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P. Mathé

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Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany



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Herausgegeben vom
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Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
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# ON OPTIMAL RANDOM NETS

PETER MATHÉ

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ABSTRACT. The possibility to approximate bounded linear mappings between Banach spaces depends on the degree of compactness. One way to measure this degree of compactness is the scale of entropy numbers, cf. [CS90]. In the usual (worst-case) setting of numerical analysis this scale has been studied for a long time.

Recent research is concerned with the study of the so-called average-case and randomized (Monte Carlo) settings. We propose the respective counterparts of the entropy numbers in these settings and obtain their behavior for Sobolev embeddings. It turns out that, at least in this situation, randomly chosen nets may not improve the approximability of operators in the Monte Carlo setting. However, we can use the results to improve previous estimates for average Kolmogorov numbers, as obtained in [Mat91].

## 1. INTRODUCTION, GENERAL

The theoretical understanding of the efficiency and optimality of numerical algorithms is widely studied. In recent years emphasis is laid on the average behavior of algorithms and on the investigations of the efficiency of stochastic methods for solving numerical problems approximately. The reader is referred to the monographs [TWW88, Nov88]. A strict formalization of the notion of efficiency and optimality led to $s$-scales or related pseudo-$s$-scales. This is well-known in Approximation Theory, see [Mat90, Pin85] and has been generalized by the author to the so-called average-case and Monte Carlo settings, see [Mat91]. This way it is possible to study model problems as e.g. function approximation using a discretization technique, introduced by Maiorov in 1975, see [Mai75], which is now basic part of the theory of $s$-numbers as developped by Pietsch, cf. [Pie80, Kön86]. It turns out that most classes of approximative numerical methods have respective counterparts within the area of $s$-numbers, see [Mat91] for details.

This note is concerned with the study of the so-called entropy numbers of operators. Although this scale has no immediate interpretation within the language of approximative numerical methods its study turned out to be important, see

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[Pie80, CS90, Kön86]. Moreover, the entropy of compact sets in various spaces has been studied since Kolmogorov and Tichomirov in the late 50s, [KT59]. The author suggests appropriate average and Monte Carlo counterparts to the entropy numbers.

The study of random nets has close connections to stochastic methods for global optimization. Detailed results in this direction can be found in Sukharev [Suk89, Chapter 4.], where quantities related to ours are studied, but in a much more special situation. Also, a special case was used by Maiorov in his recent paper [Mai91], implicitely.

Given a metric space $[Y, \rho]$, an element $y_0 \in Y$ and $\varepsilon > 0$, denote by $B(y_0, \varepsilon) := \{y \in Y, \rho(y, y_0) < \varepsilon\}$ the open $\varepsilon$-ball around $y_0$ in $Y$. Let us introduce the (usual) notion of $\varepsilon$-entropy of a (continuous) mapping $S : [X, \tau] \to [Y, \rho]$. Let $\Phi(Y)$ be the set of all finite subsets of $Y$. Any element $F \in \Phi(Y)$ is called a net. Given $F \in \Phi(Y)$ define

$$\begin{aligned} \varepsilon(S, F, Y) &:= \inf\left\{\varepsilon > 0, \quad S(X) \subset \bigcup_{y \in F} B(y, \varepsilon)\right\} \\ ( &= \inf\left\{\varepsilon > 0, \quad \sup_{x \in X} \inf_{y \in F} \rho(S(x), y) < \varepsilon\right\}) \end{aligned}$$

and for $n \in \mathbb{N}$

$$\varepsilon_n(S, Y) := \inf\{\varepsilon(S, F, Y), \quad \operatorname{card}(F) \le n\}.$$

It is the aim of this note to introduce the respective notion in the average and Monte Carlo settings, too, derive some general results and give applications to the entropy behavior of Sobolev embeddings.

As a consequence we are able to improve previous estimates, given in Mathé, [Mat91, §3, Corollary 4., §4, Corollary 2.], showing, that the logarithmic factor is superflous.

