Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

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Lorenzo Taggi

submitted: May 18, 2020

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: lorenzo.taggi@wias-berlin.de

No. 2722
Berlin 2020

2010 *Mathematics Subject Classification.* 82C22, 60K35, 82C26.

Key words and phrases. Essential enhancements, activated random walks, Abelian networks, self-organised criticality, absorbing-state phase transition.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We prove that in wide generality the critical curve of the activated random walk model is a continuous function of the deactivation rate, and we provide a bound on its slope which is uniform with respect to the choice of the graph. Moreover, we derive strict monotonicity properties for the probability of a wide class of 'increasing' events, extending previous results of Rolla and Sidoravicius (2012). Our proof method is of independent interest and can be viewed as a reformulation of the 'essential enhancements' technique – which was introduced for percolation – in the framework of Abelian networks.

1 Introduction

The activated random walk model (ARW) is a particle system with conserved number of particles. It is a special case of a class of models introduced by Spitzer in the '70s and it is not only of great mathematical interest but also physically relevant due to its connections to *self-organised criticality* [7]. The informal definition of the model is as follows. Let $G = (V, E)$ be a infinite undirected vertex-transitive graph (for example $\mathbb{Z}^d$ or a regular tree). Each particle can either be of type A (active) or of type S (sleeping, or inactive). At time zero, the number of particles is sampled according to a Poisson distribution with parameter $\mu \in [0, \infty)$ independently at every vertex, where μ is the *particle density*, and every particle is of type A. An independent exponential clock with rate $\lambda \in [0, \infty)$, the *deactivation rate,* is associated to every active particle. Every A-particle performs a continuous time simple random walk independently until its own clock rings. When this happens, the A-particle turns into the S-state. Every S-particle is at rest. Moreover, whenever a S-particle shares the vertex with an A-particle, the S-particle is instantaneously activated, i.e, it becomes an A-particle. It follows from this definition that, almost surely, a particle of type S can be observed only if it does not share the vertex with other particles.

Let $\mathbb{P}_{\lambda,\mu}$ be the probability measure of the interacting particle system defined informally above, whose existence on infinite vertex-transitive graphs was proved in [11]. A central and natural question is whether the dynamics dies out with time or whether it is sustained at all times. More precisely, we say that the system *fixates* if for every finite set $A \subset V$ there exists a time $t_A < \infty$ such that for any time $t > t_A$ no active particle jumps from a vertex of A, and that it is *active* if it does not fixate. The *critical density* is defined as,

$$\forall \lambda \in [0, \infty), \quad \mu_c(\lambda) := \inf \big\{ \mu \in \mathbb{R}_{\geq 0} \ : \ \mathbb{P}_{\lambda,\mu}(\, ARW \text{ is active }) > 0 \big\}. \tag{1.1}$$

It was proved in [9] that the probability that the model is active is either zero or one, that it does not decrease with μ and does not increase with λ. This ensures the existence of a unique transition point between the regime of a.s. local fixation and the regime of a.s. activity. In recent years significant effort has been made for proving basic properties of the critical curve, $\mu = \mu_c(\lambda)$. It is known from [13]

DOI 10.20347/WIAS.PREPRINT.2722 Berlin 2020

that $\mu_c(\lambda) \leq 1$ for any $\lambda \in [0, \infty)$ in wide generality, it was proved in [15, 17] that $\mu_c(\lambda) < 1$ for any $\lambda \in (0, \infty)$ and that $\mu_c(\lambda) \to 0$ as $\lambda \to 0$ in any vertex-transitive graph where the random walk is transient, extending previous results for biased jump distributions on $\mathbb{Z}^d$ [11, 16]. Moreover, it was proved in [1, 4] that, on $\mathbb{Z}$, $\mu_c(\lambda) = O(\sqrt{\lambda})$ in the limit as $\lambda \to 0$. It was proved in [15] that $\mu_c(\lambda) \geq \frac{\lambda}{1+\lambda}$ in any vertex-transitive graphs, generalising and extending a previous result of [14]. It was proved in [10] that the critical density is universal. Our first main theorem states a new general property of the critical curve, namely that it is a continuous function of the deactivation parameter λ.

Theorem 1.1. *On any vertex-transitive graph the two following properties hold:*

(i) $\mu_c(\lambda)$ *is a continuous function of* λ *in* $(0, \infty)$*,*

(ii) for any $\lambda \in (0, \infty)$*,* $\limsup_{\delta \to 0} \frac{\mu_c(\lambda+\delta) - \mu_c(\lambda)}{\delta} \leq \frac{1}{\lambda(1+\lambda)}$.

The property of continuity of the critical curve was proved for $\lambda = 0$ (more precisely, right-continuity, namely $\lim_{\lambda \to 0^+} \mu_c(\lambda) = \mu_c(0) = 0$) in $\mathbb{Z}^d$ when $d = 1$ [4] and $d \geq 3$ [15] and, more generally, in vertex-transitive graphs where the random walk is transient [15]. Our Theorem 1.1 extends such a continuity property to all positive values of λ and holds for any vertex-transitive graph. Even though the critical curve is expected to strongly depend on the graph, the second claim of Theorem 1.1 provides a bound on its slope which is uniform with respect to the choice of the graph. Our more general formulation of Theorem 1.1, Theorem 5.2 below, does not require the assumption that the graph is vertex-transitive and it involves a more general notion of critical density (see Section 5.2). Furthermore, such a general formulation of Theorem 1.1 does not require the assumption that all the particles at time zero are active. For example, our main result also holds if the initial particle configuration is distributed as a product of Poisson distributions with parameter μ such that only the particles at the origin are active and all the remaining particles are inactive. Under these assumptions, the activated random walk model is related to the Frog model [8], which corresponds to the special case $\lambda = 0$.

Monotonicity properties. Our first main theorem is a consequence of our second theorem, which derives new general monotonicity properties for the probability of a wide class of events called 'increasing'. This class will be defined later formally and, for example, it includes any event of the form, $\mathcal{A} = \{\forall x \in K \;\; M(x) \geq H(x)\}$, where here $K \subset V$ is any finite set of sites, $(H(x))_{x \in K}$ is any integer-valued vector, and $M(x)$ is the number of times the active particles jump from the vertex x. The derivation of monotonicity properties is very useful and allows a deeper understanding of the model. From the definition of the activated random walk dynamics it is reasonable to expect that the probability of any increasing event is *non-increasing* with respect to λ and *non-decreasing* with respect to μ. The proof of this claim is non-trivial and was derived in [9] by employing a graphical representation. Here we address the following related question: do monotonicity properties hold if we increase the deactivation rate and the particle density *at the same time*? This question is challenging since the increase of the deactivation rate and of the particle density play against each other – higher deactivation rate implies that the model is 'less active', while higher particle density implies that the model is 'more active'. Our Theorem 1.2 below provides a positive answer to this question and shows that, if we increase the deactivation rate and the particle density at the same time and the increase of the particle density occurs 'fast enough', then the probability of any increasing event is also non decreasing. More precisely, our theorem states that, if we take an arbitrary point of the phases diagram, $(\lambda, \mu) \in \mathbb{R}^2_{>0}$, and we move up-right along a semi-line line which starts from (λ, μ) and whose slope, s, satisfies $s \geq \frac{1}{\lambda(1+\lambda)}$, then the probability of the event does not decrease. Remarkably, our estimate on the minimal slope is uniform not only with respect to the choice of the graph, but also with respect to the choice of the event, provided that it is increasing. The monotonicity result of Rolla and Sidoravicius [9] can thus be viewed as corresponding to the special case $s = \infty$ of our theorem (the probability that the system is active is non-decreasing if we move upwards in the phase diagram, namely if we increase the particle density without varying the deactivation rate).

Theorem 1.2. *Let $\mathcal{A}$ be any increasing event as in the statement of Theorem 3.2 below. Let $(\lambda, \mu) \in \mathbb{R}^2_{\geq 0}$ be an arbitrary point of the phase diagram, let $\mathcal{C}_{\lambda,\mu}$ be the region above the semi-line with slope $\frac{1}{\lambda(1+\lambda)}$ which starts from (λ, μ),*

$$\mathcal{C}_{\lambda,\mu} := \Big\{(x, y) \in \mathbb{R}^2 \ : \ y \geq \frac{1}{\lambda(1+\lambda)} \ (x - \lambda) + \mu, \ x \geq \lambda \Big\}. \tag{1.2}$$

Then, for any pair $(\lambda', \mu') \in \mathcal{C}_{\lambda,\mu}$,

$$\mathbb{P}_{\lambda,\,\mu}(\mathcal{A}) \leq \mathbb{P}_{\lambda',\,\mu'}(\mathcal{A}).$$

We refer to Section 3 for a precise characterisation of the events for which our theorem holds, this is a general and natural class of events.

Proof method: Essential enhancements. Our proof method can be viewed as a reformulation of the 'Essential enhancements' technique – which was mostly employed in Percolation [2, 3] – in the framework of Abelian networks and can be employed for the the study of other Abelian models, for example the frog model [8], oil and water [5, 6], or the stochastic sandpile model [9].

Our proof uses the setting of the Diaconis-Fulton graphical representation [9], where some random instructions – operators which act on the particle configuration moving active particles to their neighbours or trying to let the A-particle turn into a S-particle – are used to mimic the dynamics without employing the variable 'time'. Such a graphical representation fulfils the fundamental Abelian property which, informally, states that the relevant quantities – for example the number of times the active particles jumps from a given vertex – do not depend on the order according to which such instructions are used. Our proof is divided into three main steps.

The first step of the proof is the derivation of a Russo's formula [12] – which is a classical formula in percolation – for activated random walks, Theorem 3.2 below. This formula relates the partial derivative with respect to λ of the probability of increasing events to the expected number of instructions which are 'sleeping essential' for the event. Such instructions will be defined later and, informally, are those instructions whose removal would cause the occurrence of the event. Similarly, such a formula relates the partial derivative with respect to μ of the probability of an increasing event to the expected number of vertices which are 'particle essential' for the event, namely vertices such that the addition of one more particle there would cause the occurrence of the event.

In the second step of the proof we derive the following differential inequality, which holds for increasing events $\mathcal{A}$,

$$-\frac{\partial}{\partial \lambda}\mathcal{P}_{\lambda,\mu}(\mathcal{A}) \leq \frac{1}{\lambda(1+\lambda)}\frac{\partial}{\partial \mu}\mathcal{P}_{\lambda,\mu}(\mathcal{A}), \tag{1.3}$$

where $\mathcal{P}_{\lambda,\mu}$ is the law of the initial particle configuration and of the random instructions. The two following properties of the *odometer* – a fundamental quantity which counts how many times the active particles jump from any vertex – are derived and used for the proof of (1.3). The first property is that the removal of a 'sleep' instruction does not affect the value of the odometer, unless such a removed instruction occupies a very specific location in the array of instructions. Such a property allows us to the deduce that, on any given vertex, at most one instruction is 'sleeping essential'. The second property is as follows: If removing a sleep instruction lets the event $\mathcal{A}$ occur, then also adding a particle at the same vertex lets the event $\mathcal{A}$ occur, provided that $\mathcal{A}$ is increasing. This leads to the conclusion that, if on a vertex we have a sleeping-essential instruction, then the vertex is also particle-essential. Such two properties combined allow the comparison between the partial derivatives and lead to (1.3).

In the third step we derive our monotonicity theorem by using the differential inequality, (1.3), and we derive our main continuity theorem by using our monotonicity theorem.

We conclude with some natural questions which might be answered by further developing such a framework. To begin, the derivation of the inverse of the inequality (1.3) (with some other positive and bounded constant uniformly in $\mathcal{A}$ in place of $\frac{1}{\lambda(1+\lambda)}$) would allow us to answer the following open question.

Open Question 1. *Prove that $\mu_c(\lambda)$ is strictly increasing with respect to λ.*

A further central open problem for the ARW model is proving that $\mu_c(\lambda) \to 0$ as $\lambda \to 0$, which was not proved in the general case yet. Since $\mu_c(0) = 0$, answering the following question would lead to the solution of this problem in great generality.

Open Question 2. *Extend the continuity property of Theorem 1.1 to $\lambda = 0^+$.*

Organisation. This paper is organised as follows. In Section 2 we recall the properties of the Diaconis-Fulton representation and present some new definitions and basic lemmas. In Section 3 we derive the equivalent of Russo's formula for activated random walk. In Section 4 present the proof of (1.3). In Section 5 we present the proof of our main theorems, Theorem 1.1 and 1.2, and the generalisation of our main continuity theorem, Theorem 5.2.

