On the spatially asymptotic structure of time-periodic solutions to the Navier–Stokes equations

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Abstract

The asymptotic behavior of weak time-periodic solutions to the Navier–Stokes equations with a drift term in the three-dimensional whole space is investigated. The velocity field is decomposed into a time-independent and a remaining part, and separate asymptotic expansions are derived for both parts and their gradients. One observes that the behavior at spatial infinity is determined by the corresponding Oseen fundamental solutions.

1 Introduction

We study the behavior for $|x| \to \infty$ of time-periodic solutions to the Navier–Stokes equations

$$\begin{cases}
\partial_t u - \Delta u - \lambda \partial_1 u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{T} \times \mathbb{R}^3, \\
\text{div } u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^3, \\
\lim_{|x| \to \infty} u(t, x) = 0 & \text{for } t \in \mathbb{T},
\end{cases}$$

(1.1)

which model the flow of a viscous incompressible fluid. Here $f: \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^3$ is an external force, and $u: \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p: \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ denote velocity and pressure fields of the fluid flow. The torus group $\mathbb{T} := \mathbb{R}/\mathbb{T} \mathbb{Z}$ serves as time axis and encodes that all involved functions are time periodic with prescribed period $T > 0$. In this paper, we consider the case $\lambda \neq 0$, which models a non-vanishing inflow velocity $\lambda e_1$ at infinity. Asymptotic properties in the case $\lambda = 0$ are different and shall not be treated here.

For $\lambda \neq 0$ the pointwise decay of time-periodic solutions to (1.1) was studied by Galdi and Sohr [10] and by Galdi and Kyed [12]. By [12] a weak solution $u$ to (1.1) satisfies

$$u(t, x) = \Gamma_0^\lambda(x) \cdot \int_\mathbb{T} \int_{\mathbb{R}^3} f(s, y) \, dy \, ds + \mathcal{R}(t, x),$$

(1.2)

where $\Gamma_0^\lambda$ is the fundamental solution to the steady-state Oseen system

$$\begin{cases}
-\Delta v - \lambda \partial_1 v + \nabla p = f & \text{in } \mathbb{R}^3, \\
\text{div } v = 0 & \text{in } \mathbb{R}^3,
\end{cases}$$

(1.3)

and the remainder term satisfies $|\mathcal{R}(t, x)| \leq C|x|^{-3/2+\epsilon}$. In particular, (1.2) shows that the asymptotic behavior of the velocity field $u$ is, in general, determined by the steady-state Oseen fundamental solution $\Gamma_0^\lambda$. Moreover, (1.2) coincides with the anisotropic expansion of weak solutions to the corresponding steady-state problem, which is due to Finn [8, 9], Babenko [1] and Galdi [11] and may be seen as a special case of the time-periodic setting.

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The main theorem of this paper, Theorem 4.3 below, extends the results from [12] in several ways. Firstly, we improve the pointwise estimate of \( \mathcal{R}(t,x) \) in such a way that it reflects the anisotropic structure of the solution. Secondly, we derive an asymptotic expansion for \( \nabla u \) by establishing pointwise estimates of \( \nabla \mathcal{R}(t,x) \). Thirdly, we decompose \( u \) into its time mean over one period \( P u \) and a time-periodic remainder \( P_\perp u = u - P u \), for which we derive separate asymptotic expansions. We shall observe that the asymptotic properties of the steady-state part \( P u \) are governed by the steady-state fundamental solution \( \Gamma^\lambda_0 \), while those of the purely periodic part \( P_\perp u \) are determined by \( \Gamma^\lambda_1 \), the (faster decaying) purely periodic part of the fundamental solution \( \Gamma^\lambda \) to the time-periodic Oseen system

\[
\begin{aligned}
\partial_t u - \Delta u - \lambda \partial_t u + \nabla p &= f & \quad \text{in } T \times \mathbb{R}^3, \\
\div u &= 0 & \quad \text{in } \mathbb{R}^3.
\end{aligned}
\tag{1.4}
\]

In particular, this shows that the purely periodic part \( P_\perp u \) decays faster than the steady-state part \( P u \) as \( |x| \to \infty \).

This paper is structured as follows. After introducing the basic notation in Section 2, we recall the fundamental solution to the time-periodic Oseen equations and collect related results in Section 3. In Section 4, we present and prove our main theorems.

2 Notation

In general, we denote points in \( T \times \mathbb{R}^3 \) by \( (t,x) \) and call \( t \in T \) time variable and \( x \in \mathbb{R}^3 \) spatial variable, respectively. For a sufficiently regular function \( u : T \times \mathbb{R}^3 \to \mathbb{R}^3 \) we write \( \partial_j u := \partial_{x_j} u \), and we set \( \Delta u := \partial_j \partial_j u \) and \( \div u := \partial_j u_j \). As in this definition, we use Einstein’s summation convention frequently. If \( U : T \times \mathbb{R}^3 \to \mathbb{R}^3 \) is matrix valued, the vector field \( \div U \) is defined by \( (\div U)_j = \partial_b U_{jk} \).

For \( R > r > 0 \) and \( x \in \mathbb{R}^3 \) we set \( B_R(x) := \{ y \in \mathbb{R}^3 \mid |x - y| < R \} \), \( B_R(x) := \{ y \in \mathbb{R}^3 \mid |x - y| > R \} \) and \( B_{r,R}(x) := B^r(x) \cap B_R(x) \). If \( x = 0 \), we simply write \( B_R := B_R(0) \), \( B^R := \overline{B^R(0)} \) and \( B_{r,R} := B_{r,R}(0) \). For vectors \( a, b \in \mathbb{R}^3 \) their tensor product \( a \otimes b \) is defined by \( (a \otimes b)_{jk} = a_j b_k \).

