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On a half-space radiation condition

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Abstract

For the Dirichlet problem of the Helmholtz equation over the half space or rough surfaces, a radiation condition is needed to guarantee a unique solution, which is physically meaningful. If the Dirichlet data is a general bounded continuous function, then the well-established Sommerfeld radiation condition, the angular spectrum representation, and the upward propagating radiation condition do not apply or require restrictions on the data, in order to define the involved integrals. In this paper a new condition based on a representation of the second derivative of the solution is proposed. The twice differentiable half-space Green's function is integrable and the corresponding radiation condition applies to general bounded functions. The condition is checked for special functions like plane waves and point source solution. Moreover, the Dirichlet problem for the half plane is discussed. Note that such a "continuous" radiation condition is helpful e.g. if finite sections of the rough-surface problem are analyzed.

1 Introduction

Throughout this paper we denote the points of the three-dimensional Euclidean space $\mathbb{R}^3$ by $\vec{x}$ and $\vec{y}$ with $\vec{x} := (x_1, x_2, x_3)^\top = (x', x_3)^\top$ and $x' := (x_1, x_2)^\top \in \mathbb{R}^2$ and $\vec{y} := (y', y_3)^\top$. For fixed numbers $x_{f,3}$, we define the half spaces $\mathbb{R}^3_{x_{f,3},+} := \{\vec{x} = (x', x_3)^\top \in \mathbb{R}^3 : \; x_3 > x_{f,3}\}$ and $\mathbb{R}^3_+ := \mathbb{R}^3_{0,+}$ and the boundary planes $\mathbb{R}^3_{x_{f,3}} := \{(x', x_{f,3})^\top : \; x' \in \mathbb{R}^2\}$. We shall consider functions u defined on a perturbed half space Ω of $\mathbb{R}^3_+$, which are solutions of the Helmholtz equation $(\Delta + k^2 I)u = 0$ for a fixed wavenumber $k > 0$. We suppose that $\Omega := \Omega_F := \{\vec{x} \in \mathbb{R}^3 : \; x_3 > F(x')\}$ with a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $-h_F \leq F(x') < 0$ holds for all $x' \in \mathbb{R}^2$ (cf. Fig. 1). The number $h_F > 0$ is fixed.

The goal is to find a general radiation condition for Helmholtz solutions on Ω which are uniformly bounded on the planes $\mathbb{R}^3_{x_{f,3}}$, but, eventually, do not decay in the directions of $\mathbb{R}^3_{x_{f,3}}$. If $\Phi(\vec{x}, \vec{y})$ is the Green's function for the Dirichlet problem of the Helmholtz equation over the upper three-dimensional half space $\mathbb{R}^3_+$, then, analogously to the upward propagating radiation condition (UPRC) in the two-dimensional case (cf. [7]), a possible choice for the radiation condition would be to fix $x_{f,3} \geq 0$ and to

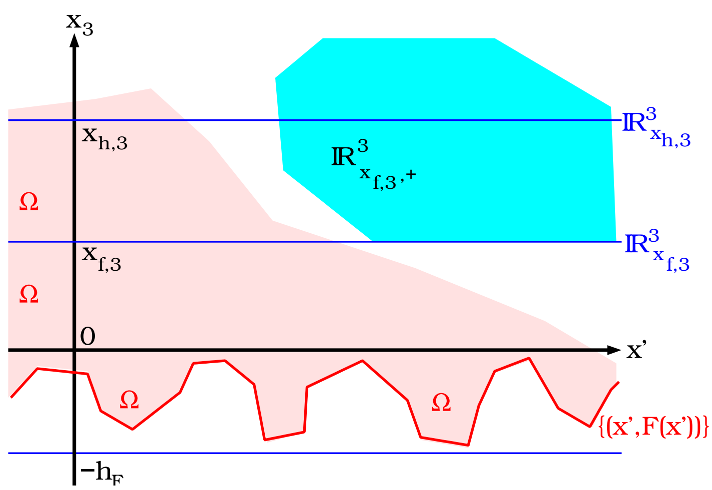

Figure 1: The geometry settings.

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require the condition (cf. [3], Chapt. 5.1.1 and [4,5])

$$u(\vec{x}\,) = \int_{\mathbb{R}^2} \partial_{y_3} \Phi\big(\vec{x}, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y' \tag{1.1}$$

for all $\vec{x} \in \mathbb{R}^3_{x_{f,3},+}$. Note that this condition is equivalent to a representation as a superposition of outgoing generalized plane waves (cf. the angular spectrum representation in [2,5,7,10]) and, in two dimensions, to the pole condition (cf. [2]). In three dimensions, the integral exists for functions from weighted L^2 or from more complicated spaces. Since we are interested in a class of solution u containing plane-wave functions, we only know that the function u restricted to the boundary plane $\mathbb{R}^3_{x_{f,3}}$ is smooth and uniformly bounded, and the existence of the integral is not guaranteed. Therefore, we formally differentiate twice to get

$$\partial^2_{x_3} u(\vec{x}\,) = \int_{\mathbb{R}^2} \partial^3_{y_3} \Phi\big(\vec{x}, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y', \quad \vec{x} \in \mathbb{R}^3_{x_{f,3},+}. \tag{1.2}$$

This will be the main part of our radiation condition. We shall see that the kernel in (1.2) satisfies $\partial^3_{y_3}\Phi(\vec{x}, \vec{y}\,) = \mathcal{O}_{x_3-y_3}(|\vec{x} - \vec{y}\,|^{-3})$ on $\mathbb{R}^3_{x_{f,3},+}$ for $|\vec{x} - \vec{y}\,| \to \infty$. Hence, the principle-value integral in (1.2) is well defined for any function u bounded and sufficiently smooth on $\mathbb{R}^3_{x_{f,3}}$. Suppose that $Op \subseteq \mathbb{R}^3_{x_{f,3},+}$ is a fixed open subset and that u is a solution of the Helmholtz equation. Then (1.2) is satisfied if and only if the equation in (1.2) holds for any $\vec{x} \in Op$. Indeed, on both sides of the equation we have solutions of the Helmholtz equation, and such analytic function coincide on $\mathbb{R}^3_{x_{f,3},+}$ if and only if they do on Op. On the other hand, it is not sufficient to require the equation for one or more planes $\mathbb{R}^3_{x_3}$ with $x_3 > x_{f,3}$. Indeed, the two sides of the equation might differ by the function $(x', x_3)^\top \mapsto \sin(k(x_3 - x_{f,3}))$ if they coincide only over $\mathbb{R}^3_{x_3}$ with $x_3 = x_{f,3} + l\pi/k$, $l = 1, 2, \ldots$. However, it would be sufficient to require the equation in (1.2) over $\mathbb{R}^3_{x_{l,3}}$ with $x_{l,3} > x_{f,3}$, $l = 1, 2$ if the homogeneous Dirichlet problem of the Helmholtz equation over the layer enclosed by $\mathbb{R}^3_{x_{1,3}}$ and $\mathbb{R}^3_{x_{2,3}}$ has the trivial solution only. In particular, this is the case for $|x_{1,3} - x_{2,3}| < \pi/k$ (cf. the subsequent Sect. 10).

For $l' \in \mathbb{Z}^2$ and $x_{h,3} > 0$, we introduce the finite subdomain $\Omega_{l',x_{h,3}}$ of Ω adjacent to the lower boundary by $\Omega_{l',x_{h,3}} := \{\vec{x} \in \Omega :\ |x' - l'| < 4\,,\ x_3 < x_{h,3}\}$. Clearly, for any $l' \in \mathbb{Z}^2$ and $x_{h,3} > 0$, we have to assume that u is in the Sobolev space $H^1(\Omega_{l',x_{h,3}})$. Unfortunately, Sobolev regularity and condition (1.2) still allow to add to u an unphysical solution of the form $u_{\mathrm{add}}(\vec{x}) := u_{\mathrm{2D}}(x')x_3$ with u_{2D} a solution of the two-dimensional Helmholtz equation $(\Delta_{x'} + k^2 I)u_{\mathrm{2D}} = 0$ (think e.g. of $u_{\mathrm{2D}}(x') = e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = k^2$). Adding such a function to u corresponds to adding the function $u_{\mathrm{2D}}(x')$ to the derivative $\partial_{x_3} u(\cdot, x_3)$. In order to exclude such addends, we augment our radiation condition by one of the next two equivalent conditions. Either we require, for fixed $\varepsilon_u > 0$ and $c_u > 0$ depending on u, the weak boundedness condition

$$\left| \frac{1}{x_3} \int_0^{x_3} \frac{x_3 - t}{x_3} u\big((x', t)^\top\big)\ \mathrm{d}t \right| < c_u x_3^{1-\varepsilon_u}, \quad \forall x_3 \geq 1 \tag{1.3}$$

or we fix the derivative $\partial_{x_3} u$ over the plane $\mathbb{R}^3_{x_{f,3}}$ by

$$\partial_{x_3} u\big((x', x_{f,3})^\top\big) = -\int_{\mathbb{R}^2} \partial^2_{y_3} \Phi\big((x', x_{f,3})^\top, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y', \quad x' \in \mathbb{R}^2. \tag{1.4}$$

The existence of the hypersingular integral in (1.4) (cf. the subsequent representation (2.8)) can be shown under additional conditions. To avoid hypersingular kernels, we can fix $\partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}}}$ by

$$\partial_{x_3} u\big((x', x_{f,3})^\top\big) = -\int_{\mathbb{R}^2} \partial^2_{y_3} \Phi\big((x', x_{f,3})^\top, (y', x_{h,3})^\top\big)\, u\big((y', x_{h,3})^\top\big)\, \mathrm{d}y', \quad x' \in \mathbb{R}^2. \tag{1.5}$$

In comparison to (1.4), in (1.5) there is no singularity in the kernel for $|x' - y'| \to 0$. However, the more serious problem in (1.4) and (1.5) is that of integrability at infinity. The weak boundedness condition (1.3) is nothing else than a restriction to an $\mathcal{O}(x_3^{3-\varepsilon_u})$ growth of the two-fold integral function $w(x', x_3) := \int_0^{x_3} (x_3 - t) u((x', t)^\top)\, \mathrm{d}t$ (i.e. w is defined by $\partial_{x_3}^2 w = u$ and $w(x', 0) = \partial_{x_3} w(x', 0) = 0$). If u is uniformly bounded or if $|u(\vec{x})| = \mathcal{O}(x_3^{1-\varepsilon_u})$ for $x_3 \to \infty$, then (1.3) is fulfilled. As we will see in Prop. 1.2, in many cases the solution u satisfying (1.2) and (1.3) is indeed uniformly bounded.

Altogether, we suggest the following outgoing radiation condition

Definition 1.1. *A solution u of $(\Delta + k^2 I)u = 0$ over Ω is said to satisfy the half-space radiation condition (HSRC) if there exist real numbers c_u, ε_u, $x_{h,3}$, and $x_{f,3}$ with $c_u > 0$, $\varepsilon_u > 0$, and $0 < x_{f,3} < x_{h,3}$ such that*

i) For any $l' \in \mathbb{Z}^2$, the restriction of u to the subdomain $\Omega_{l', x_{h,3}}$ is in the Sobolev space $H^1(\Omega_{l', x_{h,3}})$ and has a bounded norm $\|u|_{\Omega_{l', x_{h,3}}}\|_{H^1(\Omega_{l', x_{h,3}})} < c_u$.

ii) The second order derivative $\partial_{x_3}^2 u$ admits the representation (1.2).

iii) The function u satisfies the weak boundedness estimate (1.3).

For the equivalence of (1.3) and (1.4) and for the existence of solution to the Dirichlet problem for the Helmholtz equation over $\mathbb{R}_+^3$, we have to restrict the functions. We use three special classes. First, we introduce the space $C(\mathbb{R}^2)$ of continuous functions, the space $C_b(\mathbb{R}^2)$ of uniformly bounded and continuous functions and, for $v \in C(\mathbb{R}^2)$, the average function

$$av(v, x', r) \;:=\; \int_0^{2\pi} v\big(x' - r(\cos\phi, \sin\phi)^\top\big)\, \mathrm{d}\phi. \tag{1.6}$$

Furthermore, we denote the space of all function v, which together with all their derivatives upto order $l \geq 0$ are continuous and uniformly bounded by $C_b^l(\mathbb{R}^2)$. We define the first class AV_κ for $\kappa > 0$ by

$$AV_\kappa \;:=\; \big\{ v \in C_b^1(\mathbb{R}^2) : \exists c_v > 0 \text{ s.t. } |av(v, x', r)| < c_v r^{-\kappa},\ \forall r > 1,\ \forall x' \in \mathbb{R}^2 \big\}. \tag{1.7}$$

A second class is the Dirichlet data set DD_υ with $\upsilon \geq 0$ including all sums of Helmholtz solutions plus Helmholtz images, i.e.,

$$\begin{aligned} DD_\upsilon \;:=\; \big\{ v \in C_b(\mathbb{R}^2) :\ & \exists c_v > 0,\ \exists v_s \in C_b^2(\mathbb{R}^2),\ \exists v_0 \in C^2(\mathbb{R}^2),\ \exists v_i \in C_b(\mathbb{R}^2) \text{ s.t.} \\ & v = v_s + v_i,\ (\Delta_{x'} + k^2 I) v_s = 0,\ v_i = (\Delta_{x'} + k^2 I) v_0, \text{ and} \\ & |v_0(x')| \leq c_v (1 + |x'|)^\upsilon, \forall x' \in \mathbb{R}^2 \big\}. \end{aligned} \tag{1.8}$$

Finally, we introduce a class of functions v characterized by the Fourier transforms $[\mathcal{F}v]$. Unfortunately, this condition will not be easy to check. We introduce the annular domains by the formula $R_{k,\varepsilon} := \{\xi' \in \mathbb{R}^2 : k - \varepsilon < |\xi| < k + \varepsilon\}$ and set

$$FC_k \;:=\; \big\{ v \in C_b^6(\mathbb{R}^2) : \exists \varepsilon_v > 0 \text{ s.t. } [\mathcal{F}v]|_{R_{k,\varepsilon_v}} \in L^2(R_{k,\varepsilon_v}) \big\}. \tag{1.9}$$

For functions u with restrictions $v(x') := u((x', x_3)^\top)$ in such spaces, we get

Proposition 1.2. *Suppose there is an $x_{f,3}$ with $0 < x_{f,3} < x_{h,3}$ such that $v := u|_{\mathbb{R}^3_{x_{f,3}}}$ is either in AV_κ, $\kappa > 0$ in DD_υ, $0 \leq \upsilon < 1$ or in FC_k. Then, in the (HSRC), we can replace condition iii) by the equivalent condition:*

iii') The derivative $\partial_{x_3} u$ restricted to the plane $\mathbb{R}^3_{x_{f,3}}$ fulfills (1.4).