Notation

$G = (V, E)$	infinite locally-finite undirected graph
$o \in V$	a reference vertex, called *origin*
d_x	degree of the vertex x
$\eta = (\eta(x))_{x \in V}$	particle configuration
$\tau = (\tau^{x,j})_{x \in V, j \in \mathbb{N}}$	array of instructions
$\mathcal{H} \times \mathcal{I}$	set of realisations, with $\eta \in \mathcal{H}$ and $\tau \in \mathcal{I}$
$\boldsymbol{s}$	sleep instruction
τ_{xy}	instruction 'jump from x to y'
$J_\tau^{x,\ell}$	ℓth jump instruction of τ at x
$t_\tau^{x,\ell}$	index of the ℓth jump instruction of τ at x, namely $\tau^{x,t_\tau^{x,\ell}} = J_\tau^{x,\ell}$
$m_{K,\eta,\tau}$	odometer
$M_{K,\eta,\tau}$	jump-odometer
$S_\tau^{x,\ell}$	number of sleep instructions between the $\ell - 1$th and the ℓth jump instruction at x
$\eta^{(x,k)}$	particle configuration obtained from η by setting to k the number of particles at x
$\Gamma_-^{x,\ell}(\tau)$	array with no sleep instr. between the $\ell - 1$th and the ℓth jump instr.
$\Gamma_1^{x,\ell}(\tau)$	array with one sleep instr. between the $\ell - 1$th and the ℓth jump instr.
$\nu = \nu(\mu)$	distribution of the initial particle configuration
$\nu_j = \nu_j(\mu)$	probability that a site hosts $j \in \mathbb{N}$ particles at time zero.

2 Definitions and graphical representation

In this section we introduce the Diaconis-Fulton graphical representation for the dynamics of ARW, following [9], and we introduce the main definitions.

Graph. To begin, we fix a graph $G = (V, E)$, which is always assumed to be *undirected*, *infinite* and *locally finite*. For any $x \in V$, we denote by d_x the *degree of the vertex* x, which corresponds to the number of vertices which are connected to x by an edge. We choose one vertex of G arbitrarily, $o \in V$, and we call it *origin*. We write $x \sim y$ when x and y are neighbours, i.e, $\{x, y\} \in E$.

Particle configuration and array of instructions. The set of particle configurations is denoted by $\mathcal{H} = \{0, \rho, 1, 2, 3, \dots\}^V$, where a vertex being in state ρ denotes that the vertex has one S-particle, while being in state $i \in \{0, 1, 2, \dots\}$ denotes that the vertex contains i A-particles. We employ the following order on the states of a vertex: $0 < \rho < 1 < 2 < \cdots$. In a configuration $\eta \in \mathcal{H}$, a vertex $x \in V$ is called *stable* if $\eta(x) \in \{0, \rho\}$, and it is called *unstable* if $\eta(x) \geq 1$. We denote by $\mathcal{I}$ the *set of arrays of instructions*, i.e, each element of $\mathcal{I}$ is an array of instructions $\tau = \left(\tau^{x,j}\right)_{x \in V, j \in \mathbb{N}}$, where for each $x \in V$ and $j \in \mathbb{N}$,

$$\tau^{x,j} \in \{\boldsymbol{s}\} \cup \{\tau_{xy} \,:\, y \sim x\},$$

where τ_{xy} and $\boldsymbol{s}$, called *jump* and *sleep instruction* respectively, are operators acting on the particle configuration which are defined as follows. Given any configuration η such that x is unstable, performing the instruction τ_{xy} in η yields another configuration η' such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x, y\}$, $\eta'(x) = \eta(x) - \mathbb{1}\{\eta(x) \geq 1\}$, and $\eta'(y) = \eta(y) + \mathbb{1}\{\eta(x) \geq 1\}$. We use the convention that $1 + \rho = 2$. Similarly, performing the instruction $\boldsymbol{s}$ to η yields a configuration η' such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x\}$, and if $\eta(x) = 1$ we have $\eta'(x) = \rho$, otherwise $\eta'(x) = \eta(x)$.

Using instructions and stabilising a set. Fix a particle configuration $\eta \in \mathcal{H}$ and an instruction array $\tau \in \mathcal{I}$. We say that the instruction $\tau^{x,j}$ is *legal* for η if x is unstable in η, otherwise it is *illegal*. We say that we *use the instruction* (x, j), $x \in V$, $j \in \mathbb{N}$, of the array τ for η, or that we use the instruction $\tau^{x,j}$ for η, when we act on the current particle configuration η through the operator $\tau^{x,j}$. Let α be a sequence

$$\alpha = \big((x_1, n_1), (x_2, n_2), \dots, (x_k, n_k)\big),$$

define the operator $\Phi_{\alpha,\tau}$ as

$$\Phi_{\alpha,\tau} := \tau^{x_k,n_k} \;\dots\; \tau^{x_2,n_2}\, \tau^{x_1,n_1},$$

and for $1 \leq \ell \leq k$ define the subsequence $\alpha^{(\ell)} := \big((x_1, n_1), (x_2, n_2), \dots, (x_\ell, n_\ell)\big)$. We say that α is a legal sequence for η if the two following properties hold:

(i) for any $i \in \{1, \dots k-1\}$, let $j := \inf\{\ell > i : x_\ell = x_i\}$. If $j < \infty$, then $n_j = n_i + 1$,

(ii) for any $i \in \{1, \dots, k\}$, τ^{x_i,n_i} is legal for $\eta_{i-1} := \Phi_{\alpha^{(i-1)},\tau}\, \eta$.

Let $m_\alpha = \big(m_\alpha(x)\big)_{x \in V}$ be given by, $m_\alpha(x) = \sum_{i \in \{1,\dots,k\}} \mathbb{1} x_i = x$, the number of times the vertex x appears in α. Let $M_{\alpha,\tau} = \big(M_{\alpha,\tau}(x)\big)_{x \in V}$ be given by, $M_{\alpha,\tau}(x) = \sum_{i \in \{1,\dots,k\}} \mathbb{1} x_i = x, \tau^{x_i,n_i} \neq \boldsymbol{s}$, the number of jump instructions of α. Let K be a finite subset of V. A configuration η is said to be *stable* in K if all the vertices $x \in K$ are stable. We say that α *is contained in* K if $x_i \in K$ for any $i \in \{1, \dots, k\}$. We say that α *stabilizes* η in K if every $x \in K$ is stable in $\Phi_\alpha \eta$. The following lemma gives a fundamental property of the Diaconis-Fulton representation. For the proof we refer to [9].

Lemma 2.1 (Abelian Property). *Given any $K \subset V$, if α and β are both legal sequences for η that are contained in K and stabilize η in K, then $m_\alpha = m_\beta$. In particular, $\Phi_\alpha \eta = \Phi_\beta \eta$.*

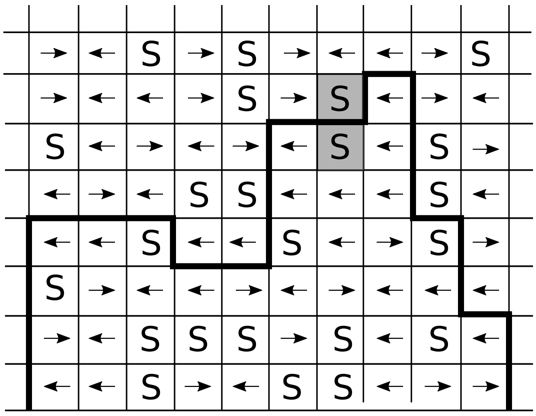

Figure 2.1: An example of array of instructions τ when $G = \mathbb{Z}$, with $t_\tau^{y,3} = 7$, $S_\tau^{y,0} = 2$, $S_\tau^{y,1} = 0$, $S_\tau^{y,3} = 2$ (recall that the index of the 'first' instruction on a vertex is zero). In the figure we assume that the instructions below the bold profile are those which have been used for the stabilisation of η in K, where $K = \{0, \dots L\}$ and η is some particle configuration, thus $M_{K,\eta,\tau}(y) = 3$. The array of instructions $\Gamma_-^{y,n}(\tau)$, with $n = M_{K,\eta,\tau}(y)$, is obtained from τ by 'removing' the two dark sleep instructions above the vertex y, which are located between the $n-1$ th and the n th jump instruction (recall again that the index of the 'first' jump instruction on a vertex is zero).

Odometers. For any subset $K \subset V$, any $x \in V$, any particle configuration η, and any array of instructions τ, we denote by $m_{K,\eta,\tau}(x)$ the number of times that x is toppled in the stabilization of K starting from configuration η and using the instructions in τ. Note that, by Lemma 2.1, we have that $m_{K,\eta,\tau}$ is well defined. We refer to $m_{K,\eta,\tau}$ as the *odometer function,* or simply *odometer*. Moreover, we define a function which counts the number of jump instructions which are used at each vertex for the stabilization of a set. More precisely, for any $K \subset V$, $\eta \in \mathcal{H}$, $\tau \in \mathcal{I}$, $x \in V$, we define,

$$M_{K,\eta,\tau}(x) := \Big| \big\{ j \in \mathbb{N} \, : \, \tau^{j,x} \neq \boldsymbol{s} \text{ and } j < m_{K,\eta,\tau}(x) \big\} \Big|. \tag{2.1}$$

We refer to the function $M_{K,\eta,\tau}$ as *jump-odometer.*

Definition of the counters $t_\tau^{x,m}$, $J_\tau^{x,m}$ and $S_\tau^{x,m}$. Fix an array of instructions $\tau \in \mathcal{I}$, a vertex $x \in V$ and an integer $m \in \mathbb{N}$. We let $J_\tau^{x,m}$ be the *m-th jump instruction at x of τ* and $t_\tau^{x,m}$ be its corresponding index. More precisely, we define $t_\tau^{x,-1} := -1$, and, for any $m \in \mathbb{N}$, we define

$$t_\tau^{x,m+1} := \min\{n > t_\tau^{x,m} \; : \; \tau^{x,n} \neq \boldsymbol{s}\} \qquad J_\tau^{x,m} := \tau^{x,t_\tau^{x,m}}. \tag{2.2}$$

Moreover, for any $m \in \mathbb{N}$ we let $S_\tau^{x,m}$ be *the number of sleep instructions of τ at x between the $m-1$ th and the m th jump instruction,*

$$S_\tau^{x,m} := \Big| \mathbb{N} \cap (\, t_\tau^{x,m-1}, \; t_\tau^{x,m} \,) \Big|, \tag{2.3}$$

(which is well-defined also when $m = 0$). See Figure 2.1 for an example.

Partial orders and monotonicity properties. We now introduce a partial order between particle configurations and arrays of instructions. Given two particle configurations $\eta, \eta' \in \mathcal{H}$, we write $\eta' \geq \eta$ if $\eta'(x) \geq \eta(x)$ for all $x \in V$. Given two arrays τ, τ', we write $\tau' \geq \tau$ if

$$\forall x \in V, \quad \forall m \in \mathbb{N}, \quad J_\tau^{x,m} = J_{\tau'}^{x,m} \quad S_\tau^{x,m} \geq S_{\tau'}^{x,m}.$$

In other words, either $\tau' = \tau$ or τ' is obtained from τ by removing some sleep instruction.

Lemma 2.2 (Monotonicity)**.** *If $K_1 \subset K_2 \subset V$, $\eta \le \eta'$, $\tau \le \tau'$, then $m_{K_1,\eta,\tau} \le m_{K_2,\eta',\tau'}$ and $M_{K_1,\eta,\tau} \le M_{K_2,\eta',\tau'}$*

By monotonicity, given any growing sequence of subsets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V$ such that $\lim_{m\to\infty} V_m = V$, the limits

$$m_{\eta,\tau} := \lim_{m\to\infty} m_{V_m,\eta,\tau}, \quad M_{\eta,\tau} := \lim_{m\to\infty} M_{V_m,\eta,\tau},$$

exist and do not depend on the particular sequence $\{V_m\}_m$.

Probability measure. We now introduce a probability measure on the space of particle configurations and arrays of instructions. The distribution of the initial particle configuration is supported in $\mathbb{N}^V \subset \mathcal{H}$ and will be denoted by ν. Precise assumptions on ν will be made later. We also introduce a *probability measure on the set of arrays of instruction*, $\mathcal{I}$. We denote by $\mathcal{P}_{\lambda,\mu}$ the probability measure according to which, for any $x \in V$ and any $j \in \mathbb{N}$, $\mathcal{P}_{\lambda,\mu}(\tau^{x,j} = \boldsymbol{s}) = \frac{\lambda}{1+\lambda}$ and $\mathcal{P}_{\lambda,\mu}(\tau^{x,j} = \tau_{xy}) = \frac{1}{d_x\,(1+\lambda)}$ for any $y \in V$ neighboring x, where d_x is the degree of the vertex $x \in V$ the $\tau^{x,j}$ are independent across different values of x or j. Finally, we denote by $\mathcal{P}^{\nu}_{\lambda,\mu} = \mathcal{P}_{\lambda,\mu} \otimes \nu$ the joint law of η and τ, where ν is a distribution on $\mathcal{H}$ giving the law of η. Let $\mathbb{P}^{\nu}_{\lambda,\mu}$ denotes the probability measure induced by the ARW process when the initial distribution of particles is given by ν. The following lemma relates the dynamics of ARW to the stability property of the representation. In [9], Lemma 2.3 was proved for $G = \mathbb{Z}^d$ and holds for vertex-transitive graphs.

