$$w \in A(y) + N_{C(u)}(y), \tag{3}$$

where $N_{C(u)}(\,\cdot\,)$ denotes the normal cone mapping (see [AF90]). We have $N_{C(u)}(y) = (C(u) - y)^\circ$. Then the solution operator $S : Y^* \times U \to Y$ is K-convex:
Setting $\bar{A}(u, y) := A(y) + N_{C(u)}(y)$, we check the conditions of Corollary 6. The strong monotonicity and the existence theory for VIs (cf. [KS80]) yield the single-valuedness of $\bar{A}(u,\cdot\,)^{-1}$.
To prove the isotonicity condition take $w_1 \geq_{K_W} w_0$ and set $y_j = S(w_j, u)$ for $j = 0, 1$. By testing with $z_1 := \max(y_0, y_1) = y_1 + (y_0 - y_1)^+ \in C(u)$ and $z_0 := \min(y_0, y_1) = y_0 - (y_0 - y_1)^+ \in C(u)$ we obtain

$$\begin{aligned}\langle A(y_1), z_1 - y_1\rangle &= \langle A(y_1), (y_0 - y_1)^+\rangle \geq \langle w_1, (y_0 - y_1)^+\rangle \geq \langle w_0, (y_0 - y_1)^+\rangle \\ &\geq \langle A(y_0), (y_0 - y_1)^+\rangle = \langle A(y_0), y_0 - z_0\rangle\end{aligned}$$

and hence $\langle A(y_1) - A(y_0), (y_0 - y_1)^+\rangle \geq 0$, which implies $y_1 \geq_K y_0$ by the strict T-monotonicity.
To show the K^+-concavity of $\bar{A}$ we use the K^+-concavity of A. Since $C(u) + K \subseteq C(u)$ we can easily show that $N_{C(u)}(y) \subseteq -K^+$ and since $0 \in N_{C(u)}(y)$ for $y \in C(u)$ we have $N_{C(u)}(y) - K^+ = -K^+$. Letting $(u_j, y_j) \in \operatorname{gph}(C)$ for $j = 0, 1$ we obtain by the convexity of the graph $ty_1 + (1-t)y_0 \in C(tu_1 + (1-t)u_0)$, which implies $N_{C(tu_1+(1-t)u_0)}(ty_1 + (1-t)y_0) \neq \emptyset$. Hence, the concavity condition reads as

$$\begin{aligned}tN_{C(u_1)}(y_1) + (1-t)N_{C(u_0)}(y_0) &\subseteq N_{C(tu_1+(1-t)u_0)}(ty_1 + (1-t)y_0) - K^+ \\ &= -K^+,\end{aligned}$$

which is fulfilled since $N_{C(u_j)}(y_j) \subseteq -K^+$. Hence, we have checked the conditions of the theorem and can deduce the K-convexity of the solution operator.

An extension of the previous results for VIs to quasi-variational inequalities (QVIs) is the scope of the following result.

Corollary 10. *Consider the following quasi-variational inequality (QVI):*

$$\langle A(y) - f, v - y\rangle \geq 0 \text{ for all } v \in C(y),$$

with A and C as in the previous Example 9, a constant right hand side $f \in Y^*$ and $U = Y$. Let S denote again the solution operator corresponding to the equation (3) *as well. Then the set of supersolutions $\{y \in Y : S(y) \leq_K y\}$ is convex.*

Proof. Denoting by $\bar{S}$ the solution operator of the previous example, we observe that $y \mapsto S(y) = \bar{S}(f, y)$ is K-convex and hence also the mapping $T(y) := S(y) - y$ is K-convex, which implies the convexity of the set of supersolutions. □

We emphasize here that Corollary 10 assumes existence of a solution to the QVI. Existence proofs for QVI, however are complex problems in general. Here we simply refer to [HR19, Chapter 1] and the references therein.

4 Convex Optimization Problems

We are now interested in reduced minimization problems of the type

$$\min f(u) + g(S(u)), \tag{4}$$

with $f : U \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \bar{\mathbb{R}}$ convex, proper and lower semi-continuous. Here, U and Y denote Banach spaces. Note that (4) might be in general non-convex due to the presence of S. We are interested in studying existence, convexity and stationarity questions for (4). Some basic properties on S and $S_K(\,\cdot\,) := S(\,\cdot\,) + K$ with $S_K : U \rightrightarrows Y$, where $K \subseteq Y$ is a non-empty, closed, convex cone in the Banach space Y are collected first.

Lemma 11. *Let $S : U \to Y$ be a locally bounded, K-convex operator. Then the normal cone to the graph of S_K is characterized as*

$$\begin{aligned} N_{\mathrm{gph}(S_K)}(u, y) = \{(h^*, d^*) \in U^* \times Y^* : d^* &\in N_K(y - S(u)), \\ h^* &\in \partial\langle -d^*, S(\,\cdot\,)\rangle(u)\}. \end{aligned}$$

Proof. Defining the set

$$\mathcal{N} := \{(h^*, d^*) \in U^* \times Y^* : d^* \in N_K(y - S(u)), h^* \in \partial\langle -d^*, S(\,\cdot\,)\rangle(u)\}$$

we have to prove $N_{\mathrm{gph}(S_K)}(u, y) = \mathcal{N}$:
Step 1: $N_{\mathrm{gph}(S_K)}(u, y) \subseteq \mathcal{N}$.
For $u \in U$ and $y \in S_K(u)$ we have by definition that $(h^*, d^*) \in N_{\mathrm{gph}(S_K)}(u, y)$ if and only if

$$\langle h^*, v - u\rangle + \langle d^*, z - y\rangle \leq 0 \text{ for all } (v, z) \in \mathrm{gph}(S_K).$$

Since $S_K(\,\cdot\,) = S(\,\cdot\,) + K$ we can write $y = S(u) + k$ and $z = S(v) + \tilde{k}$, with $k, \tilde{k} \in K$ respectively. Taking $v = u$ we obtain

$$\langle d^*, z - y\rangle = \langle d^*, \tilde{k} - k\rangle \leq 0 \text{ for all } \tilde{k} \in K,$$

which yields $d^* \in N_K(k) = N_K(y - S(u))$ $(\subseteq -K^+)$. Hence, we obtain with $z = S(v)$ and $v \in U$ that

$$\langle h^*, v - u\rangle \leq \langle -d^*, S(v) - S(u)\rangle + \langle d^*, k\rangle \leq \langle -d^*, S(v)\rangle - \langle -d^*, S(u)\rangle$$

for all $v \in U$. Since $-d^* \in K^+$, the functional $\langle -d^*, S(\,\cdot\,)\rangle : U \to \mathbb{R}$ is convex and the above inequality characterizes

$$h^* \in \partial\left(\langle -d^*, S(\,\cdot\,)\rangle\right)(u).$$

So we obtain $N_{\mathrm{gph}(S_K)}(u, y) \subseteq \mathcal{N}$.
Step 2: $\mathcal{N} \subseteq N_{\mathrm{gph}(S_K)}(u, y)$.
Take on the other hand $(h^*, d^*) \in \mathcal{N}$. Then we get for arbitrary $v \in U$ and $z = (S(v) + \tilde{k}) \in S_K(v)$ with some $\tilde{k} \in K$ that

$$\begin{aligned} \langle h^*, v - u\rangle + \langle d^*, z - y\rangle &= \langle h^*, v - u\rangle + \langle d^*, S(v) - S(u)\rangle + \langle d^*, \tilde{k} - k\rangle \\ &\leq 0 + 0 = 0, \end{aligned}$$

which proves the equality. □

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From the above lemma we are able to formulate the coderivative of the multifunction $S_K : U \rightrightarrows Y$ in $(u, y) \in \mathrm{gph}(S_K)$, see [Mor06, Definition 1.32] for the general definition of the coderivative of a mapping. In fact, we have

$$\begin{aligned} D^*S_K(u,y)(y^*) &= \{u^* \in U^* : (u^*, -y^*) \in N_{\mathrm{gph}(S_K)}(u,y)\} \\ &= \begin{cases} \partial\langle y^*, S(\cdot)\rangle(u), & \text{if } -y^* \in N_K(y - S(u)), \\ \emptyset, & \text{else.} \end{cases} \end{aligned}$$

Based on this, we define for every $u \in U$ the mapping $D^*S(u) : K^+ \rightrightarrows Y^*$ as

$$D^*S(u)(y^*) := D^*S_K(u, S(u))(y^*) = \partial\langle y^*, S(\,\cdot\,)\rangle(u).$$

and deduce from the standard sum rule, the linearity relation

$$D^*S(u)(\lambda y_1^* + y_2^*) = \lambda D^*S(u)(y_1^*) + D^*S(u)(y_2^*)$$

for all $y_1^*, y_2^* \in K^+$ and $\lambda \geq 0$. Next we establish the convexity of $g \circ S$ under K-convexity of S. Moreover, we characterize the subdifferential of this superposition.

Lemma 12. *Let U, Y be real Banach spaces, the latter one equipped with a non-empty, closed, convex cone K. Let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous, proper and K-isotone functional. Suppose $S : U \to Y$ is a locally bounded, K-convex operator. Then $g \circ S : U \to \mathbb{R} \cup \{+\infty\}$ is convex as well.*
Moreover, consider $u \in U$ with $S(u) \in \mathcal{D}(\partial g)$ and let the following constraint qualification hold

$$0 \in \mathrm{core}\,(S(U) - \mathrm{dom}\,(g))\,.$$

Then for the subdifferential it holds that

$$\partial(g \circ S)(u) = D^*S(u)\left(\partial g(S(u))\right) = \bigcup_{y^* \in \partial g(S(u))} \partial\langle y^*, S(\,\cdot\,)\rangle(u).$$

Proof. Let $u \in U$ be as above and define $M := \bigcup_{y^* \in \partial g(S(u))} \partial\langle y^*, S(\,\cdot\,)\rangle(u)$. By the assumption on u we get $\partial g(S(u)) \neq \emptyset$. Let $y^* \in \partial g(y)$ for some $y \in \mathcal{D}(\partial g)$. Then we obtain for $k \in K$, that

$$0 \geq g(y - k) - g(y) \geq \langle y^*, y - k - y\rangle = -\langle y^*, k\rangle$$

and hence we obtain $\partial g(y) \subseteq K^+$ and further the convexity of $u \mapsto \langle y^*, S(u)\rangle$. By the local boundedness of S we obtain local boundedness of $u \mapsto \langle y^*, S(u)\rangle$ as well and by [ET76, Lemma 2.1] also its continuity on all of U. So the set M is well defined in the sense of the convex subdifferential and non-empty.
Take $u^* \in M$. Then there exists $y^* \in \partial g(S(u))$ with $u^* \in \partial\langle y^*, S(\,\cdot\,)\rangle(u)$ such that

$$g(S(v)) \geq g(S(u)) + \langle y^*, S(v) - S(u)\rangle \geq g(S(u)) + \langle u^*, v - u\rangle$$

and hence $M \subseteq \partial(g \circ S)(u)$. Since $M \neq \emptyset$ we can now take $u^* \in \partial(g \circ S)(u)$ and obtain by the Fenchel-Legendre identity, that

$$\langle u^*, u\rangle = (g \circ S)(u) + (g \circ S)^*(u^*).$$

Hence, we know that $u \in \mathrm{argmin}_{v \in U}(g(S(v)) - \langle u^*, v\rangle)$. Using the K-isotonicity this is equivalent to

$$(u, S(u)) \in \mathrm{argmin}_{v \in U, y \in Y}\left(g(y) - \langle u^*, v\rangle + i_{\mathrm{gph}(S_K)}(v, y)\right).$$

Hence, the first-order condition holds at $(u, S(u))$, i.e.,

$$0 \in \partial \left(g(\mathrm{pr}_Y(\,\cdot\,)) - \langle u^*, \mathrm{pr}_U(\,\cdot\,)\rangle + i_{\mathrm{gph}(S_K)}(\,\cdot\,)\right)(u, S(u)). \tag{5}$$

By our constraint qualification we know that for every $z \in Y$ there exists a $t > 0$ such that $z \in t\,(S(U) - \mathrm{dom}\,(g))$, so there exists a pair $(u_1, y_2) \in U \times \mathrm{dom}\,(g)$ with $z = t(S(u_1) - y_2)$. For an arbitrary $v \in U$ choose $u_2 = u_1 - \frac{1}{t}v$ and $y_1 = S(u_1)$ and obtain $v = t(u_1 - u_2)$ and $z = t(y_1 - y_2)$, which means that $(v, z) \in t\,(\mathrm{gph}(S_K) - U \times \mathrm{dom}\,(g))$ and hence we get the constraint qualification $0 \in \mathrm{core}\,(\mathrm{gph}(S_K) - U \times \mathrm{dom}\,(g))$. This allows us to use the sum rule in the inclusion (5), which yields

$$0 \in \{-u^*\} \times \partial g(S(u)) + N_{\mathrm{gph}(S_K)}(u, S(u)).$$

Utilizing Lemma 11 we deduce the existence of $y^* \in \partial g(S(u))$ together with $d^* \in N_K(S(u) - S(u)) = -K^+$ as well as $h^* \in \partial\langle -d^*, S(\,\cdot\,)\rangle(u)$ such that

$$\begin{aligned} 0 &= y^* + d^*,\\ 0 &= -u^* + h^*, \end{aligned}$$

which yields $u^* = h^* \in \partial\langle -d^*, S(\,\cdot\,)\rangle(u) = \partial\langle y^*, S(\,\cdot\,)\rangle(u)$ and eventually $u^* \in M$. □

The convexity of the objective in (4) is addressed next, and a subdifferential relation is derived.

Corollary 13. *Let U, Y be Banach spaces, the latter one equipped with a closed, convex cone K. Let $f : U \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, lower semi-continuous functionals, and moreover let g be K-isotone. Consider $S : U \to Y$ a locally bounded, K-convex operator. Then the functional $f + g \circ S : U \to \mathbb{R} \cup \{+\infty\}$ is convex. Moreover, consider $u \in \mathcal{D}(\partial f)$ with $S(u) \in \mathcal{D}(\partial g)$ and let the following constraint qualification hold:*

$$0 \in \mathrm{core}\,(\mathrm{dom}\,(f) \times \mathrm{dom}\,(g) - \mathrm{gph}(S)).$$

Then the subdifferential reads as

$$\partial(f + g \circ S)(u) = \partial f(u) + D^*S(u)\left(\partial g(S(u))\right).$$

Proof. Consider the functional $h(u, y) := f(u) + g(y)$ together with the convex, closed cone $\bar{K} := \{0\} \times K$ and the operator $T : U \to U \times Y$ defined by $T(u) := (u, S(u))$. Then we see that the operator T is $\bar{K}$-convex and locally bounded, and the functional h is convex, proper, lower semi-continuous and $\bar{K}$-isotone. By assumption on u we have $(u, S(u)) = T(u) \in \mathcal{D}(\partial h) = \mathcal{D}(\partial f) \times \mathcal{D}(\partial g)$ and the constraint qualification reads as $0 \in \mathrm{core}\,(\mathrm{dom}\,(h) - T(U))$. Hence, we are in the position to use Lemma 12 and obtain with

$$\begin{aligned} D^*T(u)(u^*, y^*) &= \partial\langle (u^*, y^*), T(\,\cdot\,)\rangle(u) = \partial\langle u^*, \cdot\rangle(u) + \partial\langle y^*, S(\,\cdot\,)\rangle(u)\\ &= u^* + D^*S(u)(y^*) \end{aligned}$$

finally for the subdifferential that

$$\begin{aligned} \partial(f + g \circ S)(u) = \partial(h \circ T)(u) &= D^*T(u)\left(\partial h(T(u))\right)\\ &= D^*T(u)\left(\partial f(u) \times \partial g(S(u))\right)\\ &= \partial f(u) + D^*S(u)\left(\partial g(S(u))\right). \end{aligned}$$

□

Next we propose a variant of the previous corollary using a different constraint qualification. For this purpose we need a generalization of the Moreau-Rockafellar theorem suitable for our framework. For this sake we adapt the techniques in [BZ06, Section 4.3].

