PRESERVATION OF PROBABILISTIC LAWS THROUGH EULER METHODS FOR ORNSTEIN–UHLENBECK PROCESS

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Abstract

There is a lack of appropriate replication of the asymptotical behaviour of stationary stochastic differential equations solved by numerical methods. The paper illustrates this fact with the stationary Ornstein–Uhlenbeck process and family of implicit Euler methods. For description of occuring bias, notions of asymptotical p–th mean, mean, mean square and equilibrium preservation are introduced, due to stochasticity of stationary law. Only the trapezoidal formula among these methods is optimal in the sense of replication of exact asymptotical behaviour. We also discuss the general probabilistic law of linear Euler methods. The results can be useful for implementation of stochastic–numerical algorithms (e.g. for linear–implicit methods) in several disciplines of Natural and Environmental Sciences.

Contents

1. Introduction

In numerous fields models with additive noise are used to express uncertainty, environmental fluctuations or parameter excitations. They also serve as a possible basis for investigation of qualitative behaviour of dynamical systems, e.g. how systems behave under random perturbations which are state–independent. The **stationary Ornstein–Uhlenbeck process** is often met as system–component in statistical modelling and seems to be very useful for the purposes mentioned above. For example, in modelling of oscillation phenomena of physical and technical systems. There its 'formal derivative' is also titled as coloured noise. The dynamical behaviour of this object $\{X_t, t \geq 0\} \in \mathbb{R}^d$ can be described by stochastic differential equation (SDE)

$$dX_t \;=\; A(t)\, X_t \; dt \;+\; \sum_{j=1}^{m} \, b^j(t) \; dW_t^j \tag{1.1}$$

with an initial value $X(0) = X_0 \in \mathbb{R}^d$ (deterministic or Gaussian distributed). System (1.1) is driven by Brownian motion $W_t = (W_t^1, ..., W_t^m)$ which represents m independent, identically distributed Gaussian random variables ($\in \mathcal{N}(0,t)$). Details about this stochastic object and corresponding calculus can be found, e.g. in Karatzas and Shreve [13]. We suppose that throughout this paper $\mathbb{E}\, \|X_0\|^2 < +\infty$ and X_0 is independent of $\mathcal{F}_t^j = \sigma\{W_s^j, 0 \leq s \leq t\}$ $(j = 1, 2, ..., m)$, the σ–algebra generated by the underlying Wiener process. Matrix A as a real–valued $d \times d$ matrix in (1.1) may or may not depend on time t, however its eigenvalues have only nonpositive real parts. For the sake of simplicity, assume that vectors $b^j(t) \in \mathbb{R}^d$ and matrices $A(t)$ are deterministic. In passing, it may be noted that the case of their stochastic independence of $\mathcal{F}_t^j$ is reducable to the moment–approach presented here. However it generally leads to examination of NonGaussian distributions!
An analytic expression for the solution of (1.1) is known. Let $\Phi(t)$ denote the fundamental matrix of solution of homogeneous, **random initial value problem (RIVP)**

$$dx \;=\; A(t)\, x \, dt, \; x \;=\; X_0, \; t \;\geq\; 0\,. \tag{1.2}$$

Particularly, if $A(t)$ and $\int_0^{\hat{t}} A(s)ds$ commute at all permissible times $t, \hat{t}$, i.e.

$$A(t) \left(\int_0^{\hat{t}} A(s)ds \right) \;=\; \left(\int_0^{\hat{t}} A(s)ds \right) \; A(t) \quad \forall t, \hat{t} \geq 0, \tag{1.3}$$

$$\text{then} \qquad \Phi(t) \;=\; \exp\Big(\int_0^t A(s)ds \Big)\,. \tag{1.4}$$

This turns out to be very restrictive for nonautonomous systems! The general solution of **multi–dimensional Ornstein–Uhlenbeck process** (1.1) is

$$X_t = \Phi(t) \; \left(X_0 + \sum_{j=1}^{m} \; \int_0^t \; \Phi^{-1}(s) \; b^j(s) \; dW_s^j \right)\,. \tag{1.5}$$

However, there are **two major problems** in computational generation of these expressions. One arises from computation of matrix–valued fundamental solution (e.g. matrix integration, inversion and exponential operation, which mostly leads to series expansion), and the other, from generation of (multi–dimensional) stochastic integrals in (1.5). In general, one is tending to use stochastic–numerical methods. We shall follow this approach. One is even capable of stating its corresponding probability distribution. More precisely, with $Q_t := \Phi^{-1}(t)X_t - X_0$, it holds

$$Q_t \;\in\; \mathcal{N}\Big(0,\; \sum_{j=1}^{m} \int_0^t \; \Phi^{-1}(s)\, b^j(s)\, {b^j}^T(s)\, {\Phi^{-1}}^T(s)\; ds\Big).$$

$\mathcal{N}(\mu, \sigma^2)$ denotes the law of Gaussian distribution with mean $\mu \in \mathbb{R}^d$ and covariance $\sigma^2 \in \mathbb{R}^{d\times d}$. $(.)^T$ is the transpose of the inscribed vectors or matrices throughout this paper. Furthermore, for existence of asymptotical probabilistic law, we impose on diffusion vectors $b^j \in \mathbb{R}^d$ and matrices A that

$$Q_\infty \;:=\; \sum_{j=1}^{m} \lim_{t\to+\infty} \int_0^t \Phi(t)\, \Phi^{-1}(s)\, b^j(s)\, {b^j}^T(s)\, {\Phi^{-1}}^T(s)\, \Phi^T(t)\, ds$$

is finite.

The generation of autonomous systems (1.1) also shows computational difficulties. For illustration of this fact, assume that drift matrix A is time–independently diagonalizable. Then it exists an invertible matrix $L \neq L(t)$

$$A = L^{-1}\; D\; L \tag{1.6}$$

where D is a $d \times d$ diagonal matrix with complex–valued elements d_i. Exploiting this fact we can transform $X_t \to Z_t = L\; X_t$ and obtain the new SDE

$$d\; Z_t = D\; Z_t\; dt + \sum_{j=1}^{m} \; L\, b^j\; dW_t^j \tag{1.7}$$

starting in $Z_0 = LX_0$. Obviously system (1.7) consists of d separated components, hence for the analytical solution of this system we can separately consider its single components and find

$$d\; Z_t^i = d_i\; Z_t^i\; dt + \sum_{j=1}^{m} \; [L\, b^j]_i\; dW_t^j \tag{1.8}$$

with $d_i \in \mathbb{C}$ (space of complex numbers). The solution expression of (1.8) for autonomous systems (i.e. systems with the time–independent drift d_i and diffusion components $[L\, b^j]_i$) is very simple and found to be

$$Z_t^i = \exp(d_i t) \left(Z_0^i + \sum_{j=1}^{m} [L\, b^j]_i \int_0^t \exp(-d_i s)\, dW_s^j \right). \tag{1.9}$$

Thus we know explicit solutions of (1.1) and (1.7) as well. Despite of this fact, in expressions both (1.5) and (1.9) we have to calculate the value of stochastic integrals for pathwise evolution of processes X_t and Z_t along given Wiener paths. Note that the probability distribution of these stochastic integrals is known under complete information on underlying Wiener process.

