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Dimension and invertibility of hyperbolic endomorphisms with singularities

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1 Introduction

We introduce a class of endomorphisms which are piecewise smooth and have hyperbolic attractors. This class generalizes the class of piecewise smooth diffeomorphisms with hyperbolic attractors studied by Pesin [9], Sataev [13], and others [1]. Examples in our class are the fat Belykh map, projections of Solenoids onto cross-sections, and crossed horseshoes.

We first develop the stable manifold theory, the existence of SBR measures and the ergodic theory of our class of maps. This theory mostly parallels the invertible case so we only sketch some of the important arguments. We generally follow the outline of [9], details can be found there. Our main results, theorem 5.2 hold in the two dimensional case: if the product of the Lyapunov exponents is less than one the mapping being invertible μ_{SBR}-a.e. on the attractor is equivalent to the Young formula holding. If the mapping is not invertible a.e. we can calculate the defect in the dimension formula 5.4. If the product of the Lyapunov exponents is greater than one then the attractor is two dimensional and the mapping restricted to the attractor is not invertible on a set of positive measure. Finally, we also give an easily checkable sufficient condition for a map to belong to the general class of maps we consider. This condition is easy to check for the systems we were motivated. In particular, in [15] this theorem is applied to fat Belykh maps where the entire picture of everywhere invertibility, invertibility on the attractor, almost everywhere invertibility on the attractor and noninvertibility almost surely is understood. Kaplan and Yorke have conjectured that for a broad class of systems the dimension of the attractor equals the Lyapunov dimension for most maps from the class. The results of [15] imply that for the Belykh family the Kaplan-Yorke conjecture holds for almost all parameter values.

2 Generalized hyperbolic attractors

Let (M, d) be a Riemannian manifold, $K \subset M$ an open, bounded connected subset with compact closure and $N \subset K$ a closed subset. Let $K \backslash N = K^{(1)} \cup ... \cup K^{(r)}$ where the $K^{(i)}$ are open disjoint sets ($r < \infty$) and $f : K\backslash N \to K$. We assume the following conditions (A1) and (A2) hold.

(A1) $f|_{K^{(i)}}$ can be continued to a map $f : \overline{K}^{(i)} \to f(\overline{K}^{(i)})$ which is a C^2–diffeomorphism $i = 1, 2, \dots r$. We sometimes write f_i for $f|_{K^{(i)}}$.

Remark: Here is the main difference to [9] and [13]. They assume the map f to be a C^2-*diffeomorphism* from the open set $K\backslash N$ onto its image $f(K\backslash N)$. In our

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setting the images $f(K^{(i)})$ need not be disjoint. Furthermore let

$$\begin{aligned} N^+ &= N \cup \partial K \\ N^- &= \{y \in K : \exists z \in N^+,\ z_n \in K\backslash N^+ \text{ s.t. } z_n \to z, f(z_n) \to y\}. \end{aligned}$$

Thus N^+ consists of the singularity set and the boundary of K and N^- is the full "preimage" of N^+. Since we are interested in the asymptotic behavior of infinite orbits we have to exclude those points which are eventually mapped onto the singularities or the boundary:

(A2) $\exists C_i > 0, \alpha_i \geq 0,\ i = 1, 2$ such that

$$||D^2 f_x|| \leq C_1\, d(x, N^+)^{-\alpha_1} \qquad \forall x \in K\backslash N$$

$$||D^2 (f_i)_x^{-1}|| \leq C_2\, d(x, N^-)^{-\alpha_2} \quad \forall x \in f(K^{(i)}),\ i = 1, 2, \ldots, r.$$

Let $K^+ = \{x \in K :\ f^n x \notin N^+,\ n = 0, 1, 2, \ldots\}$ and $D = \bigcap_{n\geq 0} f^n(K^+)$. $\Lambda = \overline{D}$ is called the attractor for f.

Obviously, D is invariant: $f(D) = D$. Although no trajectory in K^+ hits the singularities N^+ it may happen that some of them come arbitrarily close to them. This makes it difficult to control their behavior in comparison to nearby points. In order to get more control we define the following filtration:

$\forall \varepsilon > 0,\ l = 1, 2, \ldots$ we define

$$D^+_{\varepsilon,l} = \{z \in K^+ : d(f^n z, N^+) \geq l^{-1} e^{-\varepsilon n},\ n = 0, 1, 2, \cdots\}$$

$$D_{\varepsilon,l} = \hat{D}_{\varepsilon,l} \cap \Lambda$$

$D_{\varepsilon,l}$ is closed (as a subset of Λ) and $D_\varepsilon = \bigcup_{l\geq 1} D_{\varepsilon,l}$ is f–invariant. Λ is called *regular* if $D_\varepsilon \neq \emptyset \quad \forall \varepsilon > 0$ small enough. The main advantage of points in D_ε is that Lyapunov charts exist for all $x \in D_\varepsilon$ [5]. The description of Lyapunov charts will be given in §3. Next we assume hyperbolic structure for our maps. This structure enables us to develop the main tools for constructing "physically" motivated measures.

Denote by $C(z, \alpha, P)$ a cone at point $z \in K$ (here $\alpha > 0$, is a number, P is a subspace in $T_z M$) consisting of all $v \in T_z M$ such that

$$\angle(v, P) \stackrel{\text{def}}{=} \min_{w\in P} \angle(v, w) \leq \alpha.$$

We say that f is *hyperbolic* if there exist constants $C > 0,\ 0 < \lambda < 1$, a function $\alpha(z)$ and two fields of subspaces $P^{(s)}(z), P^{(u)}(z) \subset T_z M$, $\dim P^{(s)}(z) = q$, $\dim P^{(u)}(z) = p - q$ $(p = \dim M)$, $z \in K\backslash N^+$ such that the cones $C^{(s)}(z) = C^{(s)}(z, \alpha(z), P^{(s)}(z))$ and $C^{(u)}(z) = C(z, \alpha(z),\ P^{(u)}(z))$ satisfy the following conditions:

$(C1)$ the angle between $C^{(s)}(z)$ and $C^{(u)}(z)$ is greater than a positive constant for all $z \in K\backslash N^+$, in particular, $C^{(s)}(z) \cap C^{(u)}(z) = 0$;

(C2) $Df\left(C^{(u)}(z)\right) \subset C^{(u)}(f(z))$ for any $z \in K \backslash N^+$ and $Df_i^{-1}\left(C^{(s)}(z)\right) \subset C^{(s)}(f_i^{-1}(z))$ for any $z \in f_i(K \backslash N^+)$ for $i = 1, 2, \ldots r$.

(C3) for any $n > 0$

(a) $||Df^n v|| \geq C^{-1}\lambda^{-n}||v||$for $z \in K^+, v \in C^{(u)}(z)$;

(b) $||Df^n v|| \leq C\lambda^n||v||$for $z \in f^n(K^+), v \in C^{(s)}(z)$.

For $z \in D$ we set

$$E^{(s)}(z) := \bigcap_{n \geq 0} Df^{-n}C^{(s)}(f^n(z)),$$

with the understanding that the backwards branch taken is the one following the orbit of z.

Note that condition (C2) means that any preimage of the stable cone is contained in the stable cone of that particular preimage. Since endomorphisms do not have a uniquely determined past we are not able to define the unstable manifold of a given point nor to use the common definition of the unstable Lyapunov exponent. In fact, Przytycki [11] proved by using inverse limit spaces that for particular examples the uncountability of unstable manifolds at a given point. Thus the corresponding definition for the unstable space

$$E^u(z) := \bigcap_{n \geq 0} Df^n C^{(u)}\left(f^{-n}(z)\right)$$

depends on the particular choice of the preimage paths of z. Nevertheless the hyperbolic structure ensures that each unstable space is contained in the cone $C^u(z)$.

For Axiom A systems there are three equivalent definitions of SBR measures.

(S1) Lebesgue a.e. point is generic for the SBR measure.

(S2) The conditional measures of the SBR measure on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on them.

(S3) The SBR measure is the limit measure for measures that are stationary for suitable small stochastic perturbations.

Measures satisfying (S1) are the ones we "see", those satisfying (S2) have inherited as much as possible geometric properties from the Lebesgue measure as possible, while those satisfying (S3) are the only ones that can be seen in the presence of noise. While it is known for Axiom A attractors all 3 conditions coincide this is not known in general. In fact, one can construct counterexamples. However, for hyperbolic diffeomorphisms with singularities it was proved by Pesin [9] and Sataev [13] that (S1) and (S2) are equivalent. Also there is a variety of literature in which there are proofs for special stochastic perturbations leading to a measure satisfying (S3) having properties (S1) and (S2). Since it is not obvious how to formulate property (S2) in the non-invertible case, we will follow condition (S1) and will give a criterion when it makes sense to formulate condition (S2).

An invariant Borel probability measure μ_{SBR} is called a *SBR* measure if for any measure ν on K which is absolutely continuous with respect to the Riemannian volume:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \tilde{\nu} = \mu_{\text{SBR}}$$

where $\tilde{\nu} := \nu\lfloor W^s\,(\text{supp}\mu_{\text{SBR}})$ and $W^s(\text{supp}\,\mu_{\text{SBR}}) = \{y \in K | d(f^n y, \text{supp}\,\mu_{\text{SBR}}) \to 0\}$ with the understanding that $\tilde{\nu} \equiv 0$ if $\nu(W^s(\text{supp}\,\mu_{\text{SBR}})) = 0$.

Let ν be Riemannian volume on K. Let $\{K_i^{(n)}\}$ be a maximal open set such that $f^n|_{K_i^{(n)}}$ is continuous and $U \subset K$ an open subset. Let

$$(f^*)^n\nu(U) := \nu_n(U) = \sum_i \nu\left(f^{-n}_{K_i^{(n)}}\left[f^n K_i^{(n)} \cap U\right]\right) := \sum_i \nu\left(K_i^{(n)} \cap f^{-n}_{K_i^{(n)}}\, U\right),$$

$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} \nu_k.$$

Let μ_{SBR} be any limit measure of μ_n. The measure μ_{SBR} is clearly supported an Λ and is f–invariant $(\nu_n(f^{-1}U) \equiv \nu_{n+1}(U))$ and a SBR measure.

However, the measure may be distributed on the singularities N^+ or on points with uncontrollable trajectories. To avoid this situation we make the following crucial assumption ([9], [13], [1]):

(A3) $\qquad \exists C > 0, q > 0$ such that $\forall \varepsilon > 0\ \forall n > 0, \quad \nu(f^{-n}U(\varepsilon, N^+)) \leq C\varepsilon^q.$

Definition 2.1 *We say Λ is a* generalized hyperbolic attractor *if Λ is regular and satisfies (A3).*

The following proposition underlines the usefulness of the above assumption. It is proved in [9] and [13]. To give an insight into the meaning of (A3) we repeat the proof which also works in the non-invertible case.

Proposition 2.2 *If Λ is a generalized hyperbolic attractor then $\mu_{\text{SBR}}(D_\varepsilon) = 1$ for all sufficiently small ε.*

Proof: $K\backslash D^+_{\varepsilon,l} \subset \{x \in K | \exists\, m \in \mathbf{Z}^+$ such that $f^m(x) \in U(\frac{1}{l}e^{-\varepsilon m}, N^+)\}$. Thus

$$\begin{aligned}\nu_n(K\backslash D^+_{\varepsilon,l}) &\leq \sum_{m=0}^{\infty} \nu_n\left(f^{-m}U\left(\frac{1}{l}e^{-\varepsilon m}, N^+\right)\right) = \sum_{m=0}^{\infty} \nu\left(f^{-(m+n)}U\left(\frac{1}{l}e^{-\varepsilon m}, N^+\right)\right) \\ &\leq \sum_{m=0}^{\infty} C\left(\frac{1}{l}e^{-\varepsilon m}\right)^q = C \cdot \frac{1}{l^q}\,\frac{1}{1-e^{-\varepsilon q}}.\end{aligned}$$

The right–hand side tends to 0 if l tends to infinity. $\square$

At this stage we define Lyapunov exponents. While in the forward direction (the positive exponent) there are no problems we have to change the definition of the smallest negative exponent because of the non-invertibility of f :

$$\chi_s(x) := \overline{\lim_{n\to\infty}}\ \frac{1}{n}\ln||Df^n(x)|| \quad \text{and} \quad \chi_u(x) := \overline{\lim_{n\to\infty}}\ \frac{1}{n}\ln||(Df^n(x))^{-1}||^{-1}. \tag{1}$$

In the invertible case such a definition of χ_u is equal to the usual definition.