Lemma 14. *Let U, Y be Banach spaces, the latter one equipped with a closed, convex cone K. Let $f : U \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, lower semi-continuous functionals and moreover let g be K-isotone. Consider $S : U \to Y$ a* demi-continuous*, K-convex operator. Suppose the following constraint qualification to be satisfied*

$$0 \in \operatorname{core}(\operatorname{dom}(g) - S(\operatorname{dom}(f))).$$

Then there exists $y^* \in Y^*$ such that for all $u \in U$ and $y \in Y$ it holds that

$$\inf_{u \in U} (f(u) + g(S(u))) \leq \Big(f(u) + \langle y^*, S(u)\rangle\Big) + \Big(g(y) - \langle y^*, y\rangle\Big).$$

Proof. The lemma and the proof are strongly based on the one of [BZ06, Lemma 4.3.1].
Define the functional $h : Y \to [-\infty, +\infty]$ by

$$h(y) := \inf_{u \in U} (f(u) + g(S(u) + y)).$$

Then h is a convex functional with $\operatorname{dom}(h) = \operatorname{dom}(g) - S(\operatorname{dom}(f))$. We show that $0 \in \operatorname{int}(\operatorname{dom}(h))$. Without loss of generality we assume $f(0) = g(S(0)) = 0$ (else take $\bar{u} \in \operatorname{dom}(f)$, $\bar{y} \in \operatorname{dom}(g)$ and consider $\bar{f}(u) := f(u + \bar{u}) - f(\bar{u})$ and $\bar{g}(y) := g(y + \bar{y} - S(0)) - g(\bar{y})$). Define the set

$$M := \bigcup_{u \in \mathbb{B}_U} \{y \in Y : f(u) + g(S(u) + y) \leq 1\}.$$

It is straightforward to argue the convexity of M. We show that M is *absorbing* and *cs-closed*. We start by showing the former. For this purpose let $y \in Y$. We need to prove, that there exists $r > 0$ such that $\lambda y \in M$ for $|\lambda| \leq r$. By the assumed constraint qualification $0 \in \operatorname{core}(\operatorname{dom}(g) - S(\operatorname{dom}(f)))$, there exists $\bar{t} > 0$ with $ty \in \operatorname{dom}(g) - S(\operatorname{dom}(f))$ for all $|t| \leq \bar{t}$. Hence, there exist $u_\pm \in \operatorname{dom}(f)$ with $S(u_\pm) \pm ty \in \operatorname{dom}(g)$ and we define

$$\alpha := \max(f(u_\pm) + g(S(u_\pm) \pm \bar{t}y), 1) < \infty.$$

Then we see for $|t| \leq \bar{t}$ with $u_t := |t|\bar{t}^{-1}u_{\operatorname{sign}(t)}$ and $u_t \in \mathbb{B}_U$ that

$$\begin{aligned} f(u_t) + g(S(u_t) + ty) &= f\left(\frac{|t|}{\bar{t}}u_{\operatorname{sign}(t)}\right) + g\left(S\left(\frac{|t|}{\bar{t}}u_{\operatorname{sign}(t)}\right) + \frac{|t|}{\bar{t}}\operatorname{sign}(t)\,\bar{t}y\right) \\ &\leq \frac{|t|}{\bar{t}}\left(f(u_{\operatorname{sign}(t)}) + g(S(u_{\operatorname{sign}(t)}) + \operatorname{sign}(t)\,\bar{t}y)\right) \leq \frac{|t|}{\bar{t}}\alpha \leq \alpha \end{aligned}$$

Choose now $m := \max(\|u_\pm\|, \alpha, 1)$. Then we see that $\frac{u_t}{m} \in \mathbb{B}_U$ and further

$$f\left(\frac{u_t}{m}\right) + g\left(S\left(\frac{u_t}{m}\right) + \frac{t}{m}y\right) \leq \frac{1}{m}(f(u_t) + g(S(u_t) + ty)) \leq \frac{\alpha}{m} \leq 1.$$

Hence, we can choose $r = \frac{\bar{t}}{m}$ to obtain $\lambda y \in M$ for all $\lambda \leq r$. To prove the cs-closedness take $y = \sum_{k=1}^\infty \lambda_k y_k$ where $\lambda_k \geq 0$, $\sum_{k=1}^\infty \lambda_k = 1$, and $(y_k)_{k\in\mathbb{N}}$ is a sequence in M. By the definition of M there exist $(u_k)_{k\in\mathbb{N}} \subseteq \mathbb{B}_U$ with

$$f(u_k) + g(S(u_k) + y_k) \leq 1 \text{ for all } k \in \mathbb{N}.$$

Since $\mathbb{B}_U$ is bounded and closed it is cs-closed and hence also cs-compact (cf. [Jam74, Theorem 22.2]). By this we set $u := \sum_{k=1}^{\infty} \lambda_k u_k$. Since the operator S is assumed to be demi-continuous and g is convex, lower semi-continuous and hence weakly lower semi-continuous we obtain

$$f(u) + g(S(u) + y) \leq 1,$$

which yields $y \in M$.
Due to the cs-closedness we obtain $\operatorname{core}(M) = \operatorname{int}(M)$ by [Sch07, Proposition 1.2.3] and since M is absorbent $0 \in \operatorname{core}(M)$, which implies $0 \in \operatorname{int}(\operatorname{dom}(h))$. From this we see that $\partial h(0) \neq \emptyset$ and take $y^* \in \partial h(0)$. Hence, we observe for all $u \in U$ and $y \in Y$ that

$$\begin{aligned}
\inf_{u \in U} \Big(f(u) + g(S(u)) \Big) &= h(0) \leq h(y - S(u)) - \langle y^*, y - S(u) \rangle \\
&\leq f(u) + g(S(u) + y - S(u)) - \langle y^*, y - S(u) \rangle \\
&\leq \Big(f(u) + \langle y^*, S(u) \rangle \Big) + \Big(g(y) - \langle y^*, y \rangle \Big),
\end{aligned}$$

which proves the assertion. □

We are now ready to state another version of a chain rule slightly different to the one given in Corollary 13.

Theorem 15. *Let U, Y be Banach spaces, the latter one equipped with a closed, convex cone K. Let $f : U \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, lower semi-continuous functionals and moreover let g be K-isotone. Suppose $S : U \to Y$ to be a demi-continuous, K-convex operator. Then the functional $f + g \circ S : U \to \mathbb{R} \cup \{+\infty\}$ is convex. Moreover, consider $u \in \mathcal{D}(\partial f)$ with $S(u) \in \mathcal{D}(\partial g)$ and let the following constraint qualification hold*

$$0 \in \operatorname{core}\left(S\left(\operatorname{dom}(f)\right) - \operatorname{dom}(g)\right).$$

Then for the subdifferential it holds that

$$\partial(f + g \circ S)(u) = \partial f(u) + D^*S(u) \Big(\partial g(S(u)) \Big).$$

Proof. The inclusion $\partial f(u) + D^*S(u)\left(\partial g(S(u))\right) \subseteq \partial(f + g \circ S)(u)$ is straightforward and its proof will therefore be omitted here. To show the reverse direction let $u^* \in \partial(f + g \circ S)(u)$. Then we obtain by the Fenchel-Legendre identity the relation

$$f(u) + g(S(u)) + (f + g \circ S)^*(u^*) = \langle u^*, u \rangle.$$

Applying Lemma 14 to $f - \langle u^*, \cdot \rangle$ (instead of f) we deduce the existence of $y^* \in Y^*$ such that for all $v \in U$ and $y \in Y$ it holds that

$$\begin{aligned}
f(u) + g(S(u)) - \langle u^*, u \rangle &= -(f + g \circ S)^*(u^*) \\
&= \inf_{w \in U} \Big(f(w) - \langle u^*, w \rangle + g(S(w)) \Big) \\
&\leq f(v) - \langle u^*, v \rangle + \langle y^*, S(v) \rangle + g(y) - \langle y^*, y \rangle.
\end{aligned}$$

On the one hand, setting $v = u$ implies

$$g(S(u)) + \langle y^*, y - S(u) \rangle \leq g(y) \text{ for all } y \in Y,$$

which yields $y^* \in \partial g(S(u))$. Since g is assumed to be K-isotone it holds that $y^* \in K^+$.
On the other hand, setting $y = S(u)$ implies

$$f(u) + \langle y^*, S(u)\rangle + \langle u^*, v - u\rangle \leq f(v) + \langle y^*, S(v)\rangle \text{ for all } v \in U.$$

Hence, we see $u^* \in \partial\left(f + \langle y^*, S(\,\cdot\,)\rangle\right)(u)$. Since S is defined on all of U, the second function has a domain equal to the entire space. Hence, we can apply the usual sum rule to deduce

$$u^* \in \partial f(u) + D^*S(u)(y^*) \subseteq \partial f(u) + D^*S(u)\left(\partial g(S(u))\right),$$

which proves the assertion. □

Next we compare the prerequisities of Theorem 15 and Corollary 13. For this purpose we need the following lemma.

Lemma 16. *Let U, Y be Banach spaces the latter one equipped with a closed, convex cone K. Then the following assertions hold:*

(i) If S is demi-continuous, then it is locally bounded.

(ii) Let $S : U \to Y$ be a K-convex operator. If S is locally bounded and K is an order cone (i.e.: $K \cap (-K) = \{0\}$), then S is demi-continuous.

Proof. ad (i): If S is not locally bounded, then there exists a point $u \in U$ such that for all $n \in \mathbb{N}$ there exists $u_n \in u + \frac{1}{n}\mathbb{B}_U$ with $\|S(u_n)\|_Y \geq n$. Then we have holds $u_n \to u$ in U and by the demi-continuity $S(u_n) \rightharpoonup S(u)$ in Y implying the boundedness of $(S(u_n))_{n\in\mathbb{N}}$ — a contradiction.
ad (ii): We consider first $y^* \in K^+$. Then the mapping $u \mapsto \langle y^*, S(u)\rangle$ is convex and locally bounded from above in every point and hence continuous by [ET76, Lemma 2.1]. Then we deduce the continuity of the functional also for $y^* \in K^+ - K^+$. Let now $y^* \in Y^*$ be arbitrary. By the calculus rules of the dual cone in Lemma 2 we see that

$$\operatorname{cl}\left(K^+ - K^+\right) = \operatorname{cl}\left(K^+ + (-K)^+\right) = (K \cap (-K))^+ = \{0\}^+ = Y^*.$$

So for every $\varepsilon > 0$ we find $y^*_\varepsilon \in K^+ - K^+$ such that $\|y^* - y^*_\varepsilon\|_{Y^*} < \varepsilon$. Taking now a convergent sequence $u_n \to u$ we get by assumption the boundedness of $S(u_n)$ by some constant B. This yields

$$\begin{aligned}
|\langle y^*, S(u_n)\rangle - \langle y^*, S(u)\rangle| &\leq |\langle y^*_\varepsilon, S(u_n)\rangle - \langle y^*_\varepsilon, S(u)\rangle| \\
&\quad + |\langle y^* - y^*_\varepsilon, S(u_n) - S(u)\rangle| \\
&\leq |\langle y^*_\varepsilon, S(u_n)\rangle - \langle y^*_\varepsilon, S(u)\rangle| + 2B\varepsilon.
\end{aligned}$$

Using the continuity of $\langle y^*_\varepsilon, S(\,\cdot\,)\rangle$ the first term tends to zero as $n \to \infty$, and we finally see that

$$0 \leq \limsup_{n\to\infty} |\langle y^*, S(u_n)\rangle - \langle y^*, S(u)\rangle| \leq 2B\varepsilon.$$

Since the choice of ε was arbitrary we deduce the desired continuity of $u \mapsto \langle y^*, S(u)\rangle$ and hence the demi-continuity of S. □

From this lemma we see that the condition on S, which appears strengthened, is traded with a slightly weaker constraint qualification. Interestingly, the above lemma can also be interpreted as a generalization of [Har77, Theorem 3, Part (a)]. This has the following consequence: Having a vector lattice Y with order cone K, we obtain that the mapping $y \mapsto y^+ = \max(0, y)$ is demi-continuous if and only if it is locally bounded (see also [Har77, Proposition 1]).

4.1 Characterization of Subdifferential for K-Convex Solution Oprators

As we have seen in the derivation of the normal cone of S_K and the chain rule, the subdifferential mapping $y^* \mapsto D^*S(u)(y^*) = \partial\langle y^*, S(\,\cdot\,)\rangle$ is of paramount importance. Returning to our initial motivation of investigating solution operators we want to derive a 'practical' characterization. We recall the generalized equation

$$w \in A(y),$$

for a K_W-concave operator with single-valued, isotone inverse $S = A^{-1} : W \to Y$. Then, for the subdifferential of the solution operator we obtain the following inversion formula.

Theorem 17. *Let $y^* \in K^+$. Then it holds that*

$$w^* \in D^*S(w)(y^*) \text{ if and only if } (-y^*, w^*) \in N_{\mathrm{gph}(A_{-K_W})}(S(w), w)$$

with $A_{-K_W}(y) := A(y) - K_W$ for $y \in Y$.

Proof. Let $w^* \in D^*S(w)(y^*)$. Then it holds that $w^* \in \partial\langle y^*, S(\,\cdot\,)\rangle(u)$ or in other words

$$\langle w^*, w' - w\rangle + \langle -y^*, S(w') - S(w)\rangle \leq 0 \text{ for all } w' \in W.$$

Taking now $w' = w - k_W$ with $k_W \in K_W$ and $w' \in A(y')$, which is the same as $y' = S(w')$, by the isotonicity of the solution operator we find $y' \leq_K S(w)$ and hence

$$\langle w^*, -k_W\rangle = \langle w^*, w' - w\rangle \leq \langle y^*, y' - S(w)\rangle \leq 0,$$

which yields $w^* \in K_W^+$. Hence, for $y' \in Y$ and $w' = \bar{w} - k_W$ with $\bar{w} \in A(y')$ and $k_W \in K_W$ we have

$$\begin{aligned}\langle w^*, w' - w\rangle + \langle -y^*, y' - S(w)\rangle = \langle w^*, -k_W\rangle &+ \langle w^*, \bar{w} - w\rangle \\ &+ \langle -y^*, S(w') - S(w)\rangle \leq 0,\end{aligned}$$

which means $(w^*, -y^*) \in N_{\mathrm{gph}(A_{-K_W})}(w, S(w))$.
For the other direction assume the latter. Then, take $w' = w - k_W \in A(S(w)) - K_W$ for an arbitrary $k_W \in K_W$. We have $\langle w^*, -k_W\rangle \leq 0$ and hence $w^* \in K_W^+$. For $w' \in A(y')$ the assumption yields

$$\langle w^*, w' - w\rangle + \langle -y^*, S(w') - S(w)\rangle \leq 0.$$

Since by assumption $y^* \in K^+$, the map $w \mapsto \langle y^*, S(w)\rangle$ is convex and hence the above reads $w^* \in \partial\langle y^*, S(\,\cdot\,)\rangle(w) = D^*S(w)(y^*)$. □

We continue by investigating the special case of $A : W \to Y$ being single-valued. In this case, we obtain the following result.