Let us now turn to the description of the average and Monte Carlo $\varepsilon$-entropy. By a random net of a subset in $Y$ we shall understand a randomly chosen set of points. It is clear that emphasis is laid on nets, where the number of points is finite. To make this precise we shall introduce the necessary notation.

Given a metric space $[Y, \rho]$ denote by $\mathfrak{B}(Y)$ the Borel $\sigma$-algebra. The set $\Phi(Y)$ of all finite subsets of $Y$ will be given the $\sigma$-algebra $\mathfrak{B}_\Phi(Y)$, generated by all sets

$$\{F, \quad \operatorname{card}(F \cap A) < t\}, \quad A \in \mathfrak{B}(Y), \quad t \in \mathbb{R},$$

which is the smallest $\sigma$-algebra, such that the mappings $F \to \operatorname{card}(F \cap A)$ are measurable for all sets $A \in \mathfrak{B}(Y)$.

Let be given a metric space $[Y, \rho]$.

**Definition 1.** A random net on $Y$ is a probability $P$ on $[\Phi(Y), \mathfrak{B}_\Phi(Y)]$

*Remark.* It is clear, that every net $F \in \Phi(Y)$ can be identified with a simple counting measure $\tilde{\Phi}$ of the form $\tilde{\Phi} = \sum_{y \in F} \delta_y$, see [MKM78, 1.2]. Using this identification we could equivalently speak of random nets as point processes on phase space $[Y, \rho]$, see again [MKM78, 1.2] for details. Observe, that the cardinality of realizations $F$ may vary.

To measure the quality of a random net $P$ on $Y$ we need some measurability result.

**Lemma 1.**

(1) *For every $x \in Y$ the mapping $F \to \inf_{y \in F} \rho(x,y)$ is $(\mathfrak{B}_\Phi(Y), \mathfrak{B}(\mathbb{R}))$-measurable.*
(2) *The cardinality $F \to \mathrm{card}(F)$ is $(\mathfrak{B}_\Phi(Y), \mathfrak{B}(\mathbb{R}))$-measurable.*

Proof: Item (2) follows immediately from the definition of the $\sigma$-algebras, while (1) follows from the simple observation

$$\{F \in \Phi(Y), \inf\{\rho(x,y),\ y \in F\} < t\} = \{F \in \Phi(Y), \mathrm{card}(F \cap B(x,t)) \geq 1\}. \quad \square$$

Given a random net $P$ on $Y$ and a mapping $S : X \to Y$ let

$$\varepsilon(S,P,Y) := \sup\left\{\int \inf\{\rho(S(x),y), y \in F\}\, dP(F), \quad x \in S\right\}$$

and

$$MC - \mathrm{card}(P) := \int \mathrm{card}(F) dP(F).$$

As above we continue to introduce the Monte Carlo $\varepsilon$-entropy as

$$\varepsilon_n^{mc}(S,Y) := \inf\{\varepsilon(S,P,Y), \quad MC - \mathrm{card}(P) \leq n\}.$$

The average counterpart can be obtained, giving any Borel probability $\mu$ on $X$ and net $F \in \Phi(Y)$ by

$$\begin{aligned}
\varepsilon(S,F,\mu) &:= \int \inf_{y \in F} \rho(S(x),y) d\mu(x), \\
\varepsilon_n(S,\mu) &:= \inf\{\varepsilon(S,F,\mu), \quad \mathrm{card}(F) \leq n\}, \\
&\text{and} \\
\varepsilon_n^{avg}(S,Y) &:= \sup\{\varepsilon_n(S,\mu), \quad \mu(X) = 1\}.
\end{aligned}$$

Let us illustrate these quantities by a simple (discrete) example.