An objective of this paper is to provide a further concept and some results for assessment of probabilistic behaviour of discrete time approximations for SDEs with additive noise. The related analysis should be done in addition to well–known convergence analysis. For example, the investigation of asymptotical behaviour of numerical solutions as integration time tends to infinity. Therein multi–dimensional Ornstein–Uhlenbeck process (1.1) also serves as test system for approximations of nonlinear SDEs with additive noise to some extent. This can be motivated by linearization of drift parts around equilibria and stochastic perturbation theory.

Moreover, there are several ways to approximate SDEs and stochastic integrals over functionals of their solutions on finite time intervals (in fact a large variety!). Instead of proceeding on with description of different generation possibilities, we want to examine the following task in particular. Given the information on the underlying Wiener path at discrete time points $(t_n)_{n\in\mathbb{N}}$,i.e. $\triangle W_n^j = W^j(t_{n+1}) - W^j(t_n)$ is known and fixed. Now we are interested in adequate replication of the long–term behaviour of Ornstein–Uhlenbeck process (1.1) by corresponding approximations along that fixed Wiener path. This interest is naturally given. If one has interest in pathwise properties like exit times (in general path–dependent functionals), computes Lyapunov exponents (see [34]), estimates parameters in drift and diffusion part of (1.1) (see [7], [17]), constructs discrete time filters (see [15]) or models stochastic oscillation phenomena (see [37]) then accurate and stable long–term integration is required. Only then one receives reasonable and reliable results.

The computation along one and the same Wiener path is particularly important when one compares stochastic integration techniques with respect to one and the same Wiener path, and one is aiming at crystallizing out an appropriate technique. For example, for parametric estimation under discretely observed diffusions while approximating continuous time models one needs some guarantee for correct replication of asymptotical behaviour of the exact solution of SDEs. There this problem mainly arises during stochastic integration which is necessary for computation of likelihood estimators under discrete observation, cf. [7] or [17]. One uses substitutions of continuous time estimators by corresponding discrete versions and supposes that these discretizations correctly provide the behaviour of continuous time estimates as integration time t tends to infinity. A general justification and proof of this approach seems to be very complicated, due to nonlinear structure of likelihood quotients. A similar effect can be observed in estimation of Lyapunov exponents. It should be clarified whether one estimates the top Lyapunov exponent of discrete or continuous time solution. Clearly, as integration time tends to zero one would theoretically obtain the correct Lyapunov exponent (of continuous time system) under sufficient smoothness conditions, cf. [34]. However, the usage of 'almost vanishing' (very small) step sizes contradicts to the requirement of 'finiteness and efficiency' on practical algorithms.

The paper is organized as follows. In section 2 we recall aspects of numerical solution of SDEs. Then notions of asymptotical preservation of probabilistic characteristics are introduced, related to stationary SDEs with additive noise. Although one exactly knows probability distribution of linear systems (1.5) and (1.9), one already arrives into troubles in order to replicate the asymptotical behaviour of exact solution process under discrete time observation of underlying Wiener path. This fact will be verified while using family of implicit Euler methods (for introduction see [16]) in section 3. Section 4 presents the general expansion and probabilistic law of these implicit Euler methods applied to multi–dimensional Ornstein–Uhlenbeck processes, supplemented by a theorem on asymptotics of nonautonomous systems in section 5. Section 6 illustrates some basic facts from presented theory with two examples, a stochastic rotation process and oscillators as often met in Mechanical Engineering. The paper is finished with some summarizing remarks and conclusions.

2. Numerical solution for SDEs (1.1) and (1.7)

It would be a natural way to make use of numerical techniques for solving of SDEs (1.1) and (1.7). They allow to get a straight forward, pathwise link between the current Wiener process increments and the procedure of stochastic integration. Let

$Y_n = Y(t_n, \Delta_{n-1})$ be the value of approximation using time step size $\Delta_{n-1} = t_n - t_{n-1} > 0$ at time point $t_n \in [0, +\infty)$. Introduce abbreviations $A_n = A(t_n)$ and $b_n^j = b^j(t_n)$. From exposition [16] (p. 158) we know the familiy of **implicit Euler methods** following the general scheme

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(\alpha\, A_{n+1}\, Y_{n+1} + (1-\alpha)\, A_n\, Y_n\Big)\Delta_n + \sum_{j=1}^{m} b_n^j \,\triangle W_n^j \qquad (2.1) \\ Y_0 &= X_0 \in \mathbb{R}^d \qquad (n = 0, 1, 2, ...) \end{aligned}$$

for system (1.1). $\alpha \in [0,1]$ represents an implicitness parameter to be chosen appropriately. For simplicity, consider equidistant approximations, i.e. $\Delta = \Delta_n$. On finite time intervals $[0,T]$ $(T < +\infty)$ one is entitled to use them as strong approximations of SDE (1.1), i.e. the criterion of **strong convergence**

$$\exists\, \delta > 0\ \forall\, \eta^{\Delta} = (t_n)_{n \in \mathbb{N}}, \Delta \leq \delta\ :\ \sup_{t_n \in \eta^{\Delta}} \mathbb{E}\, ||X_{t_n} - Y_n|| \ \leq\ K_1(T)\, \Delta^{\gamma_1} \qquad (2.2)$$

is satisfied with order $\gamma_1 = 1.0$ and positive constant $K = K_1(T)$. η^{Δ} denotes a discretization of the time axis as collection of monotonically increasing time points t_n from interval $[t_0, T]$. Moreover one also shows the validity of **mean–square convergence** towards (1.5). This criterion has the form

$$\exists\, \delta > 0\ \forall\, \eta^{\Delta} = (t_n)_{n \in \mathbb{N}}, \Delta \leq \delta\ :\ \sup_{t_n \in \eta^{\Delta}} \mathbb{E}\, ||X_{t_n} - Y_n||^2 \ \leq\ K_2^2(T)\, \Delta^{2\gamma_2} \qquad (2.3)$$

with order $\gamma_2 = 1.0$. In fact, schemes (2.1) provide us with the simplest class of numerical methods for approximation of (1.1) at discrete points t_n. Note that schemes (2.1) are identical with the family of **implicit Mil'shtein schemes** for systems with additive noise, e.g. such as (1.1), cf. [16] (p. 161). There is a large variety of further numerical methods. For references and some aspects, e.g. see [2], [16], [21], [23],[24] or [36]. In particular, Shkurko [32] and Törok [35] have already dealt with linear numerical methods. An alternative to these references is given by Kushner and Dupuis [20] via constructing Markov chain approximations for solving problems in stochastic control (Time and space are discretized for computation of control functionals). Here we follow the direct approach of references above. However, most of the suggested schemes require more smoothness on drift and diffusion functions or more information on the σ–algebra generated by the underlying Wiener process in order to achieve higher order of strong or mean square convergence. **Clark and Cameron** [4] showed that the highest possible order of mean square convergence is one, provided that only the Wiener increments are used for models with additive noise. Thus, we naturally confine to 'lower order methods'.