Corollary 18. *Let $S : W \to Y$ denote the solution operator of the equation $w = A(y)$ for an operator $A : W \to Y$ being K_W-concave with an isotone inverse. Then for $y^* \in K^+$ it holds that*

$$w^* \in D^*S(w)(y^*) \Leftrightarrow -y^* \in D^*(-A)(S(w))(w^*).$$

Proof. By Theorem 17 we have $w^* \in D^*S(w)(y^*)$ if and only if $(-y^*, w^*) \in N_{\mathrm{gph}(A_{-K_W})}(y, w)$ with $y = S(w)$. The latter is equivalent to

$$\langle -y^*, y' - y\rangle + \langle w^*, w' - w\rangle \le 0 \text{ for all } y' \in Y \text{ and } w' \in A(y') - K_W.$$

Setting $y' = y$ and $w' = w - k_W$ for an arbitrary $k_W \in K_W$ yields again $w^* \in K_W^+$. Since A is K_W-concave, the mapping $y \mapsto \langle w^*, -A(y)\rangle$ is a convex functional. Testing with arbitrary $y' \in Y$ and $w' = A(y')$, we obtain

$$\langle -y^*, y' - y\rangle + \langle w^*, -A(y)\rangle \le \langle w^*, -A(y')\rangle,$$

which yields $-y^* \in D^*(-A)(y)(w^*)$. The other direction follows as in the proof of Theorem 17. □

5 Applications

In the previous sections, our analysis has been carried out for an abstract framework. Now, we apply these theoretical findings to (generalized) equations in function spaces.

5.1 Application to a Class of Semilinear Elliptic PDEs

As a first application, the results of Section 4 are applied to a class of semilinear elliptic partial differential equations in the framework of Example 8. The characterization of the subdifferentials of the solution operator is presented in the following lemma.

Lemma 19. *Let $S : H^{-1}(\Omega) \to H_0^1(\Omega)$ denote the solution operator of the following elliptic PDE problem. Find $y \in H_0^1(\Omega)$:*

$$\begin{aligned} -\Delta y + \Phi(y) &= w \text{ in } \Omega, \\ y &= 0 \text{ on } \partial\Omega \end{aligned} \tag{6}$$

with $w \in H^{-1}(\Omega)$ given, $K = \{z \in H_0^1(\Omega) : z \ge 0 \text{ a.e. on } \Omega\}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ a continuous, non-decreasing and concave function inducing a continuous superposition operator $\Phi : L^2(\Omega) \to L^2(\Omega)$. Let $y^* \in K^+$. Then $w^* \in D^*S(w)(y^*)$ holds if and only if there exists a measurable function $m : \Omega \to \mathbb{R}$ with $-m(x) \in \partial(-\Phi)(y(x))$ a.e. such that the following PDE is satisfied

$$\begin{aligned} -\Delta w^* + m w^* &= y^* \text{ in } \Omega, \\ w^* &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{7}$$

Proof. First let $w^* \in D^*S(w)(y^*)$. Define the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$

$$\langle A(y), z\rangle_{H_0^1, H^{-1}} = (\nabla y, \nabla z)_{L^2(\Omega;\mathbb{R}^d)} + (\Phi(y), z)_{L^2(\Omega)}$$

By Corollary 18 this is equivalent to $-y^* \in D^*(-A)(y)(w^*)$, where it is also proven that $w^* \in K_W^+ = K$. Hence, $w^* \ge 0$ a.e. on Ω. For arbitrary $z \in H_0^1(\Omega)$ we obtain for y^* the following inequality:

$$\begin{aligned} \langle -y^*, z\rangle &\le -\langle -\Delta z + \Phi(y + z) - \Phi(y), w^*\rangle \\ &= -\langle \Phi(y + z) - \Phi(y), w^*\rangle - \langle -\Delta w^*, z\rangle \end{aligned} \tag{8}$$

and hence $\langle \Delta w^* + y^*, z\rangle \ge \langle \Phi(y+z) - \Phi(y), w^*\rangle$. Testing now with $z \in K$ yields $\langle \Delta w^* + y^*, z\rangle \ge 0$ by the non-decreasing nature of $\Phi : \mathbb{R} \to \mathbb{R}$. Hence, we can identify the distribution $\Delta w^* + y^*$ with a Borel measure μ. Let $E \in \mathcal{B}(\Omega)$ be a Borel set. Since $C_0^\infty(\Omega) \subseteq L^2(\Omega)$ is dense, there exists a sequence $\widetilde{\varphi}_n \in C_0^\infty(\Omega)$ with $\widetilde{\varphi}_n \to \mathbb{1}_E$ in $L^2(\Omega)$, where $\mathbb{1}_E$ denotes the characteristic function of E. Taking a subsequence we also obtain the pointwise convergence (Fischer–Riesz) and by setting $\varphi_n := \min\left(\max\left(\widetilde{\varphi}_n, 0\right), 1\right)$ we have a non-negative sequence in $H_0^1(\Omega)$ pointwise bounded by 1 and converging pointwise and in $L^2(\Omega)$ to $\mathbb{1}_E$. Using Fatou's Lemma we obtain

$$\begin{aligned} 0 \leq \mu(E) = \int_\Omega \mathbb{1}_E \mathrm{d}\mu = \int_\Omega \liminf_{n\to\infty} \varphi_n \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_\Omega \varphi_n \mathrm{d}\mu \\ \leq \liminf_{n\to\infty} \langle \Phi(y) - \Phi(y-\varphi_n), w^* \rangle = \int_E (\Phi(y) - \Phi(y-1)) w^* \mathrm{d}x < \infty, \end{aligned}$$

where we used $L^2(\Omega) \ni \Phi(y) - \Phi(y-1) \geq \Phi(y) - \Phi(y-\varphi_n) \geq 0$ a.e. on Ω as well as the continuity of $\Phi : \mathbb{R} \to \mathbb{R}$, which gives by dominated convergence the last equality. If $\lambda^d(E) = 0$, then we obtain $\mu(E) = 0$ and hence we infer that the measure μ is absolutely continuous with respect to the Lebesgue measure. Thus, by the Radon–Nikodym theorem there exists a non-negative function $\rho \in L^1(\Omega)$ with $\mu(E) = \int_E \rho \,\mathrm{d}x$ for all $E \in \mathcal{B}(\Omega)$. Testing with $E \subseteq \{w^* = 0\}$ yields as well $\mu(E) = 0$ and $\rho = 0$ on $\{w^* = 0\}$ and we rewrite $\rho = mw^*$ for a measurable function $m : \Omega \to \mathbb{R}$. Using the characterization in equation (8) we get

$$\int_\Omega \Big((-\Phi)(y+z) - (-\Phi)(y) + mz \Big) w^* \mathrm{d}x \geq 0$$

for all $z \in H_0^1(\Omega)$. Using the same density argument as before with $mw^* \in L^1(\Omega)$, we can as well test with $z = t\mathbb{1}_E \in L^\infty(\Omega)$ for $t \in \mathbb{R}$ and E again a Borel set. Hence, we find on $\{w^* > 0\}$ that for all $t \in \mathbb{R}$ it holds that

$$(-\Phi)(y+t) - (-\Phi)(y) \geq -mt \text{ a.e. on } \{w^* > 0\},$$

which, due to the convexity of $-\Phi$, implies $-m(x) \in \partial(-\Phi)(y(x))$ for almost all $x \in \{w^* > 0\}$ and since the values of m on $\{w^* = 0\}$ do not matter, we can without loss of generality deduce $-m(x) \in \partial(-\Phi)(y(x))$ on the entire domain Ω. Hence, we deduce that for all $z \in H_0^1(\Omega)$ it holds that

$$0 = \langle -\Delta w^* - y^*, z \rangle + \int_\Omega z \mathrm{d}\mu = \int_\Omega (\nabla w^* \cdot \nabla z + mw^* z) \,\mathrm{d}x - \langle y^*, z \rangle,$$

which is the weak formulation of the PDE in the assertion.

On the other hand let now m be a measurable function with $-m(x) \in \partial(-\Phi)(y(x))$ a.e. on Ω and let $w^* \in H_0^1(\Omega)$ be the solution of (7). For an arbitrary function $z \in H_0^1(\Omega)$ we find

$$\begin{aligned} (-\Phi)(y+z) - (-\Phi)(y) &\geq -mz \quad \text{a.e. on } \Omega \text{ as well as} \\ (-\Phi)(y-z) - (-\Phi)(y) &\geq mz \quad \text{a.e. on } \Omega. \end{aligned}$$

Together we get

$$\Phi(y+z) - \Phi(y) \leq mz \leq \Phi(y) - \Phi(y-z)$$

and since by assumption $\Phi : H_0^1(\Omega) \to L^2(\Omega)$ is well defined we obtain $mz \in L^2(\Omega)$. Since Φ is non-decreasing $m \geq 0$ a.e. on Ω holds, so testing (7) with $z = (-w^*)^+$ yields

$$\begin{aligned} 0 \geq -\|\nabla(-w^*)^+\|_{L^2(\Omega)}^2 \geq -\|\nabla(-w^*)^+\|_{L^2(\Omega)}^2 - \int_\Omega m \left((-w^*)^+\right)^2 \mathrm{d}x \\ = (\nabla w^*, \nabla(-w^*)^+) + \int_\Omega mw^*(-w^*)^+ \mathrm{d}x = \langle y^*, (-w)^+ \rangle \geq 0, \end{aligned}$$

from which we deduce $w^* \geq 0$. Multiplying (7) by the solution of (6) yields

$$\begin{aligned}\langle -y^*, z\rangle &= \langle \Delta w^* - m w^*, z\rangle = \langle \Delta z, w^*\rangle + \int_\Omega (-mz) w^* \mathrm{d}x \\ &\leq \langle \Delta z, w^*\rangle - \langle \Phi(y+z) - \Phi(y), w^*\rangle \\ &= -\langle -\Delta z + \Phi(y+z) - \Phi(y), w^*\rangle\end{aligned}$$

for all $z \in H_0^1(\Omega)$, which proves $y^* \in D^*A(y)(-w^*)$ and equivalently $w^* \in D^*S(w)(y^*)$. □

Lemma 19 is related to the results in [CMWC18] as follows:
Consider the case $\Phi(z) := -(-z)^+$ yielding the PDE

$$\begin{aligned}-\Delta y - (-y)^+ &= w \text{ in } \Omega, \\ y &= 0 \text{ on } \partial\Omega,\end{aligned}$$

for given $w \in H^{-1}(\Omega)$. Then for $y^* \in H^{-1}(\Omega)$ our subdifferential reads as $w^* \in D^*S(w)(y^*)$ if and only if there exists a function $m \in L^\infty(\Omega)$ with

$$\begin{aligned}0 \leq m &\leq 1 \text{ a.e. on } \{y = 0\}, \\ m &= 1 \text{ a.e. on } \{y < 0\}, \text{ and} \\ m &= 0 \text{ a.e. on } \{y > 0\},\end{aligned}$$

such that

$$\begin{aligned}-\Delta w^* + m w^* &= y^* \text{ on } \Omega, \\ w^* &= 0 \text{ on } \partial\Omega.\end{aligned}$$

This corresponds to the *strong-weak Bouligand subdifferential* calculated in [CMWC18]. For the sake of brevity we do not introduce the details of Bouligand subdifferentials here.

5.2 Applications to VI Solution Operators

In the following, special emphasis is put on the characterization of the subdifferential of the solution operator of the obstacle problem, which constitutes a specific variational inequality involving a second-order linear elliptic partial differential operator. For this purpose some aspects and results from *Capacity Theory* (cf. [BS00]) are needed.

5.2.1 Introduction to Capacity Theory and Capacitary Measures

For the sake of selfcontainment of the present work, we collect some basic definitions and results. Our exposition is strongly inspired by the one in [RW19]. For more details besides references mentioned below we refer to [BS00], [EG15].

Definition 20. (cf. [BS00, Definition 6.4.7], [BMA06, Section 5.8.2, Section 5.8.3], [DZ11, Definition 6.4])

(i) For a subset $A \subseteq \Omega$ the capacity in the sense of $H_0^1(\Omega)$ is defined by

$$\operatorname{cap}(A) := \inf\left\{\|v\|^2_{H_0^1(\Omega)} : v \in H_0^1(\Omega), v \geq 1 \text{ a.e in a neighborhood of } A\right\}.$$

(ii) A subset $\hat{\Omega} \subseteq \Omega$ is called quasi-open if for all $\varepsilon > 0$ there exists and open set $O_\varepsilon \subseteq \Omega$ such that $\hat{\Omega} \cup O_\varepsilon$ is open and $\operatorname{cap}(O_\varepsilon) < \varepsilon$ holds.

(iii) A subset $\hat{\Omega} \subseteq \Omega$ is called quasi-closed if its relative complement $\Omega \backslash A$ is quasi-open.

(iv) A function $v : \Omega \to [-\infty, +\infty]$ is called quasi-continuous (quasi lower semi-continuous, quasi upper semi-continuous) if for all $\varepsilon > 0$ there is an open set $O_\varepsilon \subseteq \Omega$ with $\operatorname{cap}(O_\varepsilon) < \varepsilon$ such that v is continuous (lower semi-continuous, upper semi-continuous) on $\Omega \backslash O_\varepsilon$.

In the same fashion as with the Lebesgue measure a pointwise property of a function on Ω is called to hold *quasi everywhere* if it holds on subsets that differ from the whole domain only by a set of capacity zero.
For two Borel sets $E_0, E_1 \in \mathcal{B}(\Omega)$ such that E_0 is a subset of E_1 up to a set of capacity zero, we also write $E_0 \subseteq E_1$. If both $E_0 \subseteq_q E_1$ and $E_1 \subseteq_q E_0$ hold, then we might also write $E_0 =_q E_1$.

Lemma 21. (cf. [BS00, p. 564, 565] with [Rud87, Theorem 2.18] for (i),(ii); [HW18, Lemmata 3.5, 3.7], [Wac14, Lemma A.4] for (iii)) *Let $\xi \in H^{-1}(\Omega)$ with $\langle \xi, v \rangle \geq 0$ for all $v \in H_0^1(\Omega)$ with $v \geq 0$ a.e. on Ω.*

(i) The functional ξ can be identified with a regular Borel measure on Ω which is finite on compact sets and which possesses the following property: For every Borel set $E \subseteq \Omega$ with $\operatorname{cap}(E) = 0$, we have $\xi(E) = 0$.

(ii) Every function $v \in H_0^1(\Omega)$ is ξ-integrable and it holds

$$\langle \xi, v \rangle_{H^{-1}, H_0^1} = \int_\Omega v \, \mathrm{d}\xi.$$

(iii) There exists a quasi-closed set $\operatorname{f-supp}(\xi) \subseteq \Omega$ with the property that for all $v \in H_0^1(\Omega)$ with $v \geq 0$ a.e. it holds that $\langle \xi, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$ if and only if $v = 0$ q.e. on $\operatorname{f-supp}(\xi)$. The set $\operatorname{f-supp}(\xi)$ is uniquely defined up to a set of zero capacity and is called the fine support *of ξ.*

One is able to extend the definition of Sobolev spaces to quasi-open subsets $\hat{\Omega} \subseteq \Omega$ by

$$H_0^1(\hat{\Omega}) = \left\{ v \in H_0^1(\Omega) : \; v = 0 \text{ q.e. on } \Omega \right\} \tag{9}$$

Definition 22. (cf. [DM87, Definition 2.1, 3.1]) Let $\mathcal{M}_0(\Omega)$ be the set of all Borel measures μ on Ω such that $\mu(E) = 0$ for every Borel set $E \subseteq \Omega$ with $\operatorname{cap}(E) = 0$ and such that μ is regular in the sense that $\mu(E) = \inf \{\mu(O) : O \text{ quasi-open }, E \subseteq_q O\}$. The set $\mathcal{M}_0(\Omega)$ is called the set of capacitary measures on Ω.