The main idea to get ergodic properties for the class of maps considered is to solve the problem to the invertible case for a related system. For this we have to "unravel" the different preimages of points. One possibility is to use the inverse limit space of the map f (see [11]). But this approach destroys the manifold structure of the set where f lives and we cannot apply Pesin's and Sataev's results. A solenoidal construction is more useful and simple for our purposes. We lift the map f to a map $\hat{f}$ which is invertible. Namely, we set for $I = [0,1], 0 < \tau < r^{-1}$

$$\hat{M} := M \times I,\ \hat{K} := K \times I,\ \hat{K}^{(i)} := K^{(i)} \times I,\ \hat{x} := (x,\omega) \in \hat{M};$$
$$\hat{f}(x,\omega) := \left(fx, \tau\omega + \frac{i-1}{r}\right),\ \text{for}\ x \in K^{(i)},\ \omega \in I,\ i = 1,...,r.$$

Let $\pi : \hat{M} \to M$ be the projection of $\hat{M}$ onto the first coordinate. Then $f\pi = \pi\hat{f}$ and $\hat{f}$ is invertible. In the next section we will briefly survey the results of [9] and [13] where it is proved that an SBR-measure $\hat{\mu}_{\text{SBR}}$ for $\hat{f}$ exists provided that the assumption (A1) - (A3) are fulfilled for the lifted system $\hat{f}$. But this is trivially true if these conditions hold for f. We then set

$$\mu = \hat{\mu}_{\text{SBR}} \circ \pi^{-1}$$

Lemma 2.3 *μ is a SBR-measure for f.*

Remark: Thus we will write μ_{SBR} instead of μ.

Proof. Let $\hat{\nu}$ and ν be the Lebesgue measure on $\hat{K}$ and K, respectively. Then $\nu = \hat{\nu} \circ \pi^{-1}$. Hence, for $U \subset K, U$ open

$$\nu \circ f^{-n}(u) = \hat{\nu} \circ \hat{f}^{-n} \circ \pi^{-1}(U) := \nu_n(U)$$

and

$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1}\nu_k = \frac{1}{n}\sum_{k=1}^{n-1}\hat{\nu}_k \cdot \pi^{-1}, \quad \hat{\nu}_k = \hat{\nu} \cdot \hat{f}^{-n}.$$

By [9] and [13] $\hat{\mu}_{\text{SBR}}$ is a limit point of $\hat{\mu}_n = \frac{1}{2}\sum_{k=0}^{n-1}\hat{\nu}_k$. Consequently, μ_{SBR} is a limit point of $\{\mu_n\}$. $\square$

We want to "project" the ergodic theory for invertible systems to the non-invertible ones. For this goal we have to pay a price: the invertible system is three-dimensional and has two-dimensional stable spaces with different contracting rates!

3 Stable manifolds and ergodic theory

In this section we give a brief overview of the theory of hyperbolic diffeomorphisms with singularities as it was developed in [5], [9], [13] and others and we discuss which parts of this theory "project" to our endomorphism.

Let $\hat{f}$ be as in §2 and assume that hypothesis (A1) and (A2) hold. Let $\hat{D}^+_{\varepsilon,l} = \pi^{-1}(D^+_{\varepsilon,l})$, $\hat{D}^+_\varepsilon = \pi^{-1}(D^+_\varepsilon)$. Since $\hat{f}$ is invertible there is no confusion about preimages and we are also interested in having control over inverse map. We define

$$\hat{N}^+ := N^+ \times [0,1];\ \hat{N}^- := N^- \times [0,1];$$
$$\hat{D}^-_{\Delta,l,n} := \left\{ \hat{x} \in \hat{Q} \middle| \exists \hat{f}^{-k}(\hat{x}) \text{ and } d(\hat{f}^{-k}(\hat{x}), N^+) \geq \frac{1}{l} e^{-\Delta k}, k = 0, 1, \ldots, n \right\};$$
$$\hat{D}^-_{\Delta,l} := \left\{ \hat{x} \in \hat{D} \middle| d(\hat{f}^{-n}(\hat{x}), \hat{N}^+) \geq \frac{1}{l} e^{-\Delta n} \right\}.$$

Then

$$\hat{D}^-_{\Delta,l} = \bigcap_{n\geq 0} \hat{D}^-_{\Delta,l,n}.$$

Let us assume that $\hat{f}$ has a hyperbolic structure with cone fields $\hat{C}^{(s)}$ and $\hat{C}^{(u)}$ defined as in §2. We now have a unique unstable space

$$\hat{E}^{(u)}(\hat{z}) = \bigcap_{n\geq 0} D\hat{f}^n \hat{C}^{(u)}(\hat{z}) \left(\hat{f}^{-n}(\hat{z})\right) \quad \hat{z} \in \hat{D}.$$

We note that if $\{C(z,\alpha,P) = C^{(s)}(z)\}$ is a stable cone-field for f then

$$\hat{C}\left((z,w)_{,\alpha,} P \times I\right) = \left\{ v \in T_{(z,w)}\hat{M} \middle| \angle(v, P \times I) \leq \alpha \right\}$$

creates a stable cone field for $\hat{f}$.

Let $\hat{E}^{(ss)}(\hat{z})$, $\hat{z} = (x_0, w_0)$ be the tangent space to the vertical line (x, w_0) at (x_0, w_0). If $\ln \tau < \chi_s(x_0)$ then $\hat{E}^{(ss)}(\hat{z})$ is the strongest stable space for $\hat{f}$. Since f (respectively $\hat{f}$) is a local diffeomorphism we always can choose $\ln \tau < \lambda < \inf_{x\in K^+} \chi_s(x)$.

Definition 3.1 *We call a point $\hat{z} = (x, w) \in \hat{K}^+$ regular if there exist numbers $\chi_s(\hat{z}) < 0 < \chi_u(\hat{z})$ such that*

(i) $\lim_{n\to+\infty} \frac{1}{n} \log ||D\hat{f}^n_{\hat{z}} v|| = \chi_s(\hat{z})$ *for* $v \in \hat{E}^{(s)}(\hat{z}) \backslash \hat{E}^{(ss)}(\hat{z})$

(ii) $\lim_{n\to\pm\infty} \frac{1}{n} \log ||D\hat{f}^n_{\hat{z}} v|| = \chi_u(\hat{z})$ *for* $v \in \hat{E}^{(u)}(\hat{z})$.

Remark: For $\hat{z} \in \hat{D}$ we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log ||D\hat{f}_{\hat{z}}^n v|| = \ln \tau = \chi_{ss} \quad \text{for } v \in \hat{E}^{(ss)}(\hat{z}).$$

For regular points the numbers $\chi_s(\hat{z})$ and $\chi_u(\hat{z})$ coincide with the Lyapunov exponents $\chi_s(x)$ and $\chi_u(x)$ for the projected non-invertible system provided $\ln \tau < \chi_s(x)$. If the set N^+ is empty - i.e. f is a diffeomorphism of a compact manifold - and μ is an invariant measure, Oseledec' theorem tells us that μ-almost every point is regular. In general this statement is no longer true in the presence of singularities. Nevertheless one can prove an analogous statement for the set $\hat{D}_\varepsilon$ [5]. More precisely let $x = (x, , x_2, x_3) \in \mathbb{R}^3, |x| = \max |x_i|$ and $R(\rho) = \{x \in \mathbb{R}^3 |\ |x| \leq \rho\}$. We fix $\varepsilon > 0$ small. Then for δ sufficiently small there exists a measurable function $r : \hat{D}_\varepsilon \to (1, \infty)$ with $r(\hat{f}^{\pm}\hat{z}) \leq e^\varepsilon r(\hat{z})$ and an embedding $\Phi_{\hat{z}} : R(r(\hat{z})^{-1}) \to M$ for each $\hat{z} \in \hat{D}_\varepsilon$ such that the following conditions hold:

(i) $\Phi_{\hat{z}} 0 = \hat{z}$ and $D\Phi_{\hat{z}}(0)$ maps $\mathbb{R} \times \{0\} \times \{0\}, \{0\} \times \mathbb{R} \times \mathbb{R}, \{0\} \times \{0\} \times \mathbb{R}$ to $\hat{E}^u(\hat{z}), \hat{E}^s(\hat{z})$ and $\hat{E}^{ss}(\hat{z})$, respectively;

(ii) $\exp_{\hat{z}}^{-1} \circ \Phi_{\hat{z}}$ coincides with $D\Phi_{\hat{z}}(0)$ on $R(r(\hat{z})^{-1})$;

(iii) For $\tilde{f}_z = \Phi_{\hat{f}_{\hat{z}}}^{-1} \circ \hat{f} \circ \Phi_{\hat{z}}$ and $u \in \{\mathbb{R}\} \times \{0\} \times \{0\}, v \in \{0\} \times \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \{0\} \times \mathbb{R}, w \in \{0\} \times \{0\} \times \mathbb{R}$

$$\begin{aligned} e^{\chi_u - \delta}|u| &\leq |D\tilde{f}_{\hat{z}}(0)u| \leq e^{\chi_u + \delta}|u|, \\ e^{\chi_s - \delta}|v| &\leq |D\tilde{f}_{\hat{z}}(0)v| \leq e^{\chi_s + \delta}|v|, \\ e^{\chi_{ss} - \delta}|w| &\leq |D\tilde{f}_{\hat{z}}(0)w| \leq e^{\chi_{ss} + \delta}|w|; \end{aligned}$$

(iv) For the Lipschitz constants L hold

$$\begin{aligned} &L\left(\tilde{f}_{\hat{z}} - D\tilde{f}_{\hat{z}}(0)\right) \leq \varepsilon \\ &L\left(\tilde{f}_{\hat{z}}^{-1} - D\tilde{f}_{\hat{z}}^{-1}(0)\right) \leq \varepsilon \quad \text{and} \\ &L(D\tilde{f}_{\hat{z}}) \leq r(\hat{z}), \quad L(D\tilde{f}_{\hat{z}})^{-1} \leq r(\hat{z}); \end{aligned}$$

(v) For all $\hat{y}, \hat{y}' \in R(r(\hat{z})^{-1})$

$$K^{-1} d(\Phi_{\hat{z}}\hat{y}, \Phi_{\hat{z}}\hat{y}') \leq |\hat{y} - \hat{y}'| \leq r(\hat{z}) d(\Phi_{\hat{z}}\hat{y}, \Phi_{\hat{z}}\hat{y}')$$

for some constant K.

The system of local charts $\{\Phi_{\hat{z}}, \hat{z} \in \hat{D}_\varepsilon\}$ is called the Lyapunov chart system.

Remark: In the situation of diffeomorphisms with an invariant probability measure with compact support Lyapunov charts exist for a.e. point.