By \( L_s^q(\Omega) \) and \( W^{k,q}(\Omega) \) we denote classical Lebesgue and Sobolev spaces, and we set

\[
\begin{aligned}
C_0(|\nabla|)(\mathbb{R}^3) :&= \{ \varphi \in C_0(\mathbb{R}^3) \mid \div \varphi = 0 \}, & D_0^{1,2}(\mathbb{R}^3) := C_0^{\infty}(\mathbb{R}^3)|\nabla| \| \cdot \|_2.
\end{aligned}
\]

Observe that \( G := T \times \mathbb{R}^3 \) is a locally compact abelian group and that its dual group can be identified with \( \hat{G} = Z \times \mathbb{R}^3 \), the elements of which we denote by \( (k, \xi) \in Z \times \mathbb{R}^3 \). We equip the group \( T \) with the normalized Haar measure given by

\[
\forall f \in C(T) : \quad \int_T f(t) \, dt = \frac{1}{T} \int_0^T f(t) \, dt,
\]

and \( G \) with the corresponding product measure. Moreover, \( \mathcal{F}_G \) denotes the Fourier transform on \( G \) with inverse \( \mathcal{F}_G^{-1} \). Then \( \mathcal{F}_G \) is an isomorphism \( \mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\hat{G}) \), where \( \mathcal{S}(G) \) is the space of tempered distributions on \( G \), which was introduced by Bruhat [2]; see also [3]. Moreover, for \( f : T \times \mathbb{R}^3 \to \mathbb{R} \) we set

\[
\mathcal{P} f(x) := \int_T f(t,x) \, dt, \quad \mathcal{P}_\perp f := f - \mathcal{P} f.
\]
such that \( f = \mathcal{P}f + \mathcal{P}_\perp f \). Since \( \mathcal{P}f \) is time independent, we call \( \mathcal{P}f \) the \textit{steady-state} part and \( \mathcal{P}_\perp f \) the \textit{purely periodic} part of \( f \). A straightforward calculation shows

\[
\mathcal{P}f = \mathcal{F}_G^{-1}[\delta_Z(k)\mathcal{F}_G[f]], \quad \mathcal{P}_\perp f = \mathcal{F}_G^{-1}[(1 - \delta_Z(k))\mathcal{F}_G[f]],
\]

where \( \delta_Z \) is the delta distribution on \( \mathbb{Z} \).

By the letter \( C \) we denote generic positive constants. In order to specify the dependence of \( C \) on quantities \( a, b, \ldots \), we write \( C(a, b, \ldots) \).

### 3 The time-periodic fundamental solution

In this section, we consider a fundamental solution \( \Gamma^\lambda \) to the time-periodic problem (1.4) such that the velocity field is given by \( u = \Gamma^\lambda * f \), where the convolution is taken with respect to the group \( G = \mathbb{T} \times \mathbb{R}^3 \). Such a fundamental solution was recently introduced in [12, 4] and is given by

\[
\Gamma^\lambda := \Gamma^\lambda_0 \otimes 1_T + \Gamma^\lambda_\perp,
\]

where

\[
\Gamma^\lambda_0 : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3 \times 3}, \quad \Gamma^\lambda_0(x) := \frac{1}{4\pi^2} \frac{1}{|\xi|^2 + i(\frac{2\pi}{T}k - \lambda \xi_1)} \left( \mathcal{F}_x^{-1} \left[ 1 - \delta_Z(k) \right] \right),
\]

\[
\Gamma^\lambda_\perp := \mathcal{F}_G^{-1} \left[ \frac{1 - \delta_Z(k)}{|\xi|^2 + i(\frac{2\pi}{T}k - \lambda \xi_1)} \right].
\]

Here the symbol \( 1_T \) denotes the constant \( 1 \) distribution on \( \mathbb{T} \), and

\[
s(x) := |x| + x_1.
\]

The function \( \Gamma^\lambda_0 \) is the fundamental solution to the steady-state Oseen problem [13]; see [11] Section VII.3. Its anisotropic behavior is reflected by the pointwise estimates

\[
\forall \alpha \in \mathbb{N}_0^3 \forall \varepsilon > 0 \exists C > 0 \forall |x| \geq \varepsilon : \quad |D^\alpha \Gamma^\lambda_0(x)| \leq C \left[ |x|(1 + s(\lambda x)) \right]^{-1 - \frac{|\alpha|}{2}};
\]

see [5] Lemma 3.2. The examination of convolutions of \( \Gamma^\lambda_0 \) with functions satisfying similar estimates was carried out by Farwig [5, 6] in dimension \( n = 3 \), and later by Kráčmar, Novotný and Pokorný [13] in the general \( n \)-dimensional case. The following theorem collects some of their results.

**Theorem 3.1.** Let \( A \in [2, \infty), B \in [0, \infty) \) and \( g \in L^\infty(\mathbb{R}^3) \) such that \( |g(x)| \leq M(1 + |x|)^{-A}(1 + s(x))^{-B} \). Then there exists \( C = C(A, B, \lambda) > 0 \) with the following properties:

1. If \( A + \min\{1, B\} > 3 \), then

\[
|\Gamma^\lambda_0 * g(x)| \leq C M \left[ (1 + |x|)(1 + s(\lambda x)) \right]^{-1}.
\]

2. If \( A + \min\{1, B\} > 3 \) and \( A + B \geq 7/2 \), then

\[
|\nabla \Gamma^\lambda_0 * g(x)| \leq C M \left[ (1 + |x|)(1 + s(\lambda x)) \right]^{-3/2}.
\]
3 If \( A + \min\{1, B\} = 3 \) and \( A + B \geq 7/2 \), then
\[
|\nabla \Gamma_0^\lambda| \ast g(x) \leq C M \left[(1 + |x|)(1 + s(\lambda x))\right]^{-3/2} \max\{1, \log |x|\}.
\]

4 If \( A + B < 3 \), then
\[
|\nabla \Gamma_0^\lambda| \ast g(x) \leq C M (1 + |x|)^{-\frac{(A+B)/2}{2}} (1 + s(\lambda x))^{-(A+B-1)/2}.
\]

**Proof.** These are special cases of [13] Theorems 3.1 and 3.2. \(\square\)

In order to derive a similar result to control convolutions with the purely periodic part \(\Gamma_0^\lambda\), we recall the following theorem established in [4].

