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An effective bulk-surface thermistor model for large-area organic light-emitting diodes

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Abstract

The existence of a weak solution for an effective system of partial differential equations describing the electrothermal behavior of large-area organic light-emitting diodes (OLEDs) is proved. The effective system consists of the heat equation in the three-dimensional bulk glass substrate and two semi-linear equations for the current flow through the electrodes coupled to algebraic equations for the continuity of the electrical fluxes through the organic layers. The electrical problem is formulated on the (curvilinear) surface of the glass substrate where the OLED is mounted. The source terms in the heat equation are due to Joule heating and are hence concentrated on the part of the boundary where the current-flow equation is posed. The existence of weak solutions to the effective system is proved via Schauder's fixed-point theorem. Moreover, since the heat sources are a priori only in L^1, the concept of entropy solutions is used.

1 Introduction

Large-area light-emitting diodes made of organic semiconductor materials are thin-film multilayer devices showing pronounced self-heating and brightness inhomogeneities at high currents [12, 35]. Since high currents are required in lighting applications, a deeper understanding of the mechanisms causing these inhomogeneities is necessary.

In organic semiconductor materials, charge carriers move via temperature-activated hopping transport through an energetically random energy landscape [22]. By applying a voltage to an organic semiconductor device a current flow is induced which leads to a power dissipation by Joule heating and hence a temperature rise. The increase in temperature improves the conductivity in organic materials leading in turn to the occurrence of higher currents [21, 23]. Thus, a positive feedback loop develops that could result in complete destruction of the device by thermal runaway if the generated heat cannot be dispersed into the environment. The temperature dependence of the conductivity is often modeled by an exponential law of Arrhenius type [13] (see also (2.5)), which features an activation energy related to the energetic disorder in the organic material. For sufficiently high activation energies, the electrothermal interaction can lead to S-shaped current-voltage characteristics with regions of negative differential resistance [12, 13]. The latter means that currents increase despite of decreasing voltages. Moreover, in [12, 20] it is demonstrated experimentally that the complex interplay of temperature-activated transport of the charge carriers and heat flow in the device leads to inhomogeneous current distributions resulting in inhomogeneous luminance. Devices, whose resistance strongly depends on temperature, are called thermistors and have attracted great interest concerning mathematical modeling, analysis, and optimization, see e.g. [37, 3, 19, 8, 36, 18].

In the current work, we prove the existence of solutions to an effective electrothermal model with bulk-surface coupling (see (2.1)). The model describes the current flow in a large-area thin-film OLED and the induced heat flow in the substrate on which the OLED is mounted. In particular, the electrically relevant Joule heat-producing processes take place in the (curvilinear) two-dimensional OLED domain, while the heat flow happens in the larger, three-dimensional domain occupied by the glass substrate and gives strong feedback to the current-flow equation in the OLED domain e.g. via an Arrhenius law. For the planar situation, this effective model was derived in [17] from a fully three-dimensional thermistor system, which in turn was introduced in [26], by considering the limit of vanishing thicknesses of the various OLED layers including the electrodes. The fully three-dimensional thermistor system in [26] is based on the heat equation for the temperature coupled to a $p(x)$-Laplace-type equation for the electric driving potential with mixed boundary conditions. It was extended in [6] to include more general conductivity laws with $p(x)$-growth. The $p(x)$-Laplacian, with discontinuous $x \mapsto p(x)$, allows us to take into account different non-Ohmic electrical behavior of the different organic layers (see, e.g., [25, 26]). The model recovers the experimentally observed S-shaped current-voltage characteristics and explains the development of luminance inhomogeneities in OLEDs [21, 20].

In the effective model, which is discussed here, the description of the current flow through the OLED is reduced to a system of two semilinear PDEs for the lateral current flow in the electrodes and algebraic equations describing the vertical flow through the several organic layers on the two-dimensional OLED domain. In particular, the nonlinear functions in the

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algebraic equations are given by the vertical component of the flux functions used in the fully three-dimensional model (see Remark 2.1 for a brief outline of the derivation). The Joule heat produced by the current flow through the actual OLED device enters as a surface source at the part of the substrate boundary where the OLED is mounted. Thus, the system falls in the class of coupled bulk-surface PDEs which are often found in the modeling of biological systems in the form of reaction-diffusion equations [2, 32, 29]. In [1], a linear, elliptic bulk-surface system on complicated geometries is analyzed using a diffuse interface type approximation. Complicated geometries also appear in cases of bent OLEDs used e.g. in the automotive sector. A similar system is considered in [11] (see also [30]), where existence and uniqueness follow from standard elliptic theory. Moreover, the numerical discretization of bulk-surface PDEs is discussed therein.

The well-posedness of solutions for the fully three-dimensional thermistor system can be found in [6, 7, 15] and in [27, 28] for the time-dependent case. The main challenge is the source term in the heat equation which is a priori only in L^1. While the results in [15] are restricted to two spatial dimensions and are based on the deriation of higher integrability properties of the driving potential to treat the right-hand side in the heat equation in L^q, with some $q > 1$, the papers [6, 7] deal with arbitrary spatial dimensions and use the theory of entropy solutions (cf. [4]) to overcome the lack of integrability of the Joule heat term. Also in the present text, the theory of entropy solutions (cf. also Definition 3.1) and Schauder's fixed point theorem are used. The fixed-point map is constructed by considering the current-flow and the heat equation separately: For a given $\widetilde{T} \in \mathcal{N}$, with $\mathcal{N}$ being a suitable subset of temperature distributions (see (5.1)), we uniquely solve the current-flow equation. Using the solution of the latter as well as $\widetilde{T}$ in the boundary Joule heat term yields an entropy solution T of the heat equation. The fixed-point of this map together with the associated unique solution of the current-flow equation is a solution of the effective thermistor system. To establish the required properties of the fixed-point map $Q : \widetilde{T} \mapsto T$ and the set $\mathcal{N}$, i.e. continuity and compactness, respectively, suitable a priori estimates for the subproblems are derived.

Finally, let us mention that a more detailed description of charge-carrier transport coupled to heat flow is achieved by considering energy-drift diffusion equations, where the Poisson equation for the electrostatic potential and continuity equations for charge carrier densities are coupled to the heat equation [9]. In the latter, additional sources due to the recombination of charge carriers appear. As demonstrated in [10], also energy-drift-diffusion models for organic semiconductor devices capture the relevant electrothermal phenomena such as S-shaped current-voltage curves with regions of negative differential resistance, for analytical results see [16].

The outline of the paper is as follows: In Section 2, we introduce the effective electrothermal model for OLEDs derived in [17]. Section 3 contains the assumptions on the data, a reformulation of the model in terms of potential differences as well as the corresponding notion of solution, and the main result of the paper, i.e. the existence result, in Theorem 3.1. Results concerning the electric and thermal subproblems are derived in Section 4. Finally, the proof of Theorem 3.1 is presented in Section 5.

2 An effective electrothermal model for OLEDs

We consider the case that a thin-film OLED is deposited on a surface part of a bulk substrate material. In particular, the latter is assumed to occupy the domain $\Omega \subset \mathbb{R}^3$ such that there exists a boundary part $\omega \subset \partial\Omega$ being C^2 regular and having positive surface measure (for more detailed assumptions see Subsection 3.1). In the situation of a flat surface ω, it was assumed in [17] that the electrically active OLED is deposited on top of ω and occupies a cylindrical domain of thickness $h > 0$ with $N > 2$ sublayers. An effective electrothermal model was derived by considering the limit of vanishing thickness $h \downarrow 0$, see Remark 2.1.

The effective model consists of two equations for the lateral current flow in the top and bottom electrode and algebraic equations for the vertical current flow through the organic layers, each given on ω, as well as the heat equation in Ω with boundary sources on ω, viz.

$$-\nabla_\omega \cdot (\sigma_{\rm sh}^- \nabla_\omega \varphi^1) - F^2(T, \varphi^2 - \varphi^1) = 0 \quad \text{on } \omega, \tag{2.1a}$$

$$F^k(T, \varphi^k - \varphi^{k-1}) - F^{k+1}(T, \varphi^{k+1} - \varphi^k) = 0 \quad \text{on } \omega,\ k = 2, \ldots, N-2, \tag{2.1b}$$

$$-\nabla_\omega \cdot (\sigma_{\rm sh}^+ \nabla_\omega \varphi^{N-1}) + F^{N-1}(T, \varphi^{N-1} - \varphi^{N-2}) = 0 \quad \text{on } \omega, \tag{2.1c}$$

$$-\nabla \cdot (\lambda(x) \nabla T) = 0 \quad \text{in } \Omega. \tag{2.1d}$$

Here, ∇_ω describes the surface gradient on ω, $\sigma_{\rm sh}^+$, $\sigma_{\rm sh}^- > 0$ are the sheet conductivities of the top and bottom electrode, and the functions $F^k : [T_{\rm a}, \infty) \times \mathbb{R} \to \mathbb{R}$ describe the vertical current flow through the different layers of the OLED, $k = 2, \ldots, N-1$, with $T_{\rm a} > 0$ being a fixed ambient temperature. In summary, we have to determine the temperature T in the glass substrate and $N-1$ potentials φ^k, $k = 1, \ldots, N-1$, representing the electrostatic potentials in the top and bottom electrodes and interfacial potentials between the different layers of the OLED.