We described the notion of Lyapunov charts in detail because they are the main tool in developing stable and unstable manifold theory. Lyapunov charts give control over stretching and contracting in the first step of iterating $\hat{f}$ while Lyapunov exponents are effective only asymptotically. Unfortunately, we have only little control over the size of the charts and in general the metric in the charts derived from the Euclidean metric is only measurable. Also we have to pay an exponentially small error in the exact stretching and contracting exponents. However it is possible to show the following stable manifold theorem.
For $C > 0, \delta > 0, \hat{z} \in \hat{D}_\varepsilon$ let

$$\begin{aligned} \hat{W}^u(\hat{z}) &= \left\{ \hat{y} \in \hat{\Lambda} \middle| d(\hat{f}^{-n}\hat{y}, \hat{f}^{-n}\hat{z}) \le C \cdot e^{-\chi_u(\hat{z})n},\ n = 0, 1, \dots \right\}, \\ \hat{W}^u_\delta(\hat{z}) &= \left\{ \hat{y} \in \hat{\Lambda} \middle| d(\hat{f}^{-n}\hat{y}, \hat{f}^{-n}\hat{z}) \le \delta,\ n = 0, 1, \dots \right\}, \\ \hat{W}^s(\hat{z}) &= \left\{ \hat{y} \in \hat{K}^+ \middle| d(\hat{f}^{n}\hat{y}, \hat{f}^{n}\hat{z}) \le C \cdot e^{\chi_s(\hat{z})n},\ n = 0, 1, \dots \right\}, \\ \hat{W}^s_\delta(\hat{z}) &= \left\{ \hat{y} \in \hat{K}^+ \middle| d(\hat{f}^{n}\hat{y}, \hat{f}^{n}\hat{z}) \le \delta,\ n = 0, 1, \dots \right\}. \end{aligned}$$

Theorem 3.2 (**[9]**, **[13]**) *Let $\hat{f}$ satisfy (A1) - (A3). Then the following assertions hold for $\hat{z} \in \hat{D}_{\varepsilon,l}$ and some $C > 0$*

(i) *There is a $\delta_0(\varepsilon, l)$ such that for $0 < \delta < \delta_0(\varepsilon, l)$ the sets $\hat{W}^u_\delta(\hat{z})$ and $\hat{W}^s_\delta(\hat{z})$ are embedded disks (if dimM=2 then $\dim \hat{W}^u_\delta = 1, \dim \hat{W}^s_\delta = 2$).*

(ii) *For $\hat{y} \in \hat{W}^u_\delta(\hat{z})\ (\hat{W}^s_\delta(\hat{z}))$ the tangent space $T_{\hat{y}}\hat{W}^u_\delta(\hat{z})\ (T_{\hat{y}}\hat{W}^s_\delta(\hat{z}))$ coincides with $\hat{E}^u(\hat{y})$*
$(\hat{E}^s(\hat{y}))$.

(iii) *$\hat{W}^u_\delta(\hat{z})(\hat{W}^s_\delta(\hat{z}))$ is a ball of radius δ inside $\hat{W}^u(\hat{z})(\hat{W}^s(\hat{z}))$ with the intrinsic metric.*

(iv) *$\hat{W}^u_\delta(\hat{z})(\hat{W}^s_\delta(\hat{z}))$ depend continuously on $\hat{z}$ on the set $\hat{D}_{\varepsilon,l}$.*

(v) *For δ_1 sufficiently small*

$$\begin{aligned} \hat{f}^{-1}\hat{W}^u_{\delta_1}(\hat{z}) &\subset \hat{W}^u_\delta(\hat{f}^{-1}\hat{z}) \\ \hat{f}\hat{W}^s_{\delta_1}(\hat{z}) &\subset \hat{W}^s_\delta(\hat{f}\,\hat{z}). \end{aligned}$$

We stated these facts because we will use them in the following sections. On the other hand they are the main tools in analyzing hyperbolic systems with singularities. To finish this section we cite two theorems compiled from Pesin [9], which give the existence of SBR-measures $\hat{\mu}_{\text{SBR}}$ for $\hat{f}$ and describe their properties. Throughout to the end of this section we assume that f satisfies conditions (A1) - (A3). Let $M^u_{\hat{f}}$ denote the class of SBR-measures defined in §2.

Theorem 3.3 *$\hat{\mu}_{\rm SBR}$ has absolutely continuous conditional measures on unstable manifolds.*

Theorem 3.4 *There exist sets $\hat{\Lambda}_n \subset \hat{\Lambda}, n = 0, 1, 2, \dots$ and measures $\hat{\mu}_n \in M_{\hat{f}}^{(u)}, n = 1, 2, \dots$ such that:*

(1) $\hat{\Lambda} = \bigcup_{n\geq 0} \hat{\Lambda}_n, \hat{\Lambda}_n \cap \hat{\Lambda}_m = \emptyset$ *for* $n \neq m$, $m,n = 0,1, \dots$;

(2) $\hat{\nu}(\hat{W}^{(s)}(\hat{\Lambda}_n)) > 0$ *for* $n > 0$ *and* $\hat{\nu}(\hat{W}^{(s)}(\hat{\Lambda}_n) \cap \hat{W}^{(s)}(\hat{\Lambda}_m)) = 0$ *for* $n \neq m$, $n, m > 0$ *(where $\hat{\nu}$ denotes the Riemannian volume in $\hat{M}$);*

(3) *for* $n > 0 : \hat{\Lambda}_n \subset \hat{D}, \hat{f}(\hat{\Lambda}_n) = \hat{\Lambda}_n, \hat{\mu}_n(\hat{\Lambda}_n) = 1$, $\hat{f}|\hat{\Lambda}_n$ *is ergodic with respect to* $\hat{\mu}_n$;

(4) *for* $n > 0$: *there exist* $k_n > 0$ *and subset* $\hat{A}_n \subset \hat{\Lambda}_n$ *such that*

   (a) *the sets* $\hat{A}_n^i = \hat{f}^i(\hat{A}_n)$ *are disjoint for* $i = 1, \dots, k_n - 1$ *and* $\hat{A}_n^{k_n} = \hat{A}_n^1$, $\hat{\Lambda}_n = \bigcup_{i=1}^{k_n-1} \hat{A}_n^i$;

   (b) $\hat{f}^{k_n}|\hat{A}_n^1$ *is isomorphic to a Bernoulli automorphism (with respect to $\hat{\mu}_n$);*

(5) *for any* $\hat{\mu} \in \hat{M}_{\hat{f}}^{(u)}$

$$\hat{\mu} = \sum_{n>0} \alpha_n \hat{\mu}_n, \qquad \alpha_n \geq 0, \sum_{n>0} \alpha_n = 1;$$

(6) *for any* $\hat{z} \in \hat{W}^{(s)}(\hat{A}_n)$ $(n > 0)$ *and any continuous function* φ *in* $\hat{M}$ *there exists*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\hat{f}^k(\hat{z})) = \frac{1}{\hat{\mu}(\hat{\Lambda}_n)} \int_{\hat{\Lambda}_n} \varphi d\hat{\mu}.$$

(7) *If $\hat{\nu}$ is a measure in $\hat{K}$ being absolutely continuous with respect to the Riemannian volume and* $\hat{\nu}_n = \hat{\nu}\lfloor \hat{W}^{(s)}(\hat{\Lambda}_n), n > 0$ *then*

$$\lim_{k\to 0} \frac{1}{k} \sum_{i=1}^{k-1} \hat{f}_*^i \hat{\nu}_n = \hat{\mu}_n.$$

(8) *for the metric entropy* $h_{\hat{\mu}(\hat{f}|\hat{\Lambda})}$ *the following formula takes place*

$$h_{\hat{\mu}(\hat{f}|\hat{\Lambda})} = \int_{\hat{\Lambda}} \sum_{i=1}^{s(\hat{x})} \chi_i(\hat{x}) d\hat{\mu}(\hat{x})$$

where $\{\chi_i(\hat{x})\}, i = 1, \dots, s(\hat{x})$ *is the collection of all positive values of the Lyapunov exponents at* $\hat{x}$.

Remark: Since $\hat{\mu}(\hat{\Lambda}) = 1$ for any $\hat{\mu} \in M_{\hat{f}}^{(u)}$ conditions (3) and (5) imply that $\hat{\mu}(\hat{\Lambda}_0) = 0$.

If our original map f would be invertible at least on a set of full measure we could "project" the assertions of theorem 3.3 - 3.4 and would get the analogous results for f rather then $\hat{f}$, directly. On the other hand if we cannot find a set of positive measure on which f is invertible there is nothing we can do about theorem 3.3. For this and many other questions mentioned before it is crucial to know when f is invertible at least when restricted to a set of full measure. The next sections are devoted to this problem.

What concerns theorem 3.4 we can derive its conclusions for f if we knew that

$$\mu_{\text{SBR}}\left(\pi(\hat{A}_i^j) \cap \pi(\hat{A}_{i'}^{j'})\right) = 0 \quad \text{for } (i,j) \neq (i',j'). \tag{2}$$

But this is always true because the projection is along the strong stable manifolds $\hat{W}_1^{(ss)}(\hat{z}) = \hat{W}_1^{(ss)}((x,w)) = \left\{(x,w')\middle|\, w' \in I\right\}$. To see this let φ be a continuous function on $\hat{M}$ which has different integrals on $\hat{\Lambda}_i^j$ and $\hat{\Lambda}_{i'}^{j'}$:

$$\int_{\hat{A}_j^i} \varphi d\hat{\mu}_{\text{SBR}} \neq \int_{\hat{A}_{j'}^{i'}} \varphi d\hat{\mu}_{\text{SBR}}. \tag{3}$$

Such a function exists since $(i,j) \neq (i',j')$. If we assume the contrary of (2) we can find two points $\hat{z}, \hat{z}' \in \hat{\Lambda}$ with $\hat{z} \in \hat{\Lambda}_j^i$, $\hat{z}' \in \hat{\Lambda}_{j'}^{i'}$, and $\pi\hat{z} = \pi\hat{z}'$. Then $d_{\hat{M}}(\hat{f}^n\hat{z}, \hat{f}^n\hat{z}') \leq \tau^n$. This yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\hat{f}^k\hat{z}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\hat{f}^k\hat{z}')$$

and by theorem 3.4 inequality (3) is impossible. Hence, the sets $\hat{\Lambda}_j^i$ are uniquely determined up to measure 0 by their projections under π and we can state the following theorems.

Theorem 3.5 *Theorem 3.4 holds for f and Λ, μ_{SBR} as well.*

Remark: Since $\mu(\Lambda) = 1$ for any $\mu \in M_f^{(u)}$ parts (3) and (5) imply that $\mu(\Lambda_0) = 0$.

4 Preparatory machinery

From now on let us assume that $\dim M = 2$. In this section we want to develop the basic machinery we have to use to prove theorem 5.2 . As we have explained we added a third dimension to overcome the difficulties of non-invertibility. The price we have to pay for it is that we cannot use the dimension theory for two-dimensional systems introduced by Young [18]. Our aim is to use the generalized theory of

Ledrappier and Young [7] which is valid in any finite dimension. This theory gives a relation between Lyapunov-exponents, Hausdorff dimension and entropy of an invariant measure. Unfortunately, we cannot apply this theory directly for two reasons. First their theory is stated for C^2-diffeomorphisms rather than systems with singularities and second their analysis works a priori only within the stable or the unstable foliation of the system. While the first reason is more or less a question of the formal statement of Ledrappier's and Young's results the second is more serious. Thus we give a brief overview of the machinery they use and extract the changes we need for our analysis. In fact, we are going to glue together the tools from [18] and [7] in our special three-dimensional case. We want to get a formula for the Hausdorff dimension for the SBR measure of our original (projected) system. In the case of a diffeomorphism with singularities Young [18] gives the answer. But in the case of an endomorphisms our lifted system has a two-dimensional stable manifold and Young's theory doesn't work any longer, because it relies on the one-dimensionality of the stable and unstable manifolds. Here we could apply [7] but the disadvantage to [18] is that we can calculate only the stable and unstable dimension of the measure and we have no information about the projection. But, fortunately, our three-dimensional lifted system has enough additional structure - both the stable and unstable foliation are Lipschitz - to find a way out.

