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für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Stationary particle systems approximating stationary solutions to the Boltzmann equation

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submitted: 30th August 1996

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Preprint No. 264
Berlin 1996

1991 Mathematics Subject Classification. 60K35, 76P05, 82C40.
Key words and phrases. Stationary Boltzmann equation, diffusive boundary condition, stochastic particle system, rate of convergence.

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ABSTRACT. We show that a regularized stationary Boltzmann equation with diffusive boundary conditions can be rigorously derived from a suitable stochastic N-particle system.

1. Introduction

Stochastic particle methods are widely used in the numerical simulation of rarefied flows, which are described at a mathematical level by the Boltzmann equation and hence convergence results for such schemes are of practical interest. From a more fundamental point of view, in the study of these problems we are naturally led to tackle subtle difficulties related to the so called propagation of chaos which is an asymptotic (in the number of particles) statistical independence. Indeed the convergence we want to establish is nothing else but a law of large numbers for (somehow weakly) dependent random variables. For this reason results in this direction are also of interest in the field of limit theorems for large systems of interacting stochastic processes. We address the reader to Ref.s [C], [BI], [LP], [W], [PWZ], [GM] for results concerning convergence of stochastic particle systems to solutions of (regularized) Boltzmann equations. Unfortunately the situation is far from being satisfactory for many reasons which we are going to illustrate.

The convergence results we mentioned above regard **time dependent problems**. Namely the empirical measure (that is a measure valued stochastic process) $\frac{1}{N}\sum_{i=1}^{N}\delta_{z_i(t)}(dz)$, where $z_i(t)$ is the state of the i–th particle at time t, is weakly converging in probability to $f(z,t)$, which is the solution of the Boltzmann equation with initial datum $f(z,0)=f_0(z)$, the distribution density of each particle at time zero, assuming also that all the particles are independently distributed. Such a convergence is not expected to hold uniformly in time. However in most of the practical applications of these stochastic codes we deal with stationary non-equilibrium situations, which we simulate in order to extract informations on the macroscopic quantities like profiles and fluxes. In other words we are interested in non-trivial **stationary solutions** to the Boltzmann equation. In this case the methods we have discussed so far are useless. In fact, even knowing the trend to a non-equilibrium stationary state for the Boltzmann dynamics (which is, incidentally, not known but for simplified models), we could not conclude anything on the particle approximation of this asymptotic state, being the two limits $N\to\infty$ and $t\to\infty$ clearly not commutable. The systematic error of some particle simulation scheme for a stationary model Boltzmann equation was studied in [B]. An alternative approach to the construction of particle schemes for the stationary Boltzmann equation has been proposed in [BS].

In this paper we face the above mentioned problem for a gas in a bounded domain with diffusive boundary conditions at a possibly not constant temperature. We consider the unique stationary measure for the N-particle system and evaluate the distance between this and the N-fold product of the unique solution to the stationary (cutoffed) Boltzmann equation with the same boundary conditions. We show that, if the mean free path inverse is sufficiently small, the L_1 difference between the k-particle distribution functions of such two measures, vanishes in the limit $N\to\infty$, for any fixed k. To do this we use a technique which we call v-functions. Such method is used in Ref. [CDPP] for time dependent problems

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related to stochastic particle systems in a lattice, in Ref. [CP] for a one-dimensional stationary problem for a model equation, and it is indeed very efficient as we shall explain in section 3.

Let us conclude by criticizing the present result. As we said, it holds for small mean free path inverse: this is consequence of the fact that we use a constructive perturbative technique. Also the existence and uniqueness for stationary solutions of the Boltzmann equation is proven under the same smallness assumption. We do not even know whether recent approaches to the existence problem (see for instance [AN] for a suitably cutoffed Boltzmann equation in a slab) can be used to obtain at least the existence of solutions for our problem without this assumption. However the uniqueness of such solutions, which would be preliminary for the convergence problem we set, seems at the moment hard to be proven, even for a regularized equation as the one we consider.

In the present paper the Boltzmann equation enjoys two regularizations. The first, and more important, is a spatial smearing, which is standard in the above quoted literature. Actually the existence theory for the true Boltzmann equation is up to now too poor to allow us to approach the real problem. In [CP] a model equation without spatial smearing has been successfully attacked, however such model is one-dimensional, that is much easier to deal with. The second type of cutoff is on the set of possible velocities, which is essentially compact and bounded away from 0. This last assumption is made to take a full advantage by the ergodic property of the Knudsen flow. We absolutely need this as a consequence of our ignorance of qualitative properties of the invariant measure for the N-particle system, which we only know to exist uniquely. In facts we think that the cutoff on large velocities is only technical: it allows us to avoid difficulties which could obscure the real essence of the approach.

2. Notations and Results

Let $\Omega \subset R^d$, $d = 2,3$ be an open set with smooth boundary in the physical space, $V \subset R^d$ a compact set not containing the origin and $[0,T]$ an interval on the real line. For $(x,v,t) \in \Omega \times V \times [0,T]$ consider the following cutoffed Boltzmann equation:

$$\partial_t p(x,v,t) + (v \cdot \nabla_x) p(x,v,t) = \lambda Q(p,p)(x,v,t) \tag{2.1}$$

with initial condition:

$$p(x,v,0) = p_0(x,v) \geq 0 \tag{2.2}$$

and boundary conditions ($n(x)$ is the outward normal in $x \in \partial\Omega$):

$$p(x,v,t) = J(x,t)M(x,v) \quad x \in \partial\Omega, \quad v \cdot n(x) \leq 0. \tag{2.3}$$

Here we have used the following symbols: λ is a real parameter,

$$Q(p,p)(x,v,t) = \int_\Omega dy \int_V dv_1 \int_{S^{d-1}} de B(v,v_1,e) h_\beta(x,y) \chi((v^*,v_1^*) \in V \times V))$$

$$\{p(x,v^*,t)p(y,v_1^*,t) - p(x,v,t)p(y,v_1,t)\} \tag{2.4}$$

e is the unit vector in R^d, χ is the characteristic function of its argument,

$$v^* = v + e \cdot (v_1 - v)e, \quad v_1^* = v_1 - e(\cdot v_1 - v)e, \tag{2.5}$$

S^{d-1} is the unit sphere.

