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Thomas Eiter

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: thomas.eiter@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

The asymptotic behavior of weak time-periodic solutions to the Navier–Stokes equations with a drift term in the three-dimensional whole space is investigated. The velocity field is decomposed into a time-independent and a remaining part, and separate asymptotic expansions are derived for both parts and their gradients. One observes that the behavior at spatial infinity is determined by the corresponding Oseen fundamental solutions.

1 Introduction

We study the behavior for $|x| \to \infty$ of time-periodic solutions to the Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u - \lambda \partial_1 u + u \cdot \nabla u + \nabla \mathfrak{p} = f & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \lim\limits_{|x|\to\infty} u(t,x) = 0 & \text{for } t \in \mathbb{T}, \end{cases} \tag{1.1}$$

which model the flow of a viscous incompressible fluid. Here $f\colon \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^3$ is an external force, and $u\colon \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathfrak{p}\colon \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ denote velocity and pressure fields of the fluid flow. The torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ serves as time axis and encodes that all involved functions are time periodic with prescribed period $\mathcal{T} > 0$. In this paper, we consider the case $\lambda \neq 0$, which models a non-vanishing inflow velocity $\lambda\,\mathrm{e}_1$ at infinity. Asymptotic properties in the case $\lambda = 0$ are different and shall not be treated here.

For $\lambda \neq 0$ the pointwise decay of time-periodic solutions to (1.1) was studied by GALDI and SOHR [10] and by GALDI and KYED [12]. By [12] a weak solution u to (1.1) satisfies

$$u(t,x) = \Gamma_0^\lambda(x) \cdot \int_{\mathbb{T}} \int_{\mathbb{R}^3} f(s,y)\,\mathrm{d}y\mathrm{d}s + \mathscr{R}(t,x), \tag{1.2}$$

where Γ_0^λ is the fundamental solution to the steady-state Oseen system

$$\begin{cases} -\Delta v - \lambda \partial_1 v + \nabla p = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

and the remainder term satisfies $|\mathscr{R}(t,x)| \leq C|x|^{-3/2+\varepsilon}$. In particular, (1.2) shows that the asymptotic behavior of the velocity field u is, in general, determined by the steady-state Oseen fundamental solution Γ_0^λ. Moreover, (1.2) coincides with the anisotropic expansion of weak solutions to the corresponding steady-state problem, which is due to FINN [8, 9], BABENKO [1] and GALDI [11] and may be seen as a special case of the time-periodic setting.

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The main theorem of this paper, Theorem 4.3 below, extends the results from [12] in several ways. Firstly, we improve the pointwise estimate of $\mathscr{R}(t,x)$ in such a way that it reflects the anisotropic structure of the solution. Secondly, we derive an asymptotic expansion for ∇u by establishing pointwise estimates of $\nabla\mathscr{R}(t,x)$. Thirdly, we decompose u into its time mean over one period $\mathcal{P}u$ and a time-periodic remainder $\mathcal{P}_\perp u = u - \mathcal{P}u$, for which we derive separate asymptotic expansions. We shall observe that the asymptotic properties of the steady-state part $\mathcal{P}u$ are governed by the steady-state fundamental solution $\varGamma_0^\lambda$, while those of the purely periodic part $\mathcal{P}_\perp u$ are determined by $\varGamma_\perp^\lambda$, the (faster decaying) purely periodic part of the fundamental solution $\varGamma^\lambda$ to the time-periodic Oseen system

$$\begin{cases} \partial_t u - \Delta u - \lambda\partial_1 u + \nabla\mathfrak{p} = f & \text{in } \mathbb{T}\times\mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T}\times\mathbb{R}^3. \end{cases} \tag{1.4}$$

In particular, this shows that the purely periodic part $\mathcal{P}_\perp u$ decays faster than the steady-state part $\mathcal{P}u$ as $|x|\to\infty$.

This paper is structured as follows. After introducing the basic notation in Section 2, we recall the fundamental solution to the time-periodic Oseen equations and collect related results in Section 3. In Section 4 we present and prove our main theorems.

2 Notation

In general, we denote points in $\mathbb{T}\times\mathbb{R}^3$ by (t,x) and call $t\in\mathbb{T}$ time variable and $x\in\mathbb{R}^3$ spatial variable, respectively. For a sufficiently regular function $u\colon \mathbb{T}\times\mathbb{R}^3\to\mathbb{R}^3$ we write $\partial_j u := \partial_{x_j} u$, and we set $\Delta u := \partial_j\partial_j u$ and $\operatorname{div} u := \partial_j u_j$. As in this definition, we use Einstein's summation convention frequently. If $U\colon \mathbb{T}\times\mathbb{R}^3\to\mathbb{R}^{3\times3}$ is matrix valued, the vector field $\operatorname{div} U$ is defined by $(\operatorname{div} U)_j = \partial_k U_{jk}$.

For $R>r>0$ and $x\in\mathbb{R}^3$ we set $\mathrm{B}_R(x) := \{y\in\mathbb{R}^3 \mid |x-y|<R\}$, $\mathrm{B}^R(x) := \{y\in\mathbb{R}^3 \mid |x-y|>R\}$ and $\mathrm{B}_{r,R}(x) := \mathrm{B}^r(x)\cap\mathrm{B}_R(x)$. If $x=0$, we simply write $\mathrm{B}_R := \mathrm{B}_R(0)$, $\mathrm{B}^R := \mathrm{B}^R(0)$ and $\mathrm{B}_{r,R} := \mathrm{B}_{r,R}(0)$. For vectors $a,b\in\mathbb{R}^3$ their tensor product $a\otimes b$ is defined by $(a\otimes b)_{jk} = a_j b_k$.

