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Optimal control of a phase field system of Caginalp type with fractional operators

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Abstract

In their recent work "Well-posedness, regularity and asymptotic analyses for a fractional phase field system" (*Asymptot. Anal.* **114** (2019), 93–128), two of the present authors have studied phase field systems of Caginalp type, which model nonconserved, nonisothermal phase transitions and in which the occurring diffusional operators are given by fractional versions in the spectral sense of unbounded, monotone, selfadjoint, linear operators having compact resolvents. In this paper, we complement this analysis by investigating distributed optimal control problems for such systems. It is shown that the associated control-to-state operator is Fréchet differentiable between suitable Banach spaces, and meaningful first-order necessary optimality conditions are derived in terms of a variational inequality and the associated adjoint state variables.

1 Introduction

The *Caginalp phase field model* is a well-known system of partial differential equations modeling the evolution of a temperature-dependent phase transition with nonconserved order parameter φ that takes place in a container $\Omega \subset \mathbb{R}^3$. A classical form that was introduced and analyzed in the seminal paper [4] (see also, e.g., [5–7]) is given by the evolutionary system

$$\rho\, C_V\, \partial_t \vartheta \,+\, \ell\, \partial_t \varphi \,-\, \kappa\, \Delta\vartheta \,=\, u,$$
$$\alpha\, \xi^2\, \partial_t \varphi \,-\, \xi^2\, \Delta\varphi \,+\, F'(\varphi) \,=\, 2\,\vartheta,$$

which is to be satisfied in the set $Q := \Omega \times (0,T)$, where $T > 0$ is a given final time. The first equation in the above system is an approximation to the universal balance law of internal energy, while the second one governs the evolution of the order parameter. The quantities $\rho, C_V, \ell, \kappa, \alpha, \xi$ are positive physical constants; in particular, ℓ is closely allied to the latent heat released or absorbed during the phase transition process, and ξ is a measure for the thickness of the transition zone between the different phases. The unknowns ϑ and φ stand for a temperature difference and the order parameter (usually a normalized fraction of one of the phases involved in the phase transition), while u represents a control (a heat source or sink) and F is a double-well potential whose derivative F' is the thermodynamic force driving the phase transition. For a derivation of the model equations from general thermodynamic principles, we refer the reader to [3, Chapter 4].

In their recent paper [11], two of the present authors have studied a variation of the Caginalp model, namely the system

$$\partial_t \vartheta \,+\, \ell(\varphi)\partial_t \varphi \,+\, A^{2\rho}\vartheta \,=\, u \quad \text{in } Q, \tag{1.1}$$
$$\partial_t \varphi \,+\, B^{2\sigma}\varphi \,+\, F'(\varphi) \,=\, \vartheta\, \ell(\varphi) \quad \text{in } Q, \tag{1.2}$$
$$\vartheta(0) \,=\, \vartheta_0, \quad \varphi(0) \,=\, \varphi_0, \quad \text{in } \Omega. \tag{1.3}$$

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The main difference to the Caginalp system (besides the fact that many physical constants are normalized to unity and that the quantity ℓ representing the latent heat is allowed to depend on the order parameter φ) is given by the fact that in (1.1)–(1.2) the expressions $A^{2\rho}$ and $B^{2\sigma}$, with $\rho > 0$ and $\sigma > 0$, denote fractional powers in the spectral sense of self-adjoint, monotone, and unbounded linear operators A and B, respectively, which are supposed to be densely defined in $H := L^2(\Omega)$ and to have compact resolvents. The standard example occurs when $A^{2\rho} = B^{2\sigma} = -\Delta$, with zero Dirichlet or Neumann boundary conditions.

The nonlinearity ℓ is assumed to be a smooth function, while F denotes a double-well potential. Typical and physically significant examples are the so-called *classical regular potential*, the *logarithmic potential*, and the *double obstacle potential*, which are given, in this order, by

$$F_{\rm reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.4}$$

$$F_{\log}(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2\,, & r \in (-1,1) \\ 2\log(2) - c_1\,, & r \in \{-1,1\} \\ +\infty\,, & r \notin [-1,1] \end{cases}\,, \tag{1.5}$$

$$F_{2{\rm obs}}(r) := c_2(1-r^2) \quad \text{if } |r| \le 1, \quad \text{and} \quad F_{2{\rm obs}}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.6}$$

Here, the constants in (1.5) and (1.6) satisfy $c_1 > 1$ and $c_2 > 0$, so that the corresponding functions are nonconvex. In cases like (1.6), one has to split F into a nondifferentiable convex part F_1 (the indicator function of $[-1,1]$, in the present example) and a smooth perturbation F_2. Accordingly, in the term $F'(\varphi)$ appearing in (1.2), one has to replace the derivative F_1' of the convex part F_1 by the subdifferential ∂F_1 and interpret (1.2) as a differential inclusion or as a variational inequality involving F_1 rather than ∂F_1.

In [11], general results on well-posedness, regularity and asymptotic behavior have been proved for the state system (1.1)–(1.3). In this paper, we complement the analysis in [11] by studying the optimal control of this system. More precisely, given nonnegative constants β_i, $1 \le i \le 5$, target functions $\varphi_\Omega, \vartheta_\Omega \in L^2(\Omega)$ and $\varphi_Q, \vartheta_Q \in L^2(Q)$, as well as threshold functions $u_{min}, u_{max} \in L^\infty(Q)$ with $u_{min} \le u_{max}$ in Q, we consider the following optimal control problem:

(CP) Minimize the tracking-type cost functional

$$\begin{aligned} \mathfrak{J}((\varphi,\vartheta),u) := {}& \frac{\beta_1}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2 + \frac{\beta_2}{2}\int_0^T\!\!\int_\Omega |\varphi-\varphi_Q|^2 + \frac{\beta_3}{2}\int_\Omega |\vartheta(T)-\vartheta_\Omega|^2 \\ & + \frac{\beta_4}{2}\int_0^T\!\!\int_\Omega |\vartheta-\vartheta_Q|^2 + \frac{\beta_5}{2}\int_Q |u|^2 \end{aligned} \tag{1.7}$$

over the set of admissible controls

$$\mathcal{U}_{\rm ad} := \{u \in L^\infty(Q) : u_{min} \le u \le u_{max} \text{ a.e. in } Q\}\,, \tag{1.8}$$

subject to the state system (1.1)–(1.3).

The optimal control problem **(CP)** constitutes a generalization of investigations for the original Caginalp phase field system with regular potential $F_{\rm reg}$ that were begun in the early nineties of the past century; in this connection, we refer the reader to the pioneering works [8,29–32] (see also the related sections in the monograph [42]). For more recent contributions, we mention the papers [1,12,13,22,35], where in [35] a thermodynamically consistent version of the phase field system was considered. We also mention the papers [23, 33, 41] that were devoted to optimal control problems for the Penrose–Fife phase field model of phase transitions with nonconserving kinetics.

The problem **(CP)** can also be seen in comparison with a class of optimal control problems for Cahn–Hilliard type systems of the form

$$\alpha\partial_t\mu + \partial_t\varphi + A^{2\rho}\mu = 0, \tag{1.9}$$
$$\beta\partial_t\varphi + B^{2\sigma}\varphi + F'(\varphi) = \mu + u, \tag{1.10}$$
$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \tag{1.11}$$

where μ represents the chemical potential and $\alpha \geq 0$ and $\beta \geq 0$. Obviously, (1.1)–(1.3) is in the special case $\ell(\varphi) \equiv \ell > 0$ of the above type (put $\mu = \vartheta$, $\alpha = 1/\ell$ and $\beta = 1$), where, however, the control u appears in the phase equation. For the case when $\alpha = 0$ and $\beta > 0$, optimal control problems for (1.9)–(1.11) have been treated in the recent papers [18, 19] and reviewed in [17]. Moreover, for the classical case when $A = B = -\Delta$, $\rho = \sigma = 1/2$, with various boundary conditions (i.e., Dirichlet, Neumann, and dynamic conditions), there exist many contributions in which optimal control problems have been studied; for a number of recent references in this direction, we refer the reader to [19].

Another closely related phase field system is given by a model for tumor growth for which control problems have recently been studied. The fractional version of this model reads as follows (cf. [20,21]):

$$\alpha\partial_t\mu + \partial_t\varphi + A^{2\rho}\mu \;=\; P(\varphi)(S-\mu), \tag{1.12}$$
$$\beta\partial_t\varphi + B^{2\sigma}\varphi + F'(\varphi) = \mu, \tag{1.13}$$
$$\partial_t S + C^{2\tau}S = -P(\varphi)(S-\mu), \tag{1.14}$$
$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad S(0) = S_0. \tag{1.15}$$

Indeed, if $P(\varphi) \equiv 0$, then (1.12), (1.13) decouple from (1.14) and attain the form (1.9), (1.10) for $u = 0$. Also for the system (1.12)–(1.15) optimal control problems have been studied for the classical case $A = B = C = -\Delta$, $\rho = \sigma = \tau = 1/2$, with zero Neumann boundary conditions. In this connection, we refer to the works [14, 36–39]. In [24], also terms modeling chemotaxis were incorporated in the model. Even more involved models have been studied in [26–28].

In this paper, in which we study the state system (1.1)–(1.3), we have to focus on the interplay between the nonlinearity F and embedding properties of the domains of the involved operators. Quite surprisingly, it turns out that in our case, where $\alpha = 1/\ell > 0$ if we consider (1.9), the situation is more delicate than in the abovementioned works where $\alpha = 0$. The reason for this is that a proper treatment of the nonlinear term $F'(\varphi)$ in the optimal control problem (in particular, the derivation of results concerning Fréchet differentiability) makes it necessary that $F'(\varphi) \in L^\infty(Q)$. This means, at least in the case of the irregular potentials $F_{\log}$ and $F_{2\text{obs}}$, that the values attained by the solution component φ must be uniformly separated from the critical values (in this case ± 1). To show such a separation condition, however, it is somehow needed that the right-hand side $\ell(\varphi)\vartheta$ of (1.2) (μ, in the case of (1.10) or (1.13)) is bounded. In terms of the expected regularities, this condition is more restrictive in our situation. Indeed, if $V_A^\rho = D(A^\rho)$ denotes the domain of the fractional operator A^ρ, then it turns out that it maximally holds $\vartheta \in L^\infty(0,T;V_A^\rho)$ in the case of the system (1.1)–(1.3), which corresponds to $\mu \in L^\infty(0,T;V_A^\rho)$ for the system (1.9)–(1.11) with $\alpha > 0$, while one can recover the better regularity $\mu \in L^\infty(0,T;V_A^{2\rho})$ if $\alpha = 0$.

It turns out that an appropriate separation property (the condition **(GB)** below) holds true in certain cases for regular potentials and singular potentials of the logarithmic type. For such cases, the Fréchet differentiability can be shown (see Section 3), and first-order necessary optimality conditions can be derived (see Section 4.2). The last Section 4.3 brings the derivation of first-order necessary conditions also for the case of the double obstacle potential $F_{2\mathrm{obs}}$. In this case, where a separation condition like **(GB)** cannot be expected to hold and where we do not have Fréchet differentiability, we apply the so-called *deep quench approximation*, taking advantage of the results established in Section 4.2 for logarithmic potentials.

Throughout this paper, we denote for a given Banach space $(X, \| \cdot \|_X)$ by X^* the dual space of X and by $\langle \cdot\,, \cdot \rangle_X$ the duality product between X^* and X. We will also make frequent use of the elementary Young inequality

$$a\,b \;\leq\; \delta a^2 + \frac{1}{4\delta}\,b^2 \quad \text{for all } \ a, b \in \mathbb{R} \ \text{ and } \ \delta > 0. \tag{1.16}$$

Finally, we denote by $W^{s,p}(\Omega)$ for $s \geq 0$ and $p \in [1, +\infty]$ the fractional Sobolev–Slobodeckij spaces defined in, e.g., [25]. We put $H^s(\Omega) := W^{s,2}(\Omega)$ for $s \geq 0$ and notice that for $s \geq 0$ and in three dimensions of space we have the continuous embeddings (cf., e.g., [25, Thms. 6.7, 8.2])

$$H^s(\Omega) \subset L^q(\Omega) \quad \text{for } 2s < 3 \text{ and } 1 \leq q \leq 6/(3-2s), \tag{1.17}$$

$$H^s(\Omega) \subset C^0(\overline{\Omega}) \quad \text{for } 2s > 3. \tag{1.18}$$

Observe that the latter embedding is compact, while the former is compact only for $1 \leq q < 6/(3-2s)$. In particular, we have

$$H^{2s}(\Omega) \subset L^4(\Omega) \ \text{ if } \ s \geq 3/8 \quad \text{and} \quad H^{4s}(\Omega) \subset L^6(\Omega) \ \text{ if } \ s \geq 1/4\,. \tag{1.19}$$

2 Statement of the problem and and the state system

In this section, we state precise assumptions and notations and present some results for the state system (1.1)–(1.3). Throughout this paper, $\Omega \subset \mathbb{R}^3$ is a bounded and connected open set with smooth boundary $\Gamma := \partial\Omega$ and volume $|\Omega|$. We denote by $\mathbf{n}$ the outward unit normal vector field and by $\partial_{\mathbf{n}}$ the outward normal derivative. We set

$$H := L^2(\Omega) \tag{2.1}$$

and denote by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ the standard norm and inner product of H. We generally assume:

(A1) $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are unbounded, monotone, self-adjoint, linear operators with compact resolvents.

Therefore, there are sequences $\{\lambda_j\}$, $\{\lambda'_j\}$ and $\{e_j\}$, $\{e'_j\}$ of eigenvalues and of corresponding eigenfunctions such that

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j, \quad \text{with} \quad (e_i, e_j) = (e'_i, e'_j) = \delta_{ij} \quad \forall\, i, j \in \mathbb{N}, \tag{2.2}$$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots, \quad 0 \leq \lambda'_1 \leq \lambda'_2 \leq \ldots, \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty, \tag{2.3}$$

$$\{e_j\} \text{ and } \{e'_j\} \text{ are complete systems in } H. \tag{2.4}$$

As a consequence, we can define the powers of A and B for arbitrary positive real exponents: we have, for $\rho > 0$,

$$V_A^\rho := D(A^\rho) = \Big\{ v \in H : \sum_{j=1}^{\infty} |\lambda_j^\rho (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.5}$$

$$A^\rho v = \sum_{j=1}^{\infty} \lambda_j^\rho (v, e_j) e_j \quad \text{for } v \in V_A^\rho, \tag{2.6}$$

the series being convergent in the strong topology of H. By endowing V_A^ρ with the graph norm, i.e., setting

$$(v, w)_{V_A^\rho} := (v, w) + (A^\rho v, A^\rho w) \quad \text{and} \quad \|v\|_{V_A^\rho} := (v, v)_{V_A^\rho}^{1/2} \quad \text{for } v, w \in V_A^\rho, \tag{2.7}$$

we obtain a Hilbert space. In the same way, we define the power B^σ for every $\sigma > 0$, starting from (2.2)–(2.4) for B. We therefore set $V_B^\sigma := D(B^\sigma)$, with the norm $\| \cdot \|_{V_B^\sigma}$ associated with the inner product

$$(v, w)_{V_B^\sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad \text{for } v, w \in V_B^\sigma. \tag{2.8}$$

Since $\lambda_j \geq 0$ and $\lambda'_j \geq 0$ for every j, one immediately deduces from the definitions that $A^\rho : V_A^\rho \subset H \to H$ and $B^\sigma : V_B^\sigma \subset H \to H$ are maximal monotone operators. Moreover, it is clear that, for every $\rho_1,\ \rho_2 > 0$, we have the Green type formula

$$(A^{\rho_1+\rho_2} v, w) = (A^{\rho_1} v, A^{\rho_2} w) \quad \text{for every } v \in V_A^{\rho_1+\rho_2} \text{ and } w \in V_A^{\rho_2}, \tag{2.9}$$

and that a similar relation holds for B. Due to these properties, we can define proper extensions of the operators that allow values in dual spaces. In particular, we can write variational formulations of (1.1) and (1.2). It is convenient to use the notations

$$V_A^{-\rho} := (V_A^\rho)^*, \quad V_B^{-\sigma} := (V_B^\sigma)^*, \quad \text{for } \rho > 0 \text{ and } \sigma > 0. \tag{2.10}$$

Then, we have that

$$A^{2\rho} \in \mathcal{L}(V_A^\rho, V_A^{-\rho}), \quad B^{2\sigma} \in \mathcal{L}(V_B^\sigma, V_B^{-\sigma}), \tag{2.11}$$

as well as

$$A^\rho \in \mathcal{L}(H, V_A^{-\rho}), \quad B^\sigma \in \mathcal{L}(H, V_B^{-\sigma}). \tag{2.12}$$

Here, we identify H with a subspace of $V_A^{-\rho}$ in the usual way, i.e., such that

$$\langle v, w \rangle_{V_A^\rho} = (v, w) \quad \text{for every } v \in H \text{ and } w \in V_A^\rho. \tag{2.13}$$

Analogously, we have $H \subset V_B^{-\sigma}$ and use corresponding notations. Observe also that the following embeddings are continuous and compact:

$$V_A^{\rho_1+\rho_2} \subset V_A^{\rho_1} \subset H, \quad V_B^{\sigma_1+\sigma_2} \subset V_B^{\sigma_1} \subset H, \quad \text{for } \rho_1 > 0, \rho_2 > 0 \text{ and } \sigma_1 > 0, \sigma_2 > 0, \tag{2.14}$$

$$V_A^\rho \subset H \subset V_A^{-\rho}, \quad V_B^\sigma \subset H \subset V_B^{-\sigma}, \quad \text{for } \rho > 0 \text{ and } \sigma > 0. \tag{2.15}$$

From now on, we generally assume for the nonlinear functions entering (1.1) and (1.2):

(F1) $F = F_1 + F_2$, where $F_1 : \mathbb{R} \to [0, +\infty]$ is convex and lower semicontinuous with $F_1(0) = 0$. Moreover, there are constants $c_1 > 0$, $c_2 > 0$, such that

$$F(s) \geq c_1 s^2 - c_2 \quad \forall\, s \in \mathbb{R}. \tag{2.16}$$

(F2) There are r_-, r_+ with $-\infty \le r_- < 0 < r_+ \le +\infty$ such that $F_1 \in C^3(r_-, r_+)$, and it holds $F_1'(0) = 0$.

