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Further regularity and uniqueness results for a non-isothermal Cahn–Hilliard equation

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Abstract

The aim of this paper is to establish new regularity results for a non-isothermal Cahn–Hilliard system in the two-dimensional setting. The main achievement is a crucial L^∞ estimate for the temperature, obtained by a suitable Moser iteration scheme. Our results in particular allow us to get a new simplified version of the uniqueness proof for the considered model.

1 Introduction

The aim of this paper is to establish new regularity properties for a mathematical model for non-isothermal Cahn–Hilliard equation in a bounded container $\Omega \subseteq \mathbb{R}^2$. For this model the existence of the solution is already known as well as some partial regularity results. Also uniqueness of the solution has been proved, see [10]. In this paper we gain additional regularity for the solution of the considered model, which in particular allows us to show also a simplest uniqueness proof.
The model we consider consists of a PDE system describing the evolution of the unknown variables φ (order parameter), μ (chemical potential) and θ (absolute temperature), which takes the form

$$\varphi_t = \Delta\mu, \tag{1.1}$$
$$\mu = -\Delta\varphi + F'(\varphi) - \theta, \tag{1.2}$$
$$\theta_t + \theta\varphi_t - \mathrm{div}\,(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2, \tag{1.3}$$

and it corresponds to the Cahn–Hilliard system for phase separation [6] (see also [23]) coupled with the internal energy equation describing the evolution of temperature. More details about the derivation of this model, which is obtained according to the general approach proposed in [14], will be provided in [19]. This is a non-linear system whose main source of difficulty is directly related to the termodynamic consistency of the model. Namely, it is represented by the quadratic term in the right-hand side of (1.3). The analysis will be carried out in the 2-dimensional torus $\Omega = [0,1] \times [0,1]$, therefore we choose periodic boundary conditions for all the unknowns. The function F, whose derivative appears in (1.2), is a possibly non-convex potential whose minima represent the least energy configuration of the phase variable. Here, we will assume that F is smooth and with power-like growth at ∞. Moreover the function $\kappa(\theta)$ in (1.3) denotes the heat conductivity coefficient, assumed to grow at ∞ as a power of θ, as it has been recently considered in several contributions, for instance [12].
Our model system is part of a more general model

$$\mathrm{div}\mathbf{u} = 0, \tag{1.4}$$
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \Delta\mathbf{u} - \mathrm{div}(\nabla\varphi\otimes\nabla\varphi), \tag{1.5}$$
$$\varphi_t + \mathbf{u}\cdot\nabla\varphi = \Delta\mu, \tag{1.6}$$
$$\mu = -\Delta\varphi + F'(\varphi) - \theta, \tag{1.7}$$
$$\theta_t + \mathbf{u}\cdot\nabla\theta + \theta\big(\varphi_t + \mathbf{u}\cdot\nabla\varphi\big) - \mathrm{div}(\kappa(\theta)\nabla\theta) = |\nabla\mathbf{u}|^2 + |\nabla\mu|^2 \tag{1.8}$$

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where the Cahn–Hilliard equation and the internal energy equation are coupled with a Navier-Stokes equation. This model has been derived and studied in [11] where the existence of solutions was shown in the 3D case (under some slightly different assumptions on the coefficients) in a very general and weak formulation. Then in [12] also the 2D case was analyzed, obtaining the existence of strong solutions. Eventually in [10] the authors were able to improve the previous results by defining a class of slightly smoother solutions and by proving that uniqueness holds in that class (and therefore well-posedness results have been proved). The authors were then able to characterize the long-time behaviour of solutions by showing that they constitute a strongly continuous dynamical process, which admits a global attractor, where the spatial mean of the initial velocity is zero.
To prove these well-posedness results, in particular a key point is the following: the right hand side of (1.8) lies exactly in $L^2(0,T;L^2(\Omega))$ and this information apparently doesn't seem to be sufficient to get additional regularity for θ, which is essential in order to be able, for instance, to test the equation for the temperature by θ_t. In particular a L^∞-bound is lacking because Moser iterations do not work for L^2 on the right hand side and this would be crucial in order to manage some coefficients growing like powers of θ. Therefore much efforts have been adopted in the already mentioned papers [12], [10] to overcome this difficulty and be able anyway to get a control of the gradient of θ in $L^2(\Omega)$, uniformly in time.
The aim of this paper is to show that, assuming a null velocity vector field, the Moser iteration scheme works, so that the crucial L^∞-estimate for θ is now available. As a consequence the proof of a uniqueness result for our non-isothermal Cahn–Hilliard model can be simplified with respect to scheme proposed in [10]. The outline of the paper is the following: Section 2 provides the setting of our problem. The core of this paper is presented in Section 3, where the main result, Theorem 3.1, namely existence and uniqueness of a solution for our problem (1.1)-(1.3), together with the additional regularity for θ, is proved.

2 Assumptions

We suppose that a two-component fluid occupies a bounded spatial domain $\Omega \subset \mathbb{R}^2$, with a sufficiently regular boundary Γ. We let $\mathbf{n}$ denote the outer normal unit vector to Γ. Moreover, $\varphi(x,t)$ is the *order parameter*, representing the concentration difference of the fluid, or the concentration of one component, and $\theta(x,t)$ is the *absolute temperature*.
In particular, we consider our PDE system taking place in the two-dimensional flat torus with periodic boundary conditions, namely $\Omega = [0,1]\times[0,1]$ and $\varphi|_{x_i=0} = \varphi|_{x_i=1}$ $i=1,2$ $\forall t \in (0,T)$.
Let us now introduce some notation we will use in the sequel. The symbol $\|\cdot\|_X$ will denote the norm in a generic Banach space. We denote as $H := L^2_{\text{per}}(\Omega)$ the space of functions in $L^2(\mathbb{R}^2)$ which are Ω-periodic (i.e., 1-periodic both in x_1 and x_2). Analogously, we set $V := H^1_{\text{per}}(\Omega)$. The spaces H and V are endowed with the norms of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. For brevity, the norm of H will be simply indicated by $\|\cdot\|$. Still for brevity, we omit the variables of integration. We will specify them when there could be a misinterpretation. The symbol $\langle\cdot,\cdot\rangle$ will indicate the duality between V' and V and $(\cdot,\cdot)$ will stand for the standard product of H. We also write $L^p(\Omega)$ instead of $L^p_{\text{per}}(\Omega)$, and the same for other spaces; indeed, no confusion should arise since periodic boundary conditions are assumed to hold for all unknowns. We denote $H^m_{\text{per}}(\Omega)$ the space of functions which are $H^m_{\text{loc}}(\Omega)$ and Ω-periodic, for $m \in \mathbb{R}, m \geq 0$. In particular, for $m=0$ we have $H^0_{\text{per}}(\Omega) = L^2_{\text{per}}(\Omega)$.

For any function $v \in H$, we set

$$v_\Omega = \frac{1}{|\Omega|}\int_\Omega v = \int_\Omega v,$$

to indicate the spatial mean of v, being $|\Omega| = 1$. If the integral is replaced with the duality, the above can be extended to $v \in V'$. We denote as H_0, V_0 and V_0' the closed subspaces of functions (or functionals) having zero mean value in H, V, and, respectively, in V'. Then

$$\|v\|_{V_0} := \left(\int_\Omega |\nabla v|^2 \mathrm{dx} \right)^{1/2},$$

represents a norm on V_0, which is equivalent to the norm inherited from V by the subsequent Poincaré-Wirtinger inequality (2.3f). In particular $\|\cdot\|_{V_0}$ is a Hilbert norm and we can introduce the associated Riesz isomorphism mapping $J : V_0 \to V_0'$ by setting, for $u, v \in V_0$,

$$\langle Ju, v \rangle := ((u, v))_{V_0} := \int_\Omega \nabla u \cdot \nabla v \quad \mathrm{dx}. \tag{2.1}$$

For $f \in H_0$ it is easy to check that $u = J^{-1} f \ \in H^2(\Omega)$. Actually, u is the (unique) solution to the elliptic problem

$$u \in H_0, \quad -\Delta u = f, \quad \nabla u \cdot \mathbf{n}_{|\Gamma} = 0.$$

Moreover, if u is as above, then

$$\langle J(u - u_\Omega), v \rangle = - \int_\Omega v \Delta u \mathrm{dx}$$

for all $v \in V_0$. Finally, we can identify H_0 with H_0' by means of the scalar product on H obtaining the Hilbert triplet $V_0 \subset H_0 \subset V_0'$, where inclusions are continuous and dense. In particular, if $z \in V$ and $v \in V_0$, it can be easily seen that

$$\int_\Omega \nabla z \cdot \nabla (J^{-1} v) \mathrm{dx} = \int_\Omega (z - z_\Omega) v \mathrm{dx} = \int_\Omega z v \mathrm{dx}. \tag{2.2}$$

