Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient

Pierre-Étienne Druet[1], Ansgar Jüngel[2]

submitted: November 22, 2019

[1] Weierstrass Institute
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: pierre-etienne.druet@wias-berlin.de

[2] Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien
Austria
E-Mail: juengel@tuwien.ac.at

No. 2646
Berlin 2019

2010 *Mathematics Subject Classification.* 35K45, 35L65, 35Q79, 35M31, 35Q92, 92C17.

Key words and phrases. Parabolic-hyperbolic system, cross diffusion, fluid mixture, existence of solutions, transport equation.

The first author was supported by the grant D1117/1-1 of the German Science Foundation (DFG). The second author acknowledges partial support from the Austrian Science Fund (FWF), grants P30000, P33010, W1245, and F65.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient

Pierre-Étienne Druet, Ansgar Jüngel

Abstract

The convective transport in a multicomponent isothermal compressible fluid subject to the mass continuity equations is considered. The velocity is proportional to the negative pressure gradient, according to Darcy's law, and the pressure is defined by a state equation imposed by the volume extension of the mixture. These model assumptions lead to a parabolic-hyperbolic system for the mass densities. The global-in-time existence of classical and weak solutions is proved in a bounded domain with no-penetration boundary conditions. The idea is to decompose the system into a porous-medium-type equation for the volume extension and transport equations for the modified number fractions. The existence proof is based on parabolic regularity theory, the theory of renormalized solutions, and an approximation of the velocity field.

1 Introduction

Multicomponent fluids are found in nature and many engineering applications, for instance, in combustion, chemical reactors, tumor growth, and gas mixtures. For efficient modeling and simulations of these applications, we need to understand the mathematical structure of the governing partial differential equations and to determine the properties of their solutions. In this paper, we analyze the mass continuity equations for the partial mass densities, subject to Darcy's law for the fluid velocity and a pressure related to the volume extension of the mixture, and we prove the global-in-time existence of smooth and weak solutions. The main novelty is the general framework of our pressure model.

1.1 Model equations

We consider the evolution of a fluid mixture of N substances in a bounded container $\Omega \subset \mathbb{R}^3$. We assume that the system is in the isothermal state. The mass densities $\rho_1, \ldots, \rho_N$ of the species are given by the conservation equations

$$\partial_t \rho_i + \operatorname{div}(\rho_i v) = 0 \quad \text{in } \Omega \times (0,T),\ i = 1, \ldots, N. \tag{1}$$

We suppose that the fluid is driven only by the thermodynamic pressure. Then the barycentric velocity v is determined by Darcy's law

$$v = -\kappa \nabla p \quad \text{in } \Omega \times (0,T), \tag{2}$$

DOI 10.20347/WIAS.PREPRINT.2646 Berlin 2019

where the porosity coefficient κ generally depends on the medium Ω and the fluid. The pressure is related to the volume extension Λ of the mixture via

$$p = G(\Lambda(\rho)), \quad \rho = (\rho_1, \ldots, \rho_N), \tag{3}$$

where G is an increasing scalar function and Λ is positively homogeneous (of degree one). Typical choices are $G(s) = c_0 s^\alpha$ with $\alpha > 1$ and $\Lambda(\rho) = \sum_{i=1}^N c_i \rho_i$, where $c_0, \ldots, c_N > 0$. We refer to Section 2 for the modeling details.

Equations (1)–(3) can be formulated as a cross-diffusion system with entropy structure. Indeed, we show in Section 2 that there exists a free energy (or entropy) $h(\rho)$ such that $\nabla p = \sum_{j=1}^N \rho_j \nabla \mu_j$, where $\mu_j = \partial h / \partial \rho_j$ are the chemical potentials (or entropy variables). Then system (1)–(3) is equivalent to the cross-diffusion system

$$\partial_t \rho_i - \operatorname{div} \sum_{j=1}^N M_{ij}(\rho) \nabla \mu_j = 0, \quad i = 1, \ldots, N, \tag{4}$$

where the kinetic (or mobility) matrix $M_{ij}(\rho) = \kappa \rho_i \rho_j$ has rank one only. Thus, we expect that there is only one "parabolic direction" and $N-1$ "hyperbolic directions". The challenge is to deal with such cross-diffusion systems possessing *incomplete diffusion*. It is sufficient to impose one boundary condition, and we choose in this paper the no-penetration (and initial) conditions

$$v \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0,T), \quad \rho_i(0) = \rho_i^0 \quad \text{in } \Omega, \ i = 1, \ldots, N, \tag{5}$$

where ν is the exterior unit normal vector to $\partial\Omega$. Alternatively, in the case that there are free inflows and outflows, we may use the pressure boundary condition

$$p = p_0 \quad \text{on } \partial\Omega \times (0,T).$$

In practical situations, a mix of free flow and impermeable portions of the boundary is often realistic. This leads to more technical problems in the mathematical analysis, so we consider mainly the boundary conditions in (5). Some remarks to treat pressure boundary conditions are given in Section 5.3.

1.2 State of the art

Our study is motivated by some related problems. First, in [23], we have analyzed the cross-diffusion system

$$\partial_t \rho_i - \operatorname{div} \sum_{j=1}^N \big(\rho_i \rho_j + \varepsilon D_{ij}(\rho) \big) \nabla \mu_j = 0, \quad i = 1, \ldots, N, \tag{6}$$

where the pressure models a Van der Waals gas mixture (this determines μ_j) and the matrix $(D_{ij}(\rho))$ is positive definite on the orthogonal complement of a one-dimensional subspace of $\mathbb{R}^N$. Because of the lack of parabolicity, the diffusion fluxes $J_i = -\varepsilon \sum_{j=1}^N D_{ij}(\rho) \nabla \mu_j$ with $\varepsilon > 0$ were needed to apply the boundedness-by-entropy method [21]. Unfortunately, the authors of [23] were not able to perform the limit $\varepsilon \to 0$. In this paper, we show that (6) admits solutions also in the case $\varepsilon = 0$ (for suitable pressure functions).

Second, consider N interacting biological species with densities ρ_i and velocities v_i governed by the continuity equations

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \quad i = 1, \dots, N.$$

When dispersal is a response to population pressure, we may assume that the dispersal of each of the species is driven by the gradient of the total population [2, 19],

$$v_i = -k_i \nabla \sum_{j=1}^{N} \rho_j, \quad i = 1, \dots, N.$$

This leads to system (4) with $M_{ij} = \rho_i \rho_j$ and $\mu_j = k_j \log \rho_j$. The model describes the evolution of the cell populations ρ_i in tissues and tumors. It was analyzed for two species $N = 2$ in [2] in one space dimension and in [3] in several space dimensions. The existence of global weak and classical solutions was proved. They possess the particular feature that they are segregated if they do so initially, i.e., their support is disjoint for all time. For related models, we refer to [15, 16].

A nonlinear pressure function $p = (\rho_1 + \rho_2)^\alpha$ with $\alpha > 1$ was considered in [20], giving (4) with $N = 2$, $M_{ij} = \rho_i \rho_j (\rho_1 + \rho_2)^{\alpha - 1}$, and $\mu_j = \log \rho_j$. The existence of weak solutions in the whole space was proved in the presence of reaction terms. The two-species system of Bertsch et a. [3] with nonlocal interaction terms models aggregation and repulsion of the species in the context of chemotaxis, opinion formation, and pedestrian dynamics (see the references in [6]).

If the pressure is the variational derivative of a certain energy functional $F(\rho)$, we may write (1) for $N = 2$ as the (formal) gradient flow

$$\partial_t \rho_1 = \operatorname{div}\left(\rho_1 \nabla \frac{\delta F}{\delta \rho_1}\right), \quad \partial_t \rho_2 = \operatorname{div}\left(\rho_2 \nabla \frac{\delta F}{\delta \rho_2}\right).$$

This relation has been exploited in [5, 10], proving the convergence of the minimizing movement scheme for quadratic functionals F [5] or general convex functionals [10], even including nonlocal terms; also see [7].

Related systems that consist of the mass continuity equation (1) and the Darcy law (2) are analyzed in the literature also in the context of fluid flows in porous media [25], often extended by the Darcy–Brinkman law for fractures in porous media [29] or for tumor growth models [17].

Surprisingly, there are almost no results for general N-species models. The work [9] studies (4) with $M_{ij} = \rho_i \rho_j / \rho_{\rm tot}$, where $\rho_{\rm tot} = \sum_{i=1}^N \rho_i$, and $\mu_j = \log \rho_j$. Furthermore, system (4) with $M_{ij} = a_{ij} \rho_i \rho_j$ and $a_{ij} > 0$ is the mean-field limit of an interacting stochastic particle system [8]. Up to our knowledge, there exists no general global existence result for (4).

In this paper, we analyze the N-species system with a rather general pressure-density relation, extending all previous existence results. In [3, 19], the problem is decomposed into a porous-medium-type equation for the total density $\rho_{\rm tot}$ (in our situation: $\Lambda(\rho)$) and transport equations for the mass fractions $\rho_i / \rho_{\rm tot}$, $i = 1, \dots, N$ (in our situation: $\rho_i / \Lambda(\rho)$). Compared to previous work, the relationship (3) between the pressure and the densities is more general, we allow for an arbitrary number of species, and we prove the uniqueness of weak and classical solutions. Special effort is necessary to allow for partially vanishing initial total densities [7, 20]; in the context of fluid dynamics, however, initial vacuum would not be meaningful.

1.3 Key ideas

As mentioned above, the idea is to decompose the equations in one parabolic equation for the function $\Lambda(\rho)$ and N transport equations for the variables $u_i = \rho_i/\Lambda(\rho)$.