Moreover B, the collision kernel, has the following form:

$$B(v, v_1, e) = \bar{B}(|v - v_1|, e)$$

where $\bar{B} : \mathbb{R} \times S^{d-1} \to \mathbb{R}^+$. We assume

$$\bar{B}(|v - v_1|, e) \leq c_1. \tag{2.6}$$

The function h_β, which acts as a spatial mollifier, belongs to L_∞ , is vanishing for $|x - y| \geq \beta$ and is such that $\int h_\beta(x, y)dy = 1$. J, the incoming flux at x, is defined as

$$J(x,t) = \int_{v \cdot n(x) \geq 0} dv \quad v \cdot n(x) p(x, v, t). \tag{2.7}$$

Finally M is a bounded positive function defined on the set

$$\{(x, v) | x \in \partial\Omega, v \in V, v \cdot n(x) \leq 0\},$$

which we require to satisfy the following normalization condition:

$$\int_{v \cdot n(x) \leq 0} dv \quad |v \cdot n(x)| M(x, v) = 1. \tag{2.8}$$

This last assumption, together with the well known properties of Q, ensures the conservation of the quantity:

$$m(t) = \int dx \int dv \quad p(x, v, t), \tag{2.9}$$

which we assume initially to be one so that we consider normalized solutions to problem (2.1)-(2.3).

From a physical point of view, eq.s (2.1-3) describe a rarefied gas in a vessel with diffusive boundary conditions at possibly not constant temperature on the boundary. The collision operator Q differs from the usual one for the cutoff on the velocities and for the presence of the smearing function h_β. The true Boltzmann equation is recovered by removing the two cutoffs, that is letting $h_\beta \to \delta$ (δ is the δ–function centered at the origin) and assuming $V = \mathbb{R}^d$.

It will be useful in the sequel to deal with the mild version of the above problem:

$$p(t, x, v) = S(t)p_0(x, v) + \lambda \int_0^t ds \quad S(t - s)Q(p, p)(x, v, s) \tag{2.10}$$

where $S(t)$ is the Knudsen semigroup, that is the solution to the initial boundary value problem :

$$[\partial_t + (v \cdot \nabla_x)]S(t)p_0(x, v) = 0 \tag{2.11}$$

$$(S(t)p_0)(x,v) = J(x,t)M(x,v) \quad x \in \partial\Omega, \quad v \cdot n(x) \le 0. \tag{2.12}$$

The solution to eq. (2.10) does exist unique, thanks to the Lipschitz continuity in $L_1(x,v)$ of Q, due to the presence of the smearing function h_β and (2.6).

Here we are interested in the corresponding stationary equation:

$$(v \cdot \nabla_x)g(x,v) = \lambda Q(g,g)(x,v) \tag{2.13}$$

with boundary conditions (2.3) and the normalization property:

$$\int dx \int dv g(x,v) = 1\,.$$

Existence and uniqueness of a solution for a slightly different formulation for such problem (under suitable smallness assumption) will be established in Thm. 2.2 below. For the moment we need a preliminary property of the **Knudsen flow** expressed by the following Theorem which will be proven in Appendix.

Theorem 2.1. *There exists a unique probability density $\bar{g}$ which is stationary under the action of the Knudsen flow i.e.*

$$S(t)\bar{g} = \bar{g} \quad for \quad all \quad t \in \mathbb{R}^+ \tag{2.14}$$

Moreover for any $\eta > 0$ there exists $T(\eta) > 0$ such that, for any $t \ge T(\eta)$ and for any probability density f, it is:

$$\|S(t)f - \bar{g}\|_{L_1} \le \eta. \tag{2.15}$$

Remark. We stress that the assumption for the velocities to stay bounded away from 0 implies the independence of $T(\eta)$ from the probability density f, which is of great importance to prove our main result.

We now establish also **existence and uniqueness for the stationary solution** of the boundary value problem (2.13):

Theorem 2.2. *If λ is sufficiently small, there exists a unique probability density g which is invariant for the flow (2.10):*

$$g = S(t)g + \lambda \int_0^t ds \quad S(t-s)Q(g,g) \qquad t \in \mathbb{R}^+ \tag{2.16}$$

and is globally attractive:

$$\|p(t) - g\|_{L_1} \le e^{-ct}\,,$$

where $p(t)$ is any solution to (2.10) and c is some constant.

The proof of this theorem, which is essentially perturbative, is given in Appendix.

Now we introduce the N-particle process which gives the approximation, in the limit $N \to \infty$, to problem (2.1)-(2.3). Let

$$Z_N = (z_1, ..., z_N) \quad z_i = (x_i, v_i) \quad i = 1, ..., N$$

and for the sake of simplicity put

$$q(z_1, z_2, e) = h_\beta(x_1, x_2) B(v_1, v_2, e) \chi((v^*, v_1^*) \in V \times V). \tag{2.17}$$

We define the generator of the N-particle process, for any function Φ as:

$$G_N(\Phi)(Z_N) = G_N^{free}(\Phi)(Z_N) + \frac{\lambda}{N} G_N^{jump}(\Phi)(Z_N) \tag{2.18}$$

where

$$G_N^{free}(\Phi)(Z_N) = \sum_{i=1}^{N} (v_i \cdot \nabla_{x_i})(\Phi)(Z_N) \tag{2.19}$$

(with diffusive boundary conditions to be specified, see eq. (2.24-25) below) and

$$G_N^{jump}(\Phi)(Z_N) = \sum_{1 \leq i < j \leq N} \int_{S^{d-1}} de [\Phi(Z_N^{(i,j)}) - \Phi(Z_N)] q(z_i, z_j, e), \tag{2.20}$$

being

$$Z_N^{(i,j)} = (z_1, \dots, z_{i-1}, x_i, v_i^*, \dots z_{j-1}, x_j, v_j^*, \dots z_N). \tag{2.21}$$

Note that G_N^{free} is the generator of N independent particles moving freely. The outgoing velocity v of each particle after a collision with the boundary at the point x, is distributed according to the probability density given in (2.8). In other words $exp\{(G_N^{free})^* t\} = S_N(t)$, where $S_N(t)$ is the product of operators acting on a single particle, namely:

$$S_N(t) = \prod_{i=1}^{N} S_i(t) \tag{2.22}$$

where $S_i(t)$ is the Knudsen semigroup associated to the particle i. Therefore the process described by the generator G_N consists in free motion (including the diffusive boundary conditions) of the N–particle system and random collisions. These last take place at random times, with random impact parameter e. The particles of the pair involved in the collision have mutual distance less than β and their outgoing velocities after the interaction follow the deterministic law (2.5). This model, introduced in Ref. [C] is sometimes called "soft balls" model.