For a given capacitary measure $m \in \mathcal{M}_0(\Omega)$ and for a quasi-continuous function $v : \Omega \to \mathbb{R}$ we write $v \in L_m^2(\Omega)$ if $\int_\Omega |v|^2 \, \mathrm{d}m < +\infty$. Let $T_m \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ denote the solution operator which maps an $f \in H^{-1}(\Omega)$ to the solution of the following equation:

$$\int_\Omega \nabla y \nabla z \mathrm{d}x + \int_\Omega yz \, \mathrm{d}m = \langle f, z \rangle_{H^{-1}, H_0^1} \text{ for all } z \in H_0^1(\Omega).$$

Definition 23. (cf. [DM87, Section 5], [RW19, Definition 3.2, Lemma 3.4]) Let a sequence of capacitary measures $(m_n) \subseteq \mathcal{M}_0(\Omega)$ be given. We say that $(m_n)_{n\in\mathbb{N}}$ γ*-converges* towards $m \in \mathcal{M}_0(\Omega)$ if the sequence of operators (T_{m_n}) converges in the *weak operator topology* towards T_m, i.e., for all $h \in H^{-1}(\Omega)$ holds $T_{m_n} h \rightharpoonup T_m h$ in $H_0^1(\Omega)$. If $(m_n)_{n\in\mathbb{N}}$ γ-converges to m we write $m_n \xrightarrow{\gamma} m$.

Lemma 24. (cf. [RW19, Corollary 3.5]) *The γ-convergence on $\mathcal{M}_0(\Omega)$ is metrizable with the metric*

$$d_{\mathcal{M}_0}(m, m') := \|T_m(1) - T_{m'}(1)\|.$$

Moreover, $(\mathcal{M}_0(\Omega), d_{\mathcal{M}_0})$ is a complete metric space.

Theorem 25. (cf. [DMM87, Proposition 4.14]) *Let $(m_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_0(\Omega)$. Then there exists a subsequence $(m_{n_k})_{k\in\mathbb{N}}$ and a measure $m \in \mathcal{M}_0(\Omega)$ such that $m_{n_k} \xrightarrow{\gamma} m$.*

5.2.2 Calculation for VI Solution Operators

Our ultimate target in this paper is to study optimization problems with VI constraints. We proceed towards this goal by first studying the solution operator of the following VI:

$$\text{Find } y \in Y \text{ such that } w \in Ay + N_{C(u)}(y) \text{ in } Y^*, \tag{VI}$$

where Y is a reflexive vector lattice equipped with an order cone K and a set-valued mapping $C : U \rightrightarrows Y$ with a convex graph and values with lower bound, i.e. $C(u) + K \subseteq C(u)$ and for $y_0, y_1 \in C(u)$ holds $\min(y_0, y_1) \in C(u)$. Moreover, $A \in \mathcal{L}(Y, Y^*)$ is a continuous, coercive as well as strictly T-monotone operator, i.e. for all $z \in Y$ with $(-z)^+ \neq 0$ we have the relation $\langle Az, (-z)^+\rangle > 0$. By Example 9 of we know that the solution operator of (VI) is a K-convex mapping. For the calculation of its subdifferential we utilize the inversion formula given in Theorem 17 and obtain the following result.

Proposition 26. *Let $S : Y^* \times U \to Y$ denote the solution operator of* (VI) *and take $y^* \in K^+$. The subdifferential of S in (w, u) reads with $y := S(w, u)$ as*

$$\begin{aligned} D^*S(w,u)(y^*) = \{(w^*, u^*) \in Y \times U^* : \; & w^* \in K \cap \{Ay - w\}^\perp \text{ and} \\ & (u^*, A^*w^* - y^*) \in N_{\mathrm{gph}(C)}(u, y)\}. \end{aligned}$$

Proof. Our aim is the use of Theorem 17. For this purpose we introduce – as in the proof of Corollary 6 – the mapping $\bar{A} : U \times Y \to Y^* \times U$ defined by $\bar{A}(u, y) := (Ay + N_{C(u)}(y)) \times \{u\}$ and obtain as solution mapping $(w, u) \mapsto \bar{S}(w, u) := (u, S(w, u))$. From the inversion formula in Theorem 17, we infer for $y^* \in K^+$ that

$$(w^*, u^*) \in D^*S(w,u)(y^*) \text{ if and only if } (w^*, u^*) \in D^*\bar{S}(w,u)(0, y^*),$$

which is equivalent to

$$(0, -y^*, w^*, u^*) \in N_{\mathrm{gph}(\bar{A}_{-K^+\times\{0\}})}(u, S(w,u), w, u).$$

Hence, it is left to calculate the normal cone of the graph of $\bar{A} - (K^+ \times \{0\})$. For this sake let $(u, y, w, u) \in \mathrm{gph}(\bar{A}_{-K^+\times\{0\}})$ and $(-v^*, -y^*, w^*, u^*) \in N_{\mathrm{gph}(\bar{A}_{-K^+\times\{0\}})}(u, y, w, u)$. Since for $u' \in U$ it holds that $C(u') + K \subseteq C(u')$ we obtain for all $y' \in C(u')$ that $N_{C(u')}(y') \subseteq -K^+$, which yields $\bar{A}(u', y') - K^+ \times \{0\} = (Ay' - K^+) \times \{u'\}$. So with $\xi := Ay - w \in K^+$ we obtain for all $(w', u') = (Ay' - \xi', u') \in (Ay' - K^+) \times \{u'\}$ that

$$\begin{aligned} 0 &\geq \langle w' - w, w^*\rangle + \langle u^*, u' - u\rangle + \langle -v^*, u' - u\rangle + \langle -y^*, y' - y\rangle \\ &= -\langle \xi' - \xi, w^*\rangle + \langle A^*w^* - y^*, y' - y\rangle + \langle u^* - v^*, u' - u\rangle \end{aligned} \tag{$*$}$$

for all $\xi' \in K^+, u' \in U, y' \in C(u')$.
First, we test $(*)$ with $y' = y$ and $u' = u$. Then we get $\langle -w^*, \xi' - \xi \rangle \leq 0$ for all $\xi' \in K^+$. By setting $\xi' = \xi + k^+$ for a $k^+ \in K^+$ we see $\langle k^+, w^* \rangle \geq 0$ and using $k^+ = \xi$ especially $\langle \xi, w^* \rangle \geq 0$. Setting $\xi' = 0$ yields $\langle \xi, w^* \rangle \leq 0$ and thus $w^* \in K \cap \{Ay - w\}^\perp$. By testing with an arbitrary $u' \in U$ with $y' \in C(u')$ and $\xi' = 0$ we get $(u^* - v^*, A^*w^* - y^*) \in N_{\mathrm{gph}(C)}(u, y)$.
On the other hand let $w^* \in K \cap \{Ay - w\}^\perp$ such that $(u^* - v^*, A^*w^* - y^*) \in N_{\mathrm{gph}(C)}(y)$ and write again $\xi = Ay - u$. Then we get

$$\begin{aligned} 0 &\geq \langle \xi' - \xi, -w^* \rangle + \langle A^*w^* - y^*, y' - y \rangle + \langle u^* - v^*, u' - u \rangle \\ &= \langle (Ay' - \xi') - (Ay - \xi), w^* \rangle + \langle u^*, u' - u \rangle + \langle -v^*, u' - u \rangle + \langle -y^*, y' - y \rangle \end{aligned}$$

for all $\xi' \in K^+, u' \in U$ and $y' \in C(u')$. This implies $(-v^*, -y^*, w^*, u^*) \in N_{\mathrm{gph}(\bar{A}_{-K^+ \times \{0\}})}(u, y, w, u)$.
Summarizing we obtain for the operator S

$$\begin{aligned} D^*S(w, u)(y^*) &= D^*\bar{S}(w, u)(0, y^*) \\ &= \{(w^*, u^*) \in Y \times U^* : w^* \in K \cap \{Ay - w\}^\perp \text{ and} \\ &\qquad (u^*, A^*w^* - y^*) \in N_{\mathrm{gph}(C)}(u, y)\}, \end{aligned}$$

which yields the assertion. □

An important subclass of VIs is associated with $C(u) = C$ for all $u \in U$ with $C \subseteq Y$ a non-empty, closed, convex set. In this case, it holds that $N_{\mathrm{gph}(C)}(u, y) = N_{U \times C}(u, y) = \{0\} \times N_C(y)$ for all $y \in C$ and $u \in U$. This yields the following corollary.

Corollary 27. *Let $S : Y^* \to Y, w \mapsto y$, denote in the setting of Proposition 26 the solution operator of the following VI:*

$$\text{Find } y \in Y : \ w \in Ay + N_C(y) \text{ in } Y^*. \tag{10}$$

Then we obtain for $y := S(w)$ that

$$D^*S(w)(y^*) = \left\{ w^* \in Y : w^* \in K \cap \{Ay - w\}^\perp \text{ and } A^*w^* - y^* \in N_C(y) \right\}.$$

Proof. We apply Proposition 26 using $C(u) = C$ for all $u \in U$. Further one observes

$$w^* \in D^*S(w)(y^*) \Leftrightarrow (w^*, 0) \in D^*\bar{S}(w, u)(y^*),$$

where $\bar{S}$ is the solution operator defined in the proof of Proposition 26. This yields $-w^* \in K \cap \{Ay - w\}^\perp$ and $(0, A^*w^* - y^*) \in N_{\mathrm{gph}(C)}(u, \bar{S}(w, u)) = \{0\} \times N_C(S(w))$ and thus the assertion. □

Next we seek to find a precise characterization for *linear* VIs of the form

$$\text{Find } y \in C : \langle Ay - u, z - y \rangle \geq 0 \text{ for all } z \in C, \tag{11}$$

where $C \subseteq Y$ is a closed, convex and lower bounded subset of a reflexive vector lattice Y with order cone K, and A is a linear, coercive, strictly T-monotone operator in $\mathcal{L}(Y, Y^*)$. An important instance are VIs involving obstacle type constraints with $K = \{z \in H_0^1(\Omega) : z \geq 0 \text{ a.e. on } \Omega\}$ and $C := \{z \in H_0^1(\Omega) : z \geq \psi \text{ a.e. on } \Omega\}$ for an obstacle function $\psi \in H^1(\Omega)$ with $\psi|_{\partial\Omega} < 0$.
Interestingly, already the assumptions on the lower boundedness of the set guarantee C to be *polyhedric*, i.e., for all $y \in C$ and $v \in N_C(y)$ it holds that

$$T_C(y) \cap \{v\}^\perp = \mathrm{cl}\left(R_C(y) \cap \{v\}^\perp\right).$$

In order to see this, we utilize the result [Wac16a, Lemma 5.4.18], which we restate here for convenience.

Lemma 28. *Let a vector lattice Y be given such that the mapping $(\,\cdot\,)^+ : Y \to Y$ is demi-continuous. Let $C := \overline{C} \cap \underline{C}$ be a closed, convex, non-empty set with $\underline{C}$ bounded from below and $\overline{C}$ bounded from above. Then, the set C is n-polyhedric for all $n \in \mathbb{N}_0$, i.e.*

$$T_C(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp = \operatorname{cl}\left(R_C(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp\right).$$

This result guarantees the polyhedricity (even n-polyhedricity see [Wac16a]) for a lower bounded subset of a vector lattice, if the mapping $y \mapsto y^+$ is demi-continuous.
To see the latter, we equip the space Y with the scalar product

$$(y_1, y_2)_A := \frac{1}{2}\Big(\langle Ay_1, y_2\rangle + \langle Ay_2, y_1\rangle\Big)$$

and obtain by the coercivity and boundedness of A, that $(Y, (\cdot,\cdot)_A)$ is indeed a Hilbert space with the same topology as its norm-topology. Next [MR95, Proposition 1.3] (see Proposition 35 in the Appendix) yields that T-monotonicity is equivalent to $\langle Az^+, (-z)^+\rangle \le 0$ for all $z \in Y$. Using now [Har77, Corollary 1] we see that the operator $y \mapsto y^+$ is bounded and hence demi-continuous.
Using the polyhedricity it is possible to show (see again [Har77] as well as [Mig76]) that the solution operator $S : Y^* \to Y$ defined by $w \mapsto y$ is directionally differentiable. In the following we use parts of our preceding analysis for the calculation of the tangent and normal cone, but this time we want to utilize the results in [HS11] to obtain a second viewpoint and better understanding.

Lemma 29. *Let W, Y be Banach spaces the latter one equipped with a closed convex cone K. Moreover, assume that S is a K-convex and Hadamard differentiable operator. Let $(w, y) \in \operatorname{gph}(S_K)$. Then the tangent cone is characterized by*

$$\begin{aligned} T_{\operatorname{gph}(S_K)}(w, y) &= \{(v, z) \in W \times Y : z \in S'(w; v) + T_K(y - S(w))\} \\ &= \operatorname{gph}(S'(w;\,\cdot\,)) + \{0\} \times T_K(y - S(w)). \end{aligned}$$

Proof. Since $T_{\operatorname{gph}(S_K)}(w, y) = T_{\operatorname{gph}(S_K)}(w, y)^{\circ\circ} = N_{\operatorname{gph}(S_K)}(w, y)^\circ$ we obtain

$$\begin{aligned} T_{\operatorname{gph}(S_K)}(w, y) = \{(v, z) \in W \times Y : \langle h^*, v\rangle + \langle d^*, z\rangle \le 0 \text{ for all} \\ (h^*, d^*) \in N_{\operatorname{gph}(S_K)}(w, y)\}. \end{aligned}$$

Let $v \in W$ and $z = S'(w; v) + \xi$ with $\xi \in T_K(y - S(w))$. Taking $(h^*, d^*) \in N_{\operatorname{gph}(S_K)}(w, y)$ we know from Lemma 11, that $d^* \in N_K(y - S(w))$ and $h^* \in \partial\langle -y^*, S(\,\cdot\,)\rangle(w)$. Hence, we see for all $v \in W$ and $t > 0$, that

$$\langle h^*, tv\rangle \le \langle -d^*, S(w + tv)\rangle - \langle -d^*, S(w)\rangle.$$

Dividing by t and letting $t \searrow 0$ yields $\langle h^*, v\rangle \le \langle -d^*, S'(w; v)\rangle$. Using $T_K(y - S(u)) = N_K(y - S(u))^\circ$ one obtains

$$\langle h^*, v\rangle + \langle d^*, S'(w; v)\rangle + \langle d^*, \xi\rangle \le 0,$$

from which we deduce $(v, S'(w; v) + \xi) \in T_{\operatorname{gph}(S_K)}(w, y)$.
On the other hand, let $(v, z) \in T_{\operatorname{gph}(S_K)}(w, y)$. Then there exist $t_n \to 0$ and $(v_n, z_n) \to (v, z) \in W \times Y$ such that $(w + t_n v_n, y + t_n z_n) \in \operatorname{gph}(S_K)$. This implies $y + t_n z_n \ge_K S(w + t_n v_n)$ and hence

$$y - S(w) + t_n\left(z_n - \frac{S(w + t_n v_n) - S(w)}{t_n}\right) \in K.$$

The Hadamard-differentiability of S then yields $z - S'(w; v) \in T_K(y - S(w))$ by the closedness of K. □

We return our attention to the VI problem. The associated solution operator is directionally differentiable and Lipschitz continuous and hence Hadamard-differentiable (see [Har77]). The corresponding derivative $z = S'(w; v)$ solves the following VI:

$$\text{Find } z \in \mathcal{K}(w) : \ \langle Az - v, z' - z\rangle \geq 0 \text{ for all } z' \in \mathcal{K}(w),$$

with the critical cone $\mathcal{K}(w) = T_C(S(w)) \cap \{w - AS(w)\}^\perp$. This VI problem is equivalent to the following *complementarity system*

$$\text{Find } z : \ \mathcal{K}(w) \ni z \perp (v - Az) \in \mathcal{K}(w)^\circ. \tag{12}$$

Motivated by the latter, we rewrite the expression for the normal cone of $\operatorname{gph}(S_K)$. For the following result and its proof compare [HS11].