In the sequel we will also frequently use the following 2D interpolation inequalities:

$$\|v\|_{L^4(\Omega)} \le c \|v\|_V^{1/2} \|v\|^{1/2}, \tag{2.3a}$$

$$\|v\|_{L^\infty(\Omega)} \le c \|v\|_{H^2(\Omega)}^{1/2} \|v\|^{1/2}, \tag{2.3b}$$

$$\|v\|_{L^r(\Omega)} \le c \|v\|^{\frac{2}{r}} \|v\|_V^{1-\frac{2}{r}}, \qquad r \in [2, \infty), \tag{2.3c}$$

$$\|v\|_{L^r(\Omega)} \le c \|v\|_{L^s(\Omega)}^{1-\alpha} \|v\|_{L^\infty(\Omega}^{\alpha}, \qquad \alpha = 1 - \frac{s}{r}, \qquad r \ge 1 \tag{2.3d}$$

$$\|v\|_{H^s(\Omega)} \le c \|v\|_{H^{s_1}(\Omega)}^{1-\theta} \|v\|_{H^{s_2}(\Omega)}^{\theta}, \quad \theta = \frac{s - s_1}{s_2 - s_1}, \tag{2.3e}$$

$$\|v - v_\Omega\| \le c \|\nabla v\|, \tag{2.3f}$$

holding for any sufficiently smooth function v and for suitable embedding constants, all denoted by the same symbol $c > 0$ for brevity. We will also use the following non linear Poincaré inequality (see [15])

$$\left\|v^{p/2}\right\|_V^2 \le c_p \left(\|v\|_{L^1(\Omega)}^p + \left\|\nabla v^{p/2}\right\|^2 \right), \tag{2.4}$$

holding for all non-negative $v \in L^1(\Omega)$ such that $\nabla v^{p/2} \in L^2(\Omega)$, and for all $p \in [2, \infty)$. We also recall that

$$\|v\| \le c \|\nabla v\|^{1/2} \|v\|_{V'}^{1/2}, \quad \forall v \in V_0, \tag{2.5}$$

as one can prove simply combining the standard interpolation inequality $\|v\| \leq c\,\|v\|_V^{1/2}\,\|v\|_{V'}^{1/2}$ with the Poincaré-Wirtinger inequality (2.3f).

Now we are ready to present our main assumptions on the nonlinear terms. We ask the configuration potential F to satisfy:

$$F \in C^3(\mathbb{R};\mathbb{R}), \quad \liminf_{|r|\to\infty} \frac{F(r)}{|r|} > 0, \tag{2.6}$$

$$F''(r) \geq -\lambda \text{ for some } \lambda \geq 0, \text{ and all } r \in \mathbb{R}, \tag{2.7}$$

$$|F'''(r)| \leq \tilde{c}_F(1+|r|^{p_F-1}) \text{ for some } \tilde{c_F} \geq 0, p_F \geq 1, \text{ and all } r \in \mathbb{R}. \tag{2.8}$$

We remark that (2.8) implies

$$|F''(r)| \leq c_F(1+|r|^{p_F}) \text{ for some } c_F \geq 0, p_F \geq 0, \text{ and all } r \in \mathbb{R}.$$

Assumption (2.6) postulates regularity and coercivity of F, (2.7) is λ-convexity and (2.8) prescribes a polynomial growth at infinity. Note that (2.6) implies that

$$F(s) \geq -c_0 \quad \forall s \in \mathbb{R}$$

for some constant $c_0 > 0$. We assume moreover the heat conductivity to be given by

$$\kappa(r) = 1 + r^q, \quad q \in [2,\infty), \quad r \geq 0. \tag{2.9}$$

Correspondingly, we define

$$K(r) := \int_0^r \kappa(s)ds = r + \frac{1}{1+q} r^{1+q}, \quad r \geq 0. \tag{2.10}$$

We then observe that, for some $k_q > 0$,

$$\int_\Omega \kappa(\theta)^2 |\nabla\theta|^2 \,\mathrm{dx} = \|\nabla K(\theta)\|^2 \geq \|\nabla\theta\|^2 + k_q \left\|\nabla\theta^{q+1}\right\|^2. \tag{2.11}$$

Then, exploiting (2.4) with the choice $p = 2$ we obtain

$$\|K(\theta)\|_V^2 \leq C\left(1 + \|\theta\|_{L^1(\Omega)}^{2(q+1)} + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2\right), \tag{2.12}$$

for some C depending on q.
The initial data are chosen according to (3.5) and (3.10), therefore

$$\varphi(t)_\Omega = \varphi(0)_\Omega \qquad \mathcal{E}(\varphi(t),\theta(t)) = \mathcal{E}(\varphi(0),\theta(0)).$$

It comes natural to define the "energy-entropy space" of data as:

$$\mathcal{H} = \left\{ z = (\varphi,\theta) \in V \times L^1(\Omega) : \theta > 0 \text{ a.e. in } \Omega,\, \log\theta \in L^1(\Omega) \right\}.$$

In our space we omitted the chemical potential μ, in view of the fact that μ can be regarded as an auxiliary variable, and sometimes, depending on the situation, it will be more convenient to "exclude"μ. This can be easily achieved rewriting the system (1.1)-(1.2) as a single equation where μ no longer appears.

Now, in agreement with [12] and [10], we define the set

$$\mathcal{V} := \left\{ z = (\varphi, \theta) \in \mathcal{H} \cap \left(H^3(\Omega) \times V \right) : K(\theta) \in V, 1/\theta \in L^1(\Omega) \right\}.$$

Because θ is the absolute temperature the condition $1/\theta \in L^1(\Omega)$ implies $\theta > 0$ (actually a strict separation from 0). Moreover, the requirement $K(\theta) \in V$ yields $\theta \in V$; however, in the definition, we stated both to have a clearer view of that space.

Eventually, we recall here a result which will be useful in order to reach regularity in Section 3.2. The proof of this Lemma can be found in [12].

Lemma 2.1. *Let $\mathcal{O}$ a smooth bounded domain in $\mathbb{R}^2$. Then, there exists $c > 0$ depending only on $\mathcal{O}$ such that*

$$\|\xi\|_{H^1(\mathcal{O})'} \le c\Big(1 + \|\xi\|_{L^1(\mathcal{O})} \log^{1/2} \left(e + \|\xi\|_{L^2(\mathcal{O})}\right)\Big) \tag{2.13}$$

for any $\xi \in L^2(\mathcal{O})$.

3 Well posedness results

3.1 Main result

We are now ready to present the main result of this work, namely

Theorem 3.1. *Let us assume* (2.6)-(2.8) *and* (2.9). *Let also $T > 0$. Then given $z_0 \in \mathcal{V}$ there exists a unique solution to our problem, namely a triple (φ, μ, θ) with the regularity*

$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;H^3(\Omega)), \tag{3.1}$$

$$\mu \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{3.2}$$

$$\theta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;V), \tag{3.3}$$

$\theta > 0$ *a.e. in* $(0,T) \times \Omega$,

$$K(\theta) \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;V), \tag{3.4}$$

satisfying equations (1.1)-(1.3) *a.e. in $(0,T) \times \Omega$ and complying with the initial conditions*

$$\varphi|_{t=0} = \varphi_0, \qquad \theta|_{t=0} = \theta_0$$

almost everywhere in Ω.

An existence result for our model problem can be proved by means of the solution of an approximating problem and then on the application of Schauder's fixed point theorem. The complete proof can be found in [12]. Here we focus on the regularity results. We distinguish two sections: in the first we recover the basic regularity already proposed in study of the general model in [12], while in the second one we achieve further regularity also with respect to [10].

3.2 Initial regularity

In this section we recover the basic regularity already proposed in study of the general model in [12]; for the reader's convenience we sketch the main points. We can observe that, considering here null velocity, as one may expect, also this part of the proof turns to be simplified. In particular this is evident in two points: in the complementary estimates (3.18) and (3.20), that can be easily derived one from the other (due to the absence of convective terms) and in the key estimate of the term (θ_t, φ_t). This last achievement is obtained by means of the control of two terms: the estimate of $\int \kappa^2(\theta)|\nabla\theta|^2$ follows exactly as in [12] while the estimate of the term $\int |\nabla\mu|^2\varphi_t$ can be heavily simplified, even if the idea of relying on conjugate functions still is necessary, for the presence of the quadratic term in the right hand side of (1.3).