If the system is initially distributed according to a probability density $f^N(Z_N)$, its time evolution is given by $f^N(t) = exp\{(G_N)^* t\} f^N$. In other words:

$$\partial_t f^N(Z_N, t) + \sum_{i=1}^{N} (v_i \cdot \nabla_{x_i}) f^N(Z_N, t) = \frac{\lambda}{N} G_N^{jump} f^N(Z_N, t) \tag{2.23}$$

with initial conditions $f^N(Z_N, 0) = f^N(Z_N)$ and with boundary conditions:

$$f^N(z_1 \dots z_i \dots z_N, t) = J_i^N(x_i, t, Z_N(i)) M(x_i, v_i) \qquad x_i \in \partial\Omega \quad v_i \cdot n(x_i) \leq 0 \tag{2.24}$$

for all $i = 1, \dots N$, with $Z_N(i) = (z_1 \dots z_{i-1}, z_{i+1} \dots z_N)$ and

$$J_i^N(x_i, t, Z_N(i)) = \int_{v_i \cdot n(x_i) \geq 0} dv_i \quad v_i \cdot n(x_i) f^N(Z_N, t). \tag{2.25}$$

If we consider the stationary version of (2.23)-(2.24), that is :

$$\sum_{i=1}^{N} (v_i \cdot \nabla_{x_i}) \tilde{f}^N(Z_N) = \frac{\lambda}{N} G_N^{jump} \tilde{f}^N(Z_N) \tag{2.26}$$

with the boundary conditions (2.24-25), we can state the following result, which is proven in Appendix:

Theorem 2.3. *For all $N > 0$ there exists a unique probability density $\tilde{f}^N = \tilde{f}^N(Z_N)$ which is invariant under the N–particle process.*

The **main goal of this paper** is to compare the stationary distribution $\tilde{f}^N$ with the one-particle stationary distribution g constructed in Thm. 2.2. To this purpose we introduce the k-particle distribution functions associated to the probability density $\tilde{f}^N$:

$$\tilde{f}_k^N(Z_k) = \int \cdots \int \tilde{f}^N(Z_N) dz_{k+1} \dots dz_N, \quad k = 1, \dots N-1. \tag{2.27}$$

Introducing analogously the k–particle distribution functions for the time dependent distribution $f^N(Z_N, t)$, we obtain from (2.23) and (2.24) the well known **BBGKY hierarchy** of equations:

$$\partial_t f_k^N(Z_k, t) + G_k^{free} f_k^N(Z_k, t) =$$

$$\frac{\lambda}{N} G_k^{jump} f_k^N(Z_k, t) + \lambda \frac{N-k}{N} C_{k,k+1} f_{k+1}^N(Z_k, t), \quad k = 1, \dots, N-1 \tag{2.28}$$

with boundary conditions:

$$f_k^N(z_1, \dots, z_i, \dots z_k, t) = J_i^N(x_i, t, Z_k(i)) M(x_i, v_i) \qquad x_i \in \partial\Omega \quad v_i \cdot n(x_i) \leq 0. \tag{2.29}$$

Here:

$$C_{k,k+1} f_{k+1}^N(Z_k, t) =$$

$$\sum_{i=1}^{k} \int \int_{S^2} dz_{k+1} de \quad q(z_i, z_{k+1}, e) [f_{k+1}^N(Z_{k+1}^{(i,k+1)}, t) - f_{k+1}^N(Z_{k+1}, t)] \tag{2.30}$$

Note that by Thm. 2.3 it follows that the unique solutions to the stationary version of problem (2.28) are those defined in (2.27).

Now we introduce the **infinite Boltzmann hierarchy**, that is the (formal) limit as $N \to \infty$ of the BBGKY hierarchy, i.e.:

$$\partial_t f_k(Z_k, t) + G_k^{free} f_k(Z_k, t) = \lambda C_{k,k+1} f_{k+1}(Z_k, t) \quad k = 1, 2, \dots \tag{2.31}$$

with initial and usual boundary conditions.

It is useful to consider the mild form of it, that is:

$$f_k(t) = S_k(t)f_k^0 + \lambda \int_0^t ds S_k(t-s)C_{k,k+1}f_{k+1}(s) \quad k = 1,2,\dots \tag{2.32}$$

We denote by $P(t)$ the solution operator of the infinite hierarchy (2.32) that is $(P(t)f^0)_k = f_k(t)$. $P(t)$ acts on sequences $f^0 = \{f_k^0\}_{k=1\dots\infty}$, $f_k^0 \in L_1((\Omega \times V)^k)$.

Analogously we can define $P^N(t)f^N = f^N(t)$ to be the solution operator of the following finite hierarchy of equations:

$$f_k^N(t) = S_k(t)f_k^N + \frac{\lambda}{N}\int_0^t ds \quad S_k(t-s)G_k^{jump}f_k^N(s) +$$

$$\lambda\frac{N-k}{N}\int_0^t ds \quad S_k(t-s)C_{k,k+1}f_{k+1}^N(s), \quad k = 1,2,\dots N. \tag{2.33}$$

Notice that eq. (2.33) for $k = N$ is the mild version of (2.23) and the full hierarchy is the mild version of (2.28)

Since (2.33) is a finite system of linear equations, it can easily be solved uniquely in $L_1(dZ_k)$, namely $f_k^N(t)$ are obtained by integrating $f^N(Z_N, t)$, unique solution to (2.23).

By iterating formula (2.32) we arrive to the following formal series expansion for the solution to equation (2.31):

$$(P(t)f^0)_k = f_k(t) = \sum_{n=0}^{\infty} \lambda^n a_{k,n}(t)f_k^0 \tag{2.34}$$

with

$$a_{k,n}(t)f_k^0 = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \dots dt_n$$

$$S_k(t-t_1)C_{k,k+1}\dots S_{k+n-1}(t_{n-1}-t_n)C_{k+n-1,k+n}S_{k+n}(t_n)f_{k+n}^0 \tag{2.35}$$

It is possible to show that the series in (2.34) converges in L_1 if the quantity λt is sufficiently small, so that, under such hypothesis, the solution to (2.32) does exist unique. The method employed is the same as in [LP] and [PWZ], inspired by the well known result due to Lanford (see [L] and [CIP]) in a L_∞-setup for the not regularized Boltzmann equation. Here we find the additional difficulty of the diffusive boundary conditions. However, working in L_1, this is not a problem, since the only property we need of the free flow is the isometry (see (2.36) below).

We will show the convergence of the series (2.34) as well as the asymptotic equivalence (for $N \to \infty$) of the operators $P^N(t)$ and $P(t)$.

