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Delayed exchange of stabilities in singularly perturbed systems

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Abstract

We consider a scalar nonautonomous singularly perturbed differential equation whose degenerate equation has two solutions which intersect at some point. These solutions represent families of equilibria of the associated equation where at least one of these families loses its stability at the intersection point. We study the behavior of the solution of an initial value problem of the singularly perturbed equation in dependence on the small parameter. We assume that the solution stays at the beginning near a stable branch of equilibria of the associated system where this branch loses its stability at some critical time t_c. By means of the method of upper and lower solutions we determine the asymptotic delay t^* of the solution for leaving the unstable branch. The obtained result holds for the case of transcritical bifurcation as well as for the case of pitchfork bifurcation. We consider some examples where we prove that a well-known result due to N.R. Lebovitz and R.J. Schaar about an immediate exchange of stabilities cannot be applied to singularly perturbed systems whose right hand side depends on ε.

1 Introduction

We consider the singularly perturbed equation

$$\varepsilon \frac{du}{dt} = g(u,t,\varepsilon) \tag{1.1}$$

and study the initial value problem

$$u(t_0,\varepsilon) = u^0, \quad t \in I_t := \{t \in R : t_0 < t < t_1\} \tag{1.2}$$

for sufficiently small ε.

The so-called associated equation to (1.1)

$$\frac{du}{d\tau} = g(u,t,0) \tag{1.3}$$

in which t has to be considered as a parameter, plays an important role in the asymptotic study of the initial value problem (1.1), (1.2) with respect to ε. If we set $\varepsilon = 0$ in (1.1) we get the degenerate equation

$$g(u,t,0) = 0. \tag{1.4}$$

A solution $u = \varphi(t)$ of the degenerate equation (1.4) represents a branch of equilibria of the associated equation (1.3). If we assume that (1.4) has an isolated solution $u = \varphi(t)$ and that u^0 lies in the region of attraction of the asymptotically stable equilibrium point $\varphi(t_0)$ of (1.3) then the asymptotic behavior of the solution $u(t,u^0,\varepsilon)$ of (1.1), (1.2) can be determined according to the standard theory of singularly perturbed differential systems [10, 16, 22, 23, 24, 25, 26] and we have

$$\lim_{\varepsilon\to 0} u(t,u^0,\varepsilon) = \varphi(t) \qquad \text{for} \quad t \in I_t.$$

In what follows we consider the case that the degenerate equation (1.4) has two solutions which intersect for $t = t_c$. Generically, this implies that the associated equation exhibits either the case of transcritical bifurcation or the case of pitchfork bifurcation (see Fig. 1 and Fig. 2, respectively, where the bold line means stability and the dotted line instability). In both cases, at least one of these solutions loses its stability at $t = t_c$ as a family of equilibria of the associated system. If we assume that u^0 belongs to the basin of attraction of a stable equilibrium point of the associated equation (1.3) for $t = t_0$ then for sufficiently small ε the solution $u(t, u^0, \varepsilon)$ of (1.1), (1.2) stays near the stable branch of equilibria of the associated equation (1.3) for $t_0 < t < t_c$. For $t > t_c$ it can stay near another stable branch intersecting the old one (immediate exchange of stabilities) or it stays for some time interval near the unstable branch and jumps then to another stable branch (delayed exchange of stabilities, delayed loss of stability) or it exhibits a failure of exchange of stabilities. Conditions for an immediate exchange of stabilities have been obtained by N.R. Lebovitz and R.J. Schaar [14, 15] for singularly perturbed systems whose right hand side does not depend on ε. In the case of transcritical bifurcation, N.N. Nefedov and K.R. Schneider [17] have derived corresponding results for more general systems by means of the technique of differential inequalities. In section 3 we show that the result of Lebovitz and Schaar mentioned before cannot be applied to systems whose right hand sides depend on ε.