3.2.1 Energy and entropy estimates

The energy estimate is obtained by testing (1.1) by μ, (1.2) by $-\varphi_t$, (1.3) by 1 and then integrating over Ω. Summing up we get

$$\frac{d}{dt}\mathcal{E}(\varphi,\theta) = 0, \qquad \text{where } \mathcal{E}(\varphi,\theta) := \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi) + \theta\right) \tag{3.5}$$

which is the *total energy* of the system, given by the sum of the *interfacial*, *configuration*, and *thermal* energies (the three terms in $\mathcal{E}$).
From relation (3.5) we infer the following *a priori* estimates

$$\|\varphi\|_{L^\infty(0,T;V)} \le c, \tag{3.6}$$
$$\|\theta\|_{L^\infty(0,T;L^1(\Omega))} \le c, \tag{3.7}$$

where we exploited (2.6) in order to obtain (3.6) and we used the nonnegativity of θ to get (3.7) from (3.5). Moreover, from (3.6) and Sobolev's embeddings, we also have

$$\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \le c_p \quad \text{for all } p \in [1,\infty). \tag{3.8}$$

On the other hand, integrating (1.1) over Ω, and using the periodic boundary conditions, we observe

$$\frac{d}{dt}\int_\Omega \varphi = 0 \quad \text{a.e. in } (0,T). \tag{3.9}$$

The entropy estimate corresponds to the entropy production principle. In order to obtain it, we test (1.3) by $-\theta^{-1}$ and integrate over Ω, therefore we infer

$$\frac{d}{dt}\int_\Omega(-\log\theta - \varphi) + \int_\Omega \frac{1}{\theta}|\nabla\mu|^2 + \int_\Omega \left(|\nabla\log\theta|^2 + k_q|\nabla\theta^{q/2}|^2\right) = 0, \tag{3.10}$$

with $k_q > 0$ only depending on the exponent q introduced in (2.9). Integrating in time and recalling (3.6)-(3.7), we get the a priori bounds

$$\|\log\theta\|_{L^\infty(0,T;L^1(\Omega))} + \|\log\theta\|_{L^2(0,T;V)} \le c, \tag{3.11}$$
$$\|\nabla\theta^{q/2}\|_{L^2(0,T;H)} \le c. \tag{3.12}$$

In particular, from (3.11) we see that the strict positivity of θ is preserved a.e. in $(0,T)\times\Omega$ also in the limit. Moreover, the combination of inequality (2.4) with estimates (3.7) and (3.12) gives

$$\|\theta^{q/2}\|_{L^2(0,T;V)} \le c,$$

which implies in particular

$$\|\theta\|_{L^2(0,T;H)} \leq c$$

and, for $q = 2$,

$$\|\theta\|_{L^2(0,T;V)} \leq c. \tag{3.13}$$

3.2.2 First estimates for μ, φ and φ_t

From equation (1.3) and periodic boundary conditions, we get

$$\int_\Omega |\nabla\mu|^2 = \frac{d}{dt}\int_\Omega \theta + \int_\Omega \theta\varphi_t. \tag{3.14}$$

Our aim is to control the terms on the right hand side.
In order to do so, we first integrate in time, and then we estimate the first one thanks to (3.7). On the other hand, by using (1.1) and Hölder's and Young's inequalities, the second integral can be controlled as follows

$$\int_\Omega \theta\varphi_t = \int_\Omega \theta\Delta\mu = -\int_\Omega \nabla\theta\cdot\nabla\mu \leq \frac{1}{2}\big(\|\nabla\mu\|^2 + \|\nabla\theta\|^2\big). \tag{3.15}$$

The first term on the right hand side is absorbed by the corresponding one on the left hand side of (3.14), while we use (3.13) to estimate the latter. Hence, we obtain

$$\|\nabla\mu\|_{L^2(0,T;H)} \leq c. \tag{3.16}$$

We now integrate (1.2) in space, combine (2.8), (3.7) and (3.8) and then take the (essential) supremum with respect to time; we infer

$$\|\mu_\Omega\|_{L^\infty(0,T)} \leq c. \tag{3.17}$$

This estimate, combined with (3.16), gives

$$\|\mu\|_{L^2(0,T;V)} \leq c. \tag{3.18}$$

Now, testing (1.1) by nonzero $v \in V$, we can notice that

$$\langle\varphi_t, v\rangle = -\int_\Omega \nabla\mu\cdot\nabla v \leq \|\nabla\mu\|\|\nabla v\| \leq c\|\nabla\mu\|\|v\|_V. \tag{3.19}$$

Hence, dividing by $\|v\|_V$, passing to the supremum with respect to $v \in V\setminus\{0\}$, squaring, integrating in time, and using (3.18), we infer

$$\|\varphi_t\|_{L^2(0,T;V')} \leq c. \tag{3.20}$$

On the other hand, testing equation (1.1) by $-\mu$ and noting that μ has zero spatial mean, the use of the Poincaré-Wirtinger inequality yields

$$\begin{aligned}\|\nabla\mu\|^2 &= -\int_\Omega \varphi_t\mu = -\int_\Omega \varphi_t(\mu-\mu_\Omega) \leq \|\mu-\mu_\Omega\|_V\|\varphi_t\|_{V'}\\ &\leq \frac{1}{2}\|\nabla\mu\|^2 + c\|\varphi_t\|^2_{V'},\end{aligned}$$

which allows us to get

$$\|\nabla\mu\|^2 \leq c\|\varphi_t\|^2_{V'}. \tag{3.21}$$

Finally, if we test (1.2) by $\Delta^2\varphi$ and integrate over Ω, by recalling (2.8), we get

$$\|\nabla\Delta\varphi\|^2 \leq c\big(1 + \|\nabla\theta\|^2 + \|\nabla\mu\|^2\big)$$

Integrating this inequality in time and using (3.13) and (3.16), we then obtain

$$\|\varphi\|_{L^2(0,T;H^3(\Omega))} \leq c. \tag{3.22}$$

3.2.3 Key estimate: control of the term (θ_t, φ_t)

First of all we take (1.1), differentiate it with respect to time, and test the result by $J\varphi_t$, where J was first introduced in (2.1). Correspondingly, we differentiate (1.2) in time and test the result by $-\varphi_t$. Summing the obtained relations, by (2.5) and (3.9), we then get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_{V'}^2 + \|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \\ = \lambda\|\varphi_t\|^2 + (\theta_t,\varphi_t) \le \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2 + (\theta_t,\varphi_t). \qquad (3.23)\end{aligned}$$

Reabsorbing, this is equivalent to

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_{V'}^2 + \frac{7}{8}\|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \le c\|\varphi_t\|_{V'}^2 + (\theta_t,\varphi_t). \qquad (3.24)$$

On the other hand, testing (1.3) by φ_t yields

$$\begin{aligned}(\theta_t,\varphi_t) + \int_\Omega \theta\varphi_t^2 &= -\int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\varphi_t + \int_\Omega |\nabla\mu|^2\varphi_t \qquad (3.25)\\ &\le \frac{1}{16}\|\nabla\varphi_t\|^2 + 4\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \int_\Omega |\nabla\mu|^2\varphi_t.\end{aligned}$$

To treat the term in $\nabla\theta$, we test (1.3) by $6K(\theta)$ introduced in (2.10) (the coefficient 6 is suitable for reabsorbing the second term in the right hand side of the previous inequality by the left hand side) and working exactly as in [12] we deduce the following estimate

$$\begin{aligned}6\frac{d}{dt}\int_\Omega \mathcal{J}(\theta) + 5\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\ \le c(1+\|\varphi_t\|_{V'}^4) + \frac{1}{8}\|\nabla\varphi_t\|^2 + 6\int_\Omega K(\theta)|\nabla\mu|^2. \qquad (3.26)\end{aligned}$$

where we set

$$\mathcal{J}(r) := \int_0^r K(s)\,\mathrm{d}s = \frac{r^2}{2} + \frac{1}{(q+1)(q+2)} r^{q+2}, \quad r \ge 0. \qquad (3.27)$$

Summing (3.25) and (3.26) we get

$$\begin{aligned}(\theta_t,\varphi_t) + \int_\Omega \theta\varphi_t^2 + 6\frac{d}{dt}\int_\Omega \mathcal{J}(\theta) + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\ \le c\big(1+\|\varphi_t\|_{V'}^4\big) + \frac{3}{16}\|\nabla\varphi_t\|^2 + \int_\Omega (6K(\theta)+\varphi_t)|\nabla\mu|^2. \qquad (3.28)\end{aligned}$$