(F3) $F_2 \in C^3(\mathbb{R})$, and F_2' is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $L > 0$.

(F4) $\ell \in C^2(\mathbb{R})$, and $\ell^{(i)} \in L^\infty(\mathbb{R})$ for $0 \le i \le 2$.

Remark 2.1. It is worth noting that all of the potentials (1.4)–(1.6) satisfy the general conditions **(F1)**–**(F3)**, where $D(F_1) = D(\partial F_1) = \mathbb{R}$ for $F = F_{\rm reg}$, while $D(F_1) = [-1,1]$ and $D(\partial F_1) = (-1,1)$ for $F = F_{\rm log}$, and $D(F_1) = D(\partial F_1) = [-1,1]$ for $F = F_{\rm 2obs}$. Here, and throughout this paper, we denote by $D(F_1)$ and $D(\partial F_1)$ the effective domains of F_1 and of its subdifferential ∂F_1, respectively. We notice that ∂F_1 is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and use the same symbol ∂F_1 for the maximal monotone operators induced in L^2 spaces. Moreover, for $r \in D(\partial F_1)$, we use the symbol $\partial F_1^\circ(r)$ for the element of $\partial F_1(r)$ having minimal modulus. If, however, ∂F_1 is single-valued (which is the case if $(r_-, r_+) = \mathbb{R}$), then we denote the sole element of the singleton $\partial F_1(r)$ by $F_1'(r)$. We also remark that **(F3)** implies that F_2' grows at most linearly on $\mathbb{R}$, while F_2 grows at most quadratically.

For the other data of the state system, we postulate:

(A2) ρ and σ are fixed positive real numbers.

(A3) $\vartheta_0 \in V_A^\rho$, $\varphi_0 \in V_B^{2\sigma}$, and there are constants r_{0-}, r_{0+} such that

$$r_- < r_{0-} \le \varphi_0 \le r_{0+} < r_+ \quad \text{a.e. in } \Omega. \tag{2.17}$$

(A4) The embeddings $V_A^\rho \subset L^4(\Omega)$ and $V_B^\sigma \subset L^4(\Omega)$ are continuous.

Remark 2.2. If, for instance, $A = -\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ (thus, with homogeneous Dirichlet conditions, but similarly for zero boundary conditions of Neumann or third kind), then $V_A^\rho \subset H^{2\rho}(\Omega)$; it then follows from (1.19) that (the first embedding in) **(A4)** holds true if $\rho \ge 3/8$. Likewise, we have in this case $V_A^\rho \subset L^6(\Omega)$ for $\rho \ge 1/2$ as well as $V_A^\rho \subset C^0(\overline{\Omega})$ provided that $\rho > 3/4$.

For the data entering the cost functional (1.7) and the admissible set $\mathcal{U}_{\rm ad}$ defined in (1.8) we generally assume:

(A5) $\vartheta_\Omega, \varphi_\Omega \in L^2(\Omega)$, $\vartheta_Q, \varphi_Q \in L^2(Q)$, $u_{min}, u_{max} \in L^\infty(Q)$ satisfy $u_{min} \le u_{max}$ a.e. in Q.

Finally, once and for all we fix some open and bounded ball in $L^\infty(Q)$ that contains the admissible set.

(A6) $R > 0$ is a constant such that $\mathcal{U}_{\rm ad} \subset \mathcal{U}_R := \{ u \in L^\infty(Q) : \|u\|_{L^\infty(Q)} < R \}$.

At this point, we are in a position to make use of (2.9) and its analogue for B to give a weak formulation of the state system (1.1)–(1.3) and to introduce our notion of solution. In particular, we present (1.2) in the form of a variational inequality. We look for a pair (ϑ, φ) satisfying

$$\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{2.18}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma), \tag{2.19}$$

$$F_1(\varphi) \in L^1(Q), \tag{2.20}$$

and solving the system

$$\partial_t \vartheta + \ell(\varphi)\partial_t \varphi + A^{2\rho}\vartheta = u \quad \text{a.e. in } Q, \tag{2.21}$$

$$(\partial_t \varphi(t), \varphi(t) - v) + (B^\sigma \varphi(t), B^\sigma(\varphi(t) - v)) + \int_\Omega F_1(\varphi(t)) \; + \; (F_2'(\varphi(t)), \varphi(t) - v)$$

$$\leq \; (\ell(\varphi(t))\vartheta(t), \varphi(t) - v) \; + \int_\Omega F_1(v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_B^\sigma, \tag{2.22}$$

$$\vartheta(0) = \vartheta_0, \quad \varphi(0) = \varphi_0. \tag{2.23}$$

Here, it is understood that $\int_\Omega F_1(v) = +\infty$ whenever $F_1(v) \notin L^1(\Omega)$. We follow a similar rule for expressions of the type $\iint_Q F_1(v)$ whenever $v \in L^2(Q)$ but $F_1(v) \notin L^1(Q)$.

We notice that (2.22) is equivalent to its time-integrated variant, that is,

$$\begin{aligned}
&\int_0^T \big(\partial_t \varphi(t), \varphi(t) - v(t)\big)\, dt + \int_0^T \big(B^\sigma \varphi(t), B^\sigma(\varphi(t) - v(t))\big)\, dt \\
&\quad + \iint_Q F_1(\varphi) + \int_0^T \big(F_2'(\varphi(t)), \varphi(t) - v(t)\big)\, dt \\
&\leq \int_0^T \big(\ell(\varphi(t))\vartheta(t), \varphi(t) - v(t)\big)\, dt + \iint_Q F_1(v) \quad \text{for all } v \in L^2(0,T;V_B^\sigma).
\end{aligned} \tag{2.24}$$

Similarly, (2.21) is equivalent to a corresponding time-integrated version with test functions $v \in L^2(0,T;V_A^\rho)$.

We have the following well-posedness result (cf. [11, Thm. 2.10]).

Theorem 2.3. *Let the assumptions* **(F1)**–**(F4)***,* **(A1)**–**(A4)***, and* **(A6)** *be fulfilled. Then the problem* (2.21)–(2.23) *has for every $u \in \mathcal{U}_R$ a unique solution (ϑ, φ) satisfying* (2.18)–(2.20)*. Moreover, there is a constant $K_1 > 0$, which depends only on R and the data of the state system, such that*

$$\|\vartheta\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho})} \; + \; \|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)} \; + \; \iint_Q F_1(\varphi) \; \leq \; K_1, \tag{2.25}$$

whenever (ϑ, φ) solves (2.21)–(2.23) *for some $u \in \mathcal{U}_R$.*

Proof. Owing to the assumptions **(F2)** and (2.17), we have that $\partial F_1^\circ(\varphi_0) = F_1'(\varphi_0) \in H$, and thus all of the conditions for the application of [11, Thm. 2.10] are fulfilled. ☐

By virtue of Theorem 2.3, the *control-to-state operator*

$$\mathcal{S} : \, \mathcal{U}_R \ni u \mapsto \mathcal{S}(u) := (\vartheta, \varphi) \tag{2.26}$$

is well defined and bounded as a mapping from $\mathcal{U}_R \subset L^\infty(Q)$ into the Banach space specified by the regularity properties (2.18), (2.19). In the following, we look for conditions that guarantee that $\mathcal{S}$

is Fréchet differentiable between suitable Banach spaces. To this end, we make use of the following global boundedness assumption which has proved to be very useful in the framework of Cahn–Hilliard type systems with fractional operators (cf. the recent works [17–19, 21]):

(GB) There are constants a_R, b_R such that

$$r_- < a_R \le \varphi \le b_R < r_+ \quad \text{a.e. in } Q \tag{2.27}$$

whenever $(\vartheta, \varphi) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$.

The condition **(GB)** is rather restrictive and has to be verified from case to case. As a rule, it cannot be satisfied for potentials of indicator function type like $F_{2\mathrm{obs}}$. It can, however, be satisfied for regular potentials like F_{reg} (see the case (ii) below) and singular potentials like $F_{\log}$ (see the case (i) below). Indeed, we have the following result, whose assumptions are commented in the next Remark 2.5.

Lemma 2.4. *Assume that the conditions* **(A1)**–**(A4)** *and* **(F1)**–**(F4)** *are satisfied and, in addition, that*

$$\psi(v) \in H \ \text{ and } \ (B^{2\sigma} v, \psi(v)) \ge 0 \quad \text{for every } v \in V_B^{2\sigma} \text{ and every monotone and Lipschitz continuous mapping } \psi : \mathbb{R} \to \mathbb{R} \text{ vanishing at the origin.} \tag{2.28}$$

In addition, assume that

$$\lim_{r \to r_-} F_1'(r) = -\infty, \quad \lim_{r \to r_+} F_1'(r) = +\infty. \tag{2.29}$$

Then **(GB)** *is satisfied in any of the following situations:*

(i) $A = -\Delta$ *with zero Neumann or Dirichlet boundary conditions, and* $\rho > \frac{3}{4}$ *or* $\rho = \frac{1}{2}$.

(ii) $(r_-, r_+) = \mathbb{R}$, $B^{2\sigma} = B = -\Delta$ *with zero Dirichlet or Neumann boundary conditions and* $\varphi_0 \in D(B)$.

Proof. Suppose that $(\vartheta, \varphi) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$. Assume first that the assumptions of (i) are satisfied. In the following, we denote by $C_i > 0$, $i \in \mathbb{N}$, constants that depend only on R and the data. We have, owing to (1.18), that (cf. also Remark 2.2) $V_A^\rho \subset H^{2\rho}(\Omega) \subset L^\infty(\Omega)$ if $\rho > 3/4$. Hence, by (2.25) it turns out that

$$\|\vartheta\|_{L^\infty(Q)} \le C_1 \tag{2.30}$$

in this case. On the other hand, if $\rho = 1/2$, then ϑ solves a standard linear parabolic problem with right-hand side $u - \ell(\varphi)\partial_t\varphi$, which is bounded in $L^\infty(0,T;H)$ for $u \in \mathcal{U}_R$. Then the validity of (2.30) follows from standard results on linear parabolic problems (see, e.g., [34, Chap. 7]). Hence, in both cases, we have that $\|\ell(\varphi)\vartheta\|_{L^\infty(Q)} \le C_2$ since ℓ is bounded, and the validity of **(GB)** with a_R, b_R satisfying $r_- < a_R \le r_{0-} \le r_{0+} \le b_R < r_+$ follows from the assumptions (2.28)–(2.29) as in the proof of [21, Thm. 2.4].

Now, let the assumptions of (ii) be fulfilled. Then, we remark that $(r_-, r_+) = \mathbb{R}$ excludes singular potentials like $F_{\log}$ and we have to prove that φ is bounded in $L^\infty(Q)$ uniformly with respect to $u \in \mathcal{U}_R$. Let, for $\lambda > 0$, $F'_{1,\lambda}$ denote the Moreau–Yosida approximation of F_1' at the level λ. It is well known (see, e.g., [2]) that in this special case, where the subdifferentials are single-valued and $F_1'(0) = 0$, the following conditions are satisfied:

$$F'_{1,\lambda} \text{ is globally Lipschitz continuous on } \mathbb{R}, \ F'_{1,\lambda}(0) = 0, \text{ and it holds } |F'_{1,\lambda}(r)| \le |F_1'(r)| \ \text{ and } \ \lim_{\lambda \searrow 0} F'_{1,\lambda}(r) = F_1'(r) \quad \text{for all } r \in \mathbb{R}. \tag{2.31}$$

In the proofs of [11, Prop. 2.4 and Prop. 2.9] (for details, see [11, Sect. 5]) it has been shown, using (2.29) and a special case of (2.28), that there is some $\Lambda > 0$ such that for every $\lambda \in (0,\Lambda]$ the general system

$$\partial_t\vartheta_\lambda + \ell(\varphi_\lambda)\partial_t\varphi_\lambda + A^{2\rho}\vartheta_\lambda \,=\, u \quad \text{a.e. in } Q, \tag{2.32}$$

$$\partial_t\varphi_\lambda + B^{2\sigma}\varphi_\lambda + F'_{1,\lambda}(\varphi_\lambda) + F'_2(\varphi_\lambda) \,=\, \ell(\varphi_\lambda)\vartheta_\lambda \quad \text{a.e. in } Q, \tag{2.33}$$

$$\vartheta_\lambda(0) = \vartheta_0, \quad \varphi_\lambda(0) = \varphi_0 \quad \text{a.e. in } \Omega, \tag{2.34}$$

has for every $u \in \mathfrak{U}_R$ a unique solution pair $(\vartheta_\lambda, \varphi_\lambda)$ such that

$$\|\vartheta_\lambda\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho})} \,+\, \|\varphi_\lambda\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)\cap L^2(0,T;V_B^{2\sigma})} \,\le\, M_1, \tag{2.35}$$

where, here and in the following, $M_i > 0$, $i \in \mathbb{N}$, denote constants that may depend on R and on the data of the system, but not on $\lambda \in (0,\Lambda]$. It was then shown in [11] that $(\vartheta_\lambda, \varphi_\lambda)$ converges to (ϑ, φ) in a suitable topology. We repeat here a part of the argument and use (2.28) with $v = \varphi_\lambda(t)$ and $\psi = F'_{1,\lambda}$. We test (2.33) by $F'_{1,\lambda}(\varphi_\lambda(t))$ to obtain for almost every $t \in (0,T)$ the identity

$$\begin{aligned} &\big(B^{2\sigma}\varphi_\lambda(t), F'_{1,\lambda}(\varphi_\lambda(t))\big) + \int_\Omega |F'_{1,\lambda}(\varphi_\lambda(t))|^2 \\ &= \int_\Omega F'_{1,\lambda}(\varphi_\lambda(t))\big(\ell(\varphi_\lambda(t))\vartheta_\lambda(t) - F'_2(\varphi_\lambda(t)) - \partial_t\varphi_\lambda(t)\big), \end{aligned} \tag{2.36}$$

where the first summand on the left-hand side is nonnegative and, by virtue of (2.35) and the general assumptions for the nonlinearities, the right-hand side is bounded by an expression of the form

$$\frac{1}{2}\int_\Omega \big|F'_{1,\lambda}(\varphi_\lambda(t))\big|^2 \,+\, M_2,$$

whence we obtain that

$$\big\|F'_{1,\lambda}(\varphi_\lambda)\big\|_{L^\infty(0,T;H)} \,\le\, M_3.$$

Thanks to our assumption on $B^{2\sigma}$ it follows that φ_λ solves a standard linear parabolic initial-boundary value problem, where the right-hand side $\ell(\varphi_\lambda)\vartheta_\lambda - F'_{1,\lambda}(\varphi_\lambda) - F'_2(\varphi_\lambda)$ is bounded in $L^\infty(0,T;H)$, independently of $\lambda \in (0,\Lambda]$. Moreover, $\varphi_0 \in D(B) \subset L^\infty(\Omega)$. Applying the classical results of [34, Chap. 7], we therefore can infer that $\|\varphi_\lambda\|_{L^\infty(Q)}$ is bounded independently of $\lambda \in (0,\Lambda]$. Hence, $\varphi_\lambda \to \varphi$ weakly-star in $L^\infty(Q)$, and the lower semicontinuity of norms yields the assertion. $\square$

Remark 2.5. The assumptions on $B^{2\sigma}$ made in (ii), and the condition (2.28) we also used in the second part of the proof, are not in contradiction with each other. In fact, the former implies the latter. Indeed, in this case, $V_B^\sigma = H_0^1(\Omega)$ or $V_B^\sigma = H^1(\Omega)$, and therefore it holds for every monotone and Lipschitz continuous function ψ vanishing at the origin that for every $v \in V_B^{2\sigma} \subset H^2(\Omega)$ we have $\psi(v) \in H^1(\Omega)$, as well as

$$(B^{2\sigma}v, \psi(v)) = (-\Delta v, \psi(v)) = \int_\Omega \psi'(v)\,|\nabla v|^2 \,\ge 0.$$

Moreover, in both (i) and (ii), the Laplacian can be replaced by more general second-order elliptic operators in divergence form with smooth coefficients complemented with more general zero boundary conditions. Furthermore, regarding A in (i), even higher order operators can be considered provided that the assumptions on ρ are modified accordingly. For instance, one can take the plate operator $A = \Delta^2$, assuming as domain the set of $v \in H^4(\Omega)$ satisfying suitable boundary conditions. Two possibilities are $v = \partial_{\mathbf{n}}v = 0$ and $\partial_{\mathbf{n}}v = \partial_{\mathbf{n}}\Delta v = 0$. In both cases, $V_A^\rho \subset H^{4\rho}(\Omega)$, and then the condition $V_A^\rho \subset L^\infty(\Omega)$ used in the above proof is satisfied if $\rho > 3/8$.