The phenomenon of delayed loss of stability in the case of Hopf bifurcation with a slowly varying bifurcation parameter was observed first by M.A. Shishkova [21] in 1973 from the school of L.S. Pontryagin. This case is equivalent to the case of pitchfork bifurcation for equations (1.1). She used the method of asymptotic expansion to estimate the delay. Her results have have been extended by S. Karimov [11] and A.I. Neishtadt [18, 19]. The occurence of delayed exchange of stabilities in the case of transcritical bifurcation was noticed first by R. Haberman [9] in 1979. His conjecture about the delay was proved by St. Schecter [20] by means differential inequalities, similar results have been obtained by F. and M. Diener [2] in the frame of canard solutions by using nonstandard analysis. Further studies on delayed exchange of stabilities can be found in [1], especially we would like to mention the investigations of T. Erneux and P. Mandel [4, 5] who have also emphasized its importance for applications in laser dynamics [6, 7, 8]. In what follows we study the phenomenon of delayed loss of stability for singularly perturbed scalar nonautonomous equations by means of the method of upper and lower solutions. This approach provides a simple but very efficient method to determine the asymptotic delay in exchange of stabilities for the case of transcritical bifurcation as well as for the case of pitchfork bifurcation. Under generic conditions, corresponding problems for higher dimensional singularly perturbed systems can be reduced to scalar equations by means of invariant manifolds.

2 Main result

We consider the initial value problem (1.1), (1.2) under the following assumptions

(A_1). $g : U \times I_t \times I_{\varepsilon_0} \to R$ is continuous and twice continuously differentiable with respect to u and ε. U is an open bounded interval containing the origin, I_{ε_0} is defined by $I_{\varepsilon_0} := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_0\}$, $\varepsilon_0 > 0$.

(A_2). $g(0,t,\varepsilon) \equiv 0$ for $(t,\varepsilon) \in \bar{I}_t \times \bar{I}_{\varepsilon_0}$ ($\bar{I}$ means the closure of I). There is a $t_c \in (t_0, t_1)$ such that

$$g_u(0,t,0) < 0 \quad \text{for } [t_0, t_c), \quad g_u(0,t,0) > 0 \quad \text{for } t \in (t_c, t_1].$$

Assumption (A_2) implies that $u \equiv 0$ is a solution of equation (1.1) in $\bar{I}_t$ for all $\varepsilon \in \bar{I}_{\varepsilon_0}$, and that $u \equiv 0$ is an equilibrium point of the associated system which is exponentially asymptotically stable for $t \in [t_0, t_c)$ and unstable for $t \in (t_c, t_1]$.
In what follows we distinguish the cases $u^0 > 0$ and $u^0 < 0$. Since the asymptotic behavior of the solution $u(t, u^0, \varepsilon)$ of (1.1), (1.2) with respect to ε strongly depends on the solution set of the degenerate equation we introduce the following assumptions on this set.

(A_3^+). The solution set of the degenerate equation $g(u,t,0) = 0$ in $(\bar{U} \times \bar{I}_t) \cap u \geq 0$ consists of the two curves $u \equiv 0$ and $u = \psi_+(t), \psi_+ \in C^0([t_c, t_1], R^+)$ with $\psi_+(t_c) = 0$.

($A_{3,t}^-$). The solution set of the degenerate equation $g(u,t,0) = 0$ in $(\bar{U} \times \bar{I}_t) \cap u \leq 0$ consists of the two curves $u \equiv 0$ and $u = \psi_{-,t}(t), \psi_{-,t} \in C^0([t_0, t_c], R^-)$ with $\psi_{-,t}(t_c) = 0$.

($A_{3,p}^-$). The solution set of the degenerate equation $g(u,t,0) = 0$ in $(\bar{U} \times \bar{I}_t) \cap u \leq 0$ consists of the two curves $u \equiv 0$ and $\psi_{-,p}(t) \in C^0([t_c, t_1], R^-)$ with with $\psi_{-,p}(t_c) = 0$.

The subindices t and p characterize the transcritical and the pitchfork bifurcation.

