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Dimension reduction of thermistor models for large-area organic light-emitting diodes

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Abstract

An effective system of partial differential equations describing the heat and current flow through a thin organic light-emitting diode (OLED) mounted on a glass substrate is rigorously derived from a recently introduced fully three-dimensional $p(x)$-Laplace thermistor model. The OLED consists of several thin layers that scale differently with respect to the multiscale parameter $\varepsilon > 0$, which is the ratio between the total thickness and the lateral extent of the OLED. Starting point of the derivation is a rescaled formulation of the current-flow equation in the OLED for the driving potential and the heat equation in OLED and glass substrate with Joule heat term concentrated in the OLED. Assuming physically motivated scalings in the electrical flux functions, uniform a priori bounds are derived for the solutions of the three-dimensional system which facilitates the extraction of converging subsequences with limits that are identified as solutions of a dimension reduced system. In the latter, the effective current-flow equation is given by two semilinear equations in the two-dimensional cross-sections of the electrodes and algebraic equations for the continuity of the electrical fluxes through the organic layers. The effective heat equation is formulated only in the glass substrate with Joule heat term on the part of the boundary where the OLED is mounted.

1 Introduction

Large-area OLEDs are a novel sustainable technology for lighting applications, e.g. in car rear lights, ceiling lights, etc. They are based on organic semiconductor materials, where charge carriers move via temperature-activated hopping transport through an energetically random energy landscape [KvdH*15]. However, with Joule self-heating this leads to a complex interplay between charge and heat flow in organic materials. In fact, it was proven experimentally that organic devices show S-shaped current-voltage relations with regions of negative differential resistance [FP*13]. Moreover, in case of large-area OLEDs this effect leads to significant brightness inhomogeneities [FK*14, FP*18] and even a saturation and decrease of brightness at high currents [KF*20].

In [LK*15] a PDE thermistor model was introduced that describes the coupling between current and heat flow in organic devices and is able to reproduce the observed S-shaped characteristics [KF*20]. It consists of a p-Laplace-type current-flow equation for the driving potential φ and the heat equation for the temperature T. The model was extended in [BGL16] by considering variable exponents $p(x)$ for the growth of the electrical flux function modeling e.g. different power laws for the dependence on the electrical field $-\nabla\varphi$ in substructures of the organic device. The existence of solutions was proven using a regularization of the Joule heat term, to overcome that it is a priori only in L^1, and a Galerkin approximation. In [BGL17] the existence of solutions is proved via the concept of entropy solutions for the heat equation with L^1 right-hand sides and Schauder's fixed-point theorem. Note that uniqueness of solutions cannot be expected for this system due to the S-shaped characteristics, where different states exists for the same applied voltage but different temperature distributions, see [FP*18].

Typically, real world large-area OLEDs are thin-film devices consisting of multiple functional layers, whose thicknesses are in the range from 20 nm (recombination layer) up to 100 nm (electrodes). In contrast, the lateral extent of the OLEDs can be in the range of several centimeters (see [FK*14]). This raises the question whether it is possible to derive an effective model from the thermistor model described above, where the description of the current and heat flow in the OLED is reduced to a two-dimensional problem.

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In the present text, we rigorously derive such an effective system for a large-area OLED occupying the domain $\Omega_\varepsilon^{\rm oled}$ and the adjacent glass substrate $\Omega^{\rm sub}$ in the limit of vanishing layer thickness. In particular, we assume a geometrically planar structure where the OLED domain $\Omega_\varepsilon^{\rm oled} = \omega \times]0, h_\varepsilon[$ is given by a cross-section $\omega \subset \mathbb{R}^2$ and total thickness $h_\varepsilon \to 0$ as $\varepsilon \to 0$ where $\varepsilon > 0$ is a dimensionless parameter describing the ratio between the thickness of the OLED and the diameter of ω.

The OLED is mounted on a glass substrate (which is not electrically active) and consists of N layers. The top and bottom layer, $i = 1$ and $i = N$, respectively, correspond to the well conducting metal electrodes between which the organic layers are sandwiched. The layers are allowed to scale differently with respect to $\varepsilon > 0$ (see (2.4)). The latter is a crucial assumption for the derivation of an effective limit for a diffusion problem in [FrL19] using evolutionary Γ-convergence, where it leads to a thermodynamically consistent model for jump processes through thin membranes. Starting point for our investigation is the system considered in [BGL16] taking the form

$$
\begin{aligned}
-\mathrm{div}\, \boldsymbol{S}_\varepsilon(x, T, \nabla\varphi) &= 0 \quad \text{in } \Omega_\varepsilon^{\rm oled},\\
-\mathrm{div}\,(\lambda(x)\nabla T) &= \begin{cases} \boldsymbol{S}_\varepsilon(x, T, \nabla\varphi)\cdot\nabla\varphi & \text{in } \Omega_\varepsilon^{\rm oled},\\ 0 & \text{in } \Omega^{\rm sub}, \end{cases}
\end{aligned}
$$

where $\boldsymbol{S}_\varepsilon : \Omega_\varepsilon^{\rm oled} \times [T_{\rm a}, \infty[\times\mathbb{R}^3 \to \mathbb{R}^3$ is the electrical flux function, which is assumed to be constant with respect to x in each sublayer of the OLED. Moreover, we suppose that $\boldsymbol{S}_\varepsilon$ scales differently in the metal electrodes and the organic layers with respect to the layer thickness, cf. (2.8). We refer to Section 2 for the concrete geometric setting, the assumptions on the data, and the statement of the main result in Theorem 2.4.

The derivation of effective models for thin structures has a long and rich history (see e.g. [CiD79, NeJ07, ScT10] for elastic plates, diffusion through thin membranes, conductive thin sheets). Especially in continuum mechanics, a hierarchy of plate and rod models was derived via Γ-convergence methods [ABP91, FJM06]. The latter is not applicable in our case since the above system cannot be formulated as a minimization principle.

For the actual limit passage, we rescale the OLED domain in Subsection 2.2 such that each layer has constant thickness 1 and the dependence on the layer thickness thus becomes explicit. The limit passage is based on the possibility to derive uniform a priori estimates for the solutions $(\varphi_\varepsilon, T_\varepsilon)$ of the rescaled thermistor system which allow us to select suitably converging subsequences. While the derivation of uniform bounds for the potential φ_ε is straightforward, the case for the heat equation is more involved. Here we use the ideas in [BGL16] and choose suitable powers of the temperature as test functions in the heat equation to obtain uniform bounds for the temperature multiplied by powers of the layer thickness. A careful bookkeeping of the appearing exponents then yields the crucial estimates, see Lemma 3.1 in Section 3.

Eventually, the limit passage is presented in Section 3 as well. The crucial point is the identification of the limits of the nonlinear flux functions and the Joule heat term, which follows from the assumed monotonicity of the flux functions. The obtained limit system is still formulated over the three-dimensional (rescaled) OLED domain. However, in the limit the derivatives of the potential with respect to the vertical direction vanish in the electrodes such that it can be identified therein with functions φ^1, φ^{N-1} on the two-dimensional domain $\Gamma_0 = \omega \times \{0\}$. In addition, due to the different scaling of the electrical fluxes in the organic layers, only derivatives $\partial_{x_3}\varphi$ appear in the limiting current-flow equation. The resulting ordinary differential equation can be solved explicitly, namely by a piecewise affine function. We call the traces of the latter on the interfaces between organic layers interface potentials and denote them by φ^i, $i = 2, \ldots, N{-}2$. Finally, the temperature is constant with respect to x_3 in the OLED and is hence identified by its trace on Γ_0.

Thus, we prove in Section 4 that in the limit $\varepsilon \to 0$ the effective PDE system for the current and heat flow in the

OLED and the glass substrate is given by

$$-\nabla' \cdot (\sigma_{\mathsf{sh}}^{-}\nabla'\varphi^1) - F^2(T, \varphi^2 - \varphi^1) = 0 \quad \text{on } \Gamma_0 \tag{1.1}$$

$$F^{i+1}(T, \varphi^{i+1} - \varphi^i) - F^i(T, \varphi^i - \varphi^{i-1}) = 0 \quad \text{on } \Gamma_0,\ i = 2, \ldots, N-2, \tag{1.2}$$

$$-\nabla' \cdot (\sigma_{\mathsf{sh}}^{+}\nabla'\varphi^{N-1}) + F^{N-1}(T, \varphi^{N-1} - \varphi^{N-2}) = 0, \ \text{on } \Gamma_0, \tag{1.3}$$

$$-\nabla \cdot (\lambda(x)\nabla T) = 0 \quad \text{in } \Omega^{\mathrm{sub}}, \tag{1.4}$$

with sheet conductivities σ_{sh}^{+}, σ_{sh}^{-} in the upper and lower electrode, F^i being the third component of the electrical flux function in the ith organic layer, where $i = 2, \ldots, N-1$. In particular, the equations in (1.2) give the continuity of the electrical current between the organic layers. The heat equation in the substrate Ω^{sub} is supplemented by the following nonlinear boundary condition taking the heating via Joule heat in addition to the Robin boundary conditions into account

$$-\lambda(x)\nabla T \cdot \nu = \begin{cases} \kappa(x)(T - T_{\mathrm{a}}) & \text{on } \partial\Omega^{\mathrm{sub}} \setminus \Gamma_0, \\ \kappa(x)(T - T_{\mathrm{a}}) - H_{\Gamma_0}(x) & \text{on } \Gamma_0, \end{cases} \tag{1.5}$$

where the surface heating $H_{\Gamma_0}(x)$ is given via

$$H_{\Gamma_0}(x) = \sigma_{\mathsf{sh}}^{-}|\nabla'\varphi^1|^2 + \sigma_{\mathsf{sh}}^{+}|\nabla'\varphi^{N-1}|^2 + \sum_{i=2}^{N-1} F^i\big(T, \varphi^i - \varphi^{i-1}\big)(\varphi^i - \varphi^{i-1}). \tag{1.6}$$

Concluding, let us remark that the derivation of effective lower dimensional models for large-area OLEDs is tremendously helpful for the efficient numerical simulation of these devices. In particular, in view of sensitivity studies with respect to parameter variation any reduction in complexity contributes to deepen the understanding of thin-film organic devices.

2 Setting and main result

In the following, we consider current and heat flow through a geometrically thin structure and denote by the dimensionless parameter $\varepsilon > 0$ the ratio between thickness and lateral extent of the structure. More precisely, we follow [BGL16] and consider the following system of equations consisting of the current-flow equation for the potential φ coupled to the heat equation for the temperature T

$$-\nabla \cdot \boldsymbol{S}_\varepsilon(x, T, \nabla\varphi) = 0 \qquad \text{in } \Omega_\varepsilon^{\mathrm{oled}} \subset \Omega_\varepsilon, \tag{2.1}$$

$$-\nabla \cdot (\lambda(x)\nabla T) = H_\varepsilon(x) \qquad \text{in } \Omega_\varepsilon, \tag{2.2}$$

where $\boldsymbol{S}_\varepsilon : \Omega_\varepsilon^{\mathrm{oled}} \times [T_{\mathsf{a}}, \infty[\times \mathbb{R}^3 \to \mathbb{R}^3$ describes the net electrical current flow through the device, λ is the thermal conductivity, and H_ε denotes the Joule heat term. The latter takes the form

$$H_\varepsilon(x) = \begin{cases} \boldsymbol{S}_\varepsilon\big(x, T(x), \nabla\varphi(x)\big) \cdot \nabla\varphi(x) & \text{if } x \in \Omega_\varepsilon^{\mathrm{oled}} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

In particular, $\Omega_\varepsilon^{\mathrm{oled}} \subset \Omega_\varepsilon \subset \mathbb{R}^3$ denotes the electrically active region, the actual OLED, while Ω_ε also includes the adjacent glass substrate $\Omega^{\mathrm{sub}} = \Omega_\varepsilon \setminus \overline{\Omega_\varepsilon^{\mathrm{oled}}}$. We assume the following planar geometric structure: The domain $\Omega_\varepsilon^{\mathrm{oled}} = \omega \times]0, h_\varepsilon[$, with cross-section $\omega \subset \mathbb{R}^2$, satisfies $\Gamma_0 := \omega \times \{0\} \subset \partial\Omega^{\mathrm{sub}}$ and consists of N layers, with $N > 2$, each with thickness h_ε^i such that the total thickness satisfies $h_\varepsilon = \sum_{i=1}^N h_\varepsilon^i$ (comp. Fig. 1). We highlight, that we take into account that the different layers shrink with different rates. More precisely, we make the following assumption concerning the layer thickness

$$h_\varepsilon^i = h_*^i \varepsilon^{\rho_i} \quad \text{with } h_*^i > 0 \text{ and } \rho_i > 0 \text{ for } i = 1, \ldots, N. \tag{2.4}$$

The first and last layer represent the electrically well conducting bottom ($i = 1$) and top ($i = N$) electrodes while the remaining layers are comprised of organic semiconductor materials ($i = 2, \ldots, N-1$) with comparatively bad conductivity properties.