Before stating Theorem 2.4 below, we stress two fairly evident estimates of the terms in the series (2.34):

$$\|S_k(t)f_k\|_{L_1} = \|f_k\|_{L_1} \tag{2.36}$$

and

$$\|C_{k,k+1}f_{k+1}\|_{L_1} \leq ka\|f_{k+1}\|_{L_1} \tag{2.37}$$

where

$$a = 2\sup_{z,z'} \int de\, q(z,z',e). \tag{2.38}$$

Theorem 2.4. *Suppose $\lambda t < \frac{1}{8a}$. Then given any sequence $\{f_k^0\}_{k=1\dots\infty}$ such that $\|f_k^0\|_{L_1} = 1$, the series (2.34) is absolutely convergent in $L_1((\Omega \times V)^k)$ for all $k > 0$. Moreover given the sequence $f^N = \{f_k^N\}_{k=1\dots N}$ of k-particle densities, we have:*

$$\|([P^N(t) - P(t)]f^N)_k\|_{L_1} \leq \frac{8^k c_2}{N} \tag{2.39}$$

for some constant c_2 independent of f^N.

Remark. Since $P(t)$ is defined as acting on infinite sequences, in (2.39) we mean $(P(t)f^N)_k = 0$ for $k > N$.

Proof. By (2.35), using (2.36) and (2.37) we have:

$$\|a_{k,n}(t)f_k^0\|_{L_1} \leq \frac{k(k+1)\dots(k+n-1)}{n!}(ta)^n\|f_{k+n}^0\|_{L_1} \leq 2^k(2ta)^n. \tag{2.40}$$

Therefore the series (2.34) converges for $2ta\lambda < 1$.

Let us define:

$$D^N(t) = [P^N(t) - P(t)]f^N, \tag{2.41}$$

$$B_k^N(t) = \frac{\lambda}{N}\int_0^t ds S_k(t-s)G_k^{jump}(P^N(s)f^N)_k\,, \tag{2.42}$$

$$E_k^N(t) = -\frac{\lambda k}{N}\int_0^t ds S_k(t-s)C_{k,k+1}(P^N(s)f^N)_{k+1}. \tag{2.43}$$

By (2.32) and (2.33) we have:

$$D_k^N(t) = B_k^N(t) + E_k^N(t) + \lambda\int_0^t ds S_k(t-s)C_{k,k+1}D_{k+1}^N(s), \quad k = 1, 2, \dots N-1. \tag{2.44}$$

Iterating (2.44) $n-1$ times, with $n \leq N-k$, we obtain:

$$D_k^N(t) = \sum_{m=0}^{n-1}\lambda^m\int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}} dt_1\dots dt_m$$

$$S_k(t-t_1)C_{k,k+1}\dots S_{k+m-1}(t_{m-1}-t_m)C_{k+m-1,k+m}(B_{k+m}^N(t_m) + E_{k+m}^N(t_m)) +$$

$$\lambda^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_1\dots dt_n$$

$$S_k(t-t_1)C_{k,k+1}\dots S_{k+n-1}(t_{n-1}-t_n)C_{k+n-1,k+n}(t_n)D_{k+n}^N(t_n) \tag{2.45}$$

By (2.34), (2.40) and the assumption $\lambda t < \frac{1}{8a}$, it follows:

$$\|D_{k+n}^N(t)\|_{L_1} \leq 1 + 2^{k+n+1} \tag{2.46}$$

so that, after elementary calculation, we can bound the L_1-norm of the last term in the right hand side of (2.45) by the quantity $4 \cdot 4^k(\frac{1}{2})^n$.

Moreover we have:

$$\|B_k^N(t)\|_{L_1} \le \frac{k^2 \lambda t a}{N} \tag{2.47}$$

and

$$\|E_k^N(t)\|_{L_1} \le \frac{k^2 \lambda t a}{N} \tag{2.48}$$

so that (2.45) implies:

$$\|D_k^N(t)\|_{L_1} \le \frac{2^{k+1}}{N} \sum_{m\ge 0} (2\lambda t a)^{m+1}(k+m)^2 + 4\cdot 4^k(\frac{1}{2})^n$$

$$\le \frac{4^k}{N}\frac{8t\lambda a}{1-4a\lambda t} + 4\cdot 4^k(\frac{1}{2})^n \tag{2.49}$$

The thesis follows by putting $n = N - k$. □

Remark. The above result can be used to show the convergence of the solutions of the N–particle system to the solution of our Boltzmann equation. Indeed Thm. 2.4 shows the existence and uniqueness of the solutions to hierarchy (2.32) for short times. Assume that the initial datum is factorizing i.e. $f_k^N = f_0^{\otimes k}$ where f_0 is some one-particle probability density. Then it is easy to show that the unique solution of the hierarchy (2.32) we have constructed is of the form $f_k(t) = f^{\otimes k}(t)$, where $f(t)$ solves the Boltzmann equation (2.1) with initial datum f_0. This property is called propagation of chaos. Thus we have shown that $f_k^N(t) \to f^{\otimes k}(t)$ for all $k > 0$, in L_1 and for short times. On the other hand t must be smaller than a numerical constant independent of f^0, so that the procedure can be iterated in time to show that the convergence is global (see [LP], [PWZ] for details).

Coming back to the stationary problem, we conclude this section by formulating the **main result of this paper** which will be proven in the next section. We recall that g denotes the stationary solution to the boundary value problem (2.13) constructed in Thm. 2.2 and we set:

$$g_k(Z_k) = \prod_{i=1}^{k} g(z_i). \tag{2.50}$$

We also recall that $\tilde{f}_k^N$ denotes the k–particle distribution of the unique invariant measure of the N-particle system. We can prove:

Theorem 2.5. *There exists $\lambda_0 > 0$ such that for any $\lambda \le \lambda_0$ and any integer $k \ge 1$ it is:*

$$\|\tilde{f}_k^N - g_k\|_{L_1} \le \frac{c^k}{N}, \quad N > k,$$

for some constant c not depending on λ, k, N.

3. Proof of Theorem 2.5

We introduce a formalism which plays a very important role in what follows. Let $I \subset \mathbb{N}$ be a bounded set of indices and let $|I|$ represent its cardinality. Given two families of symmetric functions $\phi = \{\phi_I\}_{I\subset\mathbb{N}}$ and $\psi = \{\psi_I\}_{I\subset\mathbb{N}}$, we give the following definition of $*$ product:

$$(\phi * \psi)_I(Z_I) = \sum_{J\subseteq I} \phi_J(Z_J)\psi_{I\setminus J}(Z_{I\setminus J}), \tag{3.1}$$

where we are using the notation $Z_I = \{z_i | i \in I\}$.

Let us put:

$$\phi_I^{\perp} = (-1)^{|I|}\phi_I \tag{3.2}$$

and finally let us define:

$$v_I^N = (g^{\perp} * \tilde{f}^N)_I \tag{3.3}$$

where we set $\tilde{f}_I^N(Z_I) = \tilde{f}_k^N(Z_I)$ if $|I| = k$.

