Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

On the differentiability of the minimal and maximal solution maps of elliptic quasi-variational inequalities

Amal Alphonse[1], Michael Hintermüller[1,2], Carlos N. Rautenberg[3]

submitted: September 8, 2020

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: amal.alphonse@wias-berlin.de
michael.hintermueller@wias-berlin.de

[2] Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
Germany
E-Mail: hint@math.hu-berlin.de

[3] Department of Mathematical Sciences and CMAI
George Mason University
Fairfax, VA 22030
USA
E-Mail: crautenb@gmu.edu

No. 2758
Berlin 2020

2010 *Mathematics Subject Classification.* 47J20, 49J40, 49J52, 49J50.

Key words and phrases. Quasi-variational inequality, obstacle problem, directional differentiability, minimal and maximal solutions, ordered solutions.

AA and MH were partially supported by the DFG through the DFG SPP 1962 Priority Programme *Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization* within project 10. MH and CNR acknowledge the support of Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689) within project AA4-3. In addition, MH acknowledges the support of SFB-TRR154 within subproject B02, and CNR was supported by NSF grant DMS-2012391.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

On the differentiability of the minimal and maximal solution maps of elliptic quasi-variational inequalities

Amal Alphonse, Michael Hintermüller, Carlos N. Rautenberg

Abstract

In this short note, we prove that the minimal and maximal solution maps associated to elliptic quasi-variational inequalities of obstacle type are directionally differentiable with respect to the forcing term and for directions that are signed. On the way, we show that the minimal and maximal solutions can be seen as monotone limits of solutions of certain variational inequalities and that the aforementioned directional derivatives can also be characterised as the monotone limits of sequences of directional derivatives associated to variational inequalities.

1 Introduction

Quasi-variational inequalities (QVIs) are variational inequalities (VIs) where the constraint set over which the solution is sought also depends on the solution itself. As such, QVI problems are highly nonlinear and nonconvex and in sharp contrast to the usual setting for VIs, QVIs usually possess multiple solutions. In certain situations, the set of solutions can be ordered in the sense that there exist minimal and maximal solutions. In this paper, we address the directional differentiability of the maps taking the source term of a QVI into the minimal and maximal solutions. The above-mentioned quirks of QVIs endow their study with substantial technical issues to overcome when examining questions of stability analysis and differential sensitivity.

QVIs were first formulated by Bensoussan and Lions [10, 17] in the modelling of stochastic impulse controls. Applications of QVIs are ubiquitous. Among some, we mention thermoforming processes [1], the formation and growth of lakes, rivers and sandpiles [25, 8, 24, 22, 9], generalised Nash equilibrium games [15, 13, 20], and magnetisation of superconductors [16, 7, 23, 26]. Additional details and references can be found in our survey paper [2] and the book [6].

We focus on elliptic QVIs of obstacle type (these are also known as implicit obstacle problems). The precise formulation is as follows. Let $V \subset H$ be a continuous and dense embedding of separable Hilbert spaces and suppose that there exists an ordering to elements of H via a closed convex cone H_+ that satisfies

$$H_+ = \{h \in H : (h, g) \geq 0 \quad \forall g \in H_+\}.$$

The ordering is defined by: $h_1 \leq h_2$ if and only if $h_2 - h_1 \in H_+$. This endows an ordering for V in the obvious way and we write $V_+ := \{v \in V : v \geq 0\}$. It also induces one for the dual space V^* via

$$V_+^* := \{f \in V^* : \langle f, v\rangle \geq 0 \quad \forall v \in V_+\},$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V^*,V}$ is the standard duality product. We write h^+ for the orthogonal projection of $h \in H$ onto the space H_+ and we use the decomposition $h = h^+ - h^-$. Suppose that $v \in V$ implies that $v^+ \in V$ and that there exists a constant $C > 0$ such that $\|v^+\|_V \leq C \|v\|_V$ for all $v \in V$.[1]

[1] For an example, we may take V to be the Sobolev space $V = H^1(\Omega)$ over a domain Ω with $H = L^2(\Omega)$. The ordering relation $u \leq v$ in this case is equivalent to the expected one: '$u \leq v$ a.e. in Ω'.

DOI 10.20347/WIAS.PREPRINT.2747 Berlin 2020

Let $A\colon V \to V^*$ a linear operator that satisfies the following properties for all $u, v \in V$:

$$\langle Au, v\rangle \le C_b \|u\|_V \|v\|_V , \tag{boundedness}$$

$$\langle Au, u\rangle \ge C_a \|u\|_V^2 , \tag{coercivity}$$

$$\langle Au^+, u^-\rangle \le 0. \tag{T-monotonicity}$$

Given an obstacle map $\Phi\colon H \to V$, defining the constraint set $\mathbf{K}\colon V \rightrightarrows V$ by $\mathbf{K}(y) := \{v \in V : v \le \Phi(y)\}$, and given $f \in V^*$, we consider QVIs of the form

$$\text{find } y \in \mathbf{K}(y) : \langle Ay - f, y - v\rangle \le 0 \quad \forall v \in \mathbf{K}(y). \tag{1}$$

We take Φ to be increasing, i.e., $u \le v$ implies $\Phi(u) \le \Phi(v)$, and we define $\mathbf{Q}$ to be the solution map associated to (1), so that it reads $y \in \mathbf{Q}(f)$. To prove that the set $\mathbf{Q}(f)$ is non-empty (and indeed to properly define the problem under study in this article) we need some additional details.