**Example:** Let $M$, $\mathrm{card}(M) = m$ be a finite set, considered as metric space with metric $\rho(x,y) = 1$ if $x \neq y$. $S : M \to M$ shall be the identity. It is easy to see, that

$$\varepsilon_n(S,M) = \begin{cases} 1 & n < m \\ 0 & n \geq m \end{cases},$$

while it is a good exercise to check

$$\varepsilon_n^{mc}(S,M) = \varepsilon_n^{avg}(S,M) = \max\left\{0, \frac{m-n}{m}\right\}.$$

This can be achieved by a uniform choice of $n$ distinct points from $M$ and uniform distribution on $M$, respectively. Another way of choosing a net of $n$ points on the average by chance is to choose one single point with probability $p = \frac{n-1}{m-1}$ and the whole set $M$ with probability $1-p$. Thus the MC-cardinality$(P)$ of such a random net $P$ is $= n$ while the $\varepsilon$-entropy $\varepsilon_n(S,P,M) = \frac{m-n}{m-1}$. In either case we can observe that the average and Monte Carlo counterparts yield better behavior than $\varepsilon_n(S,M)$.

Our main interest lies in the consideration of entropy numbers for linear operators acting between (real) Banach spaces. Let $X, Y$ be real Banach spaces and denote by $L(X,Y)$ the set of all bounded linear operators acting between $X$ and $Y$. Denote by $B_X$ the closed unit ball in $X$. In case of linear spaces $X$ and $Y$ it is more appropriate to have a logarithmic scale. Thus let us define the usual entropy numbers of a linear operator $S \in L(X,Y)$, cf. [Pie80], at this place.

$$e_n(S) := \inf \left\{ \varepsilon > 0, \text{ there are } q \leq 2^{n-1}, y_1, \ldots, y_q \in Y \text{ with } S(B_X) \subset \bigcup_{j=1}^{q} B(y_j, \varepsilon) \right\}.$$

With our notation this means $e_n(S) = \varepsilon_{2^{n-1}}(S,Y)$. The same way we define the $n$th Monte Carlo entropy number $e_n^{mc}(S) := \varepsilon_{2^{n-1}}^{mc}(S,Y)$, average entropy number $e_n^{avg}(S) := \varepsilon_{2^{n-1}}^{avg}(S,Y)$ and $e_n(S,\mu) = \varepsilon_{2^{n-1}}(S,\mu)$.

It is immediate from these definitions, that we have $e_n^{mc}(S) \leq e_n(S)$.

Since the function $(x,F) \in B_X \times \Phi(Y) \to \inf_{y \in F} \|Sx - y\|_Y \in \mathbb{R}^+$ may not be product measurable the following representation is helpful.

**Lemma 2.** *Let $X$ be a separable Banach space. Then we have for any operator $S \in L(X,Y)$*

$$e_n^{avg}(S) = \sup \{ e_n(S,\mu), \quad \mu(B_X) = 1, \quad \mu \textit{ discrete} \}.$$

Proof: It suffices to prove the right-hand side to be an upper bound for $e_n^{avg}(S)$. Let $\mu$ be any probability on $B_X$ and $F$ be any net on $Y$. The function

$$x \in B_X \to \inf \{ \|Sx - y\|_Y, \quad y \in F \} \in \mathbb{R}^+$$

is bounded and continuous on $B_X$, such that for any sequence of measures $\mu_n$, converging to $\mu$ in the weak topology we have

$$\int \inf \{ \|Sx - y\|_Y, \quad y \in F \} \, d\mu_n(x) \to \int \inf \{ \|Sx - y\|_Y, \quad y \in F \} \, d\mu(x).$$

Since the set of discrete probabilities is dense among all probabilities on $B_X$, see [Bil68, App. III, Thm.4] the proof can be finished. □

As a consequence we obtain the following result, relating the Monte Carlo entropy numbers to the average counterpart, in a standard manner, see [HM92, Mat91].

**Corollary 1.** *Let $X$ be a separable Banach space, $S \in L(X,Y)$ be a bounded linear operator. Then we have*

$$e_n^{mc}(S) \geq \frac{1}{2} e_{n+1}^{avg}(S).$$

The proof can be carried out as in the papers cited above, applying Fubini's Theorem only for discrete measures on $B_X$, which is always possible. An application of the above Lemma 2 gives the desired inequality. Under the above assumptions we thus have $\frac{1}{2} e_{n+1}^{avg}(S) \leq e_n^{mc}(S) \leq e_n(S)$.