Moreover, the solutions fulfilling the (HSRC) are even uniformly bounded if $v := u|_{\mathbb{R}^3_{x_{f,3}}} \in AV_\kappa$ or if $v \in DD_0$ with $v_0 \in C_b^1(\mathbb{R}^2)$.

Since (1.1) implies (1.2) and (1.4), (UPRC) implies (HSRC) with iii') replaced by iii). Prop. 1.2 and its proof (cf. (8.2) or interprete the representation $[V_k v_0]$ in part iii) of the proof to Lemma 8.1 as (1.1) evaluated by partial integration for the differential operator $\partial^2_{x_3} = -(\Delta_{x'} + k^2 I)$) means that, for $v := u|_{\mathbb{R}^3_{x_{f,3}}}$ in DD_υ, AV_κ or FC_k, the condition (HSRC) is equivalent to the (UPRC).

In Sect. 8 we shall prove Prop. 1.2 and show that the (HSRC) is independent of the choice of $x_{h,3}$ and $x_{f,3}$. For the plausibility of the (HSRC), we remark:

- Formula (1.2) can be considered to be a representation of $\partial^2_{x_3} u$ as a superposition of generalized outgoing plan-wave solutions (cf. the right-hand side of (6.5)). Outgoing for the upper half plane means that the plane wave $u(\vec{x}) := e^{\mathbf{i}(\alpha x_1 + \beta x_2 + \gamma x_3)}$ with $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$, and $\alpha^2 + \beta^2 + \gamma^2 = k^2$, is either a true plane-wave (i.e. $\alpha^2 + \beta^2 \leq k^2$) moving into the upper half plane (i.e. $\gamma \geq 0$) or a generalized plane-wave (i.e. $\alpha^2 + \beta^2 < k^2$) decaying in the x_3 direction (i.e. $\Im m\, \gamma > 0$).

- For any solution $u_{2\mathrm{D}} \in C^2_b(\mathbb{R}^2)$ of the two-dimensional equation $(\Delta_{x'} + k^2 I) u_{2\mathrm{D}} = 0$, the function $u(\vec{x}) := u_{2\mathrm{D}}(x')$ satisfies the (HSRC), and $u(\vec{x}) := u_{2\mathrm{D}}(x') x_3$ does not (cf. Section 5). In particular, $u(\vec{x}) := J_0(k|x'|)$ fulfills the (HSRC) and cylindrical wave functions like $u(\vec{x}) := \frac{\mathbf{i}}{4} H^{(1)}_0(k|x'|)$ satisfy at least (1.2) (cf. Sect. 5).

- Any generalized plane-wave solution $u(\vec{x}) := e^{\mathbf{i}(\alpha x_1 + \beta x_2 + \gamma x_3)}$ with $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$, and $\alpha^2 + \beta^2 + \gamma^2 = k^2$ satisfies the (HSRC) if and only if either $\gamma > 0$ or $\Re e\, \gamma \geq 0$ together with $\Im m\, \gamma > 0$ (cf. Sect. 6). In other words, the (HSRC) is equivalent to the well-known radiation condition for quasiperiodic functions in the theory of gratings (cf. Corollary 6.1).

- Any solution, satisfying the classical Sommerfeld radiation condition for the directions pointing into the half space, fulfills the (HSRC) (cf. Sect. 7 and cf. [8] for Sommerfeld's condition on the half space). In other words, functions $u(\vec{x}) := G(\vec{y}, \vec{x})$ and all their derivatives satisfy (HSRC), but "incoming" waves like $u(\vec{x}) := \overline{G(\vec{y}, \vec{x})}$ do not.

- Using the radiation condition (HSRC), uniqueness (cf. Prop. 9.1) and existence of the solution to the Dirichlet problem of the Helmholtz equation over the half plane $\mathbb{R}^3_+$ can be shown. However, special conditions are needed for the existence, namely the Dirichlet data is supposed to be in DD_υ, AV_κ, or FC_k (cf. Prop. 9.3 and compare [6] for the two-dimensional case with continuous data over rough surfaces and [3], Sect. 5.1.1.1 for Dirichlet data from $L^2(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2) \subset H^{1/2}_k(\mathbb{R}^2)$, and a special subspace $X' \subset C_b(\mathbb{R}^2) + L^2(\mathbb{R}^2)$).

2 Fundamental solution, Green's function, and representation formula

2.1 Denote $\vec{x}$ and $\vec{y}$ as in Sect. 1. For a complex valued wave number k with $\Re e\, k > 0$ and $\Im m\, k \geq 0$, the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ is given by

$$G(\vec{x}, \vec{y}) \ := \ G(\vec{x} - \vec{y}) \ := \ \frac{1}{4\pi} \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|}. \tag{2.1}$$

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Clearly, for $j=1,2$, the first and second order derivatives are

$$\partial_{y_3} G(\vec{x},\vec{y}) = \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{(\mathbf{i}k)(y_3-x_3)}{|\vec{x}-\vec{y}\,|^2}-\frac{y_3-x_3}{|\vec{x}-\vec{y}\,|^3}\right\}, \tag{2.2}$$

$$\partial_{y_3}\partial_{y_j} G(\vec{x},\vec{y}), = \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}(y_3-x_3)(y_j-x_j)}{4\pi|\vec{x}-\vec{y}\,|^3}\left\{(\mathbf{i}k)^2-\frac{3(\mathbf{i}k)}{|\vec{x}-\vec{y}\,|}-\frac{3}{|\vec{x}-\vec{y}\,|^2}\right\}, \tag{2.3}$$

$$\begin{aligned}\partial^2_{y_3} G(\vec{x},\vec{y}) &= \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{(\mathbf{i}k)}{|\vec{x}-\vec{y}\,|^2}-\frac{1}{|\vec{x}-\vec{y}\,|^3}+\frac{(\mathbf{i}k)^2(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^3}\right. \qquad (2.4)\\ &\qquad \left. -3\frac{(\mathbf{i}k)(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^4}+\frac{3(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^5}\right\}.\end{aligned}$$

The third order derivative is of the form $\mathcal{O}_{x_3-y_3}(|\vec{x}-\vec{y}\,|^{-3})$. Indeed,

$$\begin{aligned}\partial^3_{y_3} G(\vec{x},\vec{y}) &= \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{3(\mathbf{i}k)^2(y_3-x_3)}{|\vec{x}-\vec{y}\,|^3}-\frac{9(\mathbf{i}k)(y_3-x_3)}{|\vec{x}-\vec{y}\,|^4}+\frac{(\mathbf{i}k)^3(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^4}\right. \qquad (2.5)\\ &\qquad -\frac{6(\mathbf{i}k)^2(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^5}+\frac{9(y_3-x_3)}{|\vec{x}-\vec{y}\,|^5}+\frac{15(\mathbf{i}k)(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^6}\\ &\qquad \left.-\frac{15(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^7}\right\}\\ &= \frac{(y_3-x_3)}{4\pi|\vec{x}-\vec{y}\,|^3}\left\{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}(\mathbf{i}k)^3\frac{(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|}+\mathcal{O}(1)\right\}. \qquad (2.6)\end{aligned}$$

Note that the factor $(y_3-x_3)|\vec{x}-\vec{y}\,|^{-3}/4\pi$ is the double layer kernel for the Laplace equation. This kernel defines a uniformly bounded operator in the L^∞ space (cf. [9]).

In order to enable the computation of finite-part integrals, we shortly look at the kernel behaviour for $|\vec{x}-\vec{y}|\to 0$. By the Taylor-series expansion of $e^{\mathbf{i}k|\vec{x}-\vec{y}|}$ we get

$$\begin{aligned}G(\vec{x},\vec{y}) &= \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}+\mathcal{O}(1),\\ \partial_{y_3}G(\vec{x},\vec{y}) &= \frac{1}{4\pi}\frac{(y_3-x_3)}{|\vec{x}-\vec{y}|^3}+\mathcal{O}(1), \qquad (2.7)\\ \partial^2_{y_3}G(\vec{x},\vec{y}) &= \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|^3}-\frac{1}{4\pi}\frac{3(y_3-x_3)^2}{|\vec{x}-\vec{y}|^5}+\mathcal{O}\left(\frac{1}{|\vec{x}-\vec{y}|}\right), \qquad (2.8)\end{aligned}$$

which proves that the kernels for the Helmholtz equation, locally, are compact perturbations of the operators for the Laplace equation. Consequently, the corresponding potentials of finitely supported and smooth weight functions over $\mathbb{R}^3_{x_{f,3}}$ computed at $(x',x_3)^\top\in\mathbb{R}^3_{x_{f,3},+}$ have well-defined limits for $x_3\to x_{f,3}$. These limits can be computed by the well-known terms of the jump relation plus the values of the potential integral at $(x',x_{x_{f,3}})^\top$. In case of the kernel $\partial^2_{y_3}G(\vec{x},\vec{y})$, this integral is to be understood as the usual finite-part integral.

The Green's function for the Dirichlet problem over $\mathbb{R}^3_{x_{f,3},+}$ is chosen as

$$\Phi(\vec{x},\vec{y}\,) = G(\vec{x},\vec{y}\,) - G\big((x',2x_{x_{f,3}}-x_3)^\top,\vec{y}\big) = G(\vec{x},\vec{y}\,) - G\big(\vec{x},(y',2x_{x_{f,3}}-y_3)^\top\big). \tag{2.9}$$

We get vanishing boundary values $\Phi(\vec{x},(y',x_{x_{f,3}})^\top)=0$ and

$$\begin{aligned}\partial^3_{y_3}\Phi(\vec{x},\vec{y}\,) &= \partial^3_{y_3}G(\vec{x},\vec{y}\,)+\partial^3_{y_3}G\big(\vec{x},(y',2x_{x_{f,3}}-y_3)^\top\big),\\ \partial^3_{y_3}\Phi\big(\vec{x},(y',x_{x_{f,3}})^\top\big) &= 2\partial^3_{y_3}G\big(\vec{x},(y',x_{x_{f,3}})^\top\big), \qquad (2.10)\end{aligned}$$

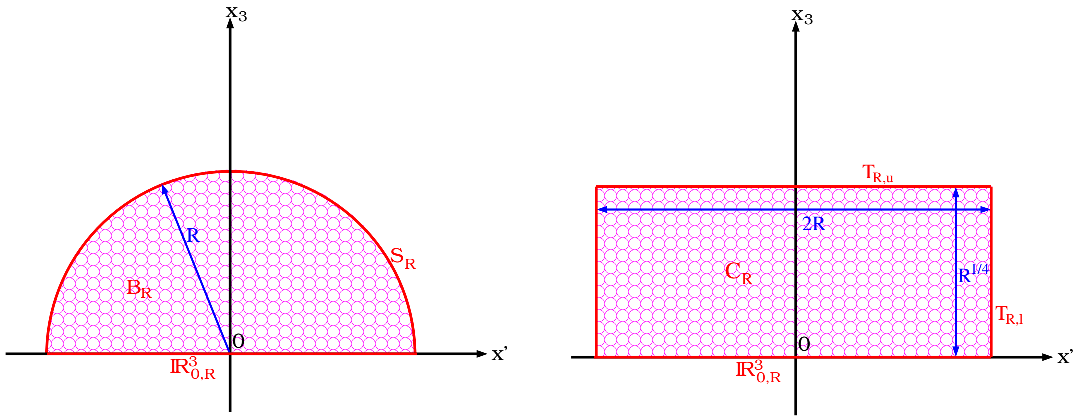

Figure 2: Half ball and cylinder.

such that $\partial^3_{y_3}\Phi(\vec{x},\vec{y}\,) = \mathcal{O}_{x_3-y_3}(|\vec{x}-\vec{y}\,|^{-3})$ on $\vec{y} \in \mathbb{R}^3_{x_{f,3}}$, $\vec{x} \in \mathbb{R}^3_{x_3}$ follows from (2.6).