1.1 Existence of (extremal) solutions

Fixing an obstacle $\varphi \in V$, consider the VI

$$u \in \mathbf{K}(\varphi) : \langle Au - f, u - v\rangle \le 0 \quad \forall v \in \mathbf{K}(\varphi)$$

and denote its solution map $S\colon V^* \times V \to V$ so that $u = S(f, \varphi)$. It follows that $\mathbf{Q}(f)$ is the set of fixed points of $\varphi \to S(f, \varphi)$. In order to show the presence of fixed points, we are going to assume the existence of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ for $S(f, \cdot)$, that is,

$$\exists \underline{u}, \overline{u} \in V \quad \text{s.t.} \quad \underline{u} \le S(f, \underline{u}) \quad \text{and} \quad \overline{u} \ge S(f, \overline{u}).$$

Remark 1.1. *For a supersolution, we can take any $\overline{u}$ satisfying $\overline{u} \ge A^{-1}f$ where the right-hand side is (by definition) the solution of the equation $Az = f$. This is a valid choice since $A^{-1}f = S(f, \infty) \ge S(f, \overline{u})$. If $f \ge 0$ and $\Phi(0) \ge 0$, then we may take $\underline{u} = 0$ to be a subsolution: $0 = S(0, 0) \le S(f, 0)$.*

Under these circumstances, we can apply the Birkhoff–Tartar theory [28, 11] (see also [5, Chapter 15.2.2] and [19, Chapter 2.5]) of fixed points in vector lattices to obtain not only that

$$\mathbf{Q}(f) \cap [\underline{u}, \overline{u}] \neq \emptyset$$

(i.e., (1) has solutions), but moreover, there exists a minimal solution $\mathsf{m}(f)$ and a maximal solution $\mathsf{M}(f)$ in this interval with respect to the ordering introduced above. These satisfy

$$\mathsf{m}(f) \le y \le \mathsf{M}(f) \quad \forall y \in \mathbf{Q}(f) \cap [\underline{u}, \overline{u}].$$

1.2 Aim of the article

In this paper, we are interested in the directional differentiability of $f \mapsto \mathsf{m}(f)$ and $f \mapsto \mathsf{M}(f)$. We will show that, under some assumptions, these maps are indeed directionally differentiable for a subset (that we will specify below) of directions belonging to V^*. That is, we prove the existence of the following limits:

$$\lim_{s\to 0^+} \frac{\mathsf{m}(f + sd) - \mathsf{m}(f)}{s} \qquad \text{and} \qquad \lim_{s\to 0^+} \frac{\mathsf{M}(f + sd) - \mathsf{M}(f)}{s}. \tag{2}$$

This builds upon our previous works [1, 4, 3] in two ways. In [1, 4] we showed that $\mathbf{Q}$ has a contingent derivative; essentially, we proved that for every $u \in \mathbf{Q}(f)$, given a direction $d \in V^*$, there exists $u^s \in \mathbf{Q}(f + sd)$ and (a directional derivative) $\alpha \in V$ such that

$$\lim_{s\to 0^+} \frac{u^s - u}{s} = \alpha$$

(with the limit taken in V). In [4], we also derived further existence results for (1) and procedures to iteratively approach solutions of the QVI. Furthermore, we also obtained stationarity systems for optimal control problems with QVI constraints. On the other hand, in [3], we studied continuity properties of the extremal operators m and M. This work then can be considered as a bridge between these two sets of papers.

The motivation of this study is twofold:

(i) the mathematical problem itself is challenging and interesting

(ii) in applications involving optimal control problem with QVI control-to-state maps, as typically there are many states associated to a single (optimal) control (due to non-uniqueness of solutions), it can be important to minimise the difference $\mathsf{M}(f) - \mathsf{m}(f)$. For example, in the case of thermoforming, manufacturers may wish to reproduce shapes or products that are within a certain acceptable tolerance value.

In the latter case, continuity properties of these maps (studied in [3]) are vital for the existence of the optimal control and differentiability properties are needed for writing down strong stationarity conditions.

The idea is to base our developments on the differentiability results obtained in [4]; let us recall this and set the scene in the next section.

Notation. Throughout the rest of the paper, we shall use the notation $o(\cdot)$ to denote a remainder term, i.e., $s^{-1}o(s) \to 0$ in V as $s \to 0^+$. The notation $B_R(z)$ will be used to mean the closed ball in V of radius R centred at z.

2 Preliminary material on QVIs

The next assumption has the consequence that the notions of capacity, quasi-continuity and related concepts are well defined, see [18, §3] and [14, §3]. Concrete examples of V (and the elliptic operator A) can be found in [1, §1.2].

Assumption 2.1. *Suppose that $H := L^2(\Omega;\mu)$ where Ω is a locally compact topological space which is σ-compact and μ is a Radon measure on Ω. We further assume that*

$$V \cap C_c(\Omega) \subset C_c(\Omega) \quad \text{and} \quad V \cap C_c(\Omega) \subset V \quad \text{are dense embeddings.}$$

Let us introduce the following notion of differentiability for operators.

Definition 2.2 ([27, §2]). A map $T\colon X \to Y$ between Banach spaces is said to be *boundedly directionally differentiable* at $x \in X$ if there exists a map $T'(x)\colon X \to Y$ such that

$$\lim_{s\to 0^+} \frac{T(x+sh) - T(x) - sT'(x)(h)}{s} = 0 \quad \text{uniformly in } h \text{ on bounded subsets of } Y.$$

Note that Fréchet differentiable operators are boundedly directionally differentiable.

The main result that we shall need is the following, which, under certain circumstances, tells us that $\mathbf{Q}$ has a contingent derivative and provides a characterisation of one such derivative. For the sake of completeness, we provide a compact proof of this theorem in Appendix A.