Lemma 30. *Let Y be a reflexive Banach space equipped with a closed convex cone K. Moreover, assume that S is the solution operator of the VI in* (11). *Let $(w, y) \in \operatorname{gph}(S_K)$. Then the normal cone is characterized by*

$$\begin{aligned} N_{\operatorname{gph}(S_K)}(w, y) = \{(h^*, d^*) \in Y \times Y^* : A^*h^* + d^* \in \mathcal{K}(w)^\circ, h^* \in \mathcal{K}(w),\\ d^* \in N_K(y - S(w))\} . \end{aligned}$$

Proof. We know by the calculus rules for the dual cone according to Lemma 2, that for two subsets A_1, A_2 of some Banach space — both containing zero — the relation $(A_1 + A_2)^\circ = (A_1 \cap A_2)^\circ$ holds. This follows upon recognizing $A_j^\circ = -A_j^+$. Since $0 \in \operatorname{gph}(S'(w; \cdot)) \cap (\{0\} \times T_K(y - S(w)))$ we find by Lemma 29 that

$$\begin{aligned} N_{\operatorname{gph}(S_K)}(w, y) &= \Big(\operatorname{gph}(S'(w; \cdot)) + \{0\} \times T_K(w, y)\Big)^\circ \\ &= \operatorname{gph}(S'(w; \cdot))^\circ \cap \Big(Y \times N_K(y - S(w))\Big) . \end{aligned}$$

Next we calculate the polar cone of the graph of the directional derivative (cf. [HS11, Proof of Theorem 4.6]). For this purpose define the sets $A_1 := \big(\mathcal{K}(w) \times \{0\}\big)$, $A_2 := \big(\{0\} \times \mathcal{K}(w)^\circ\big)$, $M := \{(\xi, z) \in Y^* \times Y : z \in \mathcal{K}(w), \xi \in \mathcal{K}(w)^\circ\}$ and $N := \{(\xi, z) \in M : \langle \xi, z\rangle = 0\}$. It obviously holds that

$$A_1 \cup A_2 \subseteq N \subseteq M = A_1 + A_2,$$

and by the calculus rules for dual cones in Lemma 2 we infer

$$A_1^\circ \cap A_2^\circ = (A_1 + A_2)^\circ = M^\circ \subseteq N^\circ \subseteq (A_1 \cup A_2)^\circ \subseteq A_1^\circ \cap A_2^\circ.$$

Hence, we have $M^\circ = N^\circ = A_1^\circ \cap A_2^\circ = \big(\mathcal{K}(w) \times \{0\}\big)^\circ \cap \big(\{0\} \times \mathcal{K}(w)^\circ\big)^\circ = \mathcal{K}(w)^\circ \times \mathcal{K}(w)$. Using (12) we eventually calculate

$$\begin{aligned} \operatorname{gph}(S'(w; \cdot))^\circ &= \{(h^*, d^*) : \langle v, h^*\rangle + \langle d^*, z\rangle \leq 0 \text{ for all } z \in \mathcal{K}(w),\\ &\qquad v - Az = \xi \in \mathcal{K}(w)^\circ, \langle \xi, z\rangle = 0\} \\ &= \{(h^*, d^*) : \langle \xi, h^*\rangle + \langle d^* + A^*h^*, z\rangle \leq 0 \text{ for all } \xi \in \mathcal{K}(w)^\circ,\\ &\qquad z \in \mathcal{K}(w), \langle \xi, z\rangle = 0\} \\ &= \{(h^*, d^*) : (d^* + A^*h^*, d^*) \in N^\circ\} \\ &= \{(h^*, d^*) : A^*h^* + d^* \in \mathcal{K}(w)^\circ, h^* \in \mathcal{K}(w)\} \end{aligned}$$

and finally

$$\begin{aligned} N_{\operatorname{gph}(S_K)}(w, y) = \{(h^*, d^*) \in Y \times Y^* : d^* \in N_K(y - S(w)), h^* \in \mathcal{K}(w),\\ A^*h^* + d^* \in \mathcal{K}(w)^\circ\}, \end{aligned}$$

which ends the proof. □

As a consequence, we are also able to derive an associated representation of $D^*S(w)$ in $y^* \in K^+$, namely

$$D^*S(w)(y^*) = \{w^* \in Y : w^* \in \mathcal{K}(w) \text{ and } A^*w^* - y^* \in \mathcal{K}(w)^\circ\}.$$

In this way we have derived two different expressions for the subdifferential of the solution operator of the VI in Corollary 27. Next we seek to get a better understanding of the relation between them and provide here a *direct* equivalence proof:

Let first $w^* \in K \cap (Ay - w)^\perp$ and $A^*w^* - y^* \in N_C(y)$. Since, by definition, $\mathcal{K}(w) \subseteq T_C(y)$ we get $A^*w^* - y^* \in N_C(y) = T_C(y)^\circ \subseteq \mathcal{K}(w)^\circ$. As $C + K \subseteq C$ by assumption we have $K \subseteq T_C(y)$ and therefore $w^* \in \mathcal{K}(w)$.

For the other direction choose $w^* \in Y$ with $w^* \in \mathcal{K}(w)$ and $A^*w^* - y^* \in \mathcal{K}(w)^\circ$. We use that the mapping $y \mapsto y^+$ is demi-continuous. The following argument is inspired by the proof of [DMG94, Proposition 2.6]. Let $d \in T_C(y)$ be arbitrarily chosen, and consider $v := \min(d, w^*)$. By the definition of the tangent cone there exist sequences $t_n \searrow 0$ and $d_n \to d$ as well as $w_n^* \to w^*$ such that $y + t_n d_n, y + t_n w_n^* \in C$. By the lower boundedness of C we obtain $\min(y + t_n d_n, y + t_n w_n^*) = y + t_n \min(d_n, w_n^*) \in C$ and using the demi-continuity as well as convexity and closedness of C and $T_C(y)$ we get $\min(d, w^*) = v \in T_C(y)$. Since $C + K \subseteq C$, we have $N_C(y) \subseteq K^+$, and hence $Ay - w \in -N_C(y)$ and $v \leq_K w^* \in K \cap \{Ay - w\}^\perp$. From this we infer

$$0 \leq \langle Ay - w, v\rangle \leq \langle Ay - w, w^*\rangle = 0,$$

and hence $v \in \mathcal{K}(w)$. Taking now $d = 0$, we get $\min(w^*, 0) \in \mathcal{K}(w)$ and

$$0 \geq \langle y^*, \min(w^*, 0)\rangle \geq \langle A^*w^*, \min(w^*, 0)\rangle = -\langle A^*w^*, (-w^*)^+\rangle$$

and by the strict T-monotonicity also $w^* \in K$, which proves the first assertion. Letting again $d \in T_C(y)$ arbitrary and setting $v_n := \min\left(\frac{1}{n}d, w^*\right) \in K$ we obtain using $y^* \in K^+$

$$\begin{aligned}
\langle y^*, d\rangle &\geq \langle y^*, nv_n\rangle \geq \langle A^*w^*, nv_n\rangle = \langle A^*w^*, \min(d, nw^*)\rangle \\
&= \langle A^*w^*, d\rangle - \langle A^*w^*, (d - nw^*)^+\rangle \\
&= \langle A^*w^*, d\rangle + \frac{1}{n}\langle A^*(d - nw^*), (d - nw^*)^+\rangle - \langle A^*d, \left(\tfrac{1}{n}d - w^*\right)^+\rangle \\
&\geq \langle A^*w^*, d\rangle - \langle A^*d, \left(\tfrac{1}{n}d - w^*\right)^+\rangle.
\end{aligned}$$

By the demi-continuity of the $\max$-operator we obtain

$$\left(\frac{1}{n}d - w^*\right)^+ \rightharpoonup (-w^*)^+ = 0.$$

Hence, we get by letting $n \to \infty$ that $\langle A^*w^* - y^*, d\rangle \leq 0$ for $d \in T_C(y)$ and hence $A^*w^* - y^* \in N_C(y)$. Next we study the special case of the obstacle problem. For this we let $Y = H_0^1(\Omega)$ equipped with the order cone $K := \{z \in H_0^1(\Omega) : z \geq 0 \text{ a.e. on } \Omega\}$. For the VI we assume $A = -\Delta$ and

$$C := \{z \in H_0^1(\Omega) : z \geq \psi \text{ a.e. on } \Omega\}.$$

with $\psi \in H^1(\Omega), \psi \leq 0$ on $\partial\Omega$. For $w \in H^{-1}(\Omega)$ we set $y = S(w)$ and define the inactive set $\mathcal{I}(y) := \{x \in \Omega : y(x) > \psi(x)\}$, the active set $\mathcal{A}(y) := \Omega \backslash \mathcal{I}(y)$ and the strictly active set as $\mathcal{A}_s(y) := \text{f-supp}(w + \Delta y)$. Then it can be shown, that the tangential cone of C in $y \in C$ reads

$$T_C(y) = \{z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } \mathcal{A}(y)\},$$

where 'q.e.' stands for 'quasi-everywhere' and the critical cone

$$\mathcal{K}(w) := \{z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } \mathcal{A}(y) \text{ and } z = 0 \text{ q.e. on } \mathcal{A}_s(y)\};$$

see [BS00] for more details. By the techniques involving capacitary measures from [RW19] and the references therein we deduce the following characterization of the subdifferential of S:

Lemma 31. *Let $w \in H^{-1}(\Omega)$ with $y = S(w)$ and $y^* \in K^+$. Then, $w^* \in D^*S(w)(y^*)$ if and only if there exists a capacitary measure $m \in \mathcal{M}_0(\Omega)$ such that*

$$m(\mathcal{I}(y)) = 0 \textit{ and } m = +\infty \textit{ on } \mathcal{A}_s(y),$$

and $w^* \in H_0^1(\Omega) \cap L_m^2(\Omega)$ solves the system

$$\begin{aligned} -\Delta w^* + m w^* &= y^* \textit{ in } \Omega, \\ w^* &= 0 \textit{ on } \partial\Omega, \ \textit{i.e.} \end{aligned} \tag{13}$$

for all $v \in H_0^1(\Omega) \cap L_m^2(\Omega)$ it holds that

$$(\nabla w^*, \nabla v)_{L^2(\Omega)} + \int_\Omega w^* v \, \mathrm{d}m = \langle y^*, v \rangle.$$

Proof. We use the first characterization of the subdifferential of the VI-solution operator, which we concluded from the inversion formula. Let first $w^* \in H_0^1(\Omega)$ be given with $w^* \in K \cap \{Ay - w\}^\perp$ and $-\Delta w^* - y^* \in N_C(y)$.
The latter implies $\Delta w^* + y^* \in K^+$ and according to Lemma 21 we can identify the functional with a non-negative Borel measure. Let $E \in \mathcal{B}(\Omega)$ be an arbitrary Borel set. We define the measure m as follows

$$m(E) := \begin{cases} \int_E \frac{1}{w^*} \mathrm{d}(y^* + \Delta w^*), & \text{if } \operatorname{cap}(E \cap \{w^* = 0\}) = 0, \\ +\infty, & \text{else.} \end{cases}$$

Since $\langle w + \Delta y, w^* \rangle = 0$ and $w^* \geq 0$ a.e. on Ω, we obtain that $\{w^* = 0\}$ q.e. on $\mathcal{A}_s(y)$ and hence $m = +\infty$ on $\mathcal{A}_s(y)$. Since $\langle y^* + \Delta w^*, v \rangle = 0$ for all $v \in H_0^1(\mathcal{I}(y))$ we see as well $\operatorname{cap}(\text{f-supp}(y^* + \Delta w^*) \cap \mathcal{I}(y)) = 0$ and hence $m(\mathcal{I}(y)) = 0$. It is left to show, that the system is fulfilled. At first we see that $w^* \in L_m^2(\Omega)$:

$$\begin{aligned} \int_\Omega w^{*2} \, \mathrm{d}m &= \int_{\{w^* \neq 0\}} w^{*2} \, \mathrm{d}m = \int_{\{w^* \neq 0\}} w^* \mathrm{d}(y^* + \Delta w^*) = \int_\Omega w^* \mathrm{d}(y^* + \Delta w^*) \\ &= \langle y^* + \Delta w^*, w^* \rangle < \infty. \end{aligned}$$

Take now $v \in H_0^1(\Omega) \cap L_m^2(\Omega)$. Then $v = 0$ q.e. on $\{w^* = 0\}$ by the construction of m, and we obtain

$$\begin{aligned} \int_\Omega w^* v \, \mathrm{d}m = \int_{\{w^* \neq 0\}} w^* v \, \mathrm{d}m &= \int_{\{w^* \neq 0\}} v \, \mathrm{d}(y^* + \Delta w^*) \\ &= \int_\Omega v \, \mathrm{d}(y^* + \Delta w^*) = \langle y^* + \Delta w^*, v \rangle, \end{aligned}$$

which proves the assertion.
To prove the other direction let now $m \in \mathcal{M}_0(\Omega)$ be a capacitary measure with $m(\mathcal{I}(y)) = 0$ and $m = +\infty$ on $\mathcal{A}_s(y)$. Let $w^* \in H_0^1(\Omega) \cap L_m^2(\Omega)$ denote the solution of the PDE involving m. Then we see that $w^* = 0$ q.e. on $\mathcal{A}_s(y)$, and since $y^* \in K^+$ we deduce by testing with $v = (-w^*)^+$ that

$$0 \geq -\|\nabla(-w^*)^+\|_{L^2(\Omega)}^2 - \int_{\{w^* < 0\}} (-w^*)^2 \, \mathrm{d}m = \langle y^*, (-w^*)^+ \rangle \geq 0,$$

and hence $w^* \geq 0$ a.e. on Ω, which proves $w^* \in K \cap \{Ay - w\}^\perp$. Let now $v \in T_C(w)$ and define similar to the proof of [DMG94, Proposition 2.6] $v_n := \min\left(\frac{1}{n}v, w^*\right)$. Then we see $0 \leq v_n \leq w^*$ q.e. on $\mathcal{A}(y)$ and $v_n = 0$ q.e. on $\mathcal{A}_s(y)$. Since $m(\mathcal{I}(y)) = 0$ we obtain

$$\int_\Omega v_n^2 \,\mathrm{d}m = \int_{\mathcal{A}(y)\setminus\mathcal{A}_s(y)} v_n^2 \,\mathrm{d}m \leq \int_{\mathcal{A}(y)\setminus\mathcal{A}_s(y)} w^{*2} \,\mathrm{d}m < \infty,$$

and hence $v_n \in H_0^1(\Omega) \cap L_m^2(\Omega)$. Testing (13) with v_n we obtain similarly to before

$$\begin{aligned}
\frac{1}{n}\langle y^*, v\rangle \geq \langle y^*, v_n\rangle &= (\nabla w^*, \nabla v_n)_{L^2(\Omega)} + \int_\Omega w^* v_n \,\mathrm{d}m \\
&= \int_{\{nw^* \leq v\}} |\nabla w^*|^2 \mathrm{d}x + \frac{1}{n}\int_{\{nw^* > v\}} \nabla w^* \cdot \nabla v \mathrm{d}x + \int_{A(y)} w^* v_n \,\mathrm{d}m \\
&\geq \frac{1}{n}\int_{\{nw^* > v\}} \nabla w^* \cdot \nabla v \mathrm{d}x.
\end{aligned}$$

We multiply by n, let $n \to \infty$ and obtain using $\nabla w^* = 0$ on $\{w^* = 0\}$ that $\langle y^* + \Delta w^*, v\rangle \geq 0$. Hence, we have $-\Delta w^* - y^* \in N_C(y)$. □

As mentioned before, also in this case the involved operator corresponds to the *strong-weak Bouligand subdifferential* (cf. [RW19]).