On open subsets γ^+, γ^- of $\partial\omega$ with positive one-dimensional Hausdorff measure, we formulate Dirichlet boundary conditions for φ^1 and φ^{N-1} in (2.1a) and (2.1c), respectively, while on the remaining part of $\partial\omega$ homogeneous Neumann are supposed, where $\nu_\omega \in \mathbb{R}^2$ denotes the outer unit normal vector on $\partial\omega$, namely

$$\varphi^1 = \varphi^D_- \text{ on } \gamma^-, \qquad \sigma^-_{\text{sh}} \nabla_\omega \varphi^1 \cdot \nu_\omega = 0 \text{ on } \partial\omega \setminus \gamma^-, \tag{2.2a}$$

$$\varphi^{N-1} = \varphi^D_+ \text{ on } \gamma^+, \quad \sigma^+_{\text{sh}} \nabla_\omega \varphi^{N-1} \cdot \nu_\omega = 0 \text{ on } \partial\omega \setminus \gamma^+. \tag{2.2b}$$

The heat equation (2.1d) in the substrate Ω is supplemented by the following nonlinear boundary conditions modeling Joule heating

$$-\lambda(x)\nabla T \cdot \nu = \begin{cases} \kappa(x)(T - T_{\text{a}}) & \text{on } \partial\Omega \setminus \omega, \\ \kappa(x)(T - T_{\text{a}}) - H_\omega & \text{on } \omega \end{cases} \tag{2.3}$$

with outer unit normal vector $\nu \in \mathbb{R}^3$ on $\partial\Omega$ and surface heating term $H_\omega = H_\omega(x, T, \varphi^1, \ldots, \varphi^{N-1})$ on that part $\omega \subset \partial\Omega$ of the surface of the substrate where the OLED is mounted,

$$H_\omega = \sigma^-_{\text{sh}} \big|\nabla_\omega \varphi^1\big|^2 + \sigma^+_{\text{sh}} \big|\nabla_\omega \varphi^{N-1}\big|^2 + \sum_{k=2}^{N-1} F^k(T, \varphi^k - \varphi^{k-1})(\varphi^k - \varphi^{k-1}). \tag{2.4}$$

Remark 2.1 *We briefly discuss the origin of the model in* (2.1)–(2.4), *for details and the derivation in the case of an OLED mounted on a flat surface part of the substate see [17]. We assume that the OLED occupies the cylindrical domain* $\Omega^{\text{OLED}} := \widetilde{\omega} \times (0, h)$, *such that* $\omega = \widetilde{\omega} \times \{0\}$. *It is further subdivided into the bottom electrode* $\Omega^1 := \widetilde{\omega} \times (0, \widehat{h}^1)$, $N-2$ *organic layers* $\Omega^k := \widetilde{\omega} \times (\widehat{h}^{k-1}, \widehat{h}^k)$, $k = 2, \ldots, N-1$, *and the top electrode* $\Omega^+ := \Omega^N = \widetilde{\omega} \times (\widehat{h}^{N-1}, \widehat{h}^N)$ *such that* $h = \widehat{h}^N$, *see Fig. 1. In this setting, the layer thicknesses* $h^k := \widehat{h}^k - \widehat{h}^{k-1}$ *are assumed to satisfy* $h^k = h^k_* \varepsilon^{\rho_k}$ *with* $\rho^k > 0$, $h^k_* > 0$, $k = 1, \ldots, N$, *where the parameter* $\varepsilon := \widehat{h}^N/\text{diam}(\widetilde{\omega})$ *denotes the ratio between thickness and diameter of the OLED. The electrical contact of the OLED is realized by Dirichlet contacts at* $\Gamma^- = \widetilde{\gamma}^- \times (0, \widehat{h}^1)$ *and* $\Gamma^+ = \widetilde{\gamma}^+ \times (\widehat{h}^{N-1}, \widehat{h}^N)$, *respectively, where* $\widetilde{\gamma}^-$, $\widetilde{\gamma}^+$ *are subsets of* $\partial\widetilde{\omega}$ *with positive one-dimensional Hausdorff measure and* $\gamma^\pm = \widetilde{\gamma}^\pm \times \{0\}$.

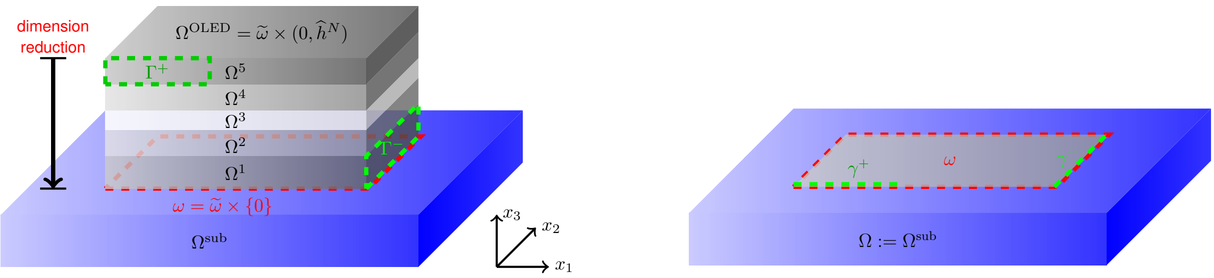

Figure 1: Left: Schematic picture of an OLED mounted on a glass substrate Ω^{sub} at $\omega = \widetilde{\omega} \times \{0\}$. The OLED domain Ω^{OLED} consists of N layers ($N = 5$ in the picture). The bottom and top layer Ω^1 and Ω^N represent the electrodes with Dirichlet boundaries Γ^- and Γ^+ (green) for the potential where the voltage is applied. Right: Considered domain in the effective model. Current flow is described by equations on the grey area ω with Ohmic contacts at γ^- and γ^+ realizing the contacting of the OLED. The heat flow equation is formulated in Ω with boundary source term at ω.

As derived in [26], the full three-dimensional $p(x)$*-Laplace thermistor model for organic LEDs has the form*

$$-\nabla \cdot \mathbf{S}^{\text{OLED}}(x, T, \nabla\varphi) = 0 \quad \textit{in } \Omega^{\text{OLED}},$$

$$-\nabla \cdot (\lambda(x)\nabla T) = \begin{cases} \mathbf{S}^{\text{OLED}}(x, T, \nabla\varphi) \cdot \nabla\varphi & \textit{in } \Omega^{\text{OLED}} \\ 0 & \textit{in } \Omega^{\text{sub}} \end{cases},$$

where the flux function $\mathbf{S}^{\text{OLED}} : \Omega^{\text{OLED}} \times [T_{\text{a}}, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ *in the different layers of the OLED is given by*

$$\mathbf{S}^{\text{OLED}}(x, T, z) = \begin{cases} \mathbf{S}^1(T, \frac{z}{h^1}) & x \in \Omega^1 \\ \mathbf{S}^k(T, h^k z) & x \in \Omega^k, \quad k = 2, \ldots, N-1, \\ \mathbf{S}^N(T, \frac{z}{h^N}) & x \in \Omega^N. \end{cases}$$

In the lower and upper electrodes Ω^1 *and* Ω^N, *respectively, there is no temperature dependence and a linear current law with so called sheet conductivities* $\sigma^-_{\text{sh}} = \sigma_0 h^1$ *and* $\sigma^+_{\text{sh}} = \sigma_0 h^N$, $\sigma_0 > 0$, *respectively, is assumed, viz.*

$$\mathbf{S}^1(T, w) = \sigma^-_{\text{sh}} w, \quad \mathbf{S}^N(T, w) = \sigma^+_{\text{sh}} w.$$

According to [26, 17], it is reasonable to assume power laws in the organic layers Ω^k, e.g. the flux functions $\mathbf{S}^k$ take the form

$$\mathbf{S}^k(T,w) = J_{\rm ref}\exp\Big[-E_{\rm a}^k\Big(\frac{1}{T}-\frac{1}{T_{\rm a}}\Big)\Big]\Big|\frac{w}{V_{\rm ref}}\Big|^{p_k-2}\frac{w}{V_{\rm ref}},\quad k=2,\dots,N-1, \tag{2.5}$$

where $J_{\rm ref}>0$ and $V_{\rm ref}>0$ are reference current density and reference voltage, respectively. Moreover, $p_k\in(1,\infty)$ denotes the power law exponent for the current flow and $E_{\rm a}^k$ is the (scaled) activation energy associated with the material of the organic layer Ω^k.

As demonstrated in [17], in the effective electrothermal model obtained for the limit $\varepsilon\to 0$, only the third component of the flux function $\widetilde{F}^k(T,w) := (\mathbf{S}^k(T,w))_3$, $w\in\mathbb{R}^3$, stays of relevance for the description of the organic layers. Moreover, in the limit of vanishing layer thickness, the potential φ becomes constant in vertical direction in the top and bottom electrode and piecewise affine in the organic layers. Thus, it can be identified with a tuple $(\varphi^1,\dots,\varphi^{N-1})$ and with the definition

$$F^k(T,v) := \widetilde{F}^k(T,(0,0,v)) = \big(\mathbf{S}^k(T,(0,0,v))\big)_3,\quad v\in\mathbb{R},$$

we arrive at the effective model (2.1) – (2.4) *investigated in the present work.*

3 Preliminaries and main result

3.1 Assumptions on the data

In this subsection, we collect the essential assumptions for the analytical investigations:

(A1) The domain $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain and $\omega\subsetneq\Gamma:=\partial\Omega$ is a compact, connected C^2 manifold having positive two-dimensional Hausdorff measure (i.e. $\mathcal{H}^2(\omega)>0$) and with boundary $\partial\omega$; γ^+, γ^- are open subsets of $\partial\omega$ with positive one-dimensional Hausdorff measure.