3. The preservation of asymptotical properties

For the purpose of classification and comparison, we introduce the notions of **asymptotical p–th mean, mean, mean square and equilibrium preservation**. Each of these notions reflects an asymptotical property of numerical solutions compared with the asymptotical behaviour of the exact solution. It also gives some information on the replication of possible equilibria of the considered stochastic systems.

Definition 3.1. Let $\{X_t, t \geq 0\} \subseteq \mathbb{R}^d$ be a stationary, ergodic stochastic process governed by SDE (1.1). Then the numerical solution $(Y_n)_{n\in\mathbb{N}}$ is said to be (**asymptotical**) **p–th mean preserving** ($p \in \mathbb{R}^1$) for SDE (1.1) if

$$\lim_{n\to+\infty} \mathbb{E}\, ||Y_n||^p = \mathbb{E}\, ||X_\infty||^p := \lim_{t\to+\infty} \mathbb{E}\, ||X_t||^p .$$

Furthermore, it is called (**asymptotical**) **mean preserving** for SDE (1.1) if

$$\lim_{n\to+\infty} \mathbb{E}\, Y_n = \mathbb{E}\, X_\infty := \lim_{t\to+\infty} \mathbb{E}\, X_t ,$$

(**asymptotical**) **mean square preserving** for SDE (1.1) if

$$\lim_{n\to+\infty} \mathbb{E}\, Y_n Y_n^T = \mathbb{E}\, X_\infty X_\infty^T := \lim_{t\to+\infty} \mathbb{E}\, X_t X_t^T$$

and (**asymptotical**) **equilibrium preserving** for SDE (1.1) if

$$\mathcal{L}\left(\lim_{n\to+\infty} Y_n\right) = \mathcal{L}\Big(X_\infty\Big) := \mathcal{L}\left(\lim_{t\to+\infty} X_t\right)$$

where $\mathcal{L}(.)$ denotes the probability law of the corresponding random variable.

The involved norm can be any chosen vector norm. For the sake of simplicity, we take the Euclidean vector norm, i.e. $||x||^2 = \sum_{i=1}^d x_i^2$ for all $x \in \mathbb{R}^d$.

Remark. The conditions of the definition above ensure probabilistic convergence of the process $(Y_n)_{n\in\mathbb{N}}$ towards the stationary solution X_∞ (equilibrium) of SDE (1.1) as time t_n tends to infinity. In contrast to deterministic analysis and to stochastic bilinear models with purely multiplicative noise, the corresponding stationary solution for nondegenerate differential systems (1.1) is a random variable which has Gaussian distribution with mean zero, hence not a simple, deterministic number. In the case of linear, multiplicative noise (i.e. state–dependent diffusion functions $b^j(x) = B^j x$ with $d \times d$ matrices B^j) the concept of asymptotical p–th mean preservation would be identical with the appearance of asymptotical p–th mean stability of null solution of both discretized and umderlying continuous time stochastic systems, cf. Khas'minskij [14] or Kozin [18]. The notion of mean preservation represents the weakest notion among the presented ones. Moreover, in case of systems with linear drift and mean square integrable diffusion parts, the concept of mean preservation reduces to the stability problem as known in deterministic numerical analysis. Thereby we may consider the concept of asymptotical preservation as an extension of stability concepts being common so far in probabilistic situation.

For the sake of simplicity, we only consider autonomous systems (1.1) in the remaining part of this section. That is, systems with time–independent drift and diffusion components. Assume that $\mathbb{E}\,[\|X_0\|^2 + \|Y_0\|^2] < +\infty$. Let (X_0, Y_0) be independent of $\mathcal{F}_t^j = \sigma\{W_s^j : 0 \le s \le t\}$ $(j = 1, 2, ..., m)$. Suppose that all real parts of eigenvalues of matrix A are negative, and A, b^j are deterministic.

Theorem 3.1. *There is only one numerical method (2.1) which exactly replicates the asymptotical behaviour of stationary Ornstein–Uhlenbeck processes governed by SDE (1.1). More precisely,* $(Y_n)_{n \in I\!N}$ *generated by (2.1) with any equidistant step size* Δ *and implicitness degree* $\alpha = 0.5$ *is asymptotical mean, p–th mean, mean square and equilibrium preserving for the model class of stationary SDEs (1.1) with diagonalizable drift matrices* A.

Proof. In analogous manner to deterministic analysis, for $\alpha \ge 0.5$ we easily verify the property of asymptotical mean preservation by (2.1) for all possible step sizes $\Delta > 0$. Now we continue with investigating the mean square evolution (variance) of implicit Euler schemes. Consider $V_n = L\ Y_n$ where $A = L^{-1}DL$ with real $d \times d$ matrices L and D ($D = I(d_i)$ is the diagonal Jordan form of A, $d_i \in \mathbb{C}$, I unit matrix of $\mathbb{R}^{d \times d}$). Then the transformed Euler scheme has the form

$$V_{n+1} = V_n + (\alpha\ D\ V_{n+1} + (1-\alpha)\ D\ V_n)\Delta + \sum_{j=1}^{m} L\, b^j\ \triangle W_n^j\ . \tag{3.1}$$

where $A = L^{-1}DL$. Because of stationarity of SDE (1.1), drift matrix A must have only eigenvalues with nonpositive real parts, hence matrix D too. Thus matrix $I - \alpha D\Delta$ is invertible for all $\alpha\Delta \geq 0$. This allows to rewrite (3.1) to

$$V_{n+1} = (I - \alpha\, D\, \Delta)^{-1} \left((I + (1-\alpha) D\, \Delta)\, V_n + \sum_{j=1}^{m} L\, b^j\, \triangle W_n^j \right) .$$

This system has completely separated components, hence we are able to treat it componentwisely. Let V_n^i denote the $i-th$ component of the approximation $V_n (i = 1, 2, ..., d)$. Then one encounters with

$$V_{n+1}^i = \frac{V_n^i\, (1 + (1-\alpha) d_i \Delta) + \sum_{j=1}^{m} \sigma_i^j\, \triangle W_n^j}{1 - \alpha d_i \Delta}$$

where $V_0^i = [L\, X_0]_i$ and $\sigma_i^j = [L\, b^j]_i$. After introducing abbreviation

$$U_{n+1}^{i,k} \;:=\; \mathbb{E}\, V_{n+1}^i V_{n+1}^k$$

for all $i, k = 1, 2, ..., d; n = 0, 1, 2, ...,$ a computation leads to the series

$$U_{n+1}^{i,k} = \nu_{i,k}\, U_n^{i,k} + \beta_{i,k} = \beta_{i,k} \sum_{l=0}^{n} (\nu_{i,k})^l + (\nu_{i,k})^{n+1}\, U_0^{i,k}$$

where

$$\nu_{i,k} = \frac{\Big(1 + (1-\alpha) d_i \Delta\Big)\Big(1 + (1-\alpha) d_k \Delta\Big)}{\Big(1 - \alpha d_i \Delta\Big)\Big(1 - \alpha d_k \Delta\Big)} \text{ and } \beta_{i,k} = \frac{\sum_{j=1}^{m} \sigma_i^j \sigma_k^j\, \Delta}{\Big(1 - \alpha d_i \Delta\Big)\Big(1 - \alpha d_k \Delta\Big)}.$$