2.2 Finally, we recall the representation formula for the second order derivative $\partial^2_{x_3} u$ of the solution u to the Helmholtz equation $\Delta u + k^2 u = 0$ over $\mathbb{R}^3_+$ (cf. the subsequent (2.11)). To slightly simplify the subsequent formulas, we set $x_{f,3} = 0$ and consider the half space $\mathbb{R}^3_+$ instead of the general $\mathbb{R}^3_{x_{f,3},+}$. For large $R > 0$, we introduce the disc $\mathbb{R}^3_{0,R} := \{(x',0)^\top \in \mathbb{R}^3 : \; |x'| < R\}$, the half ball $B_R := \{(x',x_3)^\top \in \mathbb{R}^3 : \; x_3 > 0, |\vec{x}| < R\}$ with its upper spherical boundary $S_R := \{(x',x_3)^\top \in \mathbb{R}^3 : x_3 > 0, |\vec{x}| = R\}$, and the cylinder $C_R := \{(x',x_3)^\top \in \mathbb{R}^3 : \; 0 < x_3 < R^{1/4}$ and $|x'| < R\}$ with its lateral and upper boundary (cf. Fig. 2)

$$T_R \; := \; T_{R,l} \cup T_{R,u}, \quad T_{R,l} \; := \; \left\{(x',x_3)^\top \in \mathbb{R}^3 : \; 0 < x_3 < R^{1/4} \text{ and } |x'| = R)\right\},$$
$$T_{R,u} \; := \; \left\{(x',x_3)^\top \in \mathbb{R}^3 : \; x_3 = R^{1/4} \text{ and } |x'| < R\right\}.$$

We consider either $\Omega_R := B_R$ and $\Sigma_R := S_R$ or $\Omega_R := C_R$ and $\Sigma_R := T_R$. By ν we denote the normal at the boundary $\mathbb{R}^3_{0,R} \cup \Sigma_R$ of Ω_R pointing into outward direction. We assume that condition i) in (HSRC) is fulfilled. The symmetric Green's formula applied to u and $\vec{y} \mapsto \partial^2_{y_3}\Phi(\vec{x},\vec{y})$ with fixed $\vec{x} \in \Omega_R$ leads to

$$\int_{\mathbb{R}^3_{0,R}\cup\Sigma_R} \{\partial_\nu u \partial^2_{y_3}\Phi(\vec{x},\cdot) - u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\}$$
$$= \int_{\Omega_R} \{(\Delta + k^2 I)u\partial^2_{y_3}\Phi(\vec{x},\cdot) - u(\Delta + k^2 I)\partial^2_{y_3}\Phi(\vec{x},\cdot)\}.$$

Using $\Phi(\vec{x},\cdot) \equiv 0$ over $\mathbb{R}^3_0$ and the Helmholtz equation, we get that the second order derivative $\partial^2_{y_3}\Phi(\vec{x},\cdot) = -\partial^2_{y_1}\Phi(\vec{x},\cdot) - \partial^2_{y_2}\Phi(\vec{x},\cdot) - k^2\Phi(\vec{x},\cdot)$ is zero over $\mathbb{R}^3_0$. From the Green's function property we obtain $(\Delta + k^2 I)\partial^2_{y_3}\Phi(\vec{x},\cdot) = \partial^2_{y_3}(\Delta + k^2 I)\Phi(\vec{x},\cdot) = \partial^2_{y_3}\delta_{\vec{x}}$. Thus, for a solution u of the Helmholtz equation $(\Delta + k^2 I)u = 0$, we arrive at

$$\partial^2_{x_3} u(\vec{x}\,) \; = \; \int_{\mathbb{R}^3_{0,R}} \partial^3_{y_3}\Phi(\vec{x},\cdot)u - \int_{\Sigma_R} \{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot) - u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\},$$

$$\partial^2_{x_3} u(\vec{x}\,) \; = \; [V_k u](\vec{x}\,) + I_\infty, \tag{2.11}$$

$$[V_k u](\vec{x}\,) \; := \; \int_{\mathbb{R}^2} \partial^3_{y_3}\Phi\big(\vec{x},(y',0)^\top\big)\, u\big((y',0)^\top\big)\, \mathrm{d}y', \tag{2.12}$$

$$I_\infty \; := \; \lim_{R\to\infty} \int_{\Sigma_R} \left\{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot) - u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\right\}. \tag{2.13}$$

Here $[V_k u]$ is the twice differentiated double layer potential. Altogether, to get the representation in the radiation condition (1.2) for a solution u of the Helmholtz equation over $\mathbb{R}^3_+$, we only have to suppose condition i) of (HSRC) and to show that the limit I_∞ is zero.

3 Boundedness of the potential in (1.2)

Consider a function u bounded over $\mathbb{R}^3_0$ and consider the twice differentiated double layer potential $[V_k u](\vec{x}\,)$ defined by (2.12), where Φ is defined in (2.9). Without loss of generality, we fix $x' = (0,0)^\top$ and consider the limit of $[V_k u](\vec{x}\,)$ for $\vec{x} = (x', x_3)^\top$ with $x_3 \to \infty$. Due to (2.10) we have to estimate

$$I_{x_3} \;:=\; \int_{\mathbb{R}^2} \partial^3_{y_3} G\big(\vec{x}, (y',0)^\top\big)\, f(y')\mathrm{d}y', \quad f(y') := u\big((y',0)^\top\big)\,.$$

Taking into account (2.6) and the boundedness of the integral of the double layer kernel (cf. [9]), we get $I_{x_3} = \frac{(\mathbf{i}k)^3}{4\pi} J_{x_3} + \mathcal{O}(1)$ with

$$\begin{aligned} J_{x_3} \;:=\; \int_{\mathbb{R}^2} e^{\mathbf{i}k|\vec{x}-\vec{y}\,|} \frac{(0-x_3)^3}{|\vec{x}-\vec{y}\,|^4} f(y')\mathrm{d}y' \;&=\; -\int_{\mathbb{R}^2} e^{\mathbf{i}k\sqrt{x_3^2+|y'|^2}} \frac{x_3^3}{\{x_3^2+|y'|^2\}^2} f(y')\mathrm{d}y' \\ &=\; -\,x_3 \int_{\mathbb{R}^2} \frac{e^{\mathbf{i}kx_3\sqrt{1+|z'|^2}}}{\{1+|z'|^2\}^2} f(x_3 z')\mathrm{d}z'. \end{aligned}$$

We substitute $z' = \sqrt{r^2-1}\,(\cos\phi, \sin\phi)^\top$ and $\mathrm{d}z' = r\mathrm{d}\phi\,\mathrm{d}r$ to get

$$J_{x_3} \;=\; -x_3 \int_1^\infty \frac{e^{\mathbf{i}kx_3 r}}{r^3} \int_0^{2\pi} f\left(x_3\sqrt{r^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right) \mathrm{d}\phi\,\mathrm{d}r. \tag{3.1}$$

The last integral is difficult to estimate. At least we get $|[V_k u](\vec{x}\,)| \le c|x_3|$. Here and in the following c stands for a generic positive constant, the value of which varies from instance to instance.

Next we prove that I_{x_3} fulfills at least the weak boundedness condition used in (1.3). For the terms in (4.1) and for $x' \neq y'$, we conclude

$$\begin{aligned} \int_0^{x_3} (x_3-t)\partial^3_{y_3} G\big((x',t)^\top, (y',0)^\top\big)\;\mathrm{d}t \;=\; &\partial_{y_3} G\big((x',x_3)^\top, (y',0)^\top\big) - \partial_{y_3} G\big((x',0)^\top, (y',0)^\top\big) \\ &+ x_3\,\partial^2_{y_3} G\big((x',0)^\top, (y',0)^\top\big)\,, \end{aligned} \tag{3.2}$$

where we have used $\partial^l_{x_3}\partial_{y_3} G(\vec{x},\vec{y}) = (-1)^l \partial^l_{y_3}\partial_{y_3} G(\vec{x},\vec{y})$. The formulas in (2.2) and (2.4) together with

$$\begin{aligned} e^{\mathbf{i}k\left|(x',x_3)^\top-(y',0)^\top\right|} \;&=\; e^{\mathbf{i}k|x'-y'|} e^{\mathbf{i}kx_3^2/\{|(x',x_3)^\top-(y',0)^\top|+|x'-y'|\}} \\ &=\; e^{\mathbf{i}k|x'-y'|} + \mathcal{O}\left(\frac{x_3^2}{|(x',x_3)^\top-(y',0)^\top|}\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{1}{|(x',x_3)^\top-(y',0)^\top|^l} \;&=\; \frac{1}{|x'-y'|^l} \\ &\quad + \mathcal{O}\left(\frac{x_3^2}{|x'-y'|^{l+1}\,|(x',x_3)^\top-(y',0)^\top|}\right),\; l=2,3, \end{aligned} \tag{3.4}$$

imply

$$
\begin{aligned}
&\int_0^{x_3} (x_3-t)\partial_{y_3}^3 G\big((x',t)^\top,(y',0)^\top\big)\,\mathrm{d}t\\
&= \frac{e^{\mathbf{i}k\sqrt{x_3^2+|x'-y'|^2}}}{4\pi}\left\{\frac{(\mathbf{i}k)(-x_3)}{\sqrt{x_3^2+|x'-y'|^2}^2}-\frac{-x_3}{\sqrt{x_3^2+|x'-y'|^2}^3}\right\}\\
&\quad+\frac{e^{\mathbf{i}k|x'-y'|}}{4\pi}\left\{\frac{(\mathbf{i}k)x_3}{|x'-y'|^2}-\frac{x_3}{|x'-y'|^3}\right\}\\
&= \mathcal{O}\left(\frac{x_3^3}{|x'-y'|^2\sqrt{x_3^2+|x'-y'|^2}}\right) = \mathcal{O}\left(\frac{x_3^{3-\epsilon_u}}{|x'-y'|^2\sqrt{x_3^2+|x'-y'|^2}^{1-\epsilon_u}}\right)
\end{aligned}
$$

for $|x'-y'|\to\infty$. For $|x'-y'|\to\infty$. Now we split the integral I_{x_3} into the integral over y' with $|x'-y'|<1$ and the integral over y' with $|x'-y'|\geq 1$. Due to $|\partial_{y_3}^3 G(\vec{x},\vec{y})|=\mathcal{O}(|\vec{x}-\vec{y}|^{-1})$, the first integral is uniformly bounded and consequently also weakly bounded. For the second integral we get the weak boundedness by the above kernel estimate applied to $\int_0^{x_3}(x_3-t)I_t\,\mathrm{d}t$. Repeating the same arguments as above we get

Proposition 3.1. *Suppose that f is continuous over $\mathbb{R}^2=\mathbb{R}^2_0$ and that there are constants $c_f>0$ and $0\leq\upsilon_f<1$ such that $|f(x')|<c_f(1+|x'|)^{\upsilon_f}$ holds for any $x'\in\mathbb{R}^2$. Then the potential $[V_kf]$ satisfies the weak boundedness condition, i.e., (1.3) with u replaced by $[V_kf]$.*

For special functions u, i.e., for special f we can get more. We shall assume that the gradient $\nabla f=\nabla_{x'}f$ of $f=u|_{\mathbb{R}^3_0}$ is uniformly bounded. This assumption is obviously fulfilled in the typical situation that u is a bounded solution of the Helmholtz equation over a layer $\{\vec{x}\in\mathbb{R}^3:\ -\varepsilon<x_3<\varepsilon\}$ with $\varepsilon>0$. From (3.1) we obtain

$$
\begin{aligned}
J_{x_3} &= -x_3 2\pi f(0)\int_1^\infty \frac{e^{\mathbf{i}kx_3r}}{r^3}\,\mathrm{d}r\\
&\quad -x_3\int_1^\infty \frac{e^{\mathbf{i}kx_3r}}{r^3}\int_0^{2\pi}\left[f\left(x_3\sqrt{r^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)-f(0)\right]\mathrm{d}\phi\,\mathrm{d}r\\
&= \mathcal{O}(1)\\
&\quad -\frac{1}{\mathbf{i}k}\int_1^\infty \frac{\partial_r\left(e^{\mathbf{i}kx_3r}\right)}{r^3}\int_1^r\int_0^{2\pi}\nabla f\left(x_3\sqrt{t^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)\cdot\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\mathrm{d}\phi\,\frac{t}{\sqrt{t^2-1}}\,\mathrm{d}t\,\mathrm{d}r\\
&= \mathcal{O}(1)-\frac{1}{\mathbf{i}k}\left[\frac{e^{\mathbf{i}kx_3r}}{r^3}\int_1^r\int_0^{2\pi}\nabla f\left(x_3\sqrt{t^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)\cdot\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\mathrm{d}\phi\,\frac{t}{\sqrt{t^2-1}}\,\mathrm{d}t\right]_{r=1}^\infty\\
&\quad -\frac{3}{\mathbf{i}k}\int_1^\infty \frac{e^{\mathbf{i}kx_3r}}{r^4}\int_1^r\int_0^{2\pi}\nabla f\left(x_3\sqrt{t^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)\cdot\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\mathrm{d}\phi\,\frac{t}{\sqrt{t^2-1}}\,\mathrm{d}t\,\mathrm{d}r\\
&\quad +\frac{1}{\mathbf{i}k}\int_1^\infty \frac{e^{\mathbf{i}kx_3r}}{r^2\sqrt{r^2-1}}\int_0^{2\pi}\nabla f\left(x_3\sqrt{r^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)\cdot\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\mathrm{d}\phi\,\mathrm{d}r \;=\; \mathcal{O}(1).
\end{aligned}
$$

For $f=u|_{\mathbb{R}^3_0}\in C_b^1(\mathbb{R}^2)$, we finally arrive at $|[V_ku](\vec{x})|\leq c$ for all $\vec{x}\in\mathbb{R}^3_+$.