Then, adding together (3.24) and (3.28) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_{V'}^2 + 6\frac{d}{dt}\int_\Omega \mathcal{J}(\theta) + \frac{11}{16}\|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \\ + \int_\Omega \theta\varphi_t^2 + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\ \le c\,\|\varphi_t\|_{V'}^2 + c\big(1+\|\varphi_t\|_{V'}^4\big) + \int_\Omega (6K(\theta)+\varphi_t)|\nabla\mu|^2. \qquad (3.29)\end{aligned}$$

Neglecting some positive terms in the left hand side and rearranging, we then arrive at

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_{V'}^2 + 6\frac{d}{dt}\int_\Omega \mathcal{J}(\theta) + \frac{11}{16}\|\nabla\varphi_t\|^2 + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \leq c\,(1+\|\varphi_t\|_{V'}^2)^2 + \int_\Omega (6K(\theta)+\varphi_t)|\nabla\mu|^2. \quad (3.30)$$

We now focus on controlling of the last term in the right hand side of (3.30), which represents the most difficult part of our argument. In order to do so, we use the embedding inequality (2.13) setting $\xi = |\nabla\mu|^2$. Then, exploiting (2.12), (3.9) and the Poincaré-Wirtinger inequality, we infer

$$\begin{aligned}\int_\Omega (6K(\theta)+\varphi_t)|\nabla\mu|^2 &\leq c\big(\|K(\theta)\|_V + \|\nabla\varphi_t\|\big)\big\||\nabla\mu|^2\|_{V'} \\ &\leq c + \frac{1}{2}\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\big\||\nabla\mu|^2\|_{V'}^2 \\ &\leq c + \frac{1}{2}\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \frac{1}{8}\|\nabla\varphi_t\|^2 \\ &\quad + c\,\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 \log\big(e + \big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\big). \end{aligned} \quad (3.31)$$

Next, we consider the functions $\psi(r) = e^r$, $r \in \mathbb{R}$ and $\psi^*(s) = s(\log s - 1)$, $s > 0$ (extended by continuity to $s = 0$ by setting $\psi^*(0) = 0$), which are *convex conjugate*. This means that $\forall r \in \mathbb{R}$, $s \geq 0$, it holds $rs \leq \psi(r) + \psi^*(s)$, as we can see for example in [3, Sec. 1.4]. If we now set $r = \log\big(e + \||\nabla\mu|^2\|_{L^2(\Omega)}\big)$ and $s = c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2$, we can estimate the last term in (3.31) as follows

$$\begin{aligned} & c\,\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 \log\big(e + \big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\big) \\ &\leq c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 \Big(\log\big(c\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2\big) - 1\Big) + e + \big\||\nabla\mu|^2\big\|_{L^2(\Omega)} \\ &\leq c + c\|\nabla\mu\|^4 \log\big(e + \|\nabla\mu\|^2\big) + \|\nabla\mu\|_{L^4(\Omega)}^2, \end{aligned} \quad (3.32)$$

where we used the fact that $\big\||\nabla\mu|^2\big\|_{L^1(\Omega)}^2 = \|\nabla\mu\|^4$ and elementary inequalities concerning logarithms.

The first non-constant term on the right hand side of (3.32) can be estimated by using (3.21) as follows

$$\begin{aligned} c\|\nabla\mu\|^4 \log\big(e + \|\nabla\mu\|^2\big) &\leq c\big(1 + \|\varphi_t\|_{V'}^4\big)\log\Big(e + c\big(1 + \|\varphi_t\|_{V'}^2\big)\Big) \\ &\leq c(1 + \|\varphi_t\|_{V'}^2)^2 \log\big(e + \|\varphi_t\|_{V'}^2\big) \end{aligned} \quad (3.33)$$

while the second one can be controlled by using equation (1.1) and inequalities (2.3a) and (2.5) as

$$\begin{aligned} \|\nabla\mu\|_{L^4(\Omega)}^2 &\leq c\|\nabla\mu\|\|\mu\|_{H^2(\Omega)} \leq c\|\nabla\mu\|\big(\|\mu\|_V + \|\Delta\mu\|\big) \\ &\leq c\|\mu\|_V^2 + c\|\varphi_t\|^2 \leq c\|\mu\|_V^2 + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|_{V'}^2. \end{aligned} \quad (3.34)$$

Then, plugging (3.32)-(3.34) in (3.31) and in turn the result into (3.30) we finally deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_{V'}^2 + 6\frac{d}{dt}\int_\Omega \mathcal{J}(\theta) + \frac{7}{16}\|\nabla\varphi_t\|^2 + \frac{1}{2}\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\ &\leq c(e + \|\varphi_t\|_{V'}^2)^2[1 + \log\big(e + \|\varphi_t\|_{V'}^2\big)] \\ &\leq c(e + \|\varphi_t\|_{V'}^2)^2 \log\big(e + \|\varphi_t\|_{V'}^2\big). \end{aligned} \quad (3.35)$$

Let us now set

$$\Phi(t) := \frac{1}{2}(e + \|\varphi_t\|_{V'}^2), \qquad \Theta(t) := 6\int_\Omega \mathcal{J}(\theta(t)). \tag{3.36}$$

Hence, (3.35) reads

$$\Phi'(t) + \Theta'(t) \leq \ c\,[\Phi(t)]^2 \log \Phi(t). \tag{3.37}$$

We define $Z(t) := 1 + \Phi(t) + \Theta(t)$, then we divide both hand sides of (3.37) by $Z \log Z$, therefore we get

$$\frac{d}{dt} \log\log Z(t) = \frac{Z'(t)}{Z(t)\log Z(t)} \leq \Phi(t), \tag{3.38}$$

where we recall that $\|\Phi\|_{L^1(0,T)} \leq c$ in view of the a-priori estimate (3.20). Moreover, working in a similar way as in [12] in order to estimate the initial condition, we have $Z(0) < \infty$, hence we can integrate (3.38) over $(0,T)$ to obtain

$$\|Z\|_{L^\infty(0,T)} \leq c. \tag{3.39}$$

3.2.4 Consequences

From (3.27), (3.39) reads

$$\|\varphi_t\|_{L^\infty(0,T;V')} \leq c, \tag{3.40}$$

$$\|\theta\|_{L^\infty(0,T;L^{q+2}(\Omega))} \leq c. \tag{3.41}$$

Combining (1.1) with (3.17), we get

$$\|\mu\|_{L^\infty(0,T;V)} \leq c. \tag{3.42}$$

According to the above relations and using (2.10), (3.9), after integrating (3.35) over $(0,T)$, we infer

$$\|\varphi_t\|_{L^2(0,T;V)} \leq c, \tag{3.43}$$

$$\|K(\theta)\|_{L^2(0,T;V)} \leq c. \tag{3.44}$$

Now we read (1.1) as a time-dependent family of elliptic problems. Combining standard regularity results with (3.43), we have

$$\|\mu\|_{L^2(0,T;H^3(\Omega))} \leq c. \tag{3.45}$$

We conclude by providing some estimates for the terms μ_t and φ_t. We (formally) differentiate (1.2) with respect to time and use (1.3), therefore we infer

$$\mu_t = -\Delta\varphi_t + F''(\varphi)\varphi_t + \theta\varphi_t - \Delta K(\theta) - |\nabla\mu|^2. \tag{3.46}$$

We now test the above relation by nonzero $v \in V$. Recalling the boundary conditions, we obtain

$$\begin{aligned} \langle \mu_t, v\rangle &= \int_\Omega \nabla(\varphi_t + K(\theta)) \cdot \nabla v + \big\langle F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2, v\big\rangle \\ &\leq \|v\|_V \Big(\big\|\nabla(\varphi_t + K(\theta))\big\| + \big\|F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2\big\|_{L^{3/2}(\Omega)}\Big). \end{aligned} \tag{3.47}$$

Then, dividing by $\|v\|_V$, passing to the supremum with respect to $v \in V \setminus \{0\}$, squaring, and integrating in time, we get

$$\|\mu\|_{H^1(0,T;V')} \leq c. \tag{3.48}$$

Indeed, according to (3.43)-(3.44), it holds

$$\big\|\nabla(\varphi_t + K(\theta))\big\|_{L^2(0,T;H)} \leq c. \tag{3.49}$$

and moreover it holds

$$\left\|F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2\right\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c \tag{3.50}$$

where the exponent $3/2$ is chosen just for simplicity (any number strictly greater than 1 would be allowed, indeed).
This last inequality can be proved as follows. We know that $F''(\varphi)$ grows as a power of φ, whose regularity is given by (3.6) and (3.22). Hence, according to (3.40) we infer

$$\|F''(\varphi)\varphi_t\|_{L^{3/2}(\Omega)} \leq c.$$

On the other hand, from (3.41), (3.43), Sobolev embeddings and Hölder inequality we obtain

$$\|\theta\varphi_t\|_{L^{3/2}(\Omega)} \leq c.$$

For the last term, taking advantage of (3.42) we get

$$\left\||\nabla\mu|^2\right\|_{L^{3/2}(\Omega)} \leq c.$$

Combining the previous relations and integrating in time we eventually gain (3.50).