The following observation is useful in order to apply the discretization technique. Given any operator $S \in L(X,Y)$, Banach spaces $X_0, Y_0$ and operators $T \in L(X_0, X), R \in L(Y, Y_0)$ we have

$$e_n^{avg}(RST) \leq \|R\| e_n^{avg}(S) \|T\|.$$

In fact all scales $e_n, e_n^{avg}$ and $e_n^{mc}$ form pseudo-$s$-scales, which can be proven easily, see [Mat91].

Let us close this section with the following estimate, an analogue of [Pie80, Thm. 12.3.3]. Such kind of argument is also contained in [Mai91]. It is necessary to have some further average $s$-scale, the scale of the average Kolmogorov numbers, see [Mat90, Part B, p.59] and [Mat91]. Given any linear operator $S \in L(X,Y)$, any probability $\mu$ on $B_X$ let

$$d_n(S,\mu) := \inf \left\{ \int \inf_{y \in L} \|Sx - y\|_Y d\mu(x), L \text{ is a subspace in } Y \text{ with } \dim(L) < n \right\},$$

and

$$d_n^{avg}(S) := \sup \left\{ d_n(S,\mu), \quad \mu(B_X) = 1 \right\}.$$

**Theorem 1.** *Let $S \in L(X,Y)$ be any linear operator, $\mu$ be any probability on $B_X$ and let $1 \leq 2n \leq m$. Then we have*

$$e_m(S,\mu) \leq d_n(S,\mu) + 32 \cdot 2^{-\frac{m-1}{n-1}} \int \|Sx\|_Y d\mu(x),$$

*and consequently*

$$e_m^{avg}(S) \leq d_n^{avg}(S) + 32 \cdot 2^{-\frac{m-1}{n-1}} d_1^{avg}(S).$$

Proof: Fix $\varepsilon > 0$ and choose a subspace $L_n \subset Y$ with $\dim(L_n) < n$ such that $d_n(S,\mu) \leq (1+\varepsilon) \int \inf_{y \in L_n} \|Sx - y\|_Y d\mu(x)$. Let $\alpha := 2 \int \|Sx\|_Y d\mu(x)$ and $G := 2\alpha B_Y \cap L_n$. A volume argument, cf. [Pie80, 12.1.13] yields a $\delta$-net $G_\delta \subset Y$ of $G$ with cardinality $\operatorname{card}(G_\delta) \leq (1 + \frac{4\alpha}{\delta})^{n-1}$. Using this net $G_\delta$ we obtain

$$\begin{aligned} \varepsilon(S, G_\delta, \mu) &= \int \inf_{y \in G_\delta} \|Sx - y\|_Y d\mu(x) \\ &\leq \int \inf_{y \in G_\delta} \inf_{z \in G} \{\|Sx - z\|_Y + \|z - y\|_Y\} d\mu(x) \\ &\leq \int \inf_{z \in G} \|Sx - z\|_Y d\mu(x) + \delta. \end{aligned}$$

Putting $m := \lceil \log_2 \operatorname{card}(G_\delta) \rceil + 1$ we have $\operatorname{card}(G_\delta) \leq 2^{m-1}$, hence

$$e_m(S,\mu) \leq (1+\varepsilon) d_n(S,\mu) + \delta.$$

By the definition of $m$ we obtain

$$\frac{1}{2} 2^{m-1} \leq \operatorname{card}(G_\delta) \leq \left(1 + \frac{4\alpha}{\delta}\right)^{n-1},$$

which implies $\delta \leq 16\alpha 2^{-\frac{m-1}{n-1}}$, finishing the proof. □