4 Representation of the solution by the potential operator in (1.2)

Suppose that u is a solution of the Helmholtz equation $\Delta u+k^2u=0$ over $\mathbb{R}^3_+$, that u as well as all derivatives upto order two are continuous on the closure of $\mathbb{R}^3_+$, and that the second derivative of u

w.r.t. x_3 is given as $\partial^2_{x_3}u=[V_k u]$ by the right-hand side of (1.2) with $x_{f,3}=0$. Then, due to the second order Taylor-series expansion, we get

$$u(\vec{x}) \;=\; f_1(x') + f_2(x')\,(x_3 - x_{f,3}) + \int_{x_{f,3}}^{x_3} (x_3 - t)[V_k u]\big((x',t)^\top\big)\,\mathrm{d}t, \tag{4.1}$$

where the functions $f_1(x') := u((x', x_{f,3})^\top)$ and $f_2(x') := \partial_{x_3} u((x', x_{f,3})^\top)$ are solutions of the two-dimensional Helmholtz equations

$$\Delta_{x'} f_1(x') + k^2 f_1(x') \;=\; -[V_k u]\big((x', x_{f,3})^\top\big)\,, \tag{4.2}$$

$$\Delta_{x'} f_2(x') + k^2 f_2(x') \;=\; -\partial_{x_3}[V_k u]\big((x', x_{f,3})^\top\big)\,. \tag{4.3}$$

Indeed, from $\Delta=\Delta_{x'}+\partial^2_{x_3}$ we conclude, for the Helmholtz solution u, that $\Delta_{x'}u+k^2u=-\partial^2_{x_3}u$ and $\Delta_{x'}\partial_{x_3}u+k^2\partial_{x_3}u=-\partial_{x_3}\partial^2_{x_3}u$.

Note that, for any solution $[V_k u]$ of the three-dimensional Helmholtz equation, the right-hand side of (4.1) satisfies the three-dimensional Helmholtz equation if and only if f_1 and f_2 are solutions of (4.2) and (4.3), respectively. Indeed, using the Helmholtz equation for $[V_k u]$ in the form $(\Delta_{x'}+k^2 I)[V_k u]= -\partial^2_{x_3}[V_k u]$ and the Taylor-series expansion for $[V_k u]$, we get

$$\begin{aligned}
&(\Delta + k^2 I)\left(f_1 + f_2(x_3 - x_{f_3}) + \int_{x_{f_3}}^{x_3} (x_3 - t)[V_k u]\big((\cdot,t)^\top\big)\,\mathrm{d}t\right)\\
&= (\Delta_x' + k^2 I) f_1 + (x_3 - x_{f_3})(\Delta_x' + k^2 I) f_2\\
&\quad + \int_{x_{f_3}}^{x_3} (x_3 - t)(\Delta_x' + k^2 I)[V_k u]\big((\cdot,t)^\top\big)\,\mathrm{d}t + [V_k u]\big((\cdot,x_3)^\top\big)\,.\\
&= (\Delta_x' + k^2 I) f_1 + [V_k u]\big((\cdot,0)^\top\big) + (x_3 - x_{f,3})\left[(\Delta_x' + k^2 I) f_2 + \partial_{x_3}[V_k u]\big((\cdot,0)^\top\big)\right]\\
&\quad +[V_k u]\big((\cdot,x_3)^\top\big)\\
&\quad - \left\{[V_k u]\big((\cdot,0)^\top\big) + \partial_{x_3}[V_k u]\,\big((\cdot,0)^\top\big)\,(x_3 - x_{f,3}) + \int_{x_{f,3}}^{x_3} (x_3 - t)\partial^2_{x_3}[V_k u]\big((\cdot,t)^\top\big)\,\mathrm{d}t\right\}\\
&= (\Delta_x' + k^2 I) f_1 + [V_k u]\big((\cdot,0)^\top\big) + (x_3 - x_{f,3})\left[(\Delta_x' + k^2 I) f_2 + \partial_{x_3}[V_k u]\big((\cdot,0)^\top\big)\right].
\end{aligned}$$

If a solution u is given over the plane $\mathbb{R}^3_{x_{f,3}}$, then $f_1 = u|_{\mathbb{R}^3_{x_{f,3}}}$ is known. If the function u is given over $\mathbb{R}^3_{x_{f,3}}$ and over $\mathbb{R}^3_{x_{d,3}}$ with, e.g., $x_{d,3} > x_{f,3}$, then f_2 is given by the Dirichlet-to-Neumann map

$$\begin{aligned}
f_2(x') := (D_t N f_1)(x') \;&:=\; -2\int_{\mathbb{R}^2} \partial^2_{y_3} G\big((x',x_{f,3})^\top,(y',x_{f,3})^\top\big)\,u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y'\\
&= -\frac{1}{(x_{d,3}-x_{f,3})} u\big((x',x_{d,3})^\top\big) + \int_{\mathbb{R}^2} K_{DtN}(x',y') u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y', \qquad (4.4)\\
K_{DtN}(x',y') :=\; &-2\partial^2_{y_3} G\big((x',x_{d,3})^\top,(y',x_{d,3})^\top\big) + \frac{2}{(x_{d,3}-x_{f,3})}\partial_{y_3} G\big((x',x_{d,3})^\top,(y',x_{f,3})^\top\big)\,.
\end{aligned}$$

Using the formulas (2.2) and (2.4) as well as (3.3) and (3.4), we get the estimate $K_{DtN}(x',y') = \mathcal{O}(|x'-y'|^{-3})$ for $|x'-y'|\to\infty$ such that the last integral in (4.4) converges.

Unfortunately, the solutions of (4.2) and (4.3) are not unique. Even more, (1.2) and the representation with the right-hand side in (4.4) is fulfilled for $u(\vec{x})$ replaced by the sum $u(\vec{x})+u_{2D}(x')x_3$ as well if only

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$(\Delta_{x'}+k^2I)u_{2D}=0$. For instance, the functions $u_{2D}(x'):=J_0(k\sqrt{x_1^2+x_2^2})$ or $u_{2D}(x'):=e^{\mathbf{i}(\alpha x_1+\beta x_2)}$ with $\alpha,\beta\in\mathbb{R}$ and $\alpha^2+\beta^2=k^2$ are solutions of the homogeneous equation $(\Delta_{x'}+k^2I)u_{2D}=0$. Whereas f_1 for u in (4.1) might be uniquely determined by a Dirichlet boundary condition, the function f_2 is unique due to (1.3). Indeed, if f_2 leads to a solution in (4.1) bounded as in (1.3), then any different $\tilde{f}_2$ leads to a perturbation $\vec{x}\mapsto u(\vec{x})+[\tilde{f}_2-f_2](x')x_3$ violating (1.3).

5 Radiation condition for tensor-product solutions

Suppose $u(\vec{y}\,)=u_{2D}(x')u_3(x_3)$ with a linear function u_3 and a solution u_{2D} of the two-dimensional Helmholtz equation $\Delta_{x'}u_{2D}+k^2u_{2D}=0$ over $\mathbb{R}^2$. Moreover suppose that u_{2D} and all first- and second-order derivatives are uniformly bounded at least for large $|x'|$. Clearly, u is a solution of the three-dimensional Helmholtz equation. In particular, all assumptions, except the singular behaviour at the axis $\{\vec{x}\in\mathbb{R}^3_+:\ x'=(0,0)^\top\}$, are satisfied for cylindrical waves of the form $u(\vec{x}\,)=\frac{\mathbf{i}}{4}H_1^0(k|x'|)$ and for $u(\vec{x}\,)=\frac{\mathbf{i}}{4}H_1^0(k|x'|)x_3$.

Without loss of generality, we may suppose $u_3(0)=1$ such that $u|_{\mathbb{R}^3_0}=u_{2D}$. For $j=1,2$, we obtain from the boundedness of u_{2D} and its derivatives and from the decay properties of the kernel functions in (2.2) and (2.3)

$$\begin{aligned}\int_{-R}^{R}\partial_{y_j}\big\{\partial_{y_j}\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,u_{2D}(y')-\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,\partial_{y_j}u_{2D}(y')\big\}\,dy_j\ &=\ \mathcal{O}\big(R^{-2}\big)\,,\\ \int_{-R}^{R}\Big\{\partial_{y_j}^2\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,u_{2D}(y')-\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,\partial_{y_j}^2u_{2D}(y')\Big\}\,dy_j\ &=\ \mathcal{O}\big(R^{-2}\big)\,,\end{aligned}$$

if $R\to\infty$. Consequently,

$$\begin{aligned}&\int_{-R}^{R}\int_{-R}^{R}\partial_{y_3}^3G\big(\vec{x},(y',0)^\top\big)\,u_{2D}(y')\,dy_1\,dy_2\\ &=\ -\int_{-R}^{R}\int_{-R}^{R}(\Delta_{y'}+k^2I)\partial_{y_3}G\big(\vec{x},(y',0)^\top\big)\,u_{2D}(y')\,dy_1\,dy_2\\ &=\ -\int_{-R}^{R}\int_{-R}^{R}\partial_{y_3}G\big(\vec{x},(y',0)^\top\big)\,(\Delta_{y'}+k^2I)u_{2D}(y')\,dy_1\,dy_2+\sum_{j=1}^{2}\int_{-R}^{R}\mathcal{O}\big(R^{-2}\big)\,\mathrm{d}x_j\\ &=\ \mathcal{O}\big(R^{-1}\big)\,.\end{aligned}$$

In other words, $[V_ku](\vec{x})=0$. Condition (1.2) is always fulfilled. However, the pair of radiation conditions (1.2)-(1.3) hold if and only if the linear function u_3 is a constant function.

6 Radiation condition for plane-wave functions, Fourier transform of the potential kernels

6.1 Now consider a plane-wave function $u(\vec{y}\,)=e^{\mathrm{i}(\alpha y_1+\beta y_2+\gamma y_3)}$ with $\alpha,\beta\in\mathbb{R}$ and $\alpha^2+\beta^2+\gamma^2=k^2$. For the case $\alpha^2+\beta^2=k^2$, we get $\gamma=0$ and the results of Sect. 5 apply. Thus we may suppose $\alpha^2+\beta^2\neq k^2$ and $\gamma\neq 0$. We observe that $G(\vec{x}-\vec{y}\,)=G(\vec{x},\vec{y}\,)$ and $\Phi_{x_3,y_3}(x'-y'):=\partial_{y_3}^3\Phi(\vec{x},\vec{y}\,)$ are convolution kernels. The exponential functions $y'\mapsto e^{\mathrm{i}(\alpha y_1+\beta y_2)}$ are eigenfunctions of the convolution and the eigenvalue is the value of the Fourier transform at $\xi' := (\alpha, \beta) \in \mathbb{R}^2$. Consequently, we have $[V_k u](\vec{x}\,) = e^{\mathrm{i}(\alpha x_1 + \beta x_2)} g(x_3)$ with a special function $g(x_3)$ independent of x'. However, since $V_k u$ is a solution of the Helmholtz equation, we conclude

$$[V_k u](\vec{x}\,) \;=\; \begin{cases} e^{\mathrm{i}(\alpha x_1+\beta x_2)} \left(c_1 e^{\mathrm{i}\gamma x_3} + c_2 e^{-\mathrm{i}\gamma x_3}\right) & \text{if } \gamma \neq 0 \\ e^{\mathrm{i}(\alpha x_1+\beta x_2)} \left(c_1 + c_2 x_3\right) & \text{else} \end{cases} \tag{6.1}$$

with special constants c_1 and c_2. The radiation condition (1.2) is fulfilled if and only if $c_1 = -\gamma^2$ and $c_2 = 0$. If $\Im m\,\gamma > 0$, then $e^{-\mathrm{i}\gamma x_3}$ increases exponentially for $x_3 \to \infty$, and $c_2 = 0$ due to the estimates in Sect. 3. The explicit values of c_1 and c_2 can be computed by the Fourier transform of the convolution kernels. However, we prefer to argue using (2.11) with the choice $\Omega_R = C_R,\ \Sigma_R = T_R$.