If one of real parts of d_i or $d_k \in \mathbb{C} \setminus \{0\}$ is negative, we find that $(\nu_{i,k})^{n+1} \underset{n \to +\infty}{\longrightarrow} 0$ for all step sizes $\Delta > 0$ under the assumption $\alpha \geq 0.5$. Just as well the series $\sum_{l=0}^{n} (\nu_{i,k})^l$ must converge to limit $1/(1 - \nu_{i,k})$. Consequently, it holds

$$U_{n+1}^{i,k} \underset{n \to +\infty}{\longrightarrow} \frac{\beta_{i,k}}{1 - \nu_{i,k}} =: U_\infty^{i,k}.$$

Now, we analyze $U_\infty^{i,k} = U_\infty^{i,k}(\Delta)$ and receive

$$\begin{aligned} U_\infty^{i,k}(\Delta) &= \frac{\sum_{j=1}^{m} \sigma_i^j \sigma_k^j \Delta}{(1 - \alpha d_i \Delta)(1 - \alpha d_k \Delta) - (1 + (1-\alpha) d_i \Delta)(1 + (1-\alpha) d_k \Delta)} \\ &= -\sum_{j=1}^{m} \frac{\sigma_i^j \sigma_k^j}{d_i + d_k + (1 - 2\alpha)\Delta d_i d_k}. \end{aligned}$$

Calculating second moment evolution of the exact solution one encounters with

$$\mathbb{E}\ Z^i_\infty Z^k_\infty = -\sum_{j=1}^{m} \frac{\sigma_i^j \sigma_k^j}{d_i + d_k}.$$

After comparison of latter expressions that is for all $i, k \in \{1, 2, ..., d\}$, for all step sizes $\Delta > 0$ $U^{i,k}_\infty(\Delta) = \mathbb{E}\ Z^i_\infty Z^k_\infty$ iff $\alpha = 0.5$. Thus, in another words, asymptotical mean square preservation (variance) through family of implcit Euler methods applied to class of stationary Ornstein–Uhlenbeck processes is observed iff $\alpha = 0.5$. After those steps above one transforms numerical solution $(V_n)_{n\in\mathbb{N}}$ back to $(Y_n)_{n\in\mathbb{N}}$ via relation $Y_n = L^{-1}V_n$. Besides one uses relations

$$\mathbb{E}\ Y_n Y_n^T \ = \ L^{-1}\left(\mathbb{E}\ V_n V_n^T\right) {L^{-1}}^T \quad \text{and} \quad \mathbb{E}\ X_t X_t^T \ = \ L^{-1}\left(\mathbb{E}\ Z_t Z_t^T\right) {L^{-1}}^T$$

in order to obtain the validity of

$$\lim_{n\to+\infty} \mathbb{E}\ Y_n Y_n^T \ = \ \lim_{t\to+\infty} \mathbb{E}\ X_t X_t^T$$

for $\alpha = 0.5$ under diagonalizability of matrix A. Thus asymptotical mean square preservation (variance) can be verified for the original system (1.1). Furthermore, we know that the limit distribution of (2.1) is Gaussian (cf. section 4) and Gaussian distributions are uniquely characterized by its first and second moments. Consequently, the limit distributions of exact and numerical solution are identical (preservation of the equilibrium law), i.e. the distance between the asymptotical behaviour of numerical solution (2.1) with arbitrary step sizes $\Delta > 0$ and exact solution of class (1.1) only vanishes for $\alpha = 0.5$, as claimed in the theorem. Asymptotical p–th mean preservation is obvious from the equality of limit distributions. Thereby the proof has been completed. □

4. **The general law of linear Euler methods (2.1)**

Moreover one can find the general probabilistic law of the family of Euler methods applied to linear, nonautonomous systems (1.1), also called linear Euler methods. Let $\lambda_i(t) = \lambda_i(A(t))$ be the eigenvalues of matrix $A = A(t) \in \mathbb{R}^{d\times d}$ with corresponding eigenvectors $e_i(t) = e_i(A(t))$, $\|\cdot\|_2$ the spectral norm of the inscribed matrix and $Re(\lambda_i)$ the real part of eigenvalue λ_i. $\prod(\cdot)$ denotes the forward product of matrices and $\chi_{\{\cdot\}}(\cdot)$ the indicator function of subscribed set. $l(t) \leq d$ is the maximum number of linearly independent eigenvectors $e_i(t)$. $Lin\{e_1, ..., e_l\}$ represents the set of linear combinations spanned by vectors $e_1, ..., e_l$. Without loss of generality, suppose that eigenvectors e_i are orthonormalized throughout this section. One obtains the following representation.

Theorem 4.1. *Assume that*

$$(A1) \qquad (\alpha_n)_{n\in I\!N} \text{ with } \alpha_n \in [0,1] \text{ and } (\Delta_n)_{n\in I\!N} \text{ with } \Delta_n > 0,$$

$$(A2) \qquad \mathbb{E}\,\|Y_0\|^2 < +\infty,$$

$$(A3) \qquad Y_0 \text{ independent of } \mathcal{F}_t^j = \sigma\{W_s^j : 0 \le s \le t\} \;\; (j = 1,2,...,m),$$

$$(A4) \qquad \Big(\|A(t)\|_2\Delta_n < 1 \text{ or } Re(\lambda_i(A(t_n))) \le 0 \; (\forall i = 1,2,...,d)\Big) \; (\forall n \in I\!N)\,.$$

Then $(Y_n)_{n\in I\!N}$ governed by (2.1) with implicitness $(\alpha_n)_{n\in I\!N}$ and step sizes $(\Delta_n)_{n\in I\!N}$ has the explicit expansion $Y_{n+1} =$

$$\left(\prod_{i=n}^{0} M_0(\alpha_i,\Delta_i,t_i)\right) Y_0 + \sum_{i=0}^{n}\left(\prod_{k=n}^{i} M_{0+\chi_{\{i\}}(k)}(\alpha_k,\Delta_k,t_k)\right)\left(\sum_{j=1}^{m} b_i^j \triangle W_i^j\right) \tag{4.1}$$

where $b_i^j = b^j(t_i), \triangle W_i^j = W^j(t_{i+1}) - W^j(t_i)$ and for $\alpha \in [0,1], \Delta, t \in [0,+\infty)$

$$M_1 = M_1(\alpha,\Delta,t) = \Big(I - \alpha\Delta A(t+\Delta)\Big)^{-1}, M_0(\alpha,\Delta,t) = M_1 \cdot \Big(I + (1-\alpha)\Delta A(t)\Big).$$