**Theorem 3.2.** The purely periodic velocity fundamental solution \(\Gamma_0^\lambda\) satisfies
\[
\forall q \in \left(1, \frac{5}{3}\right) : \quad \Gamma_0^\lambda \in L^q(G)^{3 \times 3}, \quad (3.5)
\]
\[
\forall q \in \left[1, \frac{5}{4}\right] : \quad \partial_j \Gamma_0^\lambda \in L^q(G)^{3 \times 3} \quad (j = 1, 2, 3), \quad (3.6)
\]
and for all \(\alpha \in \mathbb{N}_0^3\), \(r \in [1, \infty)\) and \(\varepsilon > 0\) there exists \(C > 0\) such that
\[
\forall |x| \geq \varepsilon : \quad \|D^\alpha \Gamma_0^\lambda(\cdot, x)\|_{L^r(\mathbb{T})} \leq C|x|^{-3|\alpha|}. \quad (3.7)
\]

**Proof.** See [4, Theorem 1.1]. \(\square\)

From these properties we conclude the following theorem.

**Theorem 3.3.** Let \( A \in (0, \infty) \) and \( g \in L^\infty(\mathbb{T} \times \mathbb{R}^3)\) such that \(|g(t, x)| \leq M(1 + |x|)^{-A}\). Then for any \(\varepsilon > 0\) there exists \(C = C(A, \lambda, \mathcal{T}, \varepsilon) > 0\) such that
\[
\forall |x| \geq \varepsilon : \quad |\nabla \Gamma_0^\lambda| \ast g(t, x) \leq CM(1 + |x|)^{-\min\{A, A\}}. \quad (3.8)
\]
and, if \( A > 3 \),
\[
\forall |x| \geq \varepsilon : \quad |\Gamma_0^\lambda| \ast g(t, x) \leq CM(1 + |x|)^{-3}. \quad (3.9)
\]

**Proof.** Let us focus on the derivation of (3.5). Let \( x \in \mathbb{R}^3 \), \( |x| \geq \varepsilon \) and set \( R := \varepsilon/2 \). Then we have
\[
|\nabla \Gamma_0^\lambda| \ast g(t, x) \leq M(I_1 + I_2 + I_3)
\]
with
\[
I_1 = \int_T \int_{J_{BR}} |\Gamma_0^\lambda(t - s, x - y) - (1 + |y|)^{-A} dy| ds,
\]
\[
I_2 = \int_T \int_{J_{BR}} |\Gamma_0^\lambda(t - s, x - y)| (1 + |y|)^{-A} dy| ds,
\]
\[
I_3 = \int_T \int_{J_{BR}} |\Gamma_0^\lambda(t - s, x - y)| (1 + |y|)^{-A} dy| ds.
\]

We estimate these terms separately. Since \(|y| \leq R\) implies \(|x - y| \geq |x| - |y| \geq |x|/2 = R \geq \varepsilon/2\), we can use (3.7) to estimate
\[
I_1 \leq C \int_{BR} |x - y|^{-3}(1 + |y|)^{-A} dy \leq C|x|^{-3} \int_{\mathbb{R}^3} (1 + |y|)^{-A} dy \leq C|x|^{-3}.
\]
For $I_3$ we note that $|y| \geq 4R$ implies $|x - y| \geq |y| - |x| \geq |y| - |y|/2 = |y|/2 \geq 2R \geq \varepsilon$. Therefore, (3.7) yields

$$I_2 \leq C \int_{B_{4R}} |x - y|^{-3}(1 + |y|)^{-A} \, dy \leq C \int_{B_{4R}} |y|^{-3}|y|^{-A} \, dy \leq C|x|^{-A}. $$

Furthermore, Hölder’s inequality with $q \in (1, \frac{5}{3})$ and $q' = q/(q - 1)$ implies

$$I_3 \leq |x|^{-A}\left( \int_\mathbb{T} \int_{B_{4R,4R}} 1 \, dyds \right)^{1/q'} \|\Gamma_\perp \|_q \leq C|x|^{-A}|x|^{3-\frac{3}{q}}$$

in virtue of (3.5). We now choose $q \in (1, \frac{5}{3})$ so small that $-A + 3 - \frac{3}{q} < -3$. Collecting these estimates, we obtain (3.9). A proof of (3.8) can be given in a similar way.

The next lemma can be used to conclude asymptotic expansions in the linear case, where the velocity field is given by $u = \Gamma^\lambda * f$.

**Lemma 3.4.** Let $\lambda \neq 0$ and $f \in C_0^\infty(\mathbb{T} \times \mathbb{R}^3)$ with supp $f \subset \mathbb{T} \times B_{R_0}$. Let $|\alpha| \leq 1$. Then

$$|D_x^\alpha \Gamma_0^\lambda * P f(x)| \leq C[(1 + |x|)(1 + s(\lambda x))]^{-1-|\alpha|/2}, \quad (3.10)$$

and for $|x| \geq 2R_0$ we have

$$|D_x^\alpha \Gamma_1^\lambda * P f(x)| \leq C[(1 + |x|)(1 + s(\lambda x))]^{-3-|\alpha|/2}, \quad (3.11)$$

and

$$|D_x^\alpha \Gamma_0^\lambda * P f(x) - D_x^\alpha \Gamma_0^\lambda(x) \cdot \int_{\mathbb{R}^3} P f(y) \, dy| \leq C[(1 + s(\lambda x))]^{-3-|\alpha|/2}, \quad (3.12)$$

and

$$|D_x^\alpha \Gamma_1^\lambda * P f(x) - (D_x^\alpha \Gamma_1^\lambda(x,x) * \mathcal{P}) \int_{\mathbb{R}^3} P f(y) \, dy| \leq C|x|^{-4-|\alpha|}. \quad (3.13)$$

**Proof.** Estimates (3.10) and (3.11) directly follow from Theorem 3.1 and Theorem 3.3. By the mean value theorem, we further have

$$|D_x^\alpha \Gamma_0^\lambda * P f(x) - D_x^\alpha \Gamma_0^\lambda(x) \int_{\mathbb{R}^3} P f(y) \, dy| \leq \int_{B_{\mathbb{R}}} \int_0^1 |y||\nabla D_x^\alpha \Gamma_0^\lambda(x - \theta y)||P f(y)| \, d\theta \, dy. $$

Since $|y| \leq R_0 \leq |x|/2$ implies

$$|x - \theta y| \geq |x| - \theta |y| \geq |x|/2 \geq R_0,$$

$$(1 + 2|\lambda|R_0)(1 + s(\lambda(x - \theta y))) \geq 1 + 2|\lambda|R_0 + s(\lambda(x - \theta y)) \geq 1 + s(\lambda x),$$

estimate (3.4) finally leads to (3.12). Using (3.7) instead of (3.4), we conclude (3.13) in the same way.