Theorem 2.3 ([4, Theorem 3.12]). *Let Assumption 2.1 hold and suppose that*

$$\Phi\colon V \to V \text{ is completely continuous.} \tag{3}$$

Given $f \in V^*$ and $d \in V^*$, for every $y \in \mathbf{Q}(f)$, under the local assumptions

$$\text{there exists } \epsilon > 0 \text{ s.t. } \Phi\colon V \to V \text{ is Hadamard directionally differentiable on } B_\epsilon(y), \tag{4}$$

$$\exists \epsilon > 0 : \|\Phi'(z)(v)\|_V \leq C_\Phi \|v\|_V \quad \forall z \in B_\epsilon(y), \forall v \in V, \text{ where } C_\Phi < (1 + C_a^{-1}C_b)^{-1}, \tag{5}$$

$$\Phi\colon V \to V \text{ is boundedly directionally differentiable at } y, \tag{6}$$

$$\Phi'(y)\colon V \to V \text{ is completely continuous}, \tag{7}$$

there exists $y^s \in \mathbf{Q}(f + sd) \cap B_R(y)$ (where $0 < R \leq \epsilon$) and $\alpha = \alpha(d) \in V$ such that

$$y^s = y + s\alpha + o(s),$$

where α satisfies the QVI

$$\begin{aligned} &\alpha \in \mathcal{K}^y(\alpha) : \langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y(\alpha),\\ &\qquad \mathcal{K}^y(w) := \{\varphi \in V : \varphi \leq \Phi'(y)(w) \text{ q.e. on } \mathcal{A}(y) \text{ and } \langle Ay - f, \varphi - \Phi'(y)(w)\rangle = 0\}. \end{aligned} \tag{8}$$

The directional derivative $\alpha = \alpha(d)$ is positively homogeneous in d. Furthermore, if $d \in V^*_+$ or $-d \in V^*_+$, (6) *can be omitted.*

Let us now begin the study with the minimal solution map first.

3 The minimal solution map

Our aim is to show that, given a source term f and a direction d, there exists an element $\mathsf{m}'(f)(d)$ such that

$$\mathsf{m}(f + sd) = \mathsf{m}(f) + s\mathsf{m}'(f)(d) + o(s).$$

Theorem 2.3 states that under certain assumptions, given $u \in \mathbf{Q}(f)$, there exists $u^s \in \mathbf{Q}(f + sd)$ and $\alpha \in V$ such that

$$u^s = u + s\alpha + o(s).$$

We may select u to be the minimal solution $\mathsf{m}(f)$ and it remains to prove that the selection mechanism of the theorem that furnishes the u^s is indeed such that $u^s \equiv \mathsf{m}(f + sd)$. To do this, we need to take a closer look at the method of proof of the cited theorem. The proof relies on

(i) creating an iterative sequence of solutions of VIs:

$$\begin{aligned} u_n^s &= S(f + sd, u_{n-1}^s)\\ u_0^s &= u, \end{aligned}$$

(ii) obtaining, by applying the sensitivity results for VIs by Mignot [18], expansion formulas of the type

$$u_n^s = u + s\alpha_n + o_n(s) \tag{9}$$

for these elements, and then

(iii) passing to the limit $n \to \infty$ and identifying the limits of $\{u_n^s\}$, $\{\alpha_n\}$ and $\{o_n\}$.

Thus, it is clear that we need to show that the limit of $\{u_n^s\}$ is indeed $\mathsf{m}(f + sd)$. For this purpose, we need to prove some properties of m which we shall do so in a series of lemmas.

Let us begin by defining the sequence

$$\begin{aligned} u_n &:= S(f, u_{n-1}), \\ u_0 &:= \underline{u}. \end{aligned} \tag{10}$$

We will assume (3) and that

$$\exists v_0 \in V : v_0 \leq \Phi(\underline{u}). \tag{11}$$

Remark 3.1. *The assumption* (11) *essentially asks for* $\mathbf{K}(\underline{u})$ *to be non-empty. A typical situation is where* f *is taken to be non-negative, in which case* $\underline{u} := 0$ *is a subsolution and* Φ *is taken such that* $\Phi(0) \geq 0$ *so that* $v_0 \equiv 0$ *is a possibility.*

Under these conditions, in [4, Theorem 2.3], we proved that u_n has a weak limit which belongs to $\mathbf{Q}(f)$. In fact, the sequence converges monotonically to the minimal solution as the next lemma demonstrates.

Lemma 3.2. *Assume* (3) *and* (11). *Then* $u_n \nearrow \mathsf{m}(f)$ *in* V.

Proof. By definition, $u_0 \leq \mathsf{m}(f)$. By definition of subsolution and by using the comparison principle, $u_0 \leq S(f, u_0) = u_1 \leq S(f, \mathsf{m}(f)) = \mathsf{m}(f)$. Arguing in a similar way, $u_0 \leq u_n \leq u_{n+1} \leq \mathsf{m}(f)$ for all n.