Proof. Use induction on $n \in \mathbb{N}$. For $n = 0$ one receives

$$Y_1 \;=\; M_0(\alpha_0,\Delta_0,t_0)\,Y_0 \;+\; \sum_{j=1}^{m} M_1(\alpha_0,\Delta_0,t_0)\,b_0^j\,\triangle W_0^j,$$

hence representation (4.1) holds. Suppose validity of (4.1) for fixed $n-1 \in \mathbb{N}$. Now, the induction step follows. After rewriting scheme (2.1) applied to (1.1) one gets the explicit representation

$$Y_{n+1} \;=\; M_0(\alpha_n,\Delta_n,t_n)\,Y_n \;+\; \sum_{j=1}^{m} M_1(\alpha_n,\Delta_n,t_n)\,b_n^j\,\triangle W_n^j\,.$$

This leads to Y_{n+1} =

$$= M_0(\alpha_n,\Delta_n,t_n)\left(\prod_{i=n-1}^{0} M_0(\alpha_i,\Delta_i,t_i)\right) Y_0 + M_1(\alpha_n,\Delta_n,t_n)\Big(\sum_{j=1}^{m} b_n^j \triangle W_n^j\Big)$$

$$+ M_0(\alpha_n,\Delta_n,t_n)\sum_{i=0}^{n-1}\left(\prod_{k=n-1}^{i} M_{0+\chi_{\{i\}}(k)}(\alpha_k,\Delta_k,t_k)\right)\Big(\sum_{j=1}^{m} b_i^j \triangle W_i^j\Big)$$

$$= \left(\prod_{i=n}^{0} M_0(\alpha_i,\Delta_i,t_i)\right) Y_0 + \sum_{i=0}^{n}\left(\prod_{k=n}^{i} M_{0+\chi_{\{i\}}(k)}(\alpha_k,\Delta_k,t_k)\right)\Big(\sum_{j=1}^{m} b_i^j \triangle W_i^j\Big).$$

Consequently, the proof has been completed. □

For simplicity, matrix A and vectors b^j are supposed to be deterministic in further considerations. As simple conclusion of Theorem 4.1 and due to mutual independence of Wiener process increments, the probabilistic law of the discrete time evolution of (2.1) is found to be Gaussian as well, under appropriate conditions for non-degeneracy of distribution. Set $M_0(i) = M_0(\alpha_i,\Delta_i,t_i)$ and $M_1(i) = M_1(\alpha_i,\Delta_i,t_i)$.

Theorem 4.2. *Assume $(A1)-(A4)$ and that*

(*A5*) *Y_0 has deterministic or Gaussian distributed components,*

(*A6*) $\exists k \in \{0,1,2,...,n\}\ :\ \sum_{j=1}^{m} b_k^j {b_k^j}^T$ *is positive definite.*

Then $Y_{n+1} (n \in I\!N)$ governed by (2.1) with deterministic implicitness $(\alpha_n)_{n\in I\!N}$ and deterministic step sizes $(\Delta_n)_{n\in I\!N}$ is Gaussian distributed with

$$\mathcal{L}\left(Y_{n+1} - \Big(\prod_{i=n}^{0} M_0(i)\Big)Y_0\right) = \mathcal{N}\left(0, \sum_{j=1}^{m}\sum_{i=0}^{n} c(i,n,j)c(i,n,j)^T\right) \tag{4.2}$$

where $c(i,n,j) = \sqrt{\Delta_i}\left(\prod_{k=n}^{i} M_{0+\chi_{\{i\}}(k)}(\alpha_k,\Delta_k,t_k)\right) b^j(t_i)$ *and covariances*

$$\mathbb{E}\,(Y_{n_1+1} - \mathbb{E}\,Y_{n_1+1})(Y_{n_2+1} - \mathbb{E}\,Y_{n_2+1})^T = \sum_{j=1}^{m}\sum_{i=0}^{p=min(n_1,n_2)} c(i,p,j)\,c(i,p,j)^T$$

for all $n_1, n_2 \in I\!N$.

Proof. Note that (deterministic) linear transformations of Gaussian random variables preserve the property to be Gaussian distributed. Using expansion (4.1), the remaining proof is obvious under regularity conditions $(A5)$ and $(A6)$ (for nondegeneracy of distribution) and mutual independence of $\triangle W_i^j$. $\square$

Remark. If one of the diagonal components of variance on right side of (4.2) turns out to be zero at any time step, then the corresponding solution component is deterministic, hence not random. Therefore we require $(A6)$. Generalizations to random step sizes and random implicitness can be made, but only under independence of $\mathcal{F}_t^j$ they are easier to handle.

Theorem 4.3. *Assume commutativity (1.3),* $(A1)-(A4)$ *and that*

$$(A7) \qquad e_i \, (i \in \{1,2,...,l\}) \text{ and index } l \text{ do not depend on time,}$$

$$(A8) \qquad \mathbb{E}\, Y_0 \in Lin\{e_i : i = 1,2,...,l\},\ \mathbb{E}\, Y_0 \ = \ \mathbb{E}\, X_0,$$

$$(A9) \qquad \forall k \in \{1,2,...,l\} : \prod_{i=0}^{+\infty} \frac{1+(1-\alpha_i)\Delta_i\lambda_k(t_i)}{1-\alpha_i\Delta_i\lambda_k(t_i)} = \exp\Big(\int_0^{+\infty} \lambda_k(s)ds\Big),$$

$$(A10) \qquad \forall T > 0 \ : \ \sum_{j=1}^{m} \int_0^T \|b^j(t)\|^2 \, dt \ < \ +\infty$$

Then $(Y_n)_{n\in I\!N}$ *governed by (2.1) with deterministic implicitness* $(\alpha_n)_{n\in I\!N}$ *and deterministic step sizes* $(\Delta_n)_{n\in I\!N}$ *(where* $\sum_{n=0}^{+\infty}\Delta_n = +\infty$*) is asymptotical mean preserving for SDE (1.1).*

Proof. Because of $(A8)$, there exists an expansion

$$\mathbb{E}\, Y_0 \ = \ \sum_{k=1}^{l} y_k\, e_k \, (= \ \mathbb{E}\, X_0)\,,$$

with deterministic $y_k \in \mathbb{R}^1$, as linear combination of eigenvectors e_k of A. Then

$$\mathbb{E}\, Y_{n+1} \ = \ \left(\prod_{i=n}^{0} M_0(\alpha_i,\Delta_i,t_i)\right) \mathbb{E}\, Y_0 \ = \ \sum_{k=1}^{l} y_k \left(\prod_{i=0}^{n} \frac{1+(1-\alpha_i)\Delta_i\lambda_k(t_i)}{1-\alpha_i\Delta_i\lambda_k(t_i)}\right) e_k\,.$$