First, we derive the parabolic equation. Let (ρ, v) be a differentiable solution to (1)–(3). Multiplying (1) by $\partial\Lambda/\partial\rho_i$ and summing over $i = 1, \dots, N$ leads to

$$\partial_t\Lambda(\rho) + v \cdot \nabla\Lambda(\rho) + \Lambda'(\rho) \cdot \rho \operatorname{div} v = 0.$$

The positive homogeneity of Λ implies that $\Lambda'(\rho) \cdot \rho = \Lambda(\rho)$, so Λ solves the conservation law $\partial_t\Lambda(\rho) + \operatorname{div}(\Lambda(\rho)v) = 0$. Then we infer from $v = -\kappa\nabla p = -\kappa G'(\Lambda(\rho))\nabla\Lambda(\rho)$ that the variable $w := \Lambda(\rho)$ solves the nonlinear diffusion equation

$$\partial_t w - \operatorname{div}(\kappa w G'(w)\nabla w) = 0 \quad \text{in } \Omega \times (0, T), \tag{7}$$

together with the initial and boundary conditions

$$w(0) = w^0 := \Lambda(\rho^0) \quad \text{in } \Omega, \quad \nabla w \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{8}$$

We claim that the variable $u_i = \rho_i/\Lambda(\rho)$ ($i = 1, \dots, N$), which can be interpreted as a kind of volume fraction, solves a transport equation. Indeed, with the material derivative $\dot{u} = (\partial_t + v \cdot \nabla)u$, it follows that ρ_i solves $\dot{\rho}_i = -\rho_i \operatorname{div} v$. We use the continuity equation (1) and the identity $\Lambda'(\rho) \cdot \rho = \Lambda(\rho)$ to compute

$$\dot{u}_i = \frac{\dot{\rho}_i}{\Lambda(\rho)} - \frac{\rho_i}{\Lambda(\rho)^2}\Lambda'(\rho) \cdot \dot{\rho} = \frac{\dot{\rho}_i}{\Lambda(\rho)} + \frac{\rho_i}{\Lambda(\rho)^2}\Lambda'(\rho) \cdot \rho \operatorname{div} v = 0.$$

Thus, the volume fractions u_i are just transported:

$$\dot{u}_i = 0 \quad \text{in } \Omega \times (0, T), \quad u_i(0) = u_i^0 := \frac{\rho_i^0}{\Lambda(\rho^0)} \quad \text{in } \Omega,\ i = 1, \dots, N. \tag{9}$$

The nonlinear diffusion equation (7) is solved by standard techniques: The weak maximum principle yields positive lower and upper bounds for the solution w, and parabolic regularity theory provides further a priori estimates. The velocity is then given by $v = -\kappa G'(w)\nabla w$. Because of its low regularity, we approximate the velocity field by smooth functions v_ε, such that we can solve the transport equation (9) with v replaced by v_ε. This yields the solutions u_i^ε and the approximate densities $\rho_i^\varepsilon := w u_i^\varepsilon$ for $i = 1, \dots, N$. The properties of the approximate velocity fields allow us to prove that ρ_i^ε converges to a weak solution ρ_i to the mass continuity equation as $\varepsilon \to 0$. For weak solutions and in particular for the proof of $\Lambda(\rho) = w$, the use of renormalization techniques is necessary, since we want to solve the continuity equations with a velocity field that possesses only local Sobolev regularity.

1.4 Notation

We set $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_{+,0} = [0,\infty)$. The space $C^{k+\alpha}(\Omega)$ for $k \in \mathbb{N}$, $\alpha \in (0,1]$ consists of all functions $u \in C^k(\overline{\Omega})$ whose kth partial derivatives are Hölder continuous of order α up to the boundary of Ω. Set $Q_T := \Omega \times (0,T)$ and $\overline{Q}_T := \overline{\Omega} \times [0,T]$. The space $C^{k+\alpha,\ell+\beta}(\overline{Q}_T)$ consists of all functions which are $C^{k+\alpha}$ with respect to the spatial variable and $C^{\ell+\beta}$ with respect to the time variable, where $k, \ell \in \mathbb{N}$ and $\alpha, \beta \in (0,1]$.

1.5 Main results

We make the following assumptions.

(A1) Domain: $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary.

(A2) Initial datum: $\rho^0 = (\rho_1^0, \ldots, \rho_N^0) \in L^\infty(\Omega;\mathbb{R}^N)$ satisfies $p(\rho^0) := G(\Lambda(\rho^0)) \in H^1(\Omega)$, $\rho_i^0 \geq 0$ in Ω for $i = 1,\ldots,N$, and $\sum_{i=1}^N \rho_i^0 \geq c_0 > 0$ in Ω for some $c_0 > 0$.

(A3) Function G: $G \in C^1(\mathbb{R}_+)$ is strictly increasing, i.e. $G'(s) > 0$ for all $s > 0$, and $s \mapsto sG'(s)$ is of class $C^2(\mathbb{R}_+)$.

(A4) Function Λ: $\Lambda \in C^2(\mathbb{R}_+^N) \cap C^0(\mathbb{R}_{+,0}^N)$ is nonnegative, convex, and positively homogeneous (of degree one) and there exist constants $0 < r_0 < r_1 < \infty$ such that

$$r_0|\rho| \leq \Lambda(\rho) \leq r_1|\rho| \quad \text{for all } \rho \in \mathbb{R}_+^N.$$

Assumptions (A1) and (A2) are rather natural. We already mentioned before that partially vanishing total densities are not meaningful in fluid dynamics, and we require in Assumption (A2) that $\sum_{i=1}^N \rho_i^0$ is strictly positive. The variable $w = \Lambda(\rho)$ satisfies the porous-medium-type equation (7), and the condition that $s \mapsto sG'(s)$ is of class $C^2(\mathbb{R}_+)$ in Assumption (A3) is needed to deduce classical solutions; see Section 3. The hypotheses on the volume extension Λ in Assumption (A4) guarantee that the variable $u_i = \rho_i/\Lambda(\rho)$ satisfies the transport equation (9). Moreover, the linear growth condition simplifies some estimates in Section 4.

A possible obvious extension would be to consider porosity coefficients depending on the density or pressure, $\kappa = \kappa(\Lambda(\rho))$ or $\kappa = \kappa(p)$. We comment briefly in Section 5 how Assumption (A3) can be suitably modified to treat this case.

We next formulate our main theorems on the well-posedness analysis.

Theorem 1 (Classical solutions). *Let Assumptions (A1)–(A4) hold, let $T > 0$, and $\kappa > 0$. Furthermore, let $\partial\Omega \in C^{2+\alpha}$ for some $\alpha > 0$, $G \in C^2(\mathbb{R}_+)$, and $\rho^0 \in C^{1+\alpha}(\overline{\Omega};\mathbb{R}^N)$ such that $p(\rho^0) \in C^{2+\alpha}(\overline{\Omega})$ and $\nabla p(\rho^0) \cdot \nu = 0$ on $\partial\Omega$. Then there exists a unique classical solution $\rho \in C^{1+\alpha,1}(\overline{Q}_T;\mathbb{R}_{+,0}^N)$ with $p(\rho) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ to the equations*

$$\partial_t \rho_i - \operatorname{div}(\kappa\rho_i\nabla p(\rho)) = 0 \quad \text{in } \Omega \times (0,T),\ i = 1,\ldots,N,$$

and the initial and boundary conditions (5) *are satisfied.*

Theorem 2 (Weak solutions). *Let Assumptions (A1)–(A4) hold, let $T > 0$, and $\kappa > 0$. Furthermore, let $\partial\Omega$ be piecewise of class C^2 (i.e., Ω is a curvilinear polyhedron). Then there exists a weak solution (ρ, v) to* (1)–(3) *and* (5) *such that $\rho \geq 0$ in Q_T, $\rho \in L^\infty(Q_T;\mathbb{R}^N)$, $v = -\kappa\nabla p(\rho) \in L^2(Q_T)$, and $\operatorname{div} v \in L^\sigma(Q_T)$ for some $\sigma > 4/3$, and ρ is the unique weak solution to* (1) *and* (5).

The paper is organized as follows. The pressure relation (3) and the cross-diffusion formulation (4) is motivated in Section 2. Then the proofs of Theorems 1 and 2 are presented in Sections 3 and 4, respectively. Finally, we collect in Section 5 some possible extensions of our results and open problems.

2 Modeling

In this section, we motivate system (1)–(3). We consider N chemical substances with mass densities ρ_i whose isothermal evolution is governed by the continuity equations (1), subject to Darcy's law (2). We introduce the partial number densities $n_i = \rho_i/m_i$, where the molecular masses $m_1, \ldots, m_N$ are positive constants. Then $n_{\rm tot} := \sum_{i=1}^N n_i$ is the total number density, and $X_i := n_i/n_{\rm tot}$ are the number fractions.

In order to model the thermodynamic pressure p, we follow some ideas exposed in [4, Section 15] to describe the free energy of elastic mixtures. We also refer to [12, 13] for further particular models in the case of mixtures with charged carriers. The pressure is related to the volume extension of the mixture via a state equation of the following kind:

$$p = G\big(n_{\rm tot}H(X_1, \ldots, X_N)\big), \tag{10}$$

where G and H are suitable functions. The choice $G(s) = s$, $H = 1$ gives the pressure law of (isothermal) ideal gases, namely $p = n_{\rm tot}$. Other special choices are $G(s) = s^\alpha$ with $\alpha > 1$ and $H = 1$, giving $p = n_{\rm tot}^\alpha$, while for $G(s) = s^\alpha$, $H(X) = (\sum_{i=1}^N X_i^\alpha)^{1/\alpha}$, we obtain $p = \sum_{i=1}^N n_i^\alpha$. We refer to [26] for more examples of nonlinear state equations, often refered to as Tait equations. We notice that in the last example $p = \sum_{i=1}^N n_i^\alpha$, there is possibly a relationship to the Dalton law with partial pressures of the constituents obeying $p_i = n_i^\alpha$ with uniform exponent α. This seems to be a by-product of the positive homogenity assumption for H, for suitably chosen functions G. In general, the state equation (10) yields models that are not equivalent to the Dalton law. The factor H which represents an average volume can be used to model finite-volume effects of the molecules, for instance with the linear ansatz $H = \sum_{i=1}^N V_iX_i$ (V_i are the reference partial volumes, like in [11]).