We assume that

$$\tilde{f}_\emptyset^N = g_\emptyset = v_\emptyset^N = 1\,. \tag{3.4}$$

We want to stress that, if it were:

$$\tilde{f}_I^N(Z_I) = \prod_{i\in I} f(z_i)$$

then,

$$v_I^N(Z_I) = \prod_{i=1}^{|I|}[g(z_i) - f(z_i)]. \tag{3.5}$$

This means, in a sense, that the functions v^N represent the product of the differences better than the difference of the products which we would have to deal with.

By (3.4) it follows that the definition (3.3) can be inverted to obtain:

$$\tilde{f}_I^N = (g * v^N)_I \tag{3.6}$$

and this implies, as it can be easily seen, that

$$\|\tilde{f}_k^N - g_k\|_{L_1} \leq \sum_{J\subset I,\, J\neq\emptyset} \|v_J^N\|_{L_1}\,. \tag{3.7}$$

Therefore we will prove Thm. 2.5 by estimating v^N.

As a consequence of Thm. 2.1, we have the following:

Lemma 3.1. *For any $\eta > 0$ there exists a $T(\eta)$ such that, for $t > T(\eta)$ and $k \in \mathbb{N}$ it is*

$$\|S_k(t)v^N\|_{L_1} \leq \eta^k \|v^N\|_{L_1} \tag{3.8}$$

Remark. Had we considered directly the difference $\tilde{f}_I^N - g_I$ in place of v_I^N, at the best we would have obtained $\|S_k(t)(\tilde{f}_I^N - g_I)\|_{L_1} \leq \eta\|\tilde{f}_I^N - g_I\|_{L_1}$ and this is not sufficient for our purpose.

Proof.

The Knudsen process is a collection of independent one-particle processes. Moreover it is a consequence of the definition of v^N that

$$\int dz_i \quad v_I^N(Z_I) = 0$$

for any $i = 1, 2, \dots |I|$. Thus, to prove the Lemma it is enough to prove that, for all $\eta > 0$, there exists $T(\eta)$ such that, for $t > T(\eta)$, for all $u = u(z), u \in L_1$, satisfying $\int u dz = 0$, one has:

$$\|S(t)u\|_{L_1} \leq \eta \|u\|_{L_1} \tag{3.9}$$

Indeed, denoting by u^+ and u^- the positive and negative part of u respectively, setting

$$A = \int u^+ dz = \int u^- dz \tag{3.10}$$

we have by Thm. 2.1:

$$\|S(t)u\|_{L_1} = \|S(t)u^+ - S(t)u^-\|_{L_1} \leq$$

$$A\|S(t)(\frac{u^+}{A}) - \bar{g}\|_{L_1} + A\|S(t)(\frac{u^-}{A}) - \bar{g}\|_{L_1} \leq 2A\eta = \eta\|u\|_{L_1}. \tag{3.11}$$

□

We recall that $(P^N(t)\tilde{f}^N)_I = \tilde{f}_I^N$ for all I such that $0 < |I| \leq N$ and $(P(t)g)_I = g_I$ for all I with $|I| > 0$. We extend this invariance property to the empty set, that is (see (3.4)):

$$(P^N(t)\tilde{f}^N)_\emptyset = 1 \quad (P(t)g)_\emptyset = 1. \tag{3.12}$$

We also put:

$$(P(t)\tilde{f}^N)_\emptyset = 1. \tag{3.13}$$

For any finite set of indices I, we have:

$$v_I^N = (g^\perp * \tilde{f}^N)_I = (g^\perp * P^N(t)\tilde{f}^N)_I =$$

$$(g^\perp * P(t)\tilde{f}^N)_I + (g^\perp * [P^N(t) - P(t)]\tilde{f}^N)_I. \tag{3.14}$$

Before going on in the estimate of v^N, we introduce a suitable norm. Given an infinite sequence of L_1-functions $\phi = \{\phi_k\}$ and a real number α we set:

$$\|\phi\|_\alpha = \sup_k \|\phi_k\|_{L_1} e^{-\alpha k}. \tag{3.15}$$

Putting

$$R_I^N(t) = (g^\perp * [P^N(t) - P(t)]\tilde{f}^N)_I \tag{3.16}$$

by Theorem 2.4 and (3.12) and (3.13) it follows that for $\alpha = 3log3$ (from now on fixed) and $\lambda t < \frac{1}{8a}$, we have:

$$\|R^N(t)\|_\alpha \leq \frac{c_2}{N}. \tag{3.17}$$

Indeed suppose $|I| = k$,

$$\|R_I^N(t)\|_{L_1} \leq \sum_{h=1}^{k} \binom{k}{h} \|([P^N(t) - P(t)]\tilde{f}^N)_h\|_{L_1} \leq \frac{9^k c_2}{N} \tag{3.18}$$

so that (3.17) follows.

Since $(g^\perp * g)_I = 0$ if $|I| > 0$, we have by (3.4), (3.12) and (3.14):

$$v_I^N = (g^\perp * P(t)(\tilde{f}^N - g))_I + R_I^N(t) := (g^\perp * P(t)\psi)_I + R_I^N(t) \tag{3.19}$$

where, by (3.6) we have put

$$\psi_I = \sum_{\substack{S \subset I \\ |S|>0}} v_S^N g_{I \setminus S}, \quad \psi_\emptyset = 0. \tag{3.20}$$

We prove the following result.

Lemma 3.2. *Let η be a positive real number and choose $T(\eta)$ as in Lemma 3.1. Then, for any integer $k > 0$ and $t > T(\eta)$ the following estimate holds:*

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \leq \frac{\delta_k}{1 - 2a\lambda t(e^\alpha + 1)} \|v^N\|_\alpha \tag{3.21}$$

with $\delta_k = 2^k(1 + k2^{k-1}e^\alpha)(\frac{\eta}{2} + 2a\lambda t(e^\alpha + 1))$, provided that λ and η are so small to satisfy: $e^\alpha[\frac{\eta}{2} + 2a\lambda t(e^\alpha + 1)] < 1$.