(A2) The Dirichlet data satisfy φ_+^D, $\varphi_-^D\in L^\infty(\omega)\cap H^1(\omega)$, $\sigma_{\rm sh}^-$ and $\sigma_{\rm sh}^+$ are positive constants.

(A3) Let $p_i\in(1,\infty)$, $i=2,\dots,N-1$, and $p_-:=\min\{p_i : i=2,\dots,N-1\}$, $p_+:=\max\{p_i : i=2,\dots,N-1\}$. The functions $F^i:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ are continuous and there are c_1, c_2, $c_3>0$ such that for all $T\in[T_{\rm a},\infty)$

$$F^i(T,v)\cdot v \ge c_1|v|^{p_i}-c_2\quad\forall v\in\mathbb{R}, \tag{3.1}$$

$$|F^i(T,v)|\le c_3(1+|v|)^{p_i-1}\quad\forall v\in\mathbb{R}. \tag{3.2}$$

Moreover, the functions $F^i(T,\cdot)$ are strictly monotone,

$$\big(F^i(T,v_1)-F^i(T,v_2)\big)\cdot(v_1-v_2)>0\quad\forall v_1\neq v_2\in\mathbb{R}\quad\text{and } F^i(T,0)=0, \tag{3.3}$$

for $i=2,\dots,N-1$.

(A4) The constant $T_{\rm a}>0$ is the ambient temperature. The heat conductivity λ satisfies $\lambda\in L^\infty(\Omega)$ and $\lambda\ge\lambda_0>0$ a.e. in Ω. The heat transfer coefficient κ is such that $\kappa\in L_+^\infty(\Gamma)$ and $\kappa\ge\kappa_0>0$ a.e. on ω.

For $p\in[1,\infty]$, we use the classical Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$. By $H^1(\Omega)$ we denote the usual Hilbert space. $H^1_{\gamma^-}(\omega)$ and $H^1_{\gamma^+}(\omega)$ are the subspaces of $H^1(\omega)$ functions vanishing on γ^- and γ^+, respectively. Moreover, according to assumption (A4), the estimates

$$\underline{\alpha}\|T\|_{H^1}^2 \le \int_\Omega \lambda|\nabla T|^2{\rm d}x + \int_\Gamma \kappa T^2{\rm d}\Gamma \le \overline{\alpha}\|T\|_{H^1}^2,\quad T\in H^1(\Omega) \tag{3.4}$$

with constants $\underline{\alpha}$, $\overline{\alpha}>0$ are satisfied.

In our estimates, positive constants, which may depend at most on the data of our problem, are denoted by c. In particular, we allow them to change from line to line.

3.2 Reformulation of the problem and concept of solution

Since the functions F^k used in (2.1) only depend on the potential differences (voltages)

$$v^k := \varphi^k - \varphi^{k-1}, \quad k = 2, \dots, N-1, \tag{3.5}$$

it is natural to work with the unknowns v^k and the potentials φ^1 and φ^{N-1} needed in (2.1a) and (2.1c). However, we then have N unknowns for $N-1$ equations. We eliminate one unknown v^{k_0} as follows: Let k_0 be an index with $p_{k_0} = p_-$, see assumption (A3). We obtain

$$v^{k_0} = \varphi^{k_0} - \varphi^{k_0-1} = \varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k, \tag{3.6}$$

where here and in the following '$k \neq k_0$' stands shortly for '$k = 2, \dots, N-1,\ k \neq k_0$'. For the weak formulation of the current-flow equations in (2.1) we use the vector of variables

$$z := (\varphi^1, v^2, \dots, v^{k_0-1}, v^{k_0+1}, \dots, v^{N-1}, \varphi^{N-1}) \in z^D + Z, \tag{3.7}$$

where

$$Z := H^1_{\gamma^-}(\omega) \times \prod_{k \neq k_0} L^{p_k}(\omega) \times H^1_{\gamma^+}(\omega) \quad \text{and} \quad z^D = (\varphi^D_-, 0, \dots, 0, \varphi^D_+).$$

The quantity v^{k_0} is only used as auxiliary variable. Note that according to (3.6), assumption (A2), and $H^1(\omega) \hookrightarrow L^p(\omega)$ for all $p \in [1, \infty)$ in two spatial dimensions, it is guaranteed that $v^{k_0} = \varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k \in L^{p_-}(\omega) = L^{p_{k_0}}(\omega)$ and therefore F^{k_0} is well defined for such second arguments . For arbitrary fixed T from the set of relevant (surface) temperature distributions

$$\Theta := \{T \in L^1(\omega) \,:\, T \geq T_{\mathrm{a}} \text{ a.e. in } \omega\}, \tag{3.8}$$

we introduce the operator $\mathcal{A}_T : z^D + Z \to Z^*$ and consider the following problem: Find $z \in z^D + Z$ such that

$$\begin{aligned} \langle \mathcal{A}_T(z), \overline{z} \rangle_Z := \int_\omega \Big\{ \sigma^-_{\mathrm{sh}} \nabla_\omega \varphi^1 \cdot \nabla_\omega \overline{\varphi}^1 + \sigma^+_{\mathrm{sh}} \nabla_\omega \varphi^{N-1} \cdot \nabla_\omega \overline{\varphi}^{N-1} + \sum_{k \neq k_0} F^k(T, v^k) \overline{v}^k \Big\} \, \mathrm{d}x' \\ + \int_\omega F^{k_0}\big(T, \varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k\big) \Big[\overline{\varphi}^{N-1} - \overline{\varphi}^1 - \sum_{k \neq k_0} \overline{v}^k \Big] \mathrm{d}x' = 0 \end{aligned} \tag{3.9}$$

for all $\overline{z} = (\overline{\varphi}^1, \dots, \overline{\varphi}^{N-1}) \in Z^*$ (and associated $\overline{v}^k$), which corresponds to finding a weak solution $z \in z^D + Z$ to the system (2.1a), (2.1b), (2.1c) with boundary conditions (2.2) for the fixed temperature distribution $T \in \Theta$.

We rewrite the surface heat source H_ω defined in (2.4) in the variables z introduced in (3.7) with (3.6) as

$$H_\omega = H_\omega(T, z) = \sigma^-_{\mathrm{sh}} |\nabla_\omega \varphi^1|^2 + \sigma^+_{\mathrm{sh}} |\nabla_\omega \varphi^{N-1}|^2 + \sum_{k=2}^{N-1} F^k(T, v^k) v^k. \tag{3.10}$$

To formulate our concept of solution for the full problem, we define for $m > 0$ the truncation function $C_m : \mathbb{R} \to [-m, m]$ by

$$C_m(s) := \max\{-m, \min\{s, m\}\} \tag{3.11}$$

and introduce $\mathcal{V}^{1,2}(\Omega) := \{T : \Omega \to \mathbb{R} \text{ measurable}, C_m(T) \in H^1(\Omega)\ \forall m > 0\}$.

Definition 3.1 *We call a pair (z, T) with $z \in z^D + Z$ and $T \in \mathcal{V}^{1,2}(\Omega)$ a (weak) solution to problem* (2.1), (2.2), (2.3), (3.7) *if*

(i) z solves $\langle \mathcal{A}_T(z), \overline{z} \rangle_Z = 0$ for all $\overline{z} \in Z^*$ and

(ii) T is an entropy solution to the heat equation, i.e.

$$\int_\Omega \lambda \nabla T \cdot \nabla C_m(T - \theta) \, \mathrm{d}x + \int_\Gamma \kappa\big(T - T_{\mathrm{a}}\big) C_m(T - \theta) \, \mathrm{d}\Gamma \leq \int_\omega H_\omega(T, z) C_m(T - \theta) \, \mathrm{d}x'$$

for all $m > 0$ and all $\theta \in H^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3.1 *The notion of solution introduced in Definition 3.1 is adapted from [5] (see also [31]) to the present case with nonlinear boundary conditions. We prove in Section 5 that solutions (z, T) satisfy $T \in W^{1,q}(\Omega)$ for any $q \in [1, 3/2)$. Moreover, T is a weak solution to the heat equation in the sense*

$$\int_\Omega \lambda \nabla T \cdot \psi \, \mathrm{d}x + \int_\Gamma \kappa\big(T - T_{\mathrm{a}}\big) \psi \, \mathrm{d}\Gamma = \int_\omega H_\omega(T, z) \psi \, \mathrm{d}x' \quad \forall\, \psi \in W^{1,q'}(\Omega) \cap L^\infty(\Omega), \tag{3.12}$$

where $1/q + 1/q' = 1$.