In this section we give the main definition and results from [7] and produce a measurable partition of our system which is needed for the proof of our main result: theorem 5.2.

Remark: In this paper the results are stated for C^2-diffeomorphisms but they depend only on the existence of Lyapunov charts with standard properties.

As we explained before we are in the situation that the set of regular points - i.e. the points for which Lyapunov charts exist - has full $\hat{\mu}_{\text{SBR}}$-measure. Thus we are able to apply the results of [7]. The basic idea in [7] is to consider conditional measures on the manifolds $\hat{W}_\delta^{(u)}(\hat{x}), \hat{W}_\delta^{(s)}(\hat{x})$ and $\hat{W}_\delta^{(ss)}(\hat{x})$. Unfortunately, the partitions into stable and unstable manifolds is not in general measurable. Thus we have to construct a measurable partition which gives us the information we want to have.

Let $\hat{m}$ be an ergodic measure for $\hat{f}$ with non-zero exponents such that $\hat{\Lambda}$ is regular. Let $\chi_{ss} = \ln\tau < \chi_s = \ln\lambda < 0 < \chi_u = \ln\gamma$ be its Lyapunov exponents. Let us assume that τ is so small that

$$\chi_{ss} < \chi_s - \chi_u.$$

Definition 4.1 (see [7] or [12]) *A partition ξ is measurable iff for m - a.e. x there is a normalized measure m_x^ξ living on the partition element $\xi(x)$ containing x such that for the sub-σ-algebra $\mathcal{B}_\xi$ generated by the partition ξ and A a measurable set the function $x \to m_x^\xi(A)$ is $\mathcal{B}_\xi$ - measurable and $m(A) = \int m_x^\xi(A)dm$.*

Definition 4.2 *The measures m_x^ξ are called the conditional measures of m w.r.t ξ. They are uniquely defined up to a set of measure 0.*

Definition 4.3 *A partition $\hat{\xi}$ is subordinate to the $\hat{W}^{(u)}(\hat{W}^{(s)})$ foliation if for $\hat{m}$.-a.e. $\hat{x}$*

(i) $\hat{\xi}(\hat{x}) \subset \hat{W}^{(u)}(\hat{x})$

(ii) *$\hat{\xi}(\hat{x})$ contains a neighborhood of $\hat{x}$ in $\hat{W}^{(u)}(\hat{x})$*

Remark: The partition into local stable manifolds $\hat{\xi}^{ss}(\hat{x}) = \hat{W}_1^{(ss)}((x,\omega)) = \{x,\omega')|\omega' \in I\}$ is measurable.

Partitions like this were considered by several authors ([17], [6], [7]). Their existence is ensured by:

Proposition 4.4 [7] *There exist measurable partitions $\hat{\xi}^u(\hat{\xi}^s)$ with the following properties:*

(i) *$\hat{\xi}^u(\hat{\xi}^s)$ is subordinate to $\hat{W}^{(u)}(\hat{W}^{(s)})$*

(ii) *$\hat{\xi}^u(\hat{\xi}^s)$ is generating - i.e. $\bigvee_{n=0}^{\infty} \hat{f}^{-n}\hat{\xi}$ ($\bigvee_{n=0}^{\infty} \hat{f}^{n}\hat{\xi}$) is the partition into points.*

Let $\{\hat{m}^{ss}_{\hat{x}}\}, \{\hat{m}^{s}_{\hat{x}}\}, \{\hat{m}^{u}_{\hat{x}}\}$ be fixed versions of conditional measures associated to $\hat{m}$ and $\hat{\xi}^{ss}, \hat{\xi}^s, \hat{\xi}^u$, respectively. We define

$$\underline{h}_i(\hat{x},\varepsilon,\hat{\xi}^i) = \liminf_{n\to\infty} -\tfrac{1}{n}\ln \hat{m}^i_{\hat{x}}\hat{V}^i(\hat{x},n,\varepsilon) \quad \text{and}$$
$$\overline{h}_i(\hat{x},\varepsilon,\hat{\xi}^i) = \limsup_{n\to\infty} -\tfrac{1}{n}\ln \hat{m}^i_{\hat{x}}\hat{V}^i(\hat{x},n,\varepsilon)$$

for $\hat{x} \in \hat{M}, \varepsilon > 0$, $i = ss, s, u$ with $\hat{V}^i(\hat{x},n,\varepsilon) = \left\{\hat{y} \in \hat{W}^i(\hat{x}) | d^i(\hat{f}^k\hat{x}, \hat{f}^k\hat{y}) < \varepsilon, 0 \le k \le n\right\}$ and d^i is the induced metric on $\hat{W}^i$.

The above definitions are motivated by the Shannon-McMillan-Breiman theorem on the pointwise entropy of a measure. They serve as characterizations of entropy type along the stable strong stable and unstable direction, respectively. In [7] these definition are justified by a refinement of the Shannon-McMillan-Breiman theorem. Then ([7], Prop. 7.2.1 and Cor. 7.2.2)

$$\underline{h}_i(\hat{x}) = \lim_{\varepsilon\to 0}\underline{h}_i(\hat{x},\varepsilon,\hat{\xi}^i) = \lim_{\varepsilon\to 0}\overline{h}_i(\hat{x},\varepsilon,\hat{\xi}^i) \equiv h_i, \quad i = ss, s, u$$

for $\hat{m}$ a.e. $\hat{x}$ and

$$h_u = h_m(\hat{f}) = h_s. \tag{4}$$

Our next step is to define the dimension of the measure along the stable, strong stable and unstable direction. Again this is done pointwise. We consider for the d^i-ball $B^i(\hat{x},\varepsilon)$ in $\hat{W}^i(\hat{x})$ centered at $\hat{x}$ of radius ε the quantities ($i = ss, s, u$)

$$\underline{\delta}_i(\hat{x},\hat{\xi}^i) = \liminf_{\varepsilon\to 0}\frac{\log \hat{m}^i_{\hat{x}}B^i(\hat{x},\varepsilon)}{\log\varepsilon} \quad \text{and}$$
$$\overline{\delta}_i(\hat{x},\hat{\xi}^i) = \limsup_{\varepsilon\to 0}\frac{\log \hat{m}^i_{\hat{x}}B^i(\hat{x},\varepsilon)}{\log\varepsilon}.$$

For these dimension values we also have a theorem corresponding to the Shannon-McMillan-Breiman theorem for entropies. Then ([7], Prop. 7.3.1) for $\hat{m}$-a.e. $\hat{x}$

$$\underline{\delta}_i(\hat{x}, \hat{\xi}^i) = \overline{\delta}_i(\hat{x}, \hat{\xi}^i) \equiv \delta_i \tag{5}$$

The main result of [7] is the relationship between these numbers:

$$\begin{aligned} h_{ss} &= -\delta_{ss}\chi_{ss} \\ h_s &= \chi_s(\delta_{ss} - \delta_s) + h_{ss} \\ h_u &= \delta_u \chi_u. \end{aligned} \tag{6}$$

At this stage we fix once for all the conditional measures $\{\hat{m}_{\hat{x}}^{ss}\}$, $\{\hat{m}_{\hat{x}}^{s}\}$, and $\{\hat{m}_{\hat{x}}^{u}\}$ associated to the partitions $\hat{\xi}^{ss}, \hat{\xi}^s$ and $\hat{\xi}^u$, respectively. Now we are ready to construct a finite entropy partitions $\hat{P}^n$ which simulates each $\hat{\xi}^s, \hat{\xi}^{ss}$ and $\hat{\xi}^u$ in the appropriate directions.

We are in the special situation that both foliations $\hat{W}^{(u)}$ and $\hat{W}^{(s)}$ are Lipschitz. This allows to sum up the stable and the unstable dimension ([10]). In fact, we have

Theorem 4.5 *The limit*

$$\delta \equiv \delta(\hat{z}) = \lim_{\varepsilon \to 0} \frac{\log \hat{m} B^{\hat{M}}(\hat{z}, \varepsilon)}{\log \varepsilon}$$

exists $\hat{m}$ *- a.e., ($B^{\hat{M}}$ is the ball in $\hat{M}$) and equals $\delta_u + \delta_s$.*

Now we have collected all facts we need to prove a criterion for the invertibility of the map f on a set of full measure.

5 A criterion for invertibility

As discussed before we have the Young formula for the Hausdorff dimension for invertible maps in two dimensions which says that

$$\dim_H \mu_{\text{SBR}} = h_{\mu_{\text{SBR}}} \left(\frac{1}{\chi_u} - \frac{1}{\chi_s} \right) = 1 - \frac{\chi_u}{\chi_s}.$$

A simple example shows that this is no longer true if the map is not injective. Consider the following map of the square $Q = [-1, 1] \times [-1, 1]$ into itself defined by the formula

$$g(x, y) = \begin{cases} (\frac{1}{2}x, 2y - 1) & y \geq 0 \\ (\frac{1}{2}x, 2y + 1) & y < 0 \end{cases}.$$

Then the SBR-measure is simply the normalized Lebesgue measure on the interval $\{0\} \times [-1, 1]$ and, hence, its dimension is 1. On the other hand, the Young formula gives

$$\dim_H \mu_{\text{SBR}} = \log 2 \left(\frac{1}{\log 2} - \frac{1}{\log \frac{1}{2}} \right) = 2.$$

Other examples for the failure of the Young formula were considered by J. Alexander and J. Yorke [2].
By looking at these examples and knowing that the formula holds for invertible systems we can conjecture that "gluing" together different points breaks down the dimension and vice versa. Since we are considering dimensions of measures this effect should come into account if and only if it happens on a set of positive measure.

Definition 5.1 *We call a map f μ-almost surely invertible iff we can find a measurable set E of full μ-measure such that f restricted to E is invertible.*

The main result of this article is the following criterion:

Theorem 5.2 *Let (A1)-(A3) hold for the map f. Then the following assertions are equivalent:*

i) $\dim_H \mu_{\text{SBR}} = 1 - \frac{\log \gamma}{\log \lambda} = 1 - \frac{\chi_u}{\chi_s}$;

ii) f *is* μ_{SBR} *– a.s.invertible.*

Corollary 5.3 *If $\lambda \cdot \gamma > 1$ then f is not invertible on a set of positive measure.*

Proof. We always have $\dim(\mu_{\text{SBR}}) \leq 2$. But if $\lambda \cdot \gamma > 1$ then $\delta := 1 - \frac{\log \gamma}{\log \lambda} > 2$ and the main theorem 5.2 gives the assertion. □
With a bit more work we can show the following defect formula for the dimension formula.

Theorem 5.4 *Let (A1)-(A3) hold for the map f. Then*

$$\dim_H(\mu_{\text{SBR}}) = 1 - \frac{\log \gamma}{\log \lambda} + \frac{h_{ss}}{\ln \lambda}.$$

We want to derive the criterion from two lemmata which are valid in the set up of the previous section. For this purpose suppose $\hat{m}$ is an ergodic measure for $\hat{f}$ with non-zero exponents such that $\hat{\Lambda}$ is regular. Let $\chi_{ss} = ln\tau < \chi_s < 0 < \chi_u$ be its Lyapunov exponents. Let us assume that τ is so small that

$$\chi_{ss} < \chi_s - \chi_u.$$

The first lemma is a version of the well-known fact that entropy 0 means that a process is deterministic - i.e. the future (past) is determined by the past (future) almost surely. Although our systems have positive entropy the preimages of a given point correspond to the strong stable fibers in the lifted system. In fact, the number of components of $\hat{f}^n(\hat{K}^+) \cap \hat{E}^{(ss)}(\hat{z})$ is the number of preimages of $\hat{z}$ and the partition $\hat{\xi}^{ss}$ generates them.