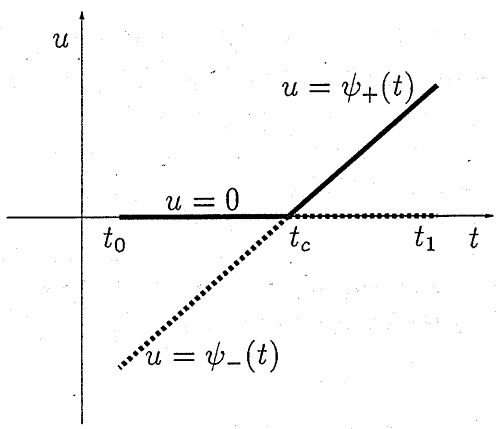

Fig. 1. Transcritical bifurcation

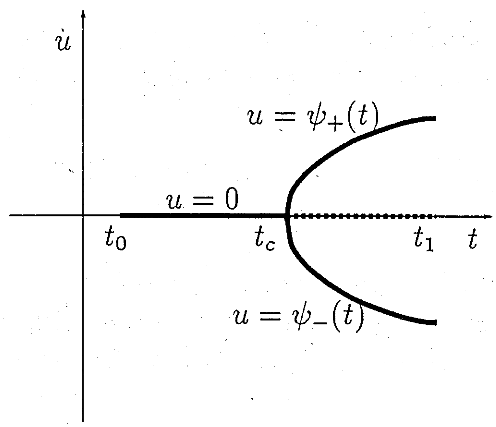

Fig. 2. Pitchfork bifurcation

In order to describe the delay in the exchange of stabilities we introduce the functions $G(t,\varepsilon)$ and $\overline{G}(t)$ by

$$G(t,\varepsilon) := \int_{t_0}^{t} g_u(0,s,\varepsilon)ds, \quad \overline{G}(t) := G(t,0). \tag{2.1}$$

From assumption (A_2) we get that $\overline{G}(t) = 0$ has at most one root $t = t^*$ in (t_0, t_1). Therefore, we assume

(A_4.) $\overline{G}(t) = 0$ has a root t^* in (t_0, t_1).

It is easy to see that t^* is such that

$$t^* > t_c, \quad \overline{G}'(t^*) > 0. \tag{2.2}$$

(A_5.) There is a positive number c_0 such that $[-c_0, c_0] \in U$ and

$$g(u,t,\varepsilon) \leq g_u(0,t,\varepsilon)u \quad \text{for} \quad t_0 \leq t \leq t^*,\ \varepsilon \in \overline{I}_{\varepsilon_0},\ |u| \leq c_0.$$

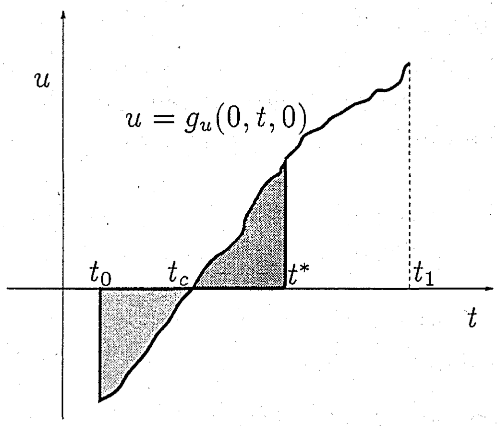

Fig. 3. $u = g_u(0,t,0)$

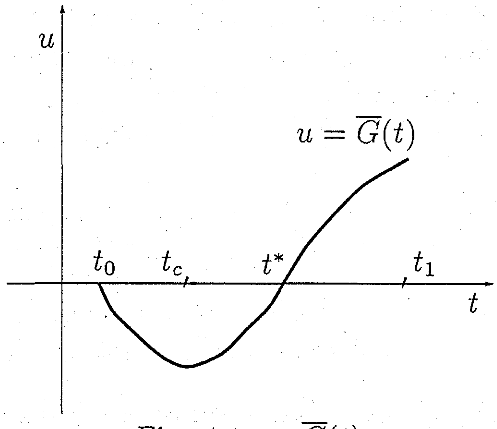

Fig. 4. $u = \overline{G}(t)$

First we consider the case $u^0 > 0$.