By $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{k,q}(\Omega)$ we denote classical Lebesgue and Sobolev spaces, and we set

$$\mathrm{C}^\infty_{0,\sigma}(\mathbb{R}^3) := \{\varphi\in\mathrm{C}_0^\infty(\mathbb{R}^3)^3 \mid \operatorname{div}\varphi = 0\}, \qquad \mathrm{D}^{1,2}_{0,\sigma}(\mathbb{R}^3) := \overline{\mathrm{C}^\infty_{0,\sigma}(\mathbb{R}^3)}^{\|\nabla\cdot\|_2}.$$

Observe that $G := \mathbb{T}\times\mathbb{R}^3$ is a locally compact abelian group and that its dual group can be identified with $\widehat{G} = \mathbb{Z}\times\mathbb{R}^3$, the elements of which we denote by $(k,\xi)\in\mathbb{Z}\times\mathbb{R}^3$. We equip the group $\mathbb{T}$ with the normalized Haar measure given by

$$\forall f\in\mathrm{C}(\mathbb{T}): \qquad \int_{\mathbb{T}} f(t)\,\mathrm{d}t = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} f(t)\,\mathrm{d}t,$$

and G with the corresponding product measure. Moreover, $\mathscr{F}_G$ denotes the Fourier transform on G with inverse $\mathscr{F}_G^{-1}$. Then $\mathscr{F}_G$ is an isomorphism $\mathscr{F}_G\colon \mathscr{S}'(G)\to\mathscr{S}'(\widehat{G})$, where $\mathscr{S}'(G)$ is the space of tempered distributions on G, which was introduced by Bruhat [2]; see also [3]. Moreover, for $f\colon \mathbb{T}\times\mathbb{R}^3\to\mathbb{R}$ we set

$$\mathcal{P}f(x) := \int_{\mathbb{T}} f(t,x)\,\mathrm{d}t, \qquad \mathcal{P}_\perp f := f - \mathcal{P}f$$

such that $f = \mathcal{P}f + \mathcal{P}_\perp f$. Since $\mathcal{P}f$ is time independent, we call $\mathcal{P}f$ the *steady-state* part and $\mathcal{P}_\perp f$ the *purely periodic* part of f. A straightforward calculation shows

$$\mathcal{P}f = \mathscr{F}_G^{-1}\big[\delta_{\mathbb{Z}}(k)\mathscr{F}_G[f]\big], \qquad \mathcal{P}_\perp f = \mathscr{F}_G^{-1}\big[(1-\delta_{\mathbb{Z}}(k))\mathscr{F}_G[f]\big],$$

where $\delta_{\mathbb{Z}}$ is the delta distribution on $\mathbb{Z}$.

By the letter C we denote generic positive constants. In order to specify the dependence of C on quantities $a, b, \ldots$, we write $C(a, b, \ldots)$.

3 The time-periodic fundamental solution

In this section, we consider a fundamental solution Γ^λ to the time-periodic problem (1.4) such that the velocity field is given by $u = \Gamma^\lambda * f$, where the convolution is taken with respect to the group $G = \mathbb{T} \times \mathbb{R}^3$. Such a fundamental solution was recently introduced in [12, 4] and is given by

$$\Gamma^\lambda := \Gamma_0^\lambda \otimes 1_{\mathbb{T}} + \Gamma_\perp^\lambda, \tag{3.1}$$

where

$$\Gamma_0^\lambda \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3\times 3}, \quad \Gamma_{0,j\ell}^\lambda(x) := \frac{1}{4\pi\lambda}\big[\delta_{j\ell}\Delta - \partial_j\partial_\ell\big] \int_0^{s(\lambda x)/2} \frac{1-\mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau, \tag{3.2}$$

$$\Gamma_\perp^\lambda := \mathscr{F}_G^{-1}\Bigg[\frac{1-\delta_{\mathbb{Z}}(k)}{|\xi|^2 + i(\frac{2\pi}{\mathcal{T}}k - \lambda\xi_1)}\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\Bigg]. \tag{3.3}$$

Here the symbol $1_{\mathbb{T}}$ denotes the constant 1 distribution on $\mathbb{T}$, and

$$s(x) := |x| + x_1.$$

The function Γ_0^λ is the fundamental solution to the steady-state Oseen problem (1.3); see [11, Section VII.3]. Its anisotropic behavior is reflected by the pointwise estimates

$$\forall\alpha\in\mathbb{N}_0^3\ \forall\varepsilon>0\ \exists C>0\ \forall|x|\geq\varepsilon: \quad |\mathrm{D}^\alpha\Gamma_0^\lambda(x)| \leq C\big[|x|(1+s(\lambda x))\big]^{-1-\frac{|\alpha|}{2}}; \tag{3.4}$$

see [5, Lemma 3.2]. The examination of convolutions of Γ_0^λ with functions satisfying similar estimates was carried out by FARWIG [5, 6] in dimension $n = 3$, and later by KRAČMAR, NOVOTNÝ and POKORNÝ [13] in the general n-dimensional case. The following theorem collects some of their results.

Theorem 3.1. *Let $A \in [2,\infty)$, $B \in [0,\infty)$ and $g \in \mathrm{L}^\infty(\mathbb{R}^3)$ such that $|g(x)| \leq M(1+|x|)^{-A}(1+s(x))^{-B}$. Then there exists $C = C(A,B,\lambda) > 0$ with the following properties:*

1 If $A + \min\{1,B\} > 3$, then

$$\big||\Gamma_0^\lambda| * g(x)\big| \leq CM\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-1}.$$

2 If $A + \min\{1,B\} > 3$ and $A + B \geq 7/2$, then

$$\big||\nabla\Gamma_0^\lambda| * g(x)\big| \leq CM\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-3/2}.$$

3 *If* $A + \min\{1, B\} = 3$ *and* $A + B \geq 7/2$, *then*

$$\big||\nabla \Gamma_0^\lambda| * g(x)\big| \leq CM\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-3/2}\max\{1, \log|x|\}.$$

4 *If* $A + B < 3$, *then*

$$\big||\nabla \Gamma_0^\lambda| * g(x)\big| \leq CM(1+|x|)^{-(A+B)/2}\big(1+s(\lambda x)\big)^{-(A+B-1)/2}.$$

Proof. These are special cases of [13, Theorems 3.1 and 3.2]. □

In order to derive a similar result to control convolutions with the purely periodic part $\Gamma_\perp^\lambda$, we recall the following theorem established in [4].

Theorem 3.2. *The purely periodic velocity fundamental solution* $\Gamma_\perp^\lambda$ *satisfies*

$$\forall q \in \Big(1, \frac{5}{3}\Big): \quad \Gamma_\perp^\lambda \in \mathrm{L}^q(G)^{3\times 3}, \tag{3.5}$$

$$\forall q \in \Big[1, \frac{5}{4}\Big): \quad \partial_j \Gamma_\perp^\lambda \in \mathrm{L}^q(G)^{3\times 3} \quad (j = 1, 2, 3), \tag{3.6}$$

and for all $\alpha \in \mathbb{N}_0^3$, $r \in [1, \infty)$ *and* $\varepsilon > 0$ *there exists* $C > 0$ *such that*

$$\forall |x| \geq \varepsilon: \ \|\mathrm{D}_x^\alpha \Gamma_\perp^\lambda(\cdot, x)\|_{\mathrm{L}^r(\mathbb{T})} \leq C|x|^{-3-|\alpha|}. \tag{3.7}$$

Proof. See [4, Theorem 1.1]. □

From these properties we conclude the following theorem.