Lemma 5.5 *There is a set $E \subset M$ of full $\hat{m}\circ\pi^{-1}$-measure such that $f|_E$ is invertible if and only if $h_{ss} = 0$.*

Proof. Let $(\hat{\xi}^{ss})_n^m = \bigvee_{k=n}^m \hat{f}^{-k}\hat{\xi}^{ss}$. The proof uses several standard facts about the entropy of a partition $H(P)$ and the conditional entropy of two partitions $H(P|Q)$ which can be found in [8] for example. In [7] it is shown that

$$h_{ss} = H(\hat{\xi}^{ss}|\hat{f}^{-1}\hat{\xi}^{ss}).$$

But

$$H(\hat{\xi}^{ss}|\hat{f}^{-1}\hat{\xi}^{ss}) = H(\hat{\xi}^{ss}|(\hat{\xi}^{ss})_0^\infty) = H((\hat{\xi}^{ss})_0^n|(\hat{\xi}^{ss})_{n+1}^\infty).$$

Thus

$$h_{ss} = 0 \leftrightarrow \mathfrak{A}\left((\hat{\xi}^{ss})_0^n\right) \subset \mathfrak{A}\left((\hat{\xi}^{ss})_{n+1}^\infty\right) \text{ a.s.} \tag{7}$$

where $\mathfrak{A}(P)$ is the σ-algebra generated by the partition P. But the right hand side of 7 is equivalent to f being invertible almost surely since $\hat{\xi}^{ss}$ is a generator for the preimages. □

The next auxiliary lemma tells us that if $h_{ss} = 0$ the projection π preserves the dimension. Note that the Ledrappier-Young formula tells us that for a.e. $\hat{z} = (x, \omega) \in \hat{M}$ the dimension δ_s is preserved under π provided $h_{ss} = 0$ but we do not know if the summation formula $\delta = \delta_u + \delta_s$ holds for the projected measure. The lemma says that the projection π "commutes" with the summation formula.

Lemma 5.6 *Let $h_{ss} = 0$. Then for a set of positive $\hat{m}$-measure the limit*

$$\dim_{\hat{m}\circ\pi^{-1}}(\pi\hat{x}) := \lim_{\varepsilon\to 0} \frac{\log \hat{m}\circ\pi^{-1}B^M(x,\varepsilon)}{\log\varepsilon}$$

exists (B^M is the ball in M) and

$$\dim_{m\circ\pi^{-1}}(\pi p) = \delta = \delta_u + \delta_s.$$

Remark: The above lemma is a verification of the Eckmann-Ruelle conjecture for 2-dimensional endomorphism. The proof actually does not depend on the dimension 2 but on the existence of the pointwise dimension for the 3-dimensional lift what is ensured by the existence for general 2-dimensional diffeomorphisms. A recent result by Barreira, Pesin and Schmeling [3] gives the existence of the pointwise dimension for the lift in **any** dimension. This together with our method seems to be enough to generalize the Eckmann-Ruelle conjecture to higherdimensional endomorphisms with regular hyperbolic measures.

Another point of view is that theorem 5.2 is a generalization of the Falconer projection theorem to the nonlinear case.

The proof of theorem 5.2 consists of several steps and uses techniques from [7]. The main technical difficulty is to get control over the sizes of partition elements as they are evolved according to $\hat{f}$. This is done in great detail in [7]. It needs a considerable amount of estimations and technicalities. Therefore we do not repeat these arguments but refer to [7] for all details.

The idea for the proof is to construct a partition such that a lot of the partition elements behave nicely w.r.t. the Lyapunov exponents of $\hat{m}$ under the iterates of $\hat{f}$. So we get control over the measure, the sizes of their faces. This will help us to estimate the Hausdorff dimension of the projected measure.

Proof of lemma 5.6. We first define a set $\hat{\Gamma}$ where the dimension and Lyapunov exponents are almost correct after N_0 iterations

$$\hat{\Gamma} = \hat{\Gamma}(N_0, \varepsilon, \Delta, l, c_1) := \{\hat{x} \in \hat{D}_{\Delta,l} : \text{ conditions (I) - (IV) listed below hold for } n \geq N_0\}$$

$$\text{(I)} \qquad c_1^{-1} e^{(\chi_u - \varepsilon)\cdot n} < ||D\hat{f}^n\big|_{\hat{E}^{(u)}(\hat{x})}|| < c_1 e^{(\chi_u + \varepsilon)\cdot n}$$

$$c_1^{-1} e^{(\chi_s - \varepsilon)\cdot n} < ||D\hat{f}^n\big|_{\hat{E}^{(s)}(\hat{x})}|| < c_1 e^{(\chi_s + \varepsilon)\cdot n}$$

where $\Delta < \min(|\chi_{ss}|, \chi_u - \chi_s + \chi_{ss}|)$ and $l \geq 1$, $c_1 > 1$.

We note that additionally for all points $\hat{x} \in \hat{M}$ in the strong stable direction the Lyapunov exponent is the actual contracting exponent:

$$||D\hat{f}^n\big|_{\hat{E}^{(ss)}(\hat{x})}|| = e^{\chi_{ss}\cdot n}.$$

We also want to sieve out all points imitating the pointwise Hausdorff dimension already within scales of large enough radius:

$$\text{(II)} \qquad \frac{\log \hat{m} B^{\hat{M}}(\hat{x}, e^{(\chi_{ss} - \chi_s + \chi_u)\cdot n})}{n(\chi_{ss} - \chi_s + \chi_u)} \geq \delta - \varepsilon$$

$$\text{(III)} \qquad \frac{\log \hat{m}^s_{\hat{x}} B^s(\hat{x}, e^{(\chi_{ss}\cdot n)})}{n \cdot \chi_{ss}} \geq \delta_s + \varepsilon$$

$$\text{(IV)} \qquad \frac{\log \hat{m}^s_{\hat{x}} B^s(\hat{x}, e^{(\chi_{ss} - \chi_s + \chi_u)\cdot n})}{n(\chi_{ss} - \chi_s + \chi_u)} \geq \delta_s - \varepsilon$$

The existence of Lyapunov exponents and (5) imply that for fixed $\varepsilon > 0$ $\hat{m}(\hat{\Gamma}) \to 1$ as $N_0 \to \infty$. The latter fact is a simple consequence of Lusin's theorem.

We are now at the point to construct suitable partitions $\hat{P}^n$ for n large. These partitions will consists of a finite number of squares and, hence, they are measurable and of finite entropy.

Claim 5.7 *Fix $\varepsilon > 0$. Let n be sufficiently large (depending on the parameters involved in the definitions of $\hat{\Gamma}$) then there is a finite partition $\hat{P}^n$ of $\hat{K}^+$ and constants $c_2, \tau > 0$ such that:*

(V) $\hat{P}^n = \{\hat{P}_1^n, \dots, \hat{P}_s^n\}$,

for $1 \le i \le r \le s$ there are points $\hat{y}_i \in \hat{\Gamma}$ with

(VI) $B^s(\hat{y}_i, c_2^{-1} e^{(\chi_{ss}-\varepsilon)n}) \subset \hat{\xi}^s(\hat{y}_i) \cap \hat{P}_i^n \subset B^s(\hat{y}_i, c_2 e^{(\chi_{ss}+\varepsilon)n})$ *and*

(VII) $B^M(\pi \circ \hat{f}^{-n}\hat{y}_i, c_2^{-1} e^{(\chi_{ss}-\chi_s)n}) \times I \subset \hat{f}^{-n}\hat{P}_i^n \subset B^M(\pi \circ \hat{f}^{-n}\hat{y}_i, c_2 e^{(\chi_{ss}-\chi_s)\cdot n}) \times I$, $1 \le i \le r$

(VIII) $\hat{m}\left(\bigcup_{i \le r} \hat{P}_i^n\right) > \tau.$

Proof of the claim. We prove for an arbitrary probability measure in $\mathbb{R}^2$ that we always can find a partition into squares such that the measure of the points located away from the boundary is still large. This ensures that we can inscribe balls around those points of radii bounded away from 0. This will give one of the conditions VIII) or VII). The remaining condition holds simply by the properties of $\hat{\Gamma}$.
Let μ be a probability measure on $\mathbb{R}^2$ and $r > 0$. We construct a partition $\tilde{P}_i$ of M and set $\hat{P}_i = \tilde{P} \times I$ simply by looking at a lattice of boxes with side length $e^{(\chi_{ss}-\chi_s)n}$ (or $ce^{\chi_{ss}n}$). What we must ensure is that a positive measure part is contained inside (in inside a box of 1/2 the side length of $\tilde{P}_i$) of the boxes. For simplicity we do this in $\mathbb{R}^2$. We look at three partitions of $\mathbb{R}^2$ into squares (A, B, C) and show that at least one of them fulfills the above conditions, i.e. the measure is not contained in the 1/2-neighborhood of the boundary. For $r > 0$ we define the partition A as

$$A := \left\{(x, y) \in \mathbb{R}^2 : |x - 3ri| \le \frac{3r}{2} \text{ and } |y - 3rj| \le \frac{3r}{2} \text{ for some } (i, j) \in 3\mathbb{Z}^2\right\}. \tag{8}$$

The 1/2-neighborhood of the boundary of A is then

$$\mathcal{A} := \left\{(x, y) \in \mathbb{R}^2 : |x - 3r\mathbb{Z}| < \frac{r}{2} \text{ or } |y - 3r\mathbb{Z}| < \frac{r}{2}\right\}. \tag{9}$$

The partitions B (respectively C) and the boundary $\mathcal{B}$ (resp. $\mathcal{C}$) are defined analogously with $3r\mathbb{Z}$ replaced by $3r\mathbb{Z} + r$ (resp. $3r\mathbb{Z} + 2r$) in the definition.
We claim that at least one of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ has measure less or equal to $\frac{1}{3}$. We have the following problem of optimization:

$$\mu(\mathcal{A}) = \mu(\mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C})) + \mu(\mathcal{A} \cap \mathcal{B}) + \mu(\mathcal{A} \cap \mathcal{C})$$
$$\mu(\mathcal{B}) = \mu(\mathcal{B} \setminus (\mathcal{A} \cup \mathcal{C})) + \mu(\mathcal{B} \cap \mathcal{A}) + \mu(\mathcal{B} \cap \mathcal{C})$$
$$\mu(\mathcal{C}) = \mu(\mathcal{C} \setminus (\mathcal{A} \cup \mathcal{B})) + \mu(\mathcal{C} \cap \mathcal{A}) + \mu(\mathcal{C} \cap \mathcal{B})$$
$$\mu(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = 0$$
$$\mu(\mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C})) + \mu(\mathcal{B} \setminus (\mathcal{A} \cup \mathcal{C})) + \mu(\mathcal{C} \setminus (\mathcal{A} \cup \mathcal{B})) + \mu(\mathcal{A} \cap \mathcal{B}) + \mu(\mathcal{A} \cap \mathcal{C}) + \mu(\mathcal{B} \cap \mathcal{C}) = 1$$

and we have to find $\min_{\mu} \max(\mu(\mathcal{A}), \mu(\mathcal{B}), \mu(\mathcal{C})) = a$ under the above conditions. Obviously, we have for the optimum

$$\mu(\mathcal{A}\backslash(\mathcal{B} \cup \mathcal{C})) = \mu(\mathcal{B}\backslash(\mathcal{A} \cup \mathcal{C})) = \mu(\mathcal{C}\backslash(\mathcal{A} \cup \mathcal{B})) = 0$$

since, otherwise one of those sets would get more mass the others would have less. By symmetry we also see that the solution