3.3 Further regularity

Thanks to the estimates obtained in the previous section, we are now able to prove the regularity presented in Theorem 3.1.
First of all we focus our attention on the estimate

$$\theta \in L^\infty(0,T;L^\infty(\Omega)) \tag{3.51}$$

obtained by a *Moser's iteration* technique, as in [20].
We start multiplying (1.3) by θ^p and then we integrate over Ω. Therefore, we have, in view of (2.9),

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega \theta^{p+1} + \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla\theta^{\frac{p+1}{2}}\right|^2 + \frac{4p}{(p+q+1)^2}\int_\Omega \left|\nabla\theta^{\frac{p+q+1}{2}}\right|^2 \\ \leq \int_\Omega |\Delta\mu|\theta^{p+1} + \int_\Omega |\nabla\mu|^2\theta^p.\end{aligned} \tag{3.52}$$

This entails, using (2.4),

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \theta^{p+1} + \frac{4p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 \\ &\quad\leq \frac{4p}{(p+1)}\|\theta\|_{L^1(\Omega)}^{p+1} + (p+1)\int_\Omega |\Delta\mu|\theta^{p+1} + (p+1)\int_\Omega |\nabla\mu|^2\theta^p \\ &\quad\leq (p+1)\int_\Omega (|\Delta\mu|+1)\theta^{p+1} + (p+1)\int_\Omega |\nabla\mu|^2\theta^p =: I_1 + I_2,\end{aligned} \tag{3.53}$$

where we observed that $\frac{4p}{p+1} \leq p+1$ and where c_p denotes the Poincaré constant in (2.4). Now,

$$\begin{aligned}I_1 &:= (p+1)\int_\Omega (|\Delta\mu|+1)\theta^{p+1} = (p+1)\int_\Omega \theta^{\frac{(p+1)}{2}}(|\Delta\mu|+1)\theta^{\frac{(p+1)}{2}} \\ &\leq (p+1)\left\|\theta^{\frac{p+1}{2}}\right\|_V \left\|\theta^{\frac{(p+1)}{2}}(|\Delta\mu|+1)\right\|_{V'} \\ &\leq \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 + C(p+1)^2\left\|\theta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{L^{6/5}(\Omega)}^2\end{aligned}$$

where the positive constant C is allowed to vary from line to line.

At this point we use Hölder's inequality with exponents 5 and $5/4$, therefore we get

$$\begin{aligned}\left\|\theta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|^2_{L^{6/5}(\Omega)} &= \left(\int_\Omega \theta^{\frac{3}{5}(p+1)}(|\Delta\mu|+1)^{\frac{6}{5}}\right)^{\frac{5}{3}}\\ &\le \||\Delta\mu|+1\|^2_{L^6(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}\\ &\le c(1+\|\mu\|^2_{H^3(\Omega)})\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}.\end{aligned}$$

Eventually we deduce

$$I_1 \le \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|^2_V + c(p)(1+\|\mu\|^2_{H^3(\Omega)})\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}. \tag{3.54}$$

On the other hand, observing that $\theta^p \le \theta^{p+1}+1$ and recalling Sobolev's embedding theorem, thanks to (3.42) we are led to

$$\begin{aligned}I_2 &:= (p+1)\int_\Omega |\nabla\mu|^2\theta^p \le (p+1)\int_\Omega |\nabla\mu|^2\theta^{p+1} + (p+1)\|\nabla\mu\|^2_{L^2(\Omega)}\\ &\le c(p+1)\left\|\theta^{\frac{p+1}{2}}\right\|_V \left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|_{V'} + C(p+1)\\ &\le \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|^2_V + C(p+1)^2\left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|^2_{L^{5/4}(\Omega)} + C(p+1).\end{aligned}$$

Now, applying Hölder's inequality with exponents $6/5,\ 6$ and interpolation inequality (2.3e) we obtain

$$\begin{aligned}\left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|^2_{L^{5/4}(\Omega)} &= \left(\int_\Omega \theta^{\frac{5}{8}(p+1)}|\nabla\mu|^{\frac{5}{2}}\right)^{\frac{8}{5}} \le \|\nabla\mu\|^4_{L^{15}(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}\\ &\le C\|\mu\|^2_{H^3(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}.\end{aligned}$$

Therefore, we have the following inequality for I_2:

$$I_2 \le \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|^2_V + C\|\mu\|^2_{H^3(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}} + C(p+1). \tag{3.55}$$

Using (3.53), (3.54) and (3.55) yields

$$\frac{d}{dt}\int_\Omega \theta^{p+1} \le C(p+1)^2\left(1+\|\mu\|^2_{H^3(\Omega)}\right)\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}} + C(p+1),$$

then, by a further integration over $(0,t), t \in (0,T]$,

$$\int_\Omega \theta^{p+1}(t) \le C\,(p+1)^2\left(\sup_{[0,T]}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}} + K_T\right), \tag{3.56}$$

where we took advantage of (3.45) and where with K_T we denote a term cointaining the information on the initial datum $\theta^{p+1}(0)$ which possibly depends on T.
In order to apply Moser's iteration, we consider the sequence $(p_k)_k$ of real numbers defined by

$$p_0 = 3, \quad p_{k+1} = \frac{4}{3}p_k, \quad k \in \mathbb{N}.$$

Let us take $p = p_{k+1} - 1$ in (3.56). We then have

$$\int_\Omega \theta^{p_{k+1}}(t) \le C\, p_{k+1}^2 \left(\sup_{[0,T]} \left(\int_\Omega \theta^{p_k} \right)^{\frac{4}{3}} + K_T \right),$$

hence

$$\sup_{[0,T]} \int_\Omega \theta^{p_{k+1}} \le C\, p_{k+1}^2 \max \left\{ \sup_{[0,T]} \left(\int_\Omega \theta^{p_k} \right)^{\frac{4}{3}}, K_T \right\}.$$

Thanks to (3.41), we already have $\theta \in L^\infty(0,T;L^{q+2}(\Omega))$, where q was introduced in (2.9). Therefore, we can apply the Moser lemma and get

$$\forall k \in \mathbb{N}, \qquad \sup_{[0,T]} \|\theta\|_{L^{p_k}(\Omega)} \le C.$$

Taking the limit as k goes to infinity leads to (3.51). This also immediately entails that $K(\theta) \in L^\infty(0,T;L^\infty(\Omega))$.
We are now able to prove that $\theta \in L^\infty(0,T;V)$. In order to do so, we formally multiply (1.3) by $\partial_t K(\theta) = \kappa(\theta)\theta_t$. We obtain

$$\begin{aligned} &\int_\Omega \left| \sqrt{\kappa(\theta)}\theta_t \right|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla K(\theta)|^2 \\ &= -\int_\Omega \theta\Delta\mu\kappa(\theta)\theta_t + \int_\Omega |\nabla\mu|^2\kappa(\theta)\theta_t = \int_\Omega \kappa(\theta)\theta_t[|\nabla\mu|^2 - \theta\,\Delta\mu]\,\mathrm{dx} \end{aligned}$$

Owing to (3.51) and (2.3a), the right hand side can be controlled as

$$\begin{aligned} &\int_\Omega \kappa(\theta)\theta_t[|\nabla\mu|^2 - \theta\,\Delta\mu]\,\mathrm{dx} \\ =&\int_\Omega \sqrt{\kappa(\theta)}\sqrt{\kappa(\theta)}\theta_t\left(\theta\Delta\mu + |\nabla\mu|^2\right) \\ \le&\|\sqrt{\kappa(\theta)}\|_{L^\infty(\Omega)}\|\sqrt{\kappa(\theta)}\theta_t\|\left(\|\theta\|_{L^\infty(\Omega)}\|\Delta\mu\| + \|\nabla\mu\|^2_{L^4(\Omega)}\right) \\ \le& c\|\sqrt{\kappa(\theta)}\theta_t\|\|\mu\|_{H^2(\Omega)} \le \frac{1}{2}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|\mu\|^2_{H^2(\Omega)}. \end{aligned}$$

Then, on account of (3.45), we get

$$\theta \in H^1(0,T;L^2(\Omega)) \qquad K(\theta) \in L^\infty(0,T;V)$$

and this last estimate entails the desired result $\theta \in L^\infty(0,T;V)$. Finally, by reading (1.2) as

$$-\Delta\varphi + F'(\varphi) = \theta + \mu \in L^\infty(0,T;V),$$

we deduce the thesis using (3.1), that is $\varphi \in L^\infty(0,T;H^3(\Omega))$.