Theorem 3.3. *Let* $A \in (0, \infty)$ *and* $g \in \mathrm{L}^\infty(\mathbb{T}\times\mathbb{R}^3)$ *such that* $|g(t,x)| \leq M(1+|x|)^{-A}$. *Then for any* $\varepsilon > 0$ *there exists* $C = C(A, \lambda, \mathcal{T}, \varepsilon) > 0$ *such that*

$$\forall |x| \geq \varepsilon: \qquad \big||\nabla \Gamma_\perp^\lambda| *_G g(t,x)\big| \leq CM(1+|x|)^{-\min\{A,4\}} \tag{3.8}$$

and, if $A > 3$,

$$\forall |x| \geq \varepsilon: \qquad \big||\Gamma_\perp^\lambda| *_G g(t,x)\big| \leq CM(1+|x|)^{-3}. \tag{3.9}$$

Proof. Let us focus on the derivation of (3.9). Let $x \in \mathbb{R}^3$, $|x| \geq \varepsilon$ and set $R := |x|/2$. Then we have

$$\big||\Gamma_\perp^\lambda| *_G g(t,x)\big| \leq M(I_1 + I_2 + I_3)$$

with

$$I_1 = \int_\mathbb{T}\int_{\mathrm{B}_R} |\Gamma_\perp^\lambda(t-s, x-y)|\,(1+|y|)^{-A}\,\mathrm{d}y\mathrm{d}s,$$

$$I_2 = \int_\mathbb{T}\int_{\mathrm{B}^{4R}} |\Gamma_\perp^\lambda(t-s, x-y)|\,(1+|y|)^{-A}\,\mathrm{d}y\mathrm{d}s,$$

$$I_3 = \int_\mathbb{T}\int_{\mathrm{B}_{R,4R}} |\Gamma_\perp^\lambda(t-s, x-y)|\,(1+|y|)^{-A}\,\mathrm{d}y\mathrm{d}s.$$

We estimate these terms separately. Since $|y| \leq R$ implies $|x-y| \geq |x| - |y| \geq |x|/2 = R \geq \varepsilon/2$, we can use (3.7) to estimate

$$I_1 \leq C\int_{\mathrm{B}_R} |x-y|^{-3}(1+|y|)^{-A}\,\mathrm{d}y \leq C|x|^{-3}\int_{\mathbb{R}^3}(1+|y|)^{-A}\,\mathrm{d}y \leq C|x|^{-3}.$$

For I_3 we note that $|y| \geq 4R$ implies $|x-y| \geq |y|-|x| \geq |y|-|y|/2 = |y|/2 \geq 2R \geq \varepsilon$. Therefore, (3.7) yields

$$I_2 \leq C \int_{\mathrm{B}^{4R}} |x-y|^{-3}(1+|y|)^{-A} \,\mathrm{d}y \leq C \int_{\mathrm{B}^{4R}} |y|^{-3}|y|^{-A} \,\mathrm{d}y \leq C|x|^{-A}.$$

Furthermore, Hölder's inequality with $q \in (1, \frac{5}{3})$ and $q' = q/(q-1)$ implies

$$I_3 \leq |x|^{-A} \bigg(\int_{\mathbb{T}} \int_{\mathrm{B}_{R,4R}} 1 \,\mathrm{d}y\mathrm{d}s \bigg)^{1/q'} \|\varGamma_\perp^\lambda\|_q \leq C|x|^{-A}|x|^{3-\frac{3}{q}}$$

in virtue of (3.5). We now choose $q \in (1, \frac{5}{3})$ so small that $-A+3-\frac{3}{q} < -3$. Collecting these estimates, we obtain (3.9). A proof of (3.8) can be given in a similar way. □

The next lemma can be used to conclude asymptotic expansions in the linear case, where the velocity field is given by $u = \varGamma^\lambda * f$.

Lemma 3.4. *Let $\lambda \neq 0$ and $f \in \mathrm{C}_0^\infty(\mathbb{T}\times\mathbb{R}^3)$ with $\operatorname{supp} f \subset \mathbb{T}\times\mathrm{B}_{R_0}$. Let $|\alpha| \leq 1$. Then*

$$\big|\mathrm{D}_x^\alpha \varGamma_0^\lambda * \mathcal{P}f(x)\big| \leq C\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-1-|\alpha|/2}, \tag{3.10}$$

$$\big|\mathrm{D}_x^\alpha \varGamma_\perp^\lambda * \mathcal{P}_\perp f(t,x)\big| \leq C(1+|x|)^{-3-|\alpha|}, \tag{3.11}$$

and for $|x| \geq 2R_0$ we have

$$\bigg|\mathrm{D}_x^\alpha \varGamma_0^\lambda * \mathcal{P}f(x) - \mathrm{D}_x^\alpha \varGamma_0^\lambda(x) \cdot \int_{\mathbb{R}^3} \mathcal{P}f(y)\,\mathrm{d}y\bigg| \leq C\big[|x|(1+s(\lambda x))\big]^{-3/2-|\alpha|/2}, \tag{3.12}$$

$$\bigg|\mathrm{D}_x^\alpha \varGamma_\perp^\lambda * \mathcal{P}_\perp f(t,x) - \bigg(\mathrm{D}_x^\alpha \varGamma_\perp^\lambda(\cdot,x) *_{\mathbb{T}} \int_{\mathbb{R}^3} \mathcal{P}_\perp f(\cdot,y)\,\mathrm{d}y\bigg)(t)\bigg| \leq C|x|^{-4-|\alpha|}. \tag{3.13}$$

Proof. Estimates (3.10) and (3.11) directly follow from Theorem 3.1 and Theorem 3.3. By the mean value theorem, we further have

$$\begin{aligned}\bigg|\mathrm{D}_x^\alpha \varGamma_0^\lambda * \mathcal{P}f(x) &- \mathrm{D}_x^\alpha \varGamma_0^\lambda(x) \int_{\mathbb{R}^3} \mathcal{P}f(y)\,\mathrm{d}y\bigg| \\ &\leq \int_{\mathrm{B}_R} \int_0^1 |y||\nabla \mathrm{D}_x^\alpha \varGamma_0^\lambda(x-\theta y)||\mathcal{P}f(y)|\,\mathrm{d}\theta\mathrm{d}y.\end{aligned}$$

Since $|y| \leq R_0 \leq |x|/2$ implies

$$|x-\theta y| \geq |x| - \theta|y| \geq |x|/2 \geq R_0,$$
$$(1+2|\lambda|R_0)(1+s(\lambda(x-\theta y))) \geq 1+2|\lambda|R_0 + s(\lambda(x-\theta y)) \geq 1+s(\lambda x),$$

estimate (3.4) finally leads to (3.12). Using (3.7) instead of (3.4), we conclude (3.13) in the same way. □

The following auxiliary result treats convolutions of functions with anisotropic decay.