$$\mu(\mathcal{A} \cap \mathcal{B}) = \mu(\mathcal{A} \cap \mathcal{C}) = \mu(\mathcal{B} \cap \mathcal{C}) = \frac{1}{3}$$

is optimal and give $a = \frac{1}{3}$. Hence, at least one of the complements, say $\mathbb{R}^2\backslash\mathcal{A}$ has measure not less then $1/3$. Let Q be the partition of $\mathbb{R}^2$ into squares built up by $(r3\mathbb{Z})^2$. If G is an arbitrary set of full measure then $G \cap (\mathcal{A}\backslash(\mathcal{B} \cap \mathcal{C}))$ has measure not less than $1/3$ and for every point $q \in G$ the ball $B(q, \frac{r}{2})$ is contained in the element $G(q)$ containing q and the ball $B(q, 6r)$ contains the element $G(q)$.
Using local coordinates for $\hat{M}$, setting $\mu = \hat{m} \cdot \pi^{-1} \lfloor \pi(\hat{\Gamma})$ - the restriction of the projected measure to the projection of $\hat{\Gamma}$ - we have for appropriate $\tilde{c}_2, \tau$ a partition $\tilde{P}^n = \{\tilde{P}_1^n, \ldots, \tilde{P}_s^n\}$ of M with

$$B^M\left(\tilde{y}_i, \tilde{c}_2^{-1} e^{(\chi_{ss} - \chi_s)n}\right) \subset \tilde{P}_i^n \subset B^M\left(\tilde{y}_i, \tilde{c}_2 e^{(\chi_{ss} - \chi_s)n}\right), \quad \tilde{y}_i \in \pi(\hat{\Gamma})$$

for $1 \leq i \leq r \leq s$, and

$$\mu\left(\bigcup_{i \geq r} \tilde{P}_i^n\right) < 1 - \tau.$$

Now let $\hat{P}_i^n = \hat{f}^{-n}(\tilde{P}_i^n) \times I, i = 1, \ldots, s$. Then $\hat{P}^n = \{\hat{P}_1^n, \ldots, \hat{P}_s^n\}$ fulfills VII) and VIII) with $\hat{y}_i \in \pi^{-1}(\tilde{y}_i) \cap \Gamma$. From I) we can derive that for n large enough and some c_2

$$B^s(\hat{y}_i, c_2^{-1} e^{(\chi_{ss} - \varepsilon)n}) \subset \hat{\xi}^s(\hat{y}_i) \cap \hat{P}_i^n \subset B^s(\hat{y}_i, c_2 e^{(\chi_{ss} + \varepsilon)n})$$

what is exactly VII). The proof of the claim is complete. □
Furthermore given $\hat{x} \in \hat{\Gamma}$ we can always choose $\hat{y}_i = \hat{x}$. Equations V) - VII) mean that $\hat{P}_i^n, 1 \leq i \leq r$ essentially look like parallelepipeds with side lengths $\tau^n, \tau^n, \tau^n \lambda^{-n} \gamma^n$ (see figure). Note that $\hat{\Gamma} \subset \hat{D}_{\Delta,l}$ ensures for large l that $\hat{f}^{-n} \hat{P}^n$ and $\hat{P}^n(\hat{x}), \hat{x} \in \hat{\Gamma}$, are contained in $\hat{K}_j^{(n)}$ or $\hat{f}^n \hat{K}_j^{(n)}$, for some j, respectively, and we don't have to care about singularities.
Let $c = \max(c_1, c_2)$.
Let $\hat{\Gamma}^n = \hat{\Gamma} \cap \bigcup_{i \leq r} \hat{P}_i^n$ and $\hat{\Gamma}' = \bigcap_{L=1}^{\infty} \bigcup_{n=L}^{\infty} \hat{\Gamma}^n$. Then by σ-additivity $\hat{m}(\hat{\Gamma}') \geq \tau$. Choosing the characteristic functions for the set $\hat{\Gamma}'$ as the function g in lemma from the appendix we have that given $\alpha > 0, c_1 > 1$, for N_0 large enough, for a.e. point $\hat{x} \in \hat{\Gamma}'$ there exists $n_1(\hat{x}) \geq N_0$ such that the following conditions hold:

$$\text{(IX)} \qquad \frac{\hat{m}\pi_{\omega}^{-1}(\hat{\Gamma}' \cap B^{\hat{M}}(\hat{f}^{-n}\hat{x}, c_1^{-1} e^{(\chi_{ss} - \chi_s)n}))}{\hat{m} \cdot \pi_{\omega}^{-1} B^{\hat{M}}(\hat{f}^{-n}\hat{x}, c_1^{-1} e^{(\chi_{ss} - \chi_s)n}))} \geq 1 - \alpha$$

$$\frac{\hat{m}\pi_\omega^{-1}(\hat{\Gamma}' \cap B^M(\hat{f}^{-n}\hat{x}, c_1 e^{(\chi_{ss}-\chi_s)n}))}{\hat{m}\cdot\pi_\omega^{-1}B^M(\hat{f}^{-n}\hat{x}, c_1 e^{(\chi_{ss}-\chi_s)n}))} \geq 1-\alpha$$

$$\text{(X)} \qquad \frac{\hat{m}^s_{\hat{x}}(\hat{\Gamma}' \cap B^s(\hat{x}, c_1^{-1}e^{(\chi_{ss}-\varepsilon)\cdot n}))}{\hat{m}^s_{\hat{x}}B^s(\hat{x}, c_1^{-1}e^{(\chi_{ss}-\varepsilon)\cdot n}))} \geq 1-\alpha$$

$$\frac{\hat{m}^s_{\hat{x}}(\hat{\Gamma}' \cap B^s(\hat{x}, c_1 e^{(\chi_{ss}+\varepsilon)\cdot n}))}{\hat{m}^s_{\hat{x}}B^s(\hat{x}, c_1 e^{(\chi_{ss}+\varepsilon)\cdot n}))} \geq 1-\alpha$$

$$\text{(XI)} \qquad \frac{\hat{m}(\hat{\Gamma}' \cap B^{\hat{M}}(\hat{x}, c_1^{-1}e^{(\chi_{ss}-\chi_s+\chi_u)n}))}{\hat{m}B^{\hat{M}}(\hat{x}, c_1^{-1}e^{(\chi_{ss}-\chi_s+\chi_u)n}))} \geq 1-\alpha$$

$$\frac{\hat{m}(\hat{\Gamma}' \cap B^{\hat{M}}(\hat{x}, c_1 e^{(\chi_{ss}-\chi_s+\chi_u)n}))}{\hat{m}B^{\hat{M}}(\hat{x}, c_1 e^{(\chi_{ss}-\chi_s+\chi_u)n}))} \geq 1-\alpha$$

This means (IX) to (XI) hold for $\hat{x} \in \hat{\Gamma}'$ infinitely often in Γ^n.
Fix $\hat{y}_1 = \hat{x} \in \hat{\Gamma}'$ and consider the box $B = B^M(\hat{x}, ce^{(\chi_{ss}-\chi_s-\chi_n)n}), n > n_1(\hat{x})$ such that (IX) to (XI) hold for $\hat{\Gamma}^n$ i.e. the box with side length equal to the longest side of the parallelepiped $\hat{P}_1^n$. We are going to calculate how many different parallelepipeds $\hat{P}_i^n (1 \leq i \leq r)$ are contained in the box B. We call this number T.
By V) to VII) good parallelepipeds $(\hat{P}_i, 1 \leq i \leq r)$ which cross the stable box $B^s(\hat{x}_i, e^{\chi_{ss}-\chi_s+\chi_u)-n})$ are contained in B:

$$\bigcup_{\text{distinct}\hat{P}_i^n, 1\leq i\leq r} \left\{\hat{P}^n(\hat{y}) \cap \hat{\xi}^s \subset B^s(\hat{x}, e^{(\chi_{ss}-\chi_s+\chi_u)\cdot n})\right\} \subset B^{\hat{M}}(\hat{x}, ce^{(\chi_{ss}-\chi_s+\chi_u)\cdot n}).$$

Clearly, the total mass of the large stable ball of radius $e^{(\chi_{ss}-\chi_s-\chi_u)n}$ is not larger than the number of smaller balls contained in it times their maximal mass.
Therefore we can estimate:

$$T \geq \#\text{distinct}\{\hat{P}^n(\hat{y}) \big| \hat{P}^n(\hat{y}) \cap \hat{\xi}^s \subset B^s(\hat{x}, e^{(\chi_{ss}-\chi_s+\chi_u)n})\} \geq$$

$$\geq \frac{\hat{m}^s_{\hat{x}}B^s(\hat{x}, c^{-1}e^{(\chi_{ss}-\chi_s+\chi_u)n})}{\max\limits_{\hat{y}\in\hat{\Gamma}'}\{\hat{m}^s_{\hat{y}}B^s(\hat{y}, ce^{\chi_{ss}\cdot n})\}} \geq$$

$$\geq (1-\alpha)\frac{\hat{m}^s_{\hat{x}}\left(B^s(\hat{x}, c^{-1}e^{(\chi_{ss}-\chi_s+\chi_u)n}) \cap \hat{\Gamma} \cap \bigcup_{1\leq i\leq r}\hat{P}_i^n\right)}{\max\limits_{\hat{y}\in\hat{\Gamma}'}\{\hat{m}^s_{\hat{y}}B^s(\hat{y}, ce^{(\chi_{ss}+\varepsilon)\cdot n})\}}$$

and by III) and IV)

$$T \geq (1-\alpha)\frac{e^{n(\chi_{ss}-\chi_s+\chi_u)(\delta_s-\varepsilon)-lnc}}{e^{n(\chi_{ss}+\varepsilon)(\delta_s+\varepsilon)}}.$$

Hence,

$$\begin{aligned}\hat{m}B^{\hat{M}}(\hat{x}, ce^{(\chi_{ss}-\chi_s+\chi_u)n}) \geq T \cdot \min_{1\leq i\leq r} \hat{m}\hat{P}_i^n \geq \\ \geq (1-\alpha)e^{n(\chi_{ss}-\chi_s+\chi_u)(\delta_s-\varepsilon)-lnc-n(\chi_{ss}+\varepsilon)(\delta_s+\varepsilon)} \cdot \min_{1\leq i\leq r} \hat{m}\hat{P}_i^n = \\ = (1-\alpha)e^{n(\chi_u-\chi_s)\delta_s-\varepsilon A_1 n - A_2} \cdot \min_{1\leq i\leq r} \hat{m}\hat{P}_i^n\end{aligned} \tag{10}$$

where A_1, A_2 are independent of ε and n.
Since $\hat{m}$ is invariant we have by VII)

$$\hat{m}\hat{P}^n(\hat{x}) = \hat{m}\hat{f}^{-n}\hat{P}^n(\hat{x}) \geq \hat{m} \circ \pi_\omega^{-1}\left(B^M(\hat{f}^{-n}\hat{x}, c^{-1}e^{(\chi_{ss}-\chi_s)n})\right). \tag{11}$$

