On a half-space radiation condition

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Abstract
For the Dirichlet problem of the Helmholtz equation over the half space or rough surfaces, a radiation condition is needed to guarantee a unique solution, which is physically meaningful. If the Dirichlet data is a general bounded continuous function, then the well-established Sommerfeld radiation condition, the angular spectrum representation, and the upward propagating radiation condition do not apply or require restrictions on the data, in order to define the involved integrals. In this paper a new condition based on a representation of the second derivative of the solution is proposed. The twice differentiable half-space Green’s function is integrable and the corresponding radiation condition applies to general bounded functions. The condition is checked for special functions like plane waves and point source solution. Moreover, the Dirichlet problem for the half plane is discussed. Note that such a “continuous” radiation condition is helpful e.g. if finite sections of the rough-surface problem are analyzed.

1 Introduction
Throughout this paper we denote the points of the three-dimensional Euclidean space $\mathbb{R}^3$ by $\vec{x}$ and $\vec{y}$ with $\vec{x} := (x_1, x_2, x_3)^T$ and $\vec{y} := (y_1, y_2, y_3)^T$. For fixed numbers $x_{f,3}$, we define the half spaces $\mathbb{R}^3_{x_{f,3},+} := \{ \vec{x} = (x', x_3)^T \in \mathbb{R}^3 : x_3 > x_{f,3} \}$ and $\mathbb{R}^3_{3,0,+} := \mathbb{R}^3_{3,0,+}$ and the boundary planes $\mathbb{R}^3_{x_{f,3}} := \{ (x', x_{f,3})^T : x' \in \mathbb{R}^2 \}$. We shall consider functions $u$ defined on a perturbed half space $\Omega$ of $\mathbb{R}^3_{x_{3},+}$, which are solutions of the Helmholtz equation $\left( \Delta + k^2 I \right) u = 0$ for a fixed wavenumber $k > 0$. We suppose that $\Omega := \Omega_F := \{ \vec{x} \in \mathbb{R}^3 : x_3 > F(x') \}$ with a function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $-h_F \leq F(x') < 0$ holds for all $x' \in \mathbb{R}^2$ (cf. Fig. 1). The number $h_F > 0$ is fixed.

The goal is to find a general radiation condition for Helmholtz solutions on $\Omega$ which are uniformly bounded on the planes $\mathbb{R}^3_{x_{f,3}}$, but, eventually, do not decay in the directions of $\mathbb{R}^3_{x_{f,3}}$. If $\Phi(\vec{x}, \vec{y})$ is the Green’s function for the Dirichlet problem of the Helmholtz equation over the upper three-dimensional half space $\mathbb{R}^3_+$, then, analogously to the upward propagating radiation condition (UPRC) in the two-dimensional case (cf. [7]), a possible choice for the radiation condition would be to fix $x_{f,3} \geq 0$ and to

![Figure 1: The geometry settings.](image-url)
require the condition (cf. [3], Chapt. 5.1.1 and [4,5])

\[ u(\vec{x}) = \int_{\mathbb{R}^2} \partial_{y_1} \Phi(\vec{x}, (y', x_{f,3})^\top) u((y', x_{f,3})^\top) \, dy' \]  

(1.1) for all \( \vec{x} \in \mathbb{R}^3_{x_{f,3}^+,+} \). Note that this condition is equivalent to a representation as a superposition of outgoing generalized plane waves (cf. the angular spectrum representation in [2],[5],[7],[10]) and, in two dimensions, to the pole condition (cf. [2]). In three dimensions, the integral exists for functions from weighted \( L^2 \) or from more complicated spaces. Since we are interested in a class of solution \( u \) containing plane-wave functions, we only know that the function \( u \) restricted to the boundary plane \( \mathbb{R}^3_{x_{f,3}} \) is smooth and uniformly bounded, and the existence of the integral is not guaranteed. Therefore, we formally differentiate twice to get

\[ \partial^2_{x_3} u(\vec{x}) = \int_{\mathbb{R}^2} \partial^2_{y_3} \Phi(\vec{x}, (y', x_{f,3})^\top) u((y', x_{f,3})^\top) \, dy', \quad \vec{x} \in \mathbb{R}^3_{x_{f,3}^+,+}. \]  

(1.2)

This will be the main part of our radiation condition. We shall see that the kernel in (1.2) satisfies

\[ \partial^2_{y_3} \Phi(\vec{x}, \vec{y}) = \mathcal{O}_{x_{f,3} - y_3}(|\vec{x} - \vec{y}|^{-3}) \quad \text{on} \quad \mathbb{R}^3_{x_{f,3}^+,+}, \quad \text{for} \quad |\vec{x} - \vec{y}| \to \infty. \]

Hence, the principle-value integral in (1.2) is well defined for any function \( u \) bounded and sufficiently smooth on \( \mathbb{R}^3_{x_{f,3}^+,+} \). Suppose that \( \mathcal{O}_p \subseteq \mathbb{R}^3_{x_{f,3}^+,+} \) is a fixed open subset and that \( u \) is a solution of the Helmholtz equation. Then (1.2) is satisfied if and only if the equation in (1.2) holds for any \( \vec{x} \in \mathcal{O}_p \). Indeed, on both sides of the equation we have solutions of the Helmholtz equation, and such analytic function coincide on \( \mathbb{R}^3_{x_{f,3}^+,+} \) if and only if they do on \( \mathcal{O}_p \). On the other hand, it is not sufficient to require the equation for one or more planes \( \mathbb{R}^3_{x_{f,3}^+} \) with \( x_3 > x_{f,3} \). Indeed, the two sides of the equation might differ by the function \( (x', x_3)^\top \mapsto \sin(k(x_3 - x_{f,3})) \) if they coincide only over \( \mathbb{R}^3_{x_{f,3}^+} \) with \( x_3 = x_{f,3} + l\pi/k \), \( l = 1, 2, \ldots \). However, it would be sufficient to require the equation in (1.2) over \( \mathbb{R}^3_{x_{f,3}^+} \) with \( x_3 > x_{f,3} \), \( l = 1, 2 \) if the homogeneous Dirichlet problem of the Helmholtz equation over the layer enclosed by \( \mathbb{R}^3_{x_{f,3}^1} \) and \( \mathbb{R}^3_{x_{f,3}^2} \) has the trivial solution only. In particular, this is the case for \( |x_{f,3} - x_3| < \pi/k \) (cf. the subsequent Sect. [10]).

For \( l' \in \mathbb{Z}^2 \) and \( x_{h,3} > 0 \), we introduce the finite subdomain \( \Omega_{l',x_{h,3}} \) of \( \Omega \) adjacent to the lower boundary by \( \Omega_{l',x_{h,3}} := \{ \vec{x} \in \Omega : |x' - l'| < 4, \ x_3 < x_{h,3} \} \). Clearly, for any \( l' \in \mathbb{Z}^2 \) and \( x_{h,3} > 0 \), we have to assume that \( u \) is in the Sobolev space \( H^{1,1}(\Omega_{l',x_{h,3}}) \). Unfortunately, Sobolev regularity and condition (1.2) still allow to add to \( u \) an unphysical solution of the form \( u_{\text{add}}(\vec{x}) := u_{2D}(x')x_3 \) with \( u_{2D} \) a solution of the two-dimensional Helmholtz equation \( (\Delta_{x'} + k^2 I)u_{2D} = 0 \) (think e.g. of \( u_{2D}(x') = e^{i(\alpha x_1 + \beta x_2)} \) with \( \alpha, \beta \in \mathbb{R} \) and \( \alpha^2 + \beta^2 = k^2 \)). Adding such a function to \( u \) corresponds to adding the function \( u_{2D}(x') \) to the derivative \( \partial_{x_3} u \). In order to exclude such addends, we augment our radiation condition by one of the next two equivalent conditions. Either we require, for fixed \( \varepsilon_u > 0 \) and \( c_u > 0 \) depending on \( u \), the weak boundedness condition

\[ \left| \frac{1}{x_3} \int_0^{x_3} \frac{x_3 - t}{x_3} u((x', t)^\top) \, dt \right| < c_u x_3^{1-\varepsilon_u}, \quad \forall x_3 \geq 1 \]  

(1.3)
or we fix the derivative \( \partial_{x_3} u \) over the plane \( \mathbb{R}^3_{x_{f,3}^-,+} \) by

\[ \partial_{x_3} u((x', x_{f,3})^\top) = -\int_{\mathbb{R}^2} \partial^2_{y_3} \Phi((x', x_{f,3})^\top, (y', x_{f,3})^\top) u((y', x_{f,3})^\top) \, dy', \quad x' \in \mathbb{R}^2. \]  

(1.4)
The existence of the hypersingular integral in (1.4) (cf. the subsequent representation (2.8)) can be shown under additional conditions. To avoid hypersingular kernels, we can fix \( \partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}^-,+}} \) by

\[ \partial_{x_3} u((x', x_{f,3})^\top) = -\int_{\mathbb{R}^2} \partial^2_{y_3} \Phi((x', x_{f,3})^\top, (y', x_{h,3})^\top) u((y', x_{h,3})^\top) \, dy', \quad x' \in \mathbb{R}^2. \]  

(1.5)
In comparison to \(1.4\), in \(1.5\) there is no singularity in the kernel for \(|x' - y'| \to 0\). However, the more serious problem in \(1.4\) and \(1.5\) is that of integrability at infinity. The weak boundedness condition \(1.3\) is nothing else than a restriction to an \(O(x_3^{-1-\varepsilon})\) growth of the two-fold integral function \(w(x', x_3) := \int_{x_3}^{x_3} (x_3 - t) u(x', t) \, dt\) (i.e. \(w\) is defined by \(\partial_{x_3}^2 w = u\) and \(w(x', 0) = \partial_{x_3} w(x', 0) = 0\)). If \(u\) is uniformly bounded or if \(|u(x)| = O(x_3^{-1-\varepsilon})\) for \(x_3 \to \infty\), then \(1.3\) is fulfilled. As we will see in Prop. \(1.2\) in many cases the solution \(u\) satisfying \(1.2\) and \(1.3\) is indeed uniformly bounded.

Altogether, we suggest the following outgoing radiation condition

**Definition 1.1.** A solution \(u\) of \((\Delta + k^2 I)u = 0\) over \(\Omega\) is said to satisfy the half-space radiation condition (HSRC) if there exist real numbers \(c_u, \varepsilon_u, x_{h,3}\), and \(x_{f,3}\) with \(c_u > 0, \varepsilon_u > 0,\) and \(0 < x_{f,3} < x_{h,3}\) such that

\[\text{i) For any } l' \in \mathbb{Z}^2, \text{ the restriction of } u \text{ to the subdomain } \Omega_{l', x_{h,3}} \text{ is in the Sobolev space } H^1(\Omega_{l', x_{h,3}}) \text{ and has a bounded norm } \|u|_{\Omega_{l', x_{h,3}}} \|H^1(\Omega_{l', x_{h,3}}) < c_u.\]

\[\text{ii) The second order derivative } \partial_{x_3}^2 u \text{ admits the representation } 1.2.\]

\[\text{iii) The function } u \text{ satisfies the weak boundedness estimate } 1.3.\]

For the equivalence of \(1.3\) and \(1.4\) and for the existence of solution to the Dirichlet problem for the Helmholtz equation over \(\mathbb{R}^3\), we have to restrict the functions. We use three special classes. First, we introduce the space \(C(\mathbb{R}^2)\) of continuous functions, the space \(C_0(\mathbb{R}^2)\) of uniformly bounded and continuous functions and, for \(v \in C(\mathbb{R}^2)\), the average function

\[av(v, x', r) := \int_0^{2\pi} v(x' - r(\cos \phi, \sin \phi)\, d\phi.\] (1.6)

Furthermore, we denote the space of all function \(v\), which together with all their derivatives up to order \(l \geq 0\) are continuous and uniformly bounded by \(C^l(\mathbb{R}^2)\). We define the first class \(AV_\kappa\) for \(\kappa > 0\) by

\[AV_\kappa := \{ v \in C^1(\mathbb{R}^2) : \exists c_v > 0 \text{ s.t. } |av(v, x', r)| < c_v r^{-\kappa}, \forall r > 1, \forall x' \in \mathbb{R}^2 \}.\] (1.7)