Remark 2.6. If condition **(GB)** is fulfilled, then it follows from the assumptions **(F2)–(F4)** that, by possibly taking a larger constant $K_1 > 0$, it holds the global bound

$$\max_{0\le i\le 3} \|F_1^{(i)}(\varphi)\|_{L^\infty(Q)} + \max_{0\le i\le 3} \|F_2^{(i)}(\varphi)\|_{L^\infty(Q)} + \max_{0\le i\le 2} \|\ell^{(i)}(\varphi)\|_{L^\infty(Q)} \le K_1 \qquad (2.37)$$

whenever $(\vartheta,\varphi) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$.

The next step in our analysis is to show that under the condition **(GB)** a rather strong stability estimate holds true for the solutions to the state system, which constitutes an important preparation for the later proof of Fréchet differentiability. We have, however, to make a further assumption:

(A7) $V_B^\sigma \cap L^\infty(\Omega)$ is dense in V_B^σ.

The condition **(A7)** is, for example, fulfilled if V_B^σ coincides with one of the Sobolev–Slobodeckij spaces $H^s(\Omega)$ for $s > 0$. We combine it with **(GB)** to prove the lemma below. Similar results were established in [11, Prop. 2.4 and Prop. 2.9] under an assumption close to (2.28).

Lemma 2.7. *Suppose that the conditions* **(F1)–(F4)***,* **(A1)–(A4)***,* **(A6)–(A7)***, and* **(GB)** *are satisfied. Then, for every $u \in \mathcal{U}_R$, the solution (ϑ,φ) to the state system satisfies the variational equation*

$$(\partial_t\varphi(t), v) + (B^\sigma\varphi(t), B^\sigma v) + (F'(\varphi(t)), v) = (\ell(\varphi(t))\vartheta(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and all } v \in V_B^\sigma. \qquad (2.38)$$

Moreover, by possibly enlarging the constant K_1 that appears in (2.25) *and* (2.37)*, we have the estimate*

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)\cap L^\infty(0,T;V_B^{2\sigma})} \le K_1\,. \qquad (2.39)$$

In particular, (ϑ,φ) is a strong solution to the state system.

Proof. By recalling **(GB)**, we set $\delta := \min\{a_R - r_-, r_+ - b_R\}$. Take now an arbitrary $w \in V_B^\sigma \cap L^\infty(\Omega)$, and let $\varepsilon_0 > 0$ be such that $\varepsilon_0\|w\|_{L^\infty(\Omega)} \le \delta/2$. Then, $v := \varphi(t) + \varepsilon w$ is for every $\varepsilon \in (0,\varepsilon_0)$ an admissible test function in (2.22), and $F_1(v) \in L^1(\Omega)$. By using it and then dividing by $-\varepsilon$, we obtain (for a.a. $t \in (0,T)$)

$$\bigl(\partial_t\varphi(t), w\bigr) + \bigl(B^\sigma\varphi(t), B^\sigma w\bigr) + \int_\Omega \frac{F_1(\varphi(t)+\varepsilon w) - F_1(\varphi(t))}{\varepsilon}$$
$$+ \bigl(F_2'(\varphi(t)), w\bigr) \ge \bigl(\ell(\varphi(t))\vartheta(t), w\bigr).$$

Since $r_- + \delta/2 \le \varphi(t) + \varepsilon w \le r_+ - \delta/2$ for every $\varepsilon \in (0,\varepsilon_0)$, and since $\varphi(t) + \varepsilon w$ converges to $\varphi(t)$ in the strong topology of $V_B^\sigma \cap L^\infty(\Omega)$ as $\varepsilon \searrow 0$, we immediately deduce that

$$\bigl(\partial_t\varphi(t), w\bigr) + \bigl(B^\sigma\varphi(t), w\bigr) + \bigl(F_1'(\varphi(t)), w\bigr) + \bigl(F_2'(\varphi(t)), w\bigr) \ge \bigl(\ell(\varphi(t))\vartheta(t), w\bigr).$$

By changing w into $-w$, we obtain the opposite inequality, and thus equality. Finally, by accounting for **(A7)**, we can remove the boundedness assumption on the test function.

Let us come to (2.39). A part of it is already given by (2.25). To derive the maximal space regularity, we notice that (2.38) can be written as

$$B^{2\sigma}\varphi = g := \ell(\varphi)\vartheta - \partial_t\varphi - F'(\varphi) \quad \text{in } V_B^{-\sigma}, \text{ a.e. in } (0,T),$$

and that g is bounded in $L^\infty(0,T;H)$ by a constant that only depends on R and the data thanks to (2.25) and (2.37). □

We derive the following result.

Theorem 2.8. *Suppose that the conditions* **(F1)–(F4)**, **(A1)–(A4)**, **(A6)–(A7)**, *and* **(GB)** *are satisfied. Then there is a constant* $K_2 > 0$, *which depends only on* R *and the data of the state system, such that the following holds true: whenever* $(\vartheta_i, \varphi_i) = \mathcal{S}(u_i)$, $i = 1, 2$, *for some controls* $u_1, u_2 \in \mathcal{U}_R$, *then, for every* $t \in (0, T]$,

$$\|\vartheta_1 - \vartheta_2\|_{H^1(0,t;H)\cap L^\infty(0,t;V_A^\rho)} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}(0,t;H)\cap H^1(0,t;V_B^\sigma)} \\ \leq K_2 \|u_1 - u_2\|_{L^2(0,t;H)}. \tag{2.40}$$

Proof. Since **(GB)** is fulfilled, the global bounds (2.25) and (2.37) are satisfied for (ϑ_i, φ_i), $i = 1, 2$. Moreover, by Lemma 2.7, we can replace the variational inequality (2.22) by the variational equation (2.38). Now let $\vartheta := \vartheta_1 - \vartheta_2$, $\varphi := \varphi_1 - \varphi_2$, and $u := u_1 - u_2$. Then it is easily seen that (ϑ, φ) is a strong solution to the system

$$\partial_t\vartheta + A^{2\rho}\vartheta + (\ell(\varphi_1) - \ell(\varphi_2))\partial_t\varphi_1 + \ell(\varphi_2)\partial_t\varphi = u \quad \text{in } Q, \tag{2.41}$$

$$\partial_t\varphi + B^{2\sigma}\varphi + F'(\varphi_1) - F'(\varphi_2) = (\ell(\varphi_1) - \ell(\varphi_2))\vartheta_1 + \ell(\varphi_2)\vartheta \quad \text{in } Q, \tag{2.42}$$

$$\vartheta(0) = 0, \quad \varphi(0) = 0, \quad \text{in } \Omega. \tag{2.43}$$

To begin with, we test (2.41) by ϑ and (2.42) by $\partial_t\varphi \in L^2(0, T; V_B^\sigma)$, add the resulting equations and integrate over $\Omega \times (0, t)$, where $t \in (0, T)$. Adding the same term $\frac{1}{2}\|\varphi(t)\|^2 = \int_0^t\int_\Omega \varphi\,\partial_t\varphi$ to both sides of the resulting identity and noting a cancellation, we arrive at the equation

$$\frac{1}{2}\|\vartheta(t)\|^2 + \frac{1}{2}\|\varphi(t)\|^2_{V_B^\sigma} + \int_0^t\int_\Omega |A^\rho\vartheta|^2 + \int_0^t\int_\Omega |\partial_t\varphi|^2 \\ = \int_0^t\int_\Omega u\vartheta - \int_0^t\int_\Omega \vartheta(\ell(\varphi_1) - \ell(\varphi_2))\partial_t\varphi_1 + \int_0^t\int_\Omega (\ell(\varphi_1) - \ell(\varphi_2))\,\vartheta_1\,\partial_t\varphi \\ - \int_0^t\int_\Omega \big(F'(\varphi_1) - F'(\varphi_2)\big)\partial_t\varphi + \int_0^t\int_\Omega \varphi\,\partial_t\varphi =: \sum_{j=1}^5 I_j\,, \tag{2.44}$$

with obvious notation. We estimate the terms on the right-hand side individually, using the Young and Hölder inequalities, the embedding conditions of **(A4)**, as well as the global bounds (2.25) and (2.37), repeatedly without further reference. In this process, C_i, $i \in \mathbb{N}$, denote constants that depend only on R and the data of the state system. Clearly, we have

$$|I_1| \leq \frac{1}{2}\int_0^t\int_\Omega \vartheta^2 + \frac{1}{2}\int_0^t\int_\Omega |u|^2\,. \tag{2.45}$$

Moreover, for every $\delta > 0$ (which is yet to be specified) it follows that

$$|I_2| \leq C_1 \int_0^t \|\vartheta(s)\|_{L^4(\Omega)}\,\|\varphi(s)\|_{L^4(\Omega)}\,\|\partial_t\varphi_1(s)\|\,ds \\ \leq \delta \int_0^t \|\vartheta(s)\|^2_{V_A^\rho}\,ds + \frac{C_2}{\delta}\int_0^t \|\partial_t\varphi_1(s)\|^2_{V_B^\sigma}\,\|\varphi(s)\|^2_{V_B^\sigma}\,ds\,. \tag{2.46}$$

In addition, we see that

$$|I_3| \leq C_3 \int_0^t \|\vartheta_1(s)\|_{L^4(\Omega)}\,\|\varphi(s)\|_{L^4(\Omega)}\,\|\partial_t\varphi(s)\|\,ds \\ \leq \delta \int_0^t |\partial_t\varphi|^2 + \frac{C_4}{\delta}\int_0^t \|\vartheta_1(s)\|^2_{V_A^\rho}\,\|\varphi(s)\|^2_{V_B^\sigma}\,ds\,. \tag{2.47}$$

Finally, owing to the Lipschitz continuity of F' in $[a_R, b_R]$, it turns out that

$$|I_4| + |I_5| \le \delta \int_0^t \int_\Omega |\partial_t \varphi|^2 + \frac{C_5}{\delta} \int_0^t \int_\Omega |\varphi|^2 . \tag{2.48}$$

Now observe that the mapping $s \mapsto \|\vartheta_1(s)\|^2_{V_A^\rho} + \|\partial_t \varphi_1(s)\|^2_{V_B^\sigma}$ belongs by (2.25) to $L^1(0,T)$. Hence, choosing $\delta = 1/4$, we can infer from Gronwall's lemma that for every $t \in (0,T)$ we have the estimate

$$\|\vartheta\|_{L^\infty(0,t;H)\cap L^2(0,t;V_A^\rho)} + \|\varphi\|_{H^1(0,t;H)\cap L^\infty(0,t;V_B^\sigma)} \le C_6 \|u\|_{L^2(0,t;H)} . \tag{2.49}$$

In the next estimate, we argue formally, noting that the arguments can be made rigorous by using, e.g., finite differences in time and the fact that $\vartheta(0) = \varphi(0) = 0$. Indeed, we formally differentiate (2.42) with respect to time to obtain the identity

$$\begin{aligned} &\partial_{tt}\varphi + B^{2\sigma}\partial_t\varphi + (F''(\varphi_1) - F''(\varphi_2))\partial_t\varphi_1 + F''(\varphi_2)\partial_t\varphi \\ &= (\ell'(\varphi_1) - \ell'(\varphi_2))\vartheta_1 \,\partial_t\varphi_1 + \ell'(\varphi_2)\vartheta_1\,\partial_t\varphi + \partial_t\vartheta_1(\ell(\varphi_1) - \ell(\varphi_2)) \\ &\quad + \vartheta\,\ell'(\varphi_2)\partial_t\varphi_2 + \ell(\varphi_2)\partial_t\vartheta . \end{aligned} \tag{2.50}$$

Let us add $\partial_t\varphi$ to both sides of (2.50). Then, we (formally) test (2.41) by $\partial_t\vartheta$ and (2.50) by $\partial_t\varphi$ and add the resulting equations. After a cancellation of terms, we obtain the identity

$$\begin{aligned} &\int_0^t\int_\Omega |\partial_t\vartheta|^2 + \frac12 \|A^\rho\vartheta(t)\|^2 + \frac12 \|\partial_t\varphi(t)\|^2 + \int_0^t \|\partial_t\varphi(s)\|^2_{V_B^\sigma} ds \\ &= \int_0^t\int_\Omega u\,\partial_t\vartheta - \int_0^t\int_\Omega (\ell(\varphi_1) - \ell(\varphi_2))\partial_t\varphi_1\partial_t\vartheta - \int_0^t\int_\Omega (F''(\varphi_1) - F''(\varphi_2))\partial_t\varphi_1\partial_t\varphi \\ &\quad + \int_0^t\int_\Omega (1 - F''(\varphi_2))|\partial_t\varphi|^2 + \int_0^t\int_\Omega (\ell'(\varphi_1) - \ell'(\varphi_2))\vartheta_1\partial_t\varphi_1\partial_t\varphi + \int_0^t\int_\Omega \ell'(\varphi_2)\vartheta_1|\partial_t\varphi|^2 \\ &\quad + \int_0^t\int_\Omega \partial_t\vartheta_1(\ell(\varphi_1) - \ell(\varphi_2))\partial_t\varphi + \int_0^t\int_\Omega \vartheta\,\ell'(\varphi_2)\partial_t\varphi_2\partial_t\varphi =: \sum_{j=1}^8 J_j , \end{aligned} \tag{2.51}$$

with obvious notation. We estimate the terms on the right-hand side individually, using the Young and Hölder inequalities, the embeddings from **(A4)**, and the estimates (2.25), (2.37), and (2.49) without further reference. Again, we denote by $C_i > 0$, $i \in \mathbb{N}$, constants that depend only on R and the data.

Now let $\delta > 0$ be arbitrary (to be chosen later). We obviously have

$$|J_1| + |J_4| \le \delta \int_0^t\int_\Omega |\partial_t\vartheta|^2 + C_1\bigl(1 + \delta^{-1}\bigr)\int_0^t\int_\Omega |u|^2. \tag{2.52}$$

Moreover, it is clear that

$$\begin{aligned} |J_2| &\le C_2 \int_0^t \|\partial_t\vartheta(s)\| \, \|\partial_t\varphi_1(s)\|_{L^4(\Omega)} \, \|\varphi(s)\|_{L^4(\Omega)} \, ds \\ &\le \delta \int_0^t\int_\Omega |\partial_t\vartheta|^2 + \frac{C_3}{\delta} \|\varphi\|^2_{L^\infty(0,t;V_B^\sigma)} \int_0^t \|\partial_t\varphi_1(s)\|^2_{V_B^\sigma} \, ds \\ &\le \delta \int_0^t\int_\Omega |\partial_t\vartheta|^2 + \frac{C_4}{\delta} \int_0^t\int_\Omega |u|^2 . \end{aligned} \tag{2.53}$$

Also, using the Lipschitz continuity of F'' in $[a_R, b_R]$ and (2.49) once more, we infer that

$$\begin{aligned} |J_3| \;&\le\; C_5 \int_0^t \|\varphi(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi_1(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|\, ds \\ &\le\; C_6\, \|\varphi\|_{L^\infty(0,t;V_B^\sigma)}\, \|\varphi_1\|_{H^1(0,t;V_B^\sigma)}\, \|\varphi\|_{H^1(0,t;H)} \\ &\le\; C_7 \int_0^t\!\!\int_\Omega |u|^2 \,. \end{aligned} \tag{2.54}$$

Similarly, in view of **(F4)** we observe that

$$\begin{aligned} |J_5| \;&\le\; C_8 \int_0^t \|\varphi(s)\|_{L^4(\Omega)}\, \|\vartheta_1(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi_1(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|_{L^4(\Omega)}\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_9}{\delta}\, \|\vartheta_1\|^2_{L^\infty(0,t;V_A^\rho)}\, \|\varphi_1\|^2_{H^1(0,t;V_B^\sigma)}\, \|\varphi\|^2_{L^\infty(0,t;V_B^\sigma)} \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+ \frac{C_{10}}{\delta} \int_0^t\!\!\int_\Omega |u|^2 \,. \end{aligned} \tag{2.55}$$

We also have

$$\begin{aligned} |J_6| \;&\le\; C_{11} \int_0^t \|\vartheta_1(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{12}}{\delta}\, \|\vartheta_1\|^2_{L^\infty(0,t;V_A^\rho)} \int_0^t \|\partial_t\varphi(s)\|^2\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{13}}{\delta} \int_0^t\!\!\int_\Omega |u|^2 \,. \end{aligned} \tag{2.56}$$

Moreover, it turns out that

$$\begin{aligned} |J_7| \;&\le\; C_{14} \int_0^t \|\partial_t\vartheta_1(s)\|\, \|\varphi(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|_{L^4(\Omega)}\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{15}}{\delta}\, \|\partial_t\vartheta_1\|^2_{L^2(0,t;H)}\, \|\varphi\|^2_{L^\infty(0,t;V_B^\sigma)} \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{16}}{\delta} \int_0^t\!\!\int_\Omega |u|^2 \,. \end{aligned} \tag{2.57}$$

Finally, we deduce that

$$\begin{aligned} |J_8| \;&\le\; C_{17} \int_0^t \|\vartheta(s)\|\, \|\partial_t\varphi_2(s)\|_{L^4(\Omega)}\, \|\partial_t\varphi(s)\|_{L^4(\Omega)}\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{18}}{\delta}\, \|\vartheta\|^2_{L^\infty(0,t;H)} \int_0^t \|\partial_t\varphi_2(s)\|^2_{V_B^\sigma}\, ds \\ &\le\; \delta \int_0^t \|\partial_t\varphi(s)\|_{V_B^\sigma}^2\, ds \,+\, \frac{C_{19}}{\delta} \int_0^t\!\!\int_\Omega |u|^2 \,. \end{aligned} \tag{2.58}$$

Summarizing the estimates (2.51)–(2.58), and choosing $\delta > 0$ small enough, we have thus shown the estimate

$$\|\vartheta\|^2_{H^1(0,t;H)\cap L^\infty(0,t;V_A^\rho)} \,+\, \|\varphi\|^2_{W^{1,\infty}(0,t;H)\cap H^1(0,t;V_B^\sigma)} \;\le\; C_{20} \int_0^t\!\!\int_\Omega |u|^2 \,. \tag{2.59}$$

This concludes the proof of the assertion. □

3 Fréchet differentiability of $\mathcal{S}$

In this section, we aim to show the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$ between suitable Banach spaces. To this end, we fix some $\overline{u} \in \mathcal{U}_R$ and set $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$. We then consider the linearized problem

$$\partial_t \eta + \ell'(\overline{\varphi})\, \partial_t \overline{\varphi}\, \xi + \ell(\overline{\varphi}) \partial_t \xi + A^{2\rho} \eta = h \quad \text{in } Q, \tag{3.1}$$

$$\partial_t \xi + B^{2\sigma} \xi + F''(\overline{\varphi}) \xi = \ell'(\overline{\varphi})\, \overline{\vartheta}\, \xi + \ell(\overline{\varphi}) \eta \quad \text{in } Q, \tag{3.2}$$

$$\eta(0) = \xi(0) = 0 \quad \text{in } \Omega. \tag{3.3}$$

The expectation is that if a Fréchet derivative $D\mathcal{S}(\overline{u})$ of $\mathcal{S}$ at $\overline{u}$ exists, then, for a given direction h, it should satisfy $D\mathcal{S}(\overline{u})[h] = (\eta, \xi)$, where (η, ξ) is the solution to (3.1)–(3.3). We first show the following result.