In the following, we will use $\widehat{h}_\varepsilon^0 := 0$ and $\widehat{h}_\varepsilon^i := \sum_{j=1}^i h_\varepsilon^j$, for $i = 1, \ldots, N$, to denote the cumulative height of the OLED stack and define the subsets corresponding to the layers via

$$\Omega_\varepsilon^i = \omega \times \big]\widehat{h}_\varepsilon^{i-1}, \widehat{h}_\varepsilon^i\big[\subset \Omega_\varepsilon^{\rm oled}, \quad \text{for } i = 1, \ldots, N.$$

For boundary subsets $\gamma_+, \gamma_- \subset \partial\omega$, we impose the following Dirichlet boundary conditions for the potential φ

$$\varphi = \varphi_-^{\rm D} \text{ on } \Gamma_\varepsilon^- := \gamma_- \times \big]\widehat{h}_\varepsilon^0, \widehat{h}_\varepsilon^1\big[\quad \text{and} \quad \varphi = \varphi_+^{\rm D} \text{ on } \Gamma_\varepsilon^+ := \gamma_+ \times \big]\widehat{h}_\varepsilon^{N-1}, \widehat{h}_\varepsilon^N\big[, \tag{2.5}$$

for some given Dirichlet data $\varphi_+^{\rm D}$, $\varphi_-^{\rm D}$. More precisely, we assume that $\varphi_+^{\rm D}, \varphi_-^{\rm D} \in {\rm W}^{1,\infty}(\omega)$ and extend them to $\Omega_\varepsilon^{\rm oled}$ by defining the interpolation

$$\varphi_\varepsilon^{\rm D}(x_1,x_2,x_3) = \begin{cases} \varphi_+^{\rm D}(x_1,x_2) & \text{for } \widehat{h}_\varepsilon^{N-1} < x_3 \le \widehat{h}_\varepsilon^N, \\ \dfrac{\varphi_+^{\rm D}(x_1,x_2) - \varphi_-^{\rm D}(x_1,x_2)}{h_\varepsilon^{\rm org}}\big(x_3 - \widehat{h}_\varepsilon^1\big) + \varphi_-^{\rm D}(x_1,x_2) & \text{for } \widehat{h}_\varepsilon^1 < x_3 \le \widehat{h}_\varepsilon^{N-1}, \\ \varphi_-^{\rm D}(x_1,x_2) & \text{for } \widehat{h}_\varepsilon^0 < x_3 \le \widehat{h}_\varepsilon^1, \end{cases} \tag{2.6}$$

where $h_\varepsilon^{\rm org} = \sum_{i=2}^{N-1} h_\varepsilon^i$ is the total thickness of the organic layers (excluding the metallic electrodes). Thus, we have that $\varphi_\varepsilon^{\rm D} \in {\rm W}^{1,\infty}(\Omega_\varepsilon^{\rm oled})$, and we can rewrite the Dirichlet boundary condition as $\varphi = \varphi_\varepsilon^{\rm D}$ on $\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$. For the remaining boundary $\partial\Omega_\varepsilon^{\rm oled} \setminus (\Gamma_\varepsilon^- \cup \Gamma_\varepsilon^+)$ we assume no-flux boundary conditions, i.e. $\boldsymbol{S}_\varepsilon(x, T, \nabla\varphi) \cdot \nu = 0$ with ν denoting the unit outer normal vector.

Finally, for the heat equation we assume Robin boundary condition on the whole boundary of Ω_ε given in terms of a transmission coefficient $\kappa(x) \ge 0$ and the ambient temperature $T_{\rm a} > 0$, viz.

$$\lambda \nabla T \cdot \nu + \kappa(x)(T - T_{\rm a}) = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{2.7}$$

We will denote by $\Gamma_\varepsilon^{\rm lat} = \partial\omega \times]0, h_\varepsilon[$ the lateral boundary of the OLED, whose contribution in the heat equation will disappear in the effective limit.

2.1 Assumptions

Concerning the constitutive equation for the flux function $\boldsymbol{S}_\varepsilon$, we assume that it is piecewise constant with respect to the spatial variable x. In particular, we assume that there exist functions $\boldsymbol{S}^i : [T_{\rm a}, \infty[\times \mathbb{R}^3 \to \mathbb{R}^3$ (independent of ε), $i = 1, \ldots, N$, such that

$$\boldsymbol{S}_\varepsilon(x, T, z) = \begin{cases} \boldsymbol{S}^1\Big(T, \dfrac{z}{h_\varepsilon^1}\Big) & \text{if } x \in \Omega_\varepsilon^1, \\ \boldsymbol{S}^i(T, h_\varepsilon^i z) & \text{if } x \in \Omega_\varepsilon^i,\ i = 2, \ldots, N-1, \\ \boldsymbol{S}^N\Big(T, \dfrac{z}{h_\varepsilon^N}\Big) & \text{if } x \in \Omega_\varepsilon^N. \end{cases} \tag{2.8}$$

We assume that in the electrodes ($i = 1, N$) we do not have any temperature dependence and a linear law, viz.

$$\boldsymbol{S}^1(T, z) = \sigma_{\rm sh}^- z \quad \text{and} \quad \boldsymbol{S}^N(T, z) = \sigma_{\rm sh}^+ z, \tag{2.9}$$

where $\sigma_{\rm sh}^+$, $\sigma_{\rm sh}^- > 0$ are the so-called sheet conductivities of the upper and lower electrode, respectively.

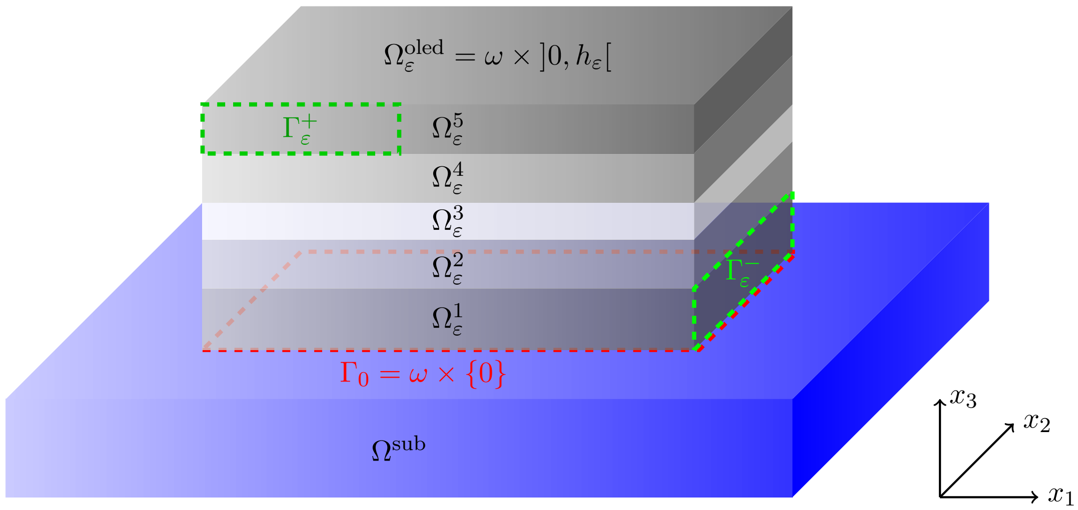

Figure 1: Sketch of the domain Ω_ε consisting of the glass substrate $\Omega^{\rm sub}$ and the OLED $\Omega_\varepsilon^{\rm oled}$. The latter consists of N layers (with $N = 5$ in the figure). The bottom and top layer Ω_ε^1 and Ω_ε^N describe the electrodes with Dirichlet boundaries Γ_ε^- and Γ_ε^+ (green) for the potential where the voltage is applied. In the effective limit, the current-flow equation reduces to coupled equations on the two-dimensional domain Γ_0 (red) and the heat equation is solved only in $\Omega^{\rm sub}$ with an additional boundary source term on Γ_0.

Remark 2.1 *The scalings in the electrical flux function $\boldsymbol{S}_\varepsilon$ in* (2.8) *are such that the sheet resistance in the electrodes is of order 1, while for the organic layers a potential difference $\varphi(\cdot, \widehat{h}_\varepsilon^i) - \varphi(\cdot, \widehat{h}_\varepsilon^{i-1})$ of order 1 with electrical field $\boldsymbol{E} = -\frac{\varphi(\cdot,\widehat{h}_\varepsilon^i)-\varphi(\cdot,\widehat{h}_\varepsilon^{i-1})}{h_\varepsilon^i} e_3$ of order $1/h_\varepsilon^i$ leads to an electrical current of order 1.*

We impose the following assumptions on the data:

(I) $\Omega^{\rm sub} \subset \mathbb{R}^3$ and $\omega \subset \mathbb{R}^2$ are bounded Lipschitz domains and $\gamma_+, \gamma_- \subset \partial\omega$ have positive one-dimensional Hausdorff measure.

(II) The sheet resistances $\sigma_{\rm sh}^+, \sigma_{\rm sh}^- > 0$ are positive constants.

(III) For $i = 2, \ldots, N{-}1$, there exists $p_i \in]1, \infty[$ as well as constants $\sigma_1 > 0$, $\sigma_2 \geq 0$ and $\sigma_3 > 0$ such that

$$\boldsymbol{S}^i(T, z) \cdot z \geq \sigma_1 |z|^{p_i} - \sigma_2 \quad \text{and} \quad |\boldsymbol{S}^i(T, z)| \leq \sigma_3 (1 + |z|)^{p_i - 1}. \tag{2.10}$$

(IV) For $i = 2, \ldots, N{-}1$, the functions $\boldsymbol{S}^i$ are continuous, $\boldsymbol{S}^i(T, 0) = 0$ for all $T \in [T_{\rm a}, \infty[$, and for all $z_1, z_2 \in \mathbb{R}^3$ with $z_1 \neq z_2$ and all $T \in [T_{\rm a}, \infty[$ we have strict monotonicity

$$\big(\boldsymbol{S}^i(T, z_1) - \boldsymbol{S}^i(T, z_2)\big) \cdot (z_1 - z_2) > 0. \tag{2.11}$$

(V) The heat conductivity satisfies $\lambda \in {\rm L}^\infty(\Omega_\varepsilon)$ and there exist constants $0 < \underline{\Lambda}_0 \leq \lambda(x) \leq \overline{\Lambda}_0 < \infty$ for almost every $x \in \Omega_\varepsilon$.

(VI) The heat transmission coefficient $\kappa \in {\rm L}^\infty_+(\partial\Omega_\varepsilon)$ is such that $\kappa(x) \geq \underline{\kappa}_0 > 0$ for almost all $x \in \partial\Omega_\varepsilon$.

(VII) The Dirichlet data satisfies $\varphi_-^{\rm D}, \varphi_+^{\rm D} \in {\rm W}^{1,\infty}(\omega)$.

We introduce the variable exponent $x \mapsto p(x) \in]1, \infty[$ by setting

$$p(x) := \begin{cases} 2 & \text{if } x \in \Omega_\varepsilon^1 \cup \Omega_\varepsilon^N, \\ p_i & \text{if } x \in \Omega_\varepsilon^i,\ i = 2, \ldots, N{-}1. \end{cases} \tag{2.12}$$

Following [KoR91, FaZ01, DH*11], we consider the standard variable exponent Lebesgue space $\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$, which consists of all measurable functions v with finite p-modular

$$\mathsf{m}_{p(\cdot)}(v) := \int_{\Omega_\varepsilon^{\mathrm{oled}}} |v(x)|^{p(x)}\,\mathrm{d}x.$$

This space is equipped with the Luxemburg norm

$$\|v\|_{\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})} := \inf\Big\{\tau > 0 :\ \mathsf{m}_{p(\cdot)}\Big(\frac{v}{\tau}\Big) \le 1\Big\},$$

for which $\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$ becomes a Banach space. In addition, we have that $\mathsf{m}_{p(\cdot)}(v) \le 1$ if and only if $\|v\|_{\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})} \le 1$.