Hence by II)

$$\begin{aligned}e^{n(\chi_{ss}-\chi_s+\chi_u)\delta+\varepsilon\cdot A_3 n+A_4} \geq \hat{m}B^{\hat{M}}(\hat{x}, ce^{(\chi_{ss}-\chi_s+\chi_u)n}) \geq \\ \geq (1-\alpha)e^{n(\chi_u\delta_s-\chi_s\delta_s)-\varepsilon\cdot n\cdot A_1-A_2} \times \\ \times \min_{1\leq i\leq r} \hat{m} \circ \pi_\omega^{-1}B^M(\hat{f}^{-n}\hat{x}_i, c^{-1}e^{(\chi_{ss}-\chi_s)n})\end{aligned} \tag{12}$$

where the right-hand inequality is obtained from (10) and (11) and A_i are constants independent of ε and n.
We are going to estimate the essential lower bound for the pointwise dimension of the measure $\hat{m} \circ \pi^{-1}$. We set for $\hat{x} \in \hat{\Gamma}'$

$$d(\hat{x}) = \liminf_{n\to\infty} \frac{\log \hat{m} \circ \pi_\omega^{-1}B^M(\hat{f}^{-n}\hat{x}e^{(\chi_{ss}-\chi_s)n})}{n(\chi_{ss}-\chi_s)}$$

and

$$d = \operatorname*{ess\,sup}_{\hat{y}\in\hat{\Gamma}'} d(\hat{y}).$$

It remains to proof that $d \geq \delta$. We now restrict our considerations to a set $\hat{\Gamma}'' \subset \hat{\Gamma}'$ such that for some $N_1 > N_0$, for all $\hat{x} \in \hat{\Gamma}''$ and $n > N_1$ conditions I) - X)

$$\hat{m} \circ \pi_\omega^{-1}B^M(\hat{f}^{-n}\hat{x}, c^{-1}e^{(\chi_{ss}-\chi_s+\chi_u)n}) \geq e^{n(\chi_{ss}-\chi_s)d-A_6} \tag{13}$$

(A_6 independent of n) hold. Then the definition of d ensures that $\hat{m}(\hat{\Gamma}'') > 0$.
Now let $h_{ss} = 0$ then (6) and theorem 4.5 read as

$$-\delta_s\chi_s = h_s = h = h_u = \delta_u\chi_u; \qquad \delta = \frac{h}{\chi_u} - \frac{h}{\chi_s}. \tag{14}$$

Inserting this and (13) into (12) by comparing exponents and dividing by n we get

$$\begin{aligned}(\chi_{ss}-\chi_s+\chi_u)\delta+\varepsilon A_3+\frac{A_4}{n} \geq (\chi_u-\chi_s)\delta_s-\varepsilon A_1-\frac{A_2}{n}+(\chi_{ss}-\chi_s)d-\frac{A_5+A_6}{n} &= \\ =(\chi_u-\chi_s)\left(-\frac{h}{\chi_s}\right)+(\chi_{ss}-\chi_s)d-\varepsilon A_1-\frac{A_8}{n} &= \\ =\chi_u\left(\frac{h}{\chi_u}-\frac{h}{\chi_s}\right)+(\chi_{ss}-\chi_s)d-\varepsilon A_7-\frac{A_8}{h} &= \qquad (15)\\ =\chi_u\delta+(\chi_{ss}-\chi_s)d-\varepsilon A_7-\frac{A_8}{n}. &\end{aligned}$$

Letting in (15) $n\to\infty$ and $\varepsilon\to 0$ we get $d\geq\delta$. The opposite inequality follows from (6) and theorem 4.5 since the projection π^{-1} is Lipschitz. We have proved lemma 5.6 □

We are now ready to prove theorem 5.2.

Proof. Let $\hat{m}=\hat{\mu}_{\text{SBR}}$. Then $h_{\hat{m}}=\log\gamma=\chi_u$ and $\chi_s=\log\lambda$. Moreover, by (6) and theorem 4.5

$$\begin{aligned}\dim_H(\mu_{\text{SBR}})\leq\dim_H(\hat{\mu}_{\text{SBR}})&=1-\frac{\log\gamma}{\log\lambda}+h_{ss}\left(\frac{1}{\log\lambda}-\frac{1}{\chi_{ss}}\right)\\ &\leq 1-\frac{\log\gamma}{\log\lambda}.\end{aligned} \tag{16}$$

Here we used that the projection π is Lipschitz. Assume that there is no set of full measure such that the restriction of f to this set is invertible. Then by lemma 5.5: $h_{ss}>0$ and by (16)

$$\dim_H(\mu_{\text{SBR}})<1-\frac{\log\gamma}{\log\lambda}.$$

On the other hand if we can find such a set then we have $h_{ss}=0$ and we can apply lemma 5.6 what gives $\hat{\mu}_{\text{SBR}}$ - a.e. point $\hat{x}$

$$\dim_{\mu_{\text{SBR}}}(\pi\hat{x})=\dim_{\hat{\mu}_{\text{SBR}}\circ\pi_\omega^{-1}}(\pi\hat{x})=\delta=\delta_u+\delta_s=\frac{h_u}{\chi_u}-\frac{h_s}{\chi_s}=1-\frac{\log\gamma}{\log\lambda}$$

($\hat{\mu}_{\text{SBR}}$-a.e. $\hat{x}$ implies μ_{SBR}-a.e. $\pi_\omega\hat{x}$). Frostman's Lemma tells us that $\dim_H(\mu_{\text{SBR}})\geq\delta$. This, together with (16), completes the proof of theorem 5.2. □

Proof of theorem 5.4. If we define h_{ss} to be the "non-invertibility degree of f with respect to m" then one can calculate the Hausdorff dimension of m in terms of the non-invertibility degree.

Proceeding in (15) we have with:

$$h=\chi_u\delta_u=-\chi_{ss}\delta_{ss}-\chi_s(\delta_s-\delta_{ss}).$$

(what is (6) rewritten) instead of (14) we have

$$(\chi_{ss}-\chi_s)\delta+\chi_u\delta \geq (\chi_u-\chi_s)\delta_s+(\chi_{ss}-\chi_s)d$$

or

$$(\chi_{ss}-\chi_s)\delta+\chi_u\delta_u+\chi_u\delta_s \geq (\chi_u-\chi_s)\delta_s+(\chi_{ss}-\chi_s)d$$

or

$$(\chi_{ss}-\chi_s)\delta-\chi_{ss}\delta_{ss}-\chi_s\delta_s-\chi_s\delta_{ss}+\chi_u\delta_s \geq (\chi_u-\chi_s)\delta_s+(\chi_{ss}-\chi_s)d$$

or

$$(\chi_{ss}-\chi_s)\delta-\chi_{ss}\delta_{ss}-\chi_s\delta_{ss} \geq (\chi_{ss}-\chi_s)d$$

or

$$(\chi_{ss}-\chi_s)(\delta-\delta_{ss}) \geq (\chi_{ss}-\chi_s)d$$

Hence $d \geq \delta-\delta_{ss}$. The opposite inequality is derived from [7] lemma 11.3.1.

$$d=\delta-\delta_{ss}=\delta_u+(\delta_s-\delta_{ss})=\frac{h}{\chi_u}-\frac{h-h_{ss}}{\chi_s}.$$

For $m=\mu_{\mathrm{SBR}}$ we get

$$\dim_H(\mu_{\mathrm{SBR}})=1-\frac{\log\gamma}{\log\lambda}+\frac{h_{ss}}{\ln\lambda}.$$

6 Verifiable conditions and some examples

We consider a number of examples of maps with generalized hyperbolic attractors in the two-dimensional case (i.e. M is a two-dimensional manifold). First we formulate some general assumptions which guarantee the validity of hypotheses (A3). Let f be a map satisfying condition (A1). Suppose that

(A4) N_{ij}, M_{ij} are smooth curves such that

$$N=\bigcup_{j=1}^{m}\bigcup_{j=1}^{r_i}N_{ij},\quad \partial K=\bigcup_{j=1}^{m}\bigcup_{j=1}^{r_i}M_{ij}\quad\text{and}$$

$$\partial K^{(i)}=\left(\bigcup_{j=1}^{r_i}N_{ij}\right)\cup\left(\bigcup_{j=1}^{q_i}M_{ij}\right)$$

(A5) f possesses two families of stable and unstable cones $C^{(s)}(z), C^{(u)}(z)$, $z\in K\backslash\bigcup_{i=1}^{m}\partial K^{(i)}$, which satisfy conditions $(1)-(3)$ and extend continuously to the boundary

(A6) the unstable cone $C^{(u)}(z)$ at z depends continuously on $z \in K^{(i)}$ and there exists $\alpha > 0$ such that for any $z \in N_{ij} \backslash \partial N_{ij}$, $\nu \in C^{(u)}(z)$ an any vector w tangent to N_{ij} we have that $\angle(\nu, w) \geq \alpha$;

(A7) there exists $\tau > 0$ such that most L singularity lines $N_{ij}^{(\tau)}$ of f^τ intersect in one point and $a^\tau > L + 1$ where

$$a = \inf_{z \in K \backslash N} \inf_{\nu \in C^{(u)}(z)} |Df\nu| > 1.$$

These conditions are the conditions introduced by Pesin for the invertible case in [9] (our condition (A7) is weaker than the comparable condition in [9]). He also stated the following theorem. Unfortunately, the proof of the theorem in [9] is incomplete and therefore we will include a proof of this statement.

Theorem 6.1 *If f satisfies conditions (A1),(A2),(A4)-(A7) then it satisfies condition (A3) for any $z \in D_0$ with constants $C, q = 1$ (C does not depend on z).*

Proof: First of all it is sufficient to prove this result for the map f^τ instead of f which still has properties (A1),(A2),(A4)-(A7) with $\tau = 1$. We say that $\gamma : [0, b] \to M$ is a u–curve if γ is smooth and $\gamma(t) \in C^{(u)}(\gamma(t)), t \in [0, b]$ (b is a positive number; we assume that $l(\gamma) = b$ is the length of the curve.) To verify condition (A3) we will show that for any u–curve γ the following holds:

$$\nu^u(\gamma \cap f^{-n} U(\varepsilon, N^+)) \leq C\varepsilon \tag{17}$$

Then using the continuity of the cone field we can continuously foliate any small enough ball in K by u–curves. Using the Fubini theorem condition (17) implies that only $C\varepsilon$ percent of the ball contributes to condition (A3).

Fix γ an arbitrary u–curve. By condition (A7) there exists a constant $c > 0$ such that if γ is a u–curve in one of the sets $K^{(i)}$ with the length $\leq c$ then $f(\gamma)$ can intersect only L singularity lines N_{pq}. Let $d = \min\{c, l(\gamma)\}$. A curve γ is called long if its length is longer than d otherwise it is called short. If γ is a u–curve then by virtue of (A4)-(A7) the curve $f(\gamma)$ for any $n > 0$ consists of a finite number of u–curves. Iterating this we get $f^n(\gamma) = \bigcup_i \gamma_{n,i}$. Moreover, for each i there exist $j = j(n, i), l = l(n, i)$ such that $\gamma_{n,i} = f(\gamma_{n-1,j} \cap K^{(l)})$. We prove (17) by induction.

We give the proof first in a special case since the general case leads to gruesome formulas. Namely assume that $|Df|$ restricted to any u–curve is equal to a constant which we call a. We call this the constant Jacobian case. This assumption does not hold for any examples since the definition of u-curves is too broad, but if we make a special choice of u–curves, such as vertical line segments for the Belykh map, then the assumption is fulfilled.