Theorem 2.1 *Assume the hypotheses* (A_1) - (A_5) with (A_3) = (A_3^+) *to be valid. Then for sufficiently small* ε *and* $u^0 > 0$ *there exists a unique solution of* (1.1), (1.2) *satisfying*

$$\lim_{\varepsilon \to 0} u(t,\varepsilon) = \begin{cases} 0 & \textit{for} \quad t \in (t_0, t^*) \\ \psi_+(t) & \textit{for} \quad t \in (t^*, t_1]. \end{cases}$$

Proof. To establish this result we apply the method of upper and lower solutions. The functions $\overline{u}(t,\varepsilon)$ and $\underline{u}(t,\varepsilon)$ are called upper and lower solutions of (1.1), (1.2), resp., on the interval $(t_0, \tilde{t})$, if they satisfy for $t = t_0$

$$\underline{u}(t_0,\varepsilon) = \underline{u}^0 \leq u^0 \leq \overline{u}^0 = \overline{u}(t_0,\varepsilon),$$

and for $t_0 < t < \tilde{t}$

$$\begin{aligned} \varepsilon \frac{d\overline{u}}{dt} &\geq g(\overline{u},t,\varepsilon), \\ \varepsilon \frac{d\underline{u}}{dt} &\leq g(\underline{u},t,\varepsilon). \end{aligned} \tag{2.3}$$

Under the assumption (A_1), the existence of an upper and lower solution of (1.1), (1.2) on $(t_0, \tilde{t})$ implies the existence of a unique solution of (1.1), (1.2) in $(t_0, \tilde{t})$.

Without loss of generality we may assume $u^0 \leq c_0$ where c_0 has been introduced in (A_5). In case $u^0 > c_0$, according to the assumptions $(A_1), (A_2)$, and $(A_3)^+$, for each arbitrarily small $\tilde{\delta} > 0$ there is a sufficiently small $\tilde{\varepsilon}$ such that the solution $u(t, u^0, \varepsilon)$ of (1.1), (1.2) exists on $(t_0, t_0 + \tilde{\delta})$ for $\varepsilon \in I_{\tilde{\varepsilon}}$ and satisfies $u(t_0 + \tilde{\delta}, u^0, \varepsilon) \leq c_0$ (see [25]).
Now we construct an upper solution $\overline{u}(t, \varepsilon)$ to (1.1), (1.2) on (t_0, t^*) in the form

$$\overline{u}(t, \varepsilon) = u^0 e^{G(t,\varepsilon)/\varepsilon} \tag{2.4}$$

where $G(t, \varepsilon)$ is introduced in (2.1). It is easy to check that $\overline{u}(t, \varepsilon)$ satisfies

$$\varepsilon \frac{d\overline{u}}{dt} = g_u(0, t, \varepsilon)\, \overline{u}, \quad \overline{u}(t_0, \varepsilon) = u^0. \tag{2.5}$$

Let ε_1, $\varepsilon_1 \leq \varepsilon_0$, be a given (small) positive number. Then, by assumption (A_1), there exists a positive number κ such that

$$G(t, \varepsilon)/\varepsilon \leq \overline{G}(t)/\varepsilon + \kappa \quad \text{for} \quad t_0 \leq t \leq t^*,\ \varepsilon \in I_{\varepsilon_1}.$$

Let ν be an arbitrarily small positive number. From the assumptions (A_2) and (A_4) it follows the existence of an $\varepsilon(\nu), 0 < \varepsilon(\nu) \leq \varepsilon_1$, such that

$$\begin{array}{rll} G(t, \varepsilon) \leq 0 & \text{for} & t_0 \leq t \leq t_0 + \nu \\ G(t, \varepsilon)/\varepsilon \leq \overline{G}(t)/\varepsilon + \kappa < 0 & \text{for} & t_0 + \nu \leq t \leq t^* - \nu,\ \varepsilon \in I_{\varepsilon(\nu)}. \end{array} \tag{2.6}$$

By (2.4) and (2.6) we have

$$\overline{u}(t, \varepsilon) \leq c_0 \quad \text{for} \quad t_0 \leq t \leq t^* - \nu,\ \varepsilon \in I_{\varepsilon(\nu)},$$

and by assumption (A_5) it holds for $t_0 \leq t \leq t_c - \nu, \varepsilon \in I_{\varepsilon(\nu)}$

$$\varepsilon \frac{d\overline{u}}{dt} - g(\overline{u}, t, \varepsilon) = g_u(0, t, \varepsilon)\overline{u} - g(\overline{u}, t, \varepsilon) \geq 0.$$