If $\alpha^2 + \beta^2 > k^2$ and if $\Im m\,\gamma > 0$, then $e^{-\mathrm{i}\gamma x_3}$ decreases exponentially for $x_3 \to \infty$. We get $|u(\vec{x}\,)|, |\partial_\nu u(\vec{x}\,)| \le c e^{-\Im m\,\gamma R^{1/4}}$ on $T_{R,t}$. The two involved kernel functions can be estimated by $|G(\vec{x}, \vec{y}\,)|, |\partial_\nu G(\vec{x}, \vec{y}\,)| \le c|\vec{x} - \vec{y}\,|^{-1}$ such that $|\partial^2_{y_3}\phi(\vec{x}, \vec{y}\,)|, |\partial^2_{y_3}\partial_\nu\phi(\vec{x}, \vec{y}\,)| \le c[R^{1/4}]^{-1}$. The area of $T_{R,t}$ is $\mathcal{O}(R^2)$. We arrive at

$$\int_{T_{R,t}} \{\partial_\nu u \partial^2_{y_3}\Phi(\vec{x}, \cdot) - u\partial_\nu\partial^2_{y_3}\Phi(\vec{x}, \cdot)\} \;=\; \mathcal{O}\Big(e^{-\Im m\,\gamma R^{1/4}} R^{7/4}\Big)\,. \tag{6.2}$$

According to the formula (2.1) the Green's function differentiated w.r.t. y_3 can be estimated by the term $\mathcal{O}((x_3 \pm y_3)\,|\vec{x} - (y', \pm y_3)^\top|^{-2})$. Therefore, on $T_{R,l}$ the two involved kernel functions satisfy $|\partial^2_{y_3}\phi(\vec{x}, \vec{y}\,)|, |\partial^2_{y_3}\partial_\nu\phi(\vec{x}, \vec{y}\,)| \le cR^{1/4}R^{-2}$, the functions u and $\partial_\nu u$ are bounded, and the area of $T_{R,l}$ is $\mathcal{O}(R^{1/4}R)$. We conclude

$$\int_{T_{R,l}} \{\partial_\nu u \partial^2_{y_3}\Phi(\vec{x}, \cdot) - u\partial_\nu\partial^2_{y_3}\Phi(\vec{x}, \cdot)\} \;=\; \mathcal{O}(R^{-1/2}). \tag{6.3}$$

The estimates (6.2) and (6.3) together with $T_R = T_{R,l} \cup T_{R,u}$ yield $I_\infty = 0$. Consequently, the radiation condition (1.2) is satisfied and we get $c_1 = -\gamma^2,\ c_2 = 0$ in (6.1).

If $\Im m\,\gamma < 0$, then we get the same $u|_{\mathbb{R}^3_0}$ as for the choice $\tilde{\gamma} = -\gamma$. From the just proved case for $\Im m\,\tilde{\gamma} > 0$, the formulas $c_1 = -\gamma^2,\ c_2 = 0$ in (6.1) for $\tilde{\gamma}$ imply $c_1 = 0,\ c_2 = -\gamma^2$ in (6.1) for γ and the radiation condition (1.2) is not satisfied.

To compute the c_1 and c_2 for $k^2 > \alpha^2 + \beta^2$, i.e. for real γ, we employ the principle of limited absorption. Choose a small $\varepsilon > 0$ and replace k by $k_\varepsilon := k + \mathrm{i}\varepsilon$. The fundamental solution $G_{k_\varepsilon}(\vec{x} - \vec{y}\,) = e^{\mathbf{i}k|\vec{x}-\vec{y}\,|-k\varepsilon|\vec{x}-\vec{y}\,|}/|\vec{x} - \vec{y}\,|$ is, in contrast to the case with real k, exponentially decaying. Then choosing $\gamma_\varepsilon := \sqrt{k_\varepsilon^2 - \alpha^2 - \beta^2}$ with $\Re e\,\gamma_\varepsilon > 0$ and $\Im m\,\gamma_\varepsilon > 0$ and following exactly the proof for the case $\alpha^2 + \beta^2 > k^2$, we obtain the representation $\partial^2_{y_3}u_\varepsilon(\vec{x}\,) = [V_{k_\varepsilon}u_\varepsilon](\vec{x}\,)$ for the function $u_\varepsilon(\vec{x}\,) = e^{\mathbf{i}(\alpha x_1 + \beta x_2 + \gamma_\varepsilon x_3)}$. In this representation we consider the limit for the parameter $\varepsilon \to 0$. Due to the $\mathcal{O}(|\vec{x} - \vec{y}\,|^{-3})$ estimate for the kernel function in V_{k_ε}, Lebesgue's theorem on dominated convergence applies. We arrive at $\partial^2_{y_3}u(\vec{x}\,) = [V_k u](\vec{x}\,)$, where $u(\vec{x}\,) = e^{\mathbf{i}(\alpha x_1 + \beta x_2 + \gamma_0 x_3)}$ with $\gamma_0 := \lim \gamma_\varepsilon$ the solution of $\gamma_0^2 = k^2 - \alpha^2 + \beta^2$, for which $\gamma_0 > 0$. Thus $c_1 = -\gamma_0^2$ and $c_2 = 0$ holds in (6.1) if $\gamma_0 > 0$ and the radiation condition (1.2) holds for the plane wave with $\gamma > 0$. Furthermore, $c_1 = 0$ and $c_2 = -\gamma^2$ holds in (6.1) if $\gamma = -\gamma_0 < 0$ and the radiation condition (1.2) does not hold for the plane wave with $\gamma < 0$.

Corollary 6.1. *Any quasiperiodic solution of the Helmholtz equation in $\mathbb{R}^3_+$ satisfies the radiation condition (1.2)-(1.3) if and only if it satisfies the classical radiation condition, i.e., if it admits a Rayleigh series expansion into a sum of outgoing plane-wave modes.*

6.2 Now we fix the formulas for the Fourier transform of the convolution kernel in V_k, and we derive a presentation of $[V_k u]$ by general plane-wave functions. For this, we introduce the function $\Phi(\vec{x}) := \Phi(\vec{x}, (0,0,0)^\top)$ and observe $\Phi(\vec{x},\vec{y}) := \Phi(\vec{x}-\vec{y})$. Consequently, $\partial^3_{y_3}\Phi(\vec{x},\vec{y}) = -\partial^3_{x_3}\Phi(\vec{x}-\vec{y})$ and $[V_k u]((x', x_3)^\top)$ is the convolution of $u|_{\mathbb{R}^3_0}$ by the function $x' \mapsto -\partial^3_{x_3}\Phi((x', x_3)^\top)$. Introducing the Fourier transform as

$$[\mathcal{F}f](\xi') \;:=\; \int_{\mathbb{R}^2} e^{-\mathrm{i}2\pi x'\cdot\xi'} f(x')\mathrm{d}\xi', \quad [\mathcal{F}^{-1}g](x') \;=\; \int_{\mathbb{R}^2} e^{\mathrm{i}2\pi x'\cdot\xi'} g(\xi')\mathrm{d}x',$$

the just proved results imply

$$\begin{aligned} [V_k u]\big((x', x_3)^\top\big) \;&=\; \Big[\mathcal{F}^{-1}\big\{m_{x_3}\big[\mathcal{F}(u|_{\mathbb{R}^3_0})\big]\big\}\Big](x'), \qquad (6.4)\\ m_{x_3}(\xi') \;&:=\; \big[\mathcal{F}\big\{-\partial^3_{x_3}\Phi((\cdot, x_3)^\top)\big\}\big](\xi') \;=\; -\big(k^2 - |\xi'|^2\big)\, e^{\mathrm{i}\sqrt{k^2-|\xi'|^2}\,x_3},\\ \sqrt{k^2-|\xi'|^2} \;&:=\; \begin{cases} \sqrt{k^2-|\xi'|^2} & \text{if } |\xi'|\le k\\ \mathrm{i}\sqrt{|\xi'|^2-k^2} & \text{if } |\xi'| > k \end{cases}. \end{aligned}$$

In this generalized sense, (6.4) means

$$[V_k u](\vec{x}) \;=\; \int_{\mathbb{R}^2} \big[\mathcal{F}(u|_{\mathbb{R}^3_0})\big](\xi')\,\partial^2_{x_3}\Big[e^{\mathrm{i}\left(\xi', \sqrt{k^2-|\xi'|^2}\right)^\top\cdot\vec{x}}\Big]\mathrm{d}\xi'. \qquad (6.5)$$

The Fourier transform can also be used to compute the limit of $[V_k u](\vec{x})$ for $x_3 \to 0$ if x' is fixed. Suppose $f := u|_{\mathbb{R}^3_0}$ is a bounded function such that all derivatives upto order five are bounded. Choose a cut-off function $y' \mapsto \chi(y')$ of the same smoothness and with χ identical to one in a neighbourhood of x'. Then $f\chi$ is in L^2 and its Fourier transform $\mathcal{F}(f\chi)(\xi')$ decays at infinity as $\mathcal{O}(|\xi'|^{-5})$. We get

$$[V_k u](\vec{x}) \;=\; \int_{\mathbb{R}^2} \partial^3_{y_3}\Phi\big(\vec{x},(y',y_3)^\top\big)\,(f\chi)(y')\,\mathrm{d}y' + \int_{\mathbb{R}^2} \partial^3_{y_3}\Phi\big(\vec{x},(y',y_3)^\top\big)\,(f[1-\chi])(y')\,\mathrm{d}y',$$

where the second term on the right-hand side tends to zero for $x_3 \to 0$ due to (2.5) and Lebesgue's theorem on dominated convergence. By Plancherel's theorem we obtain

$$[V_k u](\vec{x}) \;=\; -\int_{\mathbb{R}^2} e^{\mathrm{i}2\pi x'\cdot\xi'}\,\big(k^2-|\xi'|^2\big)\,e^{\mathrm{i}\sqrt{k^2-|\xi'|^2}\,x_3}\mathcal{F}(f\chi)(\xi')\,\mathrm{d}\xi' + o(1).$$

Again Lebesgue's theorem together with $\mathcal{F}(f\chi)(\xi') = \mathcal{O}(|\xi'|^{-5})$ for $|\xi'| \to \infty$ lead us to

$$\begin{aligned} \lim_{x_3\to 0}[V_k u](\vec{x}) \;&=\; -\int_{\mathbb{R}^2} e^{\mathrm{i}2\pi x'\cdot\xi'}\,\big(k^2-|\xi'|^2\big)\,\mathcal{F}(f\chi)(\xi')\,\mathrm{d}\xi',\\ [V_k u]\big((x',0)^\top\big) \;&=\; -\Delta_{x'}f(x') - k^2 f(x') \;=\; -\Delta_{x'}u\big((x',0)^\top\big) - k^2 u\big((x',0)^\top\big). \qquad (6.6) \end{aligned}$$

Of course, (6.6) holds under reduced smoothness assumptions on u. Namely, it is sufficient to suppose that all the derivatives of u upto order two are bounded and continuous on $\mathbb{R}^3_0$. Due to $\partial^3_{y_3}G(\vec{x},\vec{y}) = O(|\vec{x}-\vec{y}|^3)$ for $|\vec{x}-\vec{y}| \to 0$, we can fix $\vec{x} \in \mathbb{R}^3_0$ and can reduce the analysis to functions u which have a finite support. Computing classical limits of potential operators in the form of finite-part integrals, we obtain the same limits as in the smoother case considered before. Hence, if $u|_{\mathbb{R}^2_0} \in C^2_b(\mathbb{R}^2)$ with Hölder continuous second order derivatives, then we get

$$[V_k u]\big((x',0)^\top\big) \;=\; -\big(\partial^2_{x_1} + \partial^2_{x_2} + k^2 I\big)\,u\big((x',0)^\top\big) \;=\; \partial^2_{x_3}u\big((x',0)^\top\big). \qquad (6.7)$$

If $u|_{\mathbb{R}^2_0} \in C^2_b(\mathbb{R}^2)$, then the limit $[V_k u]$ in (6.7) holds locally in the L^2 sense. In particular, if the function $f := u|_{\mathbb{R}^3_0}$ is the restriction of a bounded Helmholtz solution in the half space $\{\vec{x} \in \mathbb{R}^3 : -\varepsilon < x_3\}$, then u is sufficiently smooth and (6.7) holds. By the same arguments we get even more.

Proposition 6.2. *The limit relation (6.7) holds if $u|_{\mathbb{R}^2_0} \in C^2(\mathbb{R}^2)$ and if there are constants $C > 0$ and $0 \le v < 1$ such that $|u((x',0)^\top)| < c(1+|x'|)^v$ is true for any $x' \in \mathbb{R}^2$.*

7 Radiation condition for point-source functions

Suppose u is a Helmholtz solution on $\mathbb{R}^3_+$, which is bounded together with its derivatives upto order two on the closure of $\mathbb{R}^3_+$. Similarly to Sommerfeld's condition on the full space $\mathbb{R}^3$, we define

Definition 7.1. *We shall say that a function u on $\mathbb{R}^3_+$ satisfies the outgoing Sommerfeld half-space radiation condition if*

$$\sup_{\vec{x}\in\mathbb{R}^3_+:\,|\vec{x}\,|=r} r\,|\partial_\nu u(\vec{x}\,)-\mathbf{i}ku(\vec{x}\,)| \to 0,\; r\to\infty, \qquad \sup_{\vec{x}\in\mathbb{R}^3_+:\,|\vec{x}\,|\geq R} |\vec{x}\,|\,|u(\vec{x}\,)|<\infty. \tag{7.1}$$

It is well known that, for any fixed $\vec{y}\in\mathbb{R}^3$, the Green's function $\mathbb{R}^3_+\ni\vec{x}\mapsto G(\vec{x},\vec{y}\,)$ and any derivative w.r.t. $\vec{x}$ or $\vec{y}$ satisfy Sommerfeld's radiation condition. Hence, these point source functions also satisfies (7.1).