A second class is the Dirichlet data set \(DD_v\) with \(v \geq 0\) including all sums of Helmholtz solutions plus Helmholtz images, i.e.,

\[DD_v := \{ v \in C_b(\mathbb{R}^2) : \exists c_v > 0, \exists v_0 \in C^2(\mathbb{R}^2), \exists v_i \in C_b(\mathbb{R}^2) \text{ s.t.} v = v_0 + \sum v_i, (\Delta x' + k^2 I)v_0 = 0, \quad v_i = (\Delta x' + k^2 I)v_0, \text{ and} \quad |v_0(x')| \leq c_v (1 + |x'|)\}, \forall x' \in \mathbb{R}^2\]. (1.8)

Finally, we introduce a class of functions \(v\) characterized by the Fourier transforms \([Fv]\). Unfortunately, this condition will not be easy to check. We introduce the annular domains by the formula \(R_{k,\xi} := \{ \xi' \in \mathbb{R}^2 : k - \varepsilon < |\xi| < k + \varepsilon \}\) and set

\[FC_k := \{ v \in C_b^6(\mathbb{R}^2) : \exists c_v > 0 \text{ s.t. } [Fv]|_{R_{k,\xi'}} \in L^2(R_{k,\xi'}) \}. \] (1.9)

For functions \(u\) with restrictions \(v(x') := u((x', x_3)\) in such spaces, we get

**Proposition 1.2.** Suppose there is an \(x_{f,3}\) with \(0 < x_{f,3} < x_{h,3}\) such that \(v := u|_{\mathbb{R}^3_{x_3}} \) is either in \(AV_\kappa\), \(\kappa > 0\) in \(DD_v\), \(0 \leq v < 1\) or in \(FC_k\). Then, in the (HSRC), we can replace condition \(\text{iii) by the equivalent condition:}

\[\text{iii') The derivative } \partial_{x_3} u \text{ restricted to the plane } \mathbb{R}^3_{x_{f,3}} \text{ fulfills } 1.4.\]

Moreover, the solutions fulfilling the (HSRC) are even uniformly bounded if \(v := u|_{\mathbb{R}^3_{x_3}} \in AV_\kappa\) or if \(v \in DD_0\) with \(v_0 \in C^1(\mathbb{R}^2)\).
Since (1.1) implies (1.2) and (1.4), (UPRC) implies (HSRC) with iii replaced by iii). Prop. [1.2] and its proof (cf. (8.2) or interprete the representation $V_k v_u$ in part iii) of the proof to Lemma 8.1 as evaluated by partial integration for the differential operator $\partial_{x_3}^2 = - (\Delta_{x'} + k^2 I)$ means that, for $\nu := u|_{\mathbb{R}^3_{3,3}}$ in $DD_v$, $AV_\nu$, or $FC_k$, the condition (HSRC) is equivalent to the (UPRC).

In Sect. 8 we shall prove Prop. [1.2] and show that the (HSRC) is independent of the choice of $x_{h,3}$ and $x_{f,3}$. For the plausibility of the (HSRC), we remark:

- Formula (1.2) can be considered to be a representation of $\partial_{x_3}^2 u$ as a superposition of generalized outgoing plan-wave solutions (cf. the right-hand side of (6.5)). Outgoing for the upper half plane means that the plane wave $u(\vec{x}) := e^{i(\alpha x_1 + \beta x_2 + i\gamma x_3)}$ with $\alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{C}$, and $\alpha^2 + \beta^2 + \gamma^2 = k^2$, is either a true plane-wave (i.e. $\alpha^2 + \beta^2 \leq k^2$) moving into the upper half plane (i.e. $\gamma \geq 0$) or a generalized plane-wave (i.e. $\alpha^2 + \beta^2 < k^2$) decaying in the $x_3$ direction (i.e. $\Im \gamma > 0$).

- For any solution $u_{2D} \in C^2_k(\mathbb{R}^2)$ of the two-dimensional equation \( \Delta_{x'} + k^2 I \) $u_{2D} = 0$, the function $u(\vec{x}) := u_{2D}(x')$ satisfies the (HSRC), and $u(\vec{x}) := u_{2D}(x') x_3$ does not (cf. Section 5). In particular, $u(\vec{x}) := J_0(k|x'|)$ fulfills the (HSRC) and cylindrical wave functions like $u(\vec{x}) := \frac{1}{4} H_0^{(1)}(k|x'|)$ satisfy at least (1.2) (cf. Sect. 5).

- Any generalized plane-wave solution $u(\vec{x}) := e^{i(\alpha x_1 + \beta x_2 + i\gamma x_3)}$ with $\alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{C}$, and $\alpha^2 + \beta^2 + \gamma^2 = k^2$ satisfies the (HSRC) if and only if either $\gamma > 0$ or $\Re \gamma \geq 0$ together with $\Im \gamma > 0$ (cf. Sect. 5). In other words, the (HSRC) is equivalent to the well-known radiation condition for quasiperiodic functions in the theory of gratings (cf. Corollary 6.1).

- Any solution, satisfying the classical Sommerfeld radiation condition for the directions pointing into the half space, fulfills the (HSRC) (cf. Sect. 7 and cf. [8] for Sommerfeld’s condition on the half space). In other words, functions $u(\vec{x}) := G(\vec{y}, \vec{x})$ and all their derivatives satisfy (HSRC), but “incoming” waves like $u(\vec{x}) := G(\vec{y}, \vec{x})$ do not.

- Using the radiation condition (HSRC), uniqueness (cf. Prop. 9.1) and existence of the solution to the Dirichlet problem of the Helmholtz equation over the half plane $\mathbb{R}^3_{3,3}$ can be shown. However, special conditions are needed for the existence, namely the Dirichlet data is supposed to be in $DD_v$, $AV_\nu$, or $FC_k$ (cf. Prop. 9.3 and compare [6] for the two-dimensional case with continuous data over rough surfaces and [3], Sect. 5.1.1.1 for Dirichlet data from $L^2(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2) \subset H^{1/2}_k(\mathbb{R}^2)$, and a special subspace $X' \subset C_h(\mathbb{R}^2) + L^2(\mathbb{R}^2)$).

# 2 Fundamental solution, Green’s function, and representation formula

Denote $\vec{x}$ and $\vec{y}$ as in Sect. 1. For a complex valued wave number $k$ with $\Re k > 0$ and $\Im k \geq 0$, the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ is given by

$$G(\vec{x}, \vec{y}) := G(\vec{x} - \vec{y}) := \frac{1}{4\pi} e^{ik|x-y|}. \tag{2.1}$$
In order to enable the computation of finite-part integrals, we shortly look at the kernel behaviour for the third order derivative is of the form

\[
\partial_{y_3} G(x, y) = \frac{e^{ik|x-y|}}{4\pi} \left\{ \frac{(ik)(y_3 - x_3)}{|x - y|^3} - \frac{y_3 - x_3}{|x - y|^3} \right\}, \quad (2.2)
\]

\[
\partial_{y_3} \partial_{y_3} G(x, y) = \frac{e^{ik|x-y|}}{4\pi} \left\{ (ik)^2 \left( \frac{y_3 - x_3}{|x - y|^3} - \frac{3}{|x - y|^2} \right) + \frac{3(y_3 - x_3)^2}{|x - y|^4} \right\}, \quad (2.3)
\]

\[
\partial_{y_3}^2 G(x, y) = \frac{e^{ik|x-y|}}{4\pi} \left\{ \frac{(ik)^2(y_3 - x_3)}{|x - y|^5} + \frac{3(y_3 - x_3)^2}{|x - y|^6} \right\}. \quad (2.4)
\]

The third order derivative is of the form \(O_{x_3-y_3}(|x - y|^{-3})\). Indeed,

\[
\partial_{y_3}^3 G(x, y) = \frac{e^{ik|x-y|}}{4\pi} \left\{ \frac{3(i)^3(y_3 - x_3)}{|x - y|^5} + \frac{3(i)^3(y_3 - x_3)^2}{|x - y|^6} \right\}. \quad (2.5)
\]

Note that the factor \((y_3 - x_3)|x - y|^{-3}/4\pi\) is the double layer kernel for the Laplace equation. This kernel defines a uniformly bounded operator in the \(L^\infty\) space (cf. [9]).

In order to enable the computation of finite-part integrals, we shortly look at the kernel behaviour for \(|x - y| \to 0\). By the Taylor-series expansion of \(e^{ik|x-y|}\) we get

\[
G(x, y) = \frac{1}{4\pi \frac{1}{|x - y|^3}} + O(1),
\]

\[
\partial_{y_3} G(x, y) = \frac{1}{4\pi \frac{1}{|x - y|^3}} + O(1), \quad (2.7)
\]

\[
\partial_{y_3}^2 G(x, y) = \frac{1}{4\pi \frac{1}{|x - y|^3}} - \frac{1}{4\pi \frac{3(y_3 - x_3)^2}{|x - y|^5}} + O\left(\frac{1}{|x - y|^3}\right), \quad (2.8)
\]

which proves that the kernels for the Helmholtz equation, locally, are compact perturbations of the operators for the Laplace equation. Consequently, the corresponding potentials of finitely supported and smooth weight functions over \(\mathbb{R}^3_{x_{f,3}}\), computed at \((x', x_3)^T \in \mathbb{R}^3_{x_{f,3}}\) have well-defined limits for \(x_3 \to x_{f,3}\). These limits can be computed by the well-known terms of the jump relation plus the values of the potential integral at \((x', x_{f,3})^T\). In case of the kernel \(\partial_{y_3} G(x, y)\), this integral is to be understood as the usual finite-part integral.

The Green's function for the Dirichlet problem over \(\mathbb{R}^3_{x_{f,3}}\) is chosen as

\[
\Phi(x, y) = G(x, y) - G(x', 2x_{f,3} - x_3)^T, \quad G(x', y') = G(x', y) - G(x, y') - 2G(x, y' - 2x_{f,3} - y_3)^T. \quad (2.9)
\]

We get vanishing boundary values \(\Phi(x, (y', x_{f,3})^T) = 0\) and

\[
\partial_{y_3}^3 \Phi(x, y) = \partial_{y_3}^3 G(x, y) + \partial_{y_3}^3 G(x, (y', 2x_{f,3} - y_3)^T),
\]

\[
\partial_{y_3}^3 \Phi(x, (y', x_{f,3})^T) = 2\partial_{y_3}^3 G(x, (y', x_{f,3})^T), \quad (2.10)
\]
such that $\partial^3_{y_3} \Phi(\vec{x}, \vec{y}) = O_{x_3-y_3}(|\vec{x} - \vec{y}|^{-3})$ on $\vec{y} \in \mathbb{R}^3_{x_3}, \vec{x} \in \mathbb{R}^3$ follows from (2.6).