Thus, $\overline{u}(t, \varepsilon)$ satisfies the first inequality in (2.3) and is an upper solution of (1.1), (1.2) on $(t_0, t^* - \nu)$ for $\varepsilon \in I_{\varepsilon(\nu)}$. In the case $u^0 > 0$, assumption (A_2) implies that $u \equiv 0$ is a trivial lower solution of (1.1), (1.2) in (t_0, t_1). Hence (1.1), (1.2) has a unique solution in $(t_0, t^* - \nu)$ for $\varepsilon \in I_{\varepsilon(\nu)}$.
Since ν is any small positive number we get from (2.6)

$$\lim_{\varepsilon \to 0} \overline{u}(t, \varepsilon) = 0 \quad \text{for} \quad t_0 < t < t^*.$$

Therefore, t^* is a lower bound for the time where the solution $u(t, u^0, \varepsilon)$ of (1.1), (1.2) escapes from the unstable solution $u \equiv 0$, that is, t^* yields a lower bound for the delay of the exchange of the stabilities.

To obtain an upper estimate for the escape time we construct a nontrivial lower solution of (1.1), (1.2) on some interval $(t_0, \bar{t})$ with $t^* < \bar{t} < t_1$ in the form

$$\underline{u}(t,\varepsilon,\eta,\delta) = \eta\ e^{[G(t,\varepsilon)-\delta(t-t_0)]/\varepsilon} \tag{2.7}$$

where η and δ are small positive numbers independent of ε. Concerning η we suppose

$$\eta \leq min(u^0, \psi_+(t^*)/2). \tag{2.8}$$

It is obvious that $\underline{u}(t,\varepsilon,\eta,\delta)$ fulfills

$$\begin{aligned} \underline{u}(t_0,\varepsilon,\eta,\delta) &= \eta < u^0, \\ \varepsilon\,\frac{d\underline{u}}{dt} &= (g_u(0,t,\varepsilon)-\delta)\,\underline{u}. \end{aligned} \tag{2.9}$$

Hence, we have

$$\varepsilon\,\frac{d\underline{u}}{dt} - g(\underline{u},t,\varepsilon) = g_u(0,t,\varepsilon)\underline{u} - g(\underline{u},t,\varepsilon) - \delta\underline{u}. \tag{2.10}$$

By (2.2), $\overline{G}(t)$ has a simple zero at $t = t^*$ changing its sign from minus to plus for increasing t. Thus, for sufficiently small positive δ, $\overline{G}(t) - \delta(t-t_0)$ has a simple zero at $t = t^* + \Delta(\delta)$ with $\Delta(\delta) > 0$ and $\Delta(\delta) \to 0$ as $\delta \to 0$. This implies that there is an $\varepsilon_2(\delta), \varepsilon_2(\delta) \leq \varepsilon(\nu)$, such that for $\varepsilon \in I_{\varepsilon_2}$, $G(t,\varepsilon) - \delta(t-t_0)$ has a simple root at $t = \hat{t}(\varepsilon,\delta) := t^* + \Delta(\delta) + \Omega(\varepsilon)$ where $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. We choose $\varepsilon_2(\delta)$ so small that $\Delta(\delta) + \Omega(\varepsilon) > 0$. Hence, we have

$$\underline{u}(t,\varepsilon,\eta,\delta) \leq \eta \quad \text{for} \quad t_0 \leq t \leq \hat{t}(\varepsilon,\delta) \tag{2.11}$$

with $\underline{u}(\hat{t}(\varepsilon,\delta),\varepsilon,\eta,\delta) = \eta$.