Since Φ is increasing, it follows that $v_0 \leq \Phi(u_n)$ for each n. Hence, we may test the VI for u_n with v_0 and use Young's inequality to obtain a uniform bound on u_n, which in combination with the fact that $\{u_n\}$ is monotonic, leads to

$$u_n \rightharpoonup u \in \mathbf{Q}(f)$$

(note that the convergence is for the entire sequence) with the passage to the limit (and the claim that the limit belongs to $\mathbf{Q}(f)$) handled by a standard Mosco argument thanks to (3): indeed, we test the VI for u_n with $v - \Phi(u) + \Phi(u_{n-1})$ where $v \in V$, $v \leq \Phi(u)$ is arbitrarily chosen, and then pass to the limit. It follows also that $u \in [\underline{u}, \mathsf{m}(f)]$ and therefore $u = \mathsf{m}(f)$. The strong convergence is a result of the standard continuous dependence estimate (eg., see [1, Equation (21)]) applied to u and u_n along with (3). ☐

Now let $s \geq 0$ be small and take $d \in V_+^*$. Since $\underline{u} \leq S(f, \underline{u}) \leq S(f + sd, \underline{u})$, $\underline{u}$ is also a subsolution for $S(f + sd, \cdot)$. In the other direction, we suppose that

$$\overline{u} \text{ is a supersolution for } S(f + sd, \cdot). \tag{12}$$

Then, by the argument in §1.1, we have the non-emptiness of the set $\mathbf{Q}(f + sd) \cap [\underline{u}, \overline{u}]$.

Remark 3.3. *Asking for $\overline{u}$ to be a supersolution for the perturbed problem is not a stringent requirement since any supersolution for $S(f+sd,\cdot)$ for any $s \geq 0$ is also a supersolution for $S(f,\cdot)$ and thus we may always start by taking $\overline{u}$ to be a supersolution of $S(f+s_0 d,\cdot)$ for some fixed $s_0 > 0$.*

If we define

$$\begin{aligned} y_n^s &:= S(f+sd, y_{n-1}^s), \\ y_0^s &:= \underline{u}, \end{aligned}$$

it follows that $y_n^s \nearrow \mathsf{m}(f+sd)$ in V by Lemma 3.2.

Lemma 3.4. *Let $d \in V_+^*$. Then $\mathsf{m}(f+sd) \geq \mathsf{m}(f)$.*

Proof. With u_n defined as above, we see that $y_1^s \geq u_1$ since $d \geq 0$. This implies that $y_n^s \geq u_n$ and hence, passing to the limit, we have $\mathsf{m}(f+sd) \geq \mathsf{m}(f)$. ☐

Let us define (as sketched above) a sequence starting at $\mathsf{m}(f)$ with perturbed source term as follows:

$$\begin{aligned} u_n^s &:= S(f+sd, u_{n-1}^s) \\ u_0^s &:= \mathsf{m}(f). \end{aligned}$$

Since $\mathsf{m}(f)$ acts as a subsolution, $u_n^s \to \mathsf{m}|_{[\mathsf{m}(f),\overline{u}]}(f+sd)$ by Lemma 3.2 where the notation $\mathsf{m}|_A(f+sd)$ refers to the minimal solution on $[\underline{u},\overline{u}] \cap A$. But in fact, the limit is the minimal solution on the full interval $[\underline{u},\overline{u}]$ as the next result shows. That is to say, its limit agrees with the limit of the sequence $\{y_n^s\}$ constructed above.

Lemma 3.5. *We have $u_n^s \nearrow \mathsf{m}(f+sd)$ in V.*

Proof. Since $\mathsf{m}(f) \geq \underline{u}$, we have $u_1^s \geq y_1^s$ and thus $u_n^s \geq y_n^s$. Passing to the limit,

$$u^s := \mathsf{m}|_{[\mathsf{m}(f),\overline{u}]}(f+sd) \geq \mathsf{m}(f+sd).$$

By definition, $\mathsf{m}(f+sd)$ is minimal on $[\underline{u},\overline{u}]$ and u^s is the minimal on $[\mathsf{m}(f),\overline{u}]$, but we also have that $\mathsf{m}(f+sd) \in [\mathsf{m}(f),\overline{u}]$ by Lemma 3.4. Hence it must be the case that $u^s = \mathsf{m}(f+sd)$. ☐

With all the preparations complete, we are ready to state the differentiability result.

Theorem 3.6. *Let Assumption 2.1 hold. In addition to* (3) , (11) *and* (12)*, assume the local assumptions*

$$\text{there exists } \epsilon > 0 \text{ s.t. } \Phi \colon V \to V \text{ is Hadamard directionally differentiable on } B_\epsilon(\mathsf{m}(f)), \tag{13}$$

$$\exists \epsilon > 0 : \|\Phi'(z)(v)\|_V \leq C_\Phi \|v\|_V \quad \forall z \in B_\epsilon(\mathsf{m}(f)),\ \forall v \in V, \qquad \text{where} \quad C_\Phi < C_a(C_a+C_b)^{-1}, \tag{14}$$

$$\Phi'(\mathsf{m}(f)) \colon V \to V \text{ is completely continuous.} \tag{15}$$

Then the map $\mathsf{m} \colon V^* \to V$ is directionally differentiable in every direction $d \in V_+^*$:

$$\lim_{s \to 0^+} \frac{\mathsf{m}(f+sd) - \mathsf{m}(f)}{s} = \mathsf{m}'(f)(d).$$

Furthermore, $\mathsf{m}'(f)(d)$ *satisfies the QVI*[2]

$$\begin{aligned} &\alpha \in \mathcal{K}_{\mathsf{m}}(\alpha) : \langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}_{\mathsf{m}}(\alpha), \\ &\qquad \mathcal{K}_{\mathsf{m}}(\alpha) := \{\varphi \in V : \varphi \leq \Phi'(\mathsf{m}(f))(\alpha) \text{ q.e. on } \{\mathsf{m}(f) = \Phi(\mathsf{m}(f))\} \\ &\qquad\qquad \text{and } \langle A\mathsf{m}(f) - f, \varphi - \Phi'(\mathsf{m}(f))(\alpha)\rangle = 0\}. \end{aligned} \tag{16}$$