The following auxiliary result treats convolutions of functions with anisotropic decay.

**Lemma 3.5.** Let $A \in (-2, 2)$, $B \in (1, 2]$. Then there exists $C = C(A, B) > 0$ such that for all $x \in \mathbb{R}^3 \setminus \{0\}$ it holds

$$\int_{\mathbb{R}^3} \left[ (1 + |x - y|)(1 + s(x - y)) \right]^{-2}(1 + |y|)^{-A}(1 + s(y))^{-B} \, dy \leq C(1 + |x|)^{-A}(1 + s(x))^{-B} \max\left\{1, \log \left( \frac{|x|}{1 + s(x)} \right) \right\}. $$

**Proof.** This is a consequence of the calculations in [13, Section 2].
4 Main results

We consider weak solutions to (1.1) in the following sense.

**Definition 4.1.** Let \( f \in L^1_{\text{loc}}(T \times \mathbb{R}^3)^3 \). A function \( u \in L^1_{\text{loc}}(T \times \mathbb{R}^3)^3 \) is called weak solution to (1.1) if

i. \( u \in L^2(T; D^{1/2}_{0,\sigma}(\mathbb{R}^3)) \),

ii. \( \mathcal{P}_u u \in L^\infty(T; L^2(\mathbb{R}^3))^3 \),

iii. the identity

\[
\int_{T \times \mathbb{R}^3} \left[ -u \cdot \partial_t \varphi + \nabla u : \nabla \varphi - \lambda \partial_t u \cdot \varphi + (u \cdot \nabla u) \cdot \varphi \right] \, d(t, x) = \int_{T \times \mathbb{R}^3} f \cdot \varphi \, d(t, x)
\]

holds for all test functions \( \varphi \in C_0^{\infty}(T \times \mathbb{R}^3) \).

**Remark 4.2.** The existence of a weak solution with the above properties has been shown in [14, Theorem 6.3.1] for any \( f \in L^2(T; D^{1/2}_{0,\sigma}(\mathbb{R}^3))^3 \). Therefore, this class seems to be a natural outset for further investigation. Nevertheless, at first glance, instead of ii. one would expect the condition \( u \in L^\infty(T; L^2(\Omega))^3 \), which naturally appears for weak solutions to the Navier–Stokes initial-value problem. However, this property cannot be expected for general time-periodic data \( f \). As was shown by K\( \text{YED} \) [14, Theorem 5.2.4], for smooth data \( f \in C^\infty_0(T \times \mathbb{R}^3)^3 \) one has \( u \in L^\infty(T; L^2(\mathbb{R}^3))^3 \) if and only if \( \int_{T \times \mathbb{R}^3} f \, d(t, x) = 0 \). An analogous property was established by F\( \text{INN} \) [7] for the corresponding steady-state problem.

As our main result, we establish the following asymptotic expansions.

**Theorem 4.3.** Let \( \lambda \neq 0 \) and \( f \in C^\infty_0(T \times \mathbb{R}^3)^3 \), and let \( u \) be a weak time-periodic solution to (1.1) in the sense of Definition 4.1 which satisfies

\[
\exists r \in (5, \infty) : \quad \mathcal{P}_u u \in L^r(T \times \mathbb{R}^3)^3.
\]

Then

\[
\mathcal{P}_u u(x) = \Gamma_0^\lambda(x) \cdot \int_\Omega \mathcal{P} f(y) \, dy + \mathcal{R}_0(x),
\]

\[
\mathcal{P}_u u(t, x) = \Gamma_{\perp}^\lambda(t, \cdot) \ast T \mathcal{P} f(\cdot, y) \, dy + \mathcal{R}_{\perp}(t, x)
\]

such that there exists \( C > 0 \) such that for all \( t \in T \) and \( |x| \geq 4 \) it holds

\[
|\mathcal{R}_0(x)| \leq C |x| \left( 1 + s(\lambda x) \right)^{-3/2} \log |x|,
\]

\[
|\nabla \mathcal{R}_0(x)| \leq C |x| \left( 1 + s(\lambda x) \right)^{-2} \max \left\{ 1, \log \left( \frac{|x|}{1 + s(\lambda x)} \right) \right\},
\]

\[
|\mathcal{R}_{\perp}(t, x)| \leq C |x|^{-4},
\]

\[
|\nabla \mathcal{R}_{\perp}(t, x)| \leq C |x|^{-9/2} (1 + s(\lambda x))^{-1/2}.
\]

In particular,

\[
u(t, x) = \Gamma_0^\lambda(x) \cdot \int_T \int_\Omega f(t, y) \, dy \, dt + \mathcal{R}(t, x)
\]
Proof. An application of the Fourier transform is a polynomial in each component and

\[ |\mathcal{F}(t, x)| \leq C \left[ |x| \left( 1 + s(\lambda x) \right) \right]^{-3/2} \log |x|, \quad (4.9) \]

\[ |\nabla \mathcal{F}(t, x)| \leq C \left[ |x| \left( 1 + s(\lambda x) \right) \right]^{-2} \max \left\{ 1, \log \left( \frac{|x|}{1 + s(\lambda x)} \right) \right\}. \quad (4.10) \]

Lemma 4.7. Let \( \mathcal{F} \) be a solution as in Theorem 4.5. Then \( \mathcal{F} \) satisfies

\[ |\mathcal{F}(t, x)| \leq C \left[ (1 + |x|) \left( 1 + s(\lambda x) \right) \right]^{-1}, \quad (4.11) \]

\[ |\nabla \mathcal{F}(t, x)| \leq C \left[ (1 + |x|) \left( 1 + s(\lambda x) \right) \right]^{-3/2}, \quad (4.12) \]

\[ |\mathcal{F}_{\perp}(t, x)| \leq C \left( 1 + |x| \right)^{-3}, \quad (4.13) \]

\[ |\nabla \mathcal{F}_{\perp}(t, x)| \leq C \left( 1 + |x| \right)^{-4}. \quad (4.14) \]

Remark 4.4. As explained in [12], assumption (4.1) merely appears for technical reasons. It ensures additional local regularity but does not improve spatial decay of the solution.