Lemma 3.5. *Let $A \in (-2,2]$, $B \in (1,2]$. Then there exists $C = C(A,B) > 0$ such that for all $x \in \mathbb{R}^3 \setminus \{0\}$ it holds*

$$\begin{aligned}\int_{\mathbb{R}^3} \big[(1+|x-y|)(1+s(x-y))\big]^{-2}(1+|y|)^{-A}(1+s(y))^{-B}\,\mathrm{d}y \\ \leq C(1+|x|)^{-A}(1+s(x))^{-B}\max\bigg\{1, \log\bigg(\frac{|x|}{1+s(x)}\bigg)\bigg\}.\end{aligned}$$

Proof. This is a consequence of the calculations in [13, Section 2]. □

4 Main results

We consider weak solutions to (1.1) in the following sense.

Definition 4.1. *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^3)^3$. A function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^3)^3$ is called* weak solution *to* (1.1) *if*

i. $u \in \mathrm{L}^2(\mathbb{T};\mathrm{D}^{1,2}_{0,\sigma}(\mathbb{R}^3))$,

ii. $\mathcal{P}_\perp u \in \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\mathbb{R}^3))^3$,

iii. the identity

$$\int_{\mathbb{T}\times\mathbb{R}^3}\big[-u\cdot\partial_t\varphi+\nabla u:\nabla\varphi-\lambda\partial_1 u\cdot\varphi+(u\cdot\nabla u)\cdot\varphi\big]\,\mathrm{d}(t,x)=\int_{\mathbb{T}\times\mathbb{R}^3} f\cdot\varphi\,\mathrm{d}(t,x)$$

holds for all test functions $\varphi\in\mathrm{C}^\infty_{0,\sigma}(\mathbb{T}\times\mathbb{R}^3)$.

Remark 4.2. The existence of a weak solution with the above properties has been shown in [14, Theorem 6.3.1] for any $f \in \mathrm{L}^2(\mathbb{T};\mathrm{D}^{-1,2}_0(\mathbb{R}^3))^3$. Therefore, this class seems to be a natural outset for further investigation. Nevertheless, at first glance, instead of ii. one would expect the condition $u \in \mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\Omega))^3$, which naturally appears for weak solutions to the Navier–Stokes initial-value problem. However, this property cannot be expected for general time-periodic data f. As was shown by KYED [14, Theorem 5.2.4], for smooth data $f\in\mathrm{C}^\infty_0(\mathbb{T}\times\mathbb{R}^3)^3$ one has $u\in\mathrm{L}^\infty(\mathbb{T};\mathrm{L}^2(\mathbb{R}^3))^3$ if and only if $\int_{\mathbb{T}\times\mathbb{R}^3} f\,\mathrm{d}(t,x)=0$. An analogous property was established by FINN [7] for the corresponding steady-state problem.

As our main result, we establish the following asymptotic expansions.

Theorem 4.3. *Let $\lambda\neq 0$ and $f\in\mathrm{C}^\infty_0(\mathbb{T}\times\mathbb{R}^3)^3$, and let u be a weak time-periodic solution to* (1.1) *in the sense of Definition 4.1, which satisfies*

$$\exists r\in(5,\infty):\quad \mathcal{P}_\perp u\in\mathrm{L}^r(\mathbb{T}\times\mathbb{R}^3)^3. \tag{4.1}$$

Then

$$\mathcal{P}u(x)=\varGamma^\lambda_0(x)\cdot\int_\Omega \mathcal{P}f(y)\,\mathrm{d}y+\mathscr{R}_0(x), \tag{4.2}$$

$$\mathcal{P}_\perp u(t,x)=\varGamma^\lambda_\perp(\cdot,x)*_\mathbb{T}\int_\Omega\mathcal{P}_\perp f(\cdot,y)\,\mathrm{d}y+\mathscr{R}_\perp(t,x) \tag{4.3}$$

such that there exists $C>0$ such that for all $t\in\mathbb{T}$ and $|x|\geq 4$ it holds

$$|\mathscr{R}_0(x)|\leq C\big[|x|\big(1+s(\lambda x)\big)\big]^{-3/2}\log|x|, \tag{4.4}$$

$$|\nabla\mathscr{R}_0(x)|\leq C\big[|x|\big(1+s(\lambda x)\big)\big]^{-2}\max\Big\{1,\log\Big(\frac{|x|}{1+s(\lambda x)}\Big)\Big\}, \tag{4.5}$$

$$|\mathscr{R}_\perp(t,x)|\leq C|x|^{-4}, \tag{4.6}$$

$$|\nabla\mathscr{R}_\perp(t,x)|\leq C|x|^{-9/2}(1+s(\lambda x))^{-1/2}. \tag{4.7}$$

In particular,

$$u(t,x)=\varGamma^\lambda_0(x)\cdot\int_\mathbb{T}\int_\Omega f(t,y)\,\mathrm{d}y\mathrm{d}t+\mathscr{R}(t,x) \tag{4.8}$$

with

$$|\mathscr{R}(t,x)| \leq C\big[|x|\big(1+s(\lambda x)\big)\big]^{-3/2}\log|x|, \tag{4.9}$$

$$|\nabla\mathscr{R}(t,x)| \leq C\big[|x|\big(1+s(\lambda x)\big)\big]^{-2}\max\Big\{1, \log\Big(\frac{|x|}{1+s(\lambda x)}\Big)\Big\}. \tag{4.10}$$

Remark 4.4. As explained in [12], assumption (4.1) merely appears for technical reasons. It ensures additional local regularity but does not improve spatial decay of the solution.