3.3 Main result

Theorem 3.1 *We assume (A1)–(A4). Then the problem* (2.1), (2.2), (2.3), (2.4) *has a (weak) solution* (z, T) *in the sense of Definition 3.1 with* $z \in z^D + Z$ *and* $T \in W^{1,q}(\Omega)$ *for all* $q \in [1, 3/2)$. *In particular,* T *is a weak solution in the sense of* (3.12).

The proof of Theorem 3.1, presented in Section 5, is based on the construction of a suitable fixed-point map for the temperature distribution T and Schauder's fixed-point theorem. For the construction of the fixed-point map, we consider the electric and thermal problem, i.e. (2.1a)-(2.1c) and (2.1d) separately. Results for the subproblems are collected in Section 4.

We emphasize that this method does not give uniqueness of fixed points and hence of solutions. However, for our problem at hand, uniqueness of solutions cannot be expected due to the hysteretic behavior caused by the positive feedback with respect to temperature described in the introduction. Even in the spatially homogeneous setting of self-heating in organic devices (see [13] and [26, Sect. 2.1]) S-shaped current-voltage characteristics occur. Here, in a certain range of applied voltages, three different currents are possible for the same applied voltage. Moreover, for spatially resolved electrothermal $p(x)$-Laplace thermistor models the simulations produce S-shaped current-voltage relations that coincide with experimental measurements excluding a general uniqueness result, see, e.g., [20, 25].

4 Existence results for subproblems

4.1 Unique solution to the current flow problem

Theorem 4.1 *Let* $T \in \Theta$ *(defined in* (3.8)*) be a fixed given function. Under the assumptions (A1)–(A4), problem* (3.9) *has exactly one solution* $z \in z^D + Z$. *Furthermore, there are constants* $c_\varphi > 0, c_v > 0, c_F > 0$, *and* $c_H > 0$ *depending only on the data but not on* $T \in \Theta$, *such that*

$$\|\varphi^1\|_{H^1(\omega)},\ \|\varphi^{N-1}\|_{H^1(\omega)} \le c_\varphi,\ \ \|v^k\|_{L^{p_k}(\omega)},\ \|\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k\|_{L^{p_{k_0}}(\omega)} \le c_v, \tag{4.1a}$$

$$\|F^k(T,v^k)\|_{L^{p_k'}(\omega)},\ \|F^{k_0}(T,\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k)\|_{L^{p_{k_0}'}(\omega)} \le c_F \quad \textit{with } \tfrac{1}{p_k}+\tfrac{1}{p_k'}=1, \tag{4.1b}$$

$$\|H_\omega(T,z)\|_{L^1(\omega)} \le c_H. \tag{4.1c}$$

Proof. 1. Uniform bounds. The desired bounds in (4.1a) follow from testing the equation $\mathcal{A}_T z = 0$ with $z-(\varphi_-^D, 0, \ldots, 0, \varphi_+^D)$ and taking into account the properties (3.1) and (3.2) as well as the fact that $\varphi_-^D,\ \varphi_+^D \in H^1(\omega)$. Additionally, we used for k_0 that

$$\begin{aligned}\int_\omega F^{k_0}(T,\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k)(\varphi_+^D-\varphi_-^D)\,\mathrm{d}x' \\ \le c_3\int_\omega (1+|\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k|)^{p_{k_0}-1}|\varphi_+^D-\varphi_-^D|\,\mathrm{d}x' \\ \le c\|\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k\|_{L^{p_{k_0}}}^{p_{k_0}-1}\|\varphi_+^D-\varphi_-^D\|_{L^{p_{k_0}}} + c\|\varphi_+^D-\varphi_-^D\|_{L^1}\end{aligned}$$

and applied Young's inequality. Moreover, exploiting the growth condition in (3.2) and (4.1a) we obtain the estimates in (4.1b).

The estimate of the Joule heat term $H_\omega(T,z)$ in (4.1c) defined in (3.10) then follows immediately by $\|F^k(T,v^k)v^k\|_{L^1(\omega)} \le \|F^k(T,v^k)\|_{L^{p_k'}(\omega)}\|v^k\|_{L^{p_k}(\omega)} \le c_F\, c_v$ and

$$\begin{aligned}\|H_\omega(T,z)\|_{L^1(\omega)} \le \sigma_{\mathrm{sh}}^-\|\varphi^1\|_{H^1(\omega)}^2 + \sigma_{\mathrm{sh}}^+\|\varphi^{N-1}\|_{H^1(\omega)}^2 + \sum_{k\neq k_0}\|F^k(T,v^k)v^k\|_{L^1(\omega)} \\ + \|F^{k_0}(T,\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k)(\varphi^{N-1}-\varphi^1-\sum_{k\neq k_0} v^k)\|_{L^1(\omega)} \le c_H.\end{aligned}$$

2. Unique solvability. Due to the strict monotonicity assumption in (3.3) and (A2), the operator $\mathcal{A}_T$ is strictly monotone. Moreover, $\mathcal{A}_T$ is also demi-continuous, i.e. for $z_n - z \to 0$ in Z we have $\mathcal{A}_T z_n - \mathcal{A}_T z \rightharpoonup 0$ in Z^*. Indeed, let $z_n - z \to 0$ in Z and $\overline{z} = (\overline{\varphi}^1, \overline{v}^1, \dots, \overline{v}^{k_0-1}, \overline{v}^{k_0+1}, \dots, \overline{v}^{N-1}, \overline{\varphi}^{N-1}) \in Z$ be arbitrarily chosen. Arguing as in Step 1 yields that the sets $\{F^k(T, v_n^k)\}$ are bounded and weakly compact in $L^{p_k'}(\omega)$, $k = 2, \dots, N-1$. To establish the weak convergence $F^k(T, v_n^k) \rightharpoonup F^k(T, v^k)$ in $L^{p_k'}(\omega)$, it is sufficient to verify for each convergent subsequence $\{F^k(T, v_{n_l}^k)\}$ of $\{F^k(T, v_n^k)\}$ that $F^k(T, v_{n_l}^k) \rightharpoonup F^k(T, v^k)$ in $L^{p_k'}(\omega)$, see e.g. [14, Lemma 5.4, Chapter 1]. Note that the same arguments can be applied also for k_0 and $v_{nl}^{k_0}$ defined via (3.6). Let $w \in L^{p_k'}(\omega)$ be the weak limit of such a subsequence $\{F^k(T, v_{n_l}^k)\}$. Since $v_n^k \to v^k$ in $L^{p_k}(\omega)$ there exists a further subsequence $\{v_{n_{l_j}}^k\}$ of $\{v_{n_l}^k\}$ such that $v_{n_{l_j}}^k$ converges a.e. in ω to v^k. Since F^k is a continuous function, it follows that $F^k(T, v_{n_{l_j}}^k) \to F^k(T, v^k)$ a.e. in ω. As a subsequence of $\{F^k(T, v_{n_l}^k)\}$ the sequence $\{F^k(T, v_{n_{l_j}}^k)\}$ has the weak limit w in $L^{p_k'}(\omega)$. Using [14, Lemma 1.19, Chap. 2], we obtain $F^k(T, v^k) = w$, and thus for the entire sequence $F^k(T, v_n^k) \rightharpoonup F^k(T, v^k)$ in $L^{p_k'}(\omega)$ and

$$\int_\omega \big(F^k(T, v_n^k) - F^k(T, v^k)\big)\overline{v}^k \, \mathrm{d}x' \to 0, \quad k = 2, \dots, N-1.$$

Moreover, from $\varphi_n^k \to \varphi^k$ in $H^1(\omega)$, $k = 1, N-1$, it follows that

$$\int_\omega \sigma_{\mathrm{sh}}^- \nabla_\omega(\varphi_n^1 - \varphi^1) \cdot \nabla_\omega \overline{\varphi}^1 \, \mathrm{d}x' \to 0, \quad \int_\omega \sigma_{\mathrm{sh}}^+ \nabla_\omega(\varphi_n^{N-1} - \varphi^{N-1}) \cdot \nabla_\omega \overline{\varphi}^{N-1} \, \mathrm{d}x' \to 0.$$

In summary, $\langle \mathcal{A}_T z_n - \mathcal{A}_T z, \overline{z} \rangle_Z \to 0$ for all $\overline{z} \in Z$ and thus $\mathcal{A}_T z_n - \mathcal{A}_T z \rightharpoonup 0$ in Z^* as $z_n - z \to 0$ in Z. Obviously, the demi-continuity ensures the hemi-continuity of $\mathcal{A}_T$.