5.3 Application to a Nash Equilibrium Problem with VI-Constraint

As an optimization-theoretic application of the presented techniques we focus our attention now on Nash equilibrium problems. For this purpose, let a family of Banach spaces U_ν, $\nu = 1, \ldots, N$, be given. Define the product space $U := U_1 \times \cdots \times U_N$ together with a family of real-valued functionals $\mathcal{J}_\nu : U \to \mathbb{R}$ for all $\nu = 1, \ldots, N$. With the index $-\nu$ we denote strategies, where the ν-th component has been omitted. A joint strategy $(u_1, \ldots u_{\nu-1}, v_\nu, u_{\nu+1}, \ldots, u_N) \in U$ is written as $(v_\nu, u_{-\nu})$ – with no change of the ordering.

Definition 32. A point $u \in U_{\mathrm{ad}}$ is called a *Nash equilibrium* if for all $\nu = 1, \ldots, N$ it holds that

$$u_\nu \in \operatorname{argmin}\left\{\mathcal{J}_\nu(u'_\nu, u_{-\nu}) \text{ subject to } u'_\nu \in U_{\mathrm{ad}}^\nu\right\}. \tag{NEP}$$

The problem of finding such a point is called a *Nash equilibrium problem*. If moreover the sets U_{ad}^ν are convex and the objectives $u'_\nu \mapsto \mathcal{J}_\nu(u'_\nu, u_{-\nu})$ are convex on U_{ad}^ν, the Nash equilibrium problem is called *convex*.

Let $\Omega \subseteq \mathbb{R}^d$ denote an open bounded domain. For $\nu = 1, \ldots, N$, we define the following minimization problem which is associated with the ν-th player:

$$\begin{aligned}
&\text{minimize } J_\nu(y, u) := J_\nu^1(y) + J_\nu^2(u_\nu), \text{ over } u_\nu \in U_{\mathrm{ad}}^\nu \\
&\text{subject to } U_{\mathrm{ad}}^\nu \subseteq L^2(\Omega) \text{ and } y = S\left(f + \sum\nolimits_{\nu=1}^N B_\nu u_\nu\right),
\end{aligned} \tag{$\mathbb{P}_\nu$}$$

where U_{ad}^ν are non-empty, closed, convex subsets of $L^2(\Omega)$. Again the mapping $S : H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes the solution operator of the variational inequality (11) with $A = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and obstacle constraint $C := \{z \in H_0^1(\Omega) : z \geq \underline{\psi}\}$ with a lower obstacle $\underline{\psi} \in H^1(\Omega)$

with $\underline{\psi}|_{\partial\Omega} < 0$. Without loss of generality we assume $\underline{\psi}$ to be quasi-continuous. The operators $B_\nu \in \mathcal{L}(L^2(\Omega), H^{-1}(\Omega))$, $\nu = 1, \dots, N$ are assumed to be compact. Moreover, we assume the functionals J_ν^1, J_ν^2 to be proper, convex, lower semi-continuous and additionally J_ν^1 to be K-isotone with respect to $K := \{z \in H_0^1(\Omega) : z \geq 0\}$.
Using the chain rule computed in Corollary 13 and Theorem 15 as well as the subdifferential of the obstacle operator we derive the first order system for a point $u \in U_{\mathrm{ad}}$ to be a Nash equilibrium. In the proof of the following result we employ additionally the notation $B_{-\nu}$ as $B_{-\nu}u_{-\nu} = \sum_{\substack{i=1\\ i\neq\nu}}^N B_i u_i$.

Lemma 33. *A joint strategy $u = (u_1, \dots, u_N) \in U_{\mathrm{ad}} = \prod_{\nu=1}^N U_{\mathrm{ad}}^\nu$ is a Nash equilibrium if and only if there exist $y \in H_0^1(\Omega)$, $p_1, \dots, p_N \in H_0^1(\Omega)$ as well as capacitary measures $m_1, \dots, m_N \in \mathcal{M}_0(\Omega)$ fulfilling $m_\nu(I(y)) = 0$ and $m_\nu = +\infty$ on $A_s(y)$ for $\nu = 1, \dots, N$, such that the following first-order system is fulfilled:*

$$-B_\nu^* p_\nu \in \partial J_\nu^2(u_\nu) + N_{U_{\mathrm{ad}}^\nu}(u_\nu) \qquad \text{in } L^2(\Omega), \tag{14.1}$$

$$f + \sum_{\nu=1}^N B_\nu u_\nu \in -\Delta y + N_C(y) \qquad \text{in } H^{-1}(\Omega), \tag{14.2}$$

$$-\Delta p_\nu + m_\nu p_\nu \in \partial J_\nu^1(y) \qquad \text{in } \left(H_0^1(\Omega) \cap L^2_{m_\nu}(\Omega)\right)^*. \tag{14.3}$$

Proof. Since every player solves a convex optimization problem the optimal strategy u_ν fulfils the corresponding first-order system for player ν. Hence, for $\nu = 1, \dots, N$ it holds that

$$0 \in \partial\left(J_\nu^1 \circ S_\nu + J_\nu^2 + i_{U_{\mathrm{ad}}^\nu}\right)(u_\nu),$$

with $S_\nu(u_\nu) := S(f + B_{-\nu}u_{-\nu} + B_\nu u_\nu)$, which is a K-convex operator as argued in Example 9. We write $y = S(f + Bu)$, which yields equation (14.2) by definition of S. Since the constraint qualification in Corollary 13 and Theorem 15 are trivially fulfilled, we deduce

$$0 \in D^* S_\nu(u_\nu)(\partial J_\nu^1(S_\nu(u_\nu))) + \partial J_\nu^2(u_\nu) + N_{U_{\mathrm{ad}}^\nu}(u_\nu).$$

Using standard calculus rules of convex analysis (cf. (cf. [ET76, Sch07, BC17, Roc15])) we obtain for $y^* \in K^+$ that

$$\begin{aligned} D^* S_\nu(u_\nu)(y^*) &= \partial\langle y^*, S_\nu(\,\cdot\,)\rangle(u_\nu) = \partial\langle y^*, S(f + B_{-\nu}u_{-\nu} + B_\nu\,\cdot\,)\rangle(u_\nu) \\ &= B_\nu^* \partial\langle y^*, S(\,\cdot\,)\rangle(f + Bu) = B_\nu^* D^* S(f + Bu)(y^*). \end{aligned}$$

Hence, there exists $p_\nu \in D^* S(f + Bu)(\partial J_\nu^1(y))$ with

$$0 \in B_\nu^* p_\nu + \partial J_\nu^2(u_\nu) + N_{U_{\mathrm{ad}}^\nu}(u_\nu).$$

This establishs equation (14.1).
Using the characterization of the subdifferential of the solution operator of the obstacle problem in Lemma 31 we obtain the existence of a capacitary measure $m_\nu \in \mathcal{M}_0(y)$ described as above and $p_\nu \in H_0^1(\Omega) \cap L^2_{m_\nu}(\Omega)$ such that for an element $y_\nu^* \in \partial J_\nu^1(y)$ the following variational equation is fulfilled

$$(\nabla p_\nu, \nabla v) + \int_\Omega p_\nu v \,\mathrm{d}m = \langle y_\nu^*, v\rangle \text{ for all } v \in H_0^1(\Omega) \cap L^2_{m_\nu}(\Omega),$$

which yields equation (14.3) of the assertion. ☐

In the literature, different stationarity concepts for optimization problems have been developed, see [MP84], [HK09], [HMS14], [HW18] as well as the reference therein. The system (14) can for $N=1$ be interpreted in the same way. In fact our system implies strong stationarity in comparison to [HMS14, Definition 2] or [Wac14, System (1.3)], see further [MP84, Theorem 2.2]. The additional multiplier in the adjoint state equation is in our case related to the term $mp \in H^{-1}(\Omega)$ up to a sign and the conditions on the latter as well as on the adjoint state p are implied by $p \in L^2_m(\Omega)$ under the use of $m = +\infty$ on $\mathcal{A}_s(y)$.

For numerical purposes one is interested in removing the state constraint $y \in C$ from the set of explicit constraints in ($\mathbb{P}_\nu$) and instead add a penalty functional together with an adjustable, non-negative penalty parameter. The associated solution algorithms often enjoy the desirable property of mesh independence; see [HIK02, HU04] for more on this. Here we utilize a convex C^2-penalty function $\varphi_\varepsilon : L^2(\Omega) \to [0,\infty)$ and $\varphi_\varepsilon(z) := \int_\Omega \pi_\varepsilon(z)dx$ with π_ε defined via $\pi_\varepsilon(0) = 0$ and the first derivative to be

$$\pi'_\varepsilon(z) := \begin{cases} 0 & \text{for } z \leq 0, \\ \frac{z^2}{2\varepsilon} & \text{for } z \in (0,\varepsilon), \\ z - \frac{\varepsilon}{2} & \text{for } z \geq \varepsilon, \end{cases}$$

then π_ε and π'_ε are convex functions. Hence, φ_ε is a convex functional, as well, and $D\varphi_\varepsilon : H^1_0(\Omega) \to L^2(\Omega)$ is an $L^2(\Omega)_+$-convex operator. Moreover, it holds that $\varphi_\varepsilon(z) = 0$ if and only if $z \geq 0$ a.e. on Ω. We seek to approximate solutions of the Nash equilibrium problem by considering the following sequence of regularized games involving the penalty parameter $\gamma > 0$; compare to [HSK15]:

$$\begin{aligned} &\text{minimize } J^1_\nu(y) + J^2_\nu(u_\nu), \text{ over } u_\nu \in U^\nu_{\mathrm{ad}}, \\ &\text{subject to } U^\nu_{\mathrm{ad}} \subseteq L^2(\Omega) \text{ and } y \in H^1_0(\Omega) \text{ with,} \\ &\qquad -\Delta y - \gamma D\varphi_\varepsilon(\underline{\psi} - y) = f + Bu \text{ in } \Omega, \\ &\qquad\qquad\qquad y = 0 \text{ on } \partial\Omega, \end{aligned} \tag{$\mathbb{P}^\gamma_\nu$}$$

where the underlying PDE originates from the penalized version of the obstacle problem, reading

$$\text{minimize } \frac{1}{2}\|\nabla y\|^2_{L^2(\Omega)} - \left(f + \sum_{\nu=1}^N B_\nu u_\nu, y \right) + \gamma\varphi_\varepsilon(\underline{\psi} - y) \text{ over } y \in H^1_0(\Omega). \tag{15}$$

An important aspect associated with ($\mathbb{P}^\gamma_\nu$) and the resulting Nash game is concerned with the consistency of the penalized problem, with the original problem. We therefore discuss convergence of the solutions as well as of the multipliers towards the first-order system as $\gamma \to +\infty$. The following proof is influenced by the results and techniques in [RW19].

Lemma 34. *Let $\gamma_n \to \infty$ and $(u^n)_{n\in\mathbb{N}} \subseteq U_{\mathrm{ad}}$ be a sequence of solutions of the regularized game ($\mathbb{P}^\gamma_\nu$). Assume for $\nu = 1,\ldots,N$, that the subdifferentials $\partial J^2_\nu : L^2(\Omega) \rightrightarrows L^2(\Omega)$ are bounded operators and $\partial J^1_\nu : H^1_0(\Omega) \rightrightarrows H^{-1}(\Omega)$ are locally compact operators (i.e., for all $y \in H^1_0(\Omega)$ there exists a neighborhood $\mathcal{U}$ such that $\partial J^1_\nu(\mathcal{U})$ is a relatively compact set). Then there exists a weakly convergent subsequence of $(u^n)_{n\in\mathbb{N}}$ and every limit of such a sequence is a solution of the original Nash game that fulfils the following (slightly strengthened) stationarity system: There exist $p_1,\ldots,p_N \in H^1_0(\Omega)$ as well as a capacitary measure $m \in \mathcal{M}_0(\Omega)$ with $m(\mathcal{I}(y)) = 0$ and $m = +\infty$ on $\mathcal{A}_s(y)$ such that*

$$-B^*_\nu p_\nu \in \partial J^2_\nu(u_\nu) + N_{U^\nu_{\mathrm{ad}}}(u_\nu) \qquad \text{in } L^2(\Omega), \tag{16.1}$$

$$f + \sum_{\nu=1}^N B_\nu u_\nu \in -\Delta y + N_C(y) \qquad \text{in } H^{-1}(\Omega), \tag{16.2}$$

$$-\Delta p_\nu + m p_\nu \in \partial J^1_\nu(y) \qquad \text{in } \left(H^1_0(\Omega) \cap L^2_m(\Omega)\right)^*. \tag{16.3}$$

holds.

Proof. Since U_{ad} is bounded we obtain the existence of a weakly convergent subsequence by the reflexivity of $L^2(\Omega)$. Let now such a weakly convergent subsequence be given, and let u denote its limit. Since U_{ad} is convex and closed it is weakly closed. We thus obtain $u \in U_{\mathrm{ad}}$.
We are left with showing the strengthened stationarity system. For this purpose we consider the first-order system for the regularised problem given by

$$-B_\nu^* p_\nu^n \in \partial J_\nu^2(u_\nu^n) + N_{U_{\mathrm{ad}}^\nu}(u_\nu^n) \qquad \text{in } L^2(\Omega), \tag{17.1}$$

$$-\Delta y^n - \gamma_n D\varphi_\varepsilon(\underline{\psi} - y^n) = f + \sum_{\nu=1}^N B_\nu u_\nu^n \qquad \text{in } H^{-1}(\Omega), \tag{17.2}$$

$$-\Delta p_\nu^n + \gamma_n D^2\varphi_\varepsilon(\underline{\psi} - y^n) p_\nu^n \in \partial J_\nu^1(y^n) \qquad \text{in } H^{-1}(\Omega). \tag{17.3}$$

First, we discuss the behaviour of the states originating from the regularized state equation. Take an arbitrary $y' \in C$. By the convexity of the penalty functional we obtain

$$\begin{aligned} 0 \le \varphi_\varepsilon(\underline{\psi} - y^n) &\le \varphi_\varepsilon(\underline{\psi} - y') - \langle D\varphi_\varepsilon(\underline{\psi} - y^n), y^n - y'\rangle \\ &= -\langle D\varphi_\varepsilon(\underline{\psi} - y^n), y^n - y'\rangle. \end{aligned}$$

Testing the state equation (17.2) with $y^n - y' \in H_0^1(\Omega)$ by

$$\begin{aligned} \|\nabla y^n\|^2 &\le \|\nabla y^n\|^2 - \gamma_n \langle D\varphi_\varepsilon(\underline{\psi} - y^n), y_n - y'\rangle \\ &= (\nabla y^n, \nabla y^n - \nabla y') - \gamma_n\langle D\varphi_\varepsilon(\underline{\psi} - y^n), y^n - y'\rangle + (\nabla y^n, \nabla y') \\ &= \langle f + Bu^n, y^n - y'\rangle + (\nabla y^n, \nabla y') \\ &\le \|f + Bu^n\| \left(\|y^n\| + \|y'\|\right) + \|\nabla y^n\|\|\nabla y'\| \\ &\le \left(C_F\|f + Bu^n\| + \|\nabla y'\|\right)\|\nabla y^n\| + \|f + Bu^n\|\|y'\| \\ &= \frac{1}{2}\left(C_F\|f + Bu^n\| + \|\nabla y'\|\right)^2 + \frac{1}{2}\|\nabla y^n\|^2 + \|f + Bu^n\|\|y'\| \end{aligned}$$

and hence by shifting $\frac{1}{2}\|\nabla y^n\|^2$ to the left hand side we obtain the boundedness of y^n in $H_0^1(\Omega)$. Along every subsequence exists a subsequence (not relabeled) and a limit point $y \in H_0^1(\Omega)$ with $y_n \rightharpoonup y$ in $H_0^1(\Omega)$. Since the state equation is equivalent to the minimization problem (15) we obtain

$$\gamma_n\varphi_\varepsilon(\underline{\psi} - y^n) \le \frac{1}{2}\|\nabla y'\|^2 - \frac{1}{2}\|\nabla y^n\|^2 + \langle f + Bu^n, y^n - y'\rangle.$$