## 2. The entropy of Sobolev embeddings

In this section we are going to study the behavior of the entropy numbers of Sobolev embeddings in different settings. Let $1 \leq p \leq \infty$, $\quad r,s \in \mathbb{N}$, $\quad r,s \geq 1$ and let $W_p^r([0,1]^s)$ denote the Sobolev spaces on $[0,1]^s$, see [Tri78]. If $1 \leq q \leq \infty$, and the relation $\frac{r}{s} > \frac{1}{p} - \frac{1}{q}$ is valid, then the canonical embedding $I_{r,p,q} : W_p^r([0,1]^s) \to L_q([0,1]^s)$ exists and is compact. The behavior of the (usual) entropy numbers in these cases is known, cf. [Kön86, prop. 3.c.9]:

**Theorem 2.** *Let* $1 \leq p,q \leq \infty$, $\quad r,s \in \mathbb{N}$ *If* $\frac{r}{s} > \frac{1}{p} - \frac{1}{q}$, *then*

$$e_n\left(I_{r,p,q}\right) \asymp n^{-r/s}.$$

We will rely on this result later. Such estimates are obtained by a discretization technique, allowing to reduce the problem to finite dimensional approximation problems. Lower bounds are obtained especially easy, see [Pin85, Ch. VII, Th. 2.1]. Thus we are led to consider the entropy numbers of the following finite dimensional mappings $I_{p,q}^m : l_p^m \to l_q^m$. Given $1 \leq p \leq \infty$ denote by $B_p^m := \{x \in \mathbb{R}^m, \quad \|x\|_p \leq 1\}$ the closed unit ball in $l_p^m$ and $\lambda_p^m$ the Lebesgue measure (normalized) on $B_p^m$.

**Lemma 3.** *For all* $m,n \in \mathbb{N}$ *we have*

$$e_n^{avg}\left(I_{\infty,1}^m\right) \geq \frac{m}{2e}2^{-n/m}.$$

Proof: Choose any net $F$ on $\mathbb{R}^m$ with $\mathrm{card}(F) \leq 2^{n-1}$ and let $\varepsilon := \varepsilon(I_{\infty,1}^m, F, \lambda_\infty^m)$. An application of Chebyshev's inequality yields

$$\lambda_\infty^m\left(\{x, \quad \inf\{\|x-y\|_1, \quad y \in F\} \leq 2\varepsilon\}\right) \geq \frac{1}{2},$$

hence

$$\begin{aligned} \frac{1}{2} &\leq \lambda_\infty^m\left(\{x, \quad \inf\{\|x-y\|_1, \quad y \in F\} \leq 2\varepsilon\}\right) \\ &\leq \sum_{y \in F} \lambda_\infty^m\left(y + 2\varepsilon B_1^m\right) \\ &\leq 2^{n-1}(2\varepsilon)^m \frac{\mathrm{vol}(B_1^m)}{\mathrm{vol}(B_\infty^m)} \\ &= 2^{n+m-1}\varepsilon^m\left(m!\right)^{-1} \end{aligned}$$

Using the well-known estimate $m! \geq \left(\frac{m}{e}\right)^m$ we conclude

$$\varepsilon \geq \frac{1}{2}\left(m!\right)^{1/m} 2^{-n/m} \geq \frac{m}{2e}2^{-n/m} \tag{1}$$

and consequently $e_n\left(I_{\infty,1}^m, \lambda_\infty^m\right) \geq \frac{m}{2e}2^{-n/m}$. □

The following result is a slight refinement of an estimate by [Mai91].

**Corollary 2.** *For $1 \leq p, q \leq \infty$ and $n \in \mathbb{N}$ we have*

$$e_n^{avg}\left(I_{p,q}^{2n}\right) \asymp n^{1/q-1/p}, \quad n \in \mathbb{N}.$$

Proof: The lower estimate is obtained from Lemma 3 and

$$e_n^{avg}\left(I_{\infty,1}^{m}\right) \leq \|I_{\infty,p}^{m}\| e_n^{avg}\left(I_{p,q}^{m}\right) \|I_{q,1}^{m}\|.$$

To prove the upper bound we remind of the fact, that $e_n^{avg}(I_{p,q}^m) \leq e_n(I_{p,q}^m)$. The upper bound in the case $1 \leq q \leq p \leq \infty$ can be estimated by the norm $\|I_{p,q}^m\|$, while the remaining case $1 \leq p < q \leq \infty$ is more complicated, but it has been proven by Höllig, see [Kön86, 3.c.8], that $e_n(I_{1,\infty}^m) \leq C\frac{\log(m/n)+1}{n}$ for some constant $C$ and $\log_2 m \leq n \leq m$. Therefore the result follows from a factorization argument. □

The above estimate also improves an estimate given for the average Kolmogorov numbers, as presented in [Mat91, §3, Corollary 4. (iii)].