3.4 Uniqueness

We now address the uniqueness of solution in Theorem 3.1. Let $z_0 \in \mathcal{V}$ and let $(\varphi_i, \mu_i, \theta_i), i = 1, 2,$ be a couple of (stable) solutions both emanating from z_0 over the interval $(0, T)$. Taking $(\varphi, \mu, \theta) := (\varphi_1 - \varphi_2, \mu_1 - \mu_2, \theta_1 - \theta_2)$, we can readily obtain

$$\varphi_t = \Delta\mu, \tag{3.57}$$

$$\mu = -\Delta\varphi + F'(\varphi_1) - F'(\varphi_2) - \theta, \tag{3.58}$$

$$\theta_t + \theta_1\Delta\mu + \theta\Delta\mu_2 - \Delta[K(\theta_1) - K(\theta_2)] = (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu \tag{3.59}$$

coupled with null initial data. This guarantees e.g. $\varphi_\Omega(t) = 0 \quad \forall t \geq 0$. By the regularity (3.1)-(3.3), we observe

$$\begin{aligned} &\|\varphi_i(t)\|_{H^3(\Omega)} + \|\mu_i(t)\|_V + \|\theta_i(t)\|_V \leq c, \quad t \in (0,T) \\ &\|\mu_i\|_{L^2(0,T;H^3(\Omega))} + \|\theta_i\|_{L^\infty(0,T;L^\infty(\Omega))} \leq c \end{aligned} \tag{3.60}$$

for some positive constant c depending on T and on the initial data. These properties will be frequently used in the following.
The proof we show is based on the application of Gronwall's Lemma to the functional

$$\mathcal{Z}(t) := \varepsilon\|\nabla\varphi(t)\|^2 - 2\varepsilon\langle\theta(t) - \theta_\Omega(t), \varphi(t)\rangle + \|\theta(t) - \theta_\Omega(t)\|_{V'}^2 + \theta_\Omega(t)^2, \tag{3.61}$$

which is zero for $t = 0$. We notice that for $\varepsilon > 0$ small enough,

$$\mathcal{Z}(t) \geq c_\varepsilon(\|\nabla\varphi(t)\|^2 + \|\theta(t) - \theta_\Omega(t)\|_{V'}^2 + \theta_\Omega(t)^2). \tag{3.62}$$

Therefore we need to estimate the terms $\frac{d}{dt}\|\nabla\varphi(t)\|^2$, $\frac{d}{dt}\|\theta(t) - \theta_\Omega(t)\|_{V'}^2$ and $\frac{d}{dt}\theta_\Omega(t)^2$. The first one will be addressed in Section 3.4.2, the second in Section 3.4.3 and the third in Section 3.4.4.

3.4.1 Preliminary estimates

First of all we control $\Delta\varphi$ by testing (3.58) by $-\Delta\varphi$:

$$\begin{aligned} \|\Delta\varphi\|^2 &= -\langle\mu - \mu_\Omega, \Delta\varphi\rangle - \langle\theta - \theta_\Omega, \Delta\varphi\rangle + \langle F'(\varphi_1) - F'(\varphi_2), \Delta\varphi\rangle \\ &\leq c\|\Delta\varphi\|\,(\|\nabla\mu\| + \|\theta - \theta_\Omega\| + \|F'(\varphi_1) - F'(\varphi_2)\| \end{aligned}$$

Exploiting Hölder's inequality with exponents 3 and $3/2$ and recalling (2.8),(3.60), we obtain

$$\begin{aligned} \|F'(\varphi_1) - F'(\varphi_2)\|^2 &\leq c\left\langle(1 + |\varphi_1|^{2p_F} + |\varphi_2|^{2p_F}), \varphi^2\right\rangle \\ &\leq c\|\varphi\|_{L^3(\Omega)}^2 \leq c\|\varphi\|_V^2\,. \end{aligned}$$

Eventually, according to (3.21), we conclude that

$$\|\Delta\varphi\|^2 \leq c(\|\nabla\mu\|^2 + \|\theta - \theta_\Omega\|^2 + \|\varphi\|_V^2) \leq c(\|\varphi_t\|_{V'}^2 + \|\theta - \theta_\Omega\|^2 + \|\varphi\|_V^2). \tag{3.63}$$

3.4.2 Difference in order parameters for Gronwall's argument (3.61)

Testing (3.57) by $J^{-1}\varphi_t$, we get

$$\|\varphi_t\|_{V'}^2 + \langle \mu, \varphi_t \rangle = 0,$$

where, multiplying (3.58) by φ_t, we can write the second term as

$$\langle \mu, \varphi_t \rangle = \frac{1}{2}\frac{d}{dt}(\|\nabla \varphi\|^2 - 2\,\langle \theta - \theta_\Omega, \varphi \rangle) + \langle F'(\varphi_1) - F'(\varphi_2), \varphi_t \rangle + \langle \theta_t, \varphi \rangle\,.$$

Combining the previous relations we infer

$$\begin{aligned} &\|\varphi_t\|_{V'}^2 + \frac{1}{2}\frac{d}{dt}(\|\nabla \varphi\|^2 - 2\,\langle \theta - \theta_\Omega, \varphi \rangle) \\ &= -\,\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t \rangle - \langle \theta_t, \varphi \rangle\,. \end{aligned} \tag{3.64}$$

We take care of second term on the right hand side by multiplying (3.59) by φ:

$$\begin{aligned} \langle \theta_t, \varphi \rangle = {}& \langle K(\theta_1) - K(\theta_2), \Delta\varphi \rangle + \langle \nabla\mu, \nabla\theta_1 \varphi \rangle + \langle \nabla\mu, \theta_1 \nabla\varphi \rangle \\ & - \langle \theta \Delta \mu_2, \varphi \rangle + \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi \rangle\,. \end{aligned}$$

We notice that a direct estimate of the first term in the right hand side could have been provided due to (3.63) and the regularity achieved on $K(\theta)$. However in this way, it turns to be difficult to reabsorb the term $C\|\varphi_t\|_{V'}$ in the left hand side. Therefore the gain of regularity in θ apparently does not simplify this part of the proof and we need to proceed as in [10].
We notice that, exploiting (3.58) two times, the first term on the right hand side reads

$$\begin{aligned} &\langle K(\theta_1) - K(\theta_2), \Delta\varphi \rangle \\ &= \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - \theta + \theta_\Omega \rangle - \\ &(\mu_\Omega + \theta_\Omega)\int_\Omega [K(\theta_1) - K(\theta_2)] \\ &= \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega \rangle \\ &- \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega \rangle\,. \end{aligned}$$

Therefore we have,

$$\begin{aligned} &\langle \theta_t, \varphi \rangle \\ &= -\,\langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega \rangle \\ &\quad - \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega \rangle + \langle \nabla\mu, \nabla\theta_1\varphi \rangle + \langle \nabla\mu, \theta_1\nabla\varphi \rangle - \langle \theta\Delta\mu_2, \varphi \rangle \\ &\quad + \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi \rangle\,. \end{aligned}$$

Owing (3.64) and combining the above relations, we obtain

$$\begin{aligned} &\|\varphi_t\|_{V'}^2 + \frac{1}{2}\frac{d}{dt}\bigg(\, \|\nabla\varphi\|^2 - 2\,\langle \theta - \theta_\Omega, \varphi \rangle \bigg) - \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega \rangle \\ &= -\,\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t \rangle \\ &\quad + \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega \rangle \\ &\quad - \langle \nabla\mu, \nabla\theta_1\varphi + \theta_1\nabla\varphi \rangle + \langle \theta\Delta\mu_2, \varphi \rangle \\ &\quad - \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi \rangle =: I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{3.65}$$