Proof. To prove the Lemma, we write the expansion already introduced in (2.34-35). More precisely:

$$(P(t)\psi)_k = \sum_{n \geq 0} \int_0^t \int_0^{t_1} \dots \lambda^n \int_0^{t_{n-1}} dt_1 \dots dt_n \sum_{i_1=1}^{k} \sum_{i_2=1}^{k+1} \cdots \sum_{i_n=1}^{k+n-1}$$

$$S_k(t - t_1) C_{k,k+1}^{i_1} \dots S_{k+n-1}(t_{n-1} - t_n) C_{k+n-1,k+n}^{i_n} S_{k+n}(t_n) \psi_{k+n} \tag{3.22}$$

where

$$\sum_{i_r=1}^{k+r-1} C_{k+r-1,k+r}^{i_r} = C_{k+r-1,k+r} \qquad r = 1, 2, \dots, n$$

that is $C_{k+r-1,k+r}^{i_r}$ is the contribution due to the collision of the i_r-th particle (among the $k + r - 1$ particles) with the $k + r$-th. Let us indicate by I_r the set of indices $\{1,2,\ldots,r\}$ and by $I(k,n)$ the set $I_{k+n} \setminus I_k \equiv \{k+1,\ldots,n\}$. Then by the definition of ψ it is:

$$\psi_{k+n} = \sum_{S_1 \subseteq I_k} \sum_{S_2 \subseteq I(k,n)} v^N_{S_1 \cup S_2} g_{I_{k+n} \setminus (S_1 \cup S_2)} \chi(|S_1| + |S_2| > 0). \tag{3.23}$$

We now select among the particles in S_1 those which do not interact with any other particle. To this end, we consider the set $J = S_1 \setminus \{i_1,\ldots,i_n\}$ and notice that:

$$\sum_{i_1,\ldots,i_n} \sum_{S_1 \subseteq I_k} \sum_{S_2 \subseteq I(k,n)} = \sum_{S_1 \subseteq I_k} \sum_{S_2 \subseteq I(k,n)} = \sum_{J \subseteq S_1} \sum_{i_1,\ldots,i_n} \prod_{r=1}^{n} \chi(i_r \notin J) \tag{3.24}$$

Defining $n(s_1,j) = max(s_1 - j, 1)$, (3.22) can be rewritten as:

$$(P(t)\psi)_k = S_k(t)\psi_k + \sum_{s_1=0}^{k} \sum_{\substack{S_1 \subseteq I_k \\ |S_1| = s_1}} \sum_{j=0}^{s_1} \sum_{\substack{J \subseteq S_1 \\ |J| = j}} \sum_{n > n(s_1,j)} \sum_{i_1,\ldots,i_n}$$

$$\prod_{r=1}^{n} \chi(i_r \notin J) \sum_{s_2=0}^{n} \sum_{\substack{S_2 \subseteq I(n,k) \\ |S_2| = s_2}} \chi(s_1 + s_2 > 0) \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \ldots dt_n$$

$$S_{I_k \setminus J}(t - t_1) C^{i_1}_{k-j,k-j+1} S_{I_k \setminus J \bigcup \{i_1\}}(t_1 - t_2) \ldots C^{i_k}_{k-j+n-1,k-j+n}$$

$$S_J(t) v^N_{S_1 \bigcup S_2} g_{I_{n+k} \setminus S_1 \bigcup S_2} \tag{3.25}$$

Here we are using the notation $S_A(t) = \prod_{i \in A} S_i(t)$ and hence $S_A(t)$ represents the Knudsen semigroup associated with the free motion of the particles with labels in A.

(3.25) follows from the fact that we have selected the set J of particles non interacting with the rest and hence $S_J(t)$ commutes with all other operators.

By Lemma 3.1, for $\eta > 0$ and $t > T(\eta)$ we have:

$$\|S_J(t) v^N_{S_1 \bigcup S_2}\|_{L_1} \leq \eta^j \|v^N_{S_1 \bigcup S_2}\|_{L_1} .$$

Moreover

$$\|g_{I_{n+k} \setminus S_1 \bigcup S_2}\|_{L_1} = 1 .$$

Thus, using the equality

$$\sum_{i_1,\ldots,i_n} \prod_{r=1}^{n} \chi(i_r \notin J) = \frac{(k-j+n-1)!}{(k-j)!},$$

by (2.37) we arrive to the formula

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \leq \|v^N\|_\alpha \sum_{s_1=0}^{k} \binom{k}{s_1} \sum_{j=0}^{s_1} \binom{s_1}{j} \eta^j \sum_{n > n(s_1,j)} \sum_{s_2=0}^{n} \binom{n}{s_2} \times$$

$$\frac{(2a\lambda t)^n}{n!}\frac{(k-j+n-1)!}{(k-j)!}e^{\alpha(s_1+s_2)}\chi(s_1+s_2>0)\,. \tag{3.26}$$

Now we separate from the rest the term corresponding to $s_1 = 0$ and obtain:

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \le \|v^N\|_\alpha 2^k \sum_{n\ge1}(2a\lambda t)^n \sum_{s_2=1}^{n}\binom{n}{s_2}e^{\alpha s_2}+$$

$$\|v^N\|_\alpha \sum_{s_1=1}^{k}\binom{k}{s_1}e^{\alpha s_1}\sum_{j=0}^{s_1}\binom{s_1}{j}\eta^j 2^{k-j}\sum_{n>n(s_1,j)}(2a\lambda t)^n\sum_{s_2=0}^{n}\binom{n}{s_2}e^{\alpha s_2}\,. \tag{3.27}$$

Since

$$\sum_{h=0}^{n}\binom{n}{h}a^h = (1+a)^h$$

it follows:

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \le \|v^N\|_\alpha 2^k \sum_{n\ge1}(2a\lambda t)^n(1+e^\alpha)^n+$$

$$\|v^N\|_\alpha 2^k \sum_{s_1=1}^{k}\binom{k}{s_1}e^{\alpha s_1}\sum_{j=0}^{s_1}\binom{s_1}{j}(\frac{\eta}{2})^j\sum_{n>n(s_1,j)}(2a\lambda t)^n(1+e^\alpha)^n\,. \tag{3.28}$$

By the hypothesis on λ, $2a\lambda t(1+e^\alpha)\le 1$ so that we have:

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \le \|v^N\|_\alpha 2^k\frac{2a\lambda t(1+e^\alpha)}{1-2a\lambda t(1+e^\alpha)}+$$

$$\|v^N\|_\alpha\frac{2^k}{1-2a\lambda t(1+e^\alpha)}\sum_{s_1=1}^{k}\binom{k}{s_1}e^{\alpha s_1}\sum_{j=0}^{s_1}\binom{s_1}{j}(\frac{\eta}{2})^j[2a\lambda t(1+e^\alpha)]^{s_1-j}\,. \tag{3.29}$$

After a few simple calculation, we arrive to:

$$\|(P(t)\psi)_k - S_k(t)\psi_k\|_{L_1} \le \|v^N\|_\alpha\frac{2^k}{1-2a\lambda t(1+e^\alpha)}\times$$

$$\left\{2a\lambda t(1+e^\alpha)+[1+e^\alpha(\frac{\eta}{2}+2a\lambda t(1+e^\alpha))]^k-1\right\}\,. \tag{3.30}$$