One main observation is that the asymptotic behavior of u and ∇u for $|x| \to \infty$ is governed by the time-periodic Oseen fundamental solution $\varGamma^\lambda$. In particular, the purely periodic part $\mathcal{P}_\perp u$ decays faster than the steady-state part $\mathcal{P}u$. As a direct consequence of Theorem 4.3, we obtain the following pointwise estimates, which we shall derive as intermediate results on the way to a proof of Theorem 4.3.

Theorem 4.5. *Under the assumptions of Theorem 4.5 there is* $C > 0$ *such that for all* $t \in \mathbb{T}$ *and* $x \in \mathbb{R}^3$ *the function* u *satisfies*

$$|\mathcal{P}u(x)| \leq C\big[\big(1+|x|\big)\big(1+s(\lambda x)\big)\big]^{-1}, \tag{4.11}$$

$$|\nabla\mathcal{P}u(x)| \leq C\big[\big(1+|x|\big)\big(1+s(\lambda x)\big)\big]^{-\frac{3}{2}}, \tag{4.12}$$

$$|\mathcal{P}_\perp u(t,x)| \leq C\big(1+|x|\big)^{-3}, \tag{4.13}$$

$$|\nabla\mathcal{P}_\perp u(t,x)| \leq C\big(1+|x|\big)^{-4}. \tag{4.14}$$

In order to prove these theorems, we recall the following regularity result.

Lemma 4.6. *Let* u *be a weak solution as in Theorem 4.3. Then* $u \in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{R}^3)^3$ *and*

$$\forall r \in (1,\infty),\ q \in (1,2):\ \nabla^2\mathcal{P}u \in \mathrm{L}^r(\mathbb{R}^3),\ \nabla\mathcal{P}u \in \mathrm{L}^{\frac{4q}{4-q}}(\mathbb{R}^3),\ \mathcal{P}u \in \mathrm{L}^{\frac{2q}{2-q}}(\mathbb{R}^3),$$
$$\forall q \in (1,\infty):\ \mathcal{P}_\perp u \in \mathrm{L}^q(\mathbb{T};\mathrm{W}^{2,q}(\mathbb{R}^3)) \cap \mathrm{W}^{1,q}(\mathbb{T};\mathrm{L}^q(\mathbb{R}^3)),$$

and there is a pressure function $\mathfrak{p} \in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{R}^3)$ *such that* (1.1) *is satisfied pointwise.*

Proof. We refer to [12, Lemma 5.1]. □

We also need a uniqueness statement for solutions to the linear problem (1.4).

Lemma 4.7. *Let* $(u,\mathfrak{p}) \in \mathscr{S}'(G)^{3+1}$ *be a solution to* (1.4) *for the right-hand side* $f = 0$*. Then,* $\mathcal{P}u$ *is a polynomial in each component and* $\mathcal{P}_\perp u = 0$*.*

Proof. An application of the Fourier transform $\mathscr{F}_G$ on G to $(1.4)_1$ yields

$$\big(i\tfrac{2\pi}{\mathcal{T}}k + |\xi|^2 - i\lambda\xi_1\big)\widehat{u} + i\xi\widehat{\mathfrak{p}} = 0$$

with $\widehat{u} := \mathscr{F}_G[u]$ and $\widehat{\mathfrak{p}} := \mathscr{F}_G[\mathfrak{p}]$. Multiplying this equation with $i\xi$ and using $\operatorname{div} u = 0$, we obtain $-|\xi|^2\widehat{\mathfrak{p}} = 0$, so that $\operatorname{supp}\widehat{\mathfrak{p}} \subset \mathbb{Z}\times\{0\}$. Then, the above equation yields

$$\operatorname{supp}\big[(i\tfrac{2\pi}{\mathcal{T}}k + |\xi|^2 - i\lambda\xi_1)\widehat{u}\big] = \operatorname{supp}\big[-i\xi\widehat{\mathfrak{p}}\big] \subset \mathbb{Z}\times\{0\}.$$

Because the only zero of $(k,\xi) \mapsto (i\frac{2\pi}{\mathcal{T}}k + |\xi|^2 - i\lambda\xi_1)$ is $(k,\xi) = (0,0)$, we conclude $\operatorname{supp}\widehat{u} \subset \{(0,0)\}$. Thus we obtain $\mathcal{P}_\perp u = 0$ and that $\mathcal{P}u$ is a polynomial. □

These lemmas enable us to derive the following representation formulas.

Proposition 4.8. *Let u be a weak solution as in Theorem 4.3. Then*

$$\mathrm{D}_x^\alpha u = \mathrm{D}_x^\alpha \mathit{\Gamma}^\lambda * [f - u\cdot\nabla u] \tag{4.15}$$

for all $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\leq 1$. In particular, $v:=\mathcal{P}u$ and $w:=\mathcal{P}_\perp u$ satisfy

$$\mathrm{D}_x^\alpha v = \mathrm{D}_x^\alpha \mathit{\Gamma}_0^\lambda * \big[\mathcal{P}f - v\cdot\nabla v - \mathcal{P}(w\cdot\nabla w)\big], \tag{4.16}$$
$$\mathrm{D}_x^\alpha w = \mathrm{D}_x^\alpha \mathit{\Gamma}_\perp^\lambda * \big[\mathcal{P}_\perp f - v\cdot\nabla w - w\cdot\nabla v - \mathcal{P}_\perp(w\cdot\nabla w)\big]. \tag{4.17}$$

Moreover, we have[1]

$$u = \mathit{\Gamma}^\lambda * f - \nabla\mathit{\Gamma}^\lambda * (u\otimes u), \tag{4.18}$$
$$v = \mathit{\Gamma}_0^\lambda * \mathcal{P}f - \nabla\mathit{\Gamma}_0^\lambda * \big[v\otimes v + \mathcal{P}(w\otimes w)\big], \tag{4.19}$$
$$w = \mathit{\Gamma}_\perp^\lambda * \mathcal{P}_\perp f - \nabla\mathit{\Gamma}_\perp^\lambda * \big[v\otimes w + w\otimes v + \mathcal{P}_\perp(w\otimes w)\big]. \tag{4.20}$$