Next, we verify the coercivity of $\mathcal{A}_T$. We find $\sigma_{\mathrm{sh}}^- \nabla_\omega \varphi^1 \cdot \nabla_\omega(\varphi^1 - \varphi_-^D) \geq \frac{\sigma_{\mathrm{sh}}^-}{2} |\nabla_\omega(\varphi^1 - \varphi_-^D)|^2 - c|\nabla_\omega \varphi_-^D|^2$, $\sigma_{\mathrm{sh}}^+ \nabla_\omega \varphi^{N-1} \cdot \nabla_\omega(\varphi^{N-1} - \varphi_+^D) \geq \frac{\sigma_{\mathrm{sh}}^+}{2} |\nabla_\omega(\varphi^{N-1} - \varphi_+^D)|^2 - c|\nabla_\omega \varphi_+^D|^2$, and $F^k(T, v^k) v^k \geq c_1 |v^k|^{p_k} - c_2$, $k \neq k_0$. Moreover, exploiting (3.1) and (3.2) as well as Young's inequality, we estimate

$$\begin{aligned}
&F^{k_0}\Big(T, \varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k\Big)\Big(\varphi^{N-1} - \varphi_-^D - \varphi^1 + \varphi_+^D - \sum_{k \neq k_0} v^k\Big) \\
&\geq c_1 \Big|\varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k\Big|^{p_{k_0}} - c_2 - c_3\Big(1 + \Big|\varphi^{N-1} - \varphi^1 - \sum_{k \neq k_0} v^k\Big|\Big)^{p_{k_0}-1} |\varphi_-^D - \varphi_+^D| \\
&\geq -c_2 - c|\varphi_-^D - \varphi_+^D|^{p_{k_0}} - c|\varphi_-^D - \varphi_+^D| \geq -c.
\end{aligned}$$

Combining the previous estimates, by (A1), (A2) we derive

$$\begin{aligned}
&\langle \mathcal{A}_T z, z - z^D \rangle_Z \\
&\geq \overline{c}\Big(\|\varphi^1 - \varphi_-^D\|_{H^1_{\gamma^-}(\omega)}^2 + \|\varphi^{N-1} - \varphi_+^D\|_{H^1_{\gamma^+}(\omega)}^2 + \sum_{k \neq k_0} \|v^k\|_{L^{p_k}(\omega)}^{p_k}\Big) - c.
\end{aligned} \tag{4.2}$$

Having in mind that all exponents p_k, $k = 2, \dots, N-1$, are strictly greater than 1, we divide the previous estimate (4.2) by $\|z - z^D\|_Z$ and obtain that the right-hand side tends to $+\infty$ if $\|z - z^D\|_Z \to \infty$ which ensures the coercivity of the operator $\mathcal{A}_T$.

Now we are in the position to apply the theorem of Browder and Minty (see [34, p. 65]) that ensures the existence of a solution to (3.9). Finally, the strict monotonicity of $\mathcal{A}_T$ supplies the uniqueness result. □

4.2 Unique entropy solution of the heat equation

To show the existence of solutions to the heat equation with source term $g \in L^1(\Gamma)$, $\Gamma := \partial\Omega$, in the Robin boundary condition, we work with the concept of entropy solutions. For Dirichlet boundary conditions this theory can be found in the survey [31], for nonlinear problems see [5, 24].

Let the stationary heat equation with Robin boundary conditions and right-hand side $f \in L^1(\Omega)$ as well as boundary data $g \in L^1(\Gamma)$,

$$\begin{aligned}
-\nabla \cdot (\lambda(x) \nabla T) &= f(x) && \text{in } \Omega, \\
-\lambda(x) \nabla T \cdot \nu &= \kappa(x) T - g(x) && \text{on } \Gamma
\end{aligned} \tag{4.3}$$

be given. In our case we have

$$f(x) \equiv 0, \qquad g(x) = \begin{cases} \kappa(x)T_\mathrm{a} & x \in \Gamma \setminus \omega, \\ \kappa(x)T_\mathrm{a} + H_\omega(x) & x \in \omega. \end{cases}$$

Definition 4.1 *Let $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$. A function $T \in \mathcal{V}^{1,2}(\Omega)$ is called an entropy solution to* (4.3) *if*

$$\int_\Omega \lambda \nabla T \cdot \nabla C_m(T-\theta)\,\mathrm{d}x + \int_\Gamma (\kappa T - g) C_m(T-\theta)\,\mathrm{d}\Gamma \le \int_\Omega f C_m(T-\theta)\,\mathrm{d}x \tag{4.4}$$

for all $m > 0$ and all $\theta \in H^1(\Omega) \cap L^\infty(\Omega)$. (For C_m and $\mathcal{V}^{1,2}(\Omega)$ see (3.11).)

The next two results are given in [7, Theorem 3.3 and Appendix A] and [7, Lemma 3.4].

Theorem 4.2 *We assume (A1) and (A4). Let $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$. Then, there exists a unique entropy solution T to* (4.3)*. This entropy solution belongs to $W^{1,q}(\Omega)$, for all $1 \le q < \frac{3}{2}$. Moreover, there is a constant $c_q > 0$ not depending on f and g such that*

$$\|T\|_{W^{1,q}(\Omega)} \le c_q\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}\big), \quad 1 \le q < \tfrac{3}{2}.$$

Lemma 4.1 *We assume (A1) and (A4). Let $f^l \to f$ in $L^1(\Omega)$, $g^l \to g$ in $L^1(\Gamma)$. Then the corresponding entropy solutions T^l to* (4.3) *converge weakly in $W^{1,q}(\Omega)$, $1 \le q < \frac{3}{2}$, to the entropy solution T for data f and g.*

Lemma 4.2 *We assume (A1) and (A4). Let $f \in L^1_+(\Omega)$ and $g = \kappa T_\mathrm{a} + h$ with $T_\mathrm{a} = \mathrm{const} > 0$ and $h \in L^1_+(\Gamma)$. Then, the entropy solution T to* (4.3) *satisfies $T \ge T_\mathrm{a}$ a.e. in Ω as well as $T \ge T_\mathrm{a}$ a.e. on ω.*

Proof. Let $f_n := C_n(f) \in L^\infty(\Omega)$, $g_n := C_n(g) = C_n(\kappa T_\mathrm{a} + h)$ and let $T_n \in H^1(\Omega)$ be the unique weak solution to (4.3) with data f_n and g_n. Note that $g_n \ge \kappa T_\mathrm{a}$, thus, for $-(T_n - T_\mathrm{a})^- = \min\{T_n - T_\mathrm{a}, 0\}$ we obtain

$$\begin{aligned}\int_\Gamma (\kappa T_n - g_n)\min\{T - T_\mathrm{a}, 0\}\,\mathrm{d}\Gamma &= \int_\Gamma \big\{\kappa(T_n - T_\mathrm{a}) + \kappa T_\mathrm{a} - g_n\big\}\min\{T - T_\mathrm{a}, 0\}\,\mathrm{d}\Gamma \\ &\ge \int_\Gamma \kappa[(T_n - T_\mathrm{a})^-]^2\,\mathrm{d}\Gamma.\end{aligned}$$

Therefore, the test of (4.3) for T_n by $-(T_n - T_\mathrm{a})^-$ yields

$$\int_\Omega \lambda|\nabla(T_n - T_\mathrm{a})^-|^2\,\mathrm{d}x + \int_\Gamma \kappa((T_n - T_\mathrm{a})^-)^2\,\mathrm{d}\Gamma \le 0$$

implying that $T_n \ge T_\mathrm{a}$ a.e. in Ω. Since $\kappa \ge \kappa_0 > 0$ a.e. on $\omega \subset \Gamma$ (cf. (A4)), we also find $T_n \ge T_\mathrm{a}$ a.e. on ω.

Now, we argue similar to the proof of [7, Lemma 3.5]: Let us fix $m > T_\mathrm{a} > 0$. Since $C_m(T_n) \to C_m(T)$ in $L^1(\Omega)$ and $L^1(\omega)$ as $n \to \infty$ (note that $T_n \to T$ in $L^1(\Omega)$ and $L^1(\omega)$ due to Lemma 4.1) there is a subsequence $\{n_l\}$ such that $C_m(T_{n_l}) \to C_m(T)$ a.e. in Ω and ω. Together with $T_{n_l} \ge T_\mathrm{a}$ a.e. in Ω and ω, this guarantees that $C_m(T) \ge T_\mathrm{a}$ a.e. in Ω and ω. This yields especially that $T \ge 0$ a.e. in Ω and ω. The choice of $m > T_\mathrm{a}$ ensures therefore $T \ge C_m(T) \ge T_\mathrm{a}$ a.e. in Ω and ω. □

5 Proof of the main result

Here we prove our main result, Theorem 3.1, by means of Schauder's fixed-point theorem. At first, we introduce our fixed-point map. We work with the set of $L^1(\omega)$ functions being traces of $W^{1,6/5}(\Omega)$ functions on the substrate Ω and being greater or equal to the ambient temperature T_a a.e. on the boundary part ω where the OLED is mounted,

$$\mathcal{N} := \big\{T \in L^1(\omega) : \|T\|_{W^{1,6/5}(\Omega)} \le c_Q,\, T \ge T_\mathrm{a} \text{ a.e. in } \omega\big\}, \tag{5.1}$$

where $c_Q > 0$ will be fixed in (5.3). The fixed-point map $Q : \mathcal{N} \to \mathcal{N}$ is defined as follows: For $\widetilde{T} \in \mathcal{N}$ the quantity $T = Q(\widetilde{T})$ is the unique entropy solution of

$$\begin{aligned} -\nabla \cdot (\lambda \nabla T) &= 0 && \text{in } \Omega, \\ -\lambda \nabla T \cdot \nu &= \kappa(T - T_\mathrm{a}) - H_\omega(\widetilde{T}, z) && \text{on } \Gamma, \end{aligned} \tag{5.2}$$