One main observation is that the asymptotic behavior of \( u \) and \( \nabla u \) for \( |x| \to \infty \) is governed by the time-periodic Oseen fundamental solution \( \mathcal{F}_{\perp} \). In particular, the purely periodic part \( \mathcal{F}_{\perp} \) decays faster than the steady-state part \( \mathcal{F} \). As a direct consequence of Theorem 4.3, we obtain the following pointwise estimates, which we shall derive as intermediate results on the way to a proof of Theorem 4.3.

Theorem 4.5. Under the assumptions of Theorem 4.5 there is \( C > 0 \) such that for all \( t \in \mathbb{T} \) and \( x \in \mathbb{R}^3 \) the function \( u \) satisfies

\[ |\mathcal{F}(t, x)| \leq C \left[ (1 + |x|) \left( 1 + s(\lambda x) \right) \right]^{-1}, \quad (4.11) \]

\[ |\nabla \mathcal{F}(t, x)| \leq C \left[ (1 + |x|) \left( 1 + s(\lambda x) \right) \right]^{-3/2}, \quad (4.12) \]

\[ |\mathcal{F}_{\perp}(t, x)| \leq C \left( 1 + |x| \right)^{-3}, \quad (4.13) \]

\[ |\nabla \mathcal{F}_{\perp}(t, x)| \leq C \left( 1 + |x| \right)^{-4}. \quad (4.14) \]

In order to prove these theorems, we recall the following regularity result.

Lemma 4.6. Let \( u \) be a weak solution as in Theorem 4.3 Then \( u \in C^\infty(\mathbb{T} \times \mathbb{R}^3)^3 \) and

\[ \forall r \in (1, \infty), q \in (1, 2) : \nabla^2 \mathcal{F} \in L^r(\mathbb{R}^3), \quad \nabla \mathcal{F} \in L^{4q}(\mathbb{R}^3), \quad \mathcal{F} \in L^{4q}(\mathbb{R}^3), \]

\[ \forall q \in (1, \infty) : \mathcal{F}_{\perp} \in L^{q}(\mathbb{T} ; W^{2,q}(\mathbb{R}^3)) \cap W^{1,q}(\mathbb{T} ; L^q(\mathbb{R}^3)), \]

and there is a pressure function \( p \in C^\infty(\mathbb{T} \times \mathbb{R}^3) \) such that (1.1) is satisfied pointwise.

Proof. We refer to [12, Lemma 5.1]. \( \square \)

We also need a uniqueness statement for solutions to the linear problem (1.4).

Lemma 4.7. Let \( (u, p) \in \mathcal{S}^\prime(G)^{3+1} \) be a solution to (1.4) for the right-hand side \( f = 0 \). Then, \( \mathcal{F} \) is a polynomial in each component and \( \mathcal{F}_{\perp} = 0 \).

Proof. An application of the Fourier transform \( \mathcal{F} \) on \( G \) to (1.4) yields

\[ (i\frac{2\pi}{T} k + |\xi|^2 - i\lambda \xi_1 ) \hat{u} + i\xi \hat{p} = 0 \]

with \( \hat{u} := \mathcal{F} \) and \( \hat{p} := \mathcal{F} \). Multiplying this equation with \( i\xi \) and using \( \nabla u = 0 \), we obtain

\[ -|\xi|^2 \hat{p} = 0, \quad \text{so that supp} \hat{p} \subset \mathbb{Z} \times \{0\}. \]

Then, the above equation yields

\[ \text{supp} \left[ (i\frac{2\pi}{T} k + |\xi|^2 - i\lambda \xi_1 ) \hat{u} \right] = \text{supp} \left[ -i\xi \hat{p} \right] \subset \mathbb{Z} \times \{0\}. \]

Because the only zero of \( (k, \xi) \mapsto (i\frac{2\pi}{T} k + |\xi|^2 - i\lambda \xi_1 ) \) is \( (k, \xi) = (0, 0) \), we conclude \( \text{supp} \hat{u} \subset \{ (0, 0) \} \). Thus we obtain \( \mathcal{F}_{\perp} = 0 \) and that \( \mathcal{F} \) is a polynomial. \( \square \)
These lemmas enable us to derive the following representation formulas.

**Proposition 4.8.** Let \( u \) be a weak solution as in Theorem 4.3. Then

\[
D^\alpha_x u = D^\alpha_x \Gamma^\lambda \ast [f - u \cdot \nabla u] \tag{4.15}
\]

for all \( \alpha \in \mathbb{N}_0^3 \) with \( |\alpha| \leq 1 \). In particular, \( v := \mathcal{P} u \) and \( w := \mathcal{P}_\perp u \) satisfy

\[
D^\alpha_x v = D^\alpha_x \Gamma^\lambda \ast [\mathcal{P} f - v \cdot \nabla v - \mathcal{P}(w \cdot \nabla w)],
\]

\[
D^\alpha_x w = D^\alpha_x \Gamma^\lambda \ast [\mathcal{P}_\perp f - v \cdot \nabla w - w \cdot \nabla v - \mathcal{P}_\perp (w \cdot \nabla w)].
\]