**Corollary 3.** *For $1 \leq p \leq \infty$ we have*

$$d_n^{avg}(I_{p,1}^{2n}) \asymp n^{1-1/p}, \quad n \in \mathbb{N}.$$

Proof: It suffices to prove the lower estimate. The same argument as in Corollary 2. provides

$$e_k^{avg}(I_{p,1}^m) \geq \frac{m^{1-1/p}}{2e} 2^{-k/m}, \quad k \in \mathbb{N}.$$

Since $d_1^{avg}(I_{p,1}^m) = \|I_{p,1}^m\| = m^{1-1/p}$, we may apply Theorem 1. to obtain

$$d_{n+1}^{avg}(I_{p,1}^m) \geq \frac{m^{1-1/p}}{2e}\left(2^{-k/m} - 64e2^{-n/(k-1)}\right).$$

Letting $m = 2n = 18k$ we get

$$\begin{aligned} d_n^{avg}(I_{p,1}^{2n}) &\geq d_{n+1}^{avg}(I_{p,1}^{2n}) \\ &\geq \frac{m^{1-1/p}}{2e}\left(2^{-1/18} - 64e2^{-9}\right) \\ &\geq \frac{n^{1-1/p}}{10}, \end{aligned}$$

finishing the proof of the lower bound. □

The above result also leads to a sharp lower bound for the behavior of the Monte Carlo Kolmogorov numbers, presented in [Mat91, §4, Corollary 2. (ii)]. In the case of Sobolev embeddings $I_{r,p,1}$ there was given a lower bound, which was sharp only up to a logarithmic factor. Next we show that this factor is indeed superflous.

The main results can now be stated as follows.

**Theorem 3.** *Let* $1 \le p, q \le \infty$, $r, s \in \mathbb{N}$. *If* $\frac{r}{s} > \frac{1}{p} - \frac{1}{q}$, *then*

(1)

$$e_n^{avg}(I_{r,p,q}) \asymp e_n^{mc}(I_{r,p,q}) \asymp n^{-r/s}.$$

(2) *Moreover*

$$d_n^{avg}(I_{r,p,1}) \asymp d_n^{mc}(I_{r,p,1}) \asymp d_n(I_{r,p,1}) \asymp n^{-r/s}.$$

Proof: The proof is immediate, because of the well-known inequality

$$e_n^{avg}(I_{r,p,q}) \ge c \cdot n^{-r/s+1/p-1/q} e_n^{avg}(I_{p,q}^{2n}),$$

for some $c > 0$, which can be proven for arbitrary pseudo-$s$-scales, cf. Pinkus [Pin85, Ch. VII, Th. 2.1]. Especially, we have an analogous inequality for $d_n^{avg}(I_{r,p,1})$. The upper bound for item (1) follows from Theorem 2. while the upper bound for item (2) was already given in [Mat91]. □

*Concluding Remark.* The above result has shown, that random nets will not improve the entropy behavior of Sobolev embeddings. The same arguments as above will yield the same conclusion for embedding maps between Besov spaces, see [Tri78] for the definition and [Kön86, 3.c.9] for the decrease of the (usual) entropy numbers. Thus the following question arises naturally: Are there operators between Banach spaces, such that the behavior of the Monte Carlo entropy numbers is significantly better than using deterministic nets?

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**25.** S. Seifarth: The discrete spectrum of the Dirac operators on certain symmetric spaces.

**26.** J. Schmeling: Hölder continuity of the holonomy maps for hyperbolic basic sets II.