Lemma 2.3 (Zero-one law, activity and fixation)**.** *Let $G = (V, E)$ be an undirected vertex-transitive graph and let $x \in V$ be any given vertex. Then,*

$$\mathbb{P}^{\nu}_{\lambda,\mu}(\textit{ARW fixates}) = \mathcal{P}^{\nu}_{\lambda,\mu}(m_{\eta,\tau}(x) < \infty) = \mathcal{P}^{\nu}_{\lambda,\mu}(M_{\eta,\tau}(x) < \infty) \in \{0, 1\}. \tag{2.4}$$

We shall often omit the dependence on ν by writing $\mathcal{P}_{\lambda,\mu}$ and $\mathbb{P}$ instead of $\mathcal{P}^{\nu}_{\lambda,\mu}$ and $\mathbb{P}^{\nu}$. Moreover, when we average over η and τ using the measure $\mathcal{P}_{\lambda,\mu}$, we will simply write m_K instead of $m_{K,\eta,\tau}$ and we will do the same for the other quantities that will be introduced later.

2.1 Stabilisation of a domain

Fix a finite set $K \subset V$ and consider a function $Z = \big(Z(x)\big)_{x\in K}$ such that $Z(x) \in \mathbb{N}$ or $Z(x) = \infty$ for any $x \in K$. The stabilisation of a domain is a stabilisation procedure which stops toppling any site x whenever a certain number of instructions, $Z(x)$, has been used and will serve as technical tool in several proofs. Fix a particle configuration $\eta \in \mathcal{H}$ and an array of instructions $\tau \in \mathcal{I}$. Let $\alpha = \big((x_1, n_1), (x_2, n_2), \ldots, (x_k, n_k)\big)$ be a *legal* sequence of instructions of τ which *is contained in* K. Recall that $m_\alpha(x) = \sum_{\ell\in\{1,\ldots,k\}} \mathbb{1}\{x_\ell = x\}$ is the number of times the vertex x appears in the sequence. We say that α *is contained in* (K, Z) if it is contained in K and for any $x \in K$, $m_\alpha(x) \le Z(x)$. Let $\eta' = \Phi_{\alpha,\tau}\eta \in \mathcal{H}$ be the particle configuration which is obtained using the instructions of α. We say that α *stabilises* η *in* (K, Z) if α is contained in (K, Z) and for every vertex $x \in K$, either **(1)** $\eta'(x) \in \{0, \rho\}$, or **(2)** $\eta'(x) \notin \{0, \rho\}$ and $m_\alpha(x) = Z(x)$.

Remark 2.4. *If $Z(x) = \infty$ for any $x \in K$, then the stabilisation of (K, Z) is equivalent to the stabilisation of K.*

Note that the Abelian property holds for the stabilisation of (K, Z) as well, with no change in the proof. Hence, we denote by $m_{(K,Z),\eta,\tau}(x)$ the number of times that x is toppled in the stabilisation of (K, Z) and by $M_{(K,Z),\eta,\tau}(x) = \big|\{j \in \mathbb{N} \ : \ j < m_{(K,Z),\eta,\tau}(x) \text{ and } \tau^{x,j} \ne \boldsymbol{s}\}\big|$ the number of jump instructions which are used at x during the stabilisation of (K, Z). Note that, by the Abelian property, such functions are well defined. The next lemma states the monotonicity properties for the stabilisation of (K, Z), whose proof is analogous to the proof of Lemma 2.5.

Lemma 2.5 (Monotonicity). *If $K_1 \subset K_2 \subset V$, $Z_1 = \big(Z(x)\big)_{x \in K_1}$, $Z_2 = \big(Z(x)\big)_{x \in K_2}$, $\eta_1, \eta_2 \in \mathcal{H}$, $\tau_1, \tau_2 \in \mathcal{I}$ are such that $Z_1(x) \leq Z_2(x)$ for any $x \in K_1$, $\tau_1 \leq \tau_2$, $\eta_1 \leq \eta_2$, then*

$$m_{(K_1,Z_1),\eta_1,\tau_1} \leq m_{(K_2,Z_2),\eta_2,\tau_2} \quad \textit{and} \quad M_{(K_1,Z_1),\eta_1,\tau_1} \leq M_{(K_2,Z_2),\eta_2,\tau_2}.$$

3 Essential pairs and partial derivatives

The goal of this section is to state and prove Theorem 3.2 below, which provides an explicit formula for the partial derivatives with respect to λ and μ of the probability of a wide class of events. To begin, we introduce the notion of increasing relevant events and of particle-essential and sleeping-essential pairs.

Increasing, relevant events and domain. Recall that $\mathcal{H}$ denotes the set of particle configurations and that $\mathcal{I}$ denotes the set of arrays of instructions. Let $\mathcal{S}$ be the smallest sigma-algebra generated by all the open subsets of $\mathcal{H} \times \mathcal{I}$ with respect to the natural product topology. In all definitions, $K \subset V$ and $Z = \big(Z(x)\big)_{x \in K}$ is a function such that $Z(x) \in \mathbb{N} \cup \{\infty\}$ for any $x \in K$. To begin, we say that an event $\mathcal{A} \in \mathcal{S}$ is *increasing* if

$$(\eta, \tau) \in \mathcal{A}, \quad \tilde{\eta} \geq \eta, \quad \tilde{\tau} \geq \tau \implies (\tilde{\eta}, \tilde{\tau}) \in \mathcal{A}. \tag{3.1}$$

Moreover, we say that an event $\mathcal{A} \in \mathcal{S}$ is *relevant* if for any $\eta \in H$,$\tau_1, \tau_2 \in \mathcal{I}$ satisfying the following property,

$$\forall x \in V, \ j \in \mathbb{N}, \qquad J^{x,j}_{\tau_1} = J^{x,j}_{\tau_2} \quad \text{and} \quad S^{x,j}_{\tau_1} > 0 \iff S^{x,j}_{\tau_2} > 0, \tag{3.2}$$

we have that

$$(\eta, \tau^1) \in \mathcal{A} \iff (\eta, \tau^2) \in \mathcal{A}. \tag{3.3}$$

In other words, relevant events do not depend on the precise number of sleep instructions between two consecutive jump instructions, but only on whether such a number is zero or strictly positive. We say that the event $\mathcal{A} \in \mathcal{S}$ has *domain* (K, Z) if for any $\eta_1, \eta_2 \in \mathcal{H}, \tau_1, \tau_2 \in \mathcal{I}$ such that $\eta_1(x) = \eta_2(x)$ for any $x \in K$ and $\tau_1^{x,j} = \tau_2^{x,j}$ for any $j < Z(x)$, we have that

$$(\eta_1, \tau_1) \in \mathcal{A} \iff (\eta_2, \tau_2) \in \mathcal{A}.$$

In other words, an event which has domain (K, Z) depends only on the instructions of the array τ, $\tau^{x,j}$, such that $x \in K$ and $j < Z(x)$. If an event $\mathcal{A}$ has domain (K, Z), where $|K| < \infty$ and $Z(x) < \infty$ for any $x \in K$, then we say that it has *bounded domain.*

Particle-essential pair. Let $\mathcal{A} \in \mathcal{S}$ be an arbitrary event, let $\eta \in \mathcal{H}$ be an arbitrary particle configuration, let $\eta^{(x,k)} \in \mathcal{H}$ be obtained from η as follows,

$$\eta^{(x,k)}(y) := \begin{cases} k & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x. \end{cases}$$

We define the event $\{$the pair (x, k) is *particle-essential* for the event $\mathcal{A}\}$ as the set of realisations $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ such that,

$$(\eta^{(x,k)}, \tau) \notin \mathcal{A} \ \text{ and } \ (\eta^{(x,k+1)}, \tau) \in \mathcal{A}.$$

Sometimes, we will write p-essential in place of particle-essential.

Sleeping-essential pair. We introduce two operators, $\Gamma_{-}^{y,m}, \Gamma_{1}^{y,m} : \mathcal{I} \mapsto \mathcal{I}$ as follows. For an arbitrary array $\tau \in \mathcal{I}$ and a pair $(y, m) \in V \times \mathbb{N}$, we let $\Gamma_{-}^{y,m}(\tau) \in \mathcal{I}$ be the new array of instructions which is obtained from τ by *removing all the sleep instruction between the $m-1$th and the mth jump instruction at y.* More precisely, for any $k \in \mathbb{N}$, $x \in V$,

$$\Big(\Gamma_{-}^{y,m}(\tau)\Big)^{x,k} := \begin{cases} \tau^{x,k} & \text{if } x \neq y \\ \tau^{x,k} & \text{if } x = y \text{ and } k \leq t_{\tau}^{x,m-1} \\ \tau^{x,k+S_{\tau}^{x,m}} & \text{if } x = y \text{ and } k > t_{\tau}^{x,m-1}, \end{cases} \tag{3.4}$$

where we recall that the counters $S_{\tau}^{x,m}$ and $t_{\tau}^{x,m}$ were defined in Section 2. See Figure 2.1 for an example. Moreover, given an instruction array τ and a pair $(y, m) \in V \times \mathbb{N}$, we define a new instruction array $\Gamma_{1}^{y,m}(\tau) \in \mathcal{I}$, which is is obtained from τ by *setting to one the number of sleep instructions between the $m-1$th and the mth jump instruction at y.* More precisely, for any $k \in \mathbb{N}$, $x \in V$,

$$\Big(\Gamma_{1}^{y,m}(\tau)\Big)^{x,k} := \begin{cases} \tau^{x,k} & \text{if } x \neq y \\ \tau^{x,k} & \text{if } x = y \text{ and } k < t_{\tau}^{x,m-1} \\ \boldsymbol{s} & \text{if } x = y \text{ and } k = t_{\tau}^{x,m-1} \\ \tau^{x,k+S_{\tau}^{x,m}} & \text{if } x = y \text{ and } k > t_{\tau}^{x,m-1}. \end{cases} \tag{3.5}$$

Given an arbitrary event $\mathcal{A} \in \mathcal{S}$, vertex $y \in V$, integer $k \in \mathbb{N}$, we define the event $\{$the pair (y, k) is *sleeping-essential for $\mathcal{A}$* $\} \in \mathcal{A}$ as the set of realisations $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ such that

$$(\eta, \Gamma_{1}^{y,k}(\tau)) \notin \mathcal{A} \ \text{ and } \ \big(\eta, \Gamma_{-}^{y,k}(\tau)\big) \in \mathcal{A}.$$

In other words, the event 'the pair (y, k) is sleeping essential for $\mathcal{A}$' is defined as the set of configurations (η, τ) such that, if one considers the new array τ' which is defined as the array which is obtained from τ by setting to *zero* the number of sleep instruction between the $k-1$th and the kth jump instruction at y, then $(\eta, \tau') \in \mathcal{A}$, while if one considers a new array τ'' which is is obtained from τ by setting to *one* the number of sleep instructions between the $k-1$th and the kth jump instruction at y, then $(\eta, \tau'') \notin \mathcal{A}$. Sometimes, we will write s-essential in place of sleeping-essential.

Remark 3.1. *It follows from the definitions of particle-essential and sleeping-essential pairs that, given an arbitrary event $\mathcal{A} \in \mathcal{S}$, vertex $y \in V$, and integer $k \in \mathbb{N}$, the event $\{(y, k)$ is a particle-essential pair for $\mathcal{A}\}$ is independent from the initial number of particles at y,$\eta(y)$, and that the event $\{(y, k)$ is a sleeping-essential pair for $\mathcal{A}\}$ is independent from the variable $S_{\tau}^{y,k}$.*

We now present the main theorem of this section. In the statement of the theorem, we consider the probability of an event $\mathcal{A} \in \mathcal{S}$, $\mathcal{P}_{\lambda,\mu}(\mathcal{A})$, as a function of (λ, μ) in the domain $[0, \infty) \times [0, \infty)$. For the sake of generality we state the theorem not only for product of Poisson distributions, but for a wide class of initial particle (product) distributions ν satisfying the following assumptions.