To simplify our notation, we express the pressure law for the mass densities. Indeed, noting that we can identify each function of $n_1, \ldots, n_N$ with a function of $\rho_1, \ldots, \rho_N$ via $\rho_i = m_in_i$, we express (10) in the equivalent form $p = G(\Lambda(\rho_1, \ldots, \rho_N))$, where

$$\Lambda(\rho_1, \ldots, \rho_N) := n_{\rm tot}H\left(\frac{n_1}{n_{\rm tot}}, \ldots, \frac{n_N}{n_{\rm tot}}\right)$$

describes the volume extension of the mixture. It is essential for our analysis that the mapping $\rho \mapsto \Lambda(\rho)$ is positively homogeneous of degree one. This condition is always satisfied in applications, since the function H depends only on the fractions $n_i/n_{\rm tot}$ which are homogeneous of degree zero in ρ.

We claim that system (1)–(3) can be formulated as the cross-diffusion system (4). To this end, we introduce the scalar function

$$h_M(s) = s\int_{s_0}^{s} \frac{G(\tau)}{\tau^2}d\tau + \text{const.}, \quad s > 0,$$

and the free energy $h(\rho) = h_M(\Lambda(\rho))$ for $\rho \in [0,\infty)^N$. The Hessian of h,

$$D^2 h = h_M''(D\Lambda \otimes D\Lambda) + h_M' D^2\Lambda,$$

is positive definite if h_M is increasing and convex and Λ is convex. For instance, we may assume that G is increasing, which leads to $h_M''(s) = G'(s)/s > 0$. The pressure is given by the Gibbs–Duhem relation

$$p = -h + \sum_{i=1}^{N} \rho_i \mu_i,$$

where $\mu_i := \partial h/\partial\rho_i$ are the chemical potentials. Since $sh_M'(s) - h_M(s) = G(s)$ and Λ is supposed to be positively homogeneous (implying that $\Lambda'(\rho)\cdot\rho = \Lambda(\rho)$), we find that

$$p = -h_M(\Lambda(\rho)) + \sum_{i=1}^{N} \rho_i \frac{\partial\Lambda}{\partial\rho_i}(\rho) h_M'(\Lambda(\rho)) = -h_M(\Lambda(\rho)) + h_M'(\Lambda(\rho))\Lambda(\rho) = G(\Lambda(\rho)),$$

which equals (3). Furthermore, the Gibbs–Duhem relation implies that $\nabla p = \sum_{j=1}^{N} \rho_j \nabla\mu_j$, and inserting this expression into (1)–(2) leads to the cross-diffusion system (4).

3 Proof of Theorem 1

Problem (7)–(8) is a quasilinear parabolic equation with Neumann boundary conditions. Its unique solvability in $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ follows from Theorem 7.4 in [24, Chapter V.7] (also see [14, Theorem 10.24]) if the following conditions are satisfied:

- Every classical solution to (7)–(8) is bounded from above and below;
- the coefficient $\kappa w G'(w)$ is of class $C^2(\mathbb{R}_+)$;
- the initial datum satisfies $w^0 := \Lambda(\rho^0) \in C^{2+\alpha}(\overline{\Omega})$.

These conditions are satisfied. Indeed, the weak maximum principle for parabolic equations shows that

$$0 < \Lambda_* := \inf_{x\in\Omega} \Lambda(\rho^0(x)) \le w(x,t) \le \Lambda^* := \sup_{x\in\Omega} \Lambda(\rho^0(x)) \quad \text{for all } (x,t) \in Q_T, \tag{11}$$

where $\Lambda_* = r_0 c_0 > 0$, according to Assumptions (A2) and (A4). The second condition follows from Assumption (A3) and the third condition is a consequence of the assumptions of the theorem. Thus, there exists a unique solution $w \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ to (7)–(8).

Next, we solve (9). The transport velocity $v = -\kappa G'(w)\nabla w = -\kappa\nabla G(w)$ is bounded in $L^\infty(Q_T)$ and $\operatorname{div} v$ is bounded in $C^\alpha(\overline{Q}_T)$. This allows us to find for $x \in \Omega$, $t \in [0,T]$ the characteristic curves of

$$\Phi'(s;t,x) = v(\Phi(s;t,x),\, s), \quad s \in (0,T), \quad \Phi(t;t,x) = x.$$

For all $s \in [0,T]$, the map $(x,t) \mapsto \Phi(s;x,t)$ belongs to $C^{1+\alpha,1+\alpha/2}(\overline{Q}_T)$. Thus, problem (9) admits the unique solutions

$$u_i(x,t) = u_i^0(\Phi(0;t,x)), \quad i = 1,\ldots,N.$$

Since $u^0 = \rho^0/\Lambda(\rho^0) \in C^1(\overline{\Omega};\mathbb{R}^N)$, we have $u \in C^1(\overline{Q}_T;\mathbb{R}^N)$.

We claim that $\rho = wu$ is a classical solution to (1), (5). The positive homogeneity of Λ shows that $\Lambda(\rho) = w\Lambda(u)$. Moreover, $(\Lambda(u))^\cdot = \Lambda'(u)\dot{u} = 0$ and $\Lambda(u(0)) = \Lambda(u^0) = 1$, by definition of u^0. We conclude that $\Lambda(u) = 1$. A more direct proof of this property is as follows: We multiply the equation $0 = \dot{u} = \partial_t u + v\cdot\nabla u$ by $\Lambda'(u)$, which yields $\partial_t\Lambda(u) + v\cdot\Lambda(u) = 0$. Then we compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega(\Lambda(u)-1)^2dx &= -2\int_\Omega(\Lambda(u)-1)v\cdot\nabla\Lambda(u)dx = -\int_\Omega v\cdot\nabla(\Lambda(u)-1)^2dx \\ &= \int_\Omega \operatorname{div}(v)(\Lambda(u)-1)^2dx,\end{aligned}$$

and the regularity $\operatorname{div} v \in L^1(0,T;L^\infty(\Omega))$ is sufficient to conclude fron Gronwall's lemma that $\Lambda(u) = 1$. We infer that $\Lambda(\rho) = w\Lambda(u) = w$. We know that w solves $0 = \partial_t w - \operatorname{div}(\kappa w\nabla G(w)) = \partial_t w + \operatorname{div}(wv) = \dot{w} + w\operatorname{div} v$. Then we deduce from

$$\dot{\rho}_i = \dot{w}u_i + w\dot{u}_i = \dot{w}u_i, \quad i = 1,\ldots,N,$$

and $\dot{w} = -w\operatorname{div} v$ that

$$\dot{\rho}_i = \dot{w}u_i = -wu_i\operatorname{div} v = -\rho_i\operatorname{div} v,$$

which is (1) with $v = -\kappa\nabla G(w) = -\kappa\nabla G(\Lambda(\rho))$. The initial condition in (5) is satisfied since

$$\rho_i(0) = w(0)u_i(0) = w^0u_i^0 = \Lambda(\rho^0)\frac{\rho_i^0}{\Lambda(\rho^0)} = \rho_i^0 \quad \text{in } \Omega.$$

This finishes the proof of Theorem 1.

4 Proof of Theorem 2

We show first the following technical approximation property. It is needed to construct smooth approximations of the velocity field and to identify the weak limit of the density vector.

Lemma 3. *Let the assumptions of Theorem 2 hold. Then there exists a family of functions $(\phi_m)_{m>0} \subset C^2(\overline{\Omega})$ such that $0 \le \phi_m \le 1$ in Ω, $\phi_m(x) = 0$ for all $x \in \Omega$ with $\operatorname{dist}(x,\partial\Omega) < 1/m$, $\phi_m \to 1$ locally uniformly in Ω as $m \to \infty$, and satisfying the following property: For all $0 < \delta < 1/2$, there exists $C(\delta) > 0$ such that for all $g \in H^{1/2+\delta}(\Omega;\mathbb{R}^3)$ satisfying $g\cdot\nu = 0$ on $\partial\Omega$ and for any $1 \le p < 1/(1-\delta)$,*

$$\|g\cdot\nabla\phi_m\|_{L^p(\Omega)} \le C(\delta)\|g\|_{H^{1/2+\delta}(\Omega)}.$$

Proof. Since $\partial\Omega$ consists of surfaces $S_1,\ldots,S_\ell$ of class C^2, we may assume that the distance function $d_i(x) := \operatorname{dist}(x,S_i)$ for $x\in\Omega$, $i=1,\ldots,\ell$ is also of class C^2, i.e. $d_i\in C^2(\overline{\Omega})$. Let $m>0$ and introduce a function $h\in C^2(\mathbb{R}_+)$ such that $0\le h\le 1$ in $\mathbb{R}_+$, $h_m(s)=1$ if $s>2/m$, $h_m(s)=0$ for $0\le s<1/m$, and $h_m'(s)\le 3m$ for $s\in\mathbb{R}_+$. Then $h_m'(s)\le 3m\le 6/s$ for $1/m\le s\le 2/m$ and, in fact, for all $s\in\mathbb{R}_+$ (since $h_m'=0$ otherwise).