Using the elementary inequality

$$(1+x)^k - 1 \le k2^{k-1}x, \quad x\in[0,1] \tag{3.31}$$

and again the smallness assumption on λ, we obtain:

$$(3.30) \le \|v^N\|_\alpha\frac{2^k}{1-2a\lambda t(1+e^\alpha)}[2a\lambda t(1+e^\alpha)+k2^{k-1}e^\alpha(\frac{\eta}{2}+2a\lambda t(1+e^\alpha))]\le$$

$$\|v^N\|_\alpha \frac{2^k(1+k2^{k-1}e^\alpha)}{1-2a\lambda t(1+e^\alpha)}(\frac{\eta}{2}+2a\lambda t(1+e^\alpha)] \tag{3.32}$$

and the Lemma is proven. $\square$

We have by Lemma 3.1 and recalling the definition (3.20) of ψ:

$$\|S_k(t)\psi_k\|_{L_1} \leq \sum_{j=1}^{k} \sum_{\substack{J\subseteq I_k \\ |J|=j}} \|S_k(t)v_J^N g_{I_k\setminus J}\|_{L_1} \leq$$

$$\sum_{j=1}^{k} \sum_{\substack{J\subseteq I_k \\ |J|=j}} \eta^j \|v_J^N\|_{L_1} \leq \|v^N\|_\alpha \sum_{j=1}^{k} \binom{k}{j} \eta^j e^{\alpha j} =$$

$$\|v^N\|_\alpha[(1+e^\alpha\eta)^k - 1] \leq \|v^N\|_\alpha k 2^{k-1} e^\alpha \eta\,, \tag{3.33}$$

for $e^\alpha \eta \leq 1$.

From Lemma 3.2 and (3.33) it finally follows that:

$$\|(P(t)\psi)_k\|_{L_1} \leq 2\frac{\delta_k}{1-2a\lambda t(1+e^\alpha)}\|v^N\|_\alpha\,. \tag{3.34}$$

Now the proof of the Theorem is nearly complete. The estimate (3.34) together with the fact that $(P(t)\psi)_\emptyset = 0$, imply for $2a\lambda t(1+e^\alpha) < \frac{1}{2}$:

$$\|(g^\perp * P(t)\psi)_k\|_{L_1} \leq 4\|v^N\|_\alpha \sum_{j=1}^{k} \binom{k}{j} \delta_j \leq$$

$$4\|v^N\|_\alpha(\eta + 2a\lambda t(e^\alpha+1)) \sum_{j=1}^{k} \binom{k}{j} 2^j(1+j2^{j-1}e^\alpha) \leq$$

$$4\|v^N\|_\alpha(\eta + 2a\lambda t(e^\alpha+1))2e^\alpha \sum_{j=0}^{k} \binom{k}{j} 2^{3j} \leq$$

$$8\|v^N\|_\alpha(\eta + 2a\lambda t(e^\alpha+1))e^\alpha 9^k\,. \tag{3.35}$$

Thus

$$\|(g^\perp * P(t)\psi)_k\|_{L_1} e^{-\alpha k} \leq 8\|v^N\|_\alpha(\eta + 2a\lambda t(e^\alpha+1))e^\alpha e^{(log9-\alpha)k}. \tag{3.36}$$

Now we can fix the parameters λ, T, η. We recall that $\alpha = 3log3$, and choose $\eta \leq \frac{1}{32e^\alpha}$. Consequently we fix $t = T(\eta)$ as in Lemma 3.1. Finally we choose λ in such a way that $e^\alpha 2a\lambda t(e^\alpha+1) \leq \frac{1}{32}$. Then we have:

$$\|(g^\perp * P(t)\psi)_k\|_\alpha \leq \frac{1}{2}\|v^N\|_\alpha \tag{3.37}$$

and, by (3.19) and (3.37),

$$\|v^N\|_\alpha \leq 2\|R^N(t)\|_\alpha$$

so that (3.17) concludes the proof.

APPENDIX

Proof of Theorem 2.1.

Consider $S(t)$ the Knudsen flow and $P_t(x',v';x,v)$ the transition probability densities given by:

$$\int P_t(x',v';x,v)f(x',v')dx'dv' = S(t)f(x,v) \tag{A.1}$$

for $f \in L_1(\Omega \times V)$. For any final state (x,v) trace the backward trajectories $x - sv$ up to the instant (say t) of the collision with the boundary. Denote $y = x - vt \in \partial\Omega$ the point of the collision. We set

$$\mathcal{M}(\beta) = \{(x,v) \in \Omega \times V | M(y,v)n(y) \cdot v \geq \beta > 0\}. \tag{A.2}$$

Then if $(x,v) \in \mathcal{M}(\beta)$ and for $t_0 = \frac{4d}{\delta}$,

$$\inf_{x',v'} \inf_{(x,v)\in\mathcal{M}(\beta)} P_{t_0}(x',v';x,v) \geq \gamma > 0. \tag{A.3}$$

Here δ denotes the modulus of the smallest velocity, while d is the diameter of Ω.

Inequality (A.3) is almost evident. Indeed tracing the forward trajectory $x' + tv'$ up to the instant (say t_1) of the collisions with the boundary, we denote $y_1 = x' + v't_1$ the hitting point.

Since $(x,v) \in \mathcal{M}(\beta)$ the transition $(y,v) \to (x,v)$ is performed with positive probability density. So we still have to connect the points y_1 and y within the remaining time $\tau = t_0 - t - t_1$. Note that $\frac{d}{\delta} \leq \tau \leq \frac{3d}{\delta}$. The connection can be done by choosing a sequence of intermediate points y_i on the boundary, $i = 2, \dots k$, such that $|y_i - y_{i-1}| = d/2$ and the trajectory $y_1 \to y_2, \dots y_k \to y$ is performed with velocities $v_1 \dots v_k$ such that $|v_i| \in (\delta, 2\delta)$.