Finally, we recall the representation formula for the second order derivative $\partial^2_{x_3} u$ of the solution $u$ to the Helmholtz equation $\Delta u + k^2 u = 0$ over $\mathbb{R}^3_+$ (cf. the subsequent (2.11)). To slightly simplify the subsequent formulas, we set $x_{f,3} = 0$ and consider the half space $\mathbb{R}^3_+$ instead of the general $\mathbb{R}^3_{x_3,+,}$. For large $R > 0$, we introduce the disc $\mathbb{R}^3_{0,R} := \{(x', 0)^\top \in \mathbb{R}^3 : |x'| < R\}$, the half ball $B_R := \{(x', x_3)^\top \in \mathbb{R}^3 : x_3 > 0, |\vec{x}| < R\}$ with its upper spherical boundary $S_R := \{(x', x_3)^\top \in \mathbb{R}^3 : x_3 > 0, |\vec{x}| = R\}$, and the cylinder $C_R := \{(x', x_3)^\top \in \mathbb{R}^3 : 0 < x_3 < R^{1/4} \text{ and } |x'| < R\}$ with its lateral and upper boundary (cf. Fig. 2)

$$T_R := T_{R,l} \cup T_{R,u}, \quad T_{R,l} := \left\{(x', x_3)^\top \in \mathbb{R}^3 : 0 < x_3 < R^{1/4} \text{ and } |x'| = R\right\},$$

$$T_{R,u} := \left\{(x', x_3)^\top \in \mathbb{R}^3 : x_3 = R^{1/4} \text{ and } |x'| < R\right\}.$$

We consider either $\Omega_R := B_R$ and $\Sigma_R := S_R$ or $\Omega_R := C_R$ and $\Sigma_R := T_R$. By $\nu$ we denote the normal at the boundary $\mathbb{R}^3_{0,R} \cup \Sigma_R$ of $\Omega_R$ pointing into outward direction. We assume that condition i) in (HSRC) is fulfilled. The symmetric Green’s formula applied to $u$ and $\vec{y} \mapsto \partial^2_{y_3} \Phi(\vec{x}, \vec{y})$ with fixed $\vec{x} \in \Omega_R$ leads to

$$\int_{\mathbb{R}^3_{0,R} \cup \Sigma_R} \left\{\partial_{\nu'} u \partial^2_{y_3} \Phi(\vec{x}, \cdot) - u \partial_{\nu'} \partial^2_{y_3} \Phi(\vec{x}, \cdot)\right\} = \int_{\Omega_R} \left\{\left(\Delta + k^2 I\right) u \partial^2_{y_3} \Phi(\vec{x}, \cdot) - u \left(\Delta + k^2 I\right) \partial^2_{y_3} \Phi(\vec{x}, \cdot)\right\}.$$

Using $\Phi(\vec{x}, \cdot) \equiv 0$ over $\mathbb{R}^3_0$ and the Helmholtz equation, we get that the second order derivative $\partial^2_{y_3} \Phi(\vec{x}, \cdot) = -\partial^2_{y_1} \Phi(\vec{x}, \cdot) - \partial^2_{y_2} \Phi(\vec{x}, \cdot) - k^2 \Phi(\vec{x}, \cdot)$ is zero over $\mathbb{R}^3_{0}$. From the Green’s function property we obtain $(\Delta + k^2 I) \partial^2_{y_3} \Phi(\vec{x}, \cdot) = \partial^2_{y_3} (\Delta + k^2 I) \Phi(\vec{x}, \cdot) = \partial^2_{y_3} \Phi(\vec{x}, \cdot).$ Thus, for a solution $u$ of the Helmholtz equation $(\Delta + k^2 I) u = 0$, we arrive at

$$\partial^2_{x_3} u(\vec{x}) = \int_{\mathbb{R}^3_{0,R}} \partial^3_{y_3} \Phi(\vec{x}, \cdot) u - \int_{\Omega_R} \left\{\partial_{\nu'} u \partial^2_{y_3} \Phi(\vec{x}, \cdot) - u \partial_{\nu'} \partial^2_{y_3} \Phi(\vec{x}, \cdot)\right\},$$

$$\partial^2_{x_3} u(\vec{x}) = [V_k u](\vec{x}) + I_{\infty},$$

$$[V_k u](\vec{x}) := \int_{\mathbb{R}^2} \partial^3_{y_3} \Phi(\vec{x}, (y', 0)^\top) u((y', 0)^\top) \, dy',$$

$$I_{\infty} := \lim_{R \to \infty} \int_{\Omega_R} \left\{\partial_{\nu'} u \partial^2_{y_3} \Phi(\vec{x}, \cdot) - u \partial_{\nu'} \partial^2_{y_3} \Phi(\vec{x}, \cdot)\right\}. \quad (2.11)$$

Figure 2: Half ball and cylinder.

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Here $[V_k u]$ is the twice differentiated double layer potential. Altogether, to get the representation in the radiation condition (1.2) for a solution $u$ of the Helmholtz equation over $\mathbb{R}_+^3$, we only have to suppose condition l) of (HSRC) and to show that the limit $I_{\infty}$ is zero.

3 Boundedness of the potential in (1.2)

Consider a function $u$ bounded over $\mathbb{R}_+^3$ and consider the twice differentiated double layer potential $[V_k u](\vec{x})$ defined by (2.12), where $\Phi$ is defined in (2.9). Without loss of generality, we fix $x'(0,0)^\top$ and consider the limit of $[V_k u](\vec{x})$ for $\vec{x} = (x', x_3)^\top$ with $x_3 \to \infty$. Due to (2.10) we have to estimate

$$I_{x_3} := \int_{\mathbb{R}^2} \partial_k^3 G(\vec{x}, (y', 0)^\top) f(y')dy', \quad f(y') := u((y', 0)^\top).$$

Taking into account (2.6) and the boundedness of the integral of the double layer kernel (cf. [9]), we get $I_{x_3} = (i k)^3 J_{x_3} + O(1)$ with

$$J_{x_3} := \int_{\mathbb{R}^2} e^{ik|\vec{x}-\vec{y}|} \frac{(0-x_3)^3}{|\vec{x}-\vec{y}|^4} f(y')dy' = - \int_{\mathbb{R}^2} e^{ik\sqrt{x_3^2+|y'|^2}} \frac{x_3^3}{\{x_3^2+|y'|^2\}^2} f(y')dy'$$

$$= - x_3 \int_{\mathbb{R}^2} e^{ikx_3 \sqrt{1+|z'|^2}} f(x_3z')dz'.$$

We substitute $z' = \sqrt{r^2-1} (\cos \phi, \sin \phi)^	op$ and $dz' = r d\phi dr$ to get

$$J_{x_3} = - x_3 \int_0^{2\pi} \int_1^{\infty} \frac{e^{ikx_3 r}}{r^3} \int \frac{f(x_3 \sqrt{r^2-1} (\cos \phi, \sin \phi)^	op)}{2\pi} d\phi dr.$$

The last integral is difficult to estimate. At least we get $|[V_k u](\vec{x})| \leq c|x_3|$. Here and in the following $c$ stands for a generic positive constant, the value of which varies from instance to instance.

Next we prove that $I_{x_3}$ fulfills at least the weak boundedness condition used in (1.3). For the terms in (4.1) and for $x' \neq y'$, we conclude

$$\int_0^{x_3} (x_3-t) \partial_{y_3}^3 G((x', t)^\top, (y', 0)^\top) dt = \partial_{y_3} G((x', x_3)^\top, (y', 0)^\top) - \partial_{y_3} G((x', 0)^\top, (y', 0)^\top)$$

$$+ x_3 \partial_{y_3}^2 G((x', 0)^\top, (y', 0)^\top),$$

where we have used $\partial_{x_3} \partial_{y_3} G(\vec{x}, \vec{y}) = (-1)^l \partial_{x_3}^l \partial_{y_3} G(\vec{x}, \vec{y})$. The formulas in (2.2) and (2.4) together with

$$e^{ik|x'-x_3|-(y',0)^\top} = e^{ik|x'-y'|} e^{ikx_3/|((x',x_3)^\top-(y',0)^\top)|} \delta_{(x',x_3)^\top-(y',0)^\top}$$

$$= e^{ik|x'-y'|} + O\left(\frac{x_3^2}{|((x',x_3)^\top-(y',0)^\top)|}\right),$$

$$\frac{1}{|(x', x_3)^\top-(y', 0)^\top|^l} = \frac{1}{|x'-y'|^l}$$

$$+ O\left(\frac{x_3^2}{|x'-y'|^l+1|((x',x_3)^\top-(y',0)^\top)|}\right), \quad l = 2, 3,$$
imply
\[
\int_0^{x_3} (x_3 - t) \partial_{y_3}^3 G((x', t)^T, (y', 0)^T) \, dt \\
= \frac{e^{ik\sqrt{x_3^2 + |x'|^2}}}{4\pi} \left\{ \frac{(ik)(-x_3)}{\sqrt{x_3^2 + |x'|^2}} - \frac{-x_3}{\sqrt{x_3^2 + |x'|^2}^3} \right\} \\
+ \frac{e^{ik|x'-y'|}}{4\pi} \left\{ \frac{(ik)x_3}{|x' - y'|^2} - \frac{x_3}{|x' - y'|^3} \right\} \\
= O\left( \frac{x_3^3}{|x' - y'|^2 \sqrt{x_3^2 + |x' - y'|^2}} \right) = O\left( \frac{x_3^{3-c_u}}{|x' - y|^2 \sqrt{x_3^2 + |x' - y'|^2}} \right)
\]
for $|x' - y'| \to \infty$. For $|x' - y'| \to \infty$. Now we split the integral $I_{x_3}$ into the integral over $y'$ with $|x' - y'| < 1$ and the integral over $y'$ with $|x' - y'| \geq 1$. Due to $|\partial^3 G(\bar{x}, \bar{y})| = O(|\bar{x} - \bar{y}|^{-1})$, the first integral is uniformly bounded and consequently also weakly bounded. For the second integral we get the weak boundedness by the above kernel estimate applied to $\int_0^{x_3} (x_3 - t) I_1 \, dt$. Repeating the same arguments as above we get

**Proposition 3.1.** Suppose that $f$ is continuous over $\mathbb{R}^2 = \mathbb{R}^3_0$ and that there are constants $c_f > 0$ and $0 \leq v_f < 1$ such that $|f(x')| < c_f(1 + |x'|)^{v_f}$ holds for any $x' \in \mathbb{R}^2$. Then the potential $[V_k f]$ satisfies the weak boundedness condition, i.e., \[17,3\] with $u$ replaced by $[V_k f]$.

For special functions $u$, i.e., for special $f$ we can get more. We shall assume that the gradient $\nabla f = \nabla_x f$ of $f = u|_{\mathbb{R}^3_0}$ is uniformly bounded. This assumption is obviously fulfilled in the typical situation that $u$ is a bounded solution of the Helmholtz equation over a layer $\{ x \in \mathbb{R}^3 : -\varepsilon < x_3 < \varepsilon\}$ with $\varepsilon > 0$. From (3.1) we obtain

\[
J_{x_3} = -x_3 2\pi f(0) \int_1^{\infty} \frac{e^{ikx_3r}}{r^3} r \phi \, dr \\
- x_3 \int_1^{\infty} \frac{e^{ikx_3r}}{r^3} \int_0^{2\pi} f\left( x_3 \sqrt{r^2 - 1} \left( \cos \phi \over \sin \phi \right) \right) - f(0) \, d\phi \, dr \\
= O(1) \\
- \frac{1}{ik} \int_1^{\infty} \frac{\partial_r \left( e^{ikx_3r} \right)}{r^3} \int_1^{r} \int_0^{2\pi} \nabla f\left( x_3 \sqrt{t^2 - 1} \left( \cos \phi \over \sin \phi \right) \right) \cdot \left( \cos \phi \over \sin \phi \right) \, d\phi \, \frac{t}{\sqrt{t^2 - 1}} \, dt \, dr \\
= O(1) - \frac{1}{ik} \int_1^{\infty} \frac{e^{ikx_3r}}{r^3} \int_1^{r} \int_0^{2\pi} \nabla f\left( x_3 \sqrt{t^2 - 1} \left( \cos \phi \over \sin \phi \right) \right) \cdot \left( \cos \phi \over \sin \phi \right) \, d\phi \, \frac{t}{\sqrt{t^2 - 1}} \, dt \, dr \\
- \frac{3}{ik} \int_1^{\infty} \frac{e^{ikx_3r}}{r^4} \int_1^{r} \int_0^{2\pi} \nabla f\left( x_3 \sqrt{t^2 - 1} \left( \cos \phi \over \sin \phi \right) \right) \cdot \left( \cos \phi \over \sin \phi \right) \, d\phi \, \frac{t}{\sqrt{t^2 - 1}} \, dt \, dr \\
+ \frac{1}{ik} \int_1^{\infty} \frac{e^{ikx_3r}}{r^2 \sqrt{r^2 - 1}} \int_1^{r} \int_0^{2\pi} \nabla f\left( x_3 \sqrt{r^2 - 1} \left( \cos \phi \over \sin \phi \right) \right) \cdot \left( \cos \phi \over \sin \phi \right) \, d\phi \, dr = O(1).
\]

For $f = u|_{\mathbb{R}^3_0} \in C^1_b(\mathbb{R}^2)$, we finally arrive at $|[V_k u](\vec{x})| \leq c$ for all $\vec{x} \in \mathbb{R}^3_+$. 