Theorem 3.1. *Suppose that the general assumptions* **(F1)**–**(F4)**, **(A1)**–**(A4)** *and* **(A6)** *as well as* **(GB)** *are fulfilled, and let $\overline{u} \in \mathcal{U}_R$ be arbitrary and $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$. Then the linearized system* (3.1)–(3.3) *has for every $h \in L^2(Q)$ a unique solution (η, ξ) such that*

$$\eta \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{3.4}$$

$$\xi \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma) \cap L^2(0,T;V_B^{2\sigma}). \tag{3.5}$$

Moreover, the linear mapping $h \mapsto (\eta, \xi)$ is continuous as a mapping between the spaces $L^2(Q)$ and $\big(H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho})\big) \times \big((H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma) \cap L^2(0,T;V_B^{2\sigma})\big)$.

Proof. We use a Faedo–Galerkin method. To this end, let (see (2.2)) $\{e_j\}_{j \in \mathbb{N}}$ and $\{e_j'\}_{j \in \mathbb{N}}$ be the orthonormalized eigenfunctions of A and B, respectively. We define the n-dimensional spaces $V_n := \operatorname{span}\{e_1, \ldots, e_n\}$ and $V_n' := \operatorname{span}\{e_1', \ldots, e_n'\}$ and search for every $n \in \mathbb{N}$ functions of the form

$$\eta_n(x,t) = \sum_{j=1}^n a_j(t) e_j(x), \quad \xi_n(x,t) = \sum_{j=1}^n b_j(t) e_j'(x),$$

such that

$$(\partial_t \eta_n, v) + (A^\rho \eta_n, A^\rho v) + (\ell(\overline{\varphi}) \partial_t \xi_n, v) \;=\; -(\ell'(\overline{\varphi})\, \partial_t \overline{\varphi}\, \xi_n, v) + (h, v)$$
$$\text{for every } v \in V_n \text{ and a.e. in } (0,T), \tag{3.6}$$

$$(\partial_t \xi_n, v) + (B^\sigma \xi_n, B^\sigma v) + (F''(\overline{\varphi}) \xi_n, v) \;=\; (\ell'(\overline{\varphi})\, \overline{\vartheta}\, \xi_n, v) + (\ell(\overline{\varphi}) \eta_n, v)$$
$$\text{for every } v \in V_n' \text{ and a.e. in } (0,T), \tag{3.7}$$

$$\eta_n(0) = \xi_n(0) = 0. \tag{3.8}$$

We choose $v = e_k'$, $1 \le k \le n$, in (3.7), which, thanks to the orthogonality of the eigenfunctions, leads to n explicit first-order ordinary differential equations with leading terms $\partial_t b_k$, $1 \le k \le n$. Next, we insert $v = e_k$, $1 \le k \le n$, in (3.6), and we substitute the explicit expressions for $\partial_t b_k$, $1 \le k \le n$, in the terms $(\ell(\overline{\varphi}) \partial_t \xi_n, e_k)$ for $k = 1, \ldots, n$. By doing this, we obtain from (3.6)–(3.8) a standard initial value problem for a linear system of $2n$ ordinary differential equations in the unknowns $a_1, \ldots, a_n, b_1, \ldots, b_n$. Since all of the occurring coefficient functions belong to $L^2(0,T)$, it follows from Carathéodory's theorem the existence of a unique solution $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in$

$H^1(0,T;\mathbb{R}^{2n})$ which specifies the unique solution $(\eta_n,\xi_n) \in (H^1(0,T;V_n) \times H^1(0,T;V_n'))$ to (3.6)–(3.8).

In the following, we derive some a priori estimates for the Galerkin approximations. In this process, we denote by $C_i > 0$, $i \in \mathbb{N}$, constants that may depend on R and the data of the state system, but not on $n \in \mathbb{N}$. We recall that Remark 2.6 applies to $\overline{\varphi}$. To begin with, we insert $v = \eta_n$ in (3.6) and $v = \partial_t \xi_n$ in (3.7), and add the results, which leads to a cancellation of terms. Then, we integrate over time and add to both sides of the resulting identity the same term $\frac{1}{2}\|\xi_n(t)\|^2 = \int_0^t\int_\Omega \xi_n \partial_t \xi_n$. We then obtain the equation

$$\frac{1}{2}\|\eta_n(t)\|^2 + \frac{1}{2}\|\xi_n(t)\|^2_{V_B^\sigma} + \int_0^t\int_\Omega |A^\rho \eta_n|^2 + \int_0^t\int_\Omega |\partial_t\xi_n|^2$$
$$= \int_0^t\int_\Omega h\eta_n - \int_0^t\int_\Omega \ell'(\overline{\varphi})\,\partial_t\overline{\varphi}\,\xi_n\eta_n - \int_0^t\int_\Omega F''(\overline{\varphi})\xi_n\,\partial_t\xi_n$$
$$+ \int_0^t\int_\Omega \ell'(\overline{\varphi})\,\overline{\vartheta}\,\xi_n\,\partial_t\xi_n + \int_0^t\int_\Omega \xi_n\,\partial_t\xi_n =: \sum_{j=1}^5 L_j, \tag{3.9}$$

with obvious meaning. Let $\delta > 0$ be arbitrary (to be specified later). At first, it is readily seen that

$$|L_1| + |L_3| + |L_5| \le \frac{1}{2}\int_0^t\int_\Omega (|h|^2 + |\eta_n|^2) + \delta\int_0^t\int_\Omega |\partial_t\xi_n|^2 + \frac{C_1}{\delta}\int_0^t\int_\Omega |\xi_n|^2 . \tag{3.10}$$

Moreover, we observe that

$$|L_2| \le C_2\int_0^t \|\partial_t\overline{\varphi}(s)\|_{L^4(\Omega)}\,\|\xi_n(s)\|_{L^4(\Omega)}\,\|\eta_n(s)\|\,ds$$
$$\le C_3\int_0^t\int_\Omega |\eta_n|^2 + C_4\int_0^t \|\partial_t\overline{\varphi}(s)\|^2_{V_B^\sigma}\,\|\xi_n(s)\|^2_{V_B^\sigma}\,ds . \tag{3.11}$$

Finally, we have the estimate

$$|L_4| \le C_5\int_0^t \|\overline{\vartheta}(s)\|_{L^4(\Omega)}\,\|\xi_n(s)\|_{L^4(\Omega)}\,\|\partial_t\xi_n(s)\|\,ds$$
$$\le \delta\int_0^t\int_\Omega |\partial_t\xi_n|^2 + \frac{C_6}{\delta}\int_0^t \|\overline{\vartheta}(s)\|^2_{V_A^\rho}\,\|\xi_n(s)\|^2_{V_B^\sigma}\,ds . \tag{3.12}$$

Now observe that the mapping $s \mapsto \|\partial_t\overline{\varphi}(s)\|^2_{V_B^\sigma} + \|\overline{\vartheta}(s)\|^2_{V_A^\rho}$ is known to belong to $L^1(0,T)$. Hence, combining (3.9)–(3.11), and choosing $\delta > 0$ small enough, we obtain from Gronwall's lemma the estimate

$$\|\eta_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V_A^\rho)} + \|\xi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \le C_7\|h\|_{L^2(0,T;H)} . \tag{3.13}$$

From (3.13) we can draw some consequences. Namely, invoking the general bounds (2.25) and (2.37), as well as the embeddings given by **(A4)**, we can easily verify that

$$\|\ell'(\overline{\varphi})\,\partial_t\overline{\varphi}\,\xi_n + \ell(\overline{\varphi})\,\partial_t\xi_n\|_{L^2(0,T;H)} \le C_8\|h\|_{L^2(0,T;H)},$$
$$\|\ell'(\overline{\varphi})\,\overline{\vartheta}\,\xi_n + \ell(\overline{\varphi})\,\eta_n - F''(\overline{\varphi})\,\xi_n\|_{L^2(0,T;H)} \le C_9\|h\|_{L^2(0,T;H)}.$$

But then we may insert first $v = \partial_t \eta_n$ and then $v = A^{2\rho}\eta_n$ in (3.6) to conclude that

$$\|\eta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho})} \le C_{10}\|h\|_{L^2(0,T;H)}. \tag{3.14}$$

Likewise, by inserting $v = B^{2\sigma}\xi_n$ in (3.7), we find that

$$\|\xi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)\cap L^2(0,T;V_B^{2\sigma})} \le C_{11}\|h\|_{L^2(0,T;H)}. \tag{3.15}$$

Hence, there is a pair (η,ξ) such that (first only for a subsequence, but, by the uniqueness of the limit, eventually for the entire sequence) we have the convergence properties

$$\eta_n \to \eta \quad \text{weakly-star in } H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho}),$$
$$\xi_n \to \xi \quad \text{weakly-star in } H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)\cap L^2(0,T;V_A^{2\sigma}).$$

It is then a standard matter (which needs no repetition here) to show that (η,ξ) is a strong solution to the linearized system (3.1)–(3.3), and the validity of the assertion concerning the continuity of the mapping $h \mapsto (\eta,\xi)$ follows from (3.14) and (3.15) by using the semicontinuity properties of norms.

It remains to show the uniqueness of the solution. To this end, let (η_i,ξ_i), $i = 1,2$, be two solutions enjoying the regularity properties (3.4) and (3.5), and let $\eta := \eta_1 - \eta_2$, $\xi := \xi_1 - \xi_2$. Then (η,ξ) is a strong solution to the system (3.1)–(3.3) with $h = 0$. Repeating the a priori estimates leading to (3.13) for the continuous problem, we obtain an estimate for (η,ξ) which resembles (3.13), but this time with $h = 0$ on the right-hand side. Thus, $\eta = \xi = 0$. This concludes the proof of the assertion. □

We are now ready to prove the Fréchet differentiability of the control-to-state operator. To be able to perform this analysis, we need a slightly stronger embedding condition than that of assumption **(A4)**. We have to postulate:

(A8) The embeddings $V_A^{2\rho} \subset L^6(\Omega)$ and $V_B^\sigma \subset L^6(\Omega)$ are continuous.

Remark 3.2. The second condition is more restrictive than the first one. Indeed, if $A = B = -\Delta$ with zero Dirichlet or Neumann conditions, then $V_B^\sigma \subset H^{2\sigma}(\Omega) \subset L^6(\Omega)$ if $\sigma \ge 1/2$, by (1.19). On the other hand, $V_A^{2\rho} \subset H^{4\rho}(\Omega) \subset L^6(\Omega)$ provided that $\rho \ge 1/4$. Notice that $V_A^\rho \subset L^4(\Omega)$ if $\rho \ge 3/8$ (see also Remark 2.2), so that in this case the postulate for A in **(A8)** is not more restrictive than that required in **(A4)**.

With these preparations, the road is paved for the proof of Fréchet differentiability.

Theorem 3.3. *Suppose that the conditions* **(F1)**–**(F4)**, **(A1)**–**(A4)**, **(A6)**–**(A8)**, *and* **(GB)** *are fulfilled. Then the control-to-state operator $\mathcal{S}$ is Fréchet differentiable on $\mathcal{U}_R$ as a mapping from $L^\infty(Q)$ into the Banach space*

$$\mathcal{Y} := \big(H^1(0,T;V_A^{-\rho})\cap C^0([0,T];H)\cap L^2(0,T;V_A^\rho)\big)\times\big(H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)\big). \tag{3.16}$$

Moreover, if $\overline{u} \in \mathcal{U}_R$ and $(\overline{\vartheta},\overline{\varphi}) = \mathcal{S}(\overline{u})$, then the Fréchet derivative $D\mathcal{S}(\overline{u}) \in \mathcal{L}(L^\infty(Q),\mathcal{Y})$ of $\mathcal{S}$ at $\overline{u}$, applied to $h \in L^\infty(Q)$, satisfies $D\mathcal{S}(\overline{u})[h] = (\eta,\xi)$, where (η,ξ) is the unique solution of (3.1)–(3.3).

Proof. Let $\overline{u} \in \mathcal{U}_R$ be arbitrary and $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$. Since $\mathcal{U}_R$ is open, there is some $\Lambda > 0$ such that $\overline{u} + h \in \mathcal{U}_R$ whenever $\|h\|_{L^\infty(Q)} \leq \Lambda$. In the following, we only consider such perturbations h. For any such h, we set $(\vartheta^h, \varphi^h) = \mathcal{S}(\overline{u} + h)$, and we denote by (η^h, ξ^h) the unique solution to the linearized system (3.1)–(3.3) associated with h. Moreover, we put

$$y^h := \vartheta^h - \overline{\vartheta} - \eta^h, \quad z^h := \varphi^h - \overline{\varphi} - \xi^h.$$

Since the linear mapping $h \mapsto (\eta^h, \xi^h)$ is by Theorem 3.1 continuous from $L^\infty(Q)$ into $\mathcal{Y}$, it suffices to show that there is a mapping $Z : (0, +\infty) \to (0, +\infty)$ such that

$$\|(y^h, z^h)\|_{\mathcal{Y}} \leq Z(\|h\|_{L^\infty(Q)}) \quad \text{and} \quad \lim_{s \searrow 0} \frac{Z(s)}{s} = 0. \tag{3.17}$$

To begin with, note that (y^h, z^h) satisfies the regularity properties (see also (2.39))

$$y^h \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{3.18}$$
$$z^h \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma) \cap L^2(0,T;V_B^{2\sigma}). \tag{3.19}$$

Moreover, $(\overline{\vartheta}, \overline{\varphi})$ and (ϑ^h, φ^h) satisfy the global estimates (2.25) and (2.37), and from (2.40) we have for all $t \in (0, T]$ the estimate

$$\|\vartheta^h - \overline{\vartheta}\|_{H^1(0,t;H) \cap L^\infty(0,t;V_A^\rho)} + \|\varphi^h - \overline{\varphi}\|_{W^{1,\infty}(0,t;H) \cap H^1(0,t;V_B^\sigma)} \leq K_2 \|h\|_{L^2(0,t;H)}. \tag{3.20}$$

In the following, we denote by $C > 0$ constants that may depend on R and the data of the state system, but not on the special choice of $h \in L^\infty(Q)$ with $\overline{u} + h \in \mathcal{U}_R$. Observe that the meaning of C may change from line to line within formulas.