where H_ω is introduced in (3.10) and the vector function $z = z(\widetilde{T}) \in z^D + Z$ is the unique weak solution to $\mathcal{A}_{\widetilde{T}} z = 0$ (cf. Theorem 4.1). Because of $\widetilde{T} \in \mathcal{N}$ we have that $\widetilde{T} \in \Theta$ (see (3.8)). From (4.1c) in Theorem 4.1 we find $\|H_\omega(\widetilde{T}, z)\|_{L^1(\omega)} \le c_H$, moreover, it holds that $\|\kappa T_{\mathrm{a}}\|_{L^1(\Gamma)} \le c$. With $f := 0 \in L^1(\Omega)$ and $g := \kappa T_{\mathrm{a}} + H_\omega(\widetilde{T}, z) \in L^1(\Gamma)$, Theorem 4.2 gives a unique entropy solution T of (5.2) satisfying

$$\|T\|_{W^{1,6/5}(\Omega)} \le c_{6/5} \|\kappa T_{\mathrm{a}} + H_\omega(\widetilde{T}, z)\|_{L^1(\Gamma)} \le c_{6/5}\big(\|\kappa T_{\mathrm{a}}\|_{L^1(\Gamma)} + c_H\big) =: c_Q \tag{5.3}$$

for all $z = z(\widetilde{T})$ with $\widetilde{T} \in \mathcal{N}$. Finally, by Lemma 4.2 we obtain that $T \ge T_{\mathrm{a}}$ a.e. in Ω as well as $T \ge T_{\mathrm{a}}$ a.e. on ω. Thus, it is validated that $T = Q(\widetilde{T}) \in \mathcal{N}$.

Lemma 5.1 *Under the assumptions (A1) – (A4) the fixed-point map $Q : \mathcal{N} \to \mathcal{N}$ is continuous with respect to strong convergence in $L^1(\omega)$.*

Proof. We consider $\widetilde{T}, \widetilde{T}_n \in \mathcal{N}$ with $\widetilde{T}_n \to \widetilde{T}$ in $L^1(\omega)$. We denote by $z_n \in z^D + Z$ the unique solution to $\mathcal{A}_{\widetilde{T}_n} z = 0$ (with $\widetilde{T}_n$ as fixed argument in $\mathcal{A}_\bullet$ instead of $\widetilde{T}$). We have to verify that $T_n = Q(\widetilde{T}_n) \to T = Q(\widetilde{T})$ in $L^1(\omega)$. This is carried out in four steps.

1. Convergences for subsequences. From Theorem 4.1, the growth properties of F^k (see (3.1) and (3.2)), and Theorem 4.2 we obtain for all $z_n = z(\widetilde{T}_n)$ and $T_n = Q(\widetilde{T}_n)$ the uniform estimates

$$\begin{gathered}\|\varphi_n^1\|_{H^1(\omega)},\ \|\varphi_n^{N-1}\|_{H^1(\omega)} \le c_\varphi, \quad \|v_n^k\|_{L^{p_k}(\omega)} \le c_v,\\ \|F^k(\widetilde{T}_n, v_n^k)\|_{L^{p_k'}(\omega)} \le c_F, \quad \tfrac{1}{p_k} + \tfrac{1}{p_k'} = 1, \quad \|T_n\|_{W^{1,6/5}(\Omega)} \le c_Q.\end{gathered} \tag{5.4}$$

These estimates guarantee the existence of limits $\widehat{\varphi}^1$, $\widehat{\varphi}^{N-1} \in H^1(\omega)$ with $\widehat{\varphi}^1 - \varphi_-^D \in H^1_{\gamma^-}(\omega)$, $\widehat{\varphi}^{N-1} - \varphi_+^D \in H^1_{\gamma^+}(\omega)$, and $\widehat{v}^k \in L^{p_k}(\omega)$, $\widehat{F}^k \in L^{p_k'}(\omega)$, $k = 2, \dots, N{-}1$, and $\widehat{T} \in W^{1,6/5}(\Omega)$ and, up to a non-relabeled, subsequence the weak convergences

$$\begin{gathered}\varphi_n^1 \rightharpoonup \widehat{\varphi}^1 \text{ in } H^1(\omega), \quad \varphi_n^{N-1} \rightharpoonup \widehat{\varphi}^{N-1} \text{ in } H^1(\omega), \quad v_n^k \rightharpoonup \widehat{v}^k \text{ in } L^{p_k}(\omega),\\ F^k(\widetilde{T}_n, v_n^k) \rightharpoonup \widehat{F}^k \text{ in } L^{p_k'}(\omega), \quad T_n \rightharpoonup \widehat{T} \text{ in } W^{1,6/5}(\Omega).\end{gathered} \tag{5.5}$$

Let $\widehat{z} := (\widehat{\varphi}^1, \widehat{v}^2, \dots, \widehat{v}^{k_0-1}, \widehat{v}^{k_0+1}, \dots, \widehat{v}^{N-1}, \widehat{\varphi}^{N-1}) \in z^D + Z$ denote the associated tuple. The weak convergences of φ_n^1, φ_n^{N-1}, v_n^k, for $k = 2, \dots, k_0{-}1, k_0{+}1, \dots, k_{N-1}$, lead to

$$\widehat{v}^{k_0} = \widehat{\varphi}^{N-1} - \widehat{\varphi}^1 - \sum_{k \ne k_0} \widehat{v}^k.$$

The growth condition (3.2) ensures that $|F^k(\widetilde{T}_n, \widehat{v}^k) - F^k(\widetilde{T}, \widehat{v}^k)| \le c(1 + |\widehat{v}^k|)^{p_k - 1}$. Hence, there is an integrable majorant for the integrand $|F^k(\widetilde{T}_n, \widehat{v}^k) - F^k(\widetilde{T}, \widehat{v}^k)|^{p_k'}$. Since $\widetilde{T}_n \to \widetilde{T}$ in $L^1(\omega)$ and F^k is continuous, this integrand converges to 0 a.e. on ω for an again non-relabeled subsequence. Therefore, Lebesgue's theorem on dominated convergence yields

$$\int_\omega |F^k(\widetilde{T}_n, \widehat{v}^k) - F^k(\widetilde{T}, \widehat{v}^k)|^{p_k'} \, \mathrm{d}x' \to 0, \quad k = 2, \dots, N{-}1.$$

Using the monotonicity of F^k in the second argument and that $\mathcal{A}_{\widetilde{T}_n} z_n = 0$, we derive

$$\begin{aligned} 0 &\le \langle \mathcal{A}_{\widetilde{T}_n} z_n - \mathcal{A}_{\widetilde{T}_n} \widehat{z}, z_n - \widehat{z} \rangle \\ &= \int_\omega \Big(\sigma_{\mathrm{sh}}^- |\nabla_\omega(\varphi_n^1 - \widehat{\varphi}^1)|^2 + \sigma_{\mathrm{sh}}^+ |\nabla_\omega(\varphi_n^{N-1} - \widehat{\varphi}^{N-1})|^2 \\ &\qquad + \sum_{k=2}^{N-1} \big(F^k(\widetilde{T}_n, v_n^k) - F^k(\widetilde{T}_n, \widehat{v}^k)\big)(v_n^k - \widehat{v}^k) \Big) \, \mathrm{d}x' \\ &= 0 - \int_\omega \Big(\sigma_{\mathrm{sh}}^- \nabla_\omega \widehat{\varphi}^1 \cdot \nabla_\omega(\varphi_n^1 - \widehat{\varphi}^1) + \sigma_{\mathrm{sh}}^+ \nabla_\omega \widehat{\varphi}^{N-1} \cdot \nabla_\omega(\varphi_n^{N-1} - \widehat{\varphi}^{N-1}) \\ &\qquad + \sum_{k=2}^{N-1} F^k(\widetilde{T}_n, \widehat{v}^k)\big)(v_n^k - \widehat{v}^k) \Big) \, \mathrm{d}x \quad \to 0 \end{aligned}$$

since $\nabla_\omega\varphi_n^1 \rightharpoonup \nabla_\omega\widehat{\varphi}^1$, $\nabla_\omega\varphi_n^{N-1} \rightharpoonup \nabla_\omega\widehat{\varphi}^{N-1}$ in $L^2(\omega)$, $v_n^k \rightharpoonup \widehat{v}^k$ in $L^{p_k}(\omega)$, and $F^k(\widetilde{T}_n,\widehat{v}^k) \to F^k(\widetilde{T},\widehat{v}^k)$ in $L^{p_k'}(\omega)$. Due to the (strict) monotonicity of F^k in the second argument, we obtain from the convergence of the terms on the second and third line that

$$\varphi_n^1 \to \widehat{\varphi}^1, \quad \varphi_n^{N-1} \to \widehat{\varphi}^{N-1} \quad \text{in } H^1(\omega), \tag{5.6}$$

$$\big(F^k(\widetilde{T}_n,v_n^k) - F^k(\widetilde{T}_n,\widehat{v}^k)\big)(v_n^k-\widehat{v}^k) \to 0 \quad \text{in } L^1(\omega), \quad k=2,\ldots,N{-}1, \tag{5.7}$$

(for the subsequence). Therefore, keeping in mind that $\widetilde{T}_n \to \widetilde{T}$ in $L^1(\omega)$ there exists a subsequence $\{n_l\}$ such that

$$\big(F^k(\widetilde{T}_{n_l},v_{n_l}^k) - F^k(\widetilde{T}_{n_l},\widehat{v}^k)\big)(v_{n_l}^k-\widehat{v}^k) \to 0, \quad \widetilde{T}_{n_l} \to \widetilde{T} \text{ a.e. in } \omega. \tag{5.8}$$