We introduce $p_- := \operatorname{ess\,inf}_{x\in\Omega_\varepsilon^{\mathrm{oled}}} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega_\varepsilon^{\mathrm{oled}}} p(x)$. Then, all $v \in \mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$ satisfy the following inequality (see [DH*11, Lemma 3.2.5])

$$\begin{aligned}\min\Big\{\mathsf{m}_{p(\cdot)}(v)^{\frac{1}{p_-}}, \mathsf{m}_{p(\cdot)}(v)^{\frac{1}{p_+}}\Big\} &\le \|v\|_{\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})}\\ &\le \max\Big\{\mathsf{m}_{p(\cdot)}(v)^{\frac{1}{p_-}}, \mathsf{m}_{p(\cdot)}(v)^{\frac{1}{p_+}}\Big\}. \qquad (2.13)\end{aligned}$$

Furthermore, if $p_+ < \infty$ then $\mathsf{m}_{p(\cdot)}(v_n) \to 0$ if and only if $\|v_n\|_{\mathrm{L}^{p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})} \to 0$ (see [KoR91, Eqn. (2.28)]).

Next, we focus on a proper definition of generalized Sobolev spaces that is appropriate for our problem. We emphasize here, that the spaces introduced here are not necessarily equivalent to the standard Sobolev spaces with variable exponent. The reason for such a generalization is that we do not have the proper Poincaré inequality in case that p is not continuous and therefore we will not be able to control the $\mathrm{L}^{p(\cdot)}$ norm of φ. Thus, for p as above, we introduce the generalized Sobolev space

$$\mathbb{W}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}}) := \Big\{\varphi \in \mathrm{W}^{1,p_-}(\Omega_\varepsilon^{\mathrm{oled}}) :\ \int_{\Omega_\varepsilon^{\mathrm{oled}}} |\nabla\varphi(x)|^{p(x)}\,\mathrm{d}x < \infty\Big\},$$

which we equip with the following norm

$$\|\varphi\|_{1,p(\cdot)} := \|\varphi\|_{1,p_-} + \|\nabla\varphi\|_{p(\cdot)}.$$

It is easy to see that in the case $1 < p_- \le p_+ < \infty$ the space $\mathbb{W}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$ is a separable and reflexive Banach space, since $\mathrm{L}^{p(\cdot)}$ has the same properties. Second, we introduce the subspace

$$\mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}}) := \{\varphi \in \mathbb{W}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}}) :\ \varphi = 0 \text{ on } \Gamma_\varepsilon^{\mathrm{D}}\},$$

where $\Gamma_\varepsilon^{\mathrm{D}} := \Gamma_\varepsilon^{+} \cup \Gamma_\varepsilon^{-}$. Since we assume that $\Gamma_\varepsilon^{\mathrm{D}}$ has positive two-dimensional

measure, this space can be equipped with the equivalent norm, as follows

$$C_1\|\varphi\|_{1,p(\cdot)} \le \|\nabla\varphi\|_{p(\cdot)} \le C_2\|\varphi\|_{1,p(\cdot)}.$$

Indeed, we can use the facts that the classical Sobolev space $\mathrm{W}_{\mathrm{D}}^{1,p_-}(\Omega_\varepsilon^{\mathrm{oled}})$ satisfies the Poincaré inequality and that the variable exponent Lebesgue space $\mathrm{L}^{p(\cdot)}(\Omega)$ is continuously embedded into the Lebesgue space $\mathrm{L}^{p_-}(\Omega_\varepsilon^{\mathrm{oled}})$ to obtain for arbitrary $\varphi \in \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$

$$\begin{aligned}\|\varphi\|_{1,p(\cdot)} &= \|\varphi\|_{p_-} + \|\nabla\varphi\|_{p_-} + \|\nabla\varphi\|_{p(\cdot)}\\ &\le c(\|\nabla\varphi\|_{p_-} + \|\nabla\varphi\|_{p(\cdot)}) \le c\|\nabla\varphi\|_{p(\cdot)} \le c\|\varphi\|_{1,p(\cdot)}.\end{aligned}$$

The weak formulation of the PDE system in (2.1)–(2.7) reads: Find $\varphi \in (\varphi_\varepsilon^{\mathrm{D}} + \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})$ and $T \in \mathrm{W}^{1,q}(\Omega_\varepsilon)$, $q \in [1, 3/2[$, such that

$$\int_{\Omega_\varepsilon^{\mathrm{oled}}} \boldsymbol{S}_\varepsilon(x,T,\nabla\varphi)\cdot\nabla v\,\mathrm{d}x = 0 \qquad \forall v \in \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}}), \qquad (2.14)$$

$$\begin{aligned}\int_{\Omega_\varepsilon} \lambda(x)\nabla T\cdot\nabla\theta\,\mathrm{d}x + \int_{\partial\Omega_\varepsilon} \kappa(x)(T - T_{\mathsf{a}})\theta\,\mathrm{d}a = \int_{\Omega_\varepsilon^{\mathrm{oled}}} \theta\boldsymbol{S}_\varepsilon(x,T,\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}x&\\ \forall\theta \in \mathrm{W}^{1,q'}(\Omega_\varepsilon) \cap \mathrm{L}^\infty(\Omega_\varepsilon). \qquad (2.15)&\end{aligned}$$

Theorem 2.2 ([BGL16, Theorem 2.1]) *Under assumptions (I)–(VII) there exists a weak solution* $(\varphi_\varepsilon, T_\varepsilon) \in (\varphi_\varepsilon^{\mathrm{D}} + \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_\varepsilon^{\mathrm{oled}})) \times \mathrm{W}^{1,q}(\Omega_\varepsilon)$ *for all* $q \in [1, 3/2[$ *to* (2.14) *and* (2.15) *which satisfies* $T_\varepsilon \geq T_{\mathrm{a}}$.

Remark 2.3 *1.) The uniqueness of solutions cannot be expected. In fact, due to the self-heating S-shaped current-voltage curves with regions of negative differential resistance are observed for the OLED in experiments and in simulations (see [FK* 14] and [FP* 18]). This means that for a certain applied voltage multiple solutions exist with different temperature distributions.*

2.) In [LK 15] the following power-law form of the flux function* $\boldsymbol{S}$ *was assumed for organic layers*

$$\boldsymbol{S}(T, z) = J_{\mathrm{ref}} B(T) \Big| \frac{z}{V_{\mathrm{ref}}} \Big|^{p-2} \frac{z}{V_{\mathrm{ref}}},$$

where $J_{\mathrm{ref}} > 0$ *and* $V_{\mathrm{ref}} > 0$ *are reference current density and voltage, respectively. The temperature factor is assumed to be of Arrhenius type, viz.* $B(T) = B_0 \exp[-\frac{E_{\mathrm{a}}}{k_{\mathrm{B}} T}]$ *with an activation energy* $E_{\mathrm{a}} > 0$. *For* $z = h\nabla\varphi$ *(comp.* (2.8)*), with* $h > 0$ *being the layer thickness, we arrive at the effective conductivity* $\sigma_0 = (J_{\mathrm{ref}} h)/V_{\mathrm{ref}}$. *This highlights the different scaling behavior of the conductivities in the metallic electrodes, where the conductivity is given by* σ_{sh}/h, *and the organic materials with respect to the layer thickness.*

3.) We emphasize, that the setting in [BGL16] is more general since a larger class of constitutive functions $\boldsymbol{S}$ *is allowed. In particular, the dependence on* x *is only required to be measurable and measurable exponents* $p(x)$ *are allowed in (III). Moreover, the strict monotonicity in (IV) is weakened to monotonicity.*

The subsequent sections contain the proof of the following main result which provides the convergence of subsequences of transformed solutions (see Subsection 2.2) for the system in (2.1) and (2.2) to weak solutions of an effective limit system. In particular, the transformation rescales the layers of the OLED such that each has constant thickness 1, i.e. we introduce the rescaled layers $\Omega_1^i = \omega \times]i{-}1, i[$, for $i = 1, \ldots, N$, and Ω_1^{oled} and Ω_1 correspondingly. Note that we do not rescale the substrate Ω^{sub}.

Theorem 2.4 *1. For* $\varepsilon > 0$ *let* $(\varphi_\varepsilon, T_\varepsilon) \in (\varphi_\varepsilon^{\mathrm{D}} + \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_1^{\mathrm{oled}})) \times \mathrm{W}^{1,q}(\Omega_1)$ *denote a weak solution of the transformed thermistor system. Then, up to subsequences, the solutions converge for* $\varepsilon \to 0$ *in the sense depicted in* (3.16) *to limits* $\varphi \in \mathrm{L}^{p_-}(\Omega_1^{\mathrm{oled}})$ *and* $T \in \mathrm{L}^s(\Omega_1)$ *for an* $s \in [1, 6/5[$. *The latter satisfy*

- *(i)* ***(temperature in substrate)*** $T|_{\Omega^{\mathrm{sub}}} \in \mathrm{W}^{1,q}(\Omega^{\mathrm{sub}})$;
- *(ii)* ***(temperature in OLED stack)*** $\partial_{x_3} T = 0$ *a.e. in* Ω_1^{oled} *such that* $T|_{\Omega_1^{\mathrm{oled}}} = T^{\Gamma_0}$ *a.e. in* Ω_1^{oled}, *where* $T^{\Gamma_0} \in \mathrm{L}^q(\Omega_1^{\mathrm{oled}})$ *denotes the extension of the trace of* $T|_{\Omega^{\mathrm{sub}}}$ *on* Γ_0 *to* Ω_1^{oled};
- *(iii)* ***(potential in electrodes)*** $\nabla'\varphi = (\partial_{x_1}\varphi, \partial_{x_2}\varphi)^\top \in \mathrm{L}^2(\Omega_1^1 \cup \Omega_1^N)^2$ *and* $\partial_{x_3}\varphi = 0$ *a.e. in* $\Omega_1^1 \cup \Omega_1^N$ *such that we identify* $\varphi|_{\Omega_1^1}$ *and* $\varphi|_{\Omega_1^N}$ *with functions* $\varphi^1, \varphi^{N-1} \in \mathrm{H}^1(\Gamma_0)$ *which satisfy the boundary conditions* $\varphi^1 = \varphi_-^{\mathrm{D}}$ *on* $\gamma_- \times \{0\}$ *and* $\varphi^{N-1} = \varphi_+^{\mathrm{D}}$ *on* $\gamma_+ \times \{0\}$, *respectively;*
- *(iv)* ***(potential in organic layers)*** $\partial_{x_3}\varphi \in \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)$ *and* φ *is piecewise affine with respect to* x_3 *in* $\cup_{i=2}^{N-1}\Omega_1^i$ *(the organic layers) such that* $\partial_{x_3}\varphi = \widetilde{\varphi}^i - \widetilde{\varphi}^{i-1} \in \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)$ *on* Ω_1^i, $i = 2, \ldots, N{-}1$, *where the interface potential* $\widetilde{\varphi}^i \in \mathrm{L}^{p_-}(\omega \times \{i\})$ *is the trace of* φ *on the heterointerface* $\omega \times \{i\}$, $i = 2, \ldots, N{-}1$.