Under conditions $(A9)$ and $(A10)$, it finally follows the equality

$$\lim_{n\to+\infty} \mathbb{E}\, Y_{n+1} \ = \ \lim_{t\to+\infty} \mathbb{E}\, X_t\,.$$

$\square$

Remark. This theorem only deals with results for a subspace which corresponds to deterministic and diagonalizable part of underlying stochastic dynamics. There are simple examples where requirements $(A7)$ and $(A8)$ are satisfied. However, they are already fairly restrictive. The evaluation of condition $(A9)$ turns out to be rather complicated for nonautonomous systems. Even, when one confines the analysis to the case of trapezoidal rule in drift part (i.e. $\alpha_i = 0.5$), due to general time–variation of products $\Delta_i \lambda_k(t_i)$. For example, $(\Delta_i \lambda_k(t_i))_{i \in \mathbb{N}}$ might tend too fast to zero. Condition $(A10)$ does not have to hold uniformly in t. It only guarantees the existence of finite moments at any finite time t. Besides one is entitled to take first mean operation then. For asymptotical mean preservation, one might decisively relax conditions of Theorem 4.3 under existence of second moments at finite times. For example, in general $(A9)$ can be replaced. It suffices to show that

$$\lim_{n \to +\infty} \left(\prod_{i=n}^{0} M_0(\alpha_i, \Delta_i, t_i) \right) \; \mathbb{E}\, Y_0 \;=\; \lim_{t \to +\infty} \Phi(t) \; \mathbb{E}\, X_0 \,. \tag{4.3}$$

However, the practical verification of identity (4.3) seems to be a very hard task, unless (1.3), $(A7)$ and $(A8)$ are valid.

Under further restrictions we eventually observe equilibrium preservation which is expressed in the following theorem. Define

$$d(i, k_1, k_2) := \frac{\Delta_i}{(1 - \alpha_i \Delta_i \lambda_{k_1}(t_i))(1 - \alpha_i \Delta_i \lambda_{k_2}(t_i))} \,.$$

Theorem 4.4. *Assume commutativity (1.3),* $(A1) - (A10)$ *and that*

$$(A11) \qquad b^j \in Lin\{e_i : i = 1, 2, ..., l\} \, (j \in \{1, 2, ..., m\}),$$

$$(A12) \qquad b^j \, (j \in \{1, 2, ..., m\}) \text{ do not depend on time},$$

$$(A13) \qquad \forall k_1, k_2 \in \{1, 2, ..., l\} \; :$$

$$\lim_{n \to +\infty} \left[\sum_{i=0}^{n-1} d(i, k_1, k_2) \left(\prod_{r=i+1}^{n} \frac{(1 + (1 - \alpha_r)\Delta_r \lambda_{k_1}(t_r))(1 + (1 - \alpha_r)\Delta_r \lambda_{k_2}(t_r))}{(1 - \alpha_r \Delta_r \lambda_{k_1}(t_r))(1 - \alpha_r \Delta_r \lambda_{k_2}(t_r))} \right) \right.$$

$$\left. + d(n, k_1, k_2) \right] \;=\; \lim_{t \to +\infty} \int_0^t \exp\Big(\int_s^t [\lambda_{k_1}(u) + \lambda_{k_2}(u)] du \Big) ds < +\infty \,.$$

Then $(Y_n)_{n\in I\!N}$ governed by (2.1) with deterministic implicitness $(\alpha_n)_{n\in I\!N}$ and deterministic step sizes $(\Delta_n)_{n\in I\!N}$ (where $\sum_{n=0}^{+\infty}\Delta_n = +\infty$) is asymptotical mean square, p–th mean and equilibrium preserving for SDE (1.1), provided that

$$(A8)' \qquad Y_0 = X_0 \in Lin\{e_i : i = 1, 2, ..., l\} \text{ (a.s.)}.$$

Proof. We only sketch the proof, as it is very laborious in detail. First, it is easy to see that Gaussian distribution is preserved under discretization using family of implicit Euler methods (2.1) and assumptions $(A5), (A6)$. Second, Theorem 4.3 has already indicated the property of asymptotical mean preservation for these methods. Finally, it remains to show asymptotical mean square preservation. Recall that fundamental matrix $\Phi(t) = \exp\left(\int_0^t A(s)\,ds\right)$ when (1.3). One notices that

$$\mathbb{E}\; X_t X_t^T \;=\; \Phi(t)\,\mathbb{E}\,\left[X_0 X_0^T\right]\Phi^T(t) \;+\; \sum_{j=1}^m \int_0^t \Phi(t)\,\Phi^{-1}(s)\,b^j\,{b^j}^T\,{\Phi^{-1}}^T(s)\,\Phi^T(t)\,ds,$$

$$\begin{aligned}\mathbb{E}\; Y_{n+1}Y_{n+1}^T \;&=\; \left(\prod_{i=n}^{0} M_0(i)\right)\,\mathbb{E}\,\left[Y_0 Y_0^T\right]\left(\prod_{i=0}^{n} M_0^T(i)\right) \\ &\quad + \sum_{j=1}^m\sum_{i=0}^n \Delta_i \left(\prod_{k=n}^{i} M_{0+\chi_{\{i\}}(k)}(k)\right) b_i^j {b_i^j}^T \left(\prod_{k=i}^{n} M_{0+\chi_{\{i\}}(k)}^T(k)\right).\end{aligned} \tag{4.4}$$

Let $b^j = b_i^j = \sum_{r=1}^l w_r e_r$, cf. $(A11), (A12)$. Thanks to $(A8')$, the initial value Y_0 has expansion

$$Y_0 (= X_0) = \sum_{r=1}^l y_r\, e_r \quad \text{(a.s.)}\,.$$

The remaining part is carried out by putting this linear combination in both expressions (4.4) for continuous and discrete time evolutions. After simplification, resummation and componentwise comparison one confirms the assertion, what we leave to the reader. □

Remark. The conditions of Theorem 4.4 all together are too restrictive from practical point of view. For instance, it remains to check condition $(A13)$ in practice. The solution of this task seems to be hardly possible for nonautonomous systems. Note, for autonomous systems, requirement $(A13)$ can be fulfilled by trapezoidal rule, cf. main result in section 3. Thus, the quality of assertion of Theorem 4.4

mainly reduces to that of Theorem 3.1 by consideration of stochastic dynamics on the subspace spanned by corresponding nontrivial eigenvectors of matrix A.

5. Asymptotic moments of nonautonomous systems

As a supplement, we state the limit of moments of linear, continuous time, nonautonomous systems in a very general form. The proof follows from expansion of their first and second moments (see e.g. (4.4)), hence it is omitted here. Set

$$\Phi(\infty) = \lim_{t\to+\infty} \Phi(t) \quad \text{and} \quad X_\infty = \lim_{t\to+\infty} X_t \,.$$

Let $\mathcal{F}_0 = \sigma\{X_0\}$ denote the σ–algebra generated by initial value X_0.