We define $\phi_m(x) := \prod_{i=1}^\ell h_m(d_i(x))$. By definition of h_m, we have $\phi_m(x)=0$ for $x\in\operatorname{dist}(x,\partial\Omega)<1/m$. The gradient of ϕ_m is given by

$$\nabla\phi_m = \sum_{k=1}^\ell h_m'(d_k)\nabla d_k\prod_{i\neq k} h_m(d_i).$$

Using $h_m'(d_k)\le 6d_k^{-1}$ and $h_m\le 1$, we find that

$$|g\cdot\nabla\phi_m| \le 6\sum_{k=1}^\ell d_k^{-1}|g\cdot\nabla d_k|. \tag{12}$$

We now estimate the right-hand side. The properties of the distance function imply that $\nu(x)=\nabla d_i(x)$ for $x\in S_i$. Moreover, as $d_i\in C^2(\overline{\Omega})$,

$$\begin{aligned}\|g\cdot\nabla d_i\|_{H^{1/2+\delta}(\Omega)} &\le \|\nabla d_i\|_{L^\infty(\Omega)}\|g\|_{H^{1/2+\delta}(\Omega)} + \|D^2 d_i\|_{L^\infty(\Omega)}\|g\|_{L^2(\Omega)} \\ &\le C\|g\|_{H^{1/2+\delta}(\Omega)}.\end{aligned} \tag{13}$$

The interior trace operator $\operatorname{tr}_i : H^{1/2+\delta}(\Omega)\to H^\delta(S_i)$, $\operatorname{tr}_i(v)=v|_{S_i}$, is bounded, so the boundary condition $g\cdot\nu=0$ on S_i is valid almost everywhere. This shows that

$$g\cdot\nabla d_i\in H^{1/2+\delta}_{S_i}(\Omega) := \big\{f\in H^{1/2+\delta}(\Omega) : \operatorname{tr}_i f = 0 \text{ in } H^\delta(S_i)\big\},\quad i=1,\ldots,\ell.$$

Theorem 11.3 in [28, Chapter 1, §11] states that the linear mapping $H^s_{S_i}(\Omega)\to L^2(\Omega)$, $u\mapsto d_i^{-s}u$, with $s=1/2+\delta$ is continuous. This property and estimate (13) imply that

$$\|d_i^{-s}g\cdot\nabla d_i\|_{L^2(\Omega)} \le C\|g\cdot\nabla d_i\|_{H^s(\Omega)} \le C\|g\|_{H^{1/2+\delta}(\Omega)}. \tag{14}$$

Consequently, using (12) and the Hölder inequality, we obtain for $1\le p<2/(3-2s)=1/(1-\delta)$,

$$\begin{aligned}\int_\Omega |g\cdot\nabla\phi_m|^p dx &\le C\sum_{k=1}^\ell\int_\Omega d_k^{-ps}|g\cdot\nabla d_k|^p d_k^{-p(1-s)}dx \\ &\le C\sum_{k=1}^\ell\bigg(\int_\Omega d_k^{-2s}|g\cdot\nabla d_k|^2dx\bigg)^{p/2}\bigg(\int_\Omega d_k^{-2p(1-s)/(2-p)}dx\bigg)^{1-p/2}.\end{aligned}$$

It holds that $-2p(1-s)/(2-p)>-1$ if and only if $p<1/(1-\delta)$, and under this condition, the last integral is finite. Thus, we infer from (14) that

$$\int_\Omega |g\cdot\nabla\phi_m|^p dx \le C(\delta)\sum_{k=1}^\ell \|d_k^{-s}g\cdot\nabla d_k\|^p_{L^2(\Omega)} \le C(\delta)\|g\|^p_{H^{1/2+\delta}(\Omega)},$$

which finishes the proof. □

We formulate (7) as

$$\partial_t w - \operatorname{div}(a(w)\nabla w) = 0 \quad \text{in } \Omega \times (0,T), \tag{15}$$

where $a(w) := \kappa w G'(w)$ for $w \geq 0$, and we set

$$W_2^1(Q_T) := \{f \in L^2(0,T;H^1(\Omega)) : \partial_t f \in L^2(0,T;H^1(\Omega)')\}. \tag{16}$$

Observe that we may allow for suitable porosity coefficients $\kappa = \kappa(w)$ at this point. The following result shows that problem (7)–(8) is uniquely solvable and that the vector field $v = -\kappa G'(w)\nabla w$ has some regularity properties.

Proposition 4 (Existence and regularity for (7))**.** *Under the assumptions of Theorem 2, problem* (7)–(8) *has a unique solution* $w \in W_2^1(Q_T) \cap L^\infty(0,T;L^\infty(\Omega))$ *satisfying* (11)*. Moreover,* $v = -\kappa G(w)\nabla w$ *satisfies the following properties:*

- $v \in L^{4/3}(0,T;H^1(\Omega'))$ *for any domain* Ω' *compactly included in* Ω*;*
- $v \in L^2(0,T;L^q(\Omega))$ *and* $\operatorname{div} v \in L^{2-2/q}(Q_T)$ *for some* $3 < q \leq 6$*;*
- $v \cdot \nabla\phi_m \to 0$ *strongly in* $L^1(Q_T)$ *as* $m \to \infty$ *for* (ϕ_m) *constructed in Lemma 3.*

Proof. Step 1: Existence for (7)*.* The existence of weak solutions to (7)–(8) can be shown by a standard approximation procedure, but since $a(w)$ does not need to be monotone, we need to be careful and therefore present a sketch of the proof. We introduce the truncated functions

$$a_k(s) := \begin{cases} a(k) & \text{if } s \geq k, \\ a(s) & \text{if } 1/k < s < k, \\ a(1/k) & \text{if } 0 \leq s \leq 1/k, \end{cases}$$

where $k \in \mathbb{N}$. The functions $B_k : \mathbb{R}_+ \to \mathbb{R}_+$, $B_k(s) := \int_0^s a_k(z)dz$, are bi-Lipschitz transformations. We consider the approximated problem

$$\frac{1}{a_k(B_k^{-1}(u))}\partial_t u = \Delta u, \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad u(0) = B_k(w^0) \quad \text{in } \Omega. \tag{17}$$

If a solution to this problem is given, then $w_k := B_k^{-1}(u)$ is a solution to (15) with a replaced by a_k.

Problem (17) can be solved by the Galerkin method. Indeed, let $(v_i) \subset W^{1,\infty}(\Omega)$ be an orthonormal basis of $H^1(\Omega)$ and $X_n = \operatorname{span}\{v_1,\ldots,v_n\}$ for $n \in \mathbb{N}$. The coefficients α_j of $u_n(x,t) = \sum_{j=1}^n \alpha_j(t)v_j(x)$ solve the system of ordinary differential equations

$$\sum_{j=1}^n A_{ij}(\alpha)\alpha_j' + \sum_{j=1}^n M_{ij}\alpha_j = 0 \quad \text{in } (0,T], \quad \alpha_j(0) = \alpha_{n,j}^0, \tag{18}$$

where we assumed that $u_n^0 := \sum_{j=1}^n \alpha_{n,j}^0 v_j(x)$ converges to $B_k(w^0)$ in $H^1(\Omega)$ as $n \to \infty$, and the matrices $(A_{ij}(\alpha))$ and (M_{ij}) are given by

$$A_{ij}(\alpha) = \int_\Omega \frac{v_i v_j}{a_k(B_k^{-1}(u_n))}dx, \quad M_{ij} = \int_\Omega \nabla v_i \cdot \nabla v_j dx.$$

Since the coefficients $a_k \circ B_k^{-1}$ are bounded from below and above, the properties of (v_i) imply that $(A_{ij}(\alpha))$ is strictly positive definite. Thus, (18) possesses a global solution $\alpha \in C^1([0,T];\mathbb{R}^n)$ for any $T > 0$.

For the limit $n \to \infty$, we need some uniform estimates. We multiply (18) by α_i' and sum over $i = 1, \ldots, n$ leading to

$$\int_\Omega \frac{(\partial_t u_n)^2}{a_k(B_k^{-1}(u_n))}dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_n|^2 dx = 0.$$

Integration over $(0,t)$ for $t \le T$ yields the uniform bounds

$$\|\partial_t u_n\|_{L^2(Q_T)} \le C(k)\|u_n^0\|_{H^1(\Omega)}, \quad \sup_{0<t<T}\|\nabla u_n(t)\|_{L^2(\Omega)} \le \|u_n^0\|_{H^1(\Omega)}.$$

The first estimate implies a uniform bound for u_n in $L^2(Q_T)$ since $u_n(t) = u_n(0) + \int_0^t \partial_t u_n ds$. By the Aubin–Lions lemma, there exists a subsequence which is not relabeled such that $u_n \to u$ strongly in $L^2(Q_T)$, $\partial_t u_n \rightharpoonup \partial_t u$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(Q_T)$ as $n \to \infty$, and the limit u satisfies

$$\int_0^T\int_\Omega \frac{\partial_t u\phi}{a_k(B_k^{-1}(u))}dxdt + \int_0^T\int_\Omega \nabla u \cdot \nabla\phi dxdt = 0 \tag{19}$$

for all $\phi \in L^2(0,T;H^1(\Omega))$. Moreover, the initial condition $u(0) = B_k(w^0)$ is fulfilled in $H^1(\Omega)$ and the bounds

$$\|\partial_t u\|_{L^2(Q_T)} \le C(k)\|B_k(w^0)\|_{H^1(\Omega)}, \quad \sup_{0<t<T}\|\nabla u(t)\|_{L^2(\Omega)} \le \|B_k(w^0)\|_{H^1(\Omega)} \tag{20}$$

hold. The function $w := B_k^{-1}(u) \in H^1(\Omega)$ satisfies the equation

$$\int_0^T\int_\Omega \partial_t w\phi dxdt + \int_0^T\int_\Omega a_k(w)\nabla w \cdot \nabla\phi dxdt = 0$$

for all $\phi \in L^2(0,T;H^1(\Omega))$ and the initial condition $w(0) = w^0$ in $H^1(\Omega)$. Thus, by the weak maximum principle, $\Lambda_* \le w \le \Lambda^*$ in Q_T (see (11) for the definition of Λ_* and Λ^*). Therefore, there exists $k_0 \in \mathbb{N}$, depending only on Λ_* and Λ^* such that $a_{k_0}(w) = a(w)$ in Q_T. The function $B(s) = \int_0^s a(z)dz$ is identical to B_{k_0} on the range of w, since a_{k_0} is identical to a on the range of w. We conclude that $u_* \le u \le u^*$ in Q_T, where $u_* := B(\Lambda_*)$ and $u^* := B(\Lambda^*)$.