2. Let the space for the potentials be given by

$$V_0 = \Big\{ (\varphi^1, \ldots, \varphi^{N-1}) \in \mathrm{H}^1(\Gamma_0) \times \prod_{j=2}^{N-2} \mathrm{L}^{p_-}(\Gamma_0) \times \mathrm{H}^1(\Gamma_0) \,:\, \varphi^i - \varphi^{i-1} \in \mathrm{L}^{p_i}(\Gamma_0), \\ \varphi^1 = 0 \text{ on } \gamma_- \times \{0\} \text{ and } \varphi^{N-1} = 0 \text{ on } \gamma_+ \times \{0\} \Big\}. \quad (2.16)$$

Identifying the interface potentials $\widetilde{\varphi}^i$, $i=2,\dots,N{-}1$, with functions φ^i on Γ_0, the tuple $(\varphi^1,\dots,\varphi^{N-1},T)\in((\varphi^{\rm D}_-,0,\dots,0,\varphi^{\rm D}_+)+V_0)\times{\rm W}^{1,q}(\Omega^{\rm sub})$ satisfy the effective limit system consisting of the effective current-flow equation

$$\int_{\Gamma_0}\Big\{\sigma_{\rm sh}^-\nabla'\varphi^1\cdot\nabla'v^1+\sigma_{\rm sh}^+\nabla'\varphi^{N-1}\cdot\nabla'v^{N-1}\\ +\sum_{i=2}^{N-1}F^i\big(T,\varphi^i{-}\varphi^{i-1}\big)(v^i{-}v^{i-1})\Big\}\,{\rm da}=0, \tag{2.17}$$

where $F^i=\boldsymbol{S}^i_3$, i.e. the third component of the vector-valued function $\boldsymbol{S}^i$ and the test functions are such that $(v^1,\dots,v^{N-1})\in V_0$. The effective current-flow equation is coupled to the effective heat equation with boundary source term

$$\int_{\Omega^{\rm sub}}\lambda(x)\nabla T\cdot\nabla\theta\,{\rm d}x+\int_{\partial\Omega^{\rm sub}}\kappa(x)(T{-}T_{\rm a})\theta\,{\rm da}=\int_{\Gamma_0}\Big\{\sigma_{\rm sh}^-|\nabla'\varphi^1|^2+\sigma_{\rm sh}^+|\nabla'\varphi^{N-1}|^2\\ +\sum_{i=2}^{N-1}F^i(T,\varphi^i{-}\varphi^{i-1})(\varphi^i{-}\varphi^{i-1})\Big\}\theta\,{\rm da}, \tag{2.18}$$

and $\theta\in{\rm W}^{1,q'}(\Omega^{\rm sub})\cap{\rm L}^\infty(\Omega^{\rm sub})$.

We immediately check that (2.17) and (2.18) is formally equivalent to the system in (1.1)–(1.6).

2.2 Transformation of the domain

Before passing to the limit, we transform the domain $\Omega_\varepsilon^{\rm oled}$ such that each layer has constant thickness 1. More precisely, we define the Lipschitz map $G_\varepsilon:\Omega_\varepsilon^{\rm oled}\to\Omega_1^{\rm oled}:=\omega\times\,]0,N[$ for $x=(x_1,x_2,x_3)\in\Omega_\varepsilon^{\rm oled}$ via $G_\varepsilon(x)=(x_1,x_2,g_\varepsilon(x_3))$, where

$$g_\varepsilon(x_3):=\frac{x_3-\widehat{h}_\varepsilon^{i-1}}{h_\varepsilon^i}+i{-}1\quad\text{for } x_3\in\big]\widehat{h}_\varepsilon^{i-1},\widehat{h}_\varepsilon^i\big]\text{ and } i=1,\dots,N. \tag{2.19}$$

We denote by $\Omega_1^i:=G_\varepsilon(\Omega_\varepsilon^i)$, $i=1,\dots,N$, the rescaled layers and identify functions w on $\Omega_\varepsilon^{\rm oled}$ with functions $\widetilde{w}$ on $\Omega_1^{\rm oled}$ via $w(x)=\widetilde{w}(G_\varepsilon(x))$ for $x\in\Omega_\varepsilon^{\rm oled}$. In particular, we have $\partial_{x_3}w(G_\varepsilon^{-1}(\widetilde{x}))=\frac{1}{m_\varepsilon(\widetilde{x})}\partial_{\widetilde{x}_3}\widetilde{w}(\widetilde{x})$ for $\widetilde{x}\in\Omega_1^{\rm oled}$ and $\int_{\Omega_\varepsilon^{\rm oled}}w(x)\,{\rm d}x=\int_{\Omega_1^{\rm oled}}m_\varepsilon(\widetilde{x})\widetilde{w}(\widetilde{x})\,{\rm d}\widetilde{x}$, where we introduced the piecewise constant function $m_\varepsilon(\widetilde{x})=h_\varepsilon^i$ for $\widetilde{x}\in\Omega_1^i$, $i=1,\dots,N$.

Moreover, for notational simplicity, we introduce the piecewise constant diagonal matrices $M_\varepsilon(\widetilde{x}),A_\varepsilon(\widetilde{x})\in\mathbb{R}^{3\times3}$ given via

$$M_\varepsilon(\widetilde{x})=\begin{cases}{\rm diag}\Big(1,1,\dfrac{1}{m_\varepsilon(\widetilde{x})}\Big) & \text{if } \widetilde{x}\in\Omega_1^1\cup\Omega_1^N,\\ {\rm diag}\big(m_\varepsilon(\widetilde{x}),m_\varepsilon(\widetilde{x}),1\big) & \text{if } \widetilde{x}\in\displaystyle\bigcup_{i=2}^{N-1}\Omega_1^i.\end{cases}\\ A_\varepsilon(\widetilde{x})={\rm diag}\Big(m_\varepsilon(\widetilde{x}),m_\varepsilon(\widetilde{x}),\frac{1}{m_\varepsilon(\widetilde{x})}\Big). \tag{2.20}$$

We define Ω_1 such that $\overline{\Omega_1}=\overline{\Omega^{\rm sub}}\cup\overline{\Omega_1^{\rm oled}}$. Thus, with the assumptions on $\boldsymbol{S}_\varepsilon$ in (2.8), the resulting current-flow equation for the transformed solutions $(\varphi_\varepsilon,T_\varepsilon)\in(\varphi_\varepsilon^{\rm D}{+}\mathbb{W}_{\rm D}^{1,p(\cdot)}(\Omega_1^{\rm oled}))\times{\rm W}^{1,q}(\Omega_1)$ (omitting tildes from now on) reads

$$\int_{\Omega_1^{\rm oled}}\boldsymbol{S}(x,T_\varepsilon,M_\varepsilon(x)\nabla\varphi_\varepsilon)\cdot M_\varepsilon(x)\nabla v\,{\rm d}x=0\quad\forall v\in\mathbb{W}_{\rm D}^{1,p(\cdot)}(\Omega_1^{\rm oled}), \tag{2.21}$$

where we have set $\boldsymbol{S}(x,\cdot,\cdot) \equiv \boldsymbol{S}^i$ if and only if $x \in \Omega_1^i$. Similarly, the heat equation takes the form

$$\int_{\Omega^{\rm sub}} \lambda(x)\nabla T_\varepsilon \cdot \nabla\theta \,\mathrm{d}x + \int_{\Omega_1^{\rm oled}} \lambda(x)\nabla T_\varepsilon \cdot A_\varepsilon(x)\nabla\theta\,\mathrm{d}x + B_\varepsilon(T_\varepsilon,\theta) = \int_{\Omega_1^{\rm oled}} H_\varepsilon(x)\theta\,\mathrm{d}x \qquad \forall\theta \in \mathrm{W}^{1,q'}(\Omega_1)\cap \mathrm{L}^\infty(\Omega_1), \tag{2.22}$$

where B_ε is the bilinear form representing the boundary integrals, namely

$$\begin{aligned} B_\varepsilon(T,\theta) = &\int_{\partial\Omega^{\rm sub}\setminus\Gamma_0} \kappa(x)(T-T_{\rm a})\theta\,\mathrm{da} + \int_{\omega\times\{N\}} \kappa(x)(T-T_{\rm a})\theta\,\mathrm{da} \\ &+ \int_{\partial\omega\times]0,N[} m_\varepsilon(x)\kappa(x)(T-T_{\rm a})\theta\,\mathrm{da} \quad \text{for } T\in \mathrm{W}^{1,q}(\Omega_1),\ \theta\in\mathrm{W}^{1,q'}(\Omega_1), \end{aligned}$$

and $H_\varepsilon \in \mathrm{L}^1(\Omega_1^{\rm oled})$ denotes the Joule heat

$$H_\varepsilon(x) := \boldsymbol{S}(x, T_\varepsilon(x), M_\varepsilon(x)\nabla\varphi_\varepsilon(x)) \cdot M_\varepsilon(x)\nabla\varphi_\varepsilon(x). \tag{2.23}$$

In the subsequent text, we use the notation $\Gamma^{\rm eff} := (\partial\Omega^{\rm sub}\setminus\Gamma_0)\cup(\omega\times\{N\})$ to denote the effective boundary for the Robin boundary condition for the heat equation that survives in the limit. In contrast, boundary integrals over $\Gamma_1^{\rm lat} := G_\varepsilon(\Gamma_\varepsilon^{\rm lat}) = \omega\times\,]0,N[$ are expected to vanish in the limit $\varepsilon\to 0$.

Finally, we denote by $\Gamma_1^+ := G_\varepsilon(\Gamma_1^+)$, $\Gamma_1^- := G_\varepsilon(\Gamma_1^-)$, and $\Gamma_1^{\rm D} := \Gamma_1^+\cup\Gamma_1^-$ the rescaled Dirichlet boundary for the potential. The Dirichlet function defined in (2.6) reads in rescaled coordinates

$$\varphi_\varepsilon^{\rm D}(\cdot,x_3) = \begin{cases} \varphi_+^{\rm D} & \text{for } N{-}1 < x_3 \le N, \\ \left(\varphi_+^{\rm D}-\varphi_-^{\rm D}\right)\left(\alpha_\varepsilon^i x_3 - \beta_\varepsilon^i\right) + \varphi_-^{\rm D} & \text{for } i{-}1 < x_3 \le i, \\ & \text{with } i = 2,\dots,N{-}1, \\ \varphi_-^{\rm D} & \text{for } 0 < x_3 \le 1, \end{cases} \tag{2.24}$$

where $\alpha_\varepsilon^i = h_\varepsilon^i/h_\varepsilon^{\rm org}$ and $\beta_\varepsilon^i = ((i{-}1)h_\varepsilon^i - \widehat{h}_\varepsilon^{i-1} + h_\varepsilon^1)/h_\varepsilon^{\rm org}$. Note that $h_\varepsilon^{\rm org} = \sum_{j=2}^{N-1} h_\varepsilon^j$, thus, $\alpha_\varepsilon^i \to \alpha_0^i \in [0,1]$ and $\beta_\varepsilon^i \to \beta_0^i \in [0,\infty[$. Moreover, since $(\alpha_\varepsilon^{i+1}{-}\alpha_\varepsilon^i)i = \beta_\varepsilon^{i+1}{-}\beta_\varepsilon^i$, $\alpha_\varepsilon^2 - \beta_\varepsilon^2 = 0$, and $\alpha_\varepsilon^{N-1}(N{-}1) - \beta_\varepsilon^{N-1} = 1$, the limits α_0^i, β_0^i satisfy the same identities. Thus, $\varphi_\varepsilon^{\rm D}$ converges strongly in $\mathrm{W}^{1,\infty}(\Omega_1^{\rm oled})$ to the limit $\varphi_0^{\rm D}$, which is defined as in (2.24) for $\varepsilon = 0$.

3 A priori estimates and limit passage

First, we establish uniform a priori estimates for the solutions of the rescaled thermistor systems, which will enable us to extract convergent subsequences to pass to the limit.