Moreover, we have\(^1\)

\[
u = \Gamma^\lambda \ast f - \nabla \Gamma^\lambda \ast (u \otimes u),
\]

\[
v = \Gamma^\lambda \ast [v \otimes v + \mathcal{P}(w \otimes w)],
\]

\[
w = \Gamma^\lambda \ast [v \otimes w + w \otimes v + \mathcal{P}_\perp (w \otimes w)].
\]

**Proof.** From Lemma 4.6 we conclude \( u \cdot \nabla u \in L^q(T \times \mathbb{R}^3) \) for all \( q \in (1, \infty) \). Therefore, \( U := \Gamma^\lambda \ast (f - u \cdot \nabla u) \) is well defined as a classical convolution integral, and we have \( \partial_j U = \partial_j \Gamma^\lambda \ast (f - u \cdot \nabla u) \) for \( j = 1, 2, 3 \) by the dominated convergence theorem. Since both \( U \) and \( u \) satisfy the time-periodic Oseen system (1.4) for suitable pressure functions \( p \), Lemma 4.7 implies \( \mathcal{P}_\perp u = \mathcal{P}_\perp U \) and that \( \mathcal{P} u - \mathcal{P}_\perp U \) is a polynomial in each component. With Young's inequality we obtain \( \mathcal{P} U \in L^6(\mathbb{R}^3) \) since \( \Gamma^\lambda \in \mathcal{L}^{12/5}(\mathbb{R}^3) \) by [12, Lemma 5.4]. Hence, \( \mathcal{P} u - \mathcal{P}_\perp U \in L^6(\mathbb{R}^3) \). This leads to \( \mathcal{P} u = \mathcal{P}_\perp U \) and thus \( u = U \), which yields (4.15). The remaining formulas now follow from

\[
v = \mathcal{P} u = (\Gamma^\lambda_0 \otimes 1_\mathbb{R}) \ast [f - u \cdot \nabla u] = \Gamma^\lambda_0 \ast [\mathcal{P} (f - u \cdot \nabla u)],
\]

\[
w = \mathcal{P}_\perp u = \Gamma^\lambda_3 \ast [f - u \cdot \nabla u] = \Gamma^\lambda_3 \ast [\mathcal{P}_\perp (f - u \cdot \nabla u)]
\]

together with the identity \( u \cdot \nabla u = \text{div}(u \otimes u) \) and integration by parts. \( \square \)

Based on these formulas, we can now prove Theorem 4.5 and Theorem 4.3.

**Proof of Theorem 4.5.** We split \( u = v + w \) into steady-state part \( v := \mathcal{P} u \) and purely periodic part \( w := \mathcal{P}_\perp u \). By [12, Theorem 2.2] we have (4.8) with \( |\mathcal{H}(t, x)| \leq C|x|^{-5/4} \). In virtue of (3.4) and \( u \in C^\infty(T \times \mathbb{R}^3)^3 \), this implies

\[
|v(x)| \leq C(1 + |x|)^{-1}(1 + s(\lambda x))^{-1/4},
\]

\[
|w(t, x)| \leq C(1 + |x|)^{-5/4}
\]

for all \( t \in T \) and \( x \in \mathbb{R}^3 \). This leads to

\[
|v \otimes v + \mathcal{P}[w \otimes w]|(x) \leq C(1 + |x|)^{-2}(1 + s(\lambda x))^{-1/2}.
\]

Therefore, (4.19), (3.10) and Theorem 3.1 yield

\[
|v(x)| \leq |\Gamma^\lambda_0 \ast \mathcal{P} f|(x) + |\nabla \Gamma^\lambda_0 \ast [v \otimes v + \mathcal{P}[w \otimes w]]|(x) \leq C[(1 + |x|)(1 + s(\lambda x))]^{-1},
\]

which is the desired estimate (4.11). Now (4.11) together with (4.22) leads to

\[
|v \otimes w + w \otimes v + \mathcal{P}_\perp [w \otimes w]|(t, x) \leq C(1 + |x|)^{-9/4}.
\]

\(^1\) Here we set \( (\nabla \Gamma^\lambda \ast U)_j := \delta_{jm} \Gamma^\lambda_{ij} \ast U_{jm} \) for an \( \mathbb{R}^{3 \times 3} \)-valued function \( U \).
Therefore, (4.20), (3.11) and Theorem 3.3 imply
\[
|w(t, x)| \leq |\Gamma_0^\lambda * \mathcal{P}_\perp f(t, x) + \nabla \Gamma_0^\lambda * [v \otimes w + w \otimes v + \mathcal{P}_\perp [w \otimes w]](t, x) |
\leq C \left( (1 + |x|)^{-3} + (1 + |x|)^{-9/4} \right) \leq C(1 + |x|)^{-9/4}.
\]

Using this estimate and (4.11) again, we conclude
\[
|v \otimes w + w \otimes v + \mathcal{P}_\perp [w \otimes w]|(t, x) \leq C(1 + |x|)^{-13/4}.
\tag{4.24}
\]

Repeating the above argument with (4.24) instead of (4.23), we end up with (4.13).

Now let us turn to the estimates of \( \nabla u \). Due to \( u \in C^\infty(T \times \mathbb{R}^3) \), it suffices to consider \( |x| \geq 2 \). Let \( R := |x|/2 \geq 1 \). By Proposition 4.8 we have
\[
\partial_j v = \partial_j \Gamma_0^\lambda * \mathcal{P} f - I, \quad \partial_j w = \partial_j \Gamma_0^\lambda * \mathcal{P}_\perp f - J
\]
with
\[
I := I_1 + I_2 := \partial_j \Gamma_0^\lambda * [v \cdot \nabla v] + \partial_j \Gamma_0^\lambda * [\mathcal{P}[w \cdot \nabla w]],
\]
\[
J := J_1 + J_2 + J_3 := \partial_j \Gamma_0^\lambda * [v \cdot \nabla w] + \partial_j \Gamma_0^\lambda * [w \cdot \nabla v] + \partial_j \Gamma_0^\lambda * [\mathcal{P}_\perp [w \cdot \nabla w]].
\]