### 4 Representation of the solution by the potential operator in (1.2)

Suppose that $u$ is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ over $\mathbb{R}^3_+$, that $u$ as well as all derivatives upto order two are continuous on the closure of $\mathbb{R}^3_+$, and that the second derivative of $u$
w.r.t. $x_3$ is given as $\partial^2_{x_3} u = [V_k u]$ by the right-hand side of (1.2) with $x_{f,3} = 0$. Then, due to the second order Taylor-series expansion, we get

$$u(\bar{x}) = f_1(x') + f_2(x') (x_3 - x_{f,3}) + \int_{x_{f,3}}^{x_3} (x_3 - t)[V_k u](x', t)^\top dt,$$  \hspace{1cm} (4.1)

where the functions $f_1(x') := u(x', x_{f,3})^\top$ and $f_2(x') := \partial_{x_3} u(x', x_{f,3})^\top$ are solutions of the two-dimensional Helmholtz equations

$$\Delta_{x'} f_1(x') + k^2 f_1(x') = -[V_k u](x', x_{f,3})^\top,$$ \hspace{1cm} (4.2)

$$\Delta_{x'} f_2(x') + k^2 f_2(x') = -\partial_{x_3} [V_k u](x', x_{f,3})^\top.$$ \hspace{1cm} (4.3)

Indeed, from $\Delta = \Delta_{x'} + \partial_{x_3}^2$ we conclude, for the Helmholtz solution $u$, that $\Delta_{x'} u + k^2 u = -\partial_{x_3}^2 u$ and $\Delta_{x'} \partial_{x_3} u + k^2 \partial_{x_3} u = -\partial_{x_3} \partial_{x_3}^2 u$.

Note that, for any solution $[V_k u]$ of the three-dimensional Helmholtz equation, the right-hand side of (4.1) satisfies the three-dimensional Helmholtz equation if and only if $f_1$ and $f_2$ are solutions of (4.2) and (4.3), respectively. Indeed, using the Helmholtz equation for $[V_k u]$ in the form $(\Delta_{x'} + k^2 I)[V_k u] = -\partial_{x_3}^2 [V_k u]$ and the Taylor-series expansion for $[V_k u]$, we get

$$(\Delta + k^2 I) \left( f_1 + f_2 (x_3 - x_{f,3}) + \int_{x_{f,3}}^{x_3} (x_3 - t)[V_k u](\cdot, t)^\top dt \right)$$

$$= (\Delta_{x'} + k^2 I) f_1 + (x_3 - x_{f,3})(\Delta_{x'} + k^2 I) f_2$$

$$+ \int_{x_{f,3}}^{x_3} (x_3 - t)(\Delta_{x'} + k^2 I)[V_k u](\cdot, t)^\top dt + [V_k u](\cdot, x_{f,3})^\top.$$  \hspace{1cm} (4.4)

If a solution $u$ is given over the plane $\mathbb{R}^3_{x_{f,3}}$, then $f_1 = u|_{\mathbb{R}^3_{x_{f,3}}}$ is known. If the function $u$ is given over $\mathbb{R}^3_{x_{f,3}}$ and over $\mathbb{R}^3_{x_{d,3}}$ with, e.g., $x_{d,3} > x_{f,3}$, then $f_2$ is given by the Dirichlet-to-Neumann map

$$f_2(x') := (D_{x'} f_1)(x') := -2 \int_{\mathbb{R}^2} \partial_{y_3}^2 G((x', x_{f,3})^\top, (y', x_{f,3})^\top) u((y', x_{f,3})^\top) dy'$$

$$= -\frac{1}{(x_{d,3} - x_{f,3})} u((x', x_{d,3})^\top) + \int_{\mathbb{R}^2} K_{DIN}(x', y') u((y', x_{f,3})^\top) dy',$$  \hspace{1cm} (4.4)

$$K_{DIN}(x', y') := -2 \partial_{y_3}^2 G((x', x_{d,3})^\top, (y', x_{d,3})^\top) + \frac{2}{(x_{d,3} - x_{f,3})} \partial_{y_3} G((x', x_{d,3})^\top, (y', x_{f,3})^\top).$$

Using the formulas (2.2) and (2.4) as well as (3.3) and (3.4), we get the estimate $K_{DIN}(x', y') = \mathcal{O}(|x' - y'|^{-3})$ for $|x' - y'| \to \infty$ such that the last integral in (4.4) converges.

Unfortunately, the solutions of (4.2) and (4.3) are not unique. Even more, (1.2) and the representation with the right-hand side in (4.4) is fulfilled for $u(\bar{x})$ replaced by the sum $u(\bar{x}) + u_{2D}(x') x_3$ as well if only

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With \( u \) in (4.1) we might be uniquely determined by a Dirichlet boundary condition, the function \( f_2 \) is unique due to (1.3). Indeed, if \( f_2 \) leads to a solution in (4.1) bounded as in (1.3), then any different \( f_2 \) leads to a perturbation \( \tilde{x} \mapsto u(\tilde{x})+[\tilde{f}_2-f_2](x')x_3 \) violating (1.3).

5 Radiation condition for tensor-product solutions

Suppose \( u(\hat{y}) = u_{2D}(x')u_3(x_3) \) with a linear function \( u_3 \) and a solution \( u_{2D} \) of the two-dimensional Helmholtz equation \( \Delta x_{\nu} + k^2 u_{2D} = 0 \) over \( \mathbb{R}^2 \). Moreover suppose that \( u_{2D} \) and all first- and second-order derivatives are uniformly bounded at least for large \( |x'| \). Clearly, \( u \) is a solution of the three-dimensional Helmholtz equation. In particular, all assumptions, except the singular behaviour at the axis \( \{ \tilde{x} \in \mathbb{R}^3 : x' = (0, 0, 0) \} \), are satisfied for cylindrical waves of the form \( u(x) = \frac{1}{\pi} H_0^0(k|x'|) \) and for \( u(\tilde{x}) = \frac{1}{\pi} H_0^0(k|x'|)x_3 \).

Without loss of generality, we may suppose \( u_3(0) = 1 \) such that \( u|_{x_3=0} = u_{2D} \). For \( j = 1, 2 \), we obtain from the boundedness of \( u_{2D} \) and its derivatives and from the decay properties of the kernel functions in (2.2) and (2.3)

\[
\begin{align*}
\int_{-R}^{R} \frac{\partial y_j}{\partial y_j} \{ \partial_{y_j} \partial_{y_j} g \} u_{2D}(y') & = \mathcal{O}(R^{-2}), \\
\int_{-R}^{R} \frac{\partial^2 y_j}{\partial_{y_j} \partial_{y_j} g} u_{2D}(y') & = \mathcal{O}(R^{-2}),
\end{align*}
\]

if \( R \to \infty \). Consequently,

\[
\begin{align*}
\int_{-R}^{R} \int_{-R}^{R} \frac{\partial y_j}{\partial y_j} g(x',y',0) u_{2D}(y') dy_1 & dy_2 \\
& = - \int_{-R}^{R} \int_{-R}^{R} (\Delta y' + k^2) \partial_{y_j} g(x',y',0) u_{2D}(y') dy_1 dy_2 \\
& = - \int_{-R}^{R} \int_{-R}^{R} \partial_{y_j} g(x',y',0) (\Delta y' + k^2) u_{2D}(y') dy_1 dy_2 + \sum_{j=1}^{2} \frac{2}{\pi} \int_{-R}^{R} \mathcal{O}(R^{-2}) dx_j \\
& = \mathcal{O}(R^{-1}).
\end{align*}
\]

In other words, \( [V_k u](\tilde{x}) = 0 \). Condition (1.2) is always fulfilled. However, the pair of radiation conditions (1.2)-(1.3) hold if and only if the linear function \( u_3 \) is a constant function.

6 Radiation condition for plane-wave functions,

Fourier transform of the potential kernels

Now consider a plane-wave function \( u(\hat{y}) = e^{i(\alpha y_1 + \beta y_2 + \gamma y_3)} \) with \( \alpha, \beta \in \mathbb{R} \) and \( \alpha^2 + \beta^2 + \gamma^2 = k^2 \). For the case \( \alpha^2 + \beta^2 = k^2 \), we get \( \gamma = 0 \) and the results of Sect. 5 apply. Thus we may suppose \( \alpha^2 + \beta^2 \neq k^2 \) and \( \gamma \neq 0 \). We observe that \( G(\tilde{x}-\tilde{y}) = G(x',\tilde{y}) \) and \( \Phi_{x',y_3}(x'-y') := \partial_{y_j} \Phi(x',\tilde{y}) \) are convolution kernels. The exponential functions \( y' \mapsto e^{i(\alpha y_1 + \beta y_2)} \) are eigenfunctions of the convolution
and the eigenvalue is the value of the Fourier transform at \( \xi' := (\alpha, \beta) \in \mathbb{R}^2 \). Consequently, we have \([V_k u](\vec{x}) = e^{i(\alpha x_1 + \beta x_2)} g(x_3)\) with a special function \( g(x_3) \) independent of \( x' \). However, since \( V_k u \) is a solution of the Helmholtz equation, we conclude

\[
[V_k u](\vec{x}) = \begin{cases} 
  e^{i(\alpha x_1 + \beta x_2)} \left( c_1 e^{i\gamma x_3} + c_2 e^{-i\gamma x_3} \right) & \text{if } \gamma \neq 0 \\
  e^{i(\alpha x_1 + \beta x_2)} \left( c_1 + c_2 x_3 \right) & \text{else}
\end{cases} \tag{6.1}
\]

with special constants \( c_1 \) and \( c_2 \). The radiation condition \([1.2]\) is fulfilled if and only if \( c_1 = -\gamma^2 \) and \( c_2 = 0 \). If \( \Im \gamma > 0 \), then \( e^{-i\gamma x_3} \) increases exponentially for \( x_3 \to \infty \), and \( c_2 = 0 \) due to the estimates in Sect. 3. The explicit values of \( c_1 \) and \( c_2 \) can be computed by the Fourier transform of the convolution kernels. However, we prefer to argue using \([2.11]\) with the choice \( \Omega_R = C_R, \Sigma_R = T_R \).

If \( \alpha^2 + \beta^2 > k^2 \) and if \( \Im \gamma > 0 \), then \( e^{-i\gamma x_3} \) decreases exponentially for \( x_3 \to \infty \). We get

\[
|u(\vec{x})|, |\partial_3 u(\vec{x})| \leq c e^{-\Im \gamma R^{1/4}} \quad \text{on } T_{R,t}.
\]

The two involved kernel functions can be estimated by \(|G(\vec{x}, \vec{y})|, |\partial_{y_3} G(\vec{x}, \vec{y})| \leq c |\vec{x} - \vec{y}|^{-1} \) such that \(|\partial_{y_3}^2 \phi(\vec{x}, \vec{y})|, |\partial_{y_3}^2 \partial_{\vec{x}} \phi(\vec{x}, \vec{y})| \leq c[R^{1/4}]^{-1}\). The area of \( T_{R,t} \) is \( O(R^2) \). We arrive at

\[
\int_{T_{R,t}} \{ \partial_3 u \partial_{y_3}^2 \Phi(\vec{x}, \cdot) - u \partial_3 \partial_{y_3}^2 \Phi(\vec{x}, \cdot) \} = O(e^{-\Im \gamma R^{1/4}} R^{7/4}). \tag{6.2}
\]

According to the formula \([2.1]\) the Green's function differentiated w.r.t. \( y_3 \) can be estimated by the term \( O((x_3 \pm y_3)(\vec{x} - (y_3 \pm y_3))^{-2}) \). Therefore, on \( T_{R,t} \) the two involved kernel functions satisfy

\[
|\partial_{y_3}^2 \phi(\vec{x}, \vec{y})|, |\partial_{y_3} \partial_{\vec{x}} \phi(\vec{x}, \vec{y})| \leq c R^{1/4} R^{-2}, \quad \text{the functions } u \text{ and } \partial_3 u \text{ are bounded, and the area of } T_{R,t} \text{ is } O(R^{1/4} R).
\]

We conclude

\[
\int_{T_{R,t}} \{ \partial_3 u \partial_{y_3}^2 \Phi(\vec{x}, \cdot) - u \partial_3 \partial_{y_3}^2 \Phi(\vec{x}, \cdot) \} = O(R^{-1/2}). \tag{6.3}
\]

The estimates \([6.2] \) and \([6.3] \) together with \( T_R = T_{R,t} \cup T_{R,u} \) yield \( I_\infty = 0 \). Consequently, the radiation condition \([1.2] \) is satisfied and we get \( c_1 = -\gamma^2, c_2 = 0 \) in \([6.1] \).