Lemma 3.1 *Let* $(\varphi_\varepsilon, T_\varepsilon) \in (\varphi_\varepsilon^{\rm D} + \mathbb{W}^{1,p(\cdot)}(\Omega_1^{\rm oled})) \times \mathrm{W}^{1,q}(\Omega_1)$ *be a solution to the rescaled thermistor problem in* (2.21) *and* (2.22). *Then, there exists a constant* $C > 0$, *depending on the data but not on* ε, *such that the potential* φ_ε *satisfies the estimates*

$$\sum_{j=1,N} \Big(\|\nabla'\varphi_\varepsilon\|_{\mathrm{L}^2(\Omega_1^j)} + \Big\|\frac{\partial_{x_3}\varphi_\varepsilon}{h_\varepsilon^j}\Big\|_{\mathrm{L}^2(\Omega_1^j)} \Big) \le C, \tag{3.1}$$

$$\sum_{i=2}^{N-1} \Big(\|h_\varepsilon^i \nabla'\varphi_\varepsilon\|_{\mathrm{L}^{p_i}(\Omega_1^i)} + \|\partial_{x_3}\varphi_\varepsilon\|_{\mathrm{L}^{p_i}(\Omega_1^i)} \Big) \le C, \tag{3.2}$$

$$\sum_{i=2}^{N-1} \big\| \boldsymbol{S}^i(T_\varepsilon, M_\varepsilon\nabla\varphi_\varepsilon) \big\|_{\mathrm{L}^{p_i'}(\Omega_1^i)} \le C, \tag{3.3}$$

$$\|\varphi_\varepsilon\|_{\mathrm{L}^{p_-}(\Omega_1^{\rm oled})} \le C. \tag{3.4}$$

Moreover, there exist constants $C > 0$, $\gamma > 0$, and exponents $1 < s < 6/5$ and $1 < \widehat{s} < 3/2$, depending on the data but not on ε, such that the temperature T_ε fulfils

$$\|T_\varepsilon\|_{\mathrm{W}^{1,q}(\Omega^{\mathrm{sub}})} + \|T_\varepsilon\|_{\mathrm{L}^r(\Omega^{\mathrm{sub}})} \le C, \tag{3.5}$$

$$\sum_{i=1}^N \Big\| \frac{\partial_{x_3} T_\varepsilon}{(h_\varepsilon^i)^\gamma} \Big\|_{\mathrm{L}^s(\Omega_1^i)} \le C, \tag{3.6}$$

$$\|H_\varepsilon\|_{\mathrm{L}^1(\Omega_1^{\mathrm{oled}})} \le C, \tag{3.7}$$

$$\|T_\varepsilon\|_{\mathrm{L}^s(\Omega_1^{\mathrm{oled}})} + \sum_{i=1}^N \|h_\varepsilon^i T_\varepsilon\|_{\mathrm{W}^{1,\widehat{s}}(\Omega_1^i)} \le C, \tag{3.8}$$

where $1 \le q < 3/2$ and $1 \le r < 3$ are as in Theorem 2.2.

Proof: We derive the estimates in several steps.

Step 1: Using the test function $v_\varepsilon = \varphi_\varepsilon - \varphi_\varepsilon^{\mathrm{D}} \in \mathbb{W}_{\mathrm{D}}^{1,p(\cdot)}(\Omega_1^{\mathrm{oled}})$ in the current-flow equation in (2.21), together with the growth conditions in (2.10), leads to

$$\begin{aligned}\int_{\Omega_1^{\mathrm{oled}}} \big(c|M_\varepsilon(x)\nabla\varphi_\varepsilon|^{p(x)} - \sigma_2\big)\,\mathrm{d}x &\le \int_{\Omega_1^{\mathrm{oled}}} \boldsymbol{S}(x, T_\varepsilon, M_\varepsilon(x)\nabla\varphi_\varepsilon)\cdot M_\varepsilon(x)\nabla\varphi_\varepsilon\,\mathrm{d}x \\ &= \int_{\Omega_1^{\mathrm{oled}}} \boldsymbol{S}(x, T_\varepsilon, M_\varepsilon(x)\nabla\varphi_\varepsilon)\cdot M_\varepsilon(x)\nabla\varphi_\varepsilon^{\mathrm{D}}\,\mathrm{d}x \\ &\le C\int_{\Omega_1^{\mathrm{oled}}} \big(1+|M_\varepsilon(x)\nabla\varphi_\varepsilon|\big)^{p(x)-1}|M_\varepsilon(x)\nabla\varphi_\varepsilon^{\mathrm{D}}|\,\mathrm{d}x.\end{aligned}$$

Thus, with Hölder's inequality for the variable exponent Lebesgue space $\mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})$ and the strong convergence of the Dirichlet function $\varphi_\varepsilon^{\mathrm{D}}$ in $\mathrm{W}^{1,\infty}(\Omega_\varepsilon^{\mathrm{oled}})$, we arrive at uniform estimates for $M_\varepsilon\nabla\varphi_\varepsilon$ in $\mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})$ which are precisely the estimates in (3.1) and (3.2).

Step 2. The second growth condition in (2.10) gives $|\boldsymbol{S}(x, T_\varepsilon, M_\varepsilon\nabla\varphi_\varepsilon)| \le c(1+|M_\varepsilon\nabla\varphi_\varepsilon|)^{p(x)-1}$ such that the previous estimate on $M_\varepsilon\nabla\varphi_\varepsilon$ gives the uniform estimate for the electrical flux function in (3.3). Moreover, we have also proved the uniform estimate for the Joule heat term H_ε in $\mathrm{L}^1(\Omega_1^{\mathrm{oled}})$ in (3.7).

Step 3. Using the test function $\theta \equiv 1$ in the heat equation (2.22) leads to the estimate

$$\int_{\Gamma^{\text{eff}}} \kappa(x)(T_\varepsilon - T_{\mathrm{a}})\,\mathrm{da} + \int_{\partial\omega\times]0,N[} m_\varepsilon(x)\kappa(x)(T_\varepsilon - T_{\mathrm{a}})\,\mathrm{da} = \int_{\Omega_1^{\mathrm{oled}}} H_\varepsilon\,\mathrm{d}x.$$

Due to the uniform bound for the Joule heat term in the second step and $T_\varepsilon \ge T_{\mathrm{a}} > 0$, we obtain uniform bounds for $\|\kappa T_\varepsilon\|_{\mathrm{L}^1(\Gamma^{\text{eff}})}$, and $\|m_\varepsilon\kappa T_\varepsilon\|_{\mathrm{L}^1(\Gamma_1^{\mathrm{lat}})}$.

Step 4. Proceeding as in the proof of Lemma 3.2 in [BGL16], we test the rescaled heat equation in (2.22) with the test function $\theta = T_\varepsilon^{-\delta}$, where $\delta \in]0,1[$, which is admissible since $T_\varepsilon \ge T_{\mathrm{a}} > 0$. We arrive at

$$\begin{aligned}\int_{\Omega^{\mathrm{sub}}} \delta\lambda\frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\delta}}\,\mathrm{d}x + \int_{\Omega_1^{\mathrm{oled}}} \Big\{\frac{H_\varepsilon}{T_\varepsilon^\delta} + \delta\lambda\frac{\nabla T_\varepsilon\cdot A_\varepsilon(x)\nabla T_\varepsilon}{T_\varepsilon^{1+\delta}}\Big\}\,\mathrm{d}x &= B_\varepsilon(T_\varepsilon, T_\varepsilon^{-\delta}) \\ &\le \frac{1}{T_{\mathrm{a}}^\delta}\Big(\int_{\Gamma^{\text{eff}}}\kappa T_\varepsilon\,\mathrm{da} + \int_{\Gamma_1^{\mathrm{lat}}}\kappa m_\varepsilon T_\varepsilon\,\mathrm{da}\Big).\end{aligned}$$

Thus, with the uniform estimate for the boundary integrals from the previous step, the assumptions on the heat conductivity λ, and the nonnegativity of the Joule heat term we end up with the uniform estimate

$$\int_{\Omega^{\mathrm{sub}}} \frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\delta}}\,\mathrm{d}x + \int_{\Omega_1^{\mathrm{oled}}} \Big\{\frac{h_\varepsilon^i|\nabla' T_\varepsilon|^2}{T_\varepsilon^{1+\delta}} + \frac{|\partial_{x_3}T_\varepsilon|^2}{h_\varepsilon^i T_\varepsilon^{1+\delta}}\Big\}\,\mathrm{d}x \le \frac{C}{\delta}. \tag{3.9}$$

From this, we infer that the restriction of $T_\varepsilon^{(1-\delta)/2}$ to $\Omega^{\rm sub}$ is uniformly bounded in $\mathrm{H}^1(\Omega^{\rm sub})$ (and hence in $\mathrm{L}^6(\Omega^{\rm sub})$ by Sobolev's embedding theorem) and, using the same argumentation as in [BGL16], gives a uniform bound for $T_\varepsilon|_{\Omega^{\rm sub}}$ in $\mathrm{L}^r(\Omega^{\rm sub})$ for any $1 \le r < 3$ and $\mathrm{W}^{1,q}(\Omega^{\rm sub})$ with $1 \le q < 3/2$ by choosing $\delta = 1 - r/3 \in \,]0,1[$ and using Hölder's inequality. This ensures the estimate in (3.5).

Analogously, by considering the restriction of T_ε to Ω_1^i, we obtain a uniform bound for $\varepsilon^{3\rho_i} T_\varepsilon^r|_{\Omega_1^i}$ in $\mathrm{L}^1(\Omega_1^i)$ for $1 \le r < 3$ by the continuous embedding $\mathrm{H}^1(\Omega_1^i) \subset \mathrm{L}^6(\Omega_1^i)$ (recall that $h_\varepsilon^i = h_*^i \varepsilon^{\rho_i}$, see (2.4))

$$\int_{\Omega_1^i} \varepsilon^{3\rho_i} |T_\varepsilon|^r \,\mathrm{d}x \le C. \tag{3.10}$$

Here, we estimated for sufficiently small $\varepsilon > 0$

$$\frac{4h_*^i}{(1-\delta)^2} \big|\varepsilon^{\frac{\rho_i}{2}} \nabla T_\varepsilon^{\frac{1-\delta}{2}}\big|^2 = h_*^i \varepsilon^{\rho_i} \frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\delta}} \le \frac{\nabla T_\varepsilon \cdot A_\varepsilon \nabla T_\varepsilon}{T_\varepsilon^{1+\delta}}.$$

Thus, exploiting again Hölder's inequality, we have for an arbitrary $\widetilde{\gamma} \in \mathbb{R}$ and $1 \le s < 3/2$

$$\begin{aligned} \int_{\Omega_1^i} \varepsilon^{\widetilde{\gamma}} |\partial_{x_3} T_\varepsilon|^s \,\mathrm{d}x &= \int_{\Omega_1^i} \Big(\frac{1}{\varepsilon^{\rho_i}} \frac{|\partial_{x_3} T_\varepsilon|^2}{T_\varepsilon^{1+\delta}} \Big)^{\frac{s}{2}} \varepsilon^{\frac{2\widetilde{\gamma} + \rho_i s}{2}} T_\varepsilon^{\frac{s(1+\delta)}{2}} \,\mathrm{d}x \\ &\le \left(\int_{\Omega_1^i} \frac{1}{\varepsilon^{\rho_i}} \frac{|\partial_{x_3} T_\varepsilon|^2}{T_\varepsilon^{1+\delta}} \,\mathrm{d}x \right)^{\frac{s}{2}} \left(\int_{\Omega_1^i} \varepsilon^{\frac{2\widetilde{\gamma}+\rho_i s}{2-s}} T_\varepsilon^{\frac{s(1+\delta)}{2-s}} \,\mathrm{d}x \right)^{\frac{2-s}{2}} \\ &\le C \left(\int_{\Omega_1^i} \varepsilon^{\frac{2\widetilde{\gamma}+\rho_i s}{2-s}} T_\varepsilon^{\frac{s(1+\delta)}{2-s}} \,\mathrm{d}x \right)^{\frac{2-s}{2}}, \end{aligned} \tag{3.11}$$

where we used (3.9) to estimate the first integral on the second line.

We can find a $\delta \in \,]0,1[$ such that $s(1+\delta)/(2-s) < 3$ if $s < 3/2$ (guaranteeing that $T_\varepsilon^{\frac{s(1+\delta)}{2-s}} \in \mathrm{L}^1(\Omega_1^i)$). However, in order to obtain a uniform bound for the integral on the right-hand side, $\widetilde{\gamma}$ has to satisfy $\frac{2\widetilde{\gamma}+\rho_i s}{2-s} \ge 3\rho_i$ (cf. (3.10)). This is the case if $\widetilde{\gamma} \ge \rho_i(3-2s) > 0$ since $1 \le s < 3/2$. In particular, we obtain that $\varepsilon^{\widetilde{\gamma}/s} \partial_{x_3} T_\varepsilon$ is uniformly bounded in $\mathrm{L}^s(\Omega_1^i)$ for $1 \le s < 3/2$ and $\widetilde{\gamma} \ge \rho_i(3-2s)$. This bound will be used to improve the above estimates.