Suppose $\vec{y}=(y',y_3)^\top$ with $y_3<0$. We shall prove (1.2) for the point-source function $u(\vec{x}\,):=G(\vec{x},\vec{y}\,)$ using only the properties fixed in (7.1). Choosing $\Omega_R=B_R,\ \Sigma_R=S_R$ (cf. Sect. 2), we shall employ the representation (2.11). It remains to prove $I_\infty=0$. The estimates for this, however, are exactly the same as for the full space Sommerfeld condition. Indeed, the fundamental solution G satisfies (7.1) and we get

$$\begin{aligned}\int_{\Sigma_R}\left\{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\right\} &= \int_{\Sigma_R}\left\{[\mathbf{i}k]u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\,[\mathbf{i}k]\partial^2_{y_3}\Phi(\vec{x},\cdot)\right\} \\ &\quad+\int_{\Sigma_R}o\big(R^{-2}\big) = \int_{\Sigma_R}o\big(R^{-2}\big) = o(1)\,.\end{aligned} \tag{7.2}$$

Corollary 7.2. *Any solution of the Helmholtz equation over $\mathbb{R}^3_+$ satisfying the outgoing Sommerfeld half-space radiation condition (7.1) satisfies the (HSRC) too.*

For the "incoming" point-source $u(\vec{x}\,):=\overline{G(\vec{x},\vec{y}\,)}$ with $y_3<0$, we have (7.1) but with the term $\mathbf{i}ku(\vec{x}\,)$ replaced by $-\mathbf{i}ku(\vec{x}\,)$. Instead of (7.2), we arrive at

$$\int_{\Sigma_R}\left\{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\right\} = 2(\mathbf{i}k)I_{i,R}+o(1)\,,\quad I_{i,R}:=\int_{\Sigma_R}\partial^2_{y_3}\Phi(\vec{x},\cdot)u.$$

Taking the asymptotically largest term from (2.4), we conclude

$$\begin{aligned}I_{i,R} &= \int_{\Sigma_R}(\mathbf{i}k)^2\frac{e^{\mathbf{i}k|\vec{x}-\vec{z}\,|}(z_3-x_3)^2}{4\pi|\vec{x}-\vec{z}\,|^3}\,\frac{e^{-\mathbf{i}k|\vec{y}-\vec{z}\,|}}{4\pi|\vec{y}-\vec{z}\,|}\mathrm{d}\vec{z}+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_{\Sigma_R}e^{\mathbf{i}k(\vec{x}-\vec{y})\cdot\vec{z}/|\vec{z}|}\,\frac{z_3^2}{|\vec{z}\,|^4}\mathrm{d}\vec{z}+o(1).\end{aligned}$$

Switching to spherical coordinates, we get

$$\begin{aligned}I_{i,R} &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}k(\vec{x}-\vec{y})\cdot(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)^\top}\cos^2\theta\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}k(x'-y')\cdot(\sin\theta\cos\phi,\sin\theta\sin\phi)^\top}\,\mathrm{d}\phi\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}[k|x'-y'|\sin\theta]\cos\phi}\,\mathrm{d}\phi\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{8\pi}\int_0^{\pi/2}J_0(k|x'-y'|\sin\theta)\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{8\pi}\int_0^1 e^{\mathbf{i}k(x_3-y_3)t}J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\,t^2\,\mathrm{d}t+o(1).\end{aligned}$$

Integration by parts leads us to

$$
\begin{aligned}
I_\infty &= \frac{(\mathbf{i}k)^2}{8\pi}\int_0^1 e^{\mathbf{i}k(x_3-y_3)t} J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\, t^2 \,\mathrm{d}t \\
&= \left[\frac{(\mathbf{i}k)^2}{8\pi}\frac{e^{\mathbf{i}k(x_3-y_3)t}}{\mathbf{i}k(x_3-y_3)} J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\, t^2\right]_{t=0}^1 \\
&\quad -\frac{(\mathbf{i}k)^2}{8\pi\mathbf{i}k(x_3-y_3)}\int_0^1 e^{\mathbf{i}k(x_3-y_3)t}\partial_t\Big\{ J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\, t^2\Big\}\,\mathrm{d}t \\
&= \frac{(\mathbf{i}k)^2}{8\pi}\frac{e^{\mathbf{i}k(x_3-y_3)} - J_0(k|x'-y'|)}{\mathbf{i}k(x_3-y_3)} + \mathcal{O}\Big(\frac{1}{|x_3|^2}\Big)
\end{aligned}
$$

for $|x_3|\to\infty$. Hence, the limit I_∞ is not identically zero. In other words, for the "incoming" point-source, the radiation condition (1.2) is not fulfilled.

8 Condition (HSRC) independent of $x_{h,3}$ and $x_{f,3}$, and equivalence of the conditions (1.3) and (1.4)

8.1 For the dependence of (HSRC) from $x_{h,3}$ and $x_{f,3}$, we notice that the representation (4.1) together with the decay $\mathcal{O}(|\vec{x}-\vec{y}|^{-3})$ of the kernel functions easily imply the condition i) for any fixed $x_{h,3}$. Therefore, it is sufficient to check the dependence on $x_{f,3}$. For definiteness, we compare ii) with $x_{f,3}=0$ and ii) with a fixed $x_{f,3}=y_{f,3}>0$. We denote the operator V_k defined in (2.12) with 0 replaced by $x_{f,3}$ (i.e. integration over $\mathbb{R}^3_{y_{f,3}}$ instead of $\mathbb{R}^3_0$) by $V_{k,y_{f,3}}$. The right-hand side of (1.2) is $[V_k u|_{\mathbb{R}^3_0}]$ for $x_{f,3}=0$ and $[V_{k,y_{f,3}}u|_{\mathbb{R}^3_{y_{f,3}}}]$ for $x_{f,3}=y_{f,3}$. However, by the arguments leading to (2.11) and (6.3), we get $[V_{k,x_{f,3}}u|_{\mathbb{R}^3_{x_{f,3}}}](\vec{x}) = [V_k u|_{\mathbb{R}^3_0}](\vec{x})$ for any $\vec{x}\in\mathbb{R}^3_{x_{f,3}}$. In other words, condition (1.2) with $x_{f,3}=0$ implies (1.2) with $x_{f,3}=y_{f,3}$, and (1.2) with $x_{f,3}=y_{f,3}$ implies (1.2) with $x_{f,3}=0$ at least for $\vec{x}\in\mathbb{R}^3_{x_{f,3}}$. Consequently, the analytical function on both sides of (1.2) coincide on the whole domain of analyticity, i.e., (1.2) holds on $\mathbb{R}^3_0$. (HSRC) is indeed independent of $x_{h,3}$ and $x_{f,3}$. Note that, assuming the items i) and ii) of (HSRC) and fixing an $x_{f,3}>0$, the weak boundedness condition is equivalent to $|1/(x_3-x_{f,3})\int_{x_{f,3}}^{x_3}(x_3-t)/(x_3-x_{f,3})u((x',t)^\top)\,\mathrm{d}t| < c_u(x_3-x_{f,3})^{1-\varepsilon_u}$ for all $x_3>x_{f,3}+1$. Indeed, i) and ii) imply the boundedness of solution u in $\{\vec{x}\in\mathbb{R}^3:\ 0<x_3<x_{f,3}\}$.

8.2 For the equivalence of the conditions (1.3) and (1.4), we observe that the restriction $v(x'):=u((x',x_{f,3})^\top)$ is in the space $C_b^l(\mathbb{R}^2)$, $l\geq 0$. This follows from condition i) of (HSRC), and from the inequality $x_{f,3}>\sup_{x'\in\mathbb{R}^2}|f(x')|$. We retain the notation of the spaces AV_κ, DD_υ, and $FC_k(\mathbb{R}^2)$ from (1.7), (1.8), and (1.9), respectively. Again, for simplicity, we may suppose $x_{f,3}=0$.

Lemma 8.1. *a) If $v\in FC_k$ or if $v\in AV_\kappa$ with $\kappa>0$, then $f_2:=D_tNv$ is a well-defined bounded function which is a partial solution of (4.3) with u replaced by v. Moreover, the function $u_p:=u$ of (4.1), defined with $f_1:=v$ and $f_2:=D_tNv$, satisfies (1.3).*
b) If $v\in DD_\upsilon$ with $0\leq\upsilon<1$, then the same assertions are true for $\mathbb{R}^3_0$ replaced by $\mathbb{R}^3_\varepsilon$ with $\varepsilon>0$.

Proof. i) First we suppose $v\in FC_k$. Due to this we have $v\in C_b^6(\mathbb{R}^2)$. The multiplied and differentiated functions $x'\mapsto(1+|x'|^2)^{-1}\partial^l_{x_j}v(x')$, $l=0,1,\ldots,6$ are in the space $L^2(\mathbb{R}^2)$. Equivalently, the functions $x'\mapsto\partial^l_{x_j}[(1+|x'|^2)^{-1}v(x')]$, $l=0,1,\ldots,6$ are in $L^2(\mathbb{R}^2)$ such that there exists a function $v_L\in L^2(\mathbb{R}^2)$ with $v(x')=(1+|x'|^2)(I-\Delta_{x'})^{-3}v_L(x')$ and the Fourier transform

$[\mathcal{F}v](\xi') = (I - \Delta_{\xi'})(1 + |\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')$. From $\mathcal{F}(-2\partial^2_{x_3} G((\cdot, 0)^\top) = \sqrt{k^2 - |\xi'|^2}$ (compare (6.4)), we obtain

$$\begin{aligned} D_t N v(x') &= \int_{\mathbb{R}^2} (I - \Delta_{\xi'}) \left\{ e^{\mathbf{i}2\pi x'\cdot\xi'} \sqrt{k^2 - |\xi'|^2} \right\} (1 + |\xi'|^2)^{-3} [\mathcal{F}v_L](\xi')\, \mathrm{d}\xi' \qquad (8.1) \\ &= \int_{\mathbb{R}^2} e^{\mathbf{i}2\pi x'\cdot\xi'} \left\{ (1 + 4\pi^2|x'|^2)\sqrt{k^2 - |\xi'|^2} - \frac{2\pi \mathbf{i}\, x' \cdot \xi' + 2}{\sqrt{k^2 - |\xi'|^2}} \right. \\ &\qquad \left. + \frac{|\xi'|^2}{\sqrt{k^2 - |\xi'|^2}^3} \right\} (1 + |\xi'|^2)^{-3} [\mathcal{F}v_L](\xi')\, \mathrm{d}\xi'. \end{aligned}$$

Similar formulas hold for the derivatives w.r.t. x', i.e., for

$$(\Delta_{x'} + k^2 I) D_t N v(x') = \int_{\mathbb{R}^2} (I - \Delta_{\xi'}) \left\{ e^{\mathbf{i}2\pi x'\cdot\xi'} \sqrt{k^2 - |\xi'|^2}^3 \right\} (1 + |\xi'|^2)^{-3} [\mathcal{F}v_L](\xi')\, \mathrm{d}\xi',$$

which is exactly $\partial_{x_3}[V_k u]((x', 0)^\top)$ (for the Fourier transform of the kernel $\partial^4_{y_3}\Phi(\cdot, \vec{y})$ compare (6.4)). Clearly, there are no troubles with integration for large $|\xi'|$ due to the factor $(1 + |\xi'|^2)^{-3}$.

If we suppose that the Fourier transform of v vanishes over the annular domain $R_{k,\varepsilon}$, used in the definition of (1.9)), then we get smooth and bounded values $D_t N v(x')$ and $(\Delta_{x'} + k^2 I) D_t N v(x')$ for x' in bounded domains. Shifting the coordinate system and repeating all the above arguments, we even get uniform boundedness over $\mathbb{R}^2$. In other words, $f_2 := D_t N v$ is a well-defined bounded function which is a partial solution of (4.3) with u replaced by v. Now we turn to the estimate of the function $u_p := u$ of (4.1), defined with $f_1 := v$ and $f_2 := D_t N v$. For the terms in (4.1) and for $x' \neq y'$, we conclude (3.2). In view of (3.2) and $\partial_{y_3}\Phi\big((x', 0)^\top, (y', 0)^\top\big) = \delta_{x'}(y')$ (cf. (2.7) and take into account the jump relation for the double layer kernel), we arrive at

$$\begin{aligned} u_p(\vec{x}) &:= v(x') + x_3 [D_t N v](x') + \int_0^{x_3} (x_3 - t)[V_k v]\big((x', t)^\top\big)\, \mathrm{d}t, \\ &= \int_{\mathbb{R}^2} \partial_{y_3}\Phi\big((x', x_3)^\top, (y', 0)^\top\big)\; v(y')\, \mathrm{d}y'. \end{aligned} \qquad (8.2)$$

This is the double layer integral with Green's function from (1.1). Switching to Fourier transforms (compare (6.4)), we get

$$\begin{aligned} u_p(\vec{x}) &= \int_{\mathbb{R}^2} e^{\mathbf{i}2\pi x'\cdot\xi'} e^{\mathbf{i}\sqrt{k^2 - |\xi'|^2} x_3} [\mathcal{F}v](\xi')\; \mathrm{d}\xi'. \\ &= \int_{\mathbb{R}^2} (I - \Delta_{\xi'}) \left\{ e^{\mathbf{i}2\pi x'\cdot\xi'} e^{\mathbf{i}\sqrt{k^2 - |\xi'|^2} x_3} \right\} (1 + |\xi'|^2)^{-3} [\mathcal{F}v_L](\xi')\, \mathrm{d}\xi'. \qquad (8.3) \end{aligned}$$