Proof. From Lemma 4.6 we conclude $u\cdot\nabla u\in\mathrm{L}^q(\mathbb{T}\times\mathbb{R}^3)$ for all $q\in(1,\infty)$. Therefore, $U:=\mathit{\Gamma}^\lambda*(f-u\cdot\nabla u)$ is well defined as a classical convolution integral, and we have $\partial_j U = \partial_j\mathit{\Gamma}^\lambda*(f-u\cdot\nabla u)$ for $j=1,2,3$ by the dominated convergence theorem. Since both U and u satisfy the time-periodic Oseen system (1.4) for suitable pressure functions $\mathfrak{p}$, Lemma 4.7 implies $\mathcal{P}_\perp u = \mathcal{P}_\perp U$ and that $\mathcal{P}u - \mathcal{P}U$ is a polynomial in each component. With Young's inequality we obtain $\mathcal{P}U\in\mathrm{L}^6(\mathbb{R}^3)$ since $\mathit{\Gamma}_0^\lambda\in\mathrm{L}^{12/5}(\mathbb{R}^3)$ by [12, Lemma 5.4]. Hence, $\mathcal{P}u-\mathcal{P}U\in\mathrm{L}^6(\mathbb{R}^3)$. This leads to $\mathcal{P}u=\mathcal{P}U$ and thus $u=U$, which yields (4.15). The remaining formulas now follow from

$$v = \mathcal{P}u = (\mathit{\Gamma}_0^\lambda\otimes 1_{\mathbb{T}}) * \big[f - u\cdot\nabla u\big] = \mathit{\Gamma}_0^\lambda * \big[\mathcal{P}(f-u\cdot\nabla u)\big],$$
$$w = \mathcal{P}_\perp u = \mathit{\Gamma}_\perp^\lambda * \big[f - u\cdot\nabla u\big] = \mathit{\Gamma}_\perp^\lambda * \big[\mathcal{P}_\perp(f-u\cdot\nabla u)\big]$$

together with the identity $u\cdot\nabla u = \operatorname{div}(u\otimes u)$ and integration by parts. □

Based on these formulas, we can now prove Theorem 4.5 and Theorem 4.3.

Proof of Theorem 4.5. We split $u=v+w$ into steady-state part $v:=\mathcal{P}u$ and purely periodic part $w:=\mathcal{P}_\perp u$. By [12, Theorem 2.2] we have (4.8) with $|\mathscr{R}(t,x)|\leq C|x|^{-5/4}$. In virtue of (3.4) and $u\in\mathrm{C}^\infty(\mathbb{T}\times\mathbb{R}^3)^3$, this implies

$$|v(x)| \leq C(1+|x|)^{-1}\big(1+s(\lambda x)\big)^{-1/4}, \tag{4.21}$$
$$|w(t,x)| \leq C(1+|x|)^{-5/4} \tag{4.22}$$

for all $t\in\mathbb{T}$ and $x\in\mathbb{R}^3$. This leads to

$$\big|v\otimes v + \mathcal{P}[w\otimes w]\big|(x) \leq C(1+|x|)^{-2}(1+s(\lambda x))^{-1/2}.$$

Therefore, (4.19), (3.10) and Theorem 3.1 yield

$$|v(x)| \leq \big|\mathit{\Gamma}_0^\lambda * \mathcal{P}f\big|(x) + \big|\nabla\mathit{\Gamma}_0^\lambda * \big[v\otimes v + \mathcal{P}[w\otimes w]\big]\big|(x) \leq C\big[(1+|x|)(1+s(\lambda x))\big]^{-1},$$

which is the desired estimate (4.11). Now (4.11) together with (4.22) leads to

$$\big|v\otimes w + w\otimes v + \mathcal{P}_\perp[w\otimes w]\big|(t,x) \leq C(1+|x|)^{-9/4}. \tag{4.23}$$

[1] Here we set $(\nabla\mathit{\Gamma}^\lambda * U)_j := \partial_m\mathit{\Gamma}_{j\ell}^\lambda * U_{jm}$ for an $\mathbb{R}^{3\times 3}$-valued function U.

Therefore, (4.20), (3.11) and Theorem 3.3 imply

$$\begin{aligned} |w(t,x)| &\leq \big|\varGamma_\perp^\lambda * \mathcal{P}_\perp f\big|(t,x) + \big|\nabla\varGamma_\perp^\lambda * \big[v\otimes w + w\otimes v + \mathcal{P}_\perp[w\otimes w]\big]\big|(t,x) \\ &\leq C\big((1+|x|)^{-3} + (1+|x|)^{-9/4}\big) \leq C(1+|x|)^{-9/4}. \end{aligned}$$

Using this estimate and (4.11) again, we conclude

$$\big|v\otimes w + w\otimes v + \mathcal{P}_\perp[w\otimes w]\big|(t,x) \leq C(1+|x|)^{-13/4}. \tag{4.24}$$

Repeating the above argument with (4.24) instead of (4.23), we end up with (4.13).

Now let us turn to the estimates of ∇u. Due to $u \in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{R}^3)$, it suffices to consider $|x| \geq 2$. Let $R := |x|/2 \geq 1$. By Proposition 4.8 we have

$$\partial_j v = \partial_j\varGamma_0^\lambda * \mathcal{P}f - I, \qquad \partial_j w = \partial_j\varGamma_\perp^\lambda * \mathcal{P}_\perp f - J$$

with

$$\begin{aligned} I &:= I_1 + I_2 := \partial_j\varGamma_0^\lambda * \big[v\cdot\nabla v\big] + \partial_j\varGamma_0^\lambda * \big[\mathcal{P}[w\cdot\nabla w]\big], \\ J &:= J_1 + J_2 + J_3 := \partial_j\varGamma_\perp^\lambda * \big[v\cdot\nabla w\big] + \partial_j\varGamma_\perp^\lambda * \big[w\cdot\nabla v\big] + \partial_j\varGamma_\perp^\lambda * \big[\mathcal{P}_\perp[w\cdot\nabla w]\big]. \end{aligned}$$

We estimate these terms separately. Clearly, $|I_1| \leq I_{11} + I_{12}$ with

$$\begin{aligned} I_{11}(x) &:= \int_{\mathrm{B}_R} |\partial_j\varGamma_0^\lambda(x-y)||v(y)||\nabla v(y)|\,\mathrm{d}y, \\ I_{12}(x) &:= \int_{\mathrm{B}^R} |\partial_j\varGamma_0^\lambda(x-y)||v(y)||\nabla v(y)|\,\mathrm{d}y. \end{aligned}$$