Indeed, we can use the uniform bound for $\varepsilon^{\widetilde{\gamma}/s} \partial_{x_3} T_\varepsilon$ to get also a uniform bound for $\varepsilon^{\widetilde{\gamma}/s} T_\varepsilon$ in $\mathrm{L}^s(\Omega_1^i)$ for $1 \le s < 3/2$ and any $\widetilde{\gamma} \ge \rho_i(3-2s)$ as follows: Let $T_\varepsilon|_{\Gamma_0} \in \mathrm{L}^q(\Gamma_0)$ denote the trace of $T_\varepsilon|_{\Omega^{\rm sub}} \in \mathrm{W}^{1,q}(\Omega^{\rm sub})$, $1 \le q < 3/2$, which is uniformly bounded in $\mathrm{L}^q(\Gamma_0)$. Then, for any $1 \le s \le q$ we have the estimate

$$\varepsilon^{\widetilde{\gamma}} \|T_\varepsilon\|_{\mathrm{L}^s(\Omega_1^i)}^s \le C \varepsilon^{\widetilde{\gamma}} \left(\big\|T_\varepsilon|_{\Gamma_0}\big\|_{\mathrm{L}^s(\Gamma_0)}^s + \sum_{j=1}^N \|\partial_{x_3} T_\varepsilon\|_{\mathrm{L}^s(\Omega_1^j)}^s \right). \tag{3.12}$$

Choosing $\widetilde{\gamma} = \gamma_*(s) := (3-2s) \max_{j=1,\dots,N} \rho_i$ gives a uniform bound for $\varepsilon^{\gamma_*(s)/s} T_\varepsilon$ in $\mathrm{L}^s(\Omega_1^i)$ (and hence also in $\mathrm{L}^s(\Omega_1^{\rm oled})$) for $1 \le s < 3/2$. Note that $\gamma_*(s) \to 0$ if $s \to 3/2$.

Step 5. Considering again the estimate in (3.11), we find a $\delta \in \,]0,1[$ such that $s(1+\delta)/(2-s) < 3/2$ if s satisfies $1 \le s < 6/5$. Thus, we get a uniform bound for $\varepsilon^{\widetilde{\gamma}/s} \partial_{x_3} T_\varepsilon$ in $\mathrm{L}^s(\Omega_1^i)$ if $\widetilde{\gamma}$ satisfies $\frac{2\widetilde{\gamma}+\rho_i s}{2-s} > 0$. In particular, choosing $1 \le s < 6/5$ sufficiently large, we can find a $\widetilde{\gamma} < 0$ satisfying this inequality and for $\gamma = -\widetilde{\gamma}/s > 0$ we obtain that $\frac{1}{\varepsilon^\gamma} \partial_{x_3} T_\varepsilon$ is uniformly bounded in $\mathrm{L}^s(\Omega_1^i)$, which gives the estimate in (3.6). Furthermore, with the same arguments as above we also have that T_ε is uniformly bounded in $\mathrm{L}^s(\Omega_1^{\rm oled})$ for $1 < s < 6/5$ (see (3.12) for $\widetilde{\gamma} = 0$), thus, the first part of (3.8) holds.

Step 6. It remains to show the uniform bound for $h_\varepsilon^i \nabla' T_\varepsilon$. Proceeding as for the vertical derivative and using again (3.9), we find for the lateral gradient of T_ε and an arbitrary $\widehat\gamma \in \mathbb{R}$ and $1 \le \widehat s < 3/2$

$$\begin{aligned}\int_{\Omega_1^i} \varepsilon^{\widehat\gamma}|\nabla' T_\varepsilon|^{\widehat s}\,\mathrm{d}x &= \int_{\Omega_1^i} \Big(\frac{\varepsilon^{\rho_i}|\nabla' T_\varepsilon|^2}{T_\varepsilon^{1+\widehat\delta}}\Big)^{\frac{\widehat s}{2}} \varepsilon^{\frac{2\widehat\gamma-\rho_i\widehat s}{2}} T_\varepsilon^{\frac{\widehat s(1+\widehat\delta)}{2}}\,\mathrm{d}x \\ &\le \bigg(\int_{\Omega_1^i} \frac{\varepsilon^{\rho_i}|\nabla' T_\varepsilon|^2}{T_\varepsilon^{1+\widehat\delta}}\,\mathrm{d}x\bigg)^{\frac{\widehat s}{2}} \bigg(\int_{\Omega_1^i} \varepsilon^{\frac{2\widehat\gamma-\rho_i\widehat s}{2-\widehat s}} T_\varepsilon^{\frac{\widehat s(1+\widehat\delta)}{2-\widehat s}}\,\mathrm{d}x\bigg)^{\frac{2-\widehat s}{2}} \\ &\le C\bigg(\int_{\Omega_1^i} \varepsilon^{\frac{2\widehat\gamma-\rho_i\widehat s}{2-\widehat s}} T_\varepsilon^{\frac{\widehat s(1+\widehat\delta)}{2-\widehat s}}\,\mathrm{d}x\bigg)^{\frac{2-\widehat s}{2}}.\end{aligned} \tag{3.13}$$

Thus, using the above uniform estimates in (3.12) for $\varepsilon^{\gamma_*(s)/s}T_\varepsilon$ in $\mathrm{L}^s(\Omega_1^i)$ and choosing $\widehat\gamma = \rho_i\widehat s$ and $1 < \widehat s < 3/2$ sufficiently large gives a uniform estimate for $\varepsilon^{\rho_i}\nabla' T_\varepsilon$ in $\mathrm{L}^{\widehat s}(\Omega_1^i)$ (with $\widehat\delta$ in general different from δ). ■

In order to pass to the limit, we define the following function spaces

$$V = \big\{\varphi \in \mathrm{L}^{p_-}(\Omega_1^{\mathrm{oled}}) \,:\, \varphi|_{\Omega_1^1\cup\Omega_1^N} \in \mathrm{H}^1_{\mathrm{D}}(\Omega_1^1\cup\Omega_1^N),\ \partial_{x_3}\varphi \in \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})\big\}, \tag{3.14}$$

$$W = \big\{T \in \mathrm{L}^s(\Omega_1) \,:\, T|_{\Omega^{\mathrm{sub}}} \in \mathrm{W}^{1,q}(\Omega^{\mathrm{sub}}),\ \partial_{x_3}T \in \mathrm{L}^s(\Omega_1^{\mathrm{oled}})\big\}, \tag{3.15}$$

where $\mathrm{H}^1_{\mathrm{D}}(\Omega_1^1\cup\Omega_1^N)$ denotes the space of $\mathrm{H}^1(\Omega_1^1\cup\Omega_1^N)$ functions that vanish on the boundary $\Gamma_1^+\cup\Gamma_1^-$ and q and s are as in Theorem 2.2 and Lemma 3.1, respectively.

Due to the estimates in Lemma 3.1, we find subsequences (not relabeled) and limits $\varphi \in \varphi_0^{\mathrm{D}} + V$, $T \in W$, $Z' \in \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)^2$, $\xi \in \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)^2$, $\overline{\boldsymbol S} \in \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})^3$, $Y' \in \mathrm{L}^s(\Omega_1^{\mathrm{oled}})^2$, $\eta \in \mathrm{L}^s(\Omega_1^{\mathrm{oled}})$, and $\overline H_{\Omega_1^{\mathrm{oled}}} \in \mathcal{M}(\overline{\Omega_1^{\mathrm{oled}}})$ (the space of finite Radon measures on $\overline{\Omega_1^{\mathrm{oled}}}$) such that we have for the potential

$$\varphi_\varepsilon \rightharpoonup \varphi \quad \text{in } \mathrm{L}^{p_-}(\Omega_1^{\mathrm{oled}}), \tag{3.16a}$$
$$\partial_{x_3}\varphi_\varepsilon \rightharpoonup \partial_{x_3}\varphi \quad \text{in } \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}}), \tag{3.16b}$$
$$m_\varepsilon\nabla'\varphi_\varepsilon \rightharpoonup Z' \quad \text{in } \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)^2, \tag{3.16c}$$
$$\frac{\partial_{x_3}\varphi_\varepsilon}{m_\varepsilon} \rightharpoonup \xi \quad \text{in } \mathrm{L}^{p(\cdot)}(\Omega_1^1\cup\Omega_1^N), \tag{3.16d}$$
$$\nabla'\varphi_\varepsilon \rightharpoonup \nabla'\varphi \quad \text{in } \mathrm{L}^{p(\cdot)}(\Omega_1^1\cup\Omega_1^N)^2, \tag{3.16e}$$

and for the temperature

$$T_\varepsilon \rightharpoonup T \quad \text{in } \mathrm{W}^{1,q}(\Omega^{\mathrm{sub}}), \tag{3.16f}$$
$$m_\varepsilon\nabla' T_\varepsilon \rightharpoonup Y' \quad \text{in } \mathrm{L}^{\widehat s}(\Omega_1^{\mathrm{oled}}), \tag{3.16g}$$
$$\frac{\partial_{x_3}T_\varepsilon}{(m_\varepsilon)^\gamma} \rightharpoonup \eta \quad \text{in } \mathrm{L}^s(\Omega_1^{\mathrm{oled}}). \tag{3.16h}$$

For the electrical flux functions and the Joule heat term we have

$$\boldsymbol S(\cdot, T_\varepsilon, M_\varepsilon\nabla\varphi_\varepsilon) \rightharpoonup \overline{\boldsymbol S} \quad \text{in } \mathrm{L}^{p'(\cdot)}(\Omega_1^{\mathrm{oled}})^3, \tag{3.16i}$$
$$H_\varepsilon \rightharpoonup^* \overline H_{\Omega_1^{\mathrm{oled}}} \quad \text{in } \mathcal{M}(\overline{\Omega_1^{\mathrm{oled}}}). \tag{3.16j}$$

First, we identify the limits ξ, Y', and Z'.

Lemma 3.2 *(i) The limits in* (3.16c), (3.16d)*, and* (3.16g) *satisfy $\xi \equiv 0$ a.e. in $\Omega_1^1\cup\Omega_1^N$, and $Z' \equiv 0$ a.e. in $\cup_{i=2}^{N-1}\Omega_1^i$, and $Y' \equiv 0$ a.e. in Ω_1^{oled}.*

(ii) Let T denote the limit in (3.16f) *and $T^{\Gamma_0} \in \mathrm{L}^q(\Omega_1^{\mathrm{oled}})$, $1 < q < 3/2$, the extension of its trace on Γ_0 to Ω_1^{oled}. Then, $T_\varepsilon|_{\Omega_1^{\mathrm{oled}}}$ converges strongly in $\mathrm{L}^s(\Omega_1^{\mathrm{oled}})$ to T^{Γ_0}.*

(iii) The traces of T_ε satisfy $m_\varepsilon T_\varepsilon|_{\Gamma_1^{\rm lat}} \rightharpoonup 0$ in $\mathrm{L}^s(\Gamma_1^{\rm lat})$ and $T_\varepsilon|_{\omega\times\{N\}} \to T|_{\Gamma_0}$ strongly in $\mathrm{L}^s(\omega\times\{N\})$.

Proof: *ad (i):* Let $v \in \mathbb{W}_{\rm D}^{1,p(\cdot)}(\Omega_1^{\rm oled})$ be a test function for the current-flow equation such that $\operatorname{supp} v \subset \overline{\Omega_1^1 \cup \Omega_1^N}$. Then $v_\varepsilon = h_\varepsilon^1 v$ (resp. $v_\varepsilon = h_\varepsilon^N v$) is an admissible test function, too. Using v_ε in the current-flow equation and letting $\varepsilon \to 0$ leads to $\int_{\Omega_1^1} \xi \partial_{x_3} v \,\mathrm{d}x = 0$ (resp. $\int_{\Omega_1^n} \xi \partial_{x_3} v \,\mathrm{d}x = 0$). Thus, we have that the limit ξ does not depend on x_3, i.e. $\xi(x_1,x_2,x_3) = \widetilde{\xi}(x_1,x_2)$ with $\widetilde{\xi} \in \mathrm{L}^{p(\cdot)}(\omega)$. However, since the traces of v on $\omega \times \{0,1,N{-}1,N\}$ are not fixed, we infer that $\widetilde{\xi} \equiv 0$ a.e. in $\Omega_1^1 \cup \Omega_1^N$.