To show now that also $v_{n_l}^k \to \widehat{v}^k$ a.e. in ω, we adapt the idea in [33, p. 50f]. For the subsequence $\{n_l\}$, we consider $x\in\omega$ where $[F^k(\widetilde{T}_{n_l}(x),v_{n_l}^k(x)) - F^k(\widetilde{T}_{n_l}(x),\widehat{v}^k(x))](v_{n_l}^k(x)-\widehat{v}^k(x)) \to 0$, $\widetilde{T}_{n_l}(x) \to \widetilde{T}(x)$ and $v_{n_l}^k(x)$ and $\widehat{v}^k(x)$ are finite. Indeed, if the sequence $\{v_{n_l}^k(x)\}$ was unbounded, then with $r=\widehat{v}^k(x)$ we would have

$$\begin{aligned}
&\Big(F^k\big(\widetilde{T}_{n_l}(x),v_{n_l}^k(x)\big) - F^k\big(\widetilde{T}_{n_l}(x),r\big)\Big)\big(v_{n_l}^k(x)-r\big)\\
&= F^k\big(\widetilde{T}_{n_l}(x),v_{n_l}^k(x)\big)v_{n_l}^k(x) - F^k\big(\widetilde{T}_{n_l}(x),v_{n_l}^k(x)\big)r - F^k\big(\widetilde{T}_{n_l}(x),r\big)v_{n_l}^k(x) + F^k\big(\widetilde{T}_{n_l}(x),r\big)r\\
&\geq c_1|v_{n_l}^k(x)|^{p_k} - c_2 - c_3(1+|v_{n_l}^k(x)|)^{p_k-1}r - c|v_{n_l}^k(x)| + c_1|r|^{p_k} - c_2\\
&\geq \overline{c}|v_{n_l}^k(x)|^{p_k} - c \to \infty \quad \text{for } n_l\to\infty,
\end{aligned}$$

which would contradict the first convergence in (5.8). Thus, since the sequence $\{v_{n_l}^k(x)\}$ is bounded, there is a further (non-relabeled) subsequence and an $s\in\mathbb{R}$ such that $v_{n_l}^k(x) \to s$. Using the continuity of F^k in both arguments, we obtain

$$\Big(F^k(\widetilde{T}(x),s) - F^k(\widetilde{T}(x),\widehat{v}^k(x))\Big)\big(s-\widehat{v}^k(x)\big) = 0.$$

Because of the strict monotonicity of F^k in the second argument (see (3.3)), we conclude that $s=\widehat{v}^k(x)$. Since s is determined uniquely, the entire sequence $\{v_{n_l}^k(x)\}$ converges to s, such that $v_{n_l}^k(x) \to \widehat{v}^k(x)$ a.e. in ω. Additionally, using again the continuity of F^k, this gives $F^k(\widetilde{T},\widehat{v}^k) = \widehat{F}^k$, $k=2,\ldots,N{-}1$. In summary, we obtain

$$\begin{aligned}
\langle \mathcal{A}_{\widetilde{T}}\widehat{z},\overline{z}\rangle &= \int_\omega \Big\{\sigma_{\rm sh}^-\nabla_\omega\widehat{\varphi}^1\cdot\nabla_\omega\overline{\varphi}^1 + \sigma_{\rm sh}^+\nabla_\omega\widehat{\varphi}^{N-1}\cdot\nabla_\omega\overline{\varphi}^{N-1} + \sum_{k=2}^{N-1} F(\widetilde{T},\widehat{v}^k)\overline{v}^k\Big\}\,\mathrm{d}x'\\
&= \lim_{n_l\to\infty}\int_\omega \Big\{\sigma_{\rm sh}^-\nabla_\omega\varphi_{n_l}^1\cdot\nabla_\omega\overline{\varphi}^1 + \sigma_{\rm sh}^+\nabla_\omega\varphi_{n_l}^{N-1}\cdot\nabla_\omega\overline{\varphi}^{N-1} + \sum_{k=2}^{N-1} F(\widetilde{T}_{n_l},v_{n_l}^k)\overline{v}^k\Big\}\,\mathrm{d}x'\\
&= \lim_{n_l\to\infty}\langle \mathcal{A}_{\widetilde{T}_{n_l}}z_{n_l},\overline{z}\rangle = 0 \quad \forall\,\overline{z}\in Z.
\end{aligned}$$

According to Theorem 4.1, the weak solution to $\mathcal{A}_{\widetilde{T}}z=0$ is unique, which ensures that $\widehat{\varphi}^k=\varphi^k=\varphi^k(\widetilde{T})$, for $k=1,\,N{-}1$, and $\widehat{v}^k=v^k=v^k(\widetilde{T})$, for $k=2,\ldots,N{-}1$.

2. Weak convergence $z_n-z\rightharpoonup 0$ in Z of the entire sequence. To prove the weak convergence $z_n-z\rightharpoonup 0$ in the reflexive Banach space Z for the entire sequence and not only for the subsequence given in (5.5), we use [14, Lemma 5.4, Chap. 1]. Indeed, we need to verify that for every weakly convergent subsequence $z_{n_l}-z^*\rightharpoonup 0$ with $z^*:=(\varphi^{*,1},v^{*,2},\ldots,v^{*,k_0-1},v^{*,k_0+1},\ldots,v^{*,N-1},\varphi^{*,N-1})$ the identity $z^*=z$ holds true: If there is a subsequence $z_{n_l}-z^*\rightharpoonup 0$ in Z then the arguments of Step 1 ensure again non-relabeled subsequences such that $\varphi_{n_l}^1\to\varphi^{*,1}$, $\varphi_{n_l}^{N-1}\to\varphi^{*,N-1}$ in $H^1(\omega)$, and $v_{n_l}^k\to v^{*,k}$ a.e. on ω, $F^k(\widetilde{T}_{n_l},v_{n_l}^k)\rightharpoonup F^k(\widetilde{T},v^{*,k})$ in $L^{p_k'}(\omega)$, $k=2,\ldots,N{-}1$. And z^* would be a solution to $\mathcal{A}_{\widetilde{T}}z=0$. Since the weak solution to $\mathcal{A}_{\widetilde{T}}z=0$ is unique, we conclude that $z=z^*$. Therefore, we obtain the weak convergence $z_n-z\rightharpoonup 0$ in Z for the entire sequence.

3. Improved convergence of subsequences. For a non-relabeled subsequence, we have $F^k(\widetilde{T}_n,v^k)\to F^k(\widetilde{T},v^k)$ for a.a. $x\in\omega$ (note that $\widetilde{T}_n\to\widetilde{T}$ a.e. in ω and F^k are continuous functions). For an arbitrarily given function $u\in L^\infty(\omega)$, the growth condition (3.2) gives via $|(F^k(\widetilde{T}_n,v^k)-F^k(\widetilde{T},v^k))u|^{p_k'}\leq c(1+|v^k|)^{p_k}\|u\|_{L^\infty(\omega)}^{p_k'}$ an integrable majorant. Thus, by Lebesgue's dominated convergence theorem we obtain $F^k(\widetilde{T}_n,v^k)u\to F^k(\widetilde{T},v^k)u$ in $L^{p_k'}(\omega)$ for this subsequence. Combining this with $v_n^k\rightharpoonup v^k$ in $L^{p_k}(\omega)$, by weak-strong convergence it follows

$$\int_\omega F^k(\widetilde{T}_n,v^k)(v_n^k-v^k)u\,\mathrm{d}x'\to 0.$$

Since the function $u \in L^\infty(\omega)$ was arbitrary, we get $F^k(\widetilde{T}_n, v^k)(v_n^k - v^k) \rightharpoonup 0$ in $L^1(\omega)$. This guarantees together with (5.7) and $\widehat{v}^k = v^k$ the weak convergence

$$F^k(\widetilde{T}_n, v_n^k)(v_n^k - v^k) \rightharpoonup 0 \quad \text{in } L^1(\omega). \tag{5.9}$$

Now, the weak convergence $F^k(\widetilde{T}_n, v_n^k) \rightharpoonup \widehat{F}^k = F^k(\widetilde{T}, v^k)$ in $L^{p_k'}(\omega)$, demonstrated in Step 1, and $uv^k \in L^{p_k}(\omega)$ for all $u \in L^\infty(\omega)$ lead to

$$F^k(\widetilde{T}_n, v_n^k)v^k \rightharpoonup F^k(\widetilde{T}, v^k)v^k \quad \text{in } L^1(\omega).$$

This guarantees in combination with (5.9) for the subsequence the weak convergence

$$y_n := F^k(\widetilde{T}_n, v_n^k)v_n^k \rightharpoonup F^k(\widetilde{T}, v^k)v^k =: y \quad \text{in } L^1(\omega).$$

Since a weakly-convergent sequence in L^1 is equi-integrable, the subsequence $\{F^k(\widetilde{T}_n, v_n^k)v_n^k\}$ is equi-integrable. The convergences $\widetilde{T}_n(x) \to \widetilde{T}(x),\ v_n^k(x) \to v^k(x)$ a.e. in ω and the continuity of F^k imply for the subsequence

$$F^k(\widetilde{T}_n, v_n^k)v_n^k \to F^k(\widetilde{T}, v^k)v^k \quad \text{a.e. on } \omega, \quad k = 2, \dots, N{-}1.$$