First of all using (2.8), we have

$$\begin{aligned}&\|\nabla(F'(\varphi_1)-F'(\varphi_2))\|\\&\qquad\leq c\left\||\nabla\varphi|(1+|\varphi_1^{p_F}|)+|\varphi|(1+|\varphi_1|^{p_F-1}+|\varphi_2|^{p_F}-1)|\nabla\varphi_2|\right\|\leq c\|\varphi\|_V.\end{aligned}\tag{3.66}$$

Therefore, as $\langle\varphi_t,1\rangle=0$, we deduce

$$\begin{aligned}I_3 &:= -\langle F'(\varphi_1)-F'(\varphi_2),\varphi_t\rangle\\&\leq C\|\nabla(F'(\varphi_1)-F'(\varphi_2))\|\,\|\varphi_t\|_{V'}\leq c\|\varphi\|_V\|\varphi_t\|_{V'}.\end{aligned}$$

On the other hand, by (3.51),

$$\|K(\theta_1)-K(\theta_2)\|_{3/2}\leq c(\|\theta-\theta_\Omega\|+|\theta_\Omega|),$$

and, according to (3.66), it follows

$$\begin{aligned}I_4 &:= \langle K(\theta_1)-K(\theta_2),\mu-\mu_\Omega-F'(\varphi_1)-F'(\varphi_2)-(F'(\varphi_1)-F'(\varphi_2))_\Omega\rangle\\&\leq \|K(\theta_1)-K(\theta_2)\|_{L^{3/2}(\Omega)}(\|\mu-\mu_\Omega\|_{L^3(\Omega)}\\&\quad+\|F'(\varphi_1)-F'(\varphi_2)-(F'(\varphi_1)-F'(\varphi_2))_\Omega\|_{L^3(\Omega)})\\&\leq c(\|\theta-\theta_\Omega\|+|\theta_\Omega|)(\|\nabla\mu\|+\|\varphi\|_V).\end{aligned}$$

Owing (2.3a),(2.3b) and (2.3f)

$$\begin{aligned}I_5 &:= -\langle\nabla\mu,\nabla\theta_1\varphi+\theta_1\nabla\varphi\rangle\\&\leq c\|\nabla\mu\|(\|\nabla\theta_1\|\,\|\varphi\|_{L^\infty(\Omega)}+\|\nabla\varphi\|_{L^4(\Omega)}\|\theta_1\|_{L^4(\Omega)})\\&\leq c\|\nabla\mu\|\left(\|\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}+\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}\right)\\&\leq c\|\nabla\mu\|\,\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}.\end{aligned}$$

Now, using (2.3f) and the injection $V\subset L^p(\Omega)$, for $p\geq 1$,

$$\begin{aligned}I_6 &:= \langle\theta\Delta\mu_2,\varphi\rangle=\theta_\Omega\langle\Delta\mu_2,\varphi\rangle+\langle(\theta-\theta_\Omega)\Delta\mu_2,\varphi\rangle\\&\leq c|\theta_\Omega|\,\|\nabla\mu_2\|\,\|\nabla\varphi\|+c\|\theta-\theta_\Omega\|\,\|\Delta\mu_2\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)}\\&\leq c|\theta_\Omega|\,\|\nabla\varphi\|+c\|\theta-\theta_\Omega\|\,\|\mu_2\|_{H^3(\Omega)}\|\nabla\varphi\|.\end{aligned}$$

Finally combining the previous strategy with (2.3a), we get

$$\begin{aligned}I_7 &:= -\langle(\nabla\mu_1+\nabla\mu_2)\cdot\nabla\mu,\varphi\rangle\leq\|\nabla\mu_1+\nabla\mu_2\|_{L^4(\Omega)}\|\nabla\mu\|\,\|\varphi\|_{L^4(\Omega)}\\&\leq\left(\|\mu_1\|^{1/2}_{H^2(\Omega)}+\|\mu_2\|^{1/2}_{H^2(\Omega)}\right)\|\nabla\mu\|\,\|\nabla\varphi\|.\end{aligned}$$

Eventually, the above computations and Young's inequality yield

$$\begin{aligned}&I_3+I_4+I_5+I_6+I_7\\&\leq c\|\varphi_t\|_{V'}\|\varphi\|_V+c(\|\theta-\theta_\Omega\|+|\theta_\Omega|)(1+\|\mu_2\|_{H^3(\Omega)})\|\varphi\|_V\\&+c\|\nabla\mu\|\,\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}+c(\|\theta-\theta_\Omega\|+|\theta_\Omega|)\|\nabla\mu\|\\&+c\left(1+\|\mu_1\|^{1/2}_{H^2(\Omega)}+\|\mu_2\|^{1/2}_{H^2(\Omega)}\right)\|\nabla\mu\|\,\|\nabla\varphi\|\\&\leq\frac{1}{2}\|\varphi_t\|^2_{V'}+\frac{\alpha}{2}\|\nabla\mu\|^2+\frac{c}{\alpha}\|\theta-\theta_\Omega\|^2+\frac{\alpha}{2}\|\Delta\varphi\|^2+g(t)(\|\varphi\|^2_V+\theta^2_\Omega)\\&\leq\left(\frac{1}{2}+c\alpha\right)\|\varphi_t\|^2_{V'}+c\left(\frac{1}{\alpha}+\alpha\right)\|\theta-\theta_\Omega\|^2+g(t)(\|\varphi\|^2_V+\theta^2_\Omega).\end{aligned}$$

Where in the last passage we took advantage (3.63) to estimate the term depending on $\Delta\varphi$. Moreover, we have defined

$$g(t) := c[1 + \|\theta_2\|^2_{H^2(\Omega)} + \|\mu_1\|^2_{H^3(\Omega)} + \|\mu_2\|^2_{H^3(\Omega)}], \tag{3.67}$$

with (large) constant $c > 0$ also depending on the choice of the small constant $\alpha > 0$ Combining the previous estimates with (3.65), we finally get

$$\begin{aligned}
&\|\varphi_t\|^2_{V'} + \frac{d}{dt}(\|\nabla\varphi\|^2 - 2\langle \theta - \theta_\Omega, \varphi\rangle) - 2\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle \\
&\le \left(\frac{1}{2} + c\alpha\right)\|\varphi_t\|_{V'} + c\left(\frac{1}{\alpha} + \alpha\right)\|\theta - \theta_\Omega\|^2 + g(t)(\|\varphi\|^2_V + \theta^2_\Omega).
\end{aligned} \tag{3.68}$$

As mentioned before, the aim of these calculations is to apply Gronwall's Lemma to a specific functional already introduced. In order to do that, we are trying to obtain the derivative of such functional on the left-hand side and the functional itself on the right. Thus all terms arising from that must be either integrable over $(0, T)$ (which is the $g(t)$) or they must be balanced with some term on the left side as $\|\varphi_t\|^2$ or $\|\theta - \theta^2_\Omega\|$(which will arise from $\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle$).

3.4.3 Difference of temperatures for Gronwall's argument (3.61)

We test (3.59) by $J^{-1}(\theta - \theta_\Omega)$ and integrate by parts, therefore we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta - \theta_\Omega\|^2_{V'} + \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle \\
&= \left\langle \nabla\theta_1 \cdot \nabla\mu, J^{-1}(\theta - \theta_\Omega)\right\rangle + \left\langle \theta_1\nabla\mu, \nabla J^{-1}(\theta - \theta_\Omega)\right\rangle \\
&- \left\langle (\theta - \theta_\Omega)\Delta\mu_2, J^{-1}(\theta - \theta_\Omega)\right\rangle + \theta_\Omega\langle \mu_2 - (\mu_2)_\Omega, \theta - \theta_\Omega\rangle \\
&+ \left\langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, J^{-1}(\theta - \theta_\Omega)\right\rangle =: I_8 + I_9 + I_{10}.
\end{aligned}$$

First of all we have

$$\begin{aligned}
I_8 &:= \left\langle \theta_1\nabla\mu, \nabla J^{-1}(\theta - \theta_\Omega)\right\rangle \\
&\le c\|\nabla\mu\|\left\|\nabla J^{-1}(\theta - \theta_\Omega)\right\| \le c\|\nabla\mu\|\|\theta - \theta_\Omega\|_{V'}.
\end{aligned}$$

From (2.3b),

$$\left\|J^{-1}(\theta - \theta_\Omega)\right\|_{L^\infty(\Omega)} \le c\|\theta - \theta_\Omega\|^{1/2}_{V'}\|\theta - \theta_\Omega\|^{1/2}.$$