Theorem 5.1. *Assume that process $X = \{X_t : t \ge 0\}$ satisfies (1.1) and*

(X1) $\mathbb{E}\, \|X_0\|^2 \,<\, +\infty$,

(X2) X_0 *is independent of* $\mathcal{F}_t^j = \sigma\{W_s^j : 0 \le s \le t\}$ $(\forall t \ge 0)$,

(X3) *random initial value problem (1.2) has a solution,*

(X4) $A(t), b^j(t) (j \in \{1, 2, ..., m\})$ *are nonrandom or independent of* $\mathcal{F}_t^j, \mathcal{F}_0$,

(X5) $\mathbb{E}\, \|\Phi(\infty)\|_2^2 \;=\; \lim_{t\to+\infty} \mathbb{E}\, \|\Phi(t)\|_2^2 \;<\; +\infty$,

(X6) $\displaystyle\sum_{j=1}^{m} \lim_{t\to+\infty} \left\| \int_0^t \Phi(t)\, \Phi^{-1}(s)\, b^j(s)\, {b^j}^T(s)\, {\Phi^{-1}}^T(s)\, \Phi^T(t)\, ds \right\|_2 \;<\; +\infty\,.$

Then the first two moments of stationary law of X exist and

$$\begin{aligned}
\mathbb{E}\, X_\infty &= \lim_{t\to+\infty} \mathbb{E}\, \Phi(t)\, X_0 \;=\; \mathbb{E}\, \Phi(\infty)\, X_0, \\
\mathbb{E}\, X_\infty X_\infty^T &= \mathbb{E}\, \Phi(\infty)\, X_0\, X_0^T\, \Phi^T(\infty) \,+\, \sum_{j=1}^{m} \lim_{t\to+\infty} Q^j(t) \\
\textit{where} \qquad Q^j(t) &= \mathbb{E} \int_0^t \Phi(t)\, \Phi^{-1}(s)\, b^j(s)\, {b^j}^T(s)\, {\Phi^{-1}}^T(s)\, \Phi^T(t)\, ds\,.
\end{aligned}$$

Remark. After detailed comparison one finds conditions for preservation of asymptotical probabilistic characteristics of general nonautonomous systems, e.g.

$$\mathbb{E}\ \Phi(\infty) = \lim_{n\to+\infty} \mathbb{E} \prod_{i=n}^{0} M_0(i),\ Q^j(\infty) = \lim_{n\to+\infty} \sum_{i=0}^{n} \mathbb{E}\ [\Delta_i\, q(i,n,j)\, q^T(i,n,j)]$$

$$\text{where} \qquad q(i,n,j) = \sqrt{\Delta_i} \left(\prod_{k=n}^{i} M_{0+\chi_{\{i\}}(k)}(\alpha_k, \Delta_k, t_k) \right) b^j(t_i)\,.$$

6. Two examples

The search for a nontrivial and feasible example turns out to be very laborious in the nonautonomous case. It is already illustrative to discuss the ideas presented before with very simple, low–dimensional examples. For this purpose, consider the following two processes.

6.1. **A stochastically perturbed rotation.** A system with drift–matrices possessing time–independent eigenvectors is given by the example of two–dimensional stochastically perturbed rotation. Let $(X_t, Y_t) \in \mathbb{R}^2$ satisfy

$$\begin{aligned} dX_t &= (\eta(t)\, X_t + \rho(t)\, Y_t)\, dt + \sigma_1(t)\, dW_t^1 \\ dY_t &= (\eta(t)\, Y_t - \rho(t)\, X_t)\, dt + \sigma_2(t)\, dW_t^2 \end{aligned} \tag{6.1}$$

where $\eta, \rho, \sigma_1^2, \sigma_2^2 \in L^1([0,+\infty), \mathcal{B}, \mu)$ are time–dependent, real–valued coefficients, and W_t^1, W_t^2 represent two independent standard Wiener processes. Obviously, system (6.1) has the form of (1.1) with

$$A(t) = \begin{pmatrix} \eta(t) & \rho(t) \\ -\rho(t) & \eta(t) \end{pmatrix}, b^1(t) = \begin{pmatrix} \sigma_1(t) \\ 0 \end{pmatrix} \text{ and } b^2(t) = \begin{pmatrix} 0 \\ \sigma_2(t) \end{pmatrix}. \tag{6.2}$$

This system satisfies the condition of commutation (1.3) for explicit expansion of its fundamental solution $\Phi(t)$. Besides, drift matrix $A(t)$ can time–independently be diagonalized by matrix

$$L = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

when $\rho(t) \neq 0$, where i represents the imaginary unit (i.e. $i^2 = -1$). It is worth noting that, if $\eta(t) = cos(\theta)$ and $\rho(t) = -sin(\theta)$ where $\theta \in [0, 2\pi]$, then one obtains a classic rotation matrix A.

The deterministic system related to (6.1) has asymptotically stable null solution if

$$\int_0^{+\infty} Re[\lambda(t)]\, dt = \int_0^{+\infty} \eta(t)\, dt = -\infty \tag{6.3}$$

where $\lambda(t) = \eta(t) \overset{+}{-} \rho(t)i$ is an eigenvalue of matrix A. The autonomous case or the case $\eta(t) \leq \varepsilon < 0$ (uniformly in t) easily allows to have (nonvanishing) asymptotical laws. For example, in the autonomous case, if $\eta < 0$ and $\sigma_1^2 + \sigma_2^2 > 0$ then the stationary law is Gaussian with mean zero and variance matrix

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \quad \text{where} \tag{6.4}$$

$$m_1 = -\frac{2\eta^2\sigma_1^2 + \rho^2(\sigma_1^2 + \sigma_2^2)}{4\,\eta\,(\rho^2 + \eta^2)},\ m_2 = \frac{\rho\,(\sigma_2^2 - \sigma_1^2)}{4\,(\rho^2 + \eta^2)},\ m_3 = -\frac{2\eta^2\sigma_2^2 + \rho^2(\sigma_1^2 + \sigma_2^2)}{4\,\eta\,(\rho^2 + \eta^2)}.$$