We estimate these terms separately. Clearly, \(|I_1| \leq I_{11} + I_{12}\) with
\[
I_{11}(x) := \int_{B_R} |\partial_j \Gamma_0^\lambda (x - y)| |v(y)||\nabla v(y)| \, dy,
\]
\[
I_{12}(x) := \int_{B_R} |\partial_j \Gamma_0^\lambda (x - y)| |v(y)||\nabla v(y)| \, dy.
\]

Since \(|y| \leq R\) implies \(|x - y| \geq |x|/2 = R \geq 1\), the pointwise estimate (3.4) implies
\[
I_{11}(x) \leq \int_{B_R} \left[ (1 + |x-y|)(1 + s(x-y)) \right]^{-3/2} |v(y)| ||\nabla v(y)|| \, dy
\leq C(1 + |x|)^{-3/2} \|v\|_3 \|\nabla v\|_2 \leq C(1 + |x|)^{-3/2}
\]
in view of Lemma 4.6 and \( \nabla \Gamma_0^\lambda \in L^{17/12}(\mathbb{R}^3) \) (see [12, Lemma 5.4]) and Lemma 4.6 yield
\[
I_{12}(x) \leq C \|\partial_j \Gamma_0^\lambda\|_{L^{17/12}} \|\nabla v\|_{L^{17/12}} \|v\|_{L^\infty(B_R)} \leq C(1 + |x|)^{-1}
\]
by (4.11). We thus deduce \(|I_1(x)| \leq C(1 + |x|)^{-1}\). For \( I_2 \) we proceed similarly to obtain \(|I_2(x)| \leq (1 + |x|)^{-3/2}\). From these estimates and (3.12), we conclude
\[
|\nabla v(x)| \leq C(1 + |x|)^{-1}.
\tag{4.25}
\]

Now let us turn towards \( \nabla w \). As above, we split \( J_1 \) and estimate \(|J_1| \leq J_{11} + J_{12}\) with
\[
J_{11}(t, x) := \int_T \int_{B_R} |\partial_j \Gamma_0^\lambda (t - s, x - y)| |v(y)||\nabla w(s, y)| \, dyds,
\]
\[
J_{12}(t, x) := \int_T \int_{B_R} |\partial_j \Gamma_0^\lambda (t - s, x - y)| |v(y)||\nabla w(s, y)| \, dyds.
\]

By Hölder’s inequality in space and time, from (3.7) we obtain
\[
J_{11}(t, x) \leq C \left( \int_{B_R} |x - y|^{-8} \, dy \right)^{1/2} \|v\|_4 \|\nabla w\|_4 \leq C|x|^{-5/2}
\]

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due to Lemma 4.6. Moreover, Hölder's inequality and (4.11) lead to
\[ J_{12}(t,x) \leq C \| \partial_j \Gamma^j \|_1 \| v \|_{L_\infty(B_r)} \| \nabla w \|_{\infty} \leq C |x|^{-1} \]
because \( \nabla \Gamma^j \in L^1(T \times \mathbb{R}^3) \) by (3.3) and \( \nabla w \in L^\infty(T \times \mathbb{R}^3) \) by Lemma 4.6 and Sobolev embeddings. In a similar fashion, we can use (4.13) to estimate \( J_2 \) and \( J_3 \) and obtain
\[
\begin{align*}
|J_2(t,x)| & \leq C \left( |x|^{-\frac{5}{2}} \| w \|_4 \| \nabla v \|_4 + |x|^{-3} \| \partial_j \Gamma^j \|_2 \| \nabla v \|_9 \right) \leq C |x|^{-\frac{5}{2}}, \\
|J_3(t,x)| & \leq C \left( |x|^{-\frac{5}{2}} \| w \|_4 \| \nabla w \|_4 + |x|^{-3} \| \partial_j \Gamma^j \|_1 \| \nabla w \|_{\infty} \right) \leq C |x|^{-\frac{5}{2}}.
\end{align*}
\]
Collecting the above estimates and combining them with (3.11), we end up with
\[
|\nabla w(t,x)| \leq C (1 + |x|)^{-1}.
\] (4.26)
From (4.11), (4.25), (4.13) and (4.26) we now conclude
\[
|v(x) \cdot \nabla v(x) + \mathcal{P}[w \cdot \nabla w](x)| \leq C (1 + |x|)^{-2} (1 + s(\lambda x))^{-1/2},
\]
so that
\[
|I(x)| \leq C (1 + |x|)^{-5/4} (1 + s(\lambda x))^{-3/4}
\]
by Theorem 3.1. Together with (3.10) we thus obtain
\[
|\nabla v(x)| \leq C (1 + |x|)^{-5/4} (1 + s(\lambda x))^{-3/4},
\]
so that from (4.11), (4.13) and (4.26) we deduce
\[
|v(x) \cdot \nabla v(x) + \mathcal{P}[w \cdot \nabla w](x)| \leq C (1 + |x|)^{-9/4} (1 + s(\lambda x))^{-7/4}.
\]
By another application of Theorem 3.1 and combination with (3.10), we arrive at (4.12).
For the derivation of (4.14) we proceed with a similar bootstrap argument. From (4.11), (4.12), (4.13) and (4.26) we deduce
\[
|v \cdot \nabla w + w \cdot \nabla v + \mathcal{P}_\perp [w \cdot \nabla w](t,x) \leq C (1 + |x|)^{-2},
\]
so that Theorem 3.3 implies \( |J(t,x)| \leq C (1 + |x|)^{-2} \). Combining this with (3.11), we conclude
\[
|\nabla w(t,x)| \leq C (1 + |x|)^{-2}.
\] (4.27)
We now repeat this argument with (4.27) instead of (4.26), which leads to an improved decay rate for \( \nabla w \). Iterating this procedure, we finally arrive at (4.14). \( \square \)