Step 2: Regularity. We deduce from the bounds (20) with $k = k_0$ and the lower and upper bounds for u and w that $\partial_t w$ is bounded in $L^2(Q_T)$ and ∇w is bounded in $L^\infty(0,T;L^2(\Omega))$. Furthermore, by choosing $k = k_0$ and the test function $\phi(x,t) = \zeta(t)\eta(x)$ in (19), where $\zeta(t)$ approximates the delta distribution at t, we see that u solves

$$\int_\Omega \frac{\partial_t u\eta}{a(B^{-1}(u))}dx + \int_\Omega \nabla u(t)\cdot\nabla\eta dx = 0. \tag{21}$$

Local regularity for elliptic equations [18, Section 8.3, Theorem 8.9] implies that for any domain Ω' compactly embedded in Ω, there exists $C(\Omega', u_*, u^*) > 0$ such that

$$\|D^2 u(t)\|_{L^2(\Omega')} \le C(\Omega', u_*, u^*)\|\partial_t u(t)\|_{L^2(\Omega)}, \tag{22}$$

which shows, using (20), that $u \in L^2(0,T;H^2(\Omega'))$. Applying case (c) of Theorem 1.6 in [30] with $\alpha = 1$ therein, there exists $\varepsilon > 0$ depending only on the Lipschitz constant of $\partial\Omega$ such that for all $2/(1-\varepsilon) \le q < 3/(1-\varepsilon)$, the gradient satisfies the global bound

$$\|\nabla u(t)\|_{L^q(\Omega)} \le C\|\partial_t u(t)\|_{L^{3q/(3+q)}(\Omega)}. \tag{23}$$

Since $3q/(3+q) \le 2$ for $q \le 6$, we deduce that $u \in L^2(0,T;W^{1,q}(\Omega))$.

We know from (20) that $|\nabla u|^2$ is bounded in $L^\infty(0,T;L^1(\Omega))$. Since also $|\nabla u| \in L^2(0,T;H^1_{\rm loc}(\Omega))$, the Sobolev embedding theorem yields $|\nabla u| \in L^2(0,T;L^6_{\rm loc}(\Omega))$ (recall that the space dimension is three). Interpolation with $\theta = 3/4$ then shows for $z = |\nabla u|^2$ and any Ω' compactly embedded in Ω that

$$\|z\|^{4/3}_{L^{4/3}(0,T;L^2(\Omega'))} \le \int_0^T \|v\|^{4\theta/3}_{L^3(\Omega')}\|z\|^{4(1-\theta)/3}_{L^1(\Omega')}dt \le \|z\|^{1/3}_{L^\infty(0,T;L^1(\Omega'))}\int_0^T \|z\|_{L^3(\Omega')}dt \tag{24}$$

and consequently $z = |\nabla u|^2 \in L^{4/3}(0,T;L^2_{\rm loc}(\Omega))$. In order to show a global bound, we combine the regularity properties $|\nabla u|^2 \in L^\infty(0,T;L^1(\Omega))$, $|\nabla u|^2 \in L^1(0,T;L^{q/2}(\Omega))$ and interpolate with $r = 2-2/q$ and $\theta = 1/r$ such that $1/r = (1-\theta)/1 + 2\theta/q$:

$$\|z\|^r_{L^r(Q_T)} \le \int_0^T \|z\|^{r(1-\theta)}_{L^1(\Omega)}\|z\|^{r\theta}_{L^{q/2}(\Omega)}dt \le \|z\|^{r(1-\theta)}_{L^\infty(0,T;L^1(\Omega))}\int_0^T \|z\|_{L^{q/2}(\Omega)}dt$$

and hence, $z = |\nabla u|^2 \in L^{2-2/q}(Q_T)$.

Finally, we consider $v = -\kappa G'(w)\nabla w$. Introducing $f(u) := -\kappa G'(B^{-1}(u))(B^{-1})'(u)$ and recalling that $w = B^{-1}(u)$, this means that $v = f(u)\nabla u$. The positive lower bound for u and estimate (23) then show that $v \in L^2(0,T;L^q(\Omega))$. It holds that

$$\nabla v = f(u)D^2u + f'(u)\nabla u \otimes \nabla u, \quad \operatorname{div} v = f(u)\Delta u + f'(u)|\nabla v|^2.$$

Using estimates (22) and (24) and the lower bound for u, it follows that $|\nabla v| \in L^{4/3}(0,T;L^2_{\rm loc}(\Omega))$. Furthermore, $\Delta u = \partial_t u/(a(B^{-1}(u))) \in L^2(Q_T)$ and $|\nabla u|^2 \in L^{2-2/q}(Q_T)$ imply that $\operatorname{div} v \in L^{2-2/q}(Q_T)$.

Step 3: Proof of $v\cdot\nabla\phi_m \to 0$ strongly in $L^1(Q_T)$. Proposition 3.7 in [1] shows that for some $\delta > 0$, it holds that

$$\|\nabla u(t)\|_{H^{1/2+\delta}(\Omega)} \le C(\Omega,u_*,u^*)\|\partial_t u(t)\|_{L^2(\Omega)}. \tag{25}$$

It follows that $\nabla u(t)$ is defined in $L^2(\partial\Omega)$. Then the weak formulation (21) implies that u satisfies the Neumann boundary conditions $\nabla u(t)\cdot\nu = 0$ in $L^2(S_i)$ for $i = 1,\ldots,\ell$. Recalling that $v = f(u)\nabla u$ and $f(u)$ is globally bounded in Q_T, we infer that $|v\cdot\nabla\phi_m| \le \|f(u)\|_{L^\infty(Q_T)}|\nabla u\cdot\nabla\phi_m|$. Thus, by Lemma 3 and estimate (25), for $1 \le p < 1/(1-\delta)$,

$$\begin{aligned}\|v\cdot\nabla\phi_m\|_{L^2(0,T;L^p(\Omega))} &\le C(\delta)\|f(u)\|_{L^\infty(Q_T)}\|\nabla u\|_{L^2(0,T;H^{1/2+\delta}(\Omega))} \\ &\le C(\delta)\|f(u)\|_{L^\infty(Q_T)}\|\partial_t u\|_{L^2(Q_T)} \le C.\end{aligned}$$

Again by Lemma 3, we have $\nabla\phi_m \to 0$ a.e. in Ω and in view of the previous estimate, $v\cdot\nabla\phi_m \to 0$ a.e. in Q_T. By dominated convergence, this convergence also holds in $L^1(Q_T)$. This finishes the proof. □

The next step is to approximate the velocity $v = -\kappa G'(w)\nabla w$ by smooth vector fields.

Lemma 5 (Approximation of v)**.** *There exists a family of smooth vector fields v_ε for $\varepsilon > 0$ satisfying $v_\varepsilon \cdot \nu = 0$ on $\partial\Omega \times (0,T)$ such that, as $\varepsilon \to 0$,*

$$v_\varepsilon \to v \quad \textit{strongly in } L^2(Q_T), \quad \operatorname{div} v_\varepsilon \to \operatorname{div} v \quad \textit{strongly in } L^1(Q_T).$$

Proof. Let (ϕ_m) be the family constructed in Lemma 3 and define $\psi_\varepsilon := \phi_{2/\varepsilon}$ for $\varepsilon > 0$. Then ϕ_ε takes values in $[0,1]$ and $\psi_\varepsilon(x) = 0$ for all $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < \varepsilon/2$. Furthermore, let $\Phi \in C^\infty(\overline{B_1(0)})$ satisfy $\int_{B_1(0)} \Phi(z)dz = 1$, where $B_1(0)$ is the unit ball in $\mathbb{R}^3$. Then we define

$$v_\varepsilon(x,t) = \int_{B_1(0)} \Phi(z) v(x + \varepsilon z, t)\psi_\varepsilon(x + \varepsilon z)dz.$$

Since $\psi_\varepsilon \to 1$ uniformly on every compact subset of Ω as $\varepsilon \to 0$, by Lemma 3, we have $v_\varepsilon \to v$ strongly in $L^2(Q_T)$. Moreover,

$$\begin{aligned}\operatorname{div} v_\varepsilon(x,t) &= \int_{B_1(0)} \Phi(z) \operatorname{div} v(x + \varepsilon z, t)\psi_\varepsilon(x + \varepsilon z)dz \\ &\quad + \int_{B_1(0)} \Phi(z) v(x + \varepsilon z) \cdot \nabla\psi_\varepsilon(x + \varepsilon z)dz.\end{aligned}$$

The first integral converges to $\operatorname{div} v$ strongly in $L^{2-2/q}(Q_T)$ since $\psi_\varepsilon \nearrow 1$ in Ω, and the second integral converges to zero strongly in $L^1(Q_T)$, due to the fact that $v \cdot \nabla\phi_{2/\varepsilon} \to 0$ strongly in $L^1(Q_T)$ by Lemma 4. □

We also approximate the initial densities: Let $\rho^{0,\varepsilon} \in C^1(\overline{\Omega}; \mathbb{R}^N)$ for $0 < \varepsilon < 1$ be approximations of ρ^0 such that

$$\begin{aligned}&(1-\varepsilon)\inf_{y\in\Omega} \rho_i^0(y) \le \rho_i^{0,\varepsilon}(x) \le (1+\varepsilon)\sup_{y\in\Omega} \rho_i^0(y), \quad x \in \Omega, \\ &\rho^{0,\varepsilon} \to \rho^0 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^N) \text{ as } \varepsilon \to 0.\end{aligned}$$

Furthermore, we use the characteristic curves $\Phi_\varepsilon(s; x, t)$ associated to v_ε,

$$\Phi'_\varepsilon(s; t, x) = v_\varepsilon(\Phi(s; t, x), s) \quad \text{for } s \in (0,T), \quad \Phi(t; t, x) = x$$

(see Section 3), to solve

$$\partial_t u_i^\varepsilon + v_\varepsilon \cdot \nabla u_i^\varepsilon = 0 \quad \text{in } Q_T, \quad u_i^\varepsilon(0) = u_i^{0,\varepsilon} := \frac{\rho_i^{0,\varepsilon}}{\Lambda(\rho^{0,\varepsilon})}, \quad i = 1, \ldots, N.$$