Since the right hand side is bounded we obtain the boundedness of $\gamma_n\varphi_\varepsilon(\underline{\psi} - y^n)$ and by the weak lower semi-continuity of the penalty functional consecutively

$$0 \le \varphi_\varepsilon(\underline{\psi} - y) \le \liminf_{n\to\infty} \varphi_\varepsilon(\underline{\psi} - y^n) = 0 \tag{18}$$

and hence $y \in C$. Subsequently, we obtain for an arbitrary $y' \in C$ by the the compactness of the linear operator $B_\nu \in \mathcal{L}(L^2(\Omega), H^{-1}(\Omega))$ that

$$\begin{aligned} \frac{1}{2}\|\nabla y\|^2 - \langle f + Bu, y\rangle &\le \liminf_{n\to\infty}\left(\frac{1}{2}\|\nabla y^n\|^2 - \langle f + Bu^n, y^n\rangle\right) \\ &\le \liminf_{n\to\infty}\left(\frac{1}{2}\|\nabla y^n\|^2 - \langle f + Bu^n, y^n\rangle + \gamma_n\varphi_\varepsilon(\underline{\psi} - y^n)\right) \\ &\le \liminf_{n\to\infty}\left(\frac{1}{2}\|\nabla y'\|^2 - \langle f + Bu^n, y'\rangle + \gamma_n\varphi_\varepsilon(\underline{\psi} - y')\right) \\ &\le \liminf_{n\to\infty}\left(\frac{1}{2}\|\nabla y'\|^2 - \langle f + Bu^n, y'\rangle\right) = \frac{1}{2}\|\nabla y'\|^2 - \langle f + Bu, y'\rangle \end{aligned}$$

and as thus y being the solution of the VI in (14.2) with respect to u. Since this solution is unique we obtain the weak convergence of the whole sequence.
Moreover, we observe the strong convergence by

$$\begin{aligned}\|\nabla y^n - \nabla y\|^2 &= (\nabla y^n, \nabla y^n - \nabla y) - (\nabla y, \nabla y^n - \nabla y)\\ &= \gamma_n(-D\varphi_\varepsilon(\underline{\psi} - y^n), y - y^n) + \langle f + Bu^n, y^n - y\rangle\\ &- (\nabla y, \nabla y^n - \nabla y)\\ &\leq \langle f + Bu^n, y^n - y\rangle + \gamma_n\varphi_\varepsilon(\underline{\psi} - y) - \gamma_n\varphi_\varepsilon(\underline{\psi} - y^n)\\ &- (\nabla y, \nabla y^n - \nabla y)\\ &\leq \langle f + Bu^n, y^n - y\rangle - (\nabla y, \nabla y^n - \nabla y) \to 0\end{aligned}$$

as $n \to +\infty$. Taking $y_\nu^{*,n} \in \partial J_\nu^1(y^n)$ occurring on the right hand side of equation 17.3, we deduce (for sufficiently high indices) by the assumed local compactness of ∂J_ν^1 the existence of a strongly H^{-1}-convergent subsequence $y_\nu^{*,n} \to y_\nu^*$.
Turning our attention now to the adjoint equation we observe that the term $\gamma_n D^2\varphi_\varepsilon(\underline{\psi} - y^n)$ can be identified with a capacitary measure by

$$m_n(E) := \int_E \gamma_n D^2\varphi_\varepsilon(\underline{\psi} - y^n)\mathrm{d}x \text{ for a Borel set } E \in \mathcal{B}(\Omega).$$

By Theorem 25 we infer the existence of a γ-convergent subsequence (not relabeled) and a capacitary measure $m \in \mathcal{M}_0(\Omega)$ with $m_n \xrightarrow{\gamma} m$. In other words (see Definition 23) we obtain the weak operator convergence of T_{m_n} to T_m. Hence, we obtain for the adjoint states

$$p_\nu^n = (-\Delta + m_n)^{-1}(y_\nu^{*,n}) \rightharpoonup (-\Delta + m)^{-1}y_\nu^* =: p_\nu \text{ in } H_0^1(\Omega).$$

Eventually, we characterize m.
First, we show $m(\mathcal{I}(y)) = 0$. As in the proof of [RW19, Lemma 5.1] we take an arbitrary $z \in H_0^1(\mathcal{I}(y)) \subseteq H_0^1(\Omega)$ with $\{z > 0\} =_q \mathcal{I}(y)$ and $0 \leq z \leq 1$. By a generalization of [RW19, Lemma 4.2] given in Lemma 36 in the Appendix to this work we obtain the existence of a sequence $(z^n)_{n\in\mathbb{N}}$ with $z^n \to z$ in $H_0^1(\Omega)$ with $0 \leq z_n \leq 1$ and $z^n \in H_0^1(\mathcal{I}(y^n))$. Then we see that

$$-\Delta z^n + \gamma_n D^2\varphi_\varepsilon(\underline{\psi} - y^n)z^n = -\Delta z^n \text{ in } H^{-1}(\Omega) \tag{19}$$

By $m_n \xrightarrow{\gamma} m$ we obtain using the boundedness of the operator $z^n = T_{m_n}z^n \rightharpoonup T_m z$ and hence $z = T_m z$. By testing equation (19) with z we obtain

$$\|\nabla z\|_{L^2}^2 = \|\nabla z\|_{L^2}^2 + \int_\Omega z^2 dm,$$

and hence $\int_\Omega z^2 dm = 0$. Since $\{z > 0\} =_q \mathcal{I}(y)$ we conclude $m(\mathcal{I}(y)) = 0$.
On the other hand we obtain $m = +\infty$ on $\mathcal{A}_s(y)$. To show this we use [RW19, Lemma 5.2] and obtain that the assertion holds if and only if $w_m := T_m(1)$ is zero q.e. on $\mathcal{A}_s(y)$. Since $w_m \geq 0$ q.e. on Ω it is enough to show $\langle \xi, w_m\rangle = 0$ for $\xi = -\Delta y - f - Bu$. For this purpose, consider the sequences $\xi^n = -\Delta y^n - f - Bu^n = \gamma_n D\varphi_\varepsilon(\underline{\psi} - y^n) \to \xi$ in $H^{-1}(\Omega)$ together with $w_{m_n} := T_{m_n}(1)$. Since $m_n \xrightarrow{\gamma} m$ we obtain $w_{m_n} \rightharpoonup w_m = T_m(1)$. Since π_ε' is convex, the mapping $D\varphi_\varepsilon$ is convex as well, and we obtain

$$0 = D\varphi_\varepsilon(\underline{\psi} - y) \geq D\varphi_\varepsilon(\underline{\psi} - y^n) - D^2\varphi_\varepsilon(\underline{\psi} - y^n)(y - y^n) \text{ a.e. on } \Omega,$$

and hence

$$D\varphi_\varepsilon(\underline{\psi} - y^n) \leq D^2\varphi_\varepsilon(\underline{\psi} - y^n)(y - y^n).$$

Using $\xi \in K^+$ and the definition of w_{m_n} we see

$$\begin{aligned}
0 &\leq \langle \xi, w_m \rangle = \lim_{n\to\infty} \langle \xi^n, w_{m_n} \rangle = \lim_{n\to\infty} \gamma_n \langle D\varphi_\varepsilon(\underline{\psi} - y^n), w_{m_n} \rangle \\
&\leq \lim_{n\to\infty} \langle \gamma_n D^2\varphi_\varepsilon(\underline{\psi} - y^n)(y - y^n), w_{m_n} \rangle = \lim_{n\to\infty} \int_\Omega w_{m_n}(y - y^n)\,\mathrm{d}m_n \\
&= \lim_{n\to\infty} \big((1, y - y^n)_{L^2(\Omega)} - (\nabla w_{m_n}, \nabla(y - y^n))\big) = 0
\end{aligned}$$

by the strong convergence of the states. Hence, we obtain $w_m = 0$ on $\mathcal{A}_s(y)$ and eventually $m = +\infty$ on $\mathcal{A}_s(y)$. □

Lemma 34 shows that the application of the described regularisation scheme leads in the limit to the derivation of Nash equilibria fulfilling a slightly strengthened stationarity system, where the capacitary measures coincide. To the one hand this guarantees the existence of a special Nash equilibrium fulfilling System (16). Such a phenomenon seems similar to the concept of a variational equilibrium; see [Ros65] and compare also [HSK15], but here it is derived from a smoothing technique instead of the transition from a set-valued strategy map to a single joint constraint set generated, e.g., via a state constraint. On the other hand in case of the existence of a solution that does fulfil (14) but not (16) this implies the existence of another solution that may not be generated by the method described by the penalization scheme. However, the proof of existence of such a point is left as an open question for further work.

6 Conclusion

In this paper, we investigated a class of operators fulfilling a generalized, order-based convexity concept and their properties with regard to convex analysis and optimization theory. As part of we utilized and generalized methods from non-smooth and set-valued analysis and illustrated the applicability of these concepts to a selection of operator equations and variational inequalities closely related to the types of problems discussed in the recent literature. By considering optimality systems corresponding to Nash equilibrium problems one is able to characterize equilibria even in a non-smooth setting.

A Appendix

Proof of Lemma 2. ad (i): Let $x^* \in A_2^+$. Then, $\langle x^*, x\rangle \geq 0$ for all $x \in A_2$ and hence especially for all $x \in A_1$. This yields $x^* \in A_1^+$.
ad (ii): Since it always holds, that $A \subseteq \mathrm{cl}\,(A)$ we deduce $(\mathrm{cl}\,(A))^+ \subseteq A^+$ by (i). Let now $x^* \in A^+$ and take $x \in \mathrm{cl}\,(A)$. Then there exists a sequence $x_n \to x$ with $x_n \in A$ and we obtain $\langle x^*, x\rangle = \lim_{n\to\infty}\langle x^*, x_n\rangle \geq 0$ and hence the equality.
ad (iii): see [BS00, Proposition 2.40].
ad (iv): Since $0 \in A_1 \cap A_2$ we have that $A_j \subseteq A_1 + A_2$ and hence $(A_1 + A_2)^+ \subseteq A_j^+$ for $j = 1, 2$. This yields the inclusion $(A_1 + A_2)^+ \subseteq A_1^+ \cap A_2^+$.
Let, on the other hand, $x^* \in A_1^+ \cap A_2^+$. Then we get for all $x_j \in A_j$ that $\langle x^*, x_1 + x_2\rangle = \langle x^*, x_1\rangle + \langle x^*, x_2\rangle \geq 0$, which gives $x^* \in (A_1 + A_2)^+$.

ad (v): Since A_j^+ are closed, convex cones the set $\operatorname{cl}\left(A_1^+ + A_2^+\right)$ is a closed, convex cone as well. Hence, by the application of (ii), (iii) and (iv) we obtain, that

$$\left(\operatorname{cl}\left(A_1^+ + A_2^+\right)\right)^+ = \left(A_1^+ + A_2^+\right)^+ = A_1^{++} \cap A_2^{++} = A_1 \cap A_2.$$

The subsequent application of (ii) yields

$$\operatorname{cl}\left(A_1^+ + A_2^+\right) = \left(\operatorname{cl}\left(A_1^+ + A_2^+\right)\right)^{++} = (A_1 \cap A_2)^+.$$

□

Proposition 35. *Let $A : Y \to Y^*$ be a linear, bounded operator. Then the following statements are equivalent.*

1 *A is T-monotone, i.e. for all $z \in Y$ holds $\langle Az, (-z)^+\rangle \leq 0.$*

2 *For all $z \in Y$ holds $\langle Az^+, (-z)^+\rangle \leq 0.$*

Proof. For the statement and the proof compare to (cf. [MR95, Proposition 1.3, (i)]). Here it has been reformulated for our purposes:
Consider $z \in Y$ and define $z_\alpha = z^+ - \alpha(-z)^+$. Since $\min(z^+, (-z)^+) = 0$ we obtain as well $\min(z, \alpha(-z)^+) = 0$. Therefore take $x, y \geq 0$ with $\min(x, y) = 0$ and take $\alpha > 0$ and we show $\min(\alpha x, y) = 0$. Without loss of generality we assume $\alpha < 1$ (otherwise rewrite $\min(\alpha x, y) = \alpha \min(x, \frac{1}{\alpha} y) = 0$ and change α to α^{-1} and interchange the roles of x and y). We clearly see

$$0 = \min(x, y) = -\max(-x, -y) = y - \max(y - x, 0)$$

and therefore $y = (y - x)^+$. Since $\alpha < 1$ and $x \geq 0$ we obtain

$$y - x \leq y - \alpha x \leq y$$

and hence $y = (y - x)^+ = (y - \alpha x)^+$. This implies $\min(\alpha x, y) = y - (y - \alpha x)^+ = 0.$
By the uniqueness of the decomposition of z_α into the difference of two positive disjoint elements (see [Sch74, Proposition 1.4]), we obtain $z_\alpha^+ = z^+$ and $(-z_\alpha)^+ = \alpha(-z)^+$. Hence, we see

$$0 \geq \langle Az_\alpha, (-z_\alpha)^+\rangle = \alpha\langle Az_\alpha, (-z)^+\rangle = \alpha\langle Az^+, (-z)^+\rangle - \alpha^2\langle A(-z)^+, (-z)^+\rangle$$

and hence by dividing by α and passing $\alpha \searrow 0$ that

$$\langle Az^+, (-z)^+\rangle \leq 0.$$

On the other hand let $\langle Az^+, (-z)^+\rangle \leq 0$ hold for $z \in Y$ and hence

$$\langle Az, (-z)^+\rangle = \langle Az^+, (-z)^+\rangle - \langle A(-z)^+, (-z)^+\rangle \leq 0.$$

□

Lemma 36. *Let $\Omega \subseteq \mathbb{R}^m$ be an open set. Then the following assertions hold true.*

(i) Let $y : \Omega \to \mathbb{R}^m$ be a quasi-continuous function and let an open-valued multifunction $\mathcal{U} : \Omega \rightrightarrows \mathbb{R}^m$ be given such that for all $\varepsilon > 0$ there exists $U_\varepsilon \subseteq \Omega$ with $\operatorname{cap}(U_\varepsilon) < \varepsilon$ and the mapping $(x, y) \mapsto \operatorname{dist}(y, \mathcal{U}^c(x))$ is lower semi-continuous. Then the set

$$\mathcal{I}(y) := \{x \in \Omega : y(x) \in \mathcal{U}(x)\}$$

is quasi-open.

(ii) Let $y_n \to y$ in $H^1(\Omega;\mathbb{R}^m)$ be given. Then for all $v \in H_0^1(\mathcal{I}(y))$ with $0 \le v \le 1$ there exists a sequence $v_n \in H_0^1(\mathcal{I}(y_n))$ with $0 \le v_n \le 1$ and $v_n \to v$ in $H_0^1(\Omega)$.