Assumptions on ν. We assume **(1)** that $\nu = \nu(\mu)$ is a product of identical distributions which are functions of the parameter μ and have some domain $\mathbb{D} \subset \mathbb{R}$, possibly $\mathbb{D} = \mathbb{R}$, and **(2)** that the parameter μ corresponds to the expected number of particles per vertex, also called the *particle density*. For any $j \in \mathbb{N}$, we denote by $\nu_j = \nu_j(\mu)$ the probability that a single vertex hosts j particles. This probability does not depend on the vertex by assumption. We define $\nu_{>j} := \nu_{>j}(\mu) = \sum_{\ell=j+1}^{\infty} \nu_\ell$, the probability that a given vertex hosts more than j particles. Finally, we assume **(3)** that for any $j \in \mathbb{N}$, $\nu_{>j} = \nu_{>j}(\mu)$ is non-decreasing with respect to μ and that it is differentiable with respect to μ for any $\mu \in \mathbb{D}$ and we let $\nu'_{>j} := \frac{\partial}{\partial \mu} \nu_{>j}(\mu)$ be its derivative. Such assumptions are fulfilled for a wide class of particle distributions, including Poisson and Bernoulli. For example, if ν is a product of Poisson distributions, then $\mathbb{D} = \mathbb{R}_{\geq 0}$ and $\nu_j = \frac{\mu^j}{j!} e^{-\mu}$, while, if ν is a product of Bernoulli distributions, then $\mathbb{D} = (0,1)$, $\nu_0 = 1-\mu$, $\nu_1 = \mu$ and $\nu_j = 0$ for all $j \geq 2$.

Theorem 3.2. *Let $\mathcal{A}$ be any increasing relevant event with bounded domain or an event of the form $\mathcal{A} = \{(\eta,\tau) \in \mathcal{H} \times \mathcal{I} \ : \ M_{K,\eta,\tau}(x) \geq H(x)\}$, for some finite $K \subset V$ and $H \in \mathbb{N}^K$. Assume that ν satisfies the assumptions above and that $\mu' \in \mathbb{D}$, $\lambda' \in \mathbb{R}_{>0}$. Then, the function $\mathcal{P}_{\lambda,\mu}(\mathcal{A})$ is differentiable at (λ',μ') and its partial derivatives satisfy,*

$$\frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(\mathcal{A})\Big|_{\lambda',\mu'} = -\left(\frac{1}{1+\lambda'}\right)^2 \sum_{y \in V, j \in \mathbb{N}} \mathcal{P}_{\lambda',\mu'}\big((y,j) \text{ is sleeping-essential for } \mathcal{A}\big), \tag{3.6}$$

$$\frac{\partial}{\partial \mu} \mathcal{P}_{\lambda,\mu}(\mathcal{A})\Big|_{\lambda',\mu'} = \sum_{y \in V, j \in \mathbb{N}} \mathcal{P}_{\lambda',\mu'}\big((y,j) \text{ is particle-essential for } \mathcal{A}\big) \ \nu'_{>j}(\mu'). \tag{3.7}$$

The remainder of this section is devoted to the proof of Theorem 3.2, which is divided into three subsections. In Section 3.1 we introduce a coupling which allows the comparison of ARW-systems with different values of the parameters μ and λ. In Sections 3.2 and 3.3 we will use such a coupling to present the proof of (3.6) and (3.7) respectively. From now on, we will write $\frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big)$ in place of $\frac{\partial}{\partial \lambda'} \mathcal{P}_{\lambda',\mu'}\big(\mathcal{A}\big)\big|_{\lambda,\mu}$, sometimes we will write ∂_λ for $\frac{\partial}{\partial \lambda}$, and we will do the same for the partial derivative with respect to μ.

3.1 Probability space for coupled activated random walk models

We now introduce a new probability space which allows us to couple activated random walk systems corresponding to different values of $\mu \in \mathbb{D}$ and $\lambda \in \mathbb{R}_{\geq 0}$. This new probability space will be denoted by $(\Sigma, \mathcal{F}, \boldsymbol{\mathcal{P}})$. To begin, let $\big(X_x\big)_{x \in V}$, $\big(Y_{x,m}\big)_{x \in V, m \in \mathbb{N}}$, and $\big(A_{x,m}\big)_{x \in V, m \in \mathbb{N}}$ be three sequences of independent random variables in $(\Sigma, \mathcal{F}, \boldsymbol{\mathcal{P}})$ which are distributed as follows. The variables $\big(X_x\big)_{x \in V}$ and $\big(Y_{x,m}\big)_{x \in V, m \in \mathbb{N}}$ have uniform distribution in $[0,1]$, while the variables $\big(A_{x,m}\big)_{x \in V, m \in \mathbb{N}}$ are such that, for each $x \in V$, and $m \in \mathbb{N}$, $A_{x,m}$ takes values in $\{\tau_{xy} \ : \ x \in V, y \sim x\}$ (recall from Section 2 that τ_{xy} is the instruction which lets a particle jump from x to y), and has distribution

$$\boldsymbol{\mathcal{P}}(A_{x,m} = \tau_{xy}) = \frac{1}{d_x}.$$

The variables $(X_x)_{x \in V}$ will be used to sample the initial particle configurations, the variables $(Y_{x,m})_{x \in V, m \in \mathbb{N}}$ will be used to sample the sleep instructions, and the variables $(A_{x,m})_{x \in V, m \in \mathbb{N}}$ will correspond to the jump instructions. We start with the construction of the *initial particle configuration.* Recall that $\mathcal{H}$ denotes the set of particle configurations and define the $\eta_\mu : \Sigma \to \mathcal{H}$, which is parametrised by the parameter $\mu \in \mathbb{D}$. For any $x \in V$, let $k \in \mathbb{N}$ be the unique integer such that $X_x \in [\nu_{<k}(\mu), \nu_{<k+1}(\mu))$. Then, $\eta_\mu(x) := k$. Note that, it follows by construction that,

$$\forall \mu \in \mathbb{D}, \quad \forall x \in V, \quad \boldsymbol{\mathcal{P}}\big(\eta_\mu(x) = k\big) = \nu_k(\mu), \tag{3.8}$$

and that the variables $\big(\eta_\mu(x)\big)_{x \in V}$ are independent. We now construct the *array of instructions.* Recall that $\mathcal{I}$ denotes the space of instructions and, for any $m \in \mathbb{N}$ and $x \in V$, we define the functions $R_\lambda^{x,m} : \Sigma \to \mathbb{N}$, which represent the number of sleep instructions between the $m-1$-th and the mth jump instruction at x and depend on the parameter $\lambda \in [0,\infty)$,

$$R_\lambda^{x,m} := \begin{cases} \ell & \text{if } Y_{x,m} \in \big((\frac{\lambda}{1+\lambda})^{\ell+1}, (\frac{\lambda}{1+\lambda})^{\ell} \big] \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by construction,

$$\forall \lambda \in \mathbb{R}_{\geq 0}, \quad \forall x \in V, \quad \forall m \in \mathbb{N}, \quad \forall \ell \in \mathbb{N}, \quad \boldsymbol{\mathcal{P}}\big(R_\lambda^{x,m} = \ell\big) = \frac{1}{1+\lambda} \, \big(\frac{\lambda}{1+\lambda}\big)^{\ell}, \tag{3.9}$$

and that the variables $\big(R_\lambda^{x,m}\big)_{x \in V, m \in \mathbb{N}}$ are independent. Moreover, we define the function $\tau_\lambda : \Sigma \to \mathcal{I}$, which represents the instruction array for the coupled activated random walk systems as the unique array of instructions such that

$$\forall x \in V, \quad \forall m \in \mathbb{N}, \quad J_{\tau_\lambda}^{x,m} := A_{x,m} \quad \text{and} \quad S_{\tau_\lambda}^{x,m} := R_\lambda^{x,m},$$

(recall that, for any $\tau \in \mathcal{I}$, $J_\tau^{x,m}$ corresponds to mth jump instruction at x and $S_\tau^{x,m}$ corresponds to the number of sleep instructions between the $m-1$th and the mth jump instruction at x). By construction, we proved the following proposition.

Proposition 3.3. *Let $\lambda \in [0, \infty)$ and $\mu \in [0, \infty)$. Sample the pair $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ according to $\mathcal{P}_{\lambda,\mu}^{\nu}$ and let $\eta_\lambda : \Sigma \to \mathcal{H}$ and $\tau_\lambda : \Sigma \to \mathcal{I}$ be the random variables in the probability space $(\Sigma, \mathcal{F}, \boldsymbol{\mathcal{P}})$ which have been defined above. We have that,*

$$(\eta, \tau) \stackrel{d}{=} (\eta_\mu, \tau_\lambda),$$

where '$\stackrel{d}{=}$' denotes equality in distribution. From this, we deduce that, for any event $\mathcal{A} \in \mathcal{S}$, $\boldsymbol{\mathcal{P}}\big((\eta_\mu, \tau_\lambda) \in \mathcal{A}\big) = \mathcal{P}_{\mu,\lambda}\big((\eta, \tau) \in \mathcal{A}\big)$.

In the next two subsections we will use this coupling to prove equations (3.6) and (3.7).

3.2 Proof of equation (3.6)

We will first consider an event $\mathcal{A}$ with bounded domain. To begin note that, if $\mathcal{A}$ is has bounded domain, then the quantity $\mathcal{P}_{\lambda,\mu}(\mathcal{A})$ is a polynomial of finite degree in λ and, for this reason, it is differentiable with respect to λ. In the whole proof, we let (K, Z) be the domain of the event $\mathcal{A} \in \mathcal{S}$, with $K \subset V$ finite and $Z \in \mathbb{N}^K$. To begin, we deduce from Proposition 3.3 and from the fact that $\mathcal{A}$ is an increasing event that, for any $\delta > 0$,

$$\mathcal{P}_{\lambda,\mu}(\mathcal{A}) - \mathcal{P}_{\lambda+\delta,\mu}(\mathcal{A}) = \boldsymbol{\mathcal{P}}\big(\, (\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A} \, \big). \tag{3.10}$$

Recall the coupling construction which was defined in Section 3.1 and recall the definition of the functions $R_\lambda^{y,j}$, where $y \in V$, and $j \in \mathbb{N}$. For arbitrary $(y, j) \in (K, Z)$, we define the events in the probability space $\boldsymbol{\mathcal{P}}$,

$$\mathcal{B}^{y,j,+} := \{R_{\lambda+\delta}^{y,j} > 0\} \cap \{R_\lambda^{y,j} = 0\}, \tag{3.11}$$

$$\mathcal{B}^{y,j,-} := \Big\{\forall (y', j') \in (K, Z) \, : (y', j') \neq (y, j), \;\; R_{\lambda+\delta}^{y',j'} > 0 \iff R_\lambda^{y',j'} > 0\Big\}, \tag{3.12}$$

$$\mathcal{B}^2 := \bigcup_{\substack{(y,j),(y',j') \in (K,Z): \\ (y,j) \neq (y',j')}} \Big\{R_{\lambda+\delta}^{y,j} > 0, \, R_{\lambda+\delta}^{y',j'} > 0, \, R_\lambda^{y,j} = 0, \, R_\lambda^{y',j'} = 0\Big\}. \tag{3.13}$$

Informally, the event $\mathcal{B}^{y,j,+}$ occurs *iff* an increase by δ of the parameter λ turns the variable $R_\lambda^{y,j}$ from zero to a strictly positive number, the event $\mathcal{B}^{y,j,-}$ occurs *iff* such an increase *does not* turn from zero to a positive number any variable $R_\lambda^{y',j'}$ such that (y', j') is in the domain of $\mathcal{A}$ and $(y', j') \neq (y, j)$, the event $\mathcal{B}^2$ occurs *iff* at least two variables $R_\lambda^{y,j}$ turn from zero to a strictly positive number as λ is increased by δ. Since the event $\mathcal{A} \in \mathcal{S}$ is relevant and has domain (K, Z), we obtain, by using the conditional probability formula and summing over the probability of disjoint events, that

$$\boldsymbol{\mathcal{P}}\big((\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\big) = \sum_{(y,j)\in K\times Z} \boldsymbol{\mathcal{P}}\Big(\{(\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\} \, \cap \mathcal{B}^{y,j,-} \, \Big| \, \mathcal{B}^{y,j,+} \Big) \boldsymbol{\mathcal{P}}(\mathcal{B}^{y,j,+}) + \boldsymbol{\mathcal{P}}\big(\{ (\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A} \} \cap \mathcal{B}^2 \big). \quad (3.14)$$

The first term in the right-hand side equals the probability that the event in the left-hand side occurs *and* that an increase by δ of the parameter λ turns precisely one variable $R_\lambda^{y,j}$ (and not more than one!) from zero to a strictly positive number, while the second term in the right-hand side is the probability that the event in the left-hand side occurs *and* that at least two variables $R_\lambda^{y,j}$ turn from zero to a strictly positive number as λ is increased by δ. We now re-write each term in the right-hand side of the previous expression in a more convenient form. First, observe that, in the limit as $\delta \to 0$,

$$\boldsymbol{\mathcal{P}}(\mathcal{B}^{y,j,+}) = \boldsymbol{\mathcal{P}}\Big(Y_{y,j} \in \big(\frac{\lambda}{1+\lambda}, 1\big] \setminus \big(\frac{\lambda+\delta}{1+\lambda+\delta}, 1\big]\Big) = \delta \,\big(\frac{1}{1+\lambda}\big)^2 + O(\delta^2). \quad (3.15)$$