Proof. The proof is as described at the start of this section. Indeed, a straightforward application of Theorem 2.3 gives the existence of $\alpha \in V$ such that the limit u^s of $\{u_n^s\}$ satisfies

$$u^s = \mathsf{m}(f) + s\alpha + o(s),$$

and Lemma 3.5 tells us that $u^s = \mathsf{m}(f + sd)$. □

The QVI (16) satisfied by the derivative α in general possesses multiple solutions, hence the question of how to numerically solve for the derivative naturally arises. Here, we can answer positively: the derivative is determined as the monotone limit of the sequence $\{\alpha_n\}$ (see (9)) of solutions of VIs where each α_n satisfies

$$\begin{aligned} &\alpha_n \in \mathcal{K}_{\mathsf{m}}(\alpha_{n-1}) : \langle A\alpha_n - d, \alpha_n - \varphi\rangle \leq 0 \quad \forall \varphi \in \mathcal{K}_{\mathsf{m}}(\alpha_{n-1}), \\ &\qquad \mathcal{K}_{\mathsf{m}}(\alpha_{n-1}) := \{\varphi \in V : \varphi \leq \Phi'(\mathsf{m}(f))(\alpha_{n-1}) \text{ q.e. on } \{\mathsf{m}(f) = \Phi(\mathsf{m}(f))\} \\ &\qquad\qquad \text{and } \langle A\mathsf{m}(f) - f, \varphi - \Phi'(\mathsf{m}(f))(\alpha_{n-1})\rangle = 0\}. \end{aligned}$$

A direct consequence of the monotonicity of $\{u_n^s\}$ allows us to conclude that $\alpha^n \nearrow \alpha$ in V.

4 The maximal solution map

The strategy in this section is the same as §3. Here, we reverse the sign of the direction term in order to enforce monotonicity of a certain sequence.

In (10), if we instead start with the initial iterate at a supersolution, we are able to provide analogous results. To wit, taking for $n \geq 1$, $u_n = S(f, u_{n-1})$ as before, let now

$$u_0 := \overline{u}.$$

A similar argument to the proof of Lemma 3.2 proves the next lemma.

Lemma 4.1. *Assume* (3) *and that*

$$\exists v_0 \in V : v_0 \leq \Phi(v) \quad \forall v \in V : v \leq \overline{u}. \tag{17}$$

Then $u_n \searrow \mathsf{M}(f)$ *in* V*.*

Take $s \geq 0$ to be small. Observe that for any $d \in V^*$ with $d \leq 0$, $\overline{u}$ is a supersolution for $S(f + sd, \cdot)$ too: $\overline{u} \geq S(f, \overline{u}) \geq S(f + sd, \overline{u})$ by the sign on d. Akin to the previous section, we are going to assume that

$$\underline{u} \text{ is a subsolution for } S(f + sd, \cdot). \tag{18}$$

[2]Note that the coincidence set appearing in the critical cone $\mathcal{K}_{\mathsf{m}}(\alpha)$ defined in (16) is of course taken over Ω, i.e.,

$$\{\mathsf{m}(f) = \Phi(\mathsf{m}(f))\} = \{x \in \Omega : \mathsf{m}(f)(x) = \Phi(\mathsf{m}(f))(x)\}.$$

Lemma 4.2. *Let $d \in -V_+^*$. Then $\mathsf{M}(f+sd) \le \mathsf{M}(f)$.*

Proof. Define

$$\begin{aligned} y_n^s &:= S(f+sd, y_{n-1}^s),\\ y_0^s &:= \overline{u}. \end{aligned}$$

It follows that $u_1 \ge y_1^s$ and therefore $u_n \ge y_n^s$. Passing to the limit and using the above lemma, we see that $\mathsf{M}(f) \ge \mathsf{M}(f+sd)$. □

It is not difficult to see that $\mathsf{M}(f)$ is a supersolution for $S(f+sd,\cdot)$ for non-positive d. This allows us to construct the perturbed sequence starting at $\mathsf{M}(f)$ and we obtain the next result.

Lemma 4.3. *Let $d \in -V_+^*$ and define*

$$\begin{aligned} u_n^s &:= S(f+sd, u_{n-1}^s),\\ u_0^s &:= \mathsf{M}(f). \end{aligned}$$

Then $u_n^s \searrow \mathsf{M}(f+sd)$.

Proof. We see that, using Lemma 4.2, $u_1^s \ge S(f+sd, \mathsf{M}(f+sd)) = \mathsf{M}(f+sd)$, implying $u_n^s \ge \mathsf{M}(f+sd)$. Since $\mathsf{M}(f)$ is a supersolution for $S(f+sd,\cdot)$, we obtain, by Lemma 4.1, $u_n^s \searrow u^s = \mathsf{M}_{[\underline{u},\mathsf{M}(f)]}(f+sd) \ge \mathsf{M}(f+sd)$ since $[\underline{u},\mathsf{M}(f)] \subset [\underline{u},\overline{u}]$. But Lemma 4.2 tells us that in fact $\mathsf{M}(f+sd)$ belongs to $[\underline{u},\mathsf{M}(f)]$ so we must have $u^s \le \mathsf{M}(f+sd)$ because $\mathsf{M}(f+sd)$ is also the largest element on $[\overline{u},\mathsf{M}(f)]$.