At this point, we observe that Taylor's theorem with integral remainder shows that we have almost everywhere in Q the identities

$$\ell(\varphi^h) = \ell(\overline{\varphi}) + \ell'(\overline{\varphi})(\varphi^h - \overline{\varphi}) + (\varphi^h - \overline{\varphi})^2 R_1^h, \tag{3.21}$$
$$F'(\varphi^h) = F'(\overline{\varphi}) + F''(\overline{\varphi})(\varphi^h - \overline{\varphi}) + (\varphi^h - \overline{\varphi})^2 R_2^h, \tag{3.22}$$

with the remainders

$$R_1^h = \int_0^1 (1-s)\ell''(\overline{\varphi} + s(\varphi^h - \overline{\varphi}))\, ds, \quad R_2^h = \int_0^1 (1-s)F'''(\overline{\varphi} + s(\varphi^h - \overline{\varphi}))\, ds\,.$$

By **(F2)**–**(F4)**, **(GB)** and the boundedness of F''' in $[a_R, b_R]$, we have

$$\|R_1^h\|_{L^\infty(Q)} + \|R_2^h\|_{L^\infty(Q)} \leq C. \tag{3.23}$$

Now observe that y^h and z^h are strong solutions to the system

$$\partial_t y^h + A^{2\rho} y^h = Q_1^h \quad \text{in } Q, \tag{3.24}$$
$$\partial_t z^h + B^{2\sigma} z^h = Q_2^h \quad \text{in } Q, \tag{3.25}$$
$$y^h(0) = z^h(0) = 0 \quad \text{in } \Omega, \tag{3.26}$$

where simple algebraic manipulations using (3.21) and (3.22) show that

$$Q_1^h = -(\ell(\varphi^h) - \ell(\overline{\varphi}))(\partial_t\varphi^h - \partial_t\overline{\varphi}) - \ell(\overline{\varphi})\partial_t z^h - \ell'(\overline{\varphi})z^h\,\partial_t\overline{\varphi} - R_1^h\,(\varphi^h - \overline{\varphi})^2\,\partial_t\overline{\varphi}, \tag{3.27}$$

$$Q_2^h = (\ell(\varphi^h) - \ell(\overline{\varphi}))(\vartheta^h - \overline{\vartheta}) + \ell(\overline{\varphi})y^h + \overline{\vartheta}\,\ell'(\overline{\varphi})z^h + \overline{\vartheta}\,R_1^h\,(\varphi^h - \overline{\varphi})^2 - F''(\overline{\varphi})z^h - R_2^h\,(\varphi^h - \overline{\varphi})^2\,. \tag{3.28}$$

Now we test (3.24) by y^h and (3.25) by $\partial_t z^h$, add the results (whereby two terms cancel), and add the same term $\frac{1}{2}\,\|z^h(t)\|^2 = \int_0^t\!\int_\Omega z^h\partial_t z^h$ to both sides of the resulting identity. Since the terms involving the product $y^h\partial_t z^h$ cancel out, we obtain that

$$\begin{aligned}
&\frac{1}{2}\,\|y^h(t)\|^2 + \frac{1}{2}\,\|z^h(t)\|^2_{V_B^\sigma} + \int_0^t\!\!\int_\Omega|\partial_t z^h|^2 + \int_0^t\!\!\int_\Omega|A^\rho y^h|^2\\
&= -\int_0^t\!\!\int_\Omega(\ell(\varphi^h) - \ell(\overline{\varphi}))(\partial_t\varphi^h - \partial_t\overline{\varphi})\,y^h - \int_0^t\!\!\int_\Omega\ell'(\overline{\varphi})\,\partial_t\overline{\varphi}\,z^h\,y^h\\
&\quad - \int_0^t\!\!\int_\Omega R_1^h\,(\varphi^h - \overline{\varphi})^2\,\partial_t\overline{\varphi}\,y^h + \int_0^t\!\!\int_\Omega(\ell(\varphi^h) - \ell(\overline{\varphi}))(\vartheta^h - \overline{\vartheta})\,\partial_t z^h\\
&\quad + \int_0^t\!\!\int_\Omega\overline{\vartheta}\,\ell'(\overline{\varphi})z^h\,\partial_t z^h + \int_0^t\!\!\int_\Omega\overline{\vartheta}\,R_1^h\,(\varphi^h - \overline{\varphi})^2\,\partial_t z^h\\
&\quad + \int_0^t\!\!\int_\Omega(1 - F''(\overline{\varphi}))z^h\,\partial_t z^h - \int_0^t\!\!\int_\Omega R_2^h\,(\varphi^h - \overline{\varphi})^2\,\partial_t z^h =: \sum_{j=1}^8 M_j,
\end{aligned} \tag{3.29}$$

with obvious meaning. Let $\delta > 0$ be arbitrary (to be specified later). We estimate the terms on the right-hand side individually, using the Young and Hölder inequalities, the global bounds (2.25), (2.37) and (3.23), the stability estimate (3.20), as well as the embedding conditions **(A4)** and **(A8)**, repeatedly without further reference. Here, for the sake of brevity, we often omit the argument s of the involved functions. At first, we have

$$\begin{aligned}
|M_1| &\le C\int_0^t\|\varphi^h - \overline{\varphi}\|_{L^4(\Omega)}\,\|\partial_t\varphi^h - \partial_t\overline{\varphi}\|_{L^4(\Omega)}\,\|y^h\|\,ds\\
&\le C\,\|\varphi^h - \overline{\varphi}\|^2_{L^\infty(0,t;V_B^\sigma)}\,\|\varphi^h - \overline{\varphi}\|^2_{H^1(0,t;V_B^\sigma)} + \int_0^t\|y^h\|^2\,ds\\
&\le C\,\|h\|^4_{L^2(0,t;H)} + \int_0^t\|y^h\|^2\,ds\,.
\end{aligned} \tag{3.30}$$

Moreover, we see that

$$\begin{aligned}
|M_2| &\le C\int_0^t\|\partial_t\overline{\varphi}\|_{L^4(\Omega)}\,\|z^h\|_{L^4(\Omega)}\,\|y^h\|\,ds\\
&\le C\int_0^t\|\partial_t\overline{\varphi}\|_{V_B^\sigma}\left(\|y^h\|^2 + \|z^h\|^2_{V_B^\sigma}\right)ds,
\end{aligned} \tag{3.31}$$

as well as

$$\begin{aligned}
|M_3| &\le C\int_0^t\!\!\int_\Omega\|\varphi^h - \overline{\varphi}\|^2_{L^6(\Omega)}\,\|\partial_t\overline{\varphi}\|_{L^6(\Omega)}\,\|y^h\|\,ds\\
&\le C\,\|\varphi^h - \overline{\varphi}\|^4_{L^\infty(0,T;V_B^\sigma)}\,\|\overline{\varphi}\|^2_{H^1(0,T;V_B^\sigma)} + \int_0^t\|y^h\|^2\,ds\\
&\le C\,\|h\|^4_{L^2(0,t;H)} + \int_0^t\|y^h\|^2\,ds\,.
\end{aligned} \tag{3.32}$$

Also, it follows that

$$\begin{aligned} |M_4| &\le C\int_0^t \|\varphi^h-\overline{\varphi}\|_{L^4(\Omega)}\,\|\vartheta^h-\overline{\vartheta}\|_{L^4(\Omega)}\,\|\partial_t z^h\|\,ds \\ &\le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\,\|\varphi^h-\overline{\varphi}\|^2_{L^\infty(0,t;V_B^\sigma)}\,\|\vartheta^h-\overline{\vartheta}\|^2_{L^\infty(0,t;V_A^\rho)} \\ &\le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\,\|h\|^4_{L^2(0,t;H)}\,, \end{aligned} \tag{3.33}$$

and that

$$|M_5| \le C\int_0^t \|\overline{\vartheta}\|_{L^4(\Omega)}\,\|z^h\|_{L^4(\Omega)}\,\|\partial_t z^h\|\,ds \le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\int_0^t \|z^h\|^2_{V_B^\sigma}\,ds\,, \tag{3.34}$$

as well as

$$\begin{aligned} |M_6| &\le C\int_0^t \|\overline{\vartheta}\|_{L^6(\Omega)}\,\|\varphi^h-\overline{\varphi}\|^2_{L^6(\Omega)}\,\|\partial_t z^h\|\,ds \\ &\le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\,\|\varphi^h-\overline{\varphi}\|^4_{L^\infty(0,t;V_B^\sigma)} \\ &\le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\,\|h\|^4_{L^2(0,t;H)}\,. \end{aligned} \tag{3.35}$$

Finally, we infer that

$$|M_7| \le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\int_0^t \|z^h\|^2\,ds \tag{3.36}$$

and

$$|M_8| \le C\int_0^t \|\varphi^h-\overline{\varphi}\|^2_{L^4(\Omega)}\,\|\partial_t z^h\|\,ds \le \delta\int_0^t \|\partial_t z^h\|^2\,ds + \frac{C}{\delta}\,\|h\|^4_{L^2(0,t;H)}\,. \tag{3.37}$$

At this point, we observe that the map $s \mapsto \|\partial_t\overline{\varphi}(s)\|_{V_B^\sigma}$ belongs to $L^2(0,T)$. Thus, choosing $\delta > 0$ small enough and combining the estimates (3.29)–(3.37), we conclude that

$$\|y^h\|_{L^\infty(0,T;H)\cap L^2(0,T;V_A^\rho)} + \|z^h\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \le C\,\|h\|^2_{L^2(0,T;H)}\,. \tag{3.38}$$

With this estimate shown, it is a simple comparison argument in (3.24) (which we may leave to the reader) to verify that also

$$\|y^h\|_{H^1(0,T;V_A^{-\rho})} \le C\,\|h\|^2_{L^2(0,T;H)}\,.$$

Now observe that $H^1(0,T;V_A^{-\rho})\cap L^2(0,T;V_A^\rho)$ is continuously embedded in $C^0([0,T];H)$, so that (3.17) is satisfied with a function of the form $Z(s)=\widehat{C}s^2$, for a sufficiently large $\widehat{C}>0$. This concludes the proof of the assertion. □

As an immediate consequence of Theorem 3.3, we now deduce a first necessary optimality condition for the optimal control problem **(CP)**.

Corollary 3.4. *Suppose that the conditions* **(F1)–(F4)**, **(A1)–(A8)**, *and* **(GB)** *hold true. Moreover, let* $\overline{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control for the problem* **(CP)** *and* $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$. *Then, there holds the variational inequality*

$$\beta_1 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\,\xi(T) \,+\, \beta_2 \iint_Q (\overline{\varphi} - \varphi_Q)\,\xi \,+\, \beta_3 \int_\Omega (\overline{\vartheta}(T) - \vartheta_\Omega)\,\eta(T)$$
$$+\, \beta_4 \iint_Q (\overline{\vartheta} - \vartheta_Q)\,\eta \,+\, \beta_5 \iint_Q \overline{u}\,(u - \overline{u}) \,\geq\, 0 \qquad \forall\, u \in \mathcal{U}_{\rm ad}, \tag{3.39}$$

where (η, ξ) *is the unique solution to the linearized system* (3.1)–(3.3) *associated with* $h = u - \overline{u}$.

Proof. By virtue of the quadratic form of $\mathcal{J}$ and Theorem 3.3, the reduced cost functional $\widetilde{\mathcal{J}}(u) := \mathcal{J}(\mathcal{S}(u), u)$ is Fréchet differentiable on $\mathcal{U}_R$. Since $\mathcal{U}_{\rm ad}$ is convex, we must have $D\widetilde{\mathcal{J}}(\overline{u})[u - \overline{u}] \geq 0$ for all $u \in \mathcal{U}_{\rm ad}$. The result then follows in a standard manner from the chain rule and Theorem 3.3. □

4 The optimal control problem

In this section, we investigate the optimal control problem **(CP)**.

4.1 Existence of optimal controls

We begin our analysis of **(CP)** with an existence result.

Theorem 4.1. *Suppose that* **(F1)–(F4)** *and* **(A1)–(A4)** *are fulfilled. Then* **(CP)** *has a solution.*

Proof. We pick a minimizing sequence $\{u_n\} \subset \mathcal{U}_{\rm ad}$ and set $(\vartheta_n, \varphi_n) := \mathcal{S}(u_n)$, for all $n \in \mathbb{N}$. We fix $R > 0$ such that $\mathcal{U}_{\rm ad} \subset \mathcal{U}_R$ and account for (2.25). Hence, invoking standard compactness results (cf., e.g., [40, Sect. 8, Cor. 4] for the strong compactness), we may assume that there are $u \in \mathcal{U}_{\rm ad}$ and (ϑ, φ) such that, at least for a subsequence,

$$u_n \to u \quad \text{weakly-star in } L^\infty(Q), \tag{4.1}$$
$$\vartheta_n \to \vartheta \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \quad \text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q, \tag{4.2}$$
$$\varphi_n \to \varphi \quad \text{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma), \quad \text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q. \tag{4.3}$$

We now show that $(\vartheta, \varphi) = \mathcal{S}(u)$ which implies that the pair $((\vartheta, \varphi), u)$ is admissible for **(CP)**. Once this is proved, the lower semicontinuity of norms shows that $((\vartheta, \varphi), u)$ is an optimal pair.

At first, note that obviously $\vartheta(0) = \vartheta_0$ and $\varphi(0) = \varphi_0$. In additon, by Lipschitz continuity, $F_2'(\varphi_n) \to F_2'(\varphi)$ and $\ell(\varphi_n) \to \ell(\varphi)$, both strongly in $C^0([0,T];H)$. Since $\{\ell(\varphi_n)\vartheta_n\}$ is easily seen to be bounded in $L^2(Q)$, the latter entails that $\ell(\varphi_n)\vartheta_n \to \ell(\varphi)\vartheta$ weakly in $L^2(0,T;H)$.

Now, we write the time-integrated version of (2.21), written for $u = u_n$ and $(\vartheta, \varphi) = (\vartheta_n, \varphi_n)$, with test functions $v \in L^2(0,T;V_A^\rho)$. Taking the limit as $n \to \infty$, we find that (ϑ, φ) satisfies the time-integrated version of (2.21), which is equivalent to (2.21) itself. It remains to show the validity of (2.22). To this end, we use the semicontinuity of F_1 and (4.3), which yield that $0 \le F_1(\varphi) \le \liminf_{n\to\infty} F_1(\varphi_n)$ a.e. in Q. Then, owing to Fatou's lemma and to (2.25),

$$0 \le \iint_Q F_1(\varphi) \le \liminf_{n\to\infty} \iint_Q F_1(\varphi_n) \le K_1.$$

In particular, $F_1(\varphi) \in L^1(Q)$. Now, the quadratic form $v \mapsto \int_0^T \|B^\sigma v(t)\|^2\,dt$ is lower semicontinuous on $L^2(0,T;V_B^\sigma)$. Thus, starting from (2.24) written for (ϑ_n,φ_n), we can deduce, for every $v \in L^2(0,T;V_B^\sigma)$, the following chain:

$$\begin{aligned}
&\int_0^T (B^\sigma\varphi(t), B^\sigma(\varphi(t)-v(t)))\,dt + \iint_Q F_1(\varphi)\\
&\le \liminf_{n\to\infty}\Big(\int_0^T (B^\sigma\varphi_n(t), B^\sigma(\varphi_n(t)-v(t))\,dt + \iint_Q F_1(\varphi_n)\Big)\\
&\le \liminf_{n\to\infty}\Big(\iint_Q (\ell(\varphi_n)\vartheta_n - \partial_t\varphi_n - F_2'(\varphi_n))(\varphi_n - v) + \iint_Q F_1(v)\Big)\\
&= \iint_Q (\ell(\varphi)\vartheta - \partial_t\varphi - F_2'(\varphi))(\varphi - v) + \iint_Q F_1(v)\,.
\end{aligned} \tag{4.4}$$

In other words, (ϑ,φ) satisfies (2.24), which is equivalent to (2.22). This concludes the proof of the assertion. □

4.2 Necessary optimality conditions

We now turn our interest to the derivation of first-order necessary optimality conditions. To this end, we assume that $\overline{u} \in \mathcal{U}_{\rm ad}$ is an optimal control with associated state $(\overline{\vartheta},\overline{\varphi})$, and we assume that all of the general assumptions **(F1)–(F4)**, **(A1)–(A8)**, and **(GB)** are satisfied. Hence, in particular, the double obstacle potential $F_{2\rm obs}$ is excluded from the consideration. We aim at eliminating the expressions involving (η,ξ) from the variational inequality (3.39) by means of the adjoint state variables. The adjoint system formally reads:

$$-\partial_t q - \ell(\overline{\varphi})p + A^{2\rho}q = \beta_4(\overline{\vartheta} - \vartheta_Q) \quad \text{in } Q, \tag{4.5}$$

$$-\partial_t p - \ell(\overline{\varphi})\partial_t q + B^{2\sigma}p + F''(\overline{\varphi})p - \ell'(\overline{\varphi})\overline{\vartheta}\,p = \beta_2(\overline{\varphi} - \varphi_Q) \quad \text{in } Q, \tag{4.6}$$

$$q(T) = \beta_3(\overline{\vartheta}(T) - \vartheta_\Omega), \quad p(T) = \beta_1(\overline{\varphi}(T) - \varphi_\Omega) - \beta_3\ell(\overline{\varphi}(T))(\overline{\vartheta}(T) - \vartheta_\Omega) \quad \text{in } \Omega. \tag{4.7}$$

Owing to the low regularity of the final data appearing in (4.7), we cannot expect to obtain a strong solution to this system. Indeed, it turns out that (4.6) is meaningful only in its weak form

$$\begin{aligned}
&\langle -\partial_t p(t), v\rangle_{V_B^\sigma} - (\ell(\overline{\varphi}(t))\partial_t q(t), v) + (B^\sigma p(t), B^\sigma v) + (F''(\overline{\varphi}(t))p(t), v)\\
&\quad - (\ell'(\overline{\varphi}(t))\overline{\vartheta}(t)p(t), v) = \beta_2(\overline{\varphi}(t) - \varphi_Q(t), v)\\
&\quad \text{for all } v \in V_B^\sigma \text{ and a.e. } t \in (0,T).
\end{aligned} \tag{4.8}$$

Another point is that, in order to derive a priori bounds, one would like to test (4.6) by p and (4.5) by $-\partial_t q$, which makes it necessary to assume that the associated final datum belongs to V_A^ρ. We thus postulate:

(A9) It holds $\beta_3\,\vartheta_\Omega \in V_A^\rho$.