Since $\mathcal{A}$ has bounded domain and since the event $\mathcal{B}^2$ requires that at least two variables $R_\lambda^{y,j}$ turn from zero to a strictly positive value as λ is increased by δ, we also deduce from (3.15) and by the mutual independence of the variables $(Y_{y,j})_{y\in V, j\in\mathbb{N}}$ that, in the limit as $\delta \to 0$,

$$\boldsymbol{\mathcal{P}}\big(\{ (\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A} \} \cap \mathcal{B}^2 \big) = O(\delta^2). \quad (3.16)$$

Finally, using independence, the definition of sleeping-essential pair and the important Remark 3.1 for the first inequality and the fact that $\lim_{\delta\to 0} \boldsymbol{\mathcal{P}}(\mathcal{B}^{y,j,-}) = 1$ for the second inequality, we obtain that,

$$\lim_{\delta\to 0} \boldsymbol{\mathcal{P}}\Big(\{(\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\} \, \cap \mathcal{B}^{y,j,-} \, \Big| \, \mathcal{B}^{y,j,+} \Big) = \lim_{\delta\to 0} \boldsymbol{\mathcal{P}}\big(\{(\eta_\mu, \Gamma^{y,j,-}(\tau_\lambda)) \in \mathcal{A}, (\eta_\mu, \Gamma^{y,j,+}(\tau_{\lambda+\delta})) \notin \mathcal{A}\} \, \cap \mathcal{B}^{y,j,-} \big) = \mathcal{P}_{\lambda,\mu}\big((y,j) \text{ is s-essential for } \mathcal{A}\big). \quad (3.17)$$

In the previous expression we also used the fact that the event $\mathcal{A}$ depends only on whether the variables $(R_{\tau_\lambda}^{y,j})_{y\in V, j\in\mathbb{N}}$ are zero or strictly positive, since it is relevant by assumption. For the second identity in the next expression we use (3.10), for the third identity we use (3.14), (3.15), (3.16) and (3.17), obtaining,

$$\begin{aligned}
\partial_\lambda \mathcal{P}_{\lambda,\mu}(\mathcal{A}) &= \lim_{\delta\to 0} \frac{\mathcal{P}_{\lambda+\delta,\mu}(\mathcal{A}) - \mathcal{P}_{\lambda,\mu}(\mathcal{A})}{\delta} \\
&= - \lim_{\delta\to 0} \frac{\boldsymbol{\mathcal{P}}\big((\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\big)}{\delta} \\
&= -\big(\frac{1}{1+\lambda}\big)^2 \sum_{(y,j)\in(K,Z)} \mathcal{P}_{\lambda,\mu}\big((y,j) \text{ is s-essential for } \mathcal{A}\big) \\
&= -\big(\frac{1}{1+\lambda}\big)^2 \sum_{(y,j)\in V\times\mathbb{N}} \mathcal{P}_{\lambda,\mu}\big((y,j) \text{ is s-essential for } \mathcal{A}\big),
\end{aligned}$$

which concludes the proof of (3.6) for increasing relevant events $\mathcal{A}$ with bounded domain. Consider now the event $\mathcal{A} = \{(\eta, \tau) \in \mathcal{H} \times \mathcal{I} : M_{K,\eta,\tau} \geq H\}$ for some finite $K \subset V$ and $H \in \mathbb{N}^K$. Let

$L \in \mathbb{N}$ be an arbitrary integer, define the function $Z \in \mathbb{N}^K$ as $Z(x) := L$ for any $x \in K$, and define the event, $\mathcal{A}_L = \{(\eta, \tau) \in \mathcal{H} \times \mathcal{I} : M_{(K,Z),\eta,\tau} \geq H\}$. Note that, by the Abelian property and by the fact that K is finite,

$$\lim_{L \to \infty} \mathcal{P}_{\lambda,\mu}(\mathcal{A}_L) = \mathcal{P}_{\lambda,\mu}(\mathcal{A}), \tag{3.18}$$

and, moreover, for any (x, j) such that $x \in K$ and $j \in \mathbb{N}$,

$$\lim_{L \to \infty} \mathcal{P}_{\lambda,\mu}((x, j) \text{ is } s\text{-essential for } \mathcal{A}_L) = \mathcal{P}_{\lambda,\mu}((x, j) \text{ is } s\text{-essential for } \mathcal{A}). \tag{3.19}$$

Note also that, by the Abelian property, both $\mathcal{A}_L$ and $\mathcal{A}$ are relevant and that, by Lemmas 2.2 and 2.5, they are also increasing. Using also the fact that by monotonicity, Lemma 2.5, $(\eta, \tau) \in \mathcal{A}_L$ implies that $(\eta, \tau) \in \mathcal{A}$, we deduce that,

$$\begin{aligned} &\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big) - \mathcal{P}_{\lambda+\delta,\mu}\big(\mathcal{A}\big) = \boldsymbol{\mathcal{P}}\big(\, (\eta_\mu, \tau_\lambda) \in \mathcal{A},\ (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\,\big) \\ &= \boldsymbol{\mathcal{P}}\big(\, (\eta_\mu, \tau_\lambda) \in \mathcal{A}_L, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}_L\big) \ + \boldsymbol{\mathcal{P}}\big(\, (\eta_\mu, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}_L, (\eta_\mu, \tau_{\lambda+\delta}) \notin \mathcal{A}\big). \end{aligned} \tag{3.20}$$

Now note that, since for any finite $L \in \mathbb{N}$ the event $\mathcal{A}_L$ has bounded domain, from (3.6), (3.19) and (3.20) we deduce that, for any $\epsilon > 0$ there exists L finite and large enough such that,

$$\begin{aligned} &\big(\frac{1}{1+\lambda}\big)^2 \sum_{(y,j) \in V \times \mathbb{N}} \mathcal{P}_{\lambda,\mu}\big((y, j) \text{ is s-essential for } \mathcal{A}_L\big) \leq \\ &\qquad \limsup_{\delta \to 0} \frac{\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big) - \mathcal{P}_{\lambda+\delta,\mu}\big(\mathcal{A}\big)}{\delta} \leq \\ &\qquad\qquad \big(\frac{1}{1+\lambda}\big)^2 \sum_{(y,j) \in V \times \mathbb{N}} \mathcal{P}_{\lambda,\mu}\big((y, j) \text{ is s-essential for } \mathcal{A}_L\big) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, since the previous formula holds also with the $\limsup$ replaced by the $\liminf$ and using (3.19), we deduce that (3.6) also holds for the event $\mathcal{A} = \{(\eta, \tau) \in \mathcal{H} \times \mathcal{I} : M_{K,\eta,\tau} \geq H\}$ and this concludes the proof.

3.3 Proof of equation (3.7)

We now turn to the proof of the second equality. Let $\mathcal{A}$ be an arbitrary event satisfing the assumptions of the theorem. From Proposition 3.3, by the fact that $\mathcal{A}$ is increasing and by monotonicity, Lemma 2.2, we obtain that,

$$\mathcal{P}_{\lambda,\mu+\delta}(\mathcal{A}) - \mathcal{P}_{\lambda,\mu}(\mathcal{A}) = \sum_{\tilde{\eta} \in \mathcal{H}} \boldsymbol{\mathcal{P}}\big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \eta_\mu = \tilde{\eta}\big). \tag{3.21}$$

We proceed similarly to the proof of equation (3.6), namely we identify the terms of order $O(\delta)$ in the right-hand side of the previous expression. To begin, we define the sets,

$$\begin{aligned} &\mathcal{E}^{x,+} := \{\eta_{\mu+\delta}(x) = \eta_\mu(x) + 1\}, \\ &\mathcal{E}^{x,-} := \{\forall y \in K \setminus \{x\}, \eta_{\mu+\delta}(y) = \eta_\mu(y)\}, \\ &\mathcal{E}^2 := \{\exists x_1, x_2 \in K \ : \ x_1 \neq x_2 \text{ and } \eta_{\mu+\delta}(x_1) = \eta_\mu(x_1) + 1,\ \eta_{\mu+\delta}(x_2) = \eta_\mu(x_2) + 1\ \}, \\ &\mathcal{E}^x_{\tilde{\eta}} := \{\forall y \in K \setminus \{x\} \ \ \eta_\mu(y) = \tilde{\eta}(y)\}, \\ &\mathcal{N}_L := \{\eta \in \mathcal{H} \ : \ \forall y \in K \ \eta(y) < L\}, \quad L \in \mathbb{N}, \end{aligned}$$

where the first four sets are elements of the sigma-algebra $\mathcal{F}$ and the last set is a subset of $\mathcal{H}$. To begin, from Proposition 3.3 and from a simple computation we deduce that, in the limit as $\delta \to 0$,

$$\begin{aligned}\boldsymbol{\mathcal{P}}\Big(\eta_{\mu+\delta}(x) = k+1, \eta_\mu(x) = k\Big) &= \boldsymbol{\mathcal{P}}\Big(X_x \in [\nu_{\leq k-1}(\mu), \nu_{\leq k}(\mu)) \cap [\nu_{\leq k}(\mu+\delta), \nu_{\leq k+1}(\mu+\delta)]\Big) \\ &= \delta\ \nu'_{>k} + o(\delta), \end{aligned} \tag{3.22}$$

from which we deduce that

$$\boldsymbol{\mathcal{P}}(\mathcal{E}^2) = o(\delta) \tag{3.23}$$

$$\boldsymbol{\mathcal{P}}\big(\mathcal{N}_L^c,\ (\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A},\ (\eta_\mu, \tau_\lambda) \notin \mathcal{A}\big) \leq \delta\,|K|\,\boldsymbol{\mathcal{P}}\big(\eta_\mu(o) \geq L\big), \tag{3.24}$$

where ^{c} denotes the complementary of the event and for the last inequality we used the union bound. Each term in the sum in the right-hand side of (3.21) can be written as follows,

$$\begin{aligned}&\boldsymbol{\mathcal{P}}\Big(\ (\eta_{\mu+\delta},\ \tau_\lambda) \in \mathcal{A},\ (\eta_\mu, \tau_\lambda) \notin \mathcal{A},\ \eta_\mu = \tilde{\eta}\ \Big) = \\ &\boldsymbol{\mathcal{P}}\Big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A},\ (\eta_\mu, \tau_\lambda) \notin \mathcal{A},\ \mathcal{E}^{x,-},\ \mathcal{E}^x_{\tilde{\eta}}\ \Big|\ \mathcal{E}^{x,+}, \eta_\mu(x) = \tilde{\eta}(x)\Big)\ \boldsymbol{\mathcal{P}}\Big(\mathcal{E}^{x,+}, \eta_\mu(x) = \tilde{\eta}(x)\Big) + \\ &\qquad\qquad \boldsymbol{\mathcal{P}}\Big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \mathcal{E}^2\Big). \end{aligned} \tag{3.25}$$

Since $\lim_{\delta\to 0} \boldsymbol{\mathcal{P}}\big(\ (\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{E}^{x,-}\ \big) = 1$, we deduce from the definition of particle-essential pair and from Remark 3.1 that the first factor in the first term in the right-hand side of the previous expression satisfies,

$$\begin{aligned}&\lim_{\delta\to 0} \boldsymbol{\mathcal{P}}\Big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A},\ \mathcal{E}^{x,-},\ \mathcal{E}^x_{\tilde{\eta}}\ \Big|\ \mathcal{E}^{x,+}, \eta_\mu(x) = \tilde{\eta}(x)\Big) = \\ &\qquad \boldsymbol{\mathcal{P}}\Big((\eta_\mu, \tau_\lambda) \in \big\{(x, \tilde{\eta}(x)) \text{ is p-essential for } \mathcal{A}\big\} \cap \mathcal{E}^x_{\tilde{\eta}}\Big). \end{aligned} \tag{3.26}$$