These equations correspond to (9) but with the velocity v replaced by v_ε. They can be solved explicitly in terms of the characteristic curves,

$$u_\varepsilon(x,t) = u^{0,\varepsilon}(\Phi_\varepsilon(0; t, x)) = \frac{\rho^{0,\varepsilon}(\Phi_\varepsilon(0; t, x))}{\Lambda(\rho^{0,\varepsilon}(\Phi_\varepsilon(0; t, x)))}, \quad (x,t) \in Q_T, \tag{26}$$

which shows that u_ε belongs to $C^1(\overline{Q}_T;\mathbb{R}^N)$ and $\Lambda(u_\varepsilon) = 1$ in Q_T. Furthermore, we deduce from the growth condition in Assumption (A4) that

$$r_0(1-\varepsilon)\sum_{i=1}^N \inf_{y\in\Omega}\rho_i^0(y) \le \Lambda(\rho^{0,\varepsilon}(x)) \le r_1(1+\varepsilon)\sum_{i=1}^N \sup_{y\in\Omega}\rho_i^0(y)$$

for $x \in \Omega$ and consequently

$$\frac{1-\varepsilon}{r_1(1+\varepsilon)}\frac{\inf_{y\in\Omega}\rho_i^0(y)}{\sum_{i=1}^N \sup_{y\in\Omega}\rho_i^0(y)} \le u_i^\varepsilon(x,t) \le \frac{1+\varepsilon}{r_0(1-\varepsilon)}\frac{\sup_{y\in\Omega}\rho_i^0(y)}{\sum_{i=1}^N \inf_{y\in\Omega}\rho_i^0(y)} \tag{27}$$

for $(x,t) \in Q_T$.

Next, we define the approximate densities $\rho_i^\varepsilon := wu_i^\varepsilon$, where $i = 1,\dots,N$ and w is the solution to the porous-medium-type equation (7) from Proposition 4. We prove that a subsequence of ρ_i^ε converges to a renormalized solution to the mass continuity equation (1). For the following result, we recall definition (16) of the space $W_2^1(Q_T)$.

Proposition 6 (Convergence of ρ_i^ε)**.** *The family $(\rho^\varepsilon) \subset W_2^1(Q_T;\mathbb{R}^N) \cap L^\infty(Q_T;\mathbb{R}^N)$ is bounded in $L^\infty(Q_T;\mathbb{R}^N)$ and there exists $c_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$\inf_{(x,t)\in Q_T}\rho_i^\varepsilon(x,t) \ge c_1, \quad i = 1,\dots,N.$$

There exists a subsequence of (ρ^ε) (not relabeled) such that $\rho^\varepsilon \to \rho$ strongly in $L^2(Q_T;\mathbb{R}^N)$. The limit $\rho = (\rho_1,\dots,\rho_n)$ satisfies $\rho \in L^\infty(Q_T;\mathbb{R}^N)$ with $\rho_i \ge c_1$ in Q_T, and ρ is a renormalized solution to (1)*, i.e.,*

$$\begin{aligned}
-\int_0^T\int_\Omega b(\rho)\partial_t\zeta dxdt - \int_0^T\int_\Omega b(\rho)\nabla\zeta\cdot v dxdt \\
= \int_\Omega b(\rho^0)\zeta(0)dx - \int_0^T\int_\Omega \big(b'(\rho)\cdot\rho - b(\rho)\big)\zeta \operatorname{div} v dxdt
\end{aligned}$$

is satisfied for all $\zeta \in C_0^1(\overline{Q}_T)$ and $b \in C^1(\mathbb{R}^N)$, where $v = -\kappa G'(w)\nabla w$ and w is the weak solution to (7)–(8)*.*

Proof. We know from Proposition 4 that $w \in W_2^1(Q_T)$ and from (26) that $u^\varepsilon \in C^1(\overline{\Omega};\mathbb{R}^N)$, which shows that $\rho^\varepsilon = wu^\varepsilon \in H^1(\Omega;\mathbb{R}^N)$. Moreover, since w satisfies the uniform lower and upper bounds (11) and u^ε satisfies the uniform bounds (27), also ρ_i^ε is uniformly bounded from below and above. At time $t = 0$, we have

$$\rho_i^\varepsilon(0) = w^0 u_i^{0,\varepsilon} = \Lambda(\rho^0)\frac{\rho_i^{0,\varepsilon}}{\Lambda(\rho^{0,\varepsilon})} =: \widetilde{\rho}_i^{0,\varepsilon}, \tag{28}$$

and the strong convergence $\rho^{0,\varepsilon} \to \rho^0$ in $L^1(\Omega;\mathbb{R}^N)$ implies that $\rho_i^\varepsilon(0) \to \rho^0$ strongly in $L^1(\Omega;\mathbb{R}^N)$. As u^ε solves (26) and w solves $\partial_t w + \operatorname{div}(vw) = 0$, it follows that $\rho_i^\varepsilon = wu_i^\varepsilon$ solves

$$\partial_t\rho_i^\varepsilon + v^\varepsilon\cdot\nabla\rho_i^\varepsilon = w\big(\partial_t u_i^\varepsilon + v^\varepsilon\cdot\nabla u_i^\varepsilon\big) + u_i^\varepsilon\big(\partial_t w + v^\varepsilon\cdot\nabla w\big)$$

$$\begin{aligned}
&= u_i^\varepsilon\big(\partial_t w + (v^\varepsilon - v)\cdot\nabla w + \operatorname{div}(vw) - w\operatorname{div} v\big) \\
&= u_i^\varepsilon(v^\varepsilon - v)\cdot\nabla w - u_i^\varepsilon w\operatorname{div} v = u_i^\varepsilon\operatorname{div}((v^\varepsilon - v)w) - u_i^\varepsilon w\operatorname{div} v^\varepsilon \\
&= u_i^\varepsilon\operatorname{div}((v^\varepsilon - v)w) - \rho_i^\varepsilon\operatorname{div} v^\varepsilon.
\end{aligned}$$

In view of the regularity $w \in W_2^1(Q_T)$, $u^\varepsilon \in C^1(\overline{Q}_T;\mathbb{R}^N)$, and $v \in L^{4/3}(0,T;H^1_{\mathrm{loc}}(\Omega))$ (see Proposition 4), these calculations make sense a.e. in $\Omega'\times(0,T)$ for domains Ω' compactly embedded in Ω. We infer that

$$\partial_t\rho_i^\varepsilon + \operatorname{div}(\rho_i^\varepsilon v^\varepsilon) = r_i^\varepsilon := u_i^\varepsilon\operatorname{div}((v^\varepsilon - v)w).$$

Using $\psi \in C_0^1(\overline{\Omega}\times[0,T);\mathbb{R}^N)$ as a test function and integrating by parts, we see that

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho^\varepsilon\cdot\partial_t\psi dxdt - \int_0^T\int_\Omega(\rho^\varepsilon)^\top\nabla\psi v^\varepsilon dxdt \\
&\qquad = \int_\Omega \widetilde{\rho}^{0,\varepsilon}(x)\cdot\psi(x,0)dx + \int_0^T\int_\Omega r^\varepsilon\cdot\psi dxdt,
\end{aligned} \tag{29}$$

where $\widetilde{\rho}_i^{0,\varepsilon}$ is defined in (28). We wish to pass to the limit $\varepsilon\to 0$. We deduce from Hölder's inequality and Lemma 5 that

$$\begin{aligned}
\|r^\varepsilon\|_{L^1(Q_T)} &= \|\rho^\varepsilon\operatorname{div}(v^\varepsilon - v) + u^\varepsilon(v^\varepsilon - v)\cdot\nabla w\|_{L^1(Q_T)} \\
&\le \|\rho^\varepsilon\|_{L^\infty(Q_T)}\|\operatorname{div}(v^\varepsilon - v)\|_{L^1(Q_T)} + \|u^\varepsilon\|_{L^\infty(Q_T)}\|v^\varepsilon - v\|_{L^2(Q_T)}\|\nabla w\|_{L^2(Q_T)} \\
&\to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}$$

Since (ρ^ε) is uniformly bounded in $L^\infty(Q_T)$, there exists a subsequence of (ρ^ε) which is not relabeled such that $\rho^\varepsilon \rightharpoonup^* \rho$ weakly* in $L^\infty(0,T;L^\infty(\Omega))$ and $\rho\in L^\infty(Q_T)$ satisfies $\rho_i \ge c_1$ in Q_T, $i = 1,\ldots,N$. Thus, taking into account the convergence in Lemma 5, $\rho_i^\varepsilon v_j^\varepsilon \rightharpoonup \rho_i v_j$ weakly in $L^2(Q_T)$. In view of $\rho_i^\varepsilon(0)\to\rho^0$ strongly in $L^2(\Omega;\mathbb{R}^N)$, the limit $\varepsilon\to 0$ in (29) leads to

$$-\int_0^T\int_\Omega \rho\cdot\partial_t\psi dxdt - \int_0^T\int_\Omega \rho^\top\nabla\psi v dxdt = \int_\Omega \rho^0(x)\cdot\psi(x,0)dx \tag{30}$$

for all $\psi\in C_0^1(\overline{\Omega}\times[0,T);\mathbb{R}^N)$, recalling that $v = -\kappa G'(w)\nabla w$.