If \( \Im \gamma < 0 \), then we get the same \( u|_{\Sigma_R} \) as for the choice \( \tilde{\gamma} = -\gamma \). From the just proved case for \( \Im \gamma > 0 \), the formulas \( c_1 = -\gamma^2, c_2 = 0 \) in \([6.1] \) for \( \gamma \) imply \( c_1 = 0, c_2 = -\gamma^2 \) in \([6.1] \) for \( \gamma \) and the radiation condition \([1.2] \) is not satisfied.

To compute the \( c_1 \) and \( c_2 \) for \( k^2 > \alpha^2 + \beta^2 \), i.e. for real \( \gamma \), we employ the principle of limited absorption. Choose a small \( \varepsilon > 0 \) and replace \( k \) by \( k_\varepsilon := k + i \varepsilon \). The fundamental solution \( G_{k_\varepsilon}(\vec{x} - \vec{y}) = e^{i k_{\varepsilon} |\vec{x} - \vec{y}| - k_{\varepsilon} |\vec{x} - \vec{y}|} \) is, in contrast to the case with real \( k \), exponentially decaying. Then choosing \( \gamma_{\varepsilon} := \sqrt{k_{\varepsilon}^2 - \alpha^2 - \beta^2} \) with \( \Re \gamma_{\varepsilon} > 0 \) and \( \Im \gamma_{\varepsilon} > 0 \) following exactly the proof for the case \( \alpha^2 + \beta^2 > k^2 \), we obtain the representation \( \partial_{y_3}^2 u_{\varepsilon}(\vec{x}) = [V_{k_\varepsilon} u_{\varepsilon}](\vec{x}) \) for the function \( u_{\varepsilon}(\vec{x}) = e^{i(\alpha x_1 + \beta x_2 + \gamma_{\varepsilon} x_3)} \).

In this representation we consider the limit for the parameter \( \varepsilon \to 0 \). Due to the \( O(|\vec{x} - \vec{y}|^{-3}) \) estimate for the kernel function in \( V_{k_\varepsilon} \), Lebesgue’s theorem on dominated convergence applies. We arrive at \( \partial_{y_3}^2 u_{\varepsilon}(\vec{x}) = [V_k u](\vec{x}) \), where \( u_{\varepsilon}(\vec{x}) = e^{i(\alpha x_1 + \beta x_2 + \gamma_{0} x_3)} + \gamma_{0} \) as \( \gamma_{0} \) the solution of \( \gamma_{0}^2 = k^2 - \alpha^2 - \beta^2 \), for which \( \gamma_{0} > 0 \) Thus \( c_1 = -\gamma_{0}^2 \) and \( c_2 = 0 \) holds in \([6.1] \) if \( \gamma_{0} > 0 \) and the radiation condition \([1.2] \) holds for the plane wave with \( \gamma > 0 \). Furthermore, \( c_1 = 0 \) and \( c_2 = -\gamma_{0}^2 \) holds in \([6.1] \) if \( \gamma = -\gamma_{0} < 0 \) and the radiation condition \([1.2] \) does not hold for the plane wave with \( \gamma < 0 \).

**Corollary 6.1.** Any quasiperiodic solution of the Helmholtz equation in \( \mathbb{R}^3_+ \) satisfies the radiation condition \([1.2] \) if and only if it satisfies the classical radiation condition, i.e., if it admits a Rayleigh series expansion into a sum of outgoing plane-wave modes.
Now we fix the formulas for the Fourier transform of the convolution kernel in \( V_k \), and we derive a presentation of \([V_k u]\) by general plane-wave functions. For this, we introduce the function \( \Phi(\xi) := \Phi(x, (0, 0, 0) \top) \) and observe \( \Phi(\xi, \xi g) := \Phi(x, y + x) \). Consequently, \( \partial_{x_3}^\alpha \Phi(x, \xi y) = -\partial_{x_3}^\alpha \Phi(x, y) \) and \([V_k u](x, x_3) \top\) is the convolution of \( u(x_3) \) by the function \( x' \mapsto -\partial_{x_3}^\alpha \Phi((x', x_3) \top) \). Introducing the Fourier transform as
\[
\mathcal{F}[f](\xi') := \int_{\mathbb{R}^2} e^{-2\pi i x \cdot \xi'} f(x) dx,
\]
the just proved results imply
\[
[V_k u](x', x_3) \top = \mathcal{F}^{-1}\left\{ m_{x_3} \mathcal{F}(u(x_3)) \right\}(x'), \quad (6.4)
\]
where the second term on the right-hand side tends to zero for \( x_3 \to 0 \).

In this generalized sense, (6.4) means
\[
[V_k u](\vec{x}) = \int_{\mathbb{R}^2} [\mathcal{F}(u(x_3))](\xi') \partial_{x_3}^\alpha e^{i(\xi', \sqrt{k^2 - |\xi'|^2}) \top x} dx'. \quad (6.5)
\]

The Fourier transform can also be used to compute the limit of \([V_k u](\vec{x})\) for \( x_3 \to 0 \) if \( x' \) is fixed. Suppose \( f := u(x_3) \) is a bounded function such that all derivatives up to order five are bounded. Choose a cut-off function \( y' \mapsto \chi(y') \) of the same smoothness and with \( \chi \) identical to one in a neighbourhood of \( x' \). Then \( f \chi \) is in \( L^2 \) and its Fourier transform \( \mathcal{F}(f \chi)(\xi') \) decays at infinity as \( O(|\xi'|^{-3}) \). We get
\[
[V_k u](\vec{x}) = \int_{\mathbb{R}^2} \partial_{y_3}^\beta \Phi(x, (y', y_3) \top)(f \chi)(y') dy' + \int_{\mathbb{R}^2} \partial_{y_3}^\beta \Phi(x, (y', y_3) \top)(f(1-\chi))(y') dy',
\]
where the second term on the right-hand side tends to zero for \( x_3 \to 0 \) due to (2.5) and Lebesgue’s theorem on dominated convergence. By Plancherel’s theorem we obtain
\[
[V_k u](\vec{x}) = -\int_{\mathbb{R}^2} e^{2\pi i x \cdot \xi'} (k^2 - |\xi'|^2) e^{i\sqrt{k^2 - |\xi'|^2} x_3} \mathcal{F}(f \chi)(\xi') d\xi' + o(1).
\]

Again Lebesgue’s theorem together with \( \mathcal{F}(f \chi)(\xi') = O(|\xi'|^{-3}) \) for \( |\xi'| \to \infty \) lead us to
\[
\lim_{x_3 \to 0} [V_k u](\vec{x}) = -\int_{\mathbb{R}^2} e^{2\pi i x \cdot \xi'} (k^2 - |\xi'|^2) \mathcal{F}(f \chi)(\xi') d\xi',
\]
where the second term on the right-hand side tends to zero for \( x_3 \to 0 \).

Of course, (6.6) holds under reduced smoothness assumptions on \( u \). Namely, it is sufficient to suppose that all the derivatives of \( u \) up to order two are bounded and continuous on \( \mathbb{R}_0^3 \). Due to \( \partial_{y_3}^\alpha G(x, \xi y) = O(|\xi y|^3) \) for \( |\xi y| \to 0 \), we can fix \( \vec{x} \in \mathbb{R}_0^3 \) and can reduce the analysis to functions \( u \) which have a finite support. Computing classical limits of potential operators in the form of finite-part integrals, we obtain the same limits as in the smoother case considered before. Hence, if \( u(\vec{x}) \in C_b^2(\mathbb{R}^2) \) with Hölder continuous second derivative, then we get
\[
[V_k u](x', 0) \top = -\partial_{x_1}^2 u((x', 0) \top) = \partial_{x_2}^2 u((x', 0) \top). \quad (6.7)
\]

If \( u(\vec{x}) \in C_b^2(\mathbb{R}^2) \), then the limit \([V_k u]\) in (6.7) holds locally in the \( L^2 \) sense. In particular, if the function \( f := u(\vec{x}), \vec{y} \) is the restriction of a bounded Helmholtz solution in the half space \( \{ \vec{x} \in \mathbb{R}_0^3 : -\varepsilon < x_3 \} \), then \( u \) is sufficiently smooth and (6.7) holds. By the same arguments we get even more.

**Proposition 6.2.** The limit relation (6.7) holds if \( u(\vec{x}) \in C_b^2(\mathbb{R}^2) \) and if there are constants \( C > 0 \) and \( 0 < u < 1 \) such that \( |u((x', 0) \top)| < C(1 + |x'|)^u \) is true for any \( x' \in \mathbb{R}^2 \).
7 Radiation condition for point-source functions

Suppose \( u \) is a Helmholtz solution on \( \mathbb{R}^3_+ \), which is bounded together with its derivatives up to order two on the closure of \( \mathbb{R}^3_+ \). Similarly to Sommerfeld’s condition on the full space \( \mathbb{R}^3 \), we define

**Definition 7.1.** We shall say that a function \( u \) on \( \mathbb{R}^3_+ \) satisfies the outgoing Sommerfeld half-space radiation condition if

\[
\text{sup}_{\bar{\xi} \in \mathbb{R}^3_+; |\bar{\xi}| = r} \left| \partial_{\bar{\xi}} u(\bar{\xi}) - i k u(\bar{\xi}) \right| \to 0, \quad r \to \infty, \quad \text{sup}_{\bar{\xi} \in \mathbb{R}^3_+; |\bar{\xi}| \geq R} |\bar{\xi}| |u(\bar{\xi})| < \infty. \tag{7.1}
\]

It is well known that, for any fixed \( \bar{y} \in \mathbb{R}^3 \), the Green’s function \( \mathbb{R}^3_+ \ni \bar{x} \mapsto G(\bar{x}, \bar{y}) \) and any derivative w.r.t. \( \bar{x} \) or \( \bar{y} \) satisfy Sommerfeld’s radiation condition. Hence, these point source functions also satisfy (7.1).

Suppose \( \bar{y} = (y', y_3) \) with \( y_3 < 0 \). We shall prove (1.2) for the point-source function \( u(\bar{x}) := G(\bar{x}, \bar{y}) \) using only the properties fixed in (7.1). Choosing \( \Omega_R = \mathbb{B}_R \), \( \Sigma_R = \mathbb{S}_R \) (cf. Sect. 2), we shall employ the representation (2.11). It remains to prove \( I_{\mathbb{S}} = 0 \). The estimates for this, however, are exactly the same as for the full space Sommerfeld condition. Indeed, the fundamental solution \( G \) satisfies (7.1) and we get

\[
\int_{\Sigma_R} \{ \partial_x u \partial_y \Phi(\bar{x}, \cdot) - u \partial_x \partial_y \Phi(\bar{x}, \cdot) \} = \int_{\Sigma_R} \{ [ik] u \partial_y \Phi(\bar{x}, \cdot) - u [ik] \partial_y \Phi(\bar{x}, \cdot) \} \tag{7.2}
\]

\[
+ \int_{\Sigma_R} o(R^{-2}) = \int_{\Sigma_R} o(R^{-2}) = o(1).
\]

**Corollary 7.2.** Any solution of the Helmholtz equation over \( \mathbb{R}^3_+ \) satisfying the outgoing Sommerfeld half-space radiation condition (7.1) satisfies the (HSRC) too.