Substituting (3.22), (3.23), (3.24), and (3.26) in (3.25) and substituting (3.25) in (3.21) we obtain that, for any arbitrary small $\epsilon > 0$, there exists $L_0 = L_0(\epsilon) < \infty$ such that for any $L > L_0$,

$$\begin{aligned}&\limsup_{\delta\to 0} \frac{1}{\delta} \sum_{\tilde{\eta}\in H} \boldsymbol{\mathcal{P}}\big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \eta_\mu = \tilde{\eta}\big) \\ &\leq \limsup_{\delta\to 0} \frac{1}{\delta}\Big[\,\delta\,|K|\,\boldsymbol{\mathcal{P}}(\eta_\mu(0) \geq L)\ +\ \sum_{\tilde{\eta}\in\mathcal{N}_L} \boldsymbol{\mathcal{P}}\Big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \eta_\mu = \tilde{\eta}\Big)\Big] \\ &\quad \leq \epsilon + \sum_{\tilde{\eta}\in\mathcal{N}_L} \sum_{x\in K} \boldsymbol{\mathcal{P}}\Big((\eta_\mu, \tau_\lambda) \in \big\{(x, \tilde{\eta}(x)) \text{ is p-essential for } \mathcal{A}\big\} \cap \mathcal{E}^x_{\tilde{\eta}}\Big)\ \nu'_{>\tilde{\eta}(x)} \\ &\quad = \epsilon + \sum_{x\in K} \sum_{\tilde{\eta}\in\mathcal{N}_L} \boldsymbol{\mathcal{P}}\Big((\eta_\mu, \tau_\lambda) \in \big\{(x, \tilde{\eta}(x)) \text{ is p-essential for } \mathcal{A}\big\} \cap \mathcal{E}^x_{\tilde{\eta}}\Big)\ \nu'_{>\tilde{\eta}(x)} \\ &\quad = \epsilon + \sum_{x\in K} \sum_{k=0}^{\infty} \sum_{\substack{\tilde{\eta}\in\mathcal{N}_L: \\ \tilde{\eta}(x)=k}} \boldsymbol{\mathcal{P}}\Big((\eta_\mu, \tau_\lambda) \in \big\{(x, k) \text{ is p-essential for } \mathcal{A}\big\} \cap \mathcal{E}^x_{\tilde{\eta}}\Big)\ \nu'_{>k} \\ &\qquad = \epsilon + \sum_{x\in K} \sum_{k=0}^{L} \boldsymbol{\mathcal{P}}\big((\eta_\mu, \tau_\lambda) \in \big\{(x, k) \text{ is p-essential for } \mathcal{A}\big\} \cap \mathcal{N}_L\,\big)\ \nu'_{>k}. \end{aligned} \tag{3.27}$$

Since $\epsilon > 0$ was arbitrary, taking the limit $L \to \infty$ and using that $\lim_{L\to\infty} \mathcal{P}(\mathcal{N}_L) = 1$,

$$\limsup_{\delta\to 0} \frac{1}{\delta} \sum_{\tilde\eta \in \mathcal{H}} \boldsymbol{\mathcal{P}}\big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \eta_\mu = \tilde\eta\big) \leq$$

$$\sum_{x\in K} \sum_{k=0}^{\infty} \mathcal{P}_{\lambda,\mu}\big(\{(x,k) \text{ is p-essential for } \mathcal{A}\}\big)\, \nu'_{>k}. \quad (3.28)$$

Proceeding similarly for the lower bound, we obtain that

$$\liminf_{\delta\to 0} \frac{1}{\delta} \sum_{\tilde\eta \in \mathcal{H}} \boldsymbol{\mathcal{P}}\big((\eta_{\mu+\delta}, \tau_\lambda) \in \mathcal{A}, (\eta_\mu, \tau_\lambda) \notin \mathcal{A}, \eta_\mu = \tilde\eta\big) \geq$$

$$\sum_{x\in K} \sum_{k=0}^{\infty} \boldsymbol{\mathcal{P}}\big(\{(x,k) \text{ is p-essential for } \mathcal{A}\}\big)\, \nu'_{>k}, \quad (3.29)$$

and thus, combining the two previous bounds, we conclude the proof of (3.7).

4 The key differential inequality

The goal of this section is to state and prove Theorem 4.6 below, which provides a precise formulation of the differential inequality (1.3). We start with the following question. Let $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ be an arbitrary realisation, let $K \subset V$ be a finite set, define a new instruction array $\tau' \in \mathcal{I}$, which is obtained from $\tau \in \mathcal{I}$ by removing one sleep instruction 'somewhere'. Does such a removal affect the number of times the particles jump from the vertices of the graph? More precisely, is it the case that $M_{K,\eta,\tau'}(x) > M_{K,\eta,\tau}(x)$ at some vertex $x \in K$ or do we have that $M_{K,\eta,\tau'} = M_{K,\eta,\tau}$? Our Lemma 4.1 below shows that 'typically' (but not always) the answer to the second question, and not to the first one, is positive. This property will be key for the derivation of the differential inequality. Before stating the lemma, we introduce the set $\mathcal{W}$ of pairs $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ such that, for any finite $K \subset V$ and any $x \in K$, $m_{K,\eta,\tau}(x) < \infty$. Any $(\eta, \tau) \in \mathcal{W}$ is such that the stabilisation of any finite set of sites requires the use of a finite number of instructions and, clearly, $\mathcal{P}_{\lambda,\mu}(\mathcal{W}) = 1$.

Lemma 4.1. *Consider a pair $(\eta, \tau) \in \mathcal{W}$, let $K \subset V$ be a finite set, fix an arbitrary vertex $y \in K$. For any $n \in \mathbb{N}$ such that $n \neq M_{K,\eta,\tau}(y)$, we have that*

$$M_{K,\eta,\tau} = M_{K,\eta,\Gamma_-^{y,n}(\tau)}.$$

Proof. The proof of the lemma uses the Abelian property in several places. Fix $(\eta, \tau) \in \mathcal{W}$, $y \in K$ and $n \in \mathbb{N}$. Recall that $t_\tau^{y,n}$ is the index of the n-th jump instruction at y, see also Figure 2.1. For a lighter notation we use τ' for $\Gamma_-^{y,n}(\tau)$ and u for $S_\tau^{y,n}$, moreover we assume that $u > 0$, otherwise the claim is trivial. We distinguish between two cases. **First case.** Suppose that $n > M_{K,\eta,\tau}(y)$ and let $\alpha = \big((x_1, n_1), \ldots, (x_k, n_k)\big)$ be a sequence of instructions of τ which stabilises η in K. Since we assumed that $n > M_{K,\eta,\tau}(y)$, we deduce that $\tau^{x_i,n_i} = \tau'^{x_i,n_i}$ for any $i \in \{1, \ldots, k\}$. Thus, α is also a sequence of instructions of τ' which stabilises η in K and the claim thus follows from the Abelian property. **Second case.** Suppose that $n < M_{K,\eta,\tau}(y)$. The goal is to define a legal sequence of instructions $\alpha = \big((x_1, n_1), \ldots, (x_k, n_k)\big)$ of the array τ which stabilizes η in K and a legal sequence of instructions $\alpha' = \big((x'_1, n'_1), \ldots, (x'_{k'}, n'_{k'})\big)$ of the array τ' which stabilizes η in K and to show that, for any $z \in K$,

$$\Big|\big\{\, i \in \{1, \ldots, k\} \,:\, x_i = z, \tau^{x_i,n_i} \neq \boldsymbol{s}\big\}\Big| = \Big|\big\{ i \in \{1, \ldots, k'\} \,:\, x'_i = z, \tau'^{x'_i,n'_i} \neq \boldsymbol{s}\big\}\Big|, \quad (4.1)$$

namely that the number of jump instructions used at any site is the same for both sequences, which proves the claim by the Abelian property. We define α in two steps. In the *first step*, we define $Z = (Z(x))_{x \in K}$ as $Z(x) := \infty$ for any $x \in K \setminus \{y\}$ and $Z(y) := t_{\tau}^{y,n-1} + 1$ and we stabilise η in (K, Z). By our choice of Z, the stabilisation of (K, Z) is 'forced to stop' right before using the first of the u sleep instructions which are located between the $n-1$th and the nth jump instruction at y. We let $\eta' \in \mathcal{H}$ be the particle configuration which is obtained after such a step. The crucial observation is that it is necessarily the case that,

$$\eta'(x) \begin{cases} \geq 2 & \text{if } x = y, \\ \in \{0, \rho\} & \text{if } x \neq y. \end{cases} \tag{4.2}$$

The fact that $\eta'(x) \in \{0, \rho\}$ for any $x \neq y$ follows from the definition of stabilisation of a domain and our choice of Z. The fact that $\eta'(y) \geq 2$ will be now proved by contradiction. Indeed, suppose that this was not true, namely that either **(a)** $\eta'(y) \in \{0, \rho\}$ or **(b)** $\eta'(y) = 1$. If (a) was true, then we would have stabilized η in K using $n < M_{K,\eta,\tau}(y)$ jump instructions at y, contradicting our assumption. Similarly, if (b) was true, then by using one more instruction at y for η (which is a sleep instruction since $u > 0$ by assumption!) we would have stabilized η in K using $n < M_{K,\eta,\tau}(y)$ jump instructions, contradicting again our assumptions. Hence, we conclude that $\eta'(y) \geq 2$ and conclude the proof of (4.2) ass desired. In the *second step,* we complete the stabilisation of η in K by first using the next $u = S_{\tau}^{y,n}$ instructions at y of the array τ, which by assumption are sleep instruction – noting that such u sleep instructions produce no effect on the particle configuration since $\eta'(y) \geq 2$ – and then by following an arbitrary order. Since such u sleep instructions produce no effect, they can be removed from the sequence α and from the array τ without affecting the jump odometer and the particle configuration. This leads to a new array τ' and to a new legal sequence of instructions of τ', α', which stabilises η in K and satisfies (4.1). By the Abelian property, the proof of is concluded. □

In all the next statements and until the end of the section, the set $K \subset V$ is arbitrary and finite. This will not be recalled any more.

Lemma 4.2. *Consider an arbitrary integer-valued vector, $H \in \mathbb{N}^K$, and the event, $\mathcal{A} := \big\{(\eta, \tau) \in \mathcal{H} \times \mathcal{I} \, : \, M_{K,\eta,\tau}(x) \geq H(x)\big\}$. For any $y \in K$, $n \in \mathbb{N}$,*

$$\big\{(y, n) \text{ is s-essential for } \mathcal{A}\big\} \cap \big\{(\eta, \tau) \in \mathcal{W} : \, M_{K,\eta,\tau}(y) \neq n \text{ and } S_{\tau}^{y,n} > 0\big\} = \emptyset. \tag{4.3}$$

Proof. Suppose that the pair (η, τ) is such that (y, n) is s-essential for $\mathcal{A}$ and $S_{\tau}^{y,n} > 0$. We will show that it is necessarily the case that $M_{K,\eta,\tau}(y) = n$, thus implying (4.3). To begin, we deduce by definition of sleeping-essential pair and by the fact that $S_{\tau}^{y,n} > 0$ that,

$$(\eta, \tau) \notin \mathcal{A} \quad \text{and} \quad (\eta, \Gamma_{-}^{y,n}(\tau)) \in \mathcal{A}. \tag{4.4}$$

Suppose that $M_{K,\eta,\tau}(y) \neq n$. This will lead to a contradiction. Indeed, by Lemma 4.1 we deduce that $M_{K,\eta,\tau} = M_{K,\eta,\Gamma_{-}^{y,n}(\tau)}$. Since $\mathcal{A}$ only depends on the jump-odometer by assumption, we deduce that

$$(\eta, \Gamma_{-}^{y,n}(\tau)) \notin \mathcal{A}.$$

This, however, contradicts (4.4), and, thus, it implies that $M_{K,\eta,\tau}(y) = n$, as desired. This concludes the proof. □

The next proposition provides an alternative formula for the partial derivative with respect to λ which appears in Theorem 3.2. The difference with respect to the formulation in Theorem 3.2 is that the infinite sum over $j \in \mathbb{N}$ is replaced by only one (random) term. Such a replacement is possible in light of Lemma 4.1 and Lemma 4.2. In the next statements we write $\{(y, M_K(y))$ is s-essential for $\mathcal{A}\}$ for the set of pairs $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ which belong to the event '(y, n) is s-essential for $\mathcal{A}$' and such that $n = M_{K,\eta,\tau}(y)$. Similarly, we use $\{S^{y,M_K(y)} > 0\}$ for the set of pairs $(\eta, \tau) \in \mathcal{H} \times \mathcal{I}$ such that $S^{y,n}_\tau > 0$ and $n = M_{K,\eta,\tau}(y)$.