From the assumptions (A_1) and (A_2) it follows the existence of a positive constant κ_2 such that for $t \in [t_0,t_1], \varepsilon \in \overline{I}_{\varepsilon_1}, |u| \leq c_0$

$$g_u(0,t,\varepsilon)u - g(u,t,\varepsilon) = g_u(0,t,\varepsilon)u - g(u,t,\varepsilon) - g(0,t,\varepsilon) \leq \kappa_2 u^2. \tag{2.12}$$

If we assume $\eta \leq \delta/\kappa_2$ then we get from (2.12), (2.11) and (2.10)

$$\varepsilon\,\frac{d\underline{u}}{dt} - g(\underline{u},t,\varepsilon) = -\delta\underline{u} + \kappa_2\underline{u}^2 \leq 0,$$

that is, $\underline{u}(t,\varepsilon)$ is a lower solution of (1.1), (1.2) for $t_0 \leq t \leq \hat{t}(\varepsilon,\delta)$.

Let $\overline{\Omega}(\delta)$ be the supremum of $|\Omega(\varepsilon)|$ for $\varepsilon \in I_{\varepsilon_2}(\delta)$. Then, to given sufficiently small δ, $G(t,\varepsilon) - \delta(t-t_0)$ is positive for $t > t^* + \Delta(\delta) + \overline{\Omega} := \bar{t}(\delta)$. To construct a lower solution for $\hat{t}(\varepsilon,\delta) \leq t \leq \bar{t}(\delta)$ we set

$$\underline{u}(t,\varepsilon,\eta,\delta) := \eta \quad \text{for} \quad \hat{t}(\varepsilon,\delta) \leq t \leq \bar{t}(\delta).$$

According to (2.8) and to the assumptions $(A_2), (A_3^+)$ we may choose ε_2 so small that

$$g(\eta,t,\varepsilon) \geq \gamma_0 > 0 \quad \text{for} \quad t \in (t^*,t_1),\ \varepsilon \in I_{\varepsilon_2}$$

such that the second inequality in (2.3) is fulfilled. Therefore, we have

$$\underline{u}(t, \varepsilon, \eta, \delta) \leq \eta \quad \text{for} \quad t \in (t_0, \bar{t}(\delta)),$$
$$u(\bar{t}(\delta), u^0, \varepsilon) \geq \underline{u}(\bar{t}(\delta), \varepsilon, \eta, \delta) = \eta > 0,$$

where η does not depend on ε. Since η lies in the domain of attraction of the equilibrium point $\psi_+(\bar{t}(\delta))$ of the associated equation (1.3) for $t = \bar{t}(\delta)$ we may apply Tichonov's theorem [23, 24] which says that under our hypotheses, for $t \geq \bar{t}(\delta)$, $u(t, u^0, \varepsilon)$ is attracted exponentially by the stable root $y = \psi_+(t)$. As δ does not depend on ε and can be chosen arbitrarily small, the proof of the theorem is complete.

Now we consider a negative initial value u^0 in case of pitchfork bifurcation.

Theorem 2.2 *Assume the hypotheses* (A_1) - (A_5) *with* (A_3) *to be valid. Then for sufficiently small* ε *and* $u^0 < 0$ *there exists a unique solution of* (1.1), (1.2) *satisfying*

$$\lim_{\varepsilon \to 0} u(t, \varepsilon) = \begin{cases} 0 & \textit{for} \quad t \in (t_0, t^*), \\ \psi_-(t) & \textit{for} \quad t \in (t^*, t_1]. \end{cases}$$

Proof. The proof follows the same line as the proof of Theorem 2.2. In the case $u^0 < 0$, the functions

$$\overline{v}(t, \varepsilon) := u^0 e^{G(t,\varepsilon)/\varepsilon} \quad \text{and} \quad \underline{v}(t, \varepsilon) := \eta e^{[G(t,\varepsilon)+\delta(t-t_0)]/\varepsilon}, \quad \eta \leq u^0$$

provide upper and lower solutions of (1.1), (1.2), respectively.