This condition is satisfied if $\beta_3 = 0$ or $\vartheta_\Omega \in V_A^\rho$. In the first case, there is no endpoint tracking of the temperature in the cost functional, while in the second the regularity of the target function coincides with that of the associated state (which makes sense). We have the following well-posedness result:

Theorem 4.2. *Let the assumptions* **(F1)–(F4)**, **(A1)–(A9)**, *and* **(GB)** *be fulfilled, and let* $\overline{u} \in \mathcal{U}_{\rm ad}$ *be given with associated state* $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$. *Then the adjoint problem* (4.5), (4.8), (4.7) *has a unique solution* (p, q) *such that*

$$q \in H^1(0,T;H) \cap C^0([0,T];V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{4.9}$$

$$p \in H^1(0,T;V_B^{-\sigma}) \cap C^0([0,T];H) \cap L^2(0,T;V_B^\sigma). \tag{4.10}$$

Proof. As in the proof of Theorem 3.1, we use a Faedo–Galerkin technique with the eigenfunctions of the operators A and B. With the notations used there, we look for functions of the form

$$q_n(x,t) = \sum_{j=1}^n a_j(t) e_j(x), \quad p_n(x,t) = \sum_{j=1}^n b_j(t) e_j'(x),$$

satisfying the system

$$\begin{aligned}
&-(\partial_t q_n(t), v) - (\ell(\overline{\varphi}(t)) p_n(t), v) + (A^\rho q_n(t), A^\rho v) \,=\, (g_4(t), v) \\
&\qquad \text{for all } v \in V_n \text{ and a.e. } t \in (0,T), \qquad (4.11)\\
&-(\partial_t p_n(t), v) - (\ell(\overline{\varphi}(t)) \partial_t q_n(t), v) + (B^\sigma p_n(t), B^\sigma v) + (F''(\overline{\varphi}(t)) p_n(t), v) \\
&\quad - (\ell'(\overline{\varphi}(t)) \overline{\vartheta}(t) p_n(t), v) \,=\, (g_2(t), v) \quad \text{for all } v \in V_n' \text{ and a.e. } t \in (0,T), \qquad (4.12)\\
&(q_n(T), v) = (g_3, v) \quad \forall\, v \in V_n\,, \quad (p_n(T), v) = (g_1, v) - (\ell(\overline{\varphi}(T)) g_3, v) \quad \forall\, v \in V_n'\,, \qquad (4.13)
\end{aligned}$$

where we have set

$$g_1 = \beta_1(\overline{\varphi}(T) - \varphi_\Omega), \;\; g_2 = \beta_2(\overline{\varphi} - \varphi_Q), \;\; g_3 = \beta_3(\overline{\vartheta}(T) - \vartheta_\Omega), \;\; g_4 = \beta_4(\overline{\vartheta} - \vartheta_Q). \tag{4.14}$$

Using an analogous argument as in the proof of Theorem 3.1, we can infer that the system (4.11)–(4.13) enjoys a unique solution pair $(q_n, p_n) \in (H^1(0,T;V_n) \times H^1(0,T;V_n'))$.

We now derive a priori estimates for the approximations (q_n, p_n), where we denote by C_i, $i \in \mathbb{N}$, constants that may depend on R and the data, but not on $n \in \mathbb{N}$. To begin with, we insert $v = -\partial_t q_n(t)$ in (4.11) and $v = p_n(t)$ in (4.12), add the results, and integrate over (t,T) where $t \in [0,T)$. Noting a cancellation of two terms, and adding the same quantity $\frac{1}{2}\,\|q_n(t)\|^2 = \frac{1}{2}\,\|q_n(T)\|^2 - \int_t^T\!\int_\Omega q_n \partial_t q_n$ to both sides, we arrive at the identity

$$\begin{aligned}
&\frac{1}{2}\,\|p_n(t)\|^2 + \frac{1}{2}\,\|q_n(t)\|^2_{V_A^\rho} + \int_t^T\!\!\int_\Omega |\partial_t q_n|^2 + \int_t^T\!\!\int_\Omega |B^\sigma p_n|^2 + \int_t^T\!\!\int_\Omega F_1''(\overline{\varphi}) p_n^2 \\
&= \frac{1}{2}\,\|p_n(T)\|^2 + \frac{1}{2}\,\|q_n(T)\|^2_{V_A^\rho} - \int_t^T\!\!\int_\Omega g_4\, \partial_t q_n + \int_t^T\!\!\int_\Omega g_2 p_n - \int_t^T\!\!\int_\Omega F_2''(\overline{\varphi}) p_n^2 \\
&\quad + \int_t^T\!\!\int_\Omega \ell'(\overline{\varphi}) \overline{\vartheta}\, p_n^2 - \int_t^T\!\!\int_\Omega q_n \partial_t q_n\,. \qquad (4.15)
\end{aligned}$$

Note that the fifth term on the left-hand side of (4.15) is nonnegative due to **(F1)–(F2)**. By means of the Hölder and Young inequalities, we readily conclude that the sum of the five integrals on the right-hand side, which we denote by I, satisfies

$$
\begin{aligned}
|I| \,&\le\, \frac{1}{2}\int_t^T\!\!\int_\Omega |\partial_t q_n|^2 \,+\, C_1\Big(\int_t^T\!\!\int_\Omega \big(|g_2|^2+|g_4|^2\big) \,+\, \int_t^T\!\!\int_\Omega \big(|q_n|^2+|p_n|^2\big)\Big) \\
&\quad +\, C_2\int_t^T \|\overline{\vartheta}(s)\|_{L^4(\Omega)}\,\|p_n(s)\|\,\|p_n(s)\|_{L^4(\Omega)}\,ds \\
&\le\, \frac{1}{2}\int_t^T\!\!\int_\Omega |\partial_t q_n|^2 \,+\, C_3\Big(\int_t^T\!\!\int_\Omega \big(|g_2|^2+|g_4|^2\big) \,+\, \int_t^T\!\!\int_\Omega \big(|q_n|^2+|p_n|^2\big)\Big) \\
&\quad +\, \frac{1}{2}\int_t^T\!\!\int_\Omega \big(|p_n|^2+|B^\sigma p_n|^2\big) \,+\, C_4\int_t^T \|\overline{\vartheta}(s)\|^2_{V_A^\rho}\,\|p_n(s)\|^2\,ds\,.
\end{aligned} \tag{4.16}
$$

It remains to estimate the final value terms. At first, note that the second identity in (4.13) just means that $p_n(T)$ is the H-orthogonal projection of $g_1-\ell(\overline{\varphi}(T))\,g_3$ onto V_n'. Thus, $\|p_n(T)\|\,\le\,\|g_1\|\,+\,C_5\,\|g_3\|$. By the same token, we have that $\|q_n(T)\|\,\le\,\|g_3\|$. Now observe that $\overline{\vartheta}(T)\in V_A^\rho$. Therefore, invoking **(A9)**, we have $g_3\in V_A^\rho$. But this entails that

$$
\begin{aligned}
\|A^\rho q_n(T)\|^2 &= (q_n(T),A^{2\rho}q_n(T)) = (g_3,A^{2\rho}q_n(T)) \\
&= (A^\rho g_3, A^\rho q_n(T)), \quad \text{i.e.,}\ \ \|A^\rho q_n(T)\|\le\|A^\rho g_3\|.
\end{aligned}
$$

Hence, it turns out that $\|q_n(T)\|_{V_A^\rho}\le\|g_3\|_{V_A^\rho}$. Observing that the mapping $s\mapsto\|\overline{\vartheta}(s)\|^2_{V_A^\rho}$ belongs to $L^1(0,T)$, we obtain from the above estimates, using Gronwall's lemma, that

$$
\begin{aligned}
&\|q_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)} \,+\, \|p_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V_B^\sigma)} \\
&\le\, C_6\left(\|g_1\|_{L^2(\Omega)}+\|g_2\|_{L^2(Q)}+\|g_3\|_{V_A^\rho}+\|g_4\|_{L^2(Q)}\right) \quad \forall\,n\in\mathbb{N}.
\end{aligned} \tag{4.17}
$$

Next, we insert $v\,=\,A^{2\rho}q_n$ in (4.11). Using the estimate $\|q_n(T)\|_{V_A^\rho}\le\|g_3\|_{V_A^\rho}$ once more, we can infer that also

$$
\|q_n\|_{L^2(0,T;V_A^{2\rho})}\,\le\,C_7\left(\|g_1\|_{L^2(\Omega)}+\|g_2\|_{L^2(Q)}+\|g_3\|_{V_A^\rho}+\|g_4\|_{L^2(Q)}\right) \quad \forall\,n\in\mathbb{N}, \tag{4.18}
$$

and comparison in (4.12) shows that

$$
\|p_n\|_{H^1(0,T;V_B^{-\sigma})}\,\le\,C_8\left(\|g_1\|_{L^2(\Omega)}+\|g_2\|_{L^2(Q)}+\|g_3\|_{V_A^\rho}+\|g_4\|_{L^2(Q)}\right) \quad \forall\,n\in\mathbb{N}. \tag{4.19}
$$

From the above estimates there follows the existence of a pair (q,p) such that, possibly only on a subsequence which is still indexed by n,

$$
q_n\to q \quad \text{weakly-star in}\ \ H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho}), \tag{4.20}
$$

$$
p_n\to p \quad \text{weakly-star in}\ \ H^1(0,T;V_B^{-\sigma})\cap L^\infty(0,T;H)\cap L^2(0,T;V_B^\sigma). \tag{4.21}
$$

Moreover, by continuous embedding, $q\in C^0([0,T];V_A^\rho)$ and $p\in C^0([0,T];H)$.

At this point, it is a standard argument (which needs no repetition here) to show that (q,p) is a solution to the system (4.5), (4.8), (4.7). It remains to show uniqueness. To this end, let (q_i,p_i), $i=1,2$, be two solutions, and $q=q_1-q_2$, $p=p_1-p_2$. Then (q,p) solves (4.5), (4.8), (4.7) with zero right-hand sides. We now repeat the estimates leading to (4.17) for the continuous problem, concluding that $q=p=0$. The assertion is thus proved. □

We now can eliminate the variables (η, ξ) from the variational inequality (3.39).

Theorem 4.3. *Let the assumptions* **(F1)–(F4)**, **(A1)–(A9)**, *and* **(GB)** *be fulfilled, and let* $\overline{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control of problem* **(CP)** *with associated state* $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}(\overline{u})$ *and adjoint state* (p, q). *Then it holds the variational inequality*

$$\iint_Q q(u - \overline{u}) \, + \beta_5 \iint_Q \overline{u}(u - \overline{u}) \, \geq \, 0 \quad \forall \, u \in \mathcal{U}_{\rm ad} \, . \tag{4.22}$$

Proof. We fix $u \in \mathcal{U}_{\rm ad}$ and consider the associated linearized system (3.1)–(3.3) with $h = u - \overline{u}$. We multiply (3.1) by q and (3.2) by p, add the results, and integrate over Q. We obtain

$$\begin{aligned} \iint_Q q(u - \overline{u}) \, &= \, \iint_Q \partial_t \eta \, q + \iint_Q \bigl(\ell'(\overline{\varphi}) \, \partial_t \overline{\varphi} \, \xi + \ell(\overline{\varphi}) \, \partial_t \xi \bigr) \, q + \iint_Q \eta \, A^{2\rho} q \\ &\quad + \iint_Q \partial_t \xi \, p + \int_0^T (B^\sigma p(t), B^\sigma \xi(t)) \, dt \\ &\quad + \iint_Q F''(\overline{\varphi}) \, \xi \, p - \iint_Q \ell'(\overline{\varphi}) \, \overline{\vartheta} \, \xi \, p - \iint_Q \ell(\overline{\varphi}) \, \eta \, p \, . \end{aligned}$$

By also integrating by parts with respect to time in three of the terms, we deduce that

$$\begin{aligned} \iint_Q q(u - \overline{u}) \, &= \, \int_\Omega \bigl(\eta(T) q(T) + \ell(\overline{\varphi}(T)) \xi(T) q(T) + \xi(T) p(T) \bigr) \\ &\quad + \iint_Q \eta \, \bigl[-\partial_t q + A^{2\rho} q - \ell(\overline{\varphi}) p \bigr] \, - \int_0^T \langle \partial_t p(t), \xi(t) \rangle_{V_B^\sigma} \, dt \\ &\quad + \int_0^T (B^\sigma p(t), B^\sigma \xi(t)) \, dt \, + \iint_Q \xi \, \bigl[-\ell(\overline{\varphi}) \partial_t q + F''(\overline{\varphi}) p - \ell'(\overline{\varphi}) \, \overline{\vartheta} \, p \bigr] \, . \end{aligned}$$

Thus, using the adjoint system (4.5), (4.8), (4.7), we find the identity

$$\begin{aligned} \iint_Q q(u - \overline{u}) \, &= \beta_1 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega) \xi(T) \, + \, \beta_2 \iint_Q (\overline{\varphi} - \varphi_Q) \xi \\ &\quad + \, \beta_3 \int_\Omega (\overline{\vartheta}(T) - \vartheta_\Omega) \eta(T) \, + \, \beta_4 \iint_Q (\overline{\vartheta} - \vartheta_Q) \eta \, . \end{aligned}$$

By combining this with (3.39), we obtain (4.22). □

Remark 4.4. If $\beta_5 > 0$, then (4.22) just means that $\overline{u}$ is the $L^2(Q)$-orthogonal projection of $-\beta_5^{-1} q$ onto $\mathcal{U}_{\rm ad}$, i.e., we have

$$\overline{u} \, = \, \max \bigl\{ u_{min}, \, \min \bigl\{ -\beta_5^{-1} q, u_{max} \bigr\} \bigr\} \quad \text{a.e. in } Q. \tag{4.23}$$

4.3 The double obstacle case

In this section, we study the case of the double obstacle potential $F_{2{\rm obs}}$ in which $F_1 = I_{[-1,1]}$ is the indicator function of the interval $[-1, 1]$ that is given by $I_{[-1,1]}(r) = 0$ for $r \in [-1, 1]$ and $I_{[-1,1]}(r) = +\infty$ otherwise. Then the conditions **(F1)** and **(F2)** are fulfilled with $(r_-, r_+) = (-1, 1)$.

For the other nonlinearities F_2 and ℓ we assume that **(F3)** and **(F4)** are valid. We then consider the following optimal control problem:

(CP$_0$) Minimize $\mathcal{J}((\vartheta, \varphi), u)$ over $\mathcal{U}_{\rm ad}$ subject to the state system (2.21)–(2.23) with $F_1 = I_{[-1,1]}$.

Remark 4.5. Notice that the condition $F_1(\varphi) \in L^1(Q)$ for our notion of solution can only be satisfied for $F_1 = I_{[-1,1]}$ if $\varphi \in [-1,1]$ almost everywhere, which in turn entails that the term involving $F_1(\varphi)$ on the left-hand side of (2.22) vanishes.

Since we cannot expect the condition **(GB)** to be satisfied in this case, the control theory developed in the previous section does not apply. We therefore argue by approximation, using the deep quench approximation, which has proved to be successful in a number of similar situations (see, e.g., [9,10,15, 17,19,37]). The general idea behind this approach is the following: we define the logarithmic functions

$$h(r) := \begin{cases} (1+r)\ln(1+r) + (1-r)\ln(1-r) & \text{if } r \in (-1,1) \\ 2\ln(2) & \text{if } r \in \{-1,1\} \\ +\infty & \text{if } r \notin [-1,1] \end{cases} \tag{4.24}$$

$$h_\alpha(r) := \alpha h(r) \quad \text{for } r \in \mathbb{R} \text{ and } \alpha \in (0,1]. \tag{4.25}$$

It is easily seen that

$$\lim_{\alpha \searrow 0} h_\alpha(r) = I_{[-1,1]}(r) \quad \forall\, r \in \mathbb{R}. \tag{4.26}$$

Moreover, $h'(r) = \ln(\frac{1+r}{1-r})$ and $h''(r) = \frac{2}{1-r^2}$, and thus

$$\begin{aligned} &\lim_{\alpha \searrow 0} h'_\alpha(r) = 0 \quad \text{for all } r \in (-1,1), \\ &\lim_{\alpha \searrow 0} \Big(\lim_{r \searrow -1} h'_\alpha(r) \Big) = -\infty, \quad \lim_{\alpha \searrow 0} \Big(\lim_{r \nearrow 1} h'_\alpha(r) \Big) = +\infty. \end{aligned} \tag{4.27}$$

Hence, we may regard the graphs of the single-valued functions h'_α over the interval $(-1,1)$ as approximations to the graph of the subdifferential $\partial I_{[-1,1]}$. Observe that this is an *interior* approximation defined in the interior of the domain of $\partial I_{[-1,1]}$ in contrast to the *exterior* approximation obtained via the Moreau–Yosida approach.