It remains to show that $\Lambda(\rho) = w$. To this end, we rely on techniques of renormalization for the continuity equation. Let Ω' be compactly embedded in Ω and let ϕ_m be the function constructed in Lemma 3 satisfying $\phi_m = 1$ in Ω'. (This is possible since $\phi_m\to 1$ locally uniformly in Ω as $m\to 0$.) The function $\widetilde{\rho} := \rho\phi_m$ is nonnegative and compactly supported in $\Omega\times[0,T]$. Replacing ψ by $\psi\phi_m$ in (30) yields

$$\begin{aligned}
&-\int_0^T\int_\Omega \widetilde{\rho}\cdot\partial_t\psi dxdt - \int_0^T\int_\Omega \widetilde{\rho}^\top\nabla\psi v dxdt \\
&\qquad = \int_\Omega \rho^0(x)\phi_m(x)\cdot\psi(x,0)dx + \int_0^T\int_\Omega(\rho\cdot\psi)(v\cdot\nabla\phi_m)dxdt.
\end{aligned}$$

This is the weak formulation of

$$\partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho} v) = \rho(v\cdot\nabla\phi_m) \quad \text{in } \mathcal{D}'(Q_T).$$

In view of the regularity $\widetilde{\rho} \in L^\infty(Q_T)$, $v \in L^{4/3}(0,T;H^1_{\rm loc}(\Omega))$, and $\rho(v\cdot\nabla\phi_m) \in L^{4/3}(0,T;$ $L^2_{\rm loc}(\Omega))$, we can apply the theory of renormalized solutions to the continuity equation (see, e.g., [14, Theorem 10.29]) to conclude that

$$\partial_t b(\widetilde{\rho}) + \operatorname{div}(b(\widetilde{\rho})v) + \big(b'(\widetilde{\rho})\cdot\widetilde{\rho} - b(\widetilde{\rho})\big)\operatorname{div} v = (\rho\cdot b'(\widetilde{\rho}))(v\cdot\nabla\phi_m) \quad \text{in } \mathcal{D}'(Q_T)$$

for any $b\in C^1(\mathbb{R}^N_{+,0})\cap W^{1,\infty}(\mathbb{R}^N_+)$. Using the regularity $\operatorname{div} v\in L^{2-2/q}(Q_T)$ for $3<q\le 6$ proved in Proposition 4, we can formulate this identity in the weak form

$$\begin{aligned}
-\int_0^T\int_\Omega b(\rho\phi_m)\partial_t\zeta dxdt - \int_0^T\int_\Omega b(\rho\phi_m)v\cdot\nabla\zeta dxdt \\
= \int_\Omega b(\rho^0(x)\phi_m(x))\zeta(x,0)dx - \int_0^T\int_\Omega \big(b'(\rho\phi_m)\cdot\rho\phi_m - b(\rho\phi_m)\big)(\operatorname{div} v)\zeta dxdt \\
+ \int_0^T\int_\Omega \rho\cdot b'(\widetilde{\rho})(v\cdot\nabla\phi_m)\zeta dxdt
\end{aligned} \tag{31}$$

for all $\zeta\in C^1(\overline{Q}_T)$ such that $\zeta(T)=0$. We know that $\phi_m\to 1$ locally uniformly in Ω as $m\to\infty$, by Lemma 3, and $v\cdot\nabla\phi_m\to 0$ strongly in $L^1(Q_T)$, by Proposition 4. Thus, passing to the limit $m\to\infty$ in (31), we obtain

$$\begin{aligned}
-\int_0^T\int_\Omega b(\rho)\partial_t\zeta dxdt - \int_0^T\int_\Omega b(\rho)v\cdot\nabla\zeta dxdt \\
= \int_\Omega b(\rho^0(x))\zeta(x,0)dx - \int_0^T\int_\Omega \big(b'(\rho)\cdot\rho - b(\rho)\big)(\operatorname{div} v)\zeta dxdt.
\end{aligned} \tag{32}$$

Finally, we verify that ρ is in fact a weak solution. For this, we need to prove that $w=\Lambda(\rho)$. Since Λ is continuously differentiable and the range of ρ is contained in a compact subset of $\mathbb{R}^N_+$, we can choose a C^1 function b which is equal to Λ on this range. Thus, $b=\Lambda$ is admissible in (32). Then we deduce from the positive homogeneity of Λ, i.e. $\Lambda'(\rho)\cdot\rho-\Lambda(\rho)=0$, that

$$-\int_0^T\int_\Omega \Lambda(\rho)\partial_t\zeta dxdt - \int_0^T\int_\Omega \Lambda(\rho)v\cdot\nabla\zeta dxdt = \int_\Omega \Lambda(\rho^0(x))\zeta(x,0)dx$$

for all $\zeta\in C^1(\overline{Q}_T)$ such that $\zeta(T)=0$. The function w is a solution to (7), so $z:=w-\Lambda(\rho)$ satisfies

$$-\int_0^T\int_\Omega z\partial_t\zeta dxdt - \int_0^T\int_\Omega zv\cdot\nabla\zeta dxdt = \int_0^T\int_\Omega z(x,0)\zeta(x,0)dx = 0.$$

This means that $\partial_t z+\operatorname{div}(zv)=0$ in $\mathcal{D}'(Q_T)$, $z(0)=0$ in Ω. Applying renormalization theory again, we see that $z=0$ in Q_T. Hence, $w=\Lambda(\rho)$, and ρ is a weak solution to (1). $\square$

5 Extensions and open problems

In this paper, we have deliberately kept the technicality rather low to highlight the key ideas. In this section, we briefly point out some direct extensions of our results as well as some open problems.

5.1 Reaction terms

Let us consider the mass balance equation (1) with nonvanishing right-hand sides $r_i = r_i(\rho)$ for $i = 1, \ldots, N$, modelling some bulk source of mass, possibly from chemical reactions. Under suitable technical restrictions on the data, we expect the existence and uniqueness of a classical solution as in Theorem 1. We briefly sketch the arguments.

For the variables $w := \Lambda(\rho)$ and $u := \rho/\Lambda(\rho)$ introduced in Section 1.3, we define

$$f(w, u) := r(wu) \cdot \Lambda'(u), \quad g_i(w, u) := \frac{1}{w}(r_i(wu) - u_i f(w, u)), \quad i = 1, \ldots, N. \tag{33}$$

Equations (7) for w and (9) for u change to

$$\partial_t w - \operatorname{div}(\kappa(w) w G'(w) \nabla w) = f(w, u) \quad \text{in } Q_T, \tag{34}$$

$$\dot{u}_i = g_i(w, u) \quad \text{in } Q_T,\ i = 1, \ldots, N, \tag{35}$$

where $\dot{u} = \partial_t u + v \cdot \nabla u$ denotes the material derivative with velocity $v = -\kappa(w)\nabla G(w)$.

As exposed in our key ideas, the main ingredient to prove the well-posedness of such systems is the maximum principle for (34). We will not discuss all possible growth conditions needed for Λ, κ, G and r such that the weak maximum principle in (34) is valid. Instead, we only discuss the example of a sufficiently smooth field $r \in C^1(\mathbb{R}^N_{+,0})$ under the following assumptions:

(A5) There exist $0 < w_* < w^* < +\infty$ such for all $u \in \mathbb{R}^N_+$ satisfying $\Lambda(u) \le 1$, we have $f(w, u) \ge 0$ for all $w \le w_*$ and $f(w, u) \le 0$ for all $w \ge w^*$;

(A6) The modified functions $\widetilde{g}_i(w, u) := (1/w)(r_i(wu) - u_i f(wu)/\Lambda(u))$ are quasi-positive for $i = 1, \ldots, N$, i.e., for all $w \in \mathbb{R}^N_+$, we have $\widetilde{g}_i(w, u) \ge 0$ whenever $u_i = 0$.

We claim that under Assumptions (A1)–(A6), there exist classical solutions to (34), (35) with initial and boundary conditions (8) and (9). We use a fixed-point argument. For this, let $\bar{u} \in L^\infty(Q_T; \mathbb{R}^N_{+,0})$ be given such that $\Lambda(\bar{u}) \le 1$. Then there exists a weak solution w to

$$\partial_t w - \operatorname{div}(\kappa(w) w G'(w) \nabla w) = f(w, \bar{u}) \quad \text{in } Q_T,$$
$$\nabla w \cdot \nu = 0 \quad \text{on } \partial\Omega,\ t > 0, \quad w(0) = w^0 \quad \text{in } \Omega.$$

Using the test functions $(w - w^*)^+ = \max\{0, w - w^*\}$ and $(w - w_*)^- = \min\{0, w - w_*\}$, Assumption (A5) yields lower and upper bounds for w. The Lipschitz continuity of f allows us to conclude the uniqueness of the solution.

Next, we use parabolic regularity for quasilinear equations to derive an intermediate bound for ∇w. For Neumann problems, the global gradient bound for ∇w in $L^\infty(Q_T)$ is proved, for instance, in [27, Chapter XIII, Section 2.2]. This bound only requires an L^∞ bound on the source term $f(w,\bar u)$ in Q_T. Using Assumption (A4) on Λ and the bounds $r_0|\bar u| \le \Lambda(\bar u) \le 1$, the source term can be estimated in L^∞ independently on $\bar u$ (and, of course, independently of w).

Arguing like in [3], we invoke linear parabolic theory: We obtain the Hölder continuity of w [24, Chapter V, Theorem 7.1] and, for every $1 \le p < \infty$, the estimates

$$\|w\|_{W^{2,1}_p(Q_T} \le C\big(\|w_0\|_{W^{2-2/p,p}(\Omega)} + \|f(w,\bar u)\|_{L^p(Q_T)}\big),$$

where $W^{2,1}_p(Q_T) = L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega))$ [24, Chapter 5, §9]. In three space dimensions, the embedding properties of $W^{2,1}_p(Q_T)$ yield a bound for ∇w in $C^{\beta,\beta/2}(\overline{Q}_T)$ with $\beta = 1-5/p > 0$ for $p > 5$.

Now, we can introduce the fractions. It is possible to solve globally the characteristic differential equations

$$\Phi'(s;t,x) = v(\Phi(s;t,x),s), \quad s \in (0,T), \quad \Phi(t;t,x) = x.$$

For all $s \in [0,T]$, the map $(x,t) \mapsto \Phi(s;x,t)$ belongs to $C^{1+\beta,1+\beta/2}(\overline{Q}_T)$. Since the map $u \mapsto \widetilde g_i(w,u)$ is Lipschitz continuous, we can solve the ODEs

$$\dot u_i = \widetilde g_i(w,u), \quad t > 0,\ i = 1,\ldots,N. \tag{36}$$

Thanks to Assumption (A6), the solutions u_i remain globally positive. The definition of $\widetilde g$ in Assumption (A6) implies that

$$\sum_{i=1}^N \widetilde g_i(w,u)\frac{\partial\Lambda}{\partial\rho_i}(u) = \frac{1}{w}\bigg(\sum_{i=1}^N r_i(wu)\frac{\partial\Lambda}{\partial\rho_i}(u) - \frac{\Lambda(u)}{\Lambda(u)}f(w,u)\bigg) = 0.$$

Thus, the solutions to (36) satisfy $\dot\Lambda(u) = \sum_{i=1}^N \dot u_i \partial\Lambda/\partial\rho_i = 0$, and $\Lambda(u) = 1$ is conserved.