For the “incoming” point-source \( u(\bar{x}) := \tilde{G}(\bar{x}, \bar{y}) \) with \( y_3 < 0 \), we have (7.1) but with the term \( ik u(\bar{x}) \) replaced by \( -ik u(\bar{x}) \). Instead of (7.2), we arrive at

\[
\int_{\Sigma_R} \{ \partial_x u \partial_y \Phi(\bar{x}, \cdot) - u \partial_x \partial_y \Phi(\bar{x}, \cdot) \} = 2ik I_{i,R} + o(1), \quad I_{i,R} := \int_{\Sigma_R} \partial_y \Phi(\bar{x}, \cdot) u.
\]

Taking the asymptotically largest term from (2.4), we conclude

\[
I_{i,R} = \int_{\Sigma_R} \frac{(ik)^2 e^{ik|\bar{x}-\bar{z}|}(z_3 - x_3)^2 e^{-ik|\bar{y}-\bar{z}|}}{4\pi |\bar{x} - \bar{z}|^3} \, d\bar{z} + o(1)
\]

\[
= \frac{(ik)^2}{16\pi^2} \int_{\Sigma_R} e^{ik(\bar{x}-\bar{z})} \frac{z_3^2}{|\bar{z}|^4} \, d\bar{z} + o(1).
\]

Switching to spherical coordinates, we get

\[
I_{i,R} = \frac{(ik)^2}{16\pi^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} \sin \theta \cos \phi \sin \theta \sin \phi \cos \theta \cos^2 \theta \sin \theta \, d\phi \, d\theta + o(1)
\]

\[
= \frac{(ik)^2}{16\pi^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} \cos \theta \sin \phi \sin \theta \, d\phi \, d\theta + o(1)
\]

\[
= \frac{(ik)^2}{16\pi^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} \cos \theta \sin \theta \, d\phi \, d\theta + o(1)
\]

\[
= \frac{(ik)^2}{8\pi} \int_0^{\pi/2} J_0(k|\bar{x}' - \bar{y}'| \sin \theta) e^{ik(x_3 - y_3)} \cos^2 \theta \sin \theta \, d\theta + o(1)
\]

\[
= \frac{(ik)^2}{8\pi} \int_0^1 e^{ik(x_3 - y_3)t} J_0\left(k|\bar{x}' - \bar{y}'| \sqrt{1 - t^2}\right) t^2 \, dt + o(1).
\]
Integration by parts leads us to

\[
I_{\infty} = \frac{(ik)^2}{8\pi} \int_0^1 e^{ik(x_3-y_3)t} J_0 \left( k|x'-y'|\sqrt{1-t^2} \right) t^2 \, dt
\]

\[
= \left[ \frac{(ik)^2}{8\pi} e^{ik(x_3-y_3)t} J_0 \left( k|x'-y'|\sqrt{1-t^2} \right) \right]_{t=0}^1 - \frac{(ik)^2}{8\pi} e^{ik(x_3-y_3)t} \int_0^1 e^{ik(x_3-y_3)t} \partial_t \left( J_0 \left( k|x'-y'|\sqrt{1-t^2} \right) t^2 \right) \, dt
\]

\[
= \frac{(ik)^2}{8\pi} e^{ik(x_3-y_3)} \int_0^1 e^{ik(x_3-y_3)t} \partial_t \left( J_0 \left( k|x'-y'|\sqrt{1-t^2} \right) t^2 \right) \, dt + O \left( \frac{1}{|x_3|^2} \right)
\]

for \(|x_3| \to \infty\). Hence, the limit \(I_{\infty}\) is not identically zero. In other words, for the “incoming” point-source, the radiation condition (1.2) is not fulfilled.

8 Condition (HSRC) independent of \(x_{h,3}\) and \(x_{f,3}\), and equivalence of the conditions (1.3) and (1.4)

0.1 For the dependence of (HSRC) from \(x_{h,3}\) and \(x_{f,3}\), we notice that the representation (4.1) together with the decay \(O(|\bar{x} - \bar{y}|^{-3})\) of the kernel functions easily imply the condition i) for any fixed \(x_{h,3}\). Therefore, it is sufficient to check the dependence on \(x_{f,3}\). For definiteness, we compare ii) with a fixed \(x_{f,3} = y_{f,3} > 0\). We denote the operator \(V_k\) defined in (2.12) with 0 replaced by \(x_{f,3}\) (i.e. integration over \(\mathbb{R}^3_{y_{f,3}}\) instead of \(\mathbb{R}^3_0\)) by \(V_{k,y_{f,3}}\). The right-hand side of (1.2) is \([V_k u|_{\mathbb{R}^3_{y_{f,3}}}]\) for \(x_{f,3} = 0\) and \([V_{k,y_{f,3}} u|_{\mathbb{R}^3_{y_{f,3}}}]\) for \(x_{f,3} = y_{f,3}\). However, by the arguments leading to (2.11) and (6.3), we get \([V_{k,x_{f,3}} u|_{\mathbb{R}^3_{x_{f,3}}}](\bar{x}) = [V_k u|_{\mathbb{R}^3_{y_{f,3}}}](\bar{x})\) for any \(\bar{x} \in \mathbb{R}^3_{x_{f,3}}\). In other words, condition (1.2) with \(x_{f,3} = 0\) implies (1.2) with \(x_{f,3} = y_{f,3}\), and (1.2) with \(x_{f,3} = y_{f,3}\) implies (1.2) with \(x_{f,3} = 0\) at least for \(\bar{x} \in \mathbb{R}^3_{x_{f,3}}\). Consequently, the analytical function on both sides of (1.2) coincide on the whole domain of analyticity, i.e., (1.2) holds on \(\mathbb{R}^3_{0}\). (HSRC) is indeed independent of \(x_{h,3}\) and \(x_{f,3}\). Note that, assuming the items i) and ii) of (HSRC) and fixing an \(x_{f,3} > 0\), the weak boundedness condition is equivalent to \(|1/(x_3 - x_{f,3}) \int_{x_{f,3}}^{x_3} (x_3 - t)/(x_3 - x_{f,3}) u((x',t)^+) \, dt| < c_u (x_3 - x_{f,3})^{-1-\epsilon}\) for all \(x_3 > x_{f,3} + 1\). Indeed, i) and ii) imply the boundedness of solution \(u\) in \(\{ \bar{x} \in \mathbb{R}^3 : 0 < x_3 < x_{f,3} \}\).

8.2 For the equivalence of the conditions (1.3) and (1.4), we observe that the restriction \(v'(x') := u((x',x_{f,3})^+)\) is in the space \(C_0^\infty(\mathbb{R}^2, t \geq 0)\). This follows from condition i) of (HSRC), and from the inequality \(x_{f,3} > \sup_{x' \in \mathbb{R}^2} |f(x')|\). We retain the notation of the spaces \(A_{\infty}^k, DD_{v}\), and \(FC_k(\mathbb{R}^2)\) from (1.7), (1.8), and (1.9), respectively. Again, for simplicity, we may suppose \(x_{f,3} = 0\).

Lemma 8.1. a) If \(v \in FC_k\) or if \(v \in AV_\kappa\) with \(\kappa > 0\), then \(f_2 := D_t N v\) is a well-defined bounded function which is a partial solution of (4.3) with \(u\) replaced by \(v\). Moreover, the function \(u_p := u\) of (4.1), defined with \(f_1 := u\) and \(f_2 := D_t N v\), satisfies (1.3).

b) If \(v \in DD_v\) with \(0 \leq v < 1\), then the same assertions are true for \(\mathbb{R}^3_0\) replaced by \(\mathbb{R}^3_\varepsilon\) with \(\varepsilon > 0\).

Proof. i) First we suppose \(v \in FC_k\). Due to this we have \(v \in C_0^\infty(\mathbb{R}^2)\). The multiplied and differentiated functions \(x' \mapsto (1 + |x'|^2)^{-1/2} \partial_{x'}^l v(x')\), \(l = 0, 1, \ldots, 6\) are in the space \(L^2(\mathbb{R}^2)\). Equivalently, the functions \(x' \mapsto \partial_{x'}^l [(1 + |x'|^2)^{-1/2}] v(x')\), \(l = 0, 1, \ldots, 6\) are in \(L^2(\mathbb{R}^2)\) such that there exists a function \(v_L \in L^2(\mathbb{R}^2)\) with \(v(x') = (1 + |x'|^2)(-\Delta_{x'}) v_L(x')\) and the Fourier transform
\([ \mathcal{F} v](\xi') = (I - \Delta_{\xi'})(1 + |\xi'|^2)^{-3}[\mathcal{F} v_L](\xi'). \]
From \(\mathcal{F}(-2\partial_{x_3}^2 G((\cdot, 0)\top) = \sqrt{k^2 - |\xi'|^2}\) (compare (6.4)), we obtain
\[
D_t N v(x') = \int_{\mathbb{R}^2} (I - \Delta_{\xi'}) \left\{ e^{i2\pi x' \cdot \xi'} \sqrt{k^2 - |\xi'|^2} \right\} (1 + |\xi'|^2)^{-3}[\mathcal{F} v_L](\xi') \, d\xi'.
\]
(8.1)

Similar formulas hold for the derivatives w.r.t. \(x'\), i.e., for
\[
(\Delta_{x'} + k^2 I) D_t N v(x') = \int_{\mathbb{R}^2} (I - \Delta_{\xi'}) \left\{ e^{i2\pi x' \cdot \xi'} \sqrt{k^2 - |\xi'|^2} \right\} (1 + |\xi'|^2)^{-3}[\mathcal{F} v_L](\xi') \, d\xi',
\]
which is exactly \(\partial_{x_3} [V_{k,u}](x', 0)^\top\) (for the Fourier transform of the kernel \(\partial_{y_3}^4 \Phi(\cdot, \tilde{y})\) compare (6.4)).

If we suppose that the Fourier transform of \(v\) vanishes over the annular domain \(R_{k,e}\), used in the definition of (1.9), then we get smooth and bounded values \(D_t N v(x')\) and \((\Delta_{x'} + k^2 I) D_t N v(x')\) for \(x'\) in bounded domains. Shifting the coordinate system and repeating all the above arguments, we even get uniform boundedness over \(\mathbb{R}^2\). In other words, \(f_2 := D_t N v\) is a well-defined bounded function which is a partial solution of (4.1), defined with \(f_1 := v\) and \(f_2 := D_t N v\). For the terms in (4.1) and for \(x' \neq y'\), we conclude (3.2). In view of (3.2) and \(\partial_{y_3}^4 \Phi((x', 0)^\top, (y', 0)^\top) = \delta_{x'}(y')\) (cf. (2.7) and take into account the jump relation for the double layer kernel), we arrive at
\[
u_p(\bar{x}) := v(x') + x_3[D_t N v](x') + \int_0^{x_3} (x_3 - t)[V_{k,u}](x', t)^\top \, dt,
\]
(8.2)

This is the double layer integral with Green's function from (1.1). Switching to Fourier transforms (compare (6.4)), we get
\[
u_p(\bar{x}) = \int_{\mathbb{R}^2} e^{i2\pi x' \cdot \xi'} e^{i\sqrt{k^2 - |\xi'|^2}x_3}[\mathcal{F} v](\xi') \, d\xi'.
\]
(8.3)

Now the Taylor-series expansion at \(x_3 = 0\) for the function \(x_3 \mapsto e^{i\sqrt{k^2 - |\xi'|^2}x_3}/([|\xi'|^2 - k^2])\) takes the form
\[
e^{i\sqrt{k^2 - |\xi'|^2}x_3}/([|\xi'|^2 - k^2]) = \frac{1}{|\xi'|^2 - k^2} - \frac{i}{\sqrt{k^2 - |\xi'|^2}}x_3 + \int_0^{x_3} (x_3 - t)e^{i\sqrt{k^2 - |\xi'|^2}t} \, dt,
\]

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and leads us to
\[
\int_0^{x_3} (x_3 - t) u_p((x', t)') \, dt = \int_{\mathbb{R}^2} (I - \Delta \cdot) \left\{ e^{i 2 \pi x' \cdot \xi} \left[ e^{i \sqrt{k^2 - |\xi|^2} x_3} \frac{1}{|\xi|^2 - k^2} - \frac{i}{\sqrt{k^2 - |\xi|^2}} \right] \right\} \]
\[
\left( 1 + |\xi|^2 \right)^{-3} [Fv_{I2}] (\xi') \, d\xi',
\]
where, again, the assumption $[Fv] (\xi') = 0$, $\xi' \in \mathbb{R}^{k,\varepsilon}$ frees us from any trouble with the nonsmoothness of $\sqrt{k^2 - |\xi|^2}$, and the factor $(1 + |\xi|^2)^{-3}$ guarantees integrability for large $|\xi'|$. Applying the two-dimensional Laplacian $\Delta_{\cdot}$ to the term in brackets, we get at most a factor $x_3^2$ or a factor $|x'|^2$ such that $u_p$ satisfies the weak boundedness condition (1.3) for $|x'| < c$. Shifting the $x'$ coordinates, we get the same result for any $x'$. Hence the solution $u = u_p$ of (4.1), defined with $f_1 := v$ and $f_2 := D_t N v$, satisfies (1.3).