Let us apply the family of implicit Euler methods to system (6.1). Suppose that $\eta, \rho, \sigma_1, \sigma_2$ do not depend on time t. For this case their scheme is given by

$$\begin{aligned} X_{n+1} &= X_n + [\alpha(\eta X_{n+1} + \rho Y_{n+1}) + (1-\alpha)(\eta X_n + \rho Y_n)]\Delta_n + \sigma_1 \Delta W_n^1 \\ Y_{n+1} &= Y_n + [\alpha(\eta Y_{n+1} - \rho X_{n+1}) + (1-\alpha)(\eta Y_n - \rho X_n)]\Delta_n + \sigma_2 \Delta W_n^2 \end{aligned} \tag{6.5}$$

where $\Delta W_n^1 = W^1(t_{n+1}) - W^1(t_n), \Delta W_n^2 = W^2(t_{n+1}) - W^2(t_n)$. As noted above, matrix A is diagonalizable. Suppose that systems (6.1) and (6.5) start with deterministic or Gaussian initial values. Then one knows from Theorem 3.1 that asymptotical probabilistic laws of both continuous and discrete time systems coincide when $\alpha = 0.5$ and $\eta < 0$. Moreover, provided that $\sigma_1^2 + \sigma_2^2 > 0$, system (6.5) has an explicit Gaussian expansion as in Theorem 4.1, and its limit is Gaussian (cf. Theorem 4.2) with mean zero and second moment matrix M with entries m_i satisfying (6.4). In case $\alpha > 0.5$ and $\eta < 0$ the limit law of Euler methods exists, but may significantly differ from that of underlying continuous time ones (compare stationary second moments). In this case stationary first moments coincide at least.

6.2. **Stochastically perturbed oscillators.** A more interesting example from practical point of view is performed by class of linear oscillators in Mechanical Engineering. For simplicity, we only confine to autonomous case. Let X be displacement and Y velocity of a system with one degree of freedom. Such oscillations under perturbations with additive white noise can be written as

$$\begin{aligned} dX_t &= Y_t\,dt \\ dY_t &= -[\omega^2 X_t \,+\, 2\zeta\omega\, Y_t]\,dt \,+\, \sigma\,dW_t \end{aligned} \tag{6.6}$$

where $\omega \in \mathbb{R}^+ \setminus \{0\}$ is eigenfrequency, $\zeta \in \mathbb{R}^+ \setminus \{0\}$ damping coefficient, $\sigma \in \mathbb{R}$ noise intensity and W_t one–dimensional standard Wiener process. These restrictions on parameters yield an asymptotically stable null solution for related continuous time deterministic system as well as existence of stationary Gaussian law. System (6.6) has diagonalizable drift matrix A if $\zeta \neq 1$. The matrix for diagonalization is found to be

$$L = \begin{pmatrix} 1 & 1 \\ -\omega\,(\zeta - \sqrt{\zeta^2 - 1}) & -\omega\,(\zeta + \sqrt{\zeta^2 - 1}) \end{pmatrix}.$$

Once again we can easily apply our theoretical approach presented in previous sections. The family of implicit Euler methods applied to system (6.6) has scheme

$$\begin{aligned} X_{n+1} &= X_n \,+\, [\alpha Y_{n+1} + (1-\alpha) Y_n]\Delta_n \qquad (6.7) \\ Y_{n+1} &= Y_n \,-\, [\alpha(2\zeta\omega Y_{n+1} + \omega^2 X_{n+1}) + (1-\alpha)(2\zeta\omega Y_n + \omega^2 X_n)]\Delta_n \,+\, \sigma\,\Delta W_n \end{aligned}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$. This scheme possesses asymptotically stable null solution under the absence of random perturbations for all possible equidistant step sizes, provided that $\alpha \geq 0.5$. Therefore it is assymptotical mean preserving. The whole family has an explicit Gaussian expansion while assuming some mild regularity conditions. Furthermore we know that trapezoidal method (i.e. implicit Euler method with $\alpha = 0.5$) applied to linear systems (1.1) provides the correct stationary (Gaussian) law with second moment matrix M satisfying $A\,M \,+\, M\,A^T \;=\; -\,G$ (i.e. equilibrium preservation) where 2×2 matrix $G = (g_{ij})$ has zero elements except for

$g_{22} = \sigma^2$. The solution of this matrix equation is found to be

$$M = \frac{\sigma^2}{4\,\zeta\,\omega} \begin{pmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 1 \end{pmatrix} . \tag{6.8}$$

7. Remarks and conclusions

In this paper we established several results on the probabilistic law of numerical solutions generated by Euler methods with additive noise. For the law of corresponding error processes, see the papers of Kurtz and Protter (1991) or Talay and Tubaro (1990). Our investigation was mainly aiming at asymptotical properties of these discrete time systems themselves, as time tends to infinity. A remarkable asymptotical bias between the behaviour of exact and simplest numerical solutions is observed in models with additive noise. This distance significantly depends on the step size of numerical integration. Only the half drift–implicit Euler scheme (= trapezoidal rule in drift, i.e. implicitness 0.5) could exactly replicate the asymptotical behaviour of stationary Ornstein–Uhlenbeck processes for any choice of step sizes.

We deliberately introduced the new notions of asymptotical preservation of probabilistic characteristics, instead of using well–known stability notions (cf. Kozin [18]), but we are not insisting on these new ones! Mainly, it has been done to point out the difference between numerical analysis of models with additive noise and commonly examined models with multiplicative (parametric) noise. Note that equilibria (stochastic steady states) of SDEs with additive noise are random variables, in contrast to deterministic equilibria of SDEs with multiplicative noise. With the usual stability notions and the herein introduced notions of asymptotical mean, mean square, p–th mean and equilibrium preservation one can assess to some extent the goodness of stochastic approximations with respect to their replication of the stationary behaviour of exact solutions of dynamical systems, at least in the sense of the mean, variance and absolute moments. Moreover, because the stationary numerical behaviour for the Ornstein–Uhlenbeck process is given as a Gaussian distributed random variable with corresponding mean vector and covariance matrix, we know numerical solutions providing the same stationary Gaussian probability distribution as that of the corresponding stationary, exact solution. Note that the

Gaussian distribution is uniquely described by the behaviour of first and second moments. Consequently, with the asymptotical mean and mean square preservation by the half drift–implicit Euler scheme one only receives the correct limit distribution within the class of numerical methods with lower smoothness requirements. This is mathematically clear for stationary Ornstein–Uhlenbeck processes with autonomously diagonalizable drift at least. Note, for nonautonomous or NonGaussian systems, this fact may dramatically change. A complete evaluation of conditions presented here is still open within nonautonomous framework, hence a problem of future research. The conclusion for nonlinear system analysis also remains largely unknown.

A corresponding approach to systems with multiplicative noise (i.e. with state–dependent diffusion part) is presented in [29]. There some stability analysis of the implicit Euler schemes leads to their mean square stability (hence to a preservation of deterministic equilibria) under appropriate conditions on the corresponding continuous time systems and with implicitness degree $\alpha \geq 0.5$. However, for the guarantee of algebraic constraints and other pathwise properties one has to take into account 'real' stochastic implicitness. For a contribution in this respect, see [31].

Summarizing main results of this paper and contributions [29], [31], one obtains the **superiority of half drift–implicit Euler methods** ($\alpha = 0.5$), **i.e. superiority of stochastic trapezoidal rule**, at least within mean square calculus and asymptotical analysis of linear systems of autonomous Itô SDEs.

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