**Proof of Theorem 4.3**  We keep the notation from the previous proof. We have
\[
|v \otimes v + \mathcal{P}[w \otimes w](x)| \leq C \left[ (1 + |x|) (1 + s(\lambda x)) \right]^{-2}
\] (4.28)
by Theorem 4.5, which, by Theorem 3.1, implies
\[
\left| \partial_j \Gamma^j_0 * (v \otimes v + \mathcal{P}[w \otimes w])(x) \right| \leq C \left[ (1 + |x|) (1 + s(\lambda x)) \right]^{-3/2} \log |x|.
\]
In virtue of the representation formula (4.19) and the identity
\[
\mathcal{R}_0(x) = \mathcal{P} \mathcal{R}(x) = v(x) - \Gamma^j_0(x) \int_{\mathbb{R}^3} \mathcal{P} f(y) \, dy,
\]
Furthermore, Theorem 4.5 implies (3.12) and (4.4). Moreover, by Theorem 4.5 we have
\[ |v \otimes w + w \otimes v + \mathcal{P}_\perp [w \otimes w](t,x) | \leq C(1 + |x|)^{-4}, \]
so that
\[ |\partial_j \Gamma^\lambda \ast [v \otimes w + w \otimes v + \mathcal{P}_\perp [w \otimes w]](t,x) | \leq C(1 + |x|)^{-4} \]
by Theorem 3.3. Now (4.6) is a consequence of this estimate and (3.13).
To show (4.5), at first observe that Theorem 4.5 implies (4.29).
\[ |v \cdot \nabla v(x) + \mathcal{P}[w \cdot \nabla w](x) | \leq C[(1 + |x|)(1 + s(\lambda x))]^{-5/2}. \]  
(4.29)
Let \( \chi \in C^\infty_0(\mathbb{R}^3) \) such that \( \chi(x) = 1 \) for \( |x| \leq 1 \) and \( \chi(x) = 0 \) for \( |x| \geq 2 \). We decompose
\[ I = [\chi \partial_j \Gamma_0^\lambda] \ast [v \cdot \nabla v + \mathcal{P}[w \cdot \nabla w]] + [(1 - \chi)\partial_j \Gamma_0^\lambda] \ast [v \cdot \nabla v + \mathcal{P}[w \cdot \nabla w]] =: K_1 + K_2. \]
Then
\[ |K_1| \leq C \int_{B_2(x)} |\partial_j \Gamma_0^\lambda(x - y)| [(1 + |y|)(1 + s(\lambda y))]^{-5/2} dy \]
by (4.29). As in the proof of Lemma 3.4 from \( |x - y| \leq 2 \leq |x|/2 \) we conclude \( |y| \geq |x|/2 \geq 2 \) and \( (1 + 4|\lambda|)(1 + s(\lambda y)) \geq 1 + s(\lambda x) \). Since \( \nabla \Gamma_0^\lambda \in L^1_{\text{loc}}(\mathbb{R}^3) \), this implies
\[ |K_1| \leq C[(1 + |x|)(1 + s(\lambda x))]^{-5/2} \int_{B_2(x)} |\partial_j \Gamma_0^\lambda(x - y)| dy \leq C[(1 + |x|)(1 + s(\lambda x))]^{-5/2}. \]
By integration by parts and (3.4) and (4.28), we further obtain
\[ |K_2| \leq C \int_{\mathbb{R}^3} |1 - \chi(x - y)| |\partial_j \nabla \Gamma_0^\lambda(x - y)| |v \otimes v + \mathcal{P}[w \otimes w](y)| dy \]
\[ + C \int_{\mathbb{R}^3} |\nabla \chi(x - y)||\partial_j \Gamma_0^\lambda(x - y)| |v \otimes v + \mathcal{P}[w \otimes w](y)| dy \]
\[ \leq C \int_{B_1(x)} [(x - y)(1 + s(\lambda(x - y)))^{-2}[(1 + |y|)(1 + s(\lambda y))]^{-2} dy \]
\[ + C \int_{B_1(x)} [(x - y)(1 + s(\lambda(x - y)))]^{-3/2}[(1 + |y|)(1 + s(\lambda y))]^{-2} dy. \]
For the first integral we use Lemma 3.5 and for the second one we argue as for \( K_1 \) to deduce
\[ |K_2| \leq C((1 + |x|)(1 + s(\lambda x)))^{-2} \max \left\{ 1, \log \left( \frac{|x|}{1 + s(\lambda x)} \right) \right\} \]
\[ + C[(1 + |x|)(1 + s(\lambda x))]^{-2} \int_{B_1(x)} [(x - y)(1 + s(\lambda(x - y)))^{-2} dy \]
\[ \leq C((1 + |x|)(1 + s(\lambda x)))^{-2} \max \left\{ 1, \log \left( \frac{|x|}{1 + s(\lambda x)} \right) \right\}. \]
Combining the estimates of \( K_1 \) and \( K_2 \) with (3.12), from formula (4.16) we obtain (4.5). Furthermore, Theorem 4.5 implies
\[ |v \cdot \nabla w + w \cdot \nabla v + \mathcal{P}_\perp [w \cdot \nabla w]|(t,x) \leq C(1 + |x|)^{-9/2}(1 + s(\lambda x))^{-3/2}. \]
With an argument similar to before, we now deduce

\[
|J(t, x)| \leq C \int_{B_1(x)} |\partial_j \Gamma_{\perp}^\lambda (x - y)| (1 + |y|)^{-9/2} (1 + s(\lambda y))^{-3/2} \, dy \\
+ C \int_{B_1(x)} |\partial_j \Gamma_{\perp}^\lambda (x - y)| (1 + |y|)^{-9/2} (1 + s(\lambda y))^{-3/2} \, dy \\
\leq C \left( (1 + |x|)^{-9/2} (1 + s(\lambda x))^{-3/2} \|\partial_j \Gamma_{\perp}^\lambda\|_1 + \int_{B_1(x)} |x - y|^{-4} (1 + |y|)^{-9/2} \, dy \right) \\
\leq C (1 + |x|)^{-9/2} (1 + s(\lambda x))^{-1},
\]

where we used (3.6). Combining this estimates with (3.13), formula (4.17) implies (4.7).

Finally, the asymptotic expansion (4.8) with the asserted estimates of \( \mathcal{R}(t, x) \) is a direct consequence of these results and the pointwise estimates of \( I_{\perp}^\lambda \) from (3.7).

\[ \square \]

References