For γ an arbitrary u–curve define the bad length of γ by $bl(\gamma) = \nu^u(\gamma \cap U(\epsilon, N^+))$. We define $S := \sup_K |Df|$, $C_0 := b_1 \cdot L/d$

$$C_k := C_{k-1} + \frac{b_2}{d}(L/a)^k$$

and $C = \lim_{k\to\infty} C_k$. where b_1, b_2 are constants described in the next paragraph.
In the base case of our induction there are many pieces $\gamma_{n,i}$. The original curve γ was long and assumption (A7) implies that at the mth step ($m \leq n$) at least one of the $\gamma_{m,i}$ is long. Subdivide each long piece $\gamma_{n,i}$ into pieces of length approximately d. Then each of these pieces can intersect at most L branches of N^+ and is uniformly transverse to N^+ by condition (A6). Thus there is a constant $b_1 > 0$ such that the percentage of each of these pieces which is in $U(\varepsilon, N^+)$ is of order $b_1 L\varepsilon/d$. Thus the total percent of γ which is in $U(\varepsilon, N^+)$ is of the same order. Thus we have shown that for each long u–curve $\gamma^l_{n,i}$

$$bl(\gamma^l_{n,i}) \leq \frac{b_1 \cdot L \cdot \varepsilon}{d} l(\gamma^l_{n,i}) \leq C_0 \cdot \varepsilon \cdot l(\gamma^l_{n,i}).$$

Using condition (A6) it is clear that there is a constant $b_2 > 0$ such that for any short $\gamma^s_{n,i}$ the measure of $\gamma^s_{n,i} \cap U(\varepsilon, N^+)$ is at most $b_2\varepsilon$ or

$$bl(\gamma^s_{n,i}) \leq b_2 \cdot \varepsilon. \tag{18}$$

Our inductive assumptions are as follows. On the $(k-1)$st step equations (19)-(20) hold. For each long u–curve $\gamma^l_{n-(k-1),i}$

$$bl(\gamma^l_{n-(k-1),i}) \leq C_{k-1} \cdot \varepsilon \cdot l(\gamma^l_{n-(k-1),i}). \tag{19}$$

Each short u–curve $\gamma^s_{n-(k-1),i}$ splits into two parts $\gamma^s_{n-(k-1),i} = \hat{\gamma}^s_{n-(k-1),i} \cup \tilde{\gamma}^s_{n-(k-1),i}$. Each of the pieces consists is a finite union of u–curves. The following estimate holds for each of the pieces:

$$\begin{aligned} bl(\hat{\gamma}^s_{n-(k-1),i}) &\leq b_2 \cdot (\frac{L+1}{a})^{k-1} \cdot \varepsilon \\ bl(\tilde{\gamma}^s_{n-(k-1),i}) &\leq C_{k-1} \cdot \varepsilon \cdot l(\tilde{\gamma}^s_{n-(k-1),i}). \end{aligned} \tag{20}$$

. In the base case of the induction $\gamma^s_{n,i} = \hat{\gamma}^s_{n,i}$ while $\tilde{\gamma}^s_{n,i} = \emptyset$. Now we come to the kth step.
We consider first a long piece $\gamma^l := \gamma^l_{n-k,I}$. The image $f(\gamma^l)$ can consist of several short and long pieces. Keeping the natural order in which they appear, we write

$$f(\gamma^l) = \bar{\gamma}^s_{n-(k-1),1} \cup \bar{\gamma}^l_{n-(k-1),1} \cup \cdots \cup \bar{\gamma}^s_{n-(k-1),M} \tag{21}$$

where

$$\begin{aligned} \bar{\gamma}^s_{n-(k-1),i} &= \gamma^s_{n-(k-1),j_1} \cup \cdots \cup \gamma^s_{n-(k-1),j_r} \\ \bar{\gamma}^l_{n-(k-1),i} &= \gamma^l_{n-(k-1),j_1} \cup \cdots \cup \gamma^l_{n-(k-1),j_r} \end{aligned} \tag{22}$$

Here $1 \leq j_r \leq L$ and the first and last set of short curves $\bar{\gamma}^s_{n-(k-1),1}$ and $\bar{\gamma}^s_{n-(k-1),M}$ may not appear in equation (21).
To estimate the bad length of γ^l we estimate the bad length (for each $i < M-1$) of $f^{-1}\bar{\gamma}^s_{n-(k-1),i} \cup f^{-1}\bar{\gamma}^l_{n-(k-1),i} := \tilde{\gamma}^s_i \cup \tilde{\gamma}^l_i$ and for $i = M-1$ we estimate $\tilde{\gamma}^s_{M-1} \cup \tilde{\gamma}^l_{M-1} \cup \tilde{\gamma}^s_M$. Since we have assumed constant Jacobian any percentage estimate pulls back without change. We do this with the long and the tilde short parts. The hat short parts have very small total length and do increase the total percentage only a small bit. More precisely we consider the following:

$$
\begin{aligned}
bl(\gamma^l) &= \sum_{i=1}^{M} \left\{ bl(\hat{\tilde{\gamma}}^s_i) \right\} + \sum_{i=1}^{M-1} \left[bl(\tilde{\tilde{\gamma}}^s_i) + bl(\tilde{\gamma}^l_i) \right] + bl(\tilde{\tilde{\gamma}}^s_M) \\
&\leq \sum_{i=1}^{M} \left\{ bl(\hat{\tilde{\gamma}}^s_i) \right\} + C_{k-1} \cdot \varepsilon \left\{ \sum_{i=1}^{M-1} \left[l(\tilde{\gamma}^l_i) + l(\tilde{\tilde{\gamma}}^s_i) \right] + l(\tilde{\tilde{\gamma}}^s_M) \right\}. \qquad (23)
\end{aligned}
$$

Now, by the inductive assumption, the image of each of the hat short pieces has length at most $b_2(L/a)^{k-1} \cdot \varepsilon$. There are at most L short pieces and each one is contracted by a factor of $1/a$. Thus the first term of the sum is less than or equal to $2b_2(L/a)^k \cdot \varepsilon$. Also $\sum_{i=1}^{M-1} [l(\tilde{\gamma}^l_{n-k,i}) + l(\tilde{\tilde{\gamma}}^s_i)] + l(\tilde{\tilde{\gamma}}^s_M) \leq l(\gamma^l_{n-k,i})$. Combining these remarks we have that equation (23) is less than

$$\leq b_2(L/a)^k \cdot \varepsilon + C_{k-1} \cdot \varepsilon \cdot l(\gamma^l_{n-k,i}) \leq C_k \cdot \varepsilon \cdot l(\gamma^l_{n-k,i}). \qquad (24)$$

This completes the inductive proof of equation (19). We turn to the verification of equation (20) in the kth step. Consider a short piece $\gamma^s := \gamma^s_{n-k,I}$. The image of γ^s consists of some short pieces and possibly some long pieces, namely

$$f(\gamma^s) = \bigcup_{i=1}^{R} \gamma^{s(l)}_{n-(k-1),i} \qquad (25)$$

where $\gamma^{s(l)}$ means that the piece is either short or long and $R \leq L+1$ by assumption (A7) and the definition of short pieces. We define $\hat{\gamma}^s$ by $f(\hat{\gamma}^s) = \cup_{i=1}^{R} \hat{\gamma}^s_{n-(k-1),i}$ and $\tilde{\gamma}^s = \gamma^s \backslash \hat{\gamma}^s$. Then since we have constant Jacobian, percentages do not change and we have

$$bl(\tilde{\gamma}^s) \leq C_{k-1} \cdot \varepsilon \cdot l(\tilde{\gamma}^s) \leq C_k \cdot \varepsilon \cdot l(\tilde{\gamma}^s). \qquad (26)$$

For the long pieces and the short hat pieces again using the constant Jacobian we estimate

$$bl(\hat{\gamma}^s) \leq \frac{1}{a} \sum_{i=1}^{L+1} bl(\hat{\gamma}^s_{n-(k-1),i}) \leq \frac{L+1}{a} b_2 \left(\frac{L+1}{a} \right)^{k-1} \cdot \varepsilon. \qquad (27)$$

This completes the inductive verification of equation (20). The complete induction gives (17) since at the n-th step there is only one long piece - namely γ-remaining.

Now we sketch how the proof differs without the assumption of constant Jacobian. For any $k = 0, \dots, n$ the curve $f^{-k}(\gamma_{n,i})$ lies entirely in one of the sets $K^{(j)}$ for some $j = j(n,i,k)$. Here the preimage branch taken is the one returning to γ. This allows us to write that

$$
\begin{aligned}
l(\gamma_{n,i}) &= \int\limits_{f^{-n}(\gamma_{n,i})} |Df^n(f^{-n}(\gamma_{n,i}(t))| \, dt = |Df^n(z_{n,i})| l(f^{-n}(\gamma_{n,i})) \\
l(\gamma_{n,i}) &= \int\limits_{f^{-n}(\gamma_{n,i})} |Df^n(f^{-n}(\gamma_{n,i}(t))| \, dt = |Df^n(z_{n,i})| l(f^{-n}(\gamma_{n,i}))
\end{aligned}
\tag{28}
$$

where $z_{n,i} \in f^{-n}(\gamma_{n,i}), y_{n,i} \in f^{-n}(\tilde{\gamma}_{n,i})$ are some points. First we write

$$
\frac{|Df^n(y_{n,i})|}{|Df^n(z_{n,i})|} = \prod_{k=0}^{n=1} \frac{|Df(f^k(y_{n,i}))|}{|Df(f^k(z_{n,i}))|}.
$$

Taking into consideration that $f^{-(n-k)}(\gamma_{n,i})$ is a u–curve we have the estimation for the distance between points $f^k(y_{n,i}), f^k(z_{n,i}), k = 0, ..., n-1$. This implies by virtue of condition (A2) that there is $C > 0$ independent of n and γ such that

$$
C^{-1} \leq |Df^n(y_{n,i})| / |Df^n(z_{n,i})|^{-1} \leq C. \tag{29}
$$

It is important that the estimate (29) is independent of n. Now if we apply (29) at each step then an exponential factor C^k would appear in equations (19,20). However we need to apply (29) only once. Namely, in the above argument, the bad length of γ is gotten by pulling back, one step at a time, the bad lengths of the $\gamma_{k,i}$. Rather than doing this, we modify the argument by directly pulling back the "new" bad length gotten at the kth step all the way to γ. Thus each contribution to equations (19,20) is pulled back to γ exactly once and so at the end of the induction, we get the estimate corresponding to equation (19) for $k = n$:

$$
bl(\gamma) \leq C \cdot C_n \cdot \varepsilon \cdot l(\gamma). \tag{30}
$$

□

Next we give some examples where conditions (A4)–(A7) can be verified (see [9],[13]).

6.1 The Belykh map

Consider the square $Q = [-1,1] \times [-1,1] \in \mathbb{R}^2$ and the map $f : Q \to Q$ defined by

$$
f(x,y) = \begin{cases} (\lambda x_1 + 1 - \lambda, \gamma x_2 + 1 - \gamma) & x_2 > kx_1 \\ (\lambda x_1 + (\lambda - 1), \gamma x_2 + (\gamma - 1)) & x_2 < kx_1 \end{cases}
$$

with $-1 < k < 1, 1 < \gamma \leq \frac{2}{|k|+1}, 0 < \lambda \leq 1$.

For $\lambda \geq \frac{1}{2}$. This map is not injective. See figure

This is our main example. It is most natural for non-invertibility and features a lot of properties of the other ones. It was introduced by Belykh [4] as a simple model for phase synchronization. The ergodic properties where investigate in [9] and [13]. J. Alexander and J. Yorke considered the metric properties of the SBR-measure in the special case when $k = 0, \gamma = 2$ and $\lambda \geq \frac{1}{2}$. They called this case the fat baker's transformation. This case is very special since the SBR-measure has an explicit product structure according to the cartesian coordinates. In [15], using the results of this article, similar properties are derived for the fat Belykh map in the general case.

6.2 Other models

Other examples arise from projecting higher-dimensional hyperbolic systems to two-dimensional ones. For instance, we can consider the projection of Smale's solenoidal map $f : S^1 \times \mathcal{D}^\in$

$$f(x, y, z) = (2x \mod 1, \frac{1}{4}y + \frac{1}{10}\cos(x), \frac{1}{4}z + \frac{1}{10}\sin(x))$$

to the annulus $S^1 \times I$

$$\tilde{f}(x, y) = (2x \mod 1, \frac{1}{4}y + \frac{1}{10}\cos(x)).$$

Also crossed horseshoes as they were considered by Przytycki [11] and Simon [16] belong to our class.

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