Splitting a general $v$ into a sum of two functions, one with a Fourier transform vanishing in the annular domain $R_{k,\varepsilon}$ and one with support contained in the domain $R_{k,2\varepsilon}$, it remains to prove the lemma for the latter case. This case, however, is completely analogous to the just finished place. The only difference is that we apply the assumptions of functions from $FC_{k}$ to the term in brackets. This case, however, is completely analogous to the just finished place. The only difference is that we apply the assumptions of functions from $FC_{k}$ (cf. (1.9)) on the annular domain. Thus $[Fv] (\xi') = (I - \Delta_{\cdot})(1 + |\xi'|^2)^{-3} [Fv_{I2}] (\xi')$ turns to $[Fv] (\xi') = [Fv] (\xi')$ with support in $R_{k,2\varepsilon}$ and, e.g., (8.1) and (8.3) into
\[
D_t N v(x') = \int_{R_{k,\varepsilon}} e^{i 2 \pi x' \cdot \xi} \sqrt{k^2 - |\xi|^2} [Fv] (\xi') \, d\xi',
\]
\[
u_p(x) = \int_{R_{k,\varepsilon}} e^{i 2 \pi x' \cdot \xi} e^{i \sqrt{k^2 - |\xi|^2} x_3} [Fv] (\xi') \, d\xi'.
\]
We finally get the uniform boundedness of $\nu_p$ and all the assertions of Lemma 8.1 for $v \in FC_{k}$.

ii) Now we assume $v \in AV_{k}$. The Dirichlet-to-Neumann map on the right-hand side of (1.4) is a convolution operator with kernel depending only on $|x' - y'|$. Using (1.6), its takes the form
\[
\begin{align*}
[D_t N v](x') & := -2 \int_{\mathbb{R}^2} \partial_{y_3} G((x', 0)^{\top}, (y', 0)^{\top}) \, v(y') \, dy' \\
& = -2 \int_{\mathbb{R}^2} \partial_{y_3} G((y', 0)^{\top}, (0, 0, 0)^{\top}) \, v(x' - y') \, dy' \\
& = \int_{|y'| > 1} \mathcal{O} (|y'|^{-2} + |y'|^{-3}) \, v(x' - y') \, dy' + \mathcal{O}(1),
\end{align*}
\]
\[
||D_t N v||_{L^1} \leq \int_{1}^{\infty} \mathcal{O}(r^{-1}) \, |aw(v, x', r)| \, dr + \mathcal{O}(1),
\]
where the term $\mathcal{O}(1)$ results from an integration of a finite-part integral for a sufficiently smooth function. The estimate of $aw(v, x', r)$ in the definition (1.7) of $AV_{k}$ implies the continuity and uniform boundedness of $D_t N v$. Analogously, from (8.2) we conclude
\[
|u_p(x)| \leq \int_{1}^{\infty} \mathcal{O}(r^{-1}) \, |aw(v, x', r)| \, dy' + \mathcal{O}(1),
\]
such that the estimate of $aw(v, x', r)$ in (1.7), implies the continuity and uniform boundedness of the solution $u_p$. If we apply $(\Delta_{\cdot} + k^2 I)$ to $D_t N v$ and $(\Delta + k^2 I)$ to $u_p$, the convergence of the
Proposition 9.1. The solution of problem (9.1) is unique.

iii) Finally, we assume \( v \in DD_0 \). Taking \( v_s, v_i, v_0 \) in accordance with (1.8), we define the Helmholtz solution \( u_{DD}(x',x_3)^{\top} := v_s(x') - [V_k v_0](x',x_3)^{\top} \). Using Prop. 6.2, we get the boundary value

\[
\begin{align*}
   u_{DD}(x') &= v_s(x') - [V_k v_0](x',0)^{\top} \\
   &= v_s(x') + [(\Delta_{x'} + k^2 I) v_0](x') = v_s(x') + v_i(x') = v(x').
\end{align*}
\]

Furthermore, using \( [V_k v_s] = 0 \) (cf. Sect. 5) and the fact that differentiation and convolution operator commute, we conclude

\[
\begin{align*}
   \partial_{x_3}^2 u_{DD}(\vec{x}) &= -\partial_{x_3}^2 [V_k v_0](\vec{x}) = (\Delta_{x'} + k^2 I)[V_k v_0](\vec{x}) \\
   &= [V_k(\Delta_{x'} + k^2 I) v_0](\vec{x}) = [V_k v_1](\vec{x}) = [V_k v](\vec{x})
\end{align*}
\]

such that (1.2) is fulfilled. Together with Prop. 3.1, the radiation condition (HSRC) holds for \( u_{DD} \). In other words, \( u_{DD} \) is a solution to the Dirichlet problem \( u_{DD}(\vec{x}) = v(\vec{x}), \; \vec{x} \in \mathbb{R}^3 \) of the Helmholtz equation satisfying the radiation condition (HSRC).

Setting \( f_2 := D_1 N v := \partial_{x_3} u_{DD}|_{\mathbb{R}^3} \), we get a well-defined solution of (4.3) with \( u \) replaced by \( v \). Indeed,

\[
\begin{align*}
   (\Delta_{x'} + k^2 I)f_2 &= (\Delta_{x'} + k^2 I)\partial_{x_3}[V_k v_0] = \partial_{x_3}(\Delta_{x'} + k^2 I)[V_k v_0] \\
   &= \partial_{x_3}[V_k(\Delta_{x'} + k^2 I) v_0] = \partial_{x_3}[V_k v_1] = \partial_{x_3}[V_k v].
\end{align*}
\]

Clearly, by the Taylor-series expansion, we get that the function \( u_p := u \) of (4.1), defined with \( f_1 := v \) and \( f_2 := D_1 N v \), is equal to \( u_{DD} \). Hence, it satisfies (1.3).

Now the equivalence of the conditions (1.3) and (1.4) is easy to show. The general solution of (4.3) is \( f_{2,g} = D_1 N \tilde{v} + f_{2,h} \) with \( \tilde{v} = u|_{\mathbb{R}^3} \) and with a solution \( f_{2,h} \) of the two-dimensional Helmholtz equation \( (\Delta_{x'} + k^2 I) f_{2,h} = 0 \). The solution \( u_g = u \) in (4.1), defined with \( f_1 := \tilde{v} \) and \( f_2 := f_{2,g} \), takes the form \( u_g(\vec{x}) = u_p(\vec{x}) + f_{2,h}(x',x_3) \). This, however, fulfills (1.3) if and only if \( f_{2,h} \equiv 0 \), i.e., if and only if \( \partial_{x_3} u_g = f_{2,g} = D_1 N(u|_{\mathbb{R}^3}) \). The uniform boundedness of the solution \( v \), if \( v \in AV_\nu \) or if \( v \in DD_0 \) with \( v_0 \in C^2_0(\mathbb{R}^2) \), follows from the points ii) and iii) of the proof to Lemma 8.1 and from the last argument in Sect. 3.

9 Solution of the Dirichlet problem over the half space

For a function \( u \) continuous on the closure of \( \mathbb{R}^3_+ \) and twice differentiable on \( \mathbb{R}^3_+ \), we consider the Dirichlet boundary value problem

\[
\begin{align*}
   \Delta u(\vec{x}) - k^2 u(\vec{x}) &= 0, \; \forall \vec{x} \in \mathbb{R}^3_+, \\
   u((x',0)^{\top}) &= v(x'), \; \forall x' \in \mathbb{R}^2, \\
   u &\text{ satisfies (HSRC)}
\end{align*}
\]

Proposition 9.1. The solution of problem (9.1) is unique.
Proof. For two solutions $u_1$ and $u_2$ the difference $u = u_1 - u_2$ satisfies the homogeneous problem \[ (\Delta u + k^2 I) v = 0 \], i.e., the problem with $v \equiv 0$. However, from the radiation condition we get the representation \[ (\Delta u + k^2 I) v = f_1(x') + f_2(x') x_3. \] From the boundedness condition \[ (1.3) \], we get $f_2 \equiv 0$ and, from the homogeneous Dirichlet condition, $f_1 \equiv 0$. Hence, $u \equiv 0$ and the two solutions $u_1$ and $u_2$ coincide.

It is unclear to us, whether there exists solutions of \[ (9.1) \] for any Dirichlet data in $C_b(\mathbb{R}^2)$. Even if only the items i) and ii) of the radiation condition (HSRC) are satisfied, then we get a necessary condition. Namely, there must exist solutions of \[ (4.3) \]. We do not know whether this is always fulfilled. Therefore, by $DS$ we denote the space of all $v \in C_b(\mathbb{R}^2)$ such that there exists a solution $f_2$ of \[ (4.3) \] with $f_2 \in C_b(\mathbb{R}^2)$ for all integers $l > 0$. Unfortunately, the solution $f_2$ is not unique. The general solution of \[ (4.3) \] is the sum of the partial solution $f_2$ and a homogeneous solution of the two-dimensional Helmholtz equation. We easily obtain the formal result

**Proposition 9.2.** For $v \in C_b(\mathbb{R}^2)$, there exists a solution of \[ (9.1) \], possibly without condition \[ (1.3) \], if and only if $v$ is in the space $DS$. If $v \in DS$, then the solution is given by

$$u(x) = v(x') + [D_t N_2 v]((x',0)') x_3 + \int_0^{x_3} (x_3 - t) [V_k v]((x', t)') dt,$$

$$[V_k v](x) := \int_{\mathbb{R}^2} \partial_{x_3}^2 \Phi(x, (y',0)') v(y') dy',$n

$$[D_t N_2 v]((x',0)') := f_2(x') - \int_0^{x_3} [V_k v]((x', t)') dt$$

with $f_2 \in C_b'(\mathbb{R}^2)$, $l \in \mathbb{Z}$, $l > 0$ the solution of \[ (4.3) \]. Suppose there is a linear operator $D_t N_2$ mapping $v$ to a solution $f_2$ and, for the current proposition, replace item iii) of the radiation (HSRC) by the condition $\partial_{x_3} u|_{x_3=0} = D_t N_2 u|_{x_3=0}$. Then there exists a unique solution of \[ (9.1) \], which takes the form \[ (9.2) \] with $f_2 = D_t N_2 v$.

To avoid the non-practical assumptions in Prop. 9.2, we have to restrict the condition on $v \in C_b(\mathbb{R}^2)$.

**Proposition 9.3.** Suppose that $v$ is either in $AV_N$, $\kappa > 0$, in $DD_v$, $0 \leq v < 1$ or in $FC_k$. Then there exists a unique solution of \[ (9.1) \], which is even uniformly bounded for $v \in AV_N$ and for $v \in DD_0$ with $v_0 \in C_b^1(\mathbb{R}^2)$.

Proof. In the case of $DD_v$, the function $u_{DD}$ is the solution due to part iii) of Lemma 8.1. The boundedness follows from the last argument in Sect. 3. For the case of $AV_N$ and $FC_k$, the function $u_p$ of \[ (8.2) \] is the solution in accordance with the parts i) and ii) of Lemma 8.1. Even the boundedness for the solution in the case $AV_N$ has been shown there.