Now the Taylor-series expansion at $x_3 = 0$ for the function $x_3 \mapsto e^{\mathbf{i}\sqrt{k^2 - |\xi'|^2} x_3}/[|\xi'|^2 - k^2]$ takes the form

$$\frac{e^{\mathbf{i}\sqrt{k^2 - |\xi'|^2} x_3}}{|\xi'|^2 - k^2} = \frac{1}{|\xi'|^2 - k^2} - \frac{\mathbf{i}}{\sqrt{k^2 - |\xi'|^2}} x_3 + \int_0^{x_3} (x_3 - t) e^{\mathbf{i}\sqrt{k^2 - |\xi'|^2}\, t}\, \mathrm{d}t,$$

and leads us to

$$\int_0^{x_3}(x_3-t)u_p\big((x',t)^\top\big)\,\mathrm{d}t \;=\; \int_{\mathbb{R}^2}(I-\Delta_{\xi'})\left\{e^{-\mathbf{i}2\pi x'\cdot\xi'}\left[\frac{e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}x_3}}{|\xi'|^2-k^2} -\frac{1}{|\xi'|^2-k^2}+\frac{\mathbf{i}}{\sqrt{k^2-|\xi'|^2}}x_3\right]\right\} (1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi',$$

where, again, the assumption $[\mathcal{F}v](\xi')=0,\;\xi'\in R_{k,\varepsilon}$ frees us from any trouble with the non-smoothness of $\sqrt{k^2-|\xi'|^2}$, and the factor $(1+|\xi'|^2)^{-3}$ guarantees integrability for large $|\xi'|$. Applying the two-dimensional Laplacian $\Delta_{\xi'}$ to the term in brackets, we get at most a factor x_3^2 or a factor $|x'|^2$ such that u_p satisfies the weak boundedness condition (1.3) for $|x'|<c$. Shifting the x' coordinates, we get the same result for any x'. Hence the solution $u=u_p$ of (4.1), defined with $f_1:=v$ and $f_2:=D_tNv$, satisfies (1.3).

Splitting a general v into a sum of two functions, one with a Fourier transform vanishing in the annular domain $R_{k,\varepsilon}$ and one with support contained in the domain $R_{k,2\varepsilon}$, it remains to proof the lemma for the latter case. This case, however, is completely analogous to the just finished place. The only difference is that we apply the assumptions of functions from FC_k (cf. (1.9)) on the annular domain. Thus $[\mathcal{F}v](\xi')=(I-\Delta_{\xi'})(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')$ turns to $[\mathcal{F}v](\xi')=[\mathcal{F}v](\xi')$ with support in $R_{k,2\varepsilon}$ and, e.g., (8.1) and (8.3) into

$$\begin{aligned} D_tNv(x') &= \int_{R_{k,\varepsilon}} e^{\mathbf{i}2\pi x'\cdot\xi'}\sqrt{k^2-|\xi'|^2}[\mathcal{F}v](\xi')\,\mathrm{d}\xi',\\ u_p(\vec{x}) &= \int_{R_{k,\varepsilon}} e^{\mathbf{i}2\pi x'\cdot\xi'}e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}[\mathcal{F}v](\xi')\,\mathrm{d}\xi'. \end{aligned}$$

We finally get the uniform boundedness of u_p and all the assertions of Lemma 8.1 for $v\in FC_k$.

ii) Now we assume $v\in AV_\kappa$. The Dirichlet-to-Neumann map on the right-hand side of (1.4) is a convolution operator with kernel depending only on $|x'-y'|$. Using (1.6), it takes the form

$$\begin{aligned} [D_tNv](x') &:= -2\int_{\mathbb{R}^2}\partial^2_{y_3}G\big((x',0)^\top,(y',0)^\top\big)\,v(y')\,\mathrm{d}y'\\ &= -2\int_{\mathbb{R}^2}\partial^2_{y_3}G\big((y',0)^\top,(0,0,0)^\top\big)\,v(x'-y')\,\mathrm{d}y'\\ &= \int_{y'\in\mathbb{R}^2:\,|y'|>1}\mathcal{O}\big(|y'|^{-2}+|y'|^{-3}\big)\,v(x'-y')\,\mathrm{d}y'+\mathcal{O}(1)\,,\\ |[D_tNv](x',0)| &\le \int_1^\infty \mathcal{O}\big(r^{-1}\big)\,|av(v,x',r)|\,\mathrm{d}r+\mathcal{O}(1)\,, \end{aligned} \tag{8.4}$$

where the term $\mathcal{O}(1)$ results from an integration of a finite-part integral for a sufficiently smooth function. The estimate of $av(v,x',r)$ in the definition (1.7) of AV_κ, implies the continuity and uniform boundedness of D_tNv. Analogously, from (8.2) we conclude

$$|u_p(\vec{x})| \;\le\; \int_1^\infty \mathcal{O}\big(r^{-1}\big)\,|av(v,x',r)|\,\mathrm{d}y'+\mathcal{O}(1)\,,$$

such that the estimate of $av(v,x',r)$ in (1.7), implies the continuity and uniform boundedness of the solution u_p. If we apply $(\Delta_{x'}+k^2I)$ to D_tNv and $(\Delta+k^2I)$ to u_p, the convergence of the integrals follows easily since the differentiated kernel function satisfy the same estimates as the original kernels. Using the two facts that $\vec{x} \mapsto \partial_{y_3}\Phi(\vec{x},\vec{y})$ is a solution of the Helmholtz equation and that $(\Delta_{x'}+k^2I)\partial^2_{y_3}\Phi(\vec{x},\vec{y})=-\partial^4_{y_3}\Phi(\vec{x},\vec{y})$, we get that u_p is a Helmholtz solution and that v is a solution of (4.3) with u replaced by v.

iii) Finally, we assume $v\in DD_v$. Taking v_s, v_i, and v_0 in accordance with (1.8), we define the Helmholtz solution $u_{DD}((x',x_3)^\top):=v_s(x')-[V_kv_0]((x',x_3)^\top)$. Using Prop. 6.2, we get the boundary value

$$\begin{aligned} u_{DD}\big((x',0)^\top\big) &= v_s(x') - [V_kv_0]\big((x',0)^\top\big) \\ &= v_s(x') + \big[(\Delta_{x'}+k^2I)v_0\big](x') \;=\; v_s(x')+v_i(x') \;=\; v(x'). \end{aligned}$$

Furthermore, using $[V_kv_s]=0$ (cf. Sect. 5) and the fact that differentiation and convolution operator commute, we conclude

$$\begin{aligned} \partial^2_{x_3}u_{DD}(\vec{x}) &= -\partial^2_{x_3}[V_kv_0](\vec{x}) &&= (\Delta_{x'}+k^2I)[V_kv_0](\vec{x}) \\ &= [V_k(\Delta_{y'}+k^2I)v_0](\vec{x}) &&= [V_kv_i](\vec{x}) \;=\; [V_kv](\vec{x}) \end{aligned}$$

such that (1.2) is fulfilled. Together with Prop. 3.1 the radiation condition (HSRC) holds for u_{DD}. In other words, u_{DD} is a solution to the Dirichlet problem $u_{DD}(\vec{x})=v(\vec{x}),\ \vec{x}\in\mathbb{R}^3_0$ of the Helmholtz equation satisfying the radiation condition (HSRC).

Setting $f_2:=D_tNv:=\partial_{x_3}u_{DD}|_{\mathbb{R}^3_0}$ we get a well-defined solution of (4.3) with u replaced by v. Indeed,

$$\begin{aligned} (\Delta_{x'}+k^2I)f_2 &= (\Delta_{x'}+k^2I)\partial_{x_3}[V_kv_0] \;=\; \partial_{x_3}(\Delta_{x'}+k^2I)[V_kv_0] \\ &= \partial_{x_3}[V_k(\Delta_{y'}+k^2I)v_0] \;=\; \partial_{x_3}[V_kv_i] \;=\; \partial_{x_3}[V_kv]. \end{aligned}$$

Clearly, by the Taylor-series expansion we get that the function $u_p:=u$ of (4.1), defined with $f_1:=v$ and $f_2:=D_tNv$, is equal to u_{DD}. Hence, it satisfies (1.3). □

Now the equivalence of the conditions (1.3) and (1.4) is easy to show. The general solution of (4.3) is $f_{2,g}=D_tN\tilde{v}+f_{2,h}$ with $\tilde{v}=u|_{\mathbb{R}^3_0}$ and with a solution $f_{2,h}$ of the two-dimensional Helmholtz equation $(\Delta_{x'}+k^2I)f_{2,h}=0$. The solution $u_g=u$ in (4.1), defined with $f_1:=\tilde{v}$ and $f_2:=f_{2,g}$, takes the form $u_g(\vec{x})=u_p(\vec{x})+f_{2,h}(x')x_3$. This, however, fulfills (1.3) if and only if $f_{2,h}\equiv 0$, i.e., if and only if $\partial_{x_3}u_g=f_{2,g}=D_tN(u|_{\mathbb{R}^3_0})$. The uniform boundedness of the solution v, if $v\in AV_\kappa$ or if $v\in DD_0$ with $v_0\in C^1_b(\mathbb{R}^2)$, follows from the points ii) and iii) of the proof to Lemma 8.1 and from the last argument in Sect. 3.

9 Solution of the Dirichlet problem over the half space

For a function u continuous on the closure of $\mathbb{R}^3_+$ and twice differentiable on $\mathbb{R}^3_+$, we consider the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(\vec{x})-k^2u(\vec{x}) &= 0,\ \forall\vec{x}\in\mathbb{R}^3_+, \\ u\big((x',0)^\top\big) &= v(x'),\ \forall x'\in\mathbb{R}^2, \\ &u \text{ satisfies (HSRC).} \end{aligned} \qquad (9.1)$$

Proposition 9.1. *The solution of problem (9.1) is unique.*

Proof. For two solutions u_1 and u_2 the difference $u = u_1 - u_2$ satisfies the homogeneous problem (9.1), i.e., the problem with $v \equiv 0$. However, from the radiation condition we get the representation (1.2). Using the arguments of Sect. 8.1, we can even suppose $x_{f,3} = 0$ and infer that $\partial^2_{x_3} u \equiv 0$. Hence, u is linear with respect to x_3 and $u(\vec{x}) = f_1(x') + f_2(x')x_3$. From the boundedness condition (1.3), we get $f_2 \equiv 0$ and, from the homogeneous Dirichlet condition, $f_1 \equiv 0$. Hence, $u \equiv 0$ and the two solutions u_1 and u_2 coincide. □

It is unclear to us, whether there exists solutions of (9.1) for any Dirichlet data in $C_b(\mathbb{R}^2)$. Even if only the items i) and ii) of the radiation condition (HSRC) are satisfied, then we get a necessary condition. Namely, there must exist solutions of (4.3). We do not know whether this is always fulfilled. Therefore, by DS we denote the space of all $v \in C_b(\mathbb{R}^2)$ such that there exists a solution f_2 of (4.3) with $f_2 \in C^l_b(\mathbb{R}^2)$ for all integers $l > 0$. Unfortunately, the solution f_2 is not unique. The general solution of (4.3) is the sum of the partial solution f_2 and a homogeneous solution of the two-dimensional Helmholtz equation. We easily obtain the formal result

Proposition 9.2. *For $v \in C_b(\mathbb{R}^2)$, there exists a solution of (9.1), possibly without condition (1.3), if and only if v is in the space DS. If $v \in DS$, then the solution is given by*

$$\begin{aligned} u(\vec{x}\,) &= v(x') + [D_tNv]\big((x',0)^\top\big)\, x_3 + \int_0^{x_3} (x_3 - t)\, [V_k v]\big((x',t)^\top\big)\, \mathrm{d}t, \qquad (9.2)\\ [V_k v](\vec{x}) &:= \int_{\mathbb{R}^2} \partial^3_{y_3} \Phi\big(\vec{x}, (y',0)^\top\big)\, v(y')\, \mathrm{d}y', \\ [D_tNv]\big((x',0)^\top\big) &:= f_2(x') - \int_0^{x_{f,3}} [V_k v]\, \big((x',t)^\top\big)\, \mathrm{d}t \end{aligned}$$

with $f_2 \in C^l_b(\mathbb{R}^2),\ l \in \mathbb{Z},\ l > 0$ the solution of (4.3). Suppose there is a linear operator D_tN_2 mapping v to a solution f_2 and, for the current proposition, replace item iii) of the radiation (HSRC) by the condition $\partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}}} = D_tN_2\, u|_{\mathbb{R}^3_0}$. Then there exists a unique solution of (9.1), which takes the form (9.2) with $f_2 = D_tN_2 v$.

To avoid the non-practical assumptions in Prop. 9.2, we have to restrict the condition on $v \in C_b(\mathbb{R}^2)$.