□

Theorem 4.4. *Let Assumption 2.1 hold. In addition to* (3)*,* (17)*,* (18)*, suppose that the local assumptions* (13)*,*(14) *and* (15) *(provided all instances of* $\mathsf{m}(f)$ *are replaced by* $\mathsf{M}(f)$*) hold.*

Then the map $\mathsf{M}\colon V^* \to V$ is directionally differentiable in every direction $d \in -V_+^*$:

$$\lim_{s\to 0^+} \frac{\mathsf{M}(f+sd) - \mathsf{M}(f)}{s} = \mathsf{M}'(f)(d), \tag{19}$$

and the derivative $\mathsf{M}'(f)(d)$ satisfies the QVI (16) *with all instances of* $\mathsf{m}(f)$ *replaced by* $\mathsf{M}(f)$*.*

Proof. This is again due to Theorem 2.3 and Lemma 4.3. □

In a similar way to §3, we obtain that $\alpha_n \searrow \alpha$ in V where

$$\begin{aligned} \alpha_n \in \mathcal{K}_{\mathsf{M}}(\alpha_{n-1}) : \langle A\alpha_n - d, \alpha_n - \varphi\rangle \le 0 \quad \forall \varphi \in \mathcal{K}_{\mathsf{M}}(\alpha_{n-1}),\\ \mathcal{K}_{\mathsf{M}}(\alpha_{n-1}) := \{\varphi \in V : \varphi \le \Phi'(\mathsf{M}(f))(\alpha_{n-1}) \text{ q.e. on } \{\mathsf{M}(f) = \Phi(\mathsf{M}(f))\}\\ \text{and } \langle A\mathsf{M}(f) - f, \varphi - \Phi'(\mathsf{M}(f))(\alpha_{n-1})\rangle = 0\}. \end{aligned}$$

A Sketch proof of Theorem 2.3

We give here a compact presentation of the proof of Theorem 2.3 for the convenience of the reader; full details and additional explanation can be found in [4, §3.1].

Fix an arbitrary $f \in V^*$ and take an arbitrary but fixed $y \in \mathbf{Q}(f)$. Recall the notation $B_R(y) \subset V$ to stand for the closed ball in V of radius R centred on u. Pick a direction $d \in V^*$ and construct the sequence

$$\begin{aligned} y_0^s &:= y, \\ y_n^s &:= S(f + sd, y_{n-1}^s). \end{aligned} \tag{20}$$

We obtain the following existence and convergence result.

Lemma A.1. *Given $f, d \in V^*$ and $y \in \mathbf{Q}(f)$, under the local assumptions* (4) *and* (5)*, there exists $y^s \in \mathbf{Q}(f + sd) \cap B_R(y)$ such that*

$$y_n^s \to y^s \quad \textit{in } V$$

as long as $s \leq C_a \|d\|_{V^*}^{-1} R(1 - (1 + C_b C_a^{-1})C_\Phi)$.

Proof. First, let us show that for any $0 < R \leq \epsilon$, $S(f + sd, \cdot) \colon B_R(y) \to B_R(y)$ is a contraction for s as above. Indeed, let $v \in B_R(y)$. Using continuous dependence (eg. [1, Equation (21)]) and the mean value theorem [21, §2, Proposition 2.29],

$$\begin{aligned} \|S(f + sd, v) - y\|_V &\leq (1 + C_b C_a^{-1}) \sup_{\lambda \in (0,1)} \|\Phi'(\lambda v + (1 - \lambda) y)(v - y)\|_V + C_a^{-1} s \|d\|_{V^*} \\ &\leq (1 + C_b C_a^{-1}) C_\Phi R + C_a^{-1} s \|d\|_{V^*}, \end{aligned}$$

since $\lambda v + (1 - \lambda)y \in B_R(y) \subset B_\epsilon(y)$. Using the fact that $(1 + C_b C_a^{-1})C_\Phi < 1$, the right-hand side is bounded above by R under the stated assumption. This shows that $S(f + sd, \cdot)$ maps $B_R(y)$ into itself. To see that the map is a contraction, take $v, w \in B_R(y)$ and observe that

$$\begin{aligned} \|S(f + sd, v) - S(f + sd, w)\|_V &\leq (1 + C_a^{-1} C_b) \sup_{\lambda \in (0,1)} \|\Phi'(\lambda w + (1 - \lambda) v)(w - v)\|_V \\ &\leq C_\Phi (1 + C_a^{-1} C_b) \|z_2 - z_1\|_V. \end{aligned}$$

We finish by applying the Banach fixed point theorem. □

Making use of the differentiability result for VIs provided by Mignot [18, Theorem 3.3], we can expand

$$y_1^s = y + s\delta_1 + o_1(s),$$

where $s^{-1} o_1(s) \to 0$ as $s \to 0^+$ and $\delta_1 = \partial S(f, y)(d)$ is the directional derivative of $S(f, \cdot)$ in the direction d, and this satisfies the VI

$$\begin{aligned} \delta_1 \in \mathcal{K}^y : \quad & \langle A\delta_1 - d, \delta_1 - v \rangle \leq 0 \quad \forall v \in \mathcal{K}^y, \\ \mathcal{K}^y &:= \{ w \in V : w \leq 0 \text{ q.e. on } \mathcal{A}(y) \text{ and } \langle Ay - f, w \rangle = 0 \}. \end{aligned} \tag{21}$$

To acquire an expansion formula for a general y_n^s, define

$$\delta_n := \partial S(f, y)[d - A\Phi'(y)(\Phi'(y)[...\Phi'(y)[\Phi'(y)(\delta_0) + \delta_1] + \delta_2 ...] + \delta_{n-2}] + \delta_{n-1})] \quad \text{for } n > 1$$

and

$$\alpha_n := \begin{cases} \delta_1 & \text{: if } n = 1 \\ \Phi'(y)[\Phi'(y)[...\Phi'(y)[\Phi'(y)(\delta_1) + \delta_2] + \delta_3...] + \delta_{n-1}] + \delta_n & \text{: if } n \geq 2, \end{cases}$$

and observe the recursion formula $\alpha_n = \Phi'(y)[\alpha_{n-1}] + \delta_n$ for $n > 1$. In exactly the same way as in [1, Proposition 2], we obtain the following result.