In view of (4.26)–(4.27), it is near to mind to expect that the control problem **(CP$_0$)** is closely related to the control problem (which in the following will be denoted by **(CP$_\alpha$)**) that arises when in (2.22) we choose $F_1 = h_\alpha$ for $\alpha > 0$. Indeed, by virtue of Theorem 2.3, the system (2.21)–(2.23) enjoys for both $F_1 = I_{[-1,1]}$ and $F_1 = h_\alpha$ a solution pair (ϑ, φ) and $(\vartheta_\alpha, \varphi_\alpha)$. We introduce the corresponding solution operators

$$\mathcal{S}_0 : \mathcal{U}_R \ni u \mapsto (\vartheta, \varphi), \quad \mathcal{S}_\alpha : \mathcal{U}_R \ni u \mapsto (\vartheta_\alpha, \varphi_\alpha).$$

It can be expected that $(\vartheta_\alpha, \varphi_\alpha)$ converges in a suitable topology to (ϑ, φ) as $\alpha \searrow 0$. Moreover, the optimal control problem **(CP$_\alpha$)** belongs to the class of problems for which in Section 4.2 first-order necessary optimality conditions in terms of a variational inequality and the adjoint state system have been established. One can therefore hope to perform a passage to the limit as $\alpha \searrow 0$ in the state and the adjoint state variables in order to derive meaningful first-order necessary optimality conditions also for **(CP$_0$)**.

In order to carry out this program, we now make a restrictive assumption, which still includes the classical situation:

(A10) It holds $B^{2\sigma} = B = -\Delta$ with zero Dirichlet or Neumann boundary conditions, and $A = -\Delta$ with zero Neumann or Dirichlet boundary conditions with either $\rho > \frac{3}{4}$ or $\rho = 1/2$.

Remark 4.6. If **(A10)** is valid, then the conditions **(A4)**, **(A7)** and **(A8)** are automatically satisfied.

We now assume that also the assumptions **(A1)**–**(A3)**, **(A5)** and **(A6)** are fulfilled. Now observe that under **(A10)** both the assumption (i) of Lemma 2.4 and the condition (2.28) are met (see Remark 2.5). Since the functions h_α satisfy the condition (2.29), we thus can conclude from Lemma 2.4(i) and its proof that the solutions $(\vartheta_\alpha, \varphi_\alpha)$ to the state system with $F_1 = h_\alpha$ satisfy both the boundedness condition (2.30) and the condition **(GB)** for every $\alpha > 0$. Therefore, for every $\alpha > 0$, there are constants $a_R^\alpha, b_R^\alpha, c_R^\alpha$ such that

$$-1 < a_R^\alpha \le \varphi_\alpha \le b_R^\alpha < 1 \quad \text{and} \quad |\vartheta_\alpha| \le c_R^\alpha \quad \text{a.e. in } Q, \tag{4.28}$$

whenever $(\vartheta_\alpha, \varphi_\alpha) = \mathcal{S}_\alpha(u)$ for some $u \in \mathcal{U}_R$. In addition, as it was established in Lemma 2.7, the variational inequality (2.22) takes for every $\alpha > 0$ the form of a variational equality, namely

$$\begin{aligned}&(\partial_t \varphi_\alpha(t), v) + (\nabla \varphi_\alpha(t), \nabla v) + (h'_\alpha(\varphi_\alpha(t)), v) + (F'_2(\varphi_\alpha(t)), v)\\ &\quad = (\ell(\varphi_\alpha(t))\vartheta_\alpha(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and all } v \in H^1(\Omega),\end{aligned} \tag{4.29}$$

and $(\vartheta_\alpha, \varphi_\alpha)$ is in fact a strong solution.

The approximating control problem reads:

(CP$_\alpha$) Minimize the cost functional (1.7) over $\mathcal{U}_{\rm ad}$ subject to the state system (2.21), (4.29), (2.23).

We recall Remark 4.6 and state the following approximation result.

Theorem 4.7. *Suppose that* **(F3)**, **(F4)**, **(A1)**–**(A3)**, **(A5)**–**(A6)**, *and* **(A10)** *are fulfilled, and assume that* $(\vartheta_\alpha, \varphi_\alpha) = \mathcal{S}_\alpha(u_\alpha)$ *for some* $u_\alpha \in \mathcal{U}_R$ *and* $\alpha \in (0,1]$. *Then there is some constant* $K_3 > 0$, *which depends only on* R *and the data, such that*

$$\begin{aligned}&\|\vartheta_\alpha\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho})}\\ &\quad + \|\varphi_\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;H^1(\Omega))} + \iint_Q h_\alpha(\varphi_\alpha) \le K_3.\end{aligned} \tag{4.30}$$

Moreover, there is a sequence $\{\alpha_n\} \subset (0,1]$ *with* $\alpha_n \searrow 0$ *such that*

$$u_{\alpha_n} \to u \quad \text{weakly-star in } L^\infty(Q), \tag{4.31}$$

$$\begin{aligned}\vartheta_{\alpha_n} \to \vartheta \quad &\text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}),\\ &\text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q,\end{aligned} \tag{4.32}$$

$$\begin{aligned}\varphi_{\alpha_n} \to \varphi \quad &\text{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;H^1(\Omega)),\\ &\text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q,\end{aligned} \tag{4.33}$$

where (ϑ, φ) *denotes the unique solution to the state system* (2.21)–(2.23) *for* $F_1 = I_{[-1,1]}$ *and the control* u.

Proof. The validity of the estimate (4.30) follows from a closer inspection of the derivation of the a priori estimates performed in [11]: indeed, by virtue of **(A3)** it turns out that the bounds derived there are for $F_1 = h_\alpha$ in fact independent of $\alpha \in (0,1]$. Hence, there are a sequence $\alpha_n \searrow 0$ and u, ϑ, φ satisfying (4.31)–(4.33), where the strong convergence in $C^0([0,T];H)$ follows from [40, Sect. 8, Cor. 4]. It remains to show that $(\vartheta,\varphi) = \mathcal{S}_0(u)$.

At first, it is easily seen that $\vartheta(0) = \vartheta_0$ and $\varphi(0) = \varphi_0$. Moreover, we observe that (4.33) entails, by Lipschitz continuity, that $\ell(\varphi_{\alpha_n}) \to \ell(\varphi)$ and $F_2'(\varphi_{\alpha_n}) \to F_2'(\varphi)$, both strongly in $C^0([0,T];H)$. Moreover, the sequences $\{\ell(\varphi_{\alpha_n})\partial_t\varphi_{\alpha_n}\}$ and $\{\ell(\varphi_{\alpha_n})\vartheta_{\alpha_n}\}$ are bounded in $L^2(Q)$ since ℓ is bounded. This entails that

$$\ell(\varphi_{\alpha_n})\partial_t\varphi_{\alpha_n} \to \ell(\varphi)\partial_t\varphi \quad \text{and} \quad \ell(\varphi_{\alpha_n})\vartheta_{\alpha_n} \to \ell(\varphi)\vartheta, \quad \text{both weakly in } L^2(Q).$$

Hence, we may write (2.21), with $F_1 = h_{\alpha_n}$ and control u_{α_n}, and pass to the limit as $n \to \infty$ to see that (ϑ,φ) satisfies (2.21) with control u. It remains to show (2.22) with $F_1 = I_{[-1,1]}$. We are going to prove it in the time-integrated form (2.24).

To this end, we first note that (4.30) entails that we must have $\varphi_{\alpha_n} \in [-1,1]$ a.e. in Q. Since $\varphi_{\alpha_n} \to \varphi$ pointwise a.e. in Q, also $\varphi \in [-1,1]$ a.e. in Q and thus $\iint_Q I_{[-1,1]}(\varphi) = 0$.

Now let $v \in L^2(0,T;H^1(\Omega))$ be arbitrary. If $I_{[-1,1]}(v) \notin L^1(Q)$, then the inequality is fulfilled since its right-hand side is infinite. Otherwise, we have $v \in [-1,1]$ a.e. in Q and thus $0 = I_{[-1,1]}(v) \le h_{\alpha_n}(v) \le h_1(v)$ a.e. in Q. Since, thanks to (4.26), $h_{\alpha_n}(v) \to I_{[-1,1]}(v)$ pointwise a.e. in Q, we infer from Lebesgue's dominated convergence theorem that $0 = \iint_Q I_{[-1,1]}(v) = \lim_{n\to\infty} \iint_Q h_{\alpha_n}(v)$. Therefore, using the lower semicontinuity of the quadratic form $v \mapsto \iint_Q |\nabla v|^2$ on $L^2(0,T;H^1(\Omega))$, we can infer that

$$\begin{aligned}
&\iint_Q I_{[-1,1]}(\varphi) + \iint_Q \nabla\varphi\cdot\nabla(\varphi - v) \le \liminf_{n\to\infty} \iint_Q \nabla\varphi_{\alpha_n}\cdot\nabla(\varphi_{\alpha_n} - v)\\
&\le \liminf_{n\to\infty} \Big(\iint_Q \big(\ell(\varphi_{\alpha_n})\vartheta_{\alpha_n} - \partial_t\varphi_{\alpha_n} - F_2'(\varphi_{\alpha_n})\big)(\varphi_{\alpha_n} - v) + \iint_Q h_{\alpha_n}(v)\Big)\\
&= \iint_Q \big(\ell(\varphi)\vartheta - \partial_t\varphi - F_2'(\varphi)\big)(\varphi - v) + \iint_Q I_{[-1,1]}(v)\,.
\end{aligned}$$

This finishes the proof of the assertion. □

Remark 4.8. Notice that a uniform (with respect to $\alpha \in (0,1]$) bound resembling (2.37) for $F_1 = h_\alpha$ cannot be expected to hold true, since it may well happen that $a_R^\alpha \searrow -1$ and/or $b_R^\alpha \nearrow +1$ as $\alpha \searrow 0$, so that $h_\alpha'(\varphi_\alpha)$ and $h_\alpha''(\varphi_\alpha)$ may become unbounded as $\alpha \searrow 0$.

In view of the expression (1.7) of the functional $\mathcal{J}$ and of Theorem 4.7, it is not difficult to argue that optimal controls of **(CP$_\alpha$)** are "close" to optimal controls of **(CP$_0$)**. However, from Theorems 4.7 we cannot infer sufficient information on the family of the minimizers of **(CP$_0$)**. In order to find first-order necessary optimality conditions, we recall that in the previous section we have been able to derive such conditions for the problem **(CP$_\alpha$)**. Thus, we can hope to establish corresponding results for **(CP$_0$)** by taking the limit as $\alpha \searrow 0$. However, such an approach fails since the convergence property (4.31) is too weak to pass to the limit as $\alpha \searrow 0$ in the variational inequality (4.22) (written for an optimal control $\overline{u}_\alpha$ and the corresponding adjoint state q_α). For this, we seem to need a strong convergence of $\{\overline{u}_\alpha\}$ in $L^2(Q)$.

To this end, we employ a well-known technique. Let us assume that $\overline{u} \in \mathcal{U}_{\rm ad}$ is any optimal control for **(CP$_0$)** with associated state $(\overline{\vartheta}, \overline{\varphi}) = \mathcal{S}_0(\overline{u})$. We associate with it the *adapted cost functional*

$$\widetilde{\mathcal{J}}((\vartheta,\varphi),u) := \mathcal{J}((\vartheta,\varphi),u) \, + \, \frac{1}{2}\, \|u-\overline{u}\|^2_{L^2(Q)} \tag{4.34}$$

and a corresponding *adapted optimal control problem*:

($\widetilde{\mathbf{CP}}_\alpha$) Minimize the cost functional (4.34) over $\mathcal{U}_{\rm ad}$ subject to the state system (2.21)–(2.23), where $F_1 = h_\alpha$.

With the same direct argument as in the proof of Theorem 4.1, we can show that **($\widetilde{\mathbf{CP}}_\alpha$)** has a solution. The following result indicates why the adapted control problem suits better for our intended approximation approach.

Theorem 4.9. *Suppose that* **(F3)–(F4)**, **(A1)–(A3)**, **(A5)–(A6)**, *and* **(A10)** *are fulfilled, assume that $\overline{u} \in \mathcal{U}_{\rm ad}$ is an arbitrary optimal control of* **(CP$_0$)** *with associated state $(\overline{\vartheta}, \overline{\varphi})$, and let $\{\alpha_n\} \subset (0,1]$ be any sequence such that $\alpha_n \searrow 0$ as $n \to \infty$. Then there exist a subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ of $\{\alpha_n\}$, and, for every $k \in \mathbb{N}$, an optimal control $\overline{u}_{\alpha_{n_k}} \in \mathcal{U}_{\rm ad}$ of the adapted problem* **($\widetilde{\mathbf{CP}}_{\alpha_{n_k}}$)** *with associated state $(\overline{\vartheta}_{\alpha_{n_k}}, \overline{\varphi}_{\alpha_{n_k}})$ such that, as $k \to \infty$,*

$$\overline{u}_{\alpha_{n_k}} \to \overline{u} \quad \textit{strongly in } L^2(Q), \tag{4.35}$$

and the properties (4.32) *and* (4.33) *are satisfied correspondingly. Moreover, we have*

$$\lim_{k\to\infty} \widetilde{\mathcal{J}}((\overline{\vartheta}_{\alpha_{n_k}}, \overline{\varphi}_{\alpha_{n_k}}), \overline{u}_{\alpha_{n_k}}) \, = \, \mathcal{J}((\overline{\vartheta}, \overline{\varphi}), \overline{u})\,. \tag{4.36}$$

Proof. Let $\alpha_n \searrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we pick an optimal control $u_{\alpha_n} \in \mathcal{U}_{\rm ad}$ for the adapted control problem **($\widetilde{\mathbf{CP}}_{\alpha_n}$)** and denote by $(\vartheta_{\alpha_n}, \varphi_{\alpha_n})$ the associated solution to the state system with $F_1 = h_{\alpha_n}$ and $u = u_{\alpha_n}$. By the boundedness of $\mathcal{U}_{\rm ad}$ in $L^\infty(Q)$, there is some subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that

$$u_{\alpha_{n_k}} \to u \quad \text{weakly-star in } L^\infty(Q) \quad \text{as } k \to \infty, \tag{4.37}$$

with some $u \in \mathcal{U}_{\rm ad}$, and, thanks to Theorem 4.7, the convergence properties (4.32) and (4.33) hold true with the pair $(\vartheta, \varphi) = \mathcal{S}_0(u)$. In particular, the pair $((\vartheta,\varphi),u)$ is admissible for **(CP$_0$)**.

We now aim to prove that $u = \overline{u}$. Once this is shown, it follows from the unique solvability of the state system that also $(\vartheta,\varphi) = (\overline{\vartheta}, \overline{\varphi})$, which implies that (4.32) and (4.33) hold true with (ϑ,φ) replaced by $(\overline{\vartheta}, \overline{\varphi})$.

Now observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $((\overline{\vartheta}, \overline{\varphi}), \overline{u})$ for problem **(CP$_0$)**,

$$\begin{aligned}
&\liminf_{k\to\infty} \widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \geq \mathcal{J}((\vartheta,\varphi),u) \, + \, \frac{1}{2}\, \|u-\overline{u}\|^2_{L^2(Q)} \\
&\geq \, \mathcal{J}((\overline{\vartheta}, \overline{\varphi}), \overline{u}) \, + \, \frac{1}{2}\, \|u-\overline{u}\|^2_{L^2(Q)}\,.
\end{aligned} \tag{4.38}$$

On the other hand, the optimality property of $((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}})$ for problem **($\widetilde{\mathbf{CP}}_{\alpha_{n_k}}$)** yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \, = \, \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(u_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \, \leq \, \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(\overline{u}), \overline{u})\,. \tag{4.39}$$

Finally, with the same argument used at the beginning of the proof of Theorem 4.7 to justify the estimate (4.30), one sees that $\mathcal{S}_{\alpha_{n_k}}(\overline{u})$ satisfies a similar bound, whence a subsequence (not relabeled) converges to some pair $(\underline{\vartheta}, \underline{\varphi})$ in the topologies specified in (4.32)–(4.33). As in the proof of the above-mentioned theorem, one shows that $(\underline{\vartheta}, \underline{\varphi})$ solves the original state system associated with $\overline{u}$, i.e., it coincides with $\mathcal{S}_0(\overline{u})$. Therefore, invoking the continuity properties of the cost functional with respect to the topologies of the spaces $C^0([0,T];H)$ and $L^2(Q)$, we deduce from (4.39) that

$$\begin{aligned} &\limsup_{k\to\infty} \widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \,\le\, \limsup_{k\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(\overline{u}), \overline{u}) \\ &= \widetilde{\mathcal{J}}(\mathcal{S}_0(\overline{u}), \overline{u}) \,=\, \widetilde{\mathcal{J}}((\overline{\vartheta}, \overline{\varphi}), \overline{u}) \,=\, \mathcal{J}((\overline{\vartheta}, \overline{\varphi}), \overline{u})\,. \end{aligned} \tag{4.40}$$

Combining (4.38) with (4.40), we have thus shown that $\frac{1}{2}\,\|u-\overline{u}\|^2_{L^2(Q)} = 0$, so that $u = \overline{u}$ and thus also $(\vartheta, \varphi) = (\overline{\vartheta}, \overline{\varphi})$. Moreover, (4.38) and (4.40) also imply that

$$\begin{aligned} &\mathcal{J}((\overline{\vartheta}, \overline{\varphi}), \overline{u}) \,=\, \widetilde{\mathcal{J}}((\overline{\vartheta}, \overline{\varphi}), \overline{u}) \,=\, \liminf_{k\to\infty} \widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \\ &= \limsup_{k\to\infty} \widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}}) \,=\, \lim_{k\to\infty} \widetilde{\mathcal{J}}((\vartheta_{\alpha_{n_k}}, \varphi_{\alpha_{n_k}}), u_{\alpha_{n_k}})\,, \end{aligned}$$

which proves (4.35) and (4.36) at the same time, of course along with (4.32) and (4.33). This concludes the proof of the assertion. □