Thus, we infer

$$\begin{aligned}
I_9 &:= -\left\langle \nabla\theta_1 \cdot \nabla\mu - (\theta - \theta_\Omega)\Delta\mu_2, J^{-1}(\theta - \theta_\Omega)\right\rangle \\
&\quad + \left\langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, J^{-1}(\theta - \theta_\Omega)\right\rangle \\
&\le \left\|J^{-1}(\theta - \theta_\Omega)\right\|_{L^\infty}[\|\theta_1\|_V\|\nabla\mu\| \\
&\quad + \|\mu_2\|\|\theta - \theta_\Omega\| + (\|\nabla\mu_1\| + \|\nabla\mu_2\|)\|\nabla\mu\|] \\
&\le c\|\theta - \theta_\Omega\|^{1/2}_{V'}\|\theta - \theta_\Omega\|^{1/2}\|\nabla\mu\| + c\|\mu_2\|_{H^2(\Omega)}\|\theta - \theta_\Omega\|^{1/2}_{V'}.
\end{aligned}$$

Eventually,

$$\begin{aligned}
I_{10} &:= \theta_\Omega\langle \mu_2 - (\mu_2)_\Omega, \theta - \theta_\Omega\rangle \\
&\le c|\theta_\Omega|\|\mu_2 - (\mu_2)_\Omega\|_V\|\theta - \theta_\Omega\|_{V'} \le c|\theta_\Omega|\|\theta - \theta_\Omega\|_{V'}.
\end{aligned}$$

Combining the estimates of I_8, I_9, I_{10} and exploiting Young's inequality, we finally have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta-\theta_\Omega\|_{V'}^2 + \langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle \leq \delta\varepsilon\|\theta-\theta_\Omega\|^2 \\ +\alpha\varepsilon\|\nabla\mu\|^2 + C_*(1+\|\mu_2\|^2_{H^2(\Omega)})\|\theta-\theta_\Omega\|^2_{V'} + c|\theta_\Omega|^2,\end{aligned} \tag{3.69}$$

with the (large) constant C_* depending on the small constants $\alpha, \delta, \varepsilon$ which will be specified at the end.

3.4.4 Difference of temperatures' means for Gronwall's argument 3.61

Integrating (3.59) over Ω we obtain

$$|\Omega|(\theta_\Omega)_t = \langle\nabla\theta_1,\nabla\mu\rangle - \langle\theta-\theta_\Omega,\Delta\mu_2\rangle + \langle(\nabla\mu_1+\nabla\mu_2),\nabla\mu\rangle\,, \tag{3.70}$$

and by (3.60) it yields

$$|(\theta_\Omega)_t| \leq c(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\theta-\theta_\Omega\|_{V'}).$$

Moreover, multiplying (3.70) by θ_Ω we have, for (small) $\alpha>0$ we will choose later and corresponding (large) $c>0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\theta_\Omega^2 &\leq c|\theta_\Omega|(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\theta-\theta_\Omega\|^2_{V'}) \\ &\leq \alpha\varepsilon\|\nabla\mu\|^2 + c(\theta_\Omega^2 + \|\mu_2\|^2_{H^3(\Omega)}\|\theta-\theta_\Omega\|^2_{V'}).\end{aligned} \tag{3.71}$$

3.4.5 Conclusion

We recall the definition of the functional we want to use

$$\mathcal{Z}(t) := \varepsilon\|\nabla\varphi\|^2 - 2\varepsilon\langle\theta-\theta_\Omega,\varphi\rangle + \|\theta-\theta_\Omega\|^2_{V'} + \theta_\Omega^2,$$

Summing (3.69), (3.71) and $\frac{\varepsilon}{2}$ times (3.68), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{Z} + (1-\varepsilon)\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle + \frac{\varepsilon}{2}\|\varphi_t\|^2_{V'} \\ \leq \varepsilon(c+c\alpha^2+\delta)\|\theta-\theta_\Omega\|^2 + 2\alpha\varepsilon\|\nabla\mu\|^2 + g(t)\mathcal{Z},\end{aligned}$$

where g was introduced in (3.67).

Next, we take care of the second term in the left hand side. From (2.10), we obtain

$$\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle = \|\theta-\theta_\Omega\|^2 + \frac{1}{q+1}\langle l(\theta_1)-l(\theta_2),\theta-\theta_\Omega\rangle$$

where $l(\theta_i)=\theta_i^{q+1}, i=1,2$. Now

$$\begin{aligned}l(\theta_1)-l(\theta_2) &= \int_0^1 \frac{d}{ds}l(s\theta_1+(1-s)\theta_2)ds \\ &= \int_0^1 l'(s\theta_1+(1-s)\theta_2)(\theta_1-\theta_2)ds = \omega(\theta_1,\theta_2)\theta,\end{aligned}$$

where we set

$$\omega(\theta_1, \theta_2) := \int_0^1 l'(s\theta_1 + (1-s)\theta_2)ds).$$

We observe that it holds $\omega(\theta_1, \theta_2) \geq 0$ almost everywhere. Moreover, we notice that (3.51) implies,

$$|\omega(\theta_1, \theta_2)| \leq c \tag{3.72}$$

Therefore we infer

$$\begin{aligned}
\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega \rangle &= \|\theta - \theta_\Omega\|^2 + \frac{1}{1+q} \langle l(\theta_1) - l(\theta_2), \theta - \theta_\Omega \rangle \\
&= \|\theta - \theta_\Omega\|^2 + \frac{1}{q+1} \int_\Omega \omega(\theta_1, \theta_2)\theta(\theta - \theta_\Omega) \\
&= \|\theta - \theta_\Omega\|^2 + \frac{1}{q+1} \int_\Omega \omega(\theta_1, \theta_2)|\theta - \theta_\Omega|^2 \\
&+ \frac{1}{q+1} \int_\Omega \omega(\theta_1, \theta_2)\theta_\Omega(\theta - \theta_\Omega) \\
&\geq \|\theta - \theta_\Omega\|^2 + \frac{1}{q+1} \int_\Omega \omega(\theta_1, \theta_2)\theta_\Omega(\theta - \theta_\Omega).
\end{aligned}$$

Hence, exploiting (3.72), we get

$$\left| \frac{1}{q+1} \int_\Omega \omega(\theta_1, \theta_2)\theta_\Omega(\theta - \theta_\Omega) \right| \leq c|\theta_\Omega| \, \|\theta - \theta_\Omega\| \leq \frac{1}{2} \|\theta - \theta_\Omega\|^2 + c\theta_\Omega^2,$$

and moreover

$$\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega \rangle \geq \frac{1}{2} \|\theta - \theta_\Omega\|^2 - c\theta_\Omega^2.$$

Putting everything together we finally have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{Z} + \frac{1-\varepsilon}{2} \|\theta - \theta_\Omega\|^2 + \frac{\varepsilon}{2} \|\varphi_t\|_{V'}^2 \\
&\leq \varepsilon(c + c\alpha^2 + \delta) \|\theta - \theta_\Omega\|^2 + 3\alpha\varepsilon \|\nabla\mu\|^2 g(t)\mathcal{Z} \\
&\overset{(3.21)}{\leq} \varepsilon(c + c\alpha^2 + \delta) \|\theta - \theta_\Omega\|^2 + 3\alpha\varepsilon c \|\varphi_t\|_{V'}^2 + g(t)\mathcal{Z}.
\end{aligned} \tag{3.73}$$

The constant c on the right hand side of (3.73) only depends on the regularity properties of the solutions collected in (3.60). In particular, it is *independent* of the parameters $\alpha, \delta, \varepsilon$. Therefore, chosing $\alpha > 0$ small enough, the second term on the right hand side of (3.73) can be absorbed the corresponding quantities in the left hand side. Moreover, taking ε sufficiently small (it might depend on other parameters), the first term on the right hand side can be absorbed, too. As a matter of fact, we are able to semplify (3.73) as follows

$$\frac{d}{dt}\mathcal{Z} + \kappa_0(\|\theta - \theta_\Omega\|^2 + \|\varphi_t\|_{V'}^2) \leq g(t)\mathcal{Z},$$

where $\kappa_0 > 0$ and g was defined in (3.67), hence, exploiting (3.60), it is summable over the interval $(0, T)$. Since $\mathcal{Z}(0) = 0$, then by Gronwall's Lemma and (3.62) we eventually see that $\mathcal{Z}$ is identically 0 over $(0, T)$, which gives us the assert.

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