Proposition 4.3. *Under the same assumptions as in Lemma 4.2, we deduce that, for any $\lambda \in (0, \infty)$,*

$$\frac{\partial}{\partial \lambda} \mathcal{P}_{\lambda,\mu}(\mathcal{A}) = -\frac{1}{\lambda(1+\lambda)} \sum_{y \in K} \mathcal{P}_{\lambda,\mu}\Big(\{(y, M_K(y)) \text{ is s-essential for } \mathcal{A}\} \cap \{S^{y,M_K(y)} > 0\}\Big) \tag{4.5}$$

Proof. We write that '(y, j) is s-essential' as a short form for '(y, j) is s-essential for $\mathcal{A}$'. Below we use Remark 3.1 for the first identity and Lemma 4.2 for the third identity, obtaining that, for any $y \in K$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \mathcal{P}_{\lambda,\mu}\big((y,j) \text{ is s-essential}\big) &= \frac{1+\lambda}{\lambda} \sum_{j=0}^{\infty} \mathcal{P}_{\lambda,\mu}\big(\{(y,j) \text{ is s-essential}\} \cap \{S^{y,j} > 0\}\big) \\
&= \frac{1+\lambda}{\lambda} E_{\lambda,\mu}\big(\sum_{j=0}^{\infty} \mathbb{1}_{\{(y,j) \text{ is s-essential}\} \cap \{S^{y,j}>0\}}\big) \\
&= \frac{1+\lambda}{\lambda} E_{\lambda,\mu}\big(\sum_{j=0}^{\infty} \mathbb{1}_{\{(y,j) \text{ is s-essential}\} \cap \{S^{y,j}>0\} \cap \{j = M_K(y)\}}\big) \\
&= \frac{1+\lambda}{\lambda} \mathcal{P}_{\lambda,\mu}\big(\{(y, M_K(y)) \text{ is s-essential}\} \cap \{S^{y,M_K(y)} > 0\}\big),
\end{aligned}$$

where $E_{\lambda,\mu}$ denotes the expectation with respect to $\mathcal{P}_{\lambda,\mu}$. By using the previous formula and Theorem 3.2, we conclude the proof. □

Now we will state some preparatory lemmas which will allow the comparison of the partial derivative with respect to λ and of the partial derivative with respect to μ. We denote by η^y the particle configuration which is obtained from $\eta \in \mathcal{H}$ by adding one more particle at y, i.e.,

$$\forall x \in V \qquad \eta^y(x) := \begin{cases} \eta(x) & \text{if } x \neq y \\ \eta(x) + 1 & \text{if } x = y. \end{cases}$$

The next lemma states that, under appropriate assumptions on (η, τ), adding one more particle to the initial particle configuration increases the jump odometer strictly.

Lemma 4.4. *Assume that $\mathcal{A}$ is defined the same as in Lemma 4.2, that $(\eta, \tau) \in \mathcal{W}$ belongs to the event $\big\{(y, M_{K,\eta,\tau}(y))$ is s-essential for $\mathcal{A}\big\}$ and that $S_\tau^{y,M_{K,\eta,\tau}(y)} > 0$. Then,*

$$M_{K,\eta,\tau}(y) < M_{K,\eta^y,\tau}(y) \tag{4.6}$$

Proof. Consider a pair (η, τ) as in the assumptions of the lemma and note that, by definition of sleeping-essential pair,

$$(\eta, \tau) \notin \mathcal{A}, \qquad (\eta, \Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)) \in \mathcal{A}. \tag{4.7}$$

We define $(Z(x))_{x \in K}$ as follows,

$$\forall x \in K \qquad Z(x) := \begin{cases} \infty & \text{if } x \neq y \\ t_\tau^{M_{K,\eta,\tau}(y)-1} + 1 & \text{if } x = y, \end{cases} \tag{4.8}$$

and we let α be a sequence of instructions of τ which stabilises η in (K, Z). The sequence α does not contain the $u := S_\tau^{y, M_{K,\eta,\tau}(y)} > 0$ sleep instructions which are located right after the $M_{K,\eta,\tau}(y) - 1$th jump instruction at y (in the example of Figure 2.1, these instructions are coloured by grey), since, by definition of stabilisation of (K, Z) and by our choice of Z, the stabilisation procedure is 'forced to stop' right before using such sleep instructions. Call η' the particle configuration which is obtained after using the instructions of τ in α. We claim that,

$$\forall x \in K \qquad \eta'(x) \begin{cases} = 1 & \text{if } x = y \\ \in \{0, \rho\} & \text{if } x \neq y. \end{cases} \tag{4.9}$$

First note that, since $Z(x) = \infty$ for any $x \in K$ such that $x \neq y$, by definition of stabilisation of a domain it is necessarily the case that for any $x \in K$ such that $x \neq y$, $\eta'(x) \in \{0, \rho\}$. To prove that $\eta'(y) = 1$, we argue by contradiction. Suppose first that $\eta'(y) \in \{0, \rho\}$. Then, α also stabilises η in K and η' is stable in K. For this reason, α also stabilises η in K when we use the instructions of the array $\Gamma_-^{y,n}(\tau)$, with $n = M_{K,\eta,\tau}(y)$. This in turn implies by the Abelian property that,

$$M_{K,\eta,\tau} = M_{K,\eta,\Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)}. \tag{4.10}$$

Hence, by our assumptions on $\mathcal{A}$, by (4.10) and by the fact that $(\eta, \tau) \notin \mathcal{A}$, we deduce that $(\eta, \Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)) \notin \mathcal{A}$, finding the desired contradiction with (4.7). Hence, we conclude that $\eta'(y) \notin \{0, \rho\}$. Suppose now that $\eta'(y) > 1$, we look again for a contradiction. We argue the same as before, namely we use the instructions of τ following sequence α and, after that, we use the next u instructions at y, which are sleep instructions by assumption. Note that these instructions produce no effect on the particle configuration since we have more than one active particle at y. Hence, by arguing the same as in the previous case, we deduce by the Abelian property that (4.10) holds and this leads to the desired contradiction the same as before. Hence, we proved (4.9), as desired. Define now the sequence of instructions of τ, α', in two steps. In the first step, α' coincides with α and in the second step the next instruction at y is used. Such a last instruction is of type sleep by assumption, hence by (4.9) α' stabilises η in K and $M_{\alpha,\tau} = M_{\alpha',\tau}$. Consider now the particle configuration η^y and define the sequence of instructions of τ, α'', in two steps. In the first step, α'' coincides with α. Call η'' the particle configuration which is obtained at the end of such a step and note that, by (4.9),

$$\forall x \in V \quad \eta''(x) = \begin{cases} \eta'(x) + 1 = 2 & \text{if } x = y, \\ \eta'(x) \in \{0, \rho\} & \text{if } x \neq y\ . \end{cases} \tag{4.11}$$

In the second step, we complete the stabilisation of η^y in K by first using the next u sleep instructions at y and then following an arbitrary order. Note that the next u sleep instructions at y are ineffective because of (4.11). Hence, these do not turn any active particle at y into a sleep particle. This implies that at least one more jump instruction at y is used for the stabilisation of η^y in K (the stabilisation of K cannot end with two active particles at K by definition of stabilisation!). Hence, by the Abelian property, $M_{K,\eta^y,\tau}(y) = M_{\alpha'',\tau}(y) > M_{\alpha',\tau}(y) = M_{\alpha,\tau}(y) = M_{K,\eta,\tau}(y)$, and this concludes the proof. □

The next lemma states a crucial inclusion relation between the event that a pair is s-essential and a pair is p-essential.

Lemma 4.5. *Under the same assumptions as in Lemma 4.2, for any $j \in \mathbb{N}$,*

$$\Big\{(\eta, \tau) \in \mathcal{W} \ : \ (y, M_{K,\eta,\tau}(y)) \text{ is s-essential for } \mathcal{A},\ \eta(y) = j \text{ and } S_\tau^{y,M_{K,\eta,\tau}(y)} > 0\Big\}$$
$$\subset \Big\{(\eta, \tau) \in \mathcal{W} \ : \ (y, j) \text{ is p-essential for } \mathcal{A},\ \eta(y) = j \text{ and } S_\tau^{y,M_{K,\eta,\tau}(y)} > 0\Big\}. \tag{4.12}$$

Proof. Let (η,τ) be a realisation which belongs to the event in the left-hand side of (4.12), we will show that it also belongs to the event in the right-hand side. By definition of sleeping-essential pair and by our assumptions,

$$(\eta,\tau)\notin\mathcal{A} \quad \text{and} \quad (\eta,\Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau))\in\mathcal{A}. \tag{4.13}$$

By monotonicity, Lemma 2.2, we have that,

$$M_{K,\eta,\tau}\le M_{K,\eta,\Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)}\le M_{K,\eta^y,\Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)}. \tag{4.14}$$

By Lemma 4.1 and Lemma 4.4 we have that,

$$M_{K,\eta^y,\Gamma_-^{y,M_{K,\eta,\tau}(y)}(\tau)}=M_{K,\eta^y,\tau}, \tag{4.15}$$

Note that the previous identity would not hold true if in the left-hand side $M_{K,\eta,\tau}(y)$ (in the superscript of the subscript) was replaced by $M_{K,\eta^y,\tau}(y)$, by (4.6) we have that $M_{K,\eta^y,\tau}(y)\neq M_{K,\eta,\tau}(y)$. From (4.13), (4.14) (4.15), and by our assumption on $\mathcal{A}$ we deduce that, $(\eta^y,\tau)\in\mathcal{A}$. Summarising, (η,τ) is such that **(1)** $(\eta,\tau)\notin\mathcal{A}$ by (4.13), **(2)** $\eta(y)=j$ by assumption, and **(3)** $(\eta^y,\tau)=(\eta^{y,j+1},\tau)\in\mathcal{A}$, these three facts imply that the pair (η,τ) belongs to the event '(y,j) is p-essential for $\mathcal{A}$'. From this we deduce that (η,τ) belongs to the event in the right-hand side of (4.12). This concludes the proof. □

We are now ready to state the main result of this section.

Theorem 4.6 (**Differential inequality**). *Let $G=(V,E)$ be an undirected locally-finite graph, suppose that the initial particle configuration is distributed as a product of Poisson distributions with parameter μ, let $K\subset V$ be a finite set, let $H=(H(x))_{x\in K}$ be an integer-valued vector, define the event, $\mathcal{A}:=\big\{(\eta,\tau)\in\mathcal{H}\times\mathcal{I}\,:\,M_{K,\eta,\tau}(x)\ge H(x)\big\}$. Then, for any $\lambda\in(0,\infty)$,*

$$-\frac{\partial}{\partial\lambda}\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big)\le\frac{1}{\lambda(1+\lambda)}\frac{\partial}{\partial\mu}\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big). \tag{4.16}$$

Proof. Using Proposition 4.3 for the first step, (4.12) for the second step, Remark 3.1 for the third step, the fact that for Poisson distributions $\nu'_{>k}=\nu_k$ for the fourth step, Theorem 3.2 for the fifth and last step, we obtain that,

$$\begin{aligned}
-\frac{\partial}{\partial\lambda}\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big)&=\frac{1}{\lambda(1+\lambda)}\sum_{y\in K}\sum_{j=0}^{\infty}\mathcal{P}_{\lambda,\mu}\Big(\{(y,M_K(y))\text{ is s-sessential}\}\cap\{\eta(y)=j\}\cap\{S^{y,M_K(y)}>0\}\Big)\\
&\le\frac{1}{\lambda(1+\lambda)}\sum_{y\in K}\sum_{j=0}^{\infty}\mathcal{P}_{\lambda,\mu}\big(\{(y,j)\text{ is p-essential}\}\cap\{\eta(y)=j\}\big)\\
&=\frac{1}{\lambda(1+\lambda)}\sum_{y\in K}\sum_{j=0}^{\infty}\mathcal{P}_{\lambda,\mu}\big(\{(y,j)\text{ is p-essential}\}\big)\ \nu_j(\mu)\\
&=\frac{1}{\lambda(1+\lambda)}\sum_{y\in K}\sum_{j=0}^{\infty}\mathcal{P}_{\lambda,\mu}\big(\{(y,j)\text{ is p-essential}\}\big)\ \nu'_{>j}(\mu)\\
&=\frac{1}{\lambda(1+\lambda)}\frac{\partial}{\partial\mu}\mathcal{P}_{\lambda,\mu}\big(\mathcal{A}\big).
\end{aligned}$$

This concludes the proof. □

5 Proof of Theorems 1.1 and 1.2

Recall the definition of the curve $\mathcal{C}_{\lambda,\mu}$ which was provided in (1.2). We will start with the proof of Theorem 1.2.