Remark 2.3 *Under the conditions of Theorem 2.1 and Theorem 2.2, the solution of the initial value problem* (1.1), (1.2) *stays near the solution* $u \equiv 0$ *for* $t \in (t_c, t^*)$ *which is an unstable equilibrium point of the associated equation* (1.3) *for that* t*-interval before it exponentially approaches the stable branches of equilibria* $y = \psi_+(t)$ *or* $y = \psi_-(t)$ *resp. of the associated equation* (1.3). *Both theorems describe the phenomenona of delayed exchange of stabilities in singularly perturbed systems.*

Remark 2.4 *If we assume that* g *is* C^2 *in all variables then by combining Theorem 2.1 and Theorem 2.2 with the results in* [24] *we get the following asymptotic behavior of the solution of* (1.1), (1.2)

$$u(t, u^0, \varepsilon) = \begin{cases} 0 + O(\varepsilon) & \textit{for} \quad t \in [t_0 + \nu, t^* - \nu], \\ \psi_\pm(t) + O(\varepsilon) & \textit{for} \quad t \in [t^* + \nu, t_1]. \end{cases}$$

where ν *is any small positive number.*

Finally we study the initial value problem (1.1), (1.2) with negative u^0 in case of transcritical bifurcation.

Theorem 2.5 *Assume the hypotheses* (A_1) - (A_5) with (A_3) = ($A_{3,t}^-$) *to be valid. Additionally we assume that the initial value* u^0 *belongs to the domain of attraction of the solution* $u \equiv 0$ *of the associated equation* (1.3) *for* $t = t_0$. *Then for sufficiently small* ε *and* $u^0 < 0$ *there exists a unique solution of* (1.1), (1.2) *on* (t_0, t^*) *satisfying*

$$\lim_{\varepsilon \to 0} u(t, \varepsilon) = 0 \quad \text{for } t \in (t_0, t^*),$$

for $t > t^*$ *the solution* $u(t, \varepsilon)$ *escapes from* $u \equiv 0$.

Proof. The proof is based on the same upper and lower solutions as in the proof of Theorem 2.2.

Remark 2.6 *Theorem 2.5 describes the phenomenon of delayed loss of stability in singularly perturbed equations.*

3 Examples

We start with the simple example

$$\begin{aligned} \varepsilon \frac{du}{dt} &= u(t-u), \quad t \in (-1, 2), \\ u(-1, \varepsilon) &= u^0. \end{aligned} \tag{3.1}$$

The associated equation

$$\frac{du}{d\tau} = u(t-u)$$

has two families of steady state solutions

$$u = \varphi_1(t) \equiv t \quad \text{and} \quad u = \varphi_2(t) \equiv 0.$$

The right hand side of equation (3.1) fulfills all conditions of Theorem 2.1 such that we have a delayed exchange of stabilities for $u^0 > 0$: for sufficiently small ε and $u^0 > 0$, the solution $u(t, u^0, \varepsilon)$ of (3.1) stays near $u \equiv 0$ for $t \in (-1, 1)$, and it has a transition layer near the point $t = 1$ from the unstable root $u \equiv 0$ to the stable one $u = t$. All these properties can be verified directly since (3.1) is a Riccati equation whose exact solution can be calculated.

Remark 3.1 *By means of the change of variables* $u = t - y$ *we can rewrite equation* (3.1) *in the form*

$$\begin{aligned} \varepsilon \frac{dy}{dt} &= y(y-t) + \varepsilon, \quad t \in (-1, 2), \\ y(-1, \varepsilon) &= y^0 = -1 - u^0. \end{aligned} \tag{3.2}$$

Fig. 5 shows three trajectories of equation (3.2) with $\varepsilon = 0.01$. *The trajectory with the starting point* $P_f = (t_0 = -0.8,\ y_0 = -0.3)$ *exhibits a failure of the exchange of stabilities, while the trajectory with the starting point* $P_d = (t_0 = -0.8,\ y_0 = -1.3)$ *shows a delayed exchange of stabilities. Both trajectories are separated by the trajectory* $y = t$.

If we drop ε *on the right hand side of* (3.2) *we get the equation*

$$\varepsilon \frac{dy}{dt} = y(y-t). \tag{3.3}$$

At the first glance, we would conclude that the behavior of the trajectories of the equations (3.2) *and* (3.3)*is qualitatively the same for small* ε. *But it is easy to check that equation* (3.3) *fulfills all hypotheses of a theorem of N.R. Lebovitz and R.J. Schaar* [14] *if* y^0 *satisfies* $-\infty < y^0 < 0$, *that is,* y^0 *lies in the domain of attraction of the equilibrium point* $y = -1$ *of the associated system*

$$\frac{dy}{d\tau} = y(y-t)$$

for $t = -1$. *According to that theorem there is no delayed exchange of stabilities. Therefore, we can conclude that the mentioned theorem of N.R. Lebovitz and R.J. Schaar* [14] *cannot be applied to systems whose right hand side depends on* ε.