It is readily seen that $\bar u \mapsto u$ maps the set $M := \{u \in L^\infty(Q_T;\mathbb{R}^N) : 0 \le u,\ \Lambda(u) \le 1\}$ into itself. Moreover, the solution formula for (36),

$$u(x,t) = u^0(\Phi(0;x,t)) + \int_0^t \widetilde g\big(w(\Phi(s;x,t),s), u(\Phi(s;x,t),s)\big)ds,$$

can be used to prove a bound for u in $C^1(\overline{Q}_T)$. Thus, $\bar u \mapsto u$ is compact in $L^\infty(Q_T)$, and the Schauder fixed-point theorem gives the existence of a fixed point.

This argument can be extended to weak solutions and to more general assumptions than (A5)–(A6), but we leave the details to the reader.

5.2 External forces

A second problem is the presence of external forces in (2). In the simplest case, the flow is subject to gravity, yielding

$$v = \kappa(-\nabla p + \rho_{\text{tot}}\vec{g}), \quad \rho_{\text{tot}} = \sum_{i=1}^{N} \rho_i, \tag{37}$$

where ρ_{tot} is the total mass density and $\vec{g}$ is the constant vector of earth gravitational acceleration. In terms of the variables w and u, we find that $\rho_{\text{tot}} = w\sum_{i=1}^{N} u_i$. After the change of variables, equation (7) for $w := \Lambda(\rho)$ becomes

$$\partial_t w - \operatorname{div}\left(\kappa w\left(G'(w)\nabla w - \sum_{i=1}^{N} u_i w\vec{g}\right)\right) = 0 \quad \text{in } Q_T. \tag{38}$$

The presence of the contribution $\sum_{i=1}^{N} u_i$ in the flux prevents the higher regularity of w. We therefore expect that the extension to the case of nontrivial external forces is a challenging open problem.

However, if the sum of initial fractions is constant, $\sum_{i=1}^{N} \rho_i^0/\Lambda(\rho^0) = C_0$ in Ω for some $C_0 > 0$, and if the fractions are simply transported, we may replace (38) by $\partial_t w - \operatorname{div}(\kappa w(G'(w)\nabla w - C_0 w\vec{g})) = 0$, yielding a problem which can be treated by the previous arguments.

5.3 Other boundary conditions

The last problem that we would like to discuss is the choice of boundary conditions. Instead of (5), we might impose the Dirichlet condition $p = p_0$ on $\partial\Omega \times (0,T)$, where p_0 is a given function. This type of pressure boundary condition corresponds to a free in-outflow problem. For instance, we can think of the domain Ω as a fixed control region in a larger environment occupied by the fluid.

In formulating this problem, we realize that equations (1) are not well posed on Ω. In the absence of the impermeability condition (5), the trajectories of the characteristics are clearly not confined to Ω. In order to solve this kind of boundary-value problems, we need a representation of the flow outside of Ω. Mathematically, we need an extension operator E which, for each velocity field given on $\partial\Omega$, provides its extension $E(v)$ to a larger region $\widetilde{\Omega}$. Moreover, as the trajectories are not confined to Ω, the initial state needs to be known in the larger region $\widetilde{\Omega}$.

The strategy to solve the free flow problem is again to solve the equations

$$\partial_t w - \operatorname{div}(\kappa w G'(w)\nabla w) = 0 \quad \text{in } Q_T, \quad w = G^{-1}(p_0) \quad \text{on } \partial\Omega,\ t > 0, \tag{39}$$

with initial conditions $w(0) = w^0$ in Ω. We solve the differential equations for the characteristics in the larger domain $\widetilde{\Omega} \times (0,T)$ with the extended velocity field $v = E(-\kappa G'(w)\nabla w)$. Since the flow is confined to a bounded region, it is natural to assume that $E(v)$ possesses compact support, i.e., it vanishes uniformly outside of the domain $\widetilde{\Omega}$.

Then, we transport the fractions via $u(x,t) = u^0(\Phi(0;x,t))$ for $x \in \widetilde{\Omega}$ and $t > 0$. It is readily verified that $\rho_i(x,t) := (wu_i)(x,t)$ solves $\partial_t\rho_i + \operatorname{div}(\rho_i v) = 0$ in Q_T and $\rho_i(0,x) = \rho_i^0(x)$ for $x \in \Omega$.

We will discuss these ideas in more detail in an upcoming publication devoted to the optimal control of this type of flow problems.

References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Meth. Appl. Sci.* 21 (1998), 823–864.

[2] M. Bertsch, M. Gurtin, D. Hilhorst, and L. Peletier. On interacting populations that disperse to avoid crowding: preservation of segregation. *J. Math. Biol.* 23 (1985), 1–13.

[3] M. Bertsch, D. Hilhorst, H. Izuhara, and M. Mumura. A nonlinear parabolic-hyperbolic system for contact inhibition of cell-growth. *Diff. Eqs. Appl.* 4 (2012), 137–157.

[4] D. Bothe and W. Dreyer. Continuum thermodynamics of chemically reacting fluid mixtures. *Acta Mech.* 226 (2015), 1757–1805.

[5] M. Burger, J. A. Carrillo, J.-F. Pietschmann, and M. Schmidtchen. Segregation and gap formation in cross-diffusion models. Submitted for publication, 2019. arXiv:1906.03712.

[6] M. Burger, M. Di Francesco, S. Fagioli, and A. Stevens. Sorting phenomena in a mathematical model for two mutually attracting/repelling species. *SIAM J. Math. Anal.* 50 (2018), 3210–3250.

[7] J. A. Carrillo, F. Santambrogio, and M. Schmidtchen. Splitting schemes and segregation in reaction cross-diffusion systems. *SIAM J. Math. Anal.* 50 (2018), 5695–5718.

[8] L. Chen, E. Daus, and A. Jüngel. Rigorous mean-field limit and cross diffusion. *Z. Angew. Math. Phys.* 70 (2019), no. 122, 21 pages.

[9] E. Comparini and M. Ughi. On a multidimensional model for the codiffusion of isotopes: localization and asymptotic behavior. *Math. Meth. Appl. Sci.* 39 (2016), 1680–1688.

[10] M. Di Francesco, A. Esposito, and S. Fagioli. Nonlinear degenerate cross-diffusion systems with nonlocal interaction. *Nonlin. Anal.* 169 (2018), 94–117.

[11] W. Dreyer, C. Guhlke, and M. Landstorfer. A mixture theory of electrolytes containing solvation effects. *Electrochem. Commun.* 43 (2014), 75–78.

[12] W. Dreyer, C. Guhlke, and R. Müller. Overcoming the shortcomings of the Nernst–Planck model. *Phys. Chem. Chem. Phys.*, 15 (2013), 7075–7086.

[13] W. Dreyer, C. Guhlke, and R. Müller. Bulk-surface electro-thermodynamics and applications to electrochemistry. *Entropy* 20 (2018), 939/1–939/44.

[14] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*. Birkhäuser, Basel, 2009.

[15] G. Galiano and V. Selgas. On a cross-diffusion segregation problem arising from a model of interacting particles. *Nonlin. Anal.: Real-World Appl.* 18 (2014), 34–49.

[16] G. Galiano, S. Shmarov, and J. Velasco. Existence and multiplicity of segregated solutions to a cell-growth contact inhibition problem. *Discrete Cont. Dyn. Sys.* 35 (2015), 1479–1501.

[17] H. Garcke, K. F. Lam, R. Nürnberg, and E. Sitka. A multiphase Cahn–Hilliard–Darcy model for tumour growth with necrosis. *Math. Models Meth. Appl. Sci.* 28 (2018), 525–577.

[18] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Second edition. Springer, Berlin, 1983.

[19] M. Gurtin and A. Pipkin. A note on interacting populations that disperse to avoid crowding. *Quart. Appl. Math.* 42 (1984), 87–94.

[20] P. Gwiazda, B. Perthame, and A. Świerczewska-Gwiazda. A two-species hyperbolic-parabolic model of tissue growth. *Commun. Partial Diff. Eqs.* 44 (2019), 1605–1618.

[21] A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity* 28 (2015), 1963–2001.

[22] A. Jüngel. *Entropy Methods for Diffusive Partial Differential Equations*. SpringerBriefs in Mathematics, Springer, 2016.

[23] A. Jüngel, J. Mikyška, and N. Zamponi. Existence analysis of a single-phase flow mixture model with van der Waals pressure. *SIAM J. Math. Anal.* 50 (2018), 1367–1395.

[24] O. Ladyženskaja, V. Solonnikov, and N. Ural'ceva. *Linear and Quasi-Linear Equations of Parabolic Type*. Amer. Math. Soc., USA, 1968.

[25] M. Lesinigo, C. D'Angelo, and A. Quarteroni. A multiscale Darcy–Brinkman model for fluid flow in fractured porous media. *Numer. Math.* 117 (2011), 717–752.

[26] Y.L. Li. Equation of state of water and sea water. *J. Geophys. Res.* 72 (1967), 2665–2678.

[27] G. M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific, Singapore, 1996.

[28] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems with Applications*. Vol. 1. Springer, Berlin, 1972.

[29] F. Morales and R. Showalter. A Darcy–Brinkman model of fractures in porous media. *J. Math. Anal. Appl.* 452 (2017), 1332–1358.

[30] D. Zanger. The inhomogeneous Neumann problem in Lipschitz domains. *Commun. Partial Diff. Eqs.* 25 (2000), 1771–1808.