Proposition 9.3. *Suppose that v is either in $AV_\kappa,\ \kappa > 0$, in $DD_\upsilon,\ 0 \le \upsilon < 1$ or in FC_k. Then there exists a unique solution of (9.1), which is even uniformly bounded for $v \in AV_\kappa$ and for $v \in DD_0$ with $v_0 \in C^1_b(\mathbb{R}^2)$.*

Proof. In the case of DD_υ, the function u_{DD} is the solution due to part iii) of Lemma 8.1. The boundedness follows from the last argument in Sect. 3. For the case of AV_κ and FC_k, the function u_p of (8.2) is the solution in accordance with the parts i) and ii) of Lemma 8.1. Even the boundedness for the solution in the case AV_κ has been shown there. □

Proposition 9.4. *Consider all the $v \in DD_0$ with $v_0 \in C^1_b(\mathbb{R}^2)$ (cf. (1.8)). Then the corresponding splitting $v = v_i + v_s$ is unique.*

Proof. We have to show that the Dirichlet data cannot satisfy both, the two-dimensional Helmholtz equation $(\Delta_{x'} + k^2 I)v = 0$ and the representation $v = (\Delta_{x'} + k^2 I)v_0$. We shall suppose both and show $v \equiv 0$. From the Helmholtz equation, we get, for any test function φ, that

$$0 = \big\langle v, (\Delta_{x'} + k^2 I)\varphi \big\rangle .$$

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Now we substitute the representation of v as the image of the Helmholtz operator and choose $\varphi = \chi_l v_0$, where χ_l denotes a cut-off function with $\chi_l \equiv 1$ on the disc $D_l := \{x' \in \mathbb{R}^2 : |x'| \le l\}$ and $\chi_l \equiv 0$ on the exterior $\mathbb{R}^2 \setminus D_{l+1}$ of the larger disc D_{l+1}.

$$\begin{aligned} 0 &= \big\langle (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)\,[\chi_l v_0] \big\rangle \\ &= \big\langle \chi_l (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle \\ &\quad + \big\langle (\Delta_{x'} + k^2 I)v_0, \big[v_0 \Delta_{x'} \chi_l + \sum_{j=1}^{2} \partial_{x_j} \chi_l \partial_{x_j} v_0 \big] \big\rangle \\ &= \big\langle \chi_l (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle + \mathcal{O}(1). \end{aligned}$$

Consequently,

$$\begin{aligned} \big\langle \chi_{L+1} (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle &= \big\langle \chi_1 (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle + \\ &\quad \sum_{l=1}^{L} \big\langle [\chi_{l+1} - \chi_l](\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle \\ &= \mathcal{O}(1). \end{aligned}$$

If the truncated sum of nonnegative terms is uniformly bounded, then the infinite sum is convergent, and we arrive at

$$\big\langle (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle \;<\; \infty.$$

In other words, $v = (\Delta_{x'} + k^2 I)v_0$ is square integrable over $\mathbb{R}^2$. As a solution of the Helmholtz equation the square integrable Fourier transform $\mathcal{F}[v]$ satisfies $(-|\xi'|^2 + k^2)\mathcal{F}[v](x') \equiv 0$. Thus $\mathcal{F}[v](x') \equiv 0$ and $v = 0$. □

The space of all DD_0 with $v_0 \in C_b^1(\mathbb{R}^2)$ is algebraically the direct sum of the space of Helmholtz solutions plus the space of all images of the Helmholtz operator. If the metric of the function space corresponds to the uniform convergence over bounded subdomains, then the space of Helmholtz solutions is closed. However, the space of images is not. For example, the function $x' \mapsto e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha^2 + \beta^2 = k^2$ is the limit of functions $x' \mapsto e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha^2 + \beta^2 \ne k^2$ (cf. the subsequent example i)). It would be nice to have an intrinsic description of the space DD_0. Instead, we only recall important functions belonging to the spaces DD_0 and AV_κ:

i) The space DD_0 contains all exponential functions $x' \mapsto v(x') = e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$, i.e., the traces of the plane-wave functions. For $\alpha^2 + \beta^2 = k^2$, the function $v = v_s$ is a two-dimensional Helmholtz solution and, for $\alpha^2 + \beta^2 \ne k^2$, the function $v = (\Delta_{x'} + k^2 I)v_0$ is an image with $v_0 = \frac{1}{-\alpha^2 - \beta^2 + k^2}\, v$. If $\alpha^2 + \beta^2 \ne 0$, then the exponential function is in $AV_{1/2}$ due to $av(v, x', r) = 2\pi\, e^{\mathbf{i}(\alpha x_1 + \beta x_2)} J_0(\sqrt{\alpha^2 + \beta^2}\, r)$.

ii) The space DD_0 contains all decaying functions $v \in C_b(\mathbb{R}^2)$, with $v(x') = \mathcal{O}(|x'|^{-3/2-\varepsilon})$ for $|x'| \to 0$ and fixed positive ε. Indeed, such a function is an image $v = (\Delta_{x'} + k^2 I)v_0$ with

$$v_0(x') \;=\; \frac{\mathbf{i}}{4} \int_{\mathbb{R}^2} H_0^{(1)}\big(k|x' - y'|\big) v(y')\, \mathrm{d}y'.$$

By the same argument, we even get $AV_{3/2+\varepsilon} \subset DD_0$. Obviously, the space AV_κ contains all decaying functions $v \in C_b^1(\mathbb{R}^2)$, with $v(x') = \mathcal{O}(|x'|^{-\kappa})$ for $|x'| \to 0$ and fixed positive κ.

iii) The space DD_0 contains all traces of point source functions $y' \mapsto G(\vec{x},(y',0)^\top)$ for fixed $\vec{x} \notin \mathbb{R}^3_+ \cup \mathbb{R}^3_0$. Indeed, such a trace is an image according to

$$\begin{aligned}(\Delta_{y'}+k^2I)\Big(e^{\mathbf{i}k|\vec{x}-(y',0)^\top|}\Big) &= 2(\mathbf{i}k)\frac{e^{\mathbf{i}k|\vec{x}-(y',0)^\top|}}{|\vec{x}-(y',0)^\top|}+k^2\frac{e^{\mathbf{i}k|\vec{x}-(y',0)^\top|}x_3^2}{|\vec{x}-(y',0)^\top|^2},\\ G\big(\vec{x},(y',0)^\top\big) &= (\Delta_{y'}+k^2I)\frac{1}{8\pi(\mathbf{i}k)}\Bigg\{e^{\mathbf{i}k|\vec{x}-(y',0)^\top|}\\ &\qquad -\frac{\mathbf{i}\,k^2x_3^2}{4}\int_{\mathbb{R}^2}H_0^{(1)}\big(k|y'-z'|\big)\frac{e^{\mathbf{i}k|\vec{x}-(z',0)^\top|}}{|\vec{x}-(z',0)^\top|^2}\,\mathrm{d}z'\Bigg\}.\end{aligned}$$

Similarly, the traces of all the derivatives $y' \mapsto \partial_{\vec{x}}^{\alpha_x}\partial_{\vec{y}}^{\alpha_y}G(\vec{x},(y',0)^\top)$ with multi-indices α_x and α_y are contained in DD_0. These function belong to AV_1 in accordance with example ii).

iv) By definition DD_0 contains all solutions $u \in C_b^2(\mathbb{R}^2)$ of the two-dimensional Helmholtz equation. These function are contained in $AV_{1/2}$, since $av(v,x',r)=v(x')J_0(kr)$.

The space of solutions v_s and, correspondingly, the space DD_0 could have been extended by all traces h of cylindrical waves, the axes of which are perpendicular to $\mathbb{R}^3_0$. These traces are unbounded close to the axes. However, the functions $u(\vec{x}) := v_s(x')$ form two-dimensional Helmholtz solutions away from the axes. The boundedness condition (1.3) must be modified by replacing $u((x',x_3)^\top)$ with special averages over x'.

10 Uniqueness for the Dirichlet problem on thin layers

For a height $h_L > 0$ and index pairs $l' \in \mathbb{Z}^2$, we introduce the layer $\Omega_L := \{\vec{x} \in \mathbb{R}^3 :\ 0 < x_3 < h_L\}$ and the cylindrical domain $\Omega_{L,l'} := \{\vec{x} \in \Omega_L :\ |x'-l'| < 4\}$. We consider the Dirichlet problem

$$\begin{aligned}\Delta u(\vec{x})+k^2u(\vec{x}) &= 0,\ \forall \vec{x}\in\Omega_L,\\ u\big((x',0)^\top\big) &= v_0(x'),\ \forall x'\in\mathbb{R}^3_0,\\ u\big((x',h_L)^\top\big) &= v_{h_L}(x'),\ \forall x'\in\mathbb{R}^3_{h_L},\\ \sup_{l'\in\mathbb{Z}^2}\|u\|_{H^1(\Omega_{L,l'})} &< \infty\end{aligned} \tag{10.1}$$

with prescribed bounded and continuous Dirichlet data v_0 and v_{h_L}.

Lemma 10.1. *If the positive width h_L is less than π/k and if there is a solution of the Dirichlet problem (10.1) over the layer Ω_L of thickness h_L, then this solution is unique.*

Proof. Of course, we have to prove that any solution of the homogeneous problem (10.1) is trivial. Suppose u is a solution of (10.1) with $v_0 \equiv 0$ and $v_{h_L} \equiv 0$. Then we extend u to a function over $\mathbb{R}^3$ by

$$u(\vec{x}) \ := \ \begin{cases} u\big((x',z_3)^\top\big) & \text{if } x_3 = z_3+(2m)\,h_L,\quad\ \ 0\le z_3\le h_L,\ m\in\mathbb{Z}\\ u\big((x',h_L-z_3)^\top\big) & \text{if } x_3 = z_3+(2m+1)\,h_L,\ 0\le z_3\le h_L,\ m\in\mathbb{Z}\end{cases},$$

which is $(2h_L)$ periodic w.r.t. x_3. Since this extended u and the normal derivatives $\partial_{x_3}u$ are continuous through the interface planes $\mathbb{R}^3_{mh_L}$, $m\in\mathbb{Z}$, the function u is a periodic Helmholtz solution over $\mathbb{R}^3$.

Consequently, the modulated Fourier coefficients $\hat{v}_m$, defined by

$$\begin{aligned} u\big((x',x_3)^\top\big) &= \sum_{m\in\mathbb{Z}} \hat{u}_m(x') e^{\mathbf{i}\pi m x_3/h_L}, \\ \hat{u}_m(x') &:= \int_0^{h_L} u\big((x',x_3-z_3)^\top\big)\, e^{\mathbf{i}\pi(x_3-z_3)m/h_L} \mathrm{d}z_3, \\ \hat{v}_m(x') &:= \int_0^{h_L} e^{-\mathbf{i}\pi z_3 m/h_l} u(x',x_3-z_3)\mathrm{d}z_3 \;=\; \hat{u}_m(x') e^{-\mathbf{i}\pi x_3 m/h_L}, \end{aligned} \tag{10.2}$$

are Helmholtz solution for any $m\in\mathbb{Z}$. In other words, $\hat{u}_m$ satisfies $(\Delta_{x'}-\varrho_m^2 I)\hat{u}_m = 0$, $\varrho_m := \sqrt{(\pi m/h_L)^2 - k^2}$. The average $av(\hat{u}_m, x', r)$ defined in (1.6) satisfies the corresponding Bessel equation over the real half axis and is smooth at zero. Since the Bessel function $r \mapsto Y_0(\mathbf{i}\varrho_m r)$ is singular at zero (cf. [1], Sect. 9.1.89) and since the Bessel function $r\mapsto J_0(\mathbf{i}\varrho_m r)$ is unbounded for $r\to\infty$ (cf. [1], Sect. 9.2.1), the solution $av(\hat{u}_m, x', r)$ is zero. Using (10.2) and the differentiated (10.2), we get $av(u|_{\mathbb{R}^3_{x_3}}, x', r) = 0$ and $av(\partial_{x_j} u|_{\mathbb{R}^3_{x_3}}, x', r) = 0,\; j=1,2,3$. This together with the arguments in (2.11)-(2.13) and the choice $\Omega_R = C_R$ and $\Sigma_R = T_R$ leads to (cf. (3.3) and (3.4))

$$\begin{aligned} u(\vec{x}) &= \lim_{R\to\infty}\left\{ 2\int_{\mathbb{R}^3_0} \partial_{y_3} G(\vec{x},\cdot)u - 2\int_{\mathbb{R}^3_0} G(\vec{x},\cdot)\partial_{y_3} u - 2\int_{\mathbb{R}^3_{h_L}} \partial_{y_3} G(\vec{x},\cdot)u + 2\int_{R^3_{h_L}} G(\vec{x},\cdot)\partial_{y_3} u \right\} \\ &= -2\int_{\mathbb{R}^3_0} G(\vec{x},\cdot)\partial_{y_3} u + 2\int_{\mathbb{R}^3_{h_L}} G(\vec{x},\cdot)\partial_{y_3} u \\ &= \int_0^\infty \frac{e^{\mathbf{i}k\sqrt{(x_3-h_L)^2+r^2}}}{2\pi\sqrt{(x_3-h_L)^2+r^2}}\, av(\partial_{x_3} u|_{\mathbb{R}^3_{h_L}}, x', r) r\,\mathrm{d}r \\ &\quad - \int_0^\infty \frac{e^{\mathbf{i}k\sqrt{x_3^2+r^2}}}{2\pi\sqrt{x_3^2+r^2}}\, av(\partial_{x_3} u|_{\mathbb{R}^3_0}, x', r) r\,\mathrm{d}r \qquad = \; 0. \end{aligned}$$

□

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