Since $|y| \leq R$ implies $|x-y| \geq |x|/2 = R \geq 1$, the pointwise estimate (3.4) implies

$$\begin{aligned} I_{11}(x) &\leq \int_{\mathrm{B}_R} \big[(1+|x-y|)(1+s(x-y))\big]^{-3/2}|v(y)||\nabla v(y)|\,\mathrm{d}y \\ &\leq C(1+|x|)^{-3/2}\|v\|_3\|\nabla v\|_{\frac{3}{2}} \leq C(1+|x|)^{-3/2} \end{aligned}$$

in view of Lemma 4.6, and $\nabla\varGamma_0^\lambda \in \mathrm{L}^{17/12}(\mathbb{R}^3)$ (see [12, Lemma 5.4]) and Lemma 4.6 yield

$$I_{12}(x) \leq C\|\partial_j\varGamma_0^\lambda\|_{\frac{17}{12}}\|\nabla v\|_{\frac{17}{5}}\|v\|_{\mathrm{L}^\infty(\mathrm{B}^R)} \leq C(1+|x|)^{-1}$$

by (4.11). We thus deduce $|I_1(x)| \leq C(1+|x|)^{-1}$. For I_2 we proceed similarly to obtain $|I_2(x)| \leq (1+|x|)^{-3/2}$. From these estimates and (3.12), we conclude

$$|\nabla v(x)| \leq C(1+|x|)^{-1}. \tag{4.25}$$

Now let us turn towards ∇w. As above, we split J_1 and estimate $|J_1| \leq J_{11} + J_{12}$ with

$$\begin{aligned} J_{11}(t,x) &:= \int_{\mathbb{T}}\int_{\mathrm{B}_R} |\partial_j\varGamma_\perp^\lambda(t-s,x-y)||v(y)||\nabla w(s,y)|\,\mathrm{d}y\mathrm{d}s, \\ J_{12}(t,x) &:= \int_{\mathbb{T}}\int_{\mathrm{B}^R} |\partial_j\varGamma_\perp^\lambda(t-s,x-y)||v(y)||\nabla w(s,y)|\,\mathrm{d}y\mathrm{d}s. \end{aligned}$$

By Hölder's inequality in space and time, from (3.7) we obtain

$$J_{11}(t,x) \leq C\bigg(\int_{\mathrm{B}_R} |x-y|^{-8}\,\mathrm{d}y\bigg)^{\frac{1}{2}}\|v\|_4\|\nabla w\|_4 \leq C|x|^{-5/2}$$

due to Lemma 4.6. Moreover, Hölder's inequality and (4.11) lead to

$$J_{12}(t,x) \leq C\|\partial_j \Gamma_\perp^\lambda\|_1 \|v\|_{\mathrm{L}^\infty(\mathrm{B}^R)} \|\nabla w\|_\infty \leq C|x|^{-1}$$

because $\nabla \Gamma_\perp^\lambda \in \mathrm{L}^1(\mathbb{T}\times\mathbb{R}^3)$ by (3.6) and $\nabla w \in \mathrm{L}^\infty(\mathbb{T}\times\mathbb{R}^3)$ by Lemma 4.6 and Sobolev embeddings. In a similar fashion, we can use (4.13) to estimate J_2 and J_3 and obtain

$$|J_2(t,x)| \leq C\big(|x|^{-\frac{5}{2}}\|w\|_4\|\nabla v\|_4 + |x|^{-3}\|\partial_j \Gamma_\perp^\lambda\|_{\frac{9}{8}}\|\nabla v\|_9\big) \leq C|x|^{-\frac{5}{2}},$$
$$|J_3(t,x)| \leq C\big(|x|^{-\frac{5}{2}}\|w\|_4\|\nabla w\|_4 + |x|^{-3}\|\partial_j \Gamma_\perp^\lambda\|_1\|\nabla w\|_\infty\big) \leq C|x|^{-\frac{5}{2}}.$$

Collecting the above estimates and combining them with (3.11), we end up with

$$|\nabla w(t,x)| \leq C(1+|x|)^{-1}. \tag{4.26}$$

From (4.11), (4.25), (4.13) and (4.26) we now conclude

$$\big|v(x)\cdot\nabla v(x) + \mathcal{P}[w\cdot\nabla w](x)\big| \leq C(1+|x|)^{-2}(1+s(\lambda x))^{-1/2},$$

so that

$$|I(x)| \leq C(1+|x|)^{-5/4}(1+s(\lambda x))^{-3/4}$$

by Theorem 3.1. Together with (3.10) we thus obtain

$$|\nabla v(x)| \leq C(1+|x|)^{-5/4}(1+s(\lambda x))^{-3/4},$$

so that from (4.11), (4.13) and (4.26) we deduce

$$\big|v(x)\cdot\nabla v(x) + \mathcal{P}[w\cdot\nabla w](x)\big| \leq C(1+|x|)^{-9/4}(1+s(\lambda x))^{-7/4}.$$

By another application of Theorem 3.1 and combination with (3.10), we arrive at (4.12).

For the derivation of (4.14) we proceed with a similar bootstrap argument. From (4.11), (4.12), (4.13) and (4.26) we deduce

$$\big|v\cdot\nabla w + w\cdot\nabla v + \mathcal{P}_\perp\big[w\cdot\nabla w\big]\big|(t,x) \leq C(1+|x|)^{-2},$$

so that Theorem 3.3 implies $|J(t,x)| \leq C(1+|x|)^{-2}$. Combining this with (3.11), we conclude

$$|\nabla w(t,x)| \leq C(1+|x|)^{-2}. \tag{4.27}$$

We now repeat this argument with (4.27) instead of (4.26), which leads to an improved decay rate for ∇w. Iterating this procedure, we finally arrive at (4.14). □

Proof of Theorem 4.3. We keep the notation from the previous proof. We have

$$\big|v\otimes v + \mathcal{P}[w\otimes w]\big|(x) \leq C\big[(1+|x|)(1+s(\lambda x))\big]^{-2} \tag{4.28}$$

by Theorem 4.5, which, by Theorem 3.1, implies

$$\Big|\partial_j \Gamma_0^\lambda * \big[v\otimes v + \mathcal{P}[w\otimes w]\big](x)\Big| \leq C\big[(1+|x|)\big(1+s(\lambda x)\big)\big]^{-3/2}\log|x|.$$

In virtue of the representation formula (4.19) and the identity

$$\mathscr{R}_0(x) = \mathcal{P}\mathscr{R}(x) = v(x) - \Gamma_0^\lambda(x)\int_{\mathbb{R}^3}\mathcal{P}f(y)\,\mathrm{d}y,$$

this estimate and (3.12) imply (4.4). Moreover, by Theorem 4.5 we have

$$\left|v \otimes w + w \otimes v + \mathcal{P}_\perp[w \otimes w]\right|(t,x) \leq C(1+|x|)^{-4},$$

so that

$$\left|\partial_j \varGamma_\perp^\lambda * \left[v \otimes w + w \otimes v + \mathcal{P}_\perp[w \otimes w]\right](t,x)\right| \leq C(1+|x|)^{-4}$$

by Theorem 3.3. Now (4.6) is a consequence of this estimate and (3.13).