**Proposition 9.4.** Consider all the $v \in DD_0$ with $v_0 \in C_b^1(\mathbb{R}^2)$ (cf. \[ (1.8) \]). Then the corresponding splitting $v = v_i + v_s$ is unique.

Proof. We have to show that the Dirichlet data cannot satisfy both, the two-dimensional Helmholtz equation $(\Delta v + k^2 I) v = 0$ and the representation $v = (\Delta v + k^2 I) v_0$. We shall suppose both and show $v \equiv 0$. From the Helmholtz equation, we get, for any test function $\varphi$, that

$$0 = \langle v, (\Delta v + k^2 I) \varphi \rangle.$$
Now we substitute the representation of \( v \) as the image of the Helmholtz operator and choose \( \varphi = \chi_\varepsilon v_0 \), where \( \chi_\varepsilon \) denotes a cut-off function with \( \chi_\varepsilon \equiv 1 \) on the disc \( D_l := \{ x' \in \mathbb{R}^2 : |x'| \leq l \} \) and \( \chi_\varepsilon \equiv 0 \) on the exterior \( \mathbb{R}^2 \setminus D_{l+1} \) of the larger disc \( D_{l+1} \).

\[
0 = \langle (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I) \chi_\varepsilon v_0 \rangle \\
= \langle \chi_\varepsilon (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle \\
+ \langle (\Delta_{x'} + k^2 I)v_0, [v_0 \Delta_{x'} \chi_\varepsilon + \sum_{j=1}^2 \partial_{x_j} \chi_\varepsilon \partial_{x_j} v_0] \rangle \\
= \langle \chi_\varepsilon (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle + O(1).
\]

Consequently,

\[
\langle \chi_{L+1}(\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle = \langle \chi_1(\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle + \\
\sum_{l=1}^L \langle [\chi_{l+1} - \chi_1](\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle \\
= O(1).
\]

If the truncated sum of nonnegative terms is uniformly bounded, then the infinite sum is convergent, and we arrive at

\[
\langle (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \rangle < \infty.
\]

In other words, \( v = (\Delta_{x'} + k^2 I)v_0 \) is square integrable over \( \mathbb{R}^2 \). As a solution of the Helmholtz equation the square integrable Fourier transform \( \mathcal{F}[v] \) satisfies \( (\xi'|^2 + k^2) \mathcal{F}[v](x') \equiv 0 \). Thus \( \mathcal{F}[v](x') \equiv 0 \) and \( v = 0 \).

The space of all \( DD_0 \) with \( v_0 \in C^1_b(\mathbb{R}^2) \) is algebraically the direct sum of the space of Helmholtz solutions plus the space of all images of the Helmholtz operator. If the metric of the function space corresponds to the uniform convergence over bounded subdomains, then the space of Helmholtz solutions is closed. However, the space of images is not. For example, the function \( x' \mapsto e^{i(kx_1 + \beta x_2)} \) with \( \alpha^2 + \beta^2 = k^2 \) is the limit of functions \( x' \mapsto e^{i(\alpha x_1 + \beta x_2)} \) with \( \alpha^2 + \beta^2 \neq k^2 \) (cf. the subsequent example ii)). It would be nice to have an intrinsic description of the space \( DD_0 \). Instead, we only recall important functions belonging to the spaces \( DD_0 \) and \( AV_\kappa \):

i) The space \( DD_0 \) contains all exponential functions \( x' \mapsto v(x') = e^{i(\alpha x_1 + \beta x_2)} \), i.e., the traces of the plane-wave functions. For \( \alpha^2 + \beta^2 = k^2 \), the function \( v = v_\alpha \) is a two-dimensional Helmholtz solution and, for \( \alpha^2 + \beta^2 \neq k^2 \), the function \( v = (\Delta_{x'} + k^2 I)v_0 \) is an image with \( v_0 = \frac{1}{-\alpha^2 - \beta^2 + k^2} v \). If \( \alpha^2 + \beta^2 \neq 0 \), then the exponential function is in \( AV_{1/2} \) due to \( av(v, x', r) = 2\pi e^{i(\alpha x_1 + \beta x_2)} J_0(\sqrt{\alpha^2 + \beta^2} r) \).

ii) The space \( DD_0 \) contains all decaying functions \( v \in C^1_b(\mathbb{R}^2) \), with \( v(x') = O(|x'|^{-3/2 - \varepsilon}) \) for \( |x'| \to 0 \) and fixed positive \( \varepsilon \). Indeed, such a function is an image \( v = (\Delta_{x'} + k^2 I)v_0 \) with

\[
v_0(x') = \frac{i}{4} \int_{\mathbb{R}^2} H^{(1)}_0(k|x' - y'|) v(y') \, dy'.
\]

By the same argument, we even get \( AV_{3/2+\varepsilon} \subset DD_0 \). Obviously, the space \( AV_\kappa \) contains all decaying functions \( v \in C^1_b(\mathbb{R}^2) \), with \( v(x') = O(|x'|^{-\kappa}) \) for \( |x'| \to 0 \) and fixed positive \( \kappa \).
iii) The space $DD_0$ contains all traces of point source functions $y' \mapsto G(\vec{x}, (y', 0)^\top)$ for fixed $\vec{x} \not\in \mathbb{R}^2_+ \cup \mathbb{R}^2_0$. Indeed, such a trace is an image according to

$$
(\Delta y' + k^2 I) \left( e^{ik|x-(y',0)^\top|} \right) = 2(ik) \frac{e^{ik|x-(y',0)^\top|}}{|\vec{x} - (y', 0)^\top|} + k^2 e^{ik|x-(y',0)^\top|x_3^2 \quad |\vec{x} - (y', 0)^\top| \quad |x_3^2},
$$

$$
G(\vec{x}, (y', 0)^\top) = (\Delta y' + k^2 I) \frac{1}{8\pi(ik)} \left\{ e^{ik|x-(y',0)^\top|} \right\}
- \frac{i k^2 x_3^2}{4} \int_{\mathbb{R}^2} H^{(1)}_0(k|y' - z'|) \frac{e^{ik|x-(z',0)^\top|}}{|\vec{x} - (z', 0)^\top|} \, dz' \right\}.
$$

Similarly, the traces of all the derivatives $y' \mapsto \partial_x^\alpha \partial_y^\beta G(\vec{x}, (y', 0)^\top)$ with multi-indices $\alpha_x$ and $\alpha_y$ are contained in $DD_0$. These function belong to $AV_1$ in accordance with example ii).

iv) By definition $DD_0$ contains all solutions $u \in C^2_b(\mathbb{R}^2)$ of the two-dimensional Helmholtz equation. These function are contained in $AV_{1/2}$, since $\omega(v, x', r) = v(x').j_0(kr)$.

The space of solutions $v_x$ and, correspondingly, the space $DD_0$ could have been extended by all traces $h$ of cylindrical waves, the axes of which are perpendicular to $\mathbb{R}^2_0$. These traces are unbounded close to the axes. However, the functions $u(\vec{x}) := v_x(x')$ form two-dimensional Helmholtz solutions away from the axes. The boundedness condition (1.3) must be modified by replacing $u((x', x_3)^\top)$ with special averages over $x'$.

## 10 Uniqueness for the Dirichlet problem on thin layers

For a height $h_L > 0$ and index pairs $l' \in \mathbb{Z}^2$, we introduce the layer $\Omega_L := \{ \vec{x} \in \mathbb{R}^2 : 0 < x_3 < h_L \}$ and the cylindrical domain $\Omega_{L,l'} := \{ \vec{x} \in \Omega_L : |x' - l'| < 4 \}$. We consider the Dirichlet problem

$$
\Delta u(\vec{x}) + k^2 u(\vec{x}) = 0, \quad \forall \vec{x} \in \Omega_L,
$$

$$
u((x', 0)^\top) = v_0(x'), \quad \forall x' \in \mathbb{R}_0^3,
$$

$$
u((x', h_L)^\top) = v_{h_L}(x'), \quad \forall x' \in \mathbb{R}_0^3 ,
$$

$$\sup_{l' \in \mathbb{Z}^2} \| u \|_{H^1(\Omega_{L,l'})} < \infty
$$

with prescribed bounded and continuous Dirichlet data $v_0$ and $v_{h_L}$.

**Lemma 10.1.** If the positive width $h_L$ is less than $\pi/k$ and if there is a solution of the Dirichlet problem (10.1) over the layer $\Omega_L$ of thickness $h_L$, then this solution is unique.

**Proof.** Of course, we have to prove that any solution of the homogeneous problem (10.1) is trivial. Suppose $u$ is a solution of (10.1) with $v_0 \equiv 0$ and $v_{h_L} \equiv 0$. Then we extend $u$ to a function over $\mathbb{R}^3$ by

$$u(\vec{x}) := \begin{cases} u((x', x_3)^\top) & \text{if } x_3 = z_3 + (2m) h_L, \quad 0 \leq z_3 \leq h_L, \quad m \in \mathbb{Z} \\
u((x', h_L - z_3)^\top) & \text{if } x_3 = z_3 + (2m + 1) h_L, \quad 0 \leq z_3 \leq h_L, \quad m \in \mathbb{Z} \end{cases},$$

which is $(2h_L)$ periodic w.r.t. $x_3$. Since this extended $u$ and the normal derivatives $\partial_x u$ are continuous through the interface planes $\mathbb{R}^3_{mh_L}, m \in \mathbb{Z}$, the function $u$ is a periodic Helmholtz solution over $\mathbb{R}^3$.
Consequently, the modulated Fourier coefficients $\hat{u}_m$, defined by

$$u((x',x_3)') = \sum_{m \in \mathbb{Z}} \hat{u}_m(x') e^{i\pi mx_3/h_L},$$

are Helmholtz solution for any $m \in \mathbb{Z}$. In other words, $\hat{u}_m$ satisfies $(\Delta_{x'} - \varrho_m^2 I)\hat{u}_m = 0$, $\varrho_m := \sqrt{(\pi m/h_L)^2 - k^2}$. The average $aw(\hat{u}_m, x', r)$ defined in (1.6) satisfies the corresponding Bessel equation over the real half axis and is smooth at zero. Since the Bessel function $r \mapsto J_0(i\varrho_m r)$ is singular at zero (cf. [1], Sect. 9.1.89) and since the Bessel function $r \mapsto J_0(i\varrho_m r)$ is unbounded for $r \to \infty$ (cf. [1], Sect. 9.2.1), the solution $aw(\hat{u}_m, x', r)$ is zero. Using (10.2) and the differentiated (10.2), we get $aw(u|_{\mathbb{R}^3}, x', r) = 0$ and $aw(\partial_{x_j} u|_{\mathbb{R}^3}, x', r) = 0$, $j = 1, 2, 3$. This together with the arguments in (2.11)-(2.13) and the choice $\Omega_R = C_R$ and $\Sigma_R = T_R$ leads to (cf. (3.3) and (3.4))

$$u(\vec{x}) = \lim_{R \to \infty} \left\{ 2\int_{\mathbb{R}^3} \partial_{y_3} G(\vec{x}, \cdot) u - 2 \int_{\mathbb{R}^3} G(\vec{x}, \cdot) \partial_{y_3} u - 2 \int_{\mathbb{R}^3} \partial_{y_3} G(\vec{x}, \cdot) u + 2 \int_{\mathbb{R}^3} G(\vec{x}, \cdot) \partial_{y_3} u \right\}$$

$$= -2\int_{\mathbb{R}^3} G(\vec{x}, \cdot) \partial_{y_3} u + 2 \int_{\mathbb{R}^3} G(\vec{x}, \cdot) \partial_{y_3} u$$

$$= \int_0^\infty e^{ik\sqrt{(x_3-h_L)^2 + r^2}} \frac{aw(\partial_{x_3} u|_{\mathbb{R}^3}, x', r)r}{2\pi \sqrt{(x_3-h_L)^2 + r^2}} \, dr$$

$$- \int_0^\infty e^{ik\sqrt{x_3^2 + r^2}} \frac{aw(\partial_{x_3} u|_{\mathbb{R}^3}, x', r)r}{2\pi \sqrt{x_3^2 + r^2}} \, dr = 0.$$

**References**


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