We now discuss the first-order necessary optimality conditions for **($\widetilde{\mathbf{CP}}_\alpha$)**, assuming that the general assumptions **(F3)–(F4)**, **(A1)–(A3)**, **(A5)–(A6)**, **(A9)** and **(A10)** are fulfilled. Obviously, the adjoint system is the same as for **(CP$_\alpha$)**, and Theorem 4.2 and Theorem 4.3 apply to this situation. More precisely, the adjoint state (p_α, q_α) solves the variational system

$$-\,\partial_t q_\alpha - \ell(\overline{\varphi}_\alpha)\,p_\alpha + A^{2\rho} q_\alpha \,=\, g_4^\alpha \quad \text{in } Q, \tag{4.41}$$

$$\begin{aligned} &\big(-\partial_t p_\alpha(t), v\big) \,-\, \big(\ell(\overline{\varphi}_\alpha(t))\,\partial_t q_\alpha(t), v\big) + (\nabla p_\alpha(t), \nabla v) \\ &\quad + ((\psi_1^\alpha(t) + \psi_2^\alpha(t))\,p_\alpha(t), v) - (\ell'(\overline{\varphi}_\alpha(t))\overline{\vartheta}_\alpha(t) p_\alpha(t), v) \,=\, (g_2^\alpha(t), v) \\ &\quad \text{for all } v \in H^1(\Omega) \text{ and a.e. } t \in (0,T), \end{aligned} \tag{4.42}$$

$$q_\alpha(T) = g_3^\alpha, \quad p_\alpha(T) = g_1^\alpha - \ell(\overline{\varphi}_\alpha(T))\,g_3^\alpha \quad \text{in } \Omega, \tag{4.43}$$

where, for $\alpha > 0$,

$$\begin{aligned} &\psi_1^\alpha := h_\alpha''(\overline{\varphi}_\alpha), \quad \psi_2^\alpha := F_2''(\overline{\varphi}_\alpha), \quad g_1^\alpha := \beta_1(\overline{\varphi}_\alpha(T) - \varphi_\Omega), \quad g_2^\alpha := \beta_2(\overline{\varphi}_\alpha - \varphi_Q), \\ &g_3^\alpha := \beta_3(\overline{\vartheta}_\alpha(T) - \vartheta_\Omega), \quad g_4^\alpha := \beta_4(\overline{\vartheta}_\alpha - \vartheta_Q)\,. \end{aligned} \tag{4.44}$$

By virtue of the general bounds (4.28), (4.30) and owing to **(A9)**, we have that

$$\|\psi_2^\alpha\|_{L^\infty(Q)} + \|g_1^\alpha\| + \|g_2^\alpha\|_{L^2(Q)} + \|g_3^\alpha\|_{V_A^\rho} + \|g_4^\alpha\|_{L^2(Q)} \,\le\, C_1 \quad \forall\,\alpha \in (0,1], \tag{4.45}$$

where, here and in the following, $C_i > 0$, $i \in \mathbb{N}$, denote constants that may depend on the data of the system, but not on $\alpha \in (0,1]$. Observe that a corresponding bound for ψ_1^α cannot be expected.

On the other hand, the variational inequality characterizing optimal controls is different (nevertheless, obtained using the same arguments that led to (4.22) in Theorem 4.3). Namely, if $\overline{u}_\alpha \in \mathcal{U}_{\rm ad}$ is optimal for **($\widetilde{\mathbf{CP}}_\alpha$)** and (p_α, q_α) is the associated adjoint state, then we have that

$$\iint_Q (q_\alpha + \beta_5 \overline{u}_\alpha + (\overline{u}_\alpha - \overline{u}))(u - \overline{u}_\alpha) \,\ge\, 0 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{4.46}$$

Our aim is to let α tend to zero in both the above inequality and the adjoint system. Thus, we have to derive some a priori estimates for the adjoint variables that are uniform with respect to $\alpha \in (0,1]$. To this end, we note that the estimates (4.17), (4.18), derived for the Faedo–Galerkin approximations, persist by the semicontinuity of norms under limit processes, whence we infer that

$$\begin{aligned} &\|q_\alpha\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)\cap L^2(0,T;V_A^{2\rho})} + \|p_\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \\ &\leq C_2 \Big(\|g_1^\alpha\| + \|g_2^\alpha\|_{L^2(Q)} + \|g_3^\alpha\|_{V_A^\rho} + \|g_4^\alpha\|_{L^2(Q)} \Big) \leq C_3 \quad \forall\, \alpha \in (0,1]. \end{aligned} \tag{4.47}$$

However, the comparison argument leading to (4.19) does not work in this situation, because we do not have a bound for ψ_1^α. For this reason, we introduce the space

$$\mathcal{Z} := \{ v \in H^1(0,T;H^1(\Omega)^*) \cap L^2(0,T;H^1(\Omega)) :\ v(0) = 0 \}. \tag{4.48}$$

Since the embedding $(H^1(0,T;H^1(\Omega)^*) \cap L^2(0,T;H^1(\Omega))) \subset C^0([0,T];H)$ is continuous, the zero condition for the initial value is meaningful, and $\mathcal{Z}$ is a closed subspace of $H^1(0,T;H^1(\Omega)^*) \cap L^2(0,T;H^1(\Omega))$ and thus a Banach space when endowed with the natural norm of this space. Moreover, the embedding $\mathcal{Z} \subset C^0([0,T];H)$ is continuous, and we also have the dense and continuous embedding $\mathcal{Z} \subset L^2(0,T;H) \subset \mathcal{Z}^*$, where it is understood that

$$\langle v, z \rangle_{\mathcal{Z}} = \int_0^T (v(t), z(t))\, dt \quad \text{for all } v \in L^2(0,T;H) \text{ and } z \in \mathcal{Z}. \tag{4.49}$$

Now, let $v \in \mathcal{Z}$ be arbitrary and use it as test function in (4.42). By integrating (4.42) over $(0,T)$, with the help of (4.43) we find out that

$$\begin{aligned} &\int_\Omega \big(\ell(\overline{\varphi}_\alpha(T))\, g_3^\alpha - g_1^\alpha \big) v(T) + \int_0^T \langle \partial_t v(t), p_\alpha(t) \rangle_{H^1(\Omega)} dt \\ &- \iint_Q \ell(\overline{\varphi}_\alpha)\, \partial_t q_\alpha v + \iint_Q \nabla p_\alpha \cdot \nabla v + \iint_Q \psi_1^\alpha\, p_\alpha\, v \\ &+ \iint_Q \psi_2^\alpha\, p_\alpha v - \iint_Q \ell'(\overline{\varphi}_\alpha)\, \overline{\vartheta}_\alpha\, p_\alpha\, v = \iint_Q g_2^\alpha\, v. \end{aligned} \tag{4.50}$$

Then, due to **(F4)**, (4.45) and (4.47), we have that

$$\begin{aligned} &\Big| \int_\Omega \big(\ell(\overline{\varphi}_\alpha(T))\, g_3^\alpha - g_1^\alpha \big) v(T) + \int_0^T \langle \partial_t v(t), p_\alpha(t) \rangle_{H^1(\Omega)} dt \Big| \\ &\leq \big(C_3 \|g_3^\alpha\| + \|g_1^\alpha\| \big) \|v\|_{C^0([0,T];H)} \\ &\quad + \|p_\alpha\|_{L^2(0,T;H^1(\Omega))} \|\partial_t v\|_{L^2(0,T;H^1(\Omega)^*)} \leq C_4 \|v\|_{\mathcal{Z}}\,. \end{aligned} \tag{4.51}$$

Moreover, (4.47) obviously yields that

$$\|\ell(\overline{\varphi}_\alpha)\, \partial_t q_\alpha\|_{L^2(Q)} + \|\psi_2^\alpha\, p_\alpha\|_{L^2(Q)} + \|g_2^\alpha\|_{L^2(Q)} \leq C_5. \tag{4.52}$$

Furthermore, using also (4.30) for $(\overline{\vartheta}_\alpha, \overline{\varphi}_\alpha)$ and the fact that **(A10)** implies **(A8)** (see Remark 4.6) and thus also the continuity of the embedding $V_A^{2\rho} \subset L^4(\Omega)$, we deduce that

$$\begin{aligned} &\Big| \iint_Q \nabla p_\alpha \cdot \nabla v - \iint_Q \ell'(\overline{\varphi}_\alpha) \overline{\vartheta}_\alpha\, p_\alpha\, v \Big| \\ &\leq C_6 \|v\|_{L^2(0,T;H^1(\Omega))} + C_7 \int_0^T \|\overline{\vartheta}_\alpha(t)\|_{L^4(\Omega)} \|p_\alpha(t)\|_{L^4(\Omega)} \|v(t)\|\, dt \\ &\leq C_6 \|v\|_{\mathcal{Z}} + C_8 \|\overline{\vartheta}_\alpha\|_{L^2(0,T;V_A^{2\rho})} \|p_\alpha\|_{L^2(0,T;H^1(\Omega))} \|v\|_{C^0([0,T];H)} \leq C_9 \|v\|_{\mathcal{Z}} \end{aligned} \tag{4.53}$$

for all $v \in \mathcal{Z}$. Hence, comparison in (4.50) leads to

$$\|\Lambda_\alpha\|_{\mathcal{Z}^*} \le C_{10}, \quad \text{with } \Lambda_\alpha := \psi_1^\alpha p_\alpha = \alpha\, h''(\overline{\varphi}_\alpha) p_\alpha, \quad \forall\, \alpha \in (0,1]. \tag{4.54}$$

At this point, we are in a position to show the following first-order optimality result.

Theorem 4.10. *Suppose that the conditions* **(F3)–(F4)***,* **(A1)–(A3)***,* **(A5)–(A6)***,* **(A9)** *and* **(A10)** *are fulfilled, and let $\overline{u} \in \mathcal{U}_{\rm ad}$ be an optimal control for* **(CP$_0$)** *with associated state $(\overline{\vartheta}, \overline{\varphi})$. Then there exist (q,p,Λ) such that the following statements hold true:* (i) $q \in H^1(0,T;H) \cap C^0([0,T];V_A^\rho) \cap L^2(0,T;V_A^{2\rho})$,

$p \in L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega))$, *and* $\Lambda \in \mathcal{Z}^*$.

(ii) *The adjoint system, consisting of* (4.5)*, the final condition*

$$q(T) = \beta_3(\overline{\vartheta}(T) - \vartheta_\Omega) \quad \text{in } \Omega \tag{4.55}$$

and the equation

$$\begin{aligned} &\int_\Omega \big(\beta_3 \ell(\overline{\varphi}(T))(\overline{\vartheta}(T) - \vartheta_\Omega) - \beta_1(\overline{\varphi}(T) - \varphi_\Omega)\big) v(T) \\ &+ \int_0^T \langle \partial_t v(t), p(t) \rangle_{H^1(\Omega)} dt - \iint_Q \ell(\overline{\varphi})\, \partial_t q\, v + \iint_Q \nabla p \cdot \nabla v + \langle \Lambda, v \rangle_{\mathcal{Z}} \\ &+ \iint_Q F_2''(\overline{\varphi})\, p\, v - \iint_Q \ell'(\overline{\varphi}))\overline{\vartheta}\, p\, v \;=\; \beta_2 \iint_Q (\overline{\varphi} - \varphi_Q)\, v \quad \text{for all } v \in \mathcal{Z}, \end{aligned} \tag{4.56}$$

is satisfied.

(iii) *It holds the variational inequality*

$$\iint_Q (q \,+\, \beta_5\, \overline{u})(u - \overline{u}) \;\ge 0 \quad \text{for all } u \in \mathcal{U}_{\rm ad}\,. \tag{4.57}$$

Proof. We choose any sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$. By Theorem 4.9 we may assume that there are optimal controls $\overline{u}_{\alpha_n} \in \mathcal{U}_{\rm ad}$ of the adapted problem **($\widetilde{\mathbf{CP}}_{\alpha_n}$)** with associated states $(\overline{\vartheta}_{\alpha_n}, \overline{\varphi}_{\alpha_n})$ such that (4.35) and the analogues of (4.32)–(4.33) hold true. Then, we deduce that $\overline{\vartheta}_{\alpha_n} \to \overline{\vartheta}$ weakly in $C^0([0,T];V_A^\rho)$ and $\overline{\varphi}_{\alpha_n} \to \overline{\varphi}$ strongly in $C^0([0,T];L^r(\Omega))$ for $1 \le r < 6$, by virtue of, e.g., [40, Sect. 8, Cor. 4], and it also follows that

$$\begin{aligned} &\ell(\overline{\varphi}_{\alpha_n}) \to \ell(\overline{\varphi}) \quad \text{and} \quad \ell'(\overline{\varphi}_{\alpha_n}) \to \ell'(\overline{\varphi}) \\ &\quad \text{strongly in } C^0([0,T];L^r(\Omega)) \text{ for } 1 \le r < 6, \end{aligned} \tag{4.58}$$

$$g_1^{\alpha_n} \to \beta_1(\overline{\varphi}(T) - \varphi_\Omega) \quad \text{strongly in } H, \tag{4.59}$$

$$g_2^{\alpha_n} \to \beta_2(\overline{\varphi} - \varphi_Q) \quad \text{strongly in } L^2(Q), \tag{4.60}$$

$$g_3^{\alpha_n} \to \beta_3(\overline{\vartheta}(T) - \vartheta_\Omega) \quad \text{weakly in } V_A^\rho, \tag{4.61}$$

$$g_4^{\alpha_n} \to \beta_4(\overline{\vartheta} - \vartheta_Q) \quad \text{strongly in } L^2(Q), \tag{4.62}$$

as well as

$$F_2''(\overline{\varphi}_{\alpha_n}) \to F_2''(\overline{\varphi}) \quad \text{strongly in } L^r(Q) \text{ for } 1 \le r < +\infty\,, \tag{4.63}$$

since F_2'' is continuous and bounded. Moreover, by virtue of the estimates (4.47) and (4.54), and invoking [40, Sect. 8, Cor. 4] once more, there are limits q, p, Λ such that, at least for a subsequence which is again indexed by n,

$$q_{\alpha_n} \to q \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \\ \text{weakly in } C^0([0,T];V_A^\rho) \text{ and strongly in } C^0([0,T];H), \tag{4.64}$$

$$p_{\alpha_n} \to p \quad \text{weakly-star in } L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega)), \tag{4.65}$$

$$\Lambda^{\alpha_n} \to \Lambda \quad \text{weakly in } \mathcal{Z}^*. \tag{4.66}$$

With these convergence results, it is an easy task to show that

$$\ell(\overline{\varphi}_{\alpha_n})p_{\alpha_n} \to \ell(\overline{\varphi})p, \quad \ell(\overline{\varphi}_{\alpha_n})\partial_t q_{\alpha_n} \to \ell(\overline{\varphi})\partial_t q, \quad F_2''(\overline{\varphi}_{\alpha_n})p_{\alpha_n} \to F_2''(\overline{\varphi})p, \\ \ell'(\overline{\varphi}_{\alpha_n})\overline{\vartheta}_{\alpha_n}p_{\alpha_n} \to \ell'(\overline{\varphi})\overline{\vartheta}\, p, \quad \text{all weakly in } L^1(Q), \tag{4.67}$$

by using for the latter (4.58) with $r = 4$, the strong convergence $\overline{\vartheta}_{\alpha_n} \to \overline{\vartheta}$ in $C^0([0,T];H)$, and the weak convergence $p_{\alpha_n} \to p$ in $L^\infty(0,T;L^4(\Omega))$ ensured by (4.32) and (4.65), respectively. A fortiori, since all of the sequences occurring in (4.67) are bounded in $L^2(Q)$, we even have weak convergence in $L^2(Q)$.

At this point, we write the variational inequality (4.46) for $\alpha = \alpha_n$, $n \in \mathbb{N}$, and pass to the limit as $n \to \infty$, which immediately yields the validity of (4.57). Next, we easily see that the final value condition (4.55) holds true. Moreover, writing (4.41) with $\alpha = \alpha_n$, $n \in \mathbb{N}$, and passing to the limit as $n \to \infty$, we recover (4.5). It remains to show that (4.56) is satisfied, but this can be easily achieved by taking the limit in (4.50) written for $\alpha = \alpha_n$, because of (4.59)–(4.61) and (4.65)–(4.67). With this, the assertion is proved. □

Remark 4.11. Unfortunately, we are unable to derive any complementarity slackness conditions for the Lagrange multiplier Λ. Indeed, while it is easily seen that

$$\liminf_{n\to\infty} \iint_Q \Lambda_{\alpha_n}\, p_{\alpha_n} \;=\; \liminf_{n\to\infty} \iint_Q \alpha_n\, h''(\overline{\varphi}_{\alpha_n})\, |p_{\alpha_n}|^2 \;\geq\; 0 \quad \forall\, n \in \mathbb{N},$$

the available convergence properties do not suffice to conclude that $\langle \Lambda, p \rangle_{\mathcal{Z}} \;\geq\; 0$.

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