To show (4.5), at first observe that Theorem 4.5 implies

$$\left|v(x)\cdot\nabla v(x) + \mathcal{P}[w\cdot\nabla w](x)\right| \leq C\big[(1+|x|)(1+s(\lambda x))\big]^{-5/2}. \tag{4.29}$$

Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ such that $\chi(x)=1$ for $|x|\leq 1$ and $\chi(x)=0$ for $|x|\geq 2$. We decompose

$$I = \left[\chi \partial_j \varGamma_0^\lambda\right] * \left[v\cdot\nabla v + \mathcal{P}(w\cdot\nabla w)\right] + \left[(1-\chi)\partial_j \varGamma_0^\lambda\right] * \left[v\cdot\nabla v + \mathcal{P}(w\cdot\nabla w)\right] =: K_1 + K_2.$$

Then

$$|K_1| \leq C \int_{\mathrm{B}_2(x)} \left|\partial_j \varGamma_0^\lambda(x-y)\right| \big[(1+|y|)(1+s(\lambda y))\big]^{-5/2}\,\mathrm{d}y$$

by (4.29). As in the proof of Lemma 3.4, from $|x-y|\leq 2\leq |x|/2$ we conclude $|y|\geq |x|/2 \geq 2$ and $(1+4|\lambda|)(1+s(\lambda y))\geq 1+s(\lambda x)$. Since $\nabla \varGamma_0^\lambda \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$, this implies

$$\begin{aligned}|K_1| &\leq C\big[(1+|x|)(1+s(\lambda x))\big]^{-5/2} \int_{\mathrm{B}_2(x)} |\partial_j \varGamma_0^\lambda(x-y)|\,\mathrm{d}y \\ &\leq C\big[(1+|x|)(1+s(\lambda x))\big]^{-5/2}.\end{aligned}$$

By integration by parts and (3.4) and (4.28), we further obtain

$$\begin{aligned}|K_2| &\leq C\int_{\mathbb{R}^3} |1-\chi(x-y)|\,\left|\partial_j \nabla \varGamma_0^\lambda(x-y)\right|\,\left|v\otimes v + \mathcal{P}[w\otimes w]\right|(y)\,\mathrm{d}y \\ &\qquad + C\int_{\mathbb{R}^3} |\nabla\chi(x-y)|\left|\partial_j \varGamma_0^\lambda(x-y)\right|\,\left|v\otimes v + \mathcal{P}[w\otimes w]\right|(y)\,\mathrm{d}y \\ &\leq C\int_{\mathrm{B}^1(x)} \big[|x-y|(1+s(\lambda(x-y)))\big]^{-2}\big[(1+|y|)(1+s(\lambda y))\big]^{-2}\,\mathrm{d}y \\ &\qquad + C\int_{\mathrm{B}_{1,2}(x)} \big[|x-y|(1+s(\lambda(x-y)))\big]^{-3/2}\big[(1+|y|)(1+s(\lambda y))\big]^{-2}\,\mathrm{d}y.\end{aligned}$$

For the first integral we use Lemma 3.5, and for the second one we argue as for K_1 to deduce

$$\begin{aligned}|K_2| &\leq C\big((1+|x|)(1+s(\lambda x))\big)^{-2}\max\left\{1, \log\left(\frac{|x|}{1+s(\lambda x)}\right)\right\} \\ &\qquad + C\big[(1+|x|)(1+s(\lambda x))\big]^{-2}\int_{\mathrm{B}_{1,2}(x)} \big[|x-y|(1+s(\lambda(x-y)))\big]^{-3/2}\,\mathrm{d}y \\ &\leq C\big((1+|x|)(1+s(\lambda x))\big)^{-2}\max\left\{1, \log\left(\frac{|x|}{1+s(\lambda x)}\right)\right\}.\end{aligned}$$

Combining the estimates of K_1 and K_2 with (3.12), from formula (4.16) we obtain (4.5).

Furthermore, Theorem 4.5 implies

$$\left|v\cdot\nabla w + w\cdot\nabla v + \mathcal{P}_\perp\left[w\cdot\nabla w\right]\right|(t,x) \leq C(1+|x|)^{-9/2}(1+s(\lambda x))^{-3/2}.$$

With an argument similar to before, we now deduce

$$\begin{aligned}|J(t,x)| &\leq C \int_{\mathrm{B}_1(x)} |\partial_j \Gamma_\perp^\lambda(x-y)|(1+|y|)^{-9/2}(1+s(\lambda y))^{-3/2}\,\mathrm{d}y \\ &\qquad + C \int_{\mathrm{B}^1(x)} |\partial_j \Gamma_\perp^\lambda(x-y)|(1+|y|)^{-9/2}(1+s(\lambda y))^{-3/2}\,\mathrm{d}y \\ &\leq C\bigg((1+|x|)^{-9/2}(1+s(\lambda x))^{-3/2}\|\partial_j \Gamma_\perp^\lambda\|_1 + \int_{\mathrm{B}^1(x)} |x-y|^{-4}(1+|y|)^{-9/2}\,\mathrm{d}y\bigg) \\ &\leq C(1+|x|)^{-9/2}(1+s(\lambda x))^{-1},\end{aligned}$$

where we used (3.6). Combining this estimates with (3.13), formula (4.17) implies (4.7).

Finally, the asymptotic expansion (4.8) with the asserted estimates of $\mathscr{R}(t,x)$ is a direct consequence of these results and the pointwise estimates of $\Gamma_\perp^\lambda$ from (3.7). □

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