Proof. (i) Let $\varepsilon > 0$ be arbitrary. By the quasi-continuity of y and the assumption on $\mathcal{U}$, we deduce the existence of open sets $O_\varepsilon, U_\varepsilon \subseteq \Omega$ with $\operatorname{cap}(O_\varepsilon), \operatorname{cap}(U_\varepsilon) < \frac{1}{2}\varepsilon$ such that y is continuous on O_ε^c and the mapping $(x, y) \mapsto \operatorname{dist}(y, \mathcal{U}^c(x))$ is lower semi-continuous on $U_\varepsilon^c \times \mathbb{R}^m$. Then the set $\mathcal{I}(y) \cup O_\varepsilon \cup U_\varepsilon$ is open in Ω. Indeed define the set $D_\varepsilon := \{(x, y) \in U_\varepsilon^c \times \mathbb{R}^m : \operatorname{dist}(y, \mathcal{U}^c(x)) = 0\}$. By assumption this set is closed in $\Omega \times \mathbb{R}^m$. We rewrite

$$\mathcal{I}(y) = \{x \in \Omega : y(x) \in \mathcal{U}(x)\} = \{x \in \Omega : \operatorname{dist}(y(x), \mathcal{U}^c(x)) > 0\}.$$

Hence, we see

$$\begin{aligned}(\mathcal{I}(y) \cup O_\varepsilon \cup U_\varepsilon)^c &= \mathcal{I}(y)^c \cap O_\varepsilon^c \cap U_\varepsilon^c = \{x \in O_\varepsilon^c \cap U_\varepsilon^c : \operatorname{dist}(y(x), \mathcal{U}^c(x)) = 0\} \\ &= \{x \in O_\varepsilon^c : (x, y(x)) \in D_\varepsilon\}\end{aligned}$$

is closed in Ω and therefore $\mathcal{I}(y) \cup O_\varepsilon \cup U_\varepsilon$ is open in Ω. Hence, we deduce that $\mathcal{I}(y)$ is quasi-open with corresponding sequence $O_\varepsilon \cup U_\varepsilon \subseteq \Omega$ open and $\operatorname{cap}(O_\varepsilon \cup U_\varepsilon) < \varepsilon$.
(ii): Let now $y_n \to y$ in $H^1(\Omega;\mathbb{R}^m)$ and $v \in H_0^1(\mathcal{I}(y))$ be described as above. Define the sequence $t_n := \sup_{m \ge n} \|y_n - y\|_{H_0^1(\Omega)}^{\frac{1}{2}}$. Then, it holds that $t_n \searrow 0$, and we can write $\mathcal{I}(y) = \{x \in \Omega : \operatorname{dist}(y(x), \mathcal{U}^c(x)) > 0\} = \bigcup_{n\in\mathbb{N}} \Omega_n$ with $\Omega_n := \{x \in \Omega : \operatorname{dist}(y(x), \mathcal{U}^c(x)) > t_n\}$, i.e., as the union of a sequence of increasing, quasi-open sets. The latter property can be proven in the same fashion as above. Hence, we can apply [RW19, Lemma 2.3] and obtain a sequence $\tilde{v}_n \in H_0^1(\Omega_n)$ with $0 \le \tilde{v}_n \le 1$ and $\tilde{v}_n \to v$. Furthermore we see by the triangle inequality

$$\operatorname{dist}(y, \mathcal{U}^c(\,\cdot\,)) \le \operatorname{dist}(y_n, \mathcal{U}^c(\,\cdot\,)) + |y - y_n|_{\mathbb{R}^m}$$

that

$$\begin{aligned}\operatorname{cap}(\Omega_n \backslash \mathcal{I}(y_n)) &= \operatorname{cap}(\{\operatorname{dist}(y, \mathcal{U}^c(\,\cdot\,)) > t_n\} \cap \{\operatorname{dist}(y_n, \mathcal{U}^c(\,\cdot\,)) \le 0)\}) \\ &\le \operatorname{cap}(\{|y - y_n|_{\mathbb{R}^m} > t_n\}) \le \frac{1}{t_n^2} \left\| |y - y_n|_{\mathbb{R}^m} \right\|_{H^1(\Omega)}^2 \\ &= \frac{1}{t_n^2} \|y - y_n\|_{H^1(\Omega;\mathbb{R}^m)}^2 \to 0\end{aligned}$$

holds true. Note that for the last inequality Definition 20 was used. Hence, we deduce the existence of a sequence $w_n \in H_0^1(\Omega)$ with $w_n \to 0$ and $0 \le w_n \le 1$ with $w_n = 1$ q.e. on $\Omega_n \backslash \mathcal{I}(y_n)$. By defining $v_n := (\tilde{v}_n - w_n)^+$ we meet the requirements of the assertion. □

The above Lemma includes [RW19, Lemma 4.2] as a special case when using $\mathcal{U}(x) := (\underline{\psi}(x), \infty)$. Indeed, if $\underline{\psi}$ is assumed to be quasi upper-semicontinuous, then there exists for $\varepsilon > 0$ an open set $U_\varepsilon \subseteq \Omega$ with $\underline{\psi}$ being upper semi-continuous on U_ε^c. The distance map becomes $\operatorname{dist}(y, \mathcal{U}^c(x)) = (y - \underline{\psi}(x))^+$, which is then lower semi-continuous on $U_\varepsilon^c \times \mathbb{R}$.

References

[AF90] AUBIN, J.P. ; FRANKOWSKA, H.: *Set-valued Analysis*. Springer, 1990 (Modern Birkhäuser classics). – ISBN 9780817634780

[BC17] BAUSCHKE, H.H. ; COMBETTES, P.L.: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer International Publishing, 2017 (CMS Books in Mathematics). – ISBN 9783319483115

[Bec08] BECKER, R.: *Ordered banach spaces*. Hermann, 2008 (Travaux en cours). – ISBN 9782705667214

[BK13] BORZI, A. ; KANZOW, C.: Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems. In: *SIAM Journal on Control and Optimization* 51 (2013), Nr. 1, S. 718–744

[BMA06] BUTTAZZO, G. ; MICHAILLE, G. ; ATTOUCH, H.: *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*. Society for Industrial and Applied Mathematics, 2006 (MPS-SIAM series on optimization). – ISBN 9780898718782

[BS00] BONNANS, J.F. ; SHAPIRO, A.: *Perturbation Analysis of Optimization Problems*. Springer New York, 2000 (Springer Series in Operations Research and Financial Engineering). – ISBN 9780387987057

[BZ06] BORWEIN, J. ; ZHU, Q.: *Techniques of Variational Analysis*. Springer New York, 2006 (CMS Books in Mathematics). – ISBN 9780387282718

[Cia13] CIARLET, P.G.: *Linear and Nonlinear Functional Analysis with Applications*. Society for Industrial and Applied Mathematics, 2013 (Other Titles in Applied Mathematics). – ISBN 9781611972580

[CLV13] CROUZEIX, J.P. ; LEGAZ, J.E.M. ; VOLLE, M.: *Generalized Convexity, Generalized Monotonicity: Recent Results: Recent Results*. Springer US, 2013 (Nonconvex Optimization and Its Applications). – ISBN 9781461333418

[CMWC18] CHRISTOF, C. ; MEYER, C. ; WALTHER, S. ; CLASON, C.: Optimal control of a non-smooth semilinear elliptic equation. In: *Mathematical Control & Related Fields* 8 (2018), S. 247

[DM87] DAL MASO, G.: Γ-convergence and μ-capacities. In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 14 (1987), Nr. 3, S. 423–464

[DMG94] DAL MASO, G. ; GARRONI, A.: New results on the asymptotic behavior of Dirichlet problems in perforated domains Dirichlet problems in perforated domains. In: *Mathematical Models and Methods in Applied Sciences* 4 (1994), Nr. 03, S. 373–407

[DMM87] DAL MASO, G. ; MOSCO, U.: Wiener's criterion and Γ-convergence. In: *Applied Mathematics and Optimization* 15 (1987), Nr. 1, S. 15–63

[DZ11] DELFOUR, M.C. ; ZOLESIO, J.P.: *Shapes and Geometries: Metrics, Analysis, Differential Calculus, and Optimization, Second Edition*. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 2011 (Advances in Design and Control). – ISBN 9780898719826

[EG15] EVANS, L.C. ; GARIEPY, R.F.: *Measure Theory and Fine Properties of Functions, Revised Edition*. Taylor & Francis, 2015 (Textbooks in Mathematics). – ISBN 9781482242386

[EM46] EILENBERG, S. ; MONTGOMERY, D.: Fixed point theorems for multi-valued transformations. In: *American Journal of mathematics* 68 (1946), Nr. 2, S. 214–222

[ET76] EKELAND, I. ; TEMAM, R.: *Convex Analysis and Variational Problems*. Elsevier Science, 1976 (Studies in Mathematics and its Applications). – ISBN 9780080875224

[FFP09] FACCHINEI, F. ; FISCHER, A. ; PICCIALLI, V.: Generalized Nash equilibrium problems and Newton methods. In: *Mathematical Programming* 117 (2009), Nr. 1-2, S. 163–194

[FI92] FANG, Weifu ; ITO, Kazufumi: Identifiability of semiconductor defects from LBIC images. In: *SIAM Journal on Applied Mathematics* 52 (1992), Nr. 6, S. 1611–1626

[FI94] FANG, Weifu ; ITO, Kazufumi: Reconstruction of semiconductor doping profile from laser-beam-induced current image. In: *SIAM Journal on Applied Mathematics* 54 (1994), Nr. 4, S. 1067–1082

[FK07] FACCHINEI, F. ; KANZOW, C.: Generalized Nash equilibrium problems. In: *4or* 5 (2007), Nr. 3, S. 173–210

[Gli52] GLICKSBERG, I. L.: A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. In: *Proceedings of the American Mathematical Society* 3 (1952), Nr. 1, S. 170–174

[Har77] HARAUX, A.: How to differentiate the projection on a convex set in Hilbert space. In: *Journal of the Mathematical Society of Japan* 29 (1977), Nr. 4, S. 615–631

[HIK02] HINTERMÜLLER, M. ; ITO, K. ; KUNISCH, K.: The primal-dual active set strategy as a semismooth Newton method. In: *SIAM Journal on Optimization* 13 (2002), Nr. 3, S. 865–888

[HK09] HINTERMÜLLER, M. ; KOPACKA, I.: Mathematical programs with complementarity constraints in function space: C-and strong stationarity and a path-following algorithm. In: *SIAM Journal on Optimization* 20 (2009), Nr. 2, S. 868–902

[HMS14] HINTERMÜLLER, M. ; MORDUKHOVICH, B. S. ; SUROWIEC, T. M.: Several approaches for the derivation of stationarity conditions for elliptic MPECs with upper-level control constraints. In: *Mathematical Programming* 146 (2014), Nr. 1-2, S. 555–582

[HR19] HINTERMÜLLER, M. ; RODRIGUES, J.F.: *Topics in Applied Analysis and Optimisation: Partial Differential Equations, Stochastic and Numerical Analysis*. Springer International Publishing, 2019 (CIM Series in Mathematical Sciences). – ISBN 9783030331160

[HS11] HINTERMÜLLER, M. ; SUROWIEC, T.: First-Order Optimality Conditions for Elliptic Mathematical Programs with Equilibrium Constraints via Variational Analysis. In: *SIAM Journal on Optimization* 21 (2011), Nr. 4, S. 1561–1593

[HS13] HINTERMÜLLER, M. ; SUROWIEC, T: A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints. In: *Pac. J. Optim* 9 (2013), Nr. 2, S. 251–273

[HSK15] HINTERMÜLLER, M. ; SUROWIEC, T. ; KÄMMLER, A.: Generalized Nash Equilibrium Problems in Banach Spaces: Theory, NikaidoâĂŞIsoda-Based Path-Following Methods, and Applications. In: *SIAM Journal on Optimization* (2015), S. 1826–1856

[HU04] HINTERMÜLLER, M. ; ULBRICH, M.: A mesh-independence result for semismooth Newton methods. In: *Mathematical Programming* 101 (2004), Nr. 1, S. 151–184

[HW18] HARDER, F. ; WACHSMUTH, G.: Comparison of optimality systems for the optimal control of the obstacle problem. In: *GAMM-Mitteilungen* 40 (2018), Nr. 4, S. 312–338

[IK96] ITO, Kazufumi ; KUNISCH, Karl: Augmented Lagrangian–SQP methods for nonlinear optimal control problems of tracking type. In: *SIAM Journal on Control and Optimization* 34 (1996), Nr. 3, S. 874–891

[Jam74] JAMESON, G.J.O.: *Topology and Normed Spaces [By] G.J.O. Jameson*. Chapman and Hall, 1974 (Chapman and Hall Mathematics Series)

[KKSW19] KANZOW, C. ; KARL, V. ; STECK, D. ; WACHSMUTH, D.: The multiplier-penalty method for generalized Nash equilibrium problems in Banach spaces. In: *SIAM Journal on Optimization* 29 (2019), Nr. 1, S. 767–793

[KS80] KINDERLEHRER, D. ; STAMPACCHIA, G.: *An Introduction to Variational Inequalities and Their Applications*. Elsevier Science, 1980 (ISSN). – ISBN 9780080874043

[KS20] KOURI, Drew P. ; SUROWIEC, TM: Risk-averse optimal control of semilinear elliptic PDEs. In: *ESAIM: Control, Optimisation and Calculus of Variations* 26 (2020), S. 53

[Mig76] MIGNOT, F.: Contrôle dans les inéquations variationelles elliptiques. In: *Journal of Functional Analysis* 22 (1976), Nr. 2, S. 130–185

[Mor06] MORDUKHOVICH, B.S.: *Variational Analysis and Generalized Differentiation I: Basic Theory*. Springer Berlin Heidelberg, 2006 (Grundlehren der mathematischen Wissenschaften). – ISBN 9783540312475

[MP84] MIGNOT, F. ; PUEL, J.P.: Optimal control in some variational inequalities. In: *SIAM Journal on Control and Optimization* 22 (1984), Nr. 3, S. 466–476

[MR95] MA, Z.-M. ; RÖCKNER, M.: Markov processes associated with positivity preserving coercive forms. In: *Canadian Journal of Mathematics* 47 (1995), Nr. 4, S. 817–840

[PF05] PANG, J.-S. ; FUKUSHIMA, M.: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. In: *Computational Management Science* 2 (2005), Nr. 1, S. 21–56

[Roc15] ROCKAFELLAR, R.T.: *Convex Analysis: (PMS-28)*. Princeton University Press, 2015 (Princeton Landmarks in Mathematics and Physics). – ISBN 9781400873173

[Rod87] RODRIGUES, J.F.: *Obstacle Problems in Mathematical Physics*. Elsevier Science, 1987 (ISSN). – ISBN 9780080872452

[Ros65] ROSEN, J. B.: Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. In: *Econometrica* 33 (1965), Nr. 3, S. 520–534. – ISSN 00129682, 14680262

[Rou05] ROUBICEK, T.: *Nonlinear Partial Differential Equations with Applications*. Birkhäuser Basel, 2005 (International Series of Numerical Mathematics). – ISBN 9783764372934

[Rud87] RUDIN, W.: *Real and Complex Analysis*. McGraw-Hill, 1987 (Mathematics series). – ISBN 9780071002769

[RW19] RAULS, A.-T. ; WACHSMUTH, G.: Generalized Derivatives for the Solution Operator of the Obstacle Problem. In: *Set-Valued and Variational Analysis* (2019). – ISSN 1877–0541

[RWW09] ROCKAFELLAR, R.T. ; WETS, M. ; WETS, R.J.B.: *Variational Analysis*. Springer Berlin Heidelberg, 2009 (Grundlehren der mathematischen Wissenschaften). – ISBN 9783540627722

[Sch74] SCHÄFER, H.H.: *Banach Lattices and Positive Operators*. Springer Berlin Heidelberg, 1974 (Grundlehren der mathematischen Wissenschaften). – ISBN 9783642659706

[Sch07] SCHIROTZEK, W.: *Nonsmooth Analysis*. Springer Berlin Heidelberg, 2007 (Universitext). – ISBN 9783540713333

[Wac14] WACHSMUTH, G.: Strong stationarity for optimal control of the obstacle problem with control constraints. In: *SIAM Journal on Optimization* 24 (2014), Nr. 4, S. 1914–1932

[Wac16a] WACHSMUTH, G.: *Optimization problems with complementarity constraints in infinite-dimensional spaces*. 2016. – Habilitation Thesis

[Wac16b] WACHSMUTH, G.: Towards M-stationarity for optimal control of the obstacle problem with control constraints. In: *SIAM Journal on Control and Optimization* 54 (2016), Nr. 2, S. 964–986