To show that also Z' and Y' vanish, we consider a test function $\psi \in \mathrm{C}_0^\infty(\Omega_1^{\rm oled})$. For each $i = 2,\dots,N{-}1$ (and $i = 1,\dots,N$ for Y'), we assume that $\operatorname{supp}\psi \subset \Omega_1^i$. Integrating $m_\varepsilon \nabla' \varphi_\varepsilon \psi$ (resp. $m_\varepsilon \nabla' T_\varepsilon'$) over Ω_1^i and integrating by parts gives the result since $m_\varepsilon \varphi_\varepsilon \to 0$ in $\mathrm{L}^{p_-}(\Omega_1^{\rm oled})$ and $m_\varepsilon T_\varepsilon \to 0$ in $\mathrm{L}^s(\Omega_1^{\rm oled})$ due to (3.16a) and the second part of the lemma.

ad (ii): The claim follows from the weak convergence $T_\varepsilon \rightharpoonup T$ in $\mathrm{W}^{1,q}(\Omega^{\rm sub})$ and the strong convergence $\partial_{x_3} T_\varepsilon \to 0$ in $\mathrm{L}^s(\Omega_1^{\rm oled})$. Indeed, we have for $T_\varepsilon(x',x_3) - T_\varepsilon|_{\Gamma_0}(x') = \int_0^{x_3} \partial_{x_3} T_\varepsilon(x',z)\,\mathrm{d}z$. Thus, the result follows after taking the s-th power and integration over $\Omega_1^{\rm oled}$.

ad (iii): The assertion follows from the previous results: For each layer Ω_1^i, $i = 1,\dots,N$ we consider the sequence $u_\varepsilon = h_\varepsilon^i T_\varepsilon \in \mathrm{W}^{1,q}(\Omega_1^i)$ which satisfies $u_\varepsilon \rightharpoonup 0$ in $\mathrm{W}^{1,s}(\Omega_1^i)$ such that also the traces $u_\varepsilon|_{\partial\Omega_1^i} \rightharpoonup 0$ in $\mathrm{L}^s(\partial\Omega_1^i)$.

Finally, we have that $\|T_\varepsilon|_{\omega\times\{N\}} - T_\varepsilon|_{\Gamma_0}\|_{\mathrm{L}^s(\omega)}^s \le \|\partial_{x_3} T_\varepsilon\|_{\mathrm{L}^s(\Omega_1^{\rm oled})}^s$. Thus, with (3.16h) the result follows. ■

We are now in position to pass to the limit in the current-flow and heat equation in (2.21) and (2.22). We choose test functions $v \in \mathbb{W}_{\rm D}^{1,p(\cdot)}(\Omega_1^{\rm oled})$ and $\theta \in \mathrm{W}^{1,\widehat{s}'}(\Omega_1)$ (where $\widehat{s}' = \widehat{s}/(\widehat{s}{-}1)$) such that

$$\partial_{x_3} v \equiv 0 \text{ a.e. in } \Omega_1^1 \cup \Omega_1^N \quad \text{and} \quad \partial_{x_3}\theta \equiv 0 \text{ a.e. in } \Omega_1^{\rm oled}.$$

Note that $\mathrm{W}^{1,\widehat{s}'}(\Omega_1) \subset \mathrm{C}(\overline{\Omega}_1)$ since $\widehat{s}' > 3$ for $1 \le \widehat{s} < 3/2$. Using the convergences in (3.16), we arrive at

$$\int_{\Omega_1^1 \cup \Omega_1^N} \overline{\boldsymbol{S}}' \cdot \nabla' v \,\mathrm{d}x + \int_{\cup_{i=2}^{N-1}\Omega_1^i} \overline{\boldsymbol{S}}_3 \partial_{x_3} v \,\mathrm{d}x = 0, \tag{3.17}$$

$$\int_{\Omega^{\rm sub}} \lambda(x) \nabla T \cdot \nabla\theta \,\mathrm{d}x + \int_{\Gamma^{\rm eff}} \kappa(x)(T{-}T_{\rm a})\theta \,\mathrm{da} = \langle \overline{H}_{\Omega_1^{\rm oled}}, \theta\rangle, \tag{3.18}$$

where $\overline{\boldsymbol{S}}'$ denotes the first two components and $\overline{\boldsymbol{S}}_3$ the last component of the vector-valued function $\overline{\boldsymbol{S}}$. Thus, it remains to identify the limits $\overline{\boldsymbol{S}}$, and $\overline{H}_{\Omega_1^{\rm oled}}$.

From the second part of Lemma 3.2, we infer that there exists a further non-relabeled subsequence such that

$$T_\varepsilon \to T^{\Gamma_0} \quad \text{a.e. in } \Omega_1^{\rm oled}. \tag{3.19}$$

Moreover, due to the linear relation in $\boldsymbol{S}^1$ and $\boldsymbol{S}^N$ (cf. (2.9)), the weak convergences in (3.16e) and (3.16d), and $\xi \equiv 0$ (see Lemma 3.2(i)), we immediately obtain

$$\overline{\boldsymbol{S}}(x) = \begin{pmatrix} \sigma_{\rm sh}^- \nabla'\varphi \\ 0 \end{pmatrix} \text{ on } \Omega_1^1 \quad \text{and} \quad \overline{\boldsymbol{S}}(x) = \begin{pmatrix} \sigma_{\rm sh}^+ \nabla'\varphi \\ 0 \end{pmatrix} \text{ on } \Omega_1^N. \tag{3.20}$$

It remains to identify the limits $\overline{\boldsymbol{S}}$ and $\overline{H}_{\Omega_1^{\rm oled}}$.

Lemma 3.3 *The limits in* (3.16i) *and* (3.16j) *satisfy*

$$\overline{\boldsymbol{S}}_3 = \boldsymbol{S}_3(\cdot, T, (0, \partial_{x_3}\varphi)^\top) \text{ a.e. in } \bigcup_{i=2}^{N-1} \Omega_1^i, \qquad \overline{H}_{\Omega_1^{\rm oled}} = \overline{h}\mathcal{L}^3|_{\Omega_1^{\rm oled}}, \tag{3.21}$$

where $\mathcal{L}^3|_{\Omega_1^{\mathrm{oled}}}$ denotes the Lebesgue measure in $\mathbb{R}^3$ restricted to Ω_1^{oled} and the density $\overline{h} \in \mathrm{L}^1(\Omega_1^{\mathrm{oled}})$ is given via

$$\overline{h}(x) = \begin{cases} \sigma_{\mathrm{sh}}^-|\nabla'\varphi(x)|^2 & \text{in } \Omega_1^1, \\ \boldsymbol{S}_3^i\big(T(x),(0,\partial_{x_3}\varphi(x))^\top\big)\partial_{x_3}\varphi(x) & \text{in } \Omega_1^i \text{ for } i=2,\dots,N{-}1, \\ \sigma_{\mathrm{sh}}^+|\nabla'\varphi(x)|^2 & \text{in } \Omega_1^N. \end{cases} \tag{3.22}$$

Proof: *Step 1.* First, we establish, that the limit $\overline{\boldsymbol{S}}^i$ satisfies $\overline{\boldsymbol{S}}_3^i = \boldsymbol{S}_3^i(T,(0,\partial_{x_3}\varphi)^\top)$. Using the test function $v = \varphi - \varphi_0^{\mathrm{D}}$ in (3.17) and (3.20) leads to the identity

$$\begin{aligned} &\int_{\Omega_1^1} \sigma_{\mathsf{sh}}^-|\nabla'\varphi|^2\,\mathrm{d}x + \int_{\Omega_1^N} \sigma_{\mathsf{sh}}^+|\nabla'\varphi|^2\,\mathrm{d}x + \sum_{i=2}^{N-1}\int_{\Omega_1^i} \overline{\boldsymbol{S}}_3^i\cdot\partial_{x_3}\varphi\,\mathrm{d}x \\ &\qquad = \int_{\Omega_1^1} \sigma_{\mathsf{sh}}^-\nabla'\varphi\cdot\nabla'\varphi_0^{\mathrm{D}}\,\mathrm{d}x + \int_{\Omega_1^N} \sigma_{\mathsf{sh}}^+\nabla'\varphi\cdot\nabla'\varphi_0^{\mathrm{D}}\,\mathrm{d}x + \sum_{i=2}^{N-1}\int_{\Omega_1^i} \overline{\boldsymbol{S}}_3^i\varphi_0^{\mathrm{D}}\,\mathrm{d}x. \end{aligned} \tag{3.23}$$

For notational simplicity, we introduce the vector-valued functions $Z_\varepsilon, Z_0 \in \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})^3$ defined by

$$Z_\varepsilon := M_\varepsilon\nabla\varphi_\varepsilon \quad \text{and} \quad Z_0 := \begin{cases} (\nabla'\varphi,0)^\top & \text{in } \Omega_1^1, \\ (0,\partial_{x_3}\varphi)^\top & \text{in } \Omega_1^i \text{ for } i=2,\dots,N{-}1, \\ (\nabla'\varphi,0)^\top & \text{in } \Omega_1^N. \end{cases}$$

With this and using the test function $v_\varepsilon = \varphi_\varepsilon - \varphi_\varepsilon^{\mathrm{D}}$, as in the proof of Lemma 3.1, we arrive after passing to the limit $\varepsilon \to 0$ and using (3.23) at the identity

$$\begin{aligned} &\lim_{\varepsilon\to0}\left\{\int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-|Z_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+|Z_\varepsilon|^2\,\mathrm{d}x + \sum_{i=2}^{N-1}\int_{\Omega_1^i}\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)\cdot Z_\varepsilon\,\mathrm{d}x\right\} \\ &= \lim_{\varepsilon\to0}\left\{\int_{\Omega_1^1}\sigma_{\mathsf{sh}}^- Z_\varepsilon\cdot\begin{pmatrix}\nabla'\varphi_\varepsilon^{\mathrm{D}}\\0\end{pmatrix}\mathrm{d}x + \int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+ Z_\varepsilon\cdot\begin{pmatrix}\nabla'\varphi_\varepsilon^{\mathrm{D}}\\0\end{pmatrix}\mathrm{d}x\right. \\ &\left. + \sum_{i=2}^{N-1}\int_{\Omega_1^i}\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)\cdot M_\varepsilon\nabla\varphi_\varepsilon^{\mathrm{D}}\,\mathrm{d}x\right\} \\ &= \int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-\nabla'\varphi\cdot\nabla'\varphi_0^{\mathrm{D}}\,\mathrm{d}x + \int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+\nabla'\varphi\cdot\nabla'\varphi_0^{\mathrm{D}}\,\mathrm{d}x + \sum_{i=2}^{N-1}\int_{\Omega_1^i}\overline{\boldsymbol{S}}^i\cdot\begin{pmatrix}0\\\partial_{x_3}\varphi_0^{\mathrm{D}}\end{pmatrix}\mathrm{d}x \\ &= \int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-|\nabla'\varphi|^2\,\mathrm{d}x + \int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+|\nabla'\varphi|^2\,\mathrm{d}x + \sum_{i=2}^{N-1}\int_{\Omega_1^i}\overline{\boldsymbol{S}}_3^i\partial_{x_3}\varphi\,\mathrm{d}x. \end{aligned} \tag{3.24}$$

For arbitrary $Z' \in \mathrm{L}^2(\Omega_1^1\cup\Omega_1^N)^2$ and $z \in \mathrm{L}^{p(\cdot)}(\cup_{i=2}^{N-1}\Omega_1^i)$ we define the vector-valued function $Z \in \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})^3$ via

$$Z(x) = \begin{cases} (Z'(x),0)^\top & \text{for } x \in \Omega_1^1\cup\Omega_1^N, \\ (0,z(x))^\top & \text{for } x \in \Omega_1^i \text{ and } i=2,\dots,N{-}1. \end{cases}$$