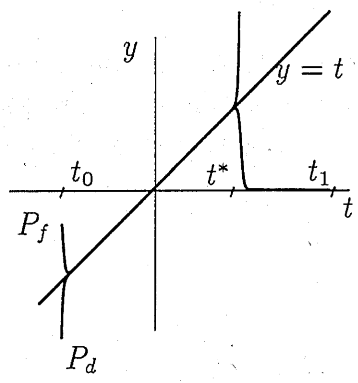

Fig. 5. Failure of exchange and delayed exchange of stabilities in equation (3.2) with $\varepsilon = 0.01$.

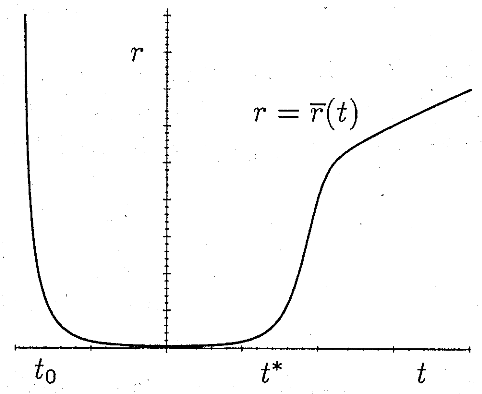

Fig. 6. Delayed Hopf bifurcation in equation (3.8) with $\varepsilon = 0.01, t_0 = -0.7, r_0 = 5$

The following example represents a slightly simplified problem considered by M.A. Shishkova [21] describing the normal form of a generic Hopf bifurcation when the bifurcation parameter is slowly changing in time.

$$\begin{aligned} \frac{dx}{d\tau} &= y + x(\varepsilon\tau - x^2 - y^2), \\ \frac{dy}{d\tau} &= -x + y(\varepsilon\tau - x^2 - y^2) \end{aligned} \tag{3.4}$$

where ε is a small parameter.
Introducing the slow time t by $\varepsilon\tau = t$ we get from (3.4)

$$
\begin{aligned}
\varepsilon \frac{dx}{dt} &= y + x(t - x^2 - y^2), \\
\varepsilon \frac{dy}{dt} &= -x + y(t - x^2 - y^2).
\end{aligned}
\tag{3.5}
$$

By means of polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$ system (3.5) can be rewritten as

$$
\begin{aligned}
\varepsilon \frac{dr}{dt} &= r(t - r^2), \\
\varepsilon \frac{d\varphi}{dt} &= 1.
\end{aligned}
\tag{3.6}
$$

Concerning (3.6) we note that the corresponding associate equation

$$
\frac{dr}{d\bar{\tau}} = r(t - r^2) \tag{3.7}
$$

has two families of steady states

$$
r = \varphi_1(t) \equiv 0,\ t = \psi_2(r) = r^2.
$$

Since we may restrict our investigation to the case $r \geq 0$, we can describe the second family of steady states of (3.7) by $r = \varphi_2(t) \equiv \sqrt{t},\ t \geq 0$. Thus, we are able to apply Theorem 2.1 and get a delayed Hopf bifurcation. If we consider the initial value problem

$$
\begin{aligned}
\varepsilon \frac{dr}{dt} &= r(t - r^2), \\
r(t_0) &= r_0,\ t_0 < 0.
\end{aligned}
\tag{3.8}
$$

then the asymptotic delay t^* satisfies $t^* = -t_0$. Fig. 6 shows the solution of (3.8) with $\varepsilon = 0.07, t_0 = -0.7, r_0 = 5$.

These examples show how efficiently the method of differential inequalities can be applied to study the phenomenon of delayed exchange of stabilities in singularly perturbed equations.

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