Proof of Theorem 1.2. Consider any event $\mathcal{A}$ satisfying the assumptions of Theorem 3.2. Let $(\lambda, \mu) \in (0,\infty) \times (0,\infty)$ be an arbitrary point in the phase diagram, take any arbitrary point $(x, y) \in \mathcal{C}_{\lambda,\mu}$, we assume that $x > \lambda$ (when $x = \lambda$, the result is already known from [9]). Let $(X(t), Y(t))_{t\in[\lambda,\infty)}$ be a curve such that $X(0) := \lambda$, $Y(0) := \mu$, and for any $t \in [\lambda, \infty)$,

$$\begin{cases} X(t) & := t, \\ Y(t) & := s(t-\lambda) + \mu, \end{cases}$$

where $s \in \mathbb{R}$ is such that there exists a positive $T \in \mathbb{R}$ such that $X(T) = x$, $Y(T) = y$. In other words, $(X(t), Y(t))$ is a semi-line in $\mathbb{R}^2$ with slope s starting from $(\lambda, \mu) \in \mathbb{R}^2$, where s is chosen in such a way that $X(T) = x$, $Y(T) = y$ for some positive $T \in \mathbb{R}$. Note that it is necessarily the case that $s \in [\frac{1}{\lambda(1+\lambda)}, \infty)$ because of our assumptions. From the fundamental theorem of calculus we deduce that,

$$\begin{aligned} \mathcal{P}_{x,y}(\mathcal{A}) &= \mathcal{P}_{\lambda,\mu}(\mathcal{A}) + \int_\lambda^x dt \;\; \nabla \mathcal{P}_{X(t),Y(t)}(\mathcal{A}) \cdot \big(\partial_t X(t), \partial_t Y(t)\big) = \\ &= \mathcal{P}_{\lambda,\mu}(\mathcal{A}) + \int_\lambda^x dt \, \Big(\partial_\lambda \mathcal{P}_{\lambda,\mu}(\mathcal{A}) \,\big|_{\lambda=t,\mu=Y(t)} \; + \; s \, \partial_\mu \mathcal{P}_{\lambda,\,\mu}(\mathcal{A}) \big|_{\lambda=t,\mu=Y(t)} \Big) \\ &\geq \mathcal{P}_{\lambda,\mu}(\mathcal{A}) + \int_\lambda^x dt \; 0 \\ &\geq \mathcal{P}_{\lambda,\mu}(\mathcal{A}), \end{aligned}$$

where for the first inequality we used Theorem 4.6 and the fact that $s \geq \frac{1}{\lambda(1+\lambda)} \geq \frac{1}{t(1+t)}$ for any $t \geq 0$. This concludes the proof. □

We now provide a more general definition of *critical density,*

$$\forall \lambda \in [0,\infty) \qquad \zeta_c(\lambda) := \inf \Big\{ \mu \in \mathbb{R}_{\geq 0} \; : \; \mathcal{P}_{\lambda,\mu}(m(o) = \infty) > 0 \Big\}. \tag{5.1}$$

By monotonicity, the random variable $m_{\eta,\tau}(o)$, which was introduced in Section 2, is well defined for every infinite graph G. Note that, whenever Lemma 2.3 holds, $\zeta_c(\lambda) = \mu_c(\lambda)$. From now on we will say that ARW *fixates* if $m(o) < \infty$ and that it is *active* otherwise. By Lemma 2.3, such a notion of activity and fixation reduces to the one which has been introduced in Section 1 if the graph is vertex-transitive. The next proposition is an immediate consequence of Theorem 1.2.

Proposition 5.1. *For any* $(\lambda, \mu) \in (0,\infty) \times (0,\infty)$, $(\lambda', \mu') \in \mathcal{C}_{\lambda,\mu}$, *we have that,*

$$\mathcal{P}_{\lambda,\mu}(\textit{ARW active}) \leq \mathcal{P}_{\lambda',\mu'}(\textit{ARW active}).$$

Proof. Define $B_L := \{x \in V \; : \; d(x,o) \leq L\}$ as the ball of radius L centred at the origin, where $d(\cdot,\cdot)$ is the graph distance, and $o \in V$ is the origin, let $H \in \mathbb{N}$ be arbitrary, define the event $\mathcal{A}_{L,H} := \{M_{B_L}(o) > H\}$. Hence, from Theorem 1.2 we deduce that, for any $L, H \in \mathbb{N}$,

$$\mathcal{P}_{\lambda,\mu}(\mathcal{A}_{L,H}) \leq \mathcal{P}_{\lambda',\mu'}(\mathcal{A}_{L,H}). \tag{5.2}$$

Thus,

$$\mathcal{P}_{\lambda,\mu}(\text{ARW active}) = \lim_{H\to\infty}\lim_{L\to\infty}\mathcal{P}_{\lambda,\mu}(\mathcal{A}_{L,H}) \le \lim_{H\to\infty}\lim_{L\to\infty}\mathcal{P}_{\lambda',\mu'}(\mathcal{A}_{L,H}) = \mathcal{P}_{\lambda',\mu'}(\text{ARW active}), \tag{5.3}$$

concluding the proof. □

5.1 Proof of Theorem 1.1

Proof. In the whole proof we use the fact that it is known from [13] that on vertex-transitive graphs the critical density is finite for any $\lambda \in [0,\infty)$ (and, more precisely, it is at most one). We start with the proof of (ii). We use an argument by contradiction. Consider an arbitrary $\lambda \in (0,\infty)$ and suppose that,

$$\varphi(\lambda) := \limsup_{\delta\to 0^+}\frac{\zeta_c(\lambda+\delta)-\zeta_c(\lambda)}{\delta} > \frac{1}{\lambda(1+\lambda)}. \tag{5.4}$$

Note that from (5.4) it follows that we can find a small enough $\Delta > 0$ such that there exists an infinite sequence $(\delta_n)_{n\in\mathbb{N}}$ convergent to zero with n such that, for any large enough n,

$$\zeta_c(\lambda+\delta_n) \ge \zeta_c(\lambda) + \delta_n\Big(\frac{1}{\lambda(1+\lambda)}+\Delta\Big). \tag{5.5}$$

The sequence $(\delta_n)_{n\in\mathbb{N}}$ and the value $\Delta > 0$ will now be kept fixed. Thus, for any $\epsilon > 0$ and $t \in [\lambda,\infty)$ define,

$$\begin{cases} X(t) := t, \\ Y_\epsilon(t) := \zeta_c(\lambda) + \epsilon + \frac{1}{\lambda(1+\lambda)}\,(t-\lambda). \end{cases}$$

From Proposition 5.1 and from the definition of the critical density, we deduce that for any $\epsilon > 0$, $\forall t \in [\lambda,\infty)$, $\zeta_c(t) \le Y_\epsilon(t)$, from which we deduce that,

$$\forall\epsilon>0,\quad \forall n\in\mathbb{N},\quad \zeta_c(\lambda+\delta_n) \le Y_\epsilon(\lambda+\delta_n) = \zeta_c(\lambda) + \epsilon + \delta_n\,\frac{1}{\lambda(1+\lambda)}.$$

This contradicts (5.5) and concludes the proof of (ii) for $\delta \to 0^+$. Note that the proof of (ii) for $\delta \to 0^-$ is analogous, hence the proof of (ii) is concluded. We now prove (i). We will use the fact that, by [9],

$$\zeta_c(\lambda) \text{ is non decreasing w.r. to } \lambda. \tag{5.6}$$

Let $\lambda_* \in (0,\infty)$ be an arbitrary point, define $\mu_1 := \lim_{\lambda\to\lambda_*^-}\zeta_c(\lambda)$ and $\mu_2 := \lim_{\lambda\to\lambda_*^+}\zeta_c(\lambda)$. These limits are well defined because of (5.6). Suppose that $\mu_1 \neq \mu_2$. We will look for a contradiction. First note that, by (5.6), it can only be that $\mu_1 < \mu_2$. Let $\epsilon > 0$ be arbitrary, consider the following curve, for $t \in [\lambda-\epsilon,\infty)$,

$$\begin{cases} X(t) := t, \\ Y_\epsilon(t) := \mu_1 + \epsilon + (t-\lambda+\epsilon)\frac{1}{\lambda(1+\lambda)} \end{cases}$$

By definition of critical particle density, by our assumptions and by (5.6), we deduce that ARW with sleeping rate $\lambda-\epsilon$ and particle density $\mu_1+\epsilon$ is active with some positive probability $p_\epsilon > 0$. By Proposition 5.1, ARW with sleeping rate t and particle density $Y_\epsilon(t)$ is active with probability at least $p_\epsilon > 0$ for any $t \ge \lambda-\epsilon$. In particular, by choosing $t = \lambda+\epsilon$, we deduce that ARW with sleeping rate $\lambda+\epsilon$ and particle density $Y_\epsilon(\lambda+\epsilon)$ is active with probability at least $p_\epsilon > 0$. This holds true for any $\epsilon > 0$. However, note that if ϵ is small enough, then $Y_\epsilon(\lambda+\epsilon) < \mu_2$. Thus, we concluded that ARW with sleeping rate $\lambda+\epsilon$ and particle density strictly below μ_2 is active. By by (5.6), this violates the definition of critical density and leads to the desired contradiction. Thus, we proved that $\zeta_c(\lambda)$ is a continuous function of λ at $\lambda = \lambda^*$. Since λ^* was an arbitrary point in $(0,\infty)$, we proved that $\zeta_c(\lambda)$ is a continuous function of λ in $(0,\infty)$. □

5.2 Extensions

While the assumption that the initial number of particles at any vertex has Poisson distribution independently is necessary for our main theorem, the assumptions that all the particles at time zero are active and that the graph is vertex-transitive are not. In this section we provide a more general formulation of our main theorem.

We represent the initial particle configuration as a pair, $(\eta, \varphi) \in \mathbb{N}^V \times \{0,1\}^V$, where $\eta(x)$ represents the initial number of particles in $x \in V$ (these might be of type A or S); moreover, if $\varphi(x) = 1$, then *all the particles* which start from x are of type S, while if $\varphi(x) = 0$, then *all the particles* which start from x are of type S. The definition of the model remains the same as before, namely A-particles perform a continuous time simple random walk, they turn into the S-state with rate λ and, whenever a A-particle visits a vertex where one or more than one S-particles are located, such S-particles turn into the A-state immediately and simultaneously. It follows from this definition that the particles located on any vertex are either *all* of type A or *all* of type S almost surely. We define the *activity function*, $\xi \in [0,1]^V$, and, for any $x \in V$, we assume that $\varphi(x)$ is an independent Bernoulli variable with parameter $\xi(x)$, and that $\eta(x)$ has Poisson distribution with parameter $\mu \in [0,\infty)$ independently. The Diaconis-Futon representation can be adapted to this setting, Lemma 2.1,holds as well, Lemma 2.2 can be adapted, the critical density (5.1) is well defined, and the results presented in the previous sections do not require the assumptions that all the particles at time zero are active, as long as the activity function ξ does not depend on the parameters μ and λ. When the activity function satisfies $\xi = \delta_{x,o}$, and, additionally, $\lambda = 0$, such a setting corresponds to the Frog model [8].

Our general theorem states a dichotomy: either **(a):** $\zeta_c(\lambda) = \infty$ for any $\lambda \in (0,\infty)$, or **(b):** $\zeta_c = \zeta_c(\lambda)$ is *finite* and *continuous* in the *whole interval* $(0,\infty)$ and, moreover, a general upper bound on its slope (corresponding to the point *(ii)* of the theorem) also holds.

Theorem 5.2. *Let $G = (V,E)$ be an arbitrary undirected infinite locally-finite graph, suppose that the initial number of particles per vertex has independent Poisson distribution with parameter μ, let $\xi \in [0,1]^V$ be an arbitrary activity function. Then, either $\zeta_c(\lambda) = \infty$ for any $\lambda \in (0,\infty)$, or the properties (i) and (ii) below hold:*

(i) $\zeta_c(\lambda)$ *is a finite continuous function of* λ *in* $(0,\infty)$*,*

(ii) *for any* $\lambda \in (0,\infty)$*,* $\limsup_{\delta\to 0} \frac{\zeta_c(\lambda+\delta)-\zeta_c(\lambda)}{\delta} \leq \frac{1}{\lambda(1+\lambda)}$.

In particular, our theorem implies that, if the critical density is finite for a – even arbitrarily small (but positive) – value of λ, then it is finite for *any* – even arbitrarily large – value of $\lambda \in (0,\infty)$. As far as we know, proving the finiteness of the critical density is a non-trivial open problem when the activity function satisfies $\xi = \delta_{o,x}$ and the graph G is non-amenable, a solution to this problem has been provided in the special case of trees and $\lambda = 0$, corresponding to the Frog model [8]. Instead, when the activity function is identically one, it is easy to deduce that the critical density is finite for any value of $\lambda \in (0,\infty)$ at least on every graph with bounded degree by using the technique of ghost walks [13].

Proof of Theorem 5.2. Fix an arbitrary activity function $\xi \in [0,1]^V$ and suppose that the first claim in the theorem does not hold, namely there exists $\lambda' \in (0,\infty)$ such that $\zeta_c(\lambda') < \infty$. Then, the ARW model with particle density $\mu = \zeta_c(\lambda') + \epsilon$ and deactivation rate λ' is active with some probability $p_\epsilon > 0$. By Proposition 5.1, this implies that for any $(\lambda'', \mu') \in \mathcal{C}_{\lambda', \zeta_c(\lambda')+\epsilon}$, the activated random walk is active with probability at least p_ϵ, namely $\zeta_c(\lambda'') < \infty$. By monotonicity, (5.6), we deduce that the critical density is finite for any for any $\lambda'' \in [0,\infty)$. The proof of (i) and (ii) is now the same as the proof of Theorem 1.1, where finiteness is used. This concludes the proof. □

Acknowledgements

This work started as the author was affiliated to Technische Universität Darmstadt, it has been carried on while the author was affiliated to the University of Bath and it was concluded as the author was affiliated to the Weierstrass Institute, Berlin. The author acknowledges support from DFG German Research Foundation BE/5267/1 and from EPSRC Early Career Fellowship EP/N004566/1.

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