Proposition A.2. *Under the assumptions of the previous lemma, for $n \geq 1$,*

$$y_n^s = y + s\alpha_n + o_n(s) \tag{22}$$

where $\alpha_n = \alpha_n(d)$ is positively homogeneous in the direction d and satisfies the VI

$$\begin{aligned} \alpha_n \in \mathcal{K}^y(\alpha_{n-1}) : \langle A\alpha_n - d, \alpha_n - \varphi \rangle \leq 0 \qquad \forall \varphi \in \mathcal{K}^y(\alpha_{n-1}), \\ \mathcal{K}^y(\alpha_{n-1}) := \{\varphi \in V : \varphi \leq \Phi'(y)(\alpha_{n-1}) \text{ q.e. on } \mathcal{A}(y) \\ \text{and } \langle Ay - f, \varphi - \Phi'(y)(\alpha_{n-1}) \rangle = 0\}, \end{aligned}$$

with $s^{-1}o_n(s) \to 0$ as $s \to 0^+$.

It remains then to pass to the limit in (22) and to identify the corresponding limits. To this end, observe that $s\alpha_n + o_n(s) = y_n^s - y \to y^s - y$ in V. Assumption (5) provides the existence of a constant $c > 0$ such that

$$\|\Phi'(y)(v)\|_V \leq \frac{C_a - c}{C_b} \|v\|_V ,$$

and thus the sequence $\{\alpha_n\}$ is bounded exactly as shown in the proof of [1, Theorem 6] and we have the existence of a subsequence $\{n_j\}$ with

$$\alpha_{n_j} \rightharpoonup \alpha \text{ in } V \qquad \text{and} \qquad o_{n_j}(s) \rightharpoonup o^*(s) \text{ in } V.$$

We can pass to the limit in (22) along this subsequence to obtain

$$y^s = y + s\alpha + o^*(s), \tag{23}$$

and it is left for us to show that o^* is a remainder term and to characterise α suitably. For this, we need some more notation. Let $S_0 \colon V^* \to V$ be the map $f \mapsto u$ of the following VI with trivial lower obstacle:

$$u \in V_+ : \langle Au - f, u - v \rangle \leq 0 \quad \forall v \in V_+, \tag{24}$$

and denote the remainder term associated to the derivative formula of S_0 by $o(\cdot, \cdot\,; \cdot)$, that is,

$$o(s, h; f) := \frac{S_0(f + sh) - S_0(f) - sS_0'(f)(h)}{s}.$$

Similarly, we denote the remainder term associated to Φ by $l(\cdot, \cdot\,; \cdot)$.

Now we adapt the proof of [1, Lemma 14] under this context.

Proposition A.3. *Assume* (4)*,* (5)*,* (6) *and* (7)*. Then $s^{-1}o^*(s) \to 0$ as $s \to 0$.*

Proof. Define
$$a_n(s) := \|o_n(s)\|_V$$
and
$$b_n(s) := (1 + C_a^{-1}C_b)\,\|l(s,\alpha_n;y)\|_V + \|o(s, A\Phi'(y)(\alpha_n); A\Phi(y) - f)\|_V\,.$$
From the proof of [1, Lemma 14], we see that a_n satisfies
$$a_n(s) \le C^{n-1}a_1(s) + C^{n-2}b_1(s) + C^{n-3}b_2(s) + ... + Cb_{n-2}(s) + b_{n-1}(s). \tag{25}$$
where the constant $C < 1$ by the assumption on C_Φ in (5). Consider
$$\frac{b_{n-1}(s)}{s} = \frac{(1 + C_a^{-1}C_b)\,\|l(s,\alpha_{n-1};y)\|_V}{s} + \frac{\|o(s, A\Phi'(y)(\alpha_{n-1}); A\Phi(y) - f)\|_V}{s}.$$
Since $\{\alpha_n\}$ is bounded, the first term on the right-hand side converges to zero uniformly in n due to (6). The compactness of $\Phi'(y)(\cdot)\colon V \to V$ implies that $\overline{\{A\Phi'(y)(\alpha_{n-1})\}}$ is a compact set in V^*. Since the remainder term o above arises from the Hadamard differentiability of the solution map associated to VIs, it follows that $o(s,h)/s \to 0$ uniformly for h belonging to the compact set $\overline{\{A\Phi'(y)(\alpha_{n-1})\}}$. It follows that
$$\frac{b_{n-1}(s)}{s} \to 0 \quad \text{uniformly in } n.$$
These facts along with (25) imply that $s^{-1}o_n(s) \to 0$ as $s \to 0^+$ uniformly in n. Finally, using the weak convergence of the subsequence o_{n_j}, taking the liminf as $n_j \to \infty$ and using the weak lower semicontinuity of norms in the above inequality for $n = n_j$, we deduce the result. □

As a byproduct of the above result, we find that the whole sequence $\{\alpha_n\}$ indeed converges.