As a consequence of (A.3) and the fact that $\mathcal{M}(\beta)$ has a positive measure, we have:

$$\inf_{x',v';x,v} \int_{\mathcal{M}(\beta)} \min(P_{t_0}(x',v';y,w), P_{t_0}(x,v;y,w))dydw \geq \varepsilon > 0. \tag{A.4}$$

Furthermore setting $(z = (x,v))$:

$$P(z',z) = P_{t_0}(x',v';x,v), \quad Sf(z) = \int dz' P(z',z)f(z') = S(t_0)f(z), \tag{A.5}$$

we construct a joint representation SR of Sf and Sg, in terms of a joint representation R of f and g, by putting:

$$\int P(z',Z';z,Z)R(z',Z')dz'dZ' = SR(z;Z) \tag{A.6}$$

where:

$$P(z',Z';z,Z) = \lambda(z;z',Z')\delta(z-Z) +$$

$$\frac{(P(z',z)-\lambda(z;z',Z'))(P(Z',Z)-\lambda(Z;z',Z'))}{1-\int dz\lambda(z;z',Z')}. \tag{A.7}$$

Inserting (A.7) in (A.6) it is easy to verify that SR is indeed a joint representation of Sf and Sg, provided that R is a joint representation of f and g. Introducing now the discrete distance d, namely $d(z,Z)=1$ if $z\neq Z$ and $d(z,Z)=0$ if $Z=z$, by the inequality:

$$1-\int dz\lambda(z;z',Z')\leq 1-\varepsilon \tag{A.8}$$

which is a consequence of (A.4), we easily find that:

$$\int d(z,Z)SR(z,Z)\leq(1-\varepsilon)\int d(z,Z)R(z,Z). \tag{A.9}$$

The above inequality can be iterated and since

$$\|Sf-Sg\|_{L_1}\leq\int d(z,Z)SR(z,Z), \tag{A.10}$$

we get

$$\|S^n f-S^n g\|_{L_1}\leq e^{-Cn} \tag{A.11}$$

where the constant C is independent of $f\in L_1(\Omega\times V)$.

Finally we conclude that there exists a unique stationary probability density $\bar{g}$ for which:

$$\|S(t)f-\bar{g}\|_{L_1}\leq e^{-bt} \tag{A.12}$$

where $b=\frac{C}{t_0}$.

Thus (A.12) implies (2.15). $\square$

Proof of Theorem 2.2. Let $p=p(x,v,t)$ and $l=l(x,v,t)$ be two solutions of the initial boundary value problem (2.1-3), with initial conditions p_0 and l_0 respectively. Writing the evolution equation in mild form (2.10), we have:

$$p(t)-l(t)=S(t)(p_0-l_0)+\lambda\int_0^t ds\,S(t-s)Q(p(s)+l(s),p(s)-l(s)) \tag{A.13}$$

where $Q(f,g)$ is the symmetrized collision operator (2.4).

By (A.12) and the same argument leading to (3.11) ,we have that, if $h=h(x,v)$ has the property $\int h=0$, then

$$\|S(t)h\|_{L_1}\leq e^{-bt}\|h\|_{L_1}. \tag{A.14}$$

Since $\int Q(f,g)=0$ for any pair of functions f and g, we have (cf. (2.38)):

$$\|p(t)-l(t)\|_{L_1}\leq e^{-bt}\|p_0-l_0\|_{L_1}+2a\lambda\int_0^t e^{-b(t-s)})\|p(s)-l(s)\|_{L_1}, \tag{A.15}$$

so that, using the Gronwall lemma:

$$\|p(t)-l(t)\|_{L_1}\leq\|p_0-l_0\|_{L_1}e^{-(b-2a\lambda)t}. \tag{A.16}$$

Therefore, if $\lambda < \frac{b}{2a}$, there exists a probability density g which is the unique global attracting point for the flow described by eq. (2.1) and also the unique invariant solution for such evolution problem. $\square$

Remark. It is not hard to show that g solves the stationary equation $(G^{free})^* g + \lambda Q(g,g) = 0$ and also the boundary value problem (2.13). In particular the trace of g on the boundary does exist. These considerations are not relevant for the present analysis so that we do not go further.

Proof of Theorem 2.3. We follow closely ref. [GLP] where the same result has been obtained in a more difficult context.

The existence of $\tilde{f}^N$ is obvious by the compactness of the state space of the process. Indeed the ergodic mean is weakly relatively compact and any cluster point cannot fail to be invariant. The uniqueness is a consequence of the same arguments used for the Knudsen flow. Indeed, it is enough to observe that for a fixed time t, the probability of each particle of the system to perform a collisionless motion is strictly positive.

References

[AN] L.Arkeryd, A.Nouri, *A compactness result related to the stationary Boltzmann equation in a slab, with applications to the existence theory*, to appear, Indiana Univ. Math. J. (1995).

[B] H.Babovsky, *Time averages of simulation schemes as approximations to stationary kinetic equations*, Eur. J. Mech., B/Fluids **11** (1992), 199-212.

[BI] H.Babovsky, R.Illner, *A convergence proof for Nanbu's simulation method for the full Boltzmann equation*, SIAM J. Numer. Anal. **26** (1989), 45-65.

[BS] A.V.Bobylev, J.Struckmeier, *Numerical simulation of the stationary one-dimensional Boltzmann equation by particle methods*, Eur. J. Mech., B/Fluids **15** (1996), 103-118.

[C] C.Cercignani, *The Grad limit for a system of soft spheres*, Comm. Pure Appl. Math. **36** (1983), 479-494.

[CDPP] S.Caprino, A.De Masi, E.Presutti, M.Pulvirenti, *A derivation of the Broadwell equation*, Comm. Math. Phys. **135** (1991), 443–465.

[CIP] C.Cercignani, R.Illner and M.Pulvirenti, *The mathematical theory of dilute gases*, Springer series in Appl. Math., vol. 106, 1994.

[CP] S.Caprino, M.Pulvirenti, *The Boltzmann-Grad limit for a one-dimensional Boltzmann equation in a stationary state*, Comm. Math. Phys. **177** (1996), 63-81.

[GLP] S.Goldstein, J.L.Lebowitz and E.Presutti, *Mechanical systems with stochastic boundaries*, Random Fields, Esztergom, Colloquia Mathematica Societatis Janos Bolyai 27 (1981), 403-419.

[GM] C.Graham, S.Méléard, *Convergence rate on path space for stochastic particle approximations to the Boltzmann equation*, Preprint (Laboratoire de Probabilités, Université Paris VI) 249 (1994).

[L] O.Lanford III, *The evolution of large classical system*, Lect. Notes in Physics. Springer, (J.Moser ed.), vol. 35, 1975, pp. 1-111.

[LP] M.Lachowicz, M.Pulvirenti, *A stochastic system of particles modelling the Euler equation*, Arch. Rat. Mech. Anal. **109** (1990), 81-93.

[PWZ] M.Pulvirenti, W.Wagner, M.B.Zavelani Rossi, *Convergence of particle schemes for the Boltzmann equation*, Eur. J. Mech., B/Fluids **13** (1994), 339–351.

[W] W.Wagner, *A convergence proof for Bird's direct simulation Monte Carlo method for the Boltzmann equation*, J. Statist. Phys. **66** (1992), 1011-1044.

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