For a sequence $y_n \to y$ a.e. on ω meaning y_n converges in measure to y, Vitali's theorem ensures that the following two properties are equivalent: (i) The sequence $\{y_n\}$ is equi-integrable and (ii) $y_n \to y$ in $L^1(\omega)$. Therefore, we have

$$F^k(\widetilde{T}_n, v_n^k)v_n^k \to F^k(\widetilde{T}, v^k)v^k \quad \text{in } L^1(\omega), \quad k = 2, \dots, N{-}1.$$

This gives, together with (5.6) and $\widehat{\varphi}^k = \varphi^k$, $k = 1, N{-}1$, from Step 1, for

$$\widetilde{H}_{\omega,n} := \sigma_{\text{sh}}^-|\nabla_\omega \varphi_n^1|^2 + \sigma_{\text{sh}}^+|\nabla_\omega \varphi_n^{N-1}|^2 + \sum_{k=2}^{N-1} F^k(\widetilde{T}_n, v_n^k)v_n^k,$$
$$\widetilde{H}_\omega := \sigma_{\text{sh}}^-|\nabla_\omega \varphi^1|^2 + \sigma_{\text{sh}}^+|\nabla_\omega \varphi^{N-1}|^2 + \sum_{k=2}^{N-1} F^k(\widetilde{T}, v^k)v^k$$

that $\widetilde{H}_{\omega,n} \to \widetilde{H}_\omega$ in $L^1(\omega)$. This, in turn, ensures for the functions

$$g_n(x) = \begin{cases} \kappa(x)T_\text{a} & \text{for } x \in \Gamma \setminus \omega \\ \kappa(x)T_\text{a} + \widetilde{H}_{\omega n}(x) & \text{for } x \in \omega \end{cases}, \quad g(x) = \begin{cases} \kappa(x)T_\text{a} & \text{for } x \in \Gamma \setminus \omega \\ \kappa(x)T_\text{a} + \widetilde{H}_\omega(x) & \text{for } x \in \omega \end{cases}$$

that $g_n \to g$ in $L^1(\Gamma)$. With Lemma 4.1, we find for the entropy solutions T_n and T of (5.2) with right-hand sides $f_n = f = 0$, and boundary functions g_n and g, respectively, the weak convergence $T_n \rightharpoonup T$ in $W^{1,6/5}(\Omega)$. According to Theorem 4.2 the solution to (5.2) with right-hand side $f = 0$ and boundary function g is unique. Therefore, with (5.5) we obtain $T_n = Q(\widetilde{T}_n) \rightharpoonup \widehat{T} = T = Q(\widetilde{T})$ in $W^{1,6/5}(\Omega)$, for the subsequence in (5.5).

4. Weak convergence $T_n \rightharpoonup T$ in $W^{1,6/5}(\Omega)$ for the entire sequence and continuity of the fixed point operator Q. Analogously to Step 2, we have to justify that for each weakly convergent subsequence $T_{n_k} \rightharpoonup T^*$ in $W^{1,6/5}(\Omega)$ (reflexive Banach space) the identity $T^* = T$ is fulfilled. We can proceed as in Step 3 to find for not-relabeled subsequences that $H_\omega(\widetilde{T}_{n_l}, z_{n_l}) \to H_\omega(\widetilde{T}, z)$ in $L^1(\Omega)$. Then the uniqueness result of Theorem 4.2 and Lemma 4.1 guarantee $T_{n_l} \rightharpoonup T = T^*$ in $W^{1,6/5}(\Omega)$. This ensures the weak convergence of the entire sequence $T_n \rightharpoonup T$ in $W^{1,6/5}(\Omega)$. With the compact embedding of $W^{1,6/5}(\Omega)$ into $L^1(\omega)$, we end up with the strong convergence of the entire sequence $T_n \to T$ in $L^1(\omega)$ which was needed for the continuity of the operator Q. □

Proof of Theorem 3.1. According to the definition of the fixed point set $\mathcal{N}$ in (5.1), for all $T \in \mathcal{N}$ the norm $\|T\|_{W^{1,6/5}(\Omega)}$ is uniformly bounded, and the compact embedding of $W^{1,6/5}(\Omega)$ in $L^1(\omega)$ ensures the desired compactness property of the convex and nonempty set $\mathcal{N} \subset L^1(\omega)$. The continuity of the map $Q : \mathcal{N} \to \mathcal{N}$ was demonstrated in Lemma 5.1. Thus, Schauder's fixed-point theorem guarantees a fixed point $T = Q(T) \in \mathcal{N}$. For this T we obtain a unique solution $z \in z^D + Z$ to $\mathcal{A}_T z = 0$. By Theorem 4.2 the entropy solution $T = Q(T)$ of (5.2) belongs to $W^{1,q}(\Omega)$ for all $q \in [1, \frac{3}{2})$. Therefore, the pair (z, T) is a solution of the coupled problem (2.1), (2.2), (2.3), (3.7).

It remains to show that T is also a weak solution in the sense of Remark 3.1. We proceed as in [31, Theorem 6.5] and fix $h > 0$ and $\psi \in W^{1,q'}(\Omega) \cap L^\infty(\Omega)$. Choosing $\theta = C_h(T) - \psi$ as test function in the entropy formulation in the second part of Definition 3.1 yields for all $m > 0$

$$\int_\Omega \lambda \nabla T \cdot \nabla C_m\big(T - C_h(T) + \psi\big)\,\mathrm{d}x + \int_\Gamma \kappa\big(T - T_\text{a}\big)C_m\big(T - C_h(T) + \psi\big)\,\mathrm{d}\Gamma$$
$$\leq \int_\omega H_\omega(T, z)C_m\big(T - C_h(T) + \psi\big)\,\mathrm{d}x'.$$

Setting $m = \|\psi\|_{L^\infty(\Omega)}$ and using Lebesgue's theorem, we obtain for the right-hand side

$$\lim_{h\to\infty}\int_\omega H_\omega(T,z)C_m\big(T-C_h(T)+\psi\big)\,\mathrm{d}x' = \int_\omega H_\omega(T,z)C_m(\psi)\,\mathrm{d}x' = \int_\omega H_\omega(T,z)\psi\,\mathrm{d}x'.$$

The second term on the left-hand side is treated analogously. Next, we consider the first term on the left-hand side. We can rewrite the latter (with the same choice for m as before) to get

$$\int_{\{|T|\le h\}}\lambda\nabla T\cdot\nabla\psi\,\mathrm{d}x + \int_{\{|T|\ge h\}}\nabla T\cdot\nabla C_m\big(T-C_h(T)+\psi\big)\,\mathrm{d}x.$$

Since we have $T\in W^{1,q}(\Omega)$, we can pass to the limit $h\to\infty$ in the first term to obtain with Lebesgue's theorem

$$\lim_{h\to\infty}\int_{\{|T|\le h\}}\lambda\nabla T\cdot\nabla\psi\,\mathrm{d}x = \int_\Omega\lambda\nabla T\cdot\nabla\psi\,\mathrm{d}x.$$

For the second term, we note that $\{|T-C_h(T)+\psi|\le m,\ |T|\ge h\}\subseteq\{h-2m\le|T|\le h+2m\}$. Thus, we get

$$\int_{\{|T|\ge m\}}\lambda|\nabla T\cdot\nabla C_m\big(T-C_h(T)+\psi\big)|\,\mathrm{d}x \le c\int_{\{h-2m\le|T|\le h+2m\}}|\nabla T|(|\nabla T|+|\nabla\psi|)\,\mathrm{d}x.$$

We claim that right-hand side vanishes as $h\to\infty$. Indeed, using the test function $\theta = C_h(T)$ in the entropy formulation gives after rewriting

$$\begin{aligned}\int_{\{|T|\le h+m\}}\lambda\nabla T\cdot\nabla T\,\mathrm{d}x &\le \int_{\Gamma\cap\{|T|\ge h\}}\kappa\big(T_\mathrm{a}-T\big)C_m\big(T-C_h(T)\big)\,\mathrm{d}\Gamma\\ &\quad+\int_{\omega\cap\{|T|\ge h\}}H_\omega(T,z)C_m\big(T-C_h(T)\big)\,\mathrm{d}x'\\ &\le m\int_{\Gamma\cap\{|T|\ge h\}}\kappa|T_\mathrm{a}-T|\,\mathrm{d}\Gamma + m\int_{\omega\cap\{|T|\ge h\}}|H_\omega(T,z)|\,\mathrm{d}x'.\end{aligned}$$

The right-hand side vanishes for $h\to\infty$ (and fixed m) since the integrands are in L^1, which proves the claim. Putting everything together gives

$$\int_\Omega\lambda\nabla T\cdot\psi\,\mathrm{d}x + \int_\Gamma\kappa\big(T-T_\mathrm{a}\big)\psi\,\mathrm{d}\Gamma \le \int_\omega H_\omega(T,z)\psi\,\mathrm{d}x'.$$

Finally, exchanging ψ with $-\psi$ shows that actually equality holds, which finishes the proof of Theorem 3.1. □

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