Lemma A.4. *Under the assumptions of the previous proposition, $\alpha_n \to \alpha$ in V (for the whole sequence).*

Proof. Defining
$$r_n(s) := \alpha_n + \frac{o_n(s)}{s} = \frac{y_n^s - y}{s},$$
we see that, thanks to the strong convergence of y_n^s and (23),
$$\lim_{n\to\infty} r_n(s) = \alpha + \frac{o^*(s)}{s}.$$
We claim that
$$\lim_{s\to 0^+} r_n(s) = \alpha_n \quad \text{uniformly in } n.$$
This follows because the quantity $r_n(s) - \alpha_n = o_n(s)/s$ converges to zero as $s \to 0^+$ uniformly in n as we have seen in the proof of Proposition A.3, and the Moore–Osgood theorem [12, §I.7, Lemma 6] then applies, giving the existence of iterated limits as well as commutability and we get
$$\alpha = \lim_{s\to 0^+}\left(\alpha + \frac{o^*(s)}{s}\right) = \lim_{s\to 0^+}\lim_{n\to\infty} r_n(s) = \lim_{n\to\infty}\lim_{s\to 0^+} r_n(s) = \lim_{n\to\infty}\alpha_n.$$
□

This strong convergence allows for the characterisation of the directional derivative as stated in the theorem — namely, it allows us to pass to the limit in the recurrence formula for α_n (see above), which is given in terms of α_{n-1} (for which arguments using convergences of subsequences would not be viable). See §5.1 and §5.2 in [1] for more details.

References

[1] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Directional differentiability for elliptic quasi-variational inequalities of obstacle type. *Calc. Var. Partial Differential Equations*, 58(1):Art. 39, 47, 2019.

[2] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Recent trends and views on elliptic quasi-variational inequalities. In M. Hintermüller and J. F. Rodrigues, editors, *Topics in Applied Analysis and Optimisation*, pages 1–31, Cham, 2019. Springer International Publishing.

[3] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Stability of the solution set of quasi-variational inequalities and optimal control. *arXiv e-prints*, 2019.

[4] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Optimal control and directional differentiability for elliptic quasi-variational inequalities. *arXiv e-prints*, page arXiv:2008.13586, Aug. 2020. Available at `https://arxiv.org/abs/2008.13586`.

[5] J.-P. Aubin. *Mathematical methods of game and economic theory*. Dover Publications, Inc., Mineola, NY, revised edition, 2007. With a new preface by the author.

[6] C. Baiocchi and A. Capelo. *Variational and Quasivariational Inequalities*. Wiley-Interscience, 1984.

[7] J. W. Barrett and L. Prigozhin. A quasi-variational inequality problem in superconductivity. *Math. Models Methods Appl. Sci.*, 20(5):679–706, 2010.

[8] J. W. Barrett and L. Prigozhin. A quasi-variational inequality problem arising in the modeling of growing sandpiles. *ESAIM Math. Model. Numer. Anal.*, 47(4):1133–1165, 2013.

[9] J. W. Barrett and L. Prigozhin. Lakes and rivers in the landscape: a quasi-variational inequality approach. *Interfaces Free Bound.*, 16(2):269–296, 2014.

[10] A. Bensoussan, M. Goursat, and J.-L. Lions. Contrôle impulsionnel et inéquations quasi-variationnelles stationnaires. *C. R. Acad. Sci. Paris Sér. A-B*, 276:A1279–A1284, 1973.

[11] G. Birkhoff. *Lattice theory*, volume 25 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., third edition, 1979.

[12] N. Dunford and J. T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[13] F. Facchinei and C. Kanzow. Generalized nash equilibrium problems. *4OR*, 5(3):173–210, Sep 2007.

[14] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *J. Math. Soc. Japan*, 29(4):615–631, 1977.

[15] P. T. Harker. Generalized nash games and quasi-variational inequalities. *European Journal of Operational Research*, 54(1):81 – 94, 1991.

[16] M. Kunze and J. F. Rodrigues. An elliptic quasi-variational inequality with gradient constraints and some of its applications. *Math. Methods Appl. Sci.*, 23(10):897–908, 2000.

[17] J.-L. Lions. Sur le côntrole optimal des systemes distribuées. *Enseigne*, 19:125–166, 1973.

[18] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *J. Functional Analysis*, 22(2):130–185, 1976.

[19] U. Mosco. Implicit variational problems and quasi variational inequalities. In *Nonlinear Operators and the Calculus of Variations*, pages 83–156. 1976.

[20] J.-S. Pang and M. Fukushima. Quasi-variational inequalities, generalized nash equilibria, and multi-leader-follower games. *Computational Management Science*, 2(1):21–56, Jan 2005.

[21] J.-P. Penot. *Calculus without derivatives*, volume 266 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[22] L. Prigozhin. Sandpiles and river networks: extended systems with non-local interactions. *Phys. Rev. E*, 49:1161–1167, 1994.

[23] L. Prigozhin. On the Bean critical-state model in superconductivity. *European Journal of Applied Mathematics*, 7:237–247, 1996.

[24] L. Prigozhin. Sandpiles, river networks, and type-ii superconductors. *Free Boundary Problems News*, 10:2–4, 1996.

[25] L. Prigozhin. Variational model of sandpile growth. *European J. Appl. Math.*, 7(3):225–235, 1996.

[26] J. F. Rodrigues and L. Santos. A parabolic quasi-variational inequality arising in a superconductivity model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(1):153–169, 2000.

[27] A. Shapiro. On concepts of directional differentiability. *Journal of Optimization Theory and Applications*, 66(3):477–487, 1990.

[28] L. Tartar. Inéquations quasi variationnelles abstraites. *C. R. Ac. Sci. Paris*, 178:1193–1196, 1974.