Note that the almost everywhere convergence of T_ε in Ω_1^{oled} (see (3.19)) also implies that $\boldsymbol{S}^i(T_\varepsilon,Z)$ converges almost everywhere in Ω_1^i due to the continuity of $(T,z) \mapsto \boldsymbol{S}^i(T,Z)$. Hence, the growth condition for $\boldsymbol{S}^i$ in (2.10) leads with Lebesgue's dominated convergence theorem to the strong convergence of $\boldsymbol{S}^i(T_\varepsilon,Z)$ to

$\boldsymbol{S}^i(T,Z)$ in $\mathrm{L}^{p_i'}(\Omega_1^i)^3$. Thus, with (3.24), the strong convergence of $\boldsymbol{S}^i(T_\varepsilon,Z)$, and the weak convergence of Z_ε to Z_0 and $\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)$ to $\overline{\boldsymbol{S}}^i$ we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\bigg\{&\int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-|Z_\varepsilon-Z|^2\,\mathrm{d}x+\int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+|Z_\varepsilon-Z|^2\,\mathrm{d}x\\&\qquad+\sum_{i=2}^{N-1}\int_{\Omega_1^i}\big(\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)-\boldsymbol{S}^i(T_\varepsilon,Z)\big)\cdot(Z_\varepsilon-Z)\,\mathrm{d}x\bigg\}\\=\int_{\Omega_1^1}&\sigma_{\mathsf{sh}}^-|\nabla'\varphi-Z'|^2\,\mathrm{d}x+\int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+|\nabla'\varphi-Z'|^2\,\mathrm{d}x+\sum_{i=2}^{N-1}\int_{\Omega_1^i}\big(\overline{\boldsymbol{S}}_3^i-\boldsymbol{S}_3^i(T,Z)\big)(\partial_{x_3}\varphi-z)\,\mathrm{d}x.\end{aligned}\tag{3.25}$$

The strict monotonicity of $\boldsymbol{S}^i$ gives $\big(\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)-\boldsymbol{S}^i(T_\varepsilon,Z)\big)\cdot(Z_\varepsilon-Z)>0$. Integrating this inequality over Ω_1^i and summing over $i=2,\dots,N-1$ gives after passing to the limit

$$0\le\int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-|\nabla'\varphi-Z'|^2\,\mathrm{d}x+\int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+|\nabla'\varphi-Z'|^2\,\mathrm{d}x+\sum_{i=2}^{N-1}\int_{\Omega_1^i}\big(\overline{\boldsymbol{S}}_3^i-\boldsymbol{S}_3^i(T,Z)\big)(\partial_{x_3}\varphi-z)\,\mathrm{d}x.$$

Setting $Z'=\nabla'\varphi$ and $z=\partial_{x_3}\varphi\pm\delta w$ with an arbitrary $w\in\mathrm{L}^{p_i}(\Omega_1^{\mathrm{oled}})$ satisfying supp $w\subset\Omega_1^i$ and $\delta>0$ we get after dividing by δ and letting $\delta\to 0$ the identity $\overline{\boldsymbol{S}}_3^i=\boldsymbol{S}_3^i(T,(0,\partial_{x_3}\varphi)^\top)$, where we also used the continuity of $z\mapsto\boldsymbol{S}^i(T,z)$.

Step 2. It remains to show that the limit $\overline{H}_{\Omega_1^{\mathrm{oled}}}$ has a density $\overline{h}\in\mathrm{L}^1(\Omega_1^{\mathrm{oled}})$ with respect to the Lebesgue measure on Ω_1^{oled} which is given by (3.22).

Indeed, using that $\overline{\boldsymbol{S}}_3^i=\boldsymbol{S}_3^i(T,(0,\partial_{x_3}\varphi)^\top)$, the monotonicity of $\boldsymbol{S}$ and choosing $Z=Z_0$ in (3.25) we obtain the strong convergence $|Z_\varepsilon-Z_0|^2\to 0$ in $\mathrm{L}^1(\Omega_1^1\cup\Omega_1^N)$ which gives the Joule heat contribution in the electrodes. Moreover, we have $(\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)-\boldsymbol{S}^i(T_\varepsilon,Z_0))\cdot(Z_\varepsilon-Z_0)\to 0$ in $\mathrm{L}^1(\Omega_1^i)$, for $i=2,\dots,N-1$. Since, $\boldsymbol{S}^i(T_\varepsilon,Z_0)$ converges strongly in $\mathrm{L}^{p_i'}(\Omega_1^i)$ and Z_ε converges weakly in $\mathrm{L}^{p_i}(\Omega_1^i)$ we infer that $\boldsymbol{S}^i(T_\varepsilon,Z_0)\cdot(Z_\varepsilon-Z_0)$ converges weakly to 0 in $\mathrm{L}^1(\Omega_1^i)$. This, however, implies that also $\boldsymbol{S}^i(T_\varepsilon,Z_\varepsilon)\cdot Z_\varepsilon$ converges weakly to $\boldsymbol{S}^i(T,Z_0)\cdot Z_0$ in $\mathrm{L}^1(\Omega_1^i)$. This finishes the proof. ∎

4 The effective model

In this section we identify the effective limit system. In the last section, we showed that the limits in (3.16) satisfy the following system of equations for test functions $v\in\mathbb{W}^{1,p(\cdot)}(\Omega_1^{\mathrm{oled}})$ and $\theta\in\mathrm{W}^{1,q}(\Omega_1)\cap\mathrm{L}^\infty(\Omega_1)$ with $\partial_{x_3}v=0$ a.e. in $\Omega_1^1\cup\Omega_1^N$ and $\partial_{x_3}\theta=0$ in Ω_1^{oled}

$$\begin{aligned}&\int_{\Omega_1^1}\sigma_{\mathsf{sh}}^-\nabla'\varphi\cdot\nabla'v\,\mathrm{d}x+\int_{\Omega_1^N}\sigma_{\mathsf{sh}}^+\nabla'\varphi\cdot\nabla'v\,\mathrm{d}x+\int_{\cup_{i=2}^{N-1}\Omega_1^i}\boldsymbol{S}_3(T,(0,\partial_{x_3}\varphi)^\top)\partial_{x_3}v\,\mathrm{d}x=0,\\&\int_{\Omega^{\mathrm{sub}}}\lambda(x)\nabla T\cdot\nabla\theta\,\mathrm{d}x+\int_{\Gamma^{\mathsf{eff}}}\kappa(x)(T-T_{\mathrm{a}})\theta\,\mathrm{d}a=\int_{\Omega_1^1}\sigma_{\mathrm{sh}}^-|\nabla'\varphi|^2\,\mathrm{d}x+\int_{\Omega_1^N}\sigma_{\mathrm{sh}}^+|\nabla'\varphi|^2\,\mathrm{d}x\\&\qquad\qquad+\int_{\cup_{i=2}^{N-1}\Omega_1^i}\boldsymbol{S}_3\big(T,(0,\partial_{x_3}\varphi)^\top\big)\partial_{x_3}\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Due to (3.16d) we infer that the limit φ satisfies $\partial_{x_3}\varphi=0$ a.e. in $\Omega_1^1\cup\Omega_1^N$. Thus we can identify $\varphi|_{\Omega_1^1}$ and $\varphi|_{\Omega_1^N}$ with functions $\varphi^1\in\mathrm{H}^1(\Gamma_0)$ and $\varphi^{N-1}\in\mathrm{H}^1(\Gamma_0)$ on the boundary $\Gamma_0=\omega\times\{0\}$ such that

$$\varphi(x_1,x_2,x_3)=\begin{cases}\varphi^1(x_1,x_2,0) & \text{for } (x_1,x_2,x_3)\in\Omega_1^1,\\ \varphi^{N-1}(x_1,x_2,0) & \text{for } (x_1,x_2,x_3)\in\Omega_1^N.\end{cases}\tag{4.1}$$

Moreover, for $i = 2,\dots,N-2$ let us denote the trace of φ on $\overline{\Omega_1^i} \cap \overline{\Omega_1^{i+1}}$ by φ^i, which is well defined due to $\partial_{x_3}\varphi \in \mathrm{L}^{p(\cdot)}(\Omega_1^{\mathrm{oled}})$ and is identified with a function in $\mathrm{L}^{p_-}(\Gamma_0)$. In particular, we identify φ with a tuple $(\varphi^1,\dots,\varphi^{N-1}) \in (\varphi_-^{\mathrm{D}},0,\dots,0,\varphi_+^{\mathrm{D}}) + V_0$, where the space of interface potentials V_0 is defined in (2.16). We proceed analogously with the test function v, i.e. we identify it with a tuple $(v^1,\dots,v^{N-1}) \in V_0$.

Next, we highlight, that φ is a weak solution in Ω_1^i, $i = 2,\dots,N-1$, of the (ordinary differential) equation $\partial_{x_3}\boldsymbol{S}_3^i(T,(0,\partial_{x_3}\varphi)^\top) = 0$ subject to the boundary conditions $\varphi(\cdot,i{-}1) = \varphi^{i-1}$ and $\varphi(x_1,x_2,i) = \varphi^i(x_1,x_2)$ with φ^i denoting the interface potentials. However, due to the strict monotonicity of $\boldsymbol{S}^i$ the unique solution of the ODE (for a fixed temperature T) is given by the affine function

$$\varphi(\cdot,x_3) = (\varphi^i - \varphi^{i-1})(x_3 - i + 1) + \varphi^{i-1} \qquad \text{such that} \qquad \partial_{x_3}\varphi = \varphi^i - \varphi^{i-1}. \tag{4.2}$$

Finally, we remark that due to (3.16h) T is given in Ω_1^{oled} by the trace of $T|_{\Omega^{\mathrm{sub}}}$ on Γ_0.

Using the above identifications in the limit PDE system leads to the system

$$\begin{aligned}
&\int_{\Gamma_0} \Big\{\sigma_{\mathrm{sh}}^-\nabla'\varphi^1\cdot\nabla' v^1 + \sigma_{\mathrm{sh}}^+\nabla'\varphi^{N-1}\cdot\nabla' v^{N-1} + \sum_{i=2}^{N-1} F^i\big(T,\varphi^i{-}\varphi^{i-1}\big)(v^i{-}v^{i-1})\Big\}\,\mathrm{da} = 0\\
&\int_{\Omega^{\mathrm{sub}}} \lambda(x)\nabla T\cdot\nabla\theta\,\mathrm{d}x + \int_{\partial\Omega^{\mathrm{sub}}} \kappa(x)(T{-}T_{\mathrm{a}})\theta\,\mathrm{da} = \int_{\Gamma_0} \Big\{\sigma_{\mathrm{sh}}^-|\nabla'\varphi^1|^2 + \sigma_{\mathrm{sh}}^+|\nabla'\varphi^{N-1}|^2\\
&\qquad\qquad + \sum_{i=2}^{N-1} F^i(T,\varphi^i{-}\varphi^{i-1})(\varphi^i{-}\varphi^{i-1})\Big\}\theta\,\mathrm{da},
\end{aligned}$$

where we have set $F^i(T,z) = \boldsymbol{S}_3^i(T,(0,z)^\top)$. The test functions satisfy $(v^1,\dots,v^{N-1}) \in V_0$ and $\theta \in \mathrm{W}^{1,\widehat{s}'}(\Omega^{\mathrm{sub}})$. However, the weak formulation above is still well-defined for $\theta \in \mathrm{W}^{1,q'}(\Omega^{\mathrm{sub}})$. Hence, we have proved Theorem 2.4.

5 Conclusion

We rigorously derived an effective system of equations from the thermistor model introduced in [LK*15] (see also [LF*17]) governing the heat and current flow through a large-area, thin film OLED device mounted on a glass substrate. The effective equations were derived by obtaining certain a priori bounds for the lateral and vertical components of the electrostatic potential and the gradient of the temperature, respectively.

Furthermore, in the vanishing thickness limit of the different layers, the Joule heat term that was present in the domain, has manifested itself as a boundary source term for the heat equation in the substrate on the part of the boundary where the OLED is mounted.

As a concluding remark we point out that the novelty of the new effective constitutive law for the current-flow, which is of reduced dimension, is that lends itself to easier implementation of numerical simulations that could provide greater insight concerning the behavior of the aforementioned devices. In particular, the case of geometrically curved device structures, e.g. used in car rear lights, is not feasible in the full three-dimensional setting due to the large anisotropy of the meshes used in the numerical approximation.

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