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Klaus R. Schneider[1], Alexander Grin[2]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: klaus.schneider@wias-berlin.de

[2] Yanka Kupala State University of Grodno
Ozheshko Street 22
230023 Grodno
Belarus
E-Mail: grin@grsu.by

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We present a new approach for the global bifurcation analysis of limit cycles for a generalized van der Pol system. It is based on the existence of a Dulac-Cherkas function and on applying two topologically equivalent systems: one of them is a rotated vector field, the other one is a singularly perturbed system.

1 Introduction

We consider a class of planar autonomous differential systems

$$\frac{dx}{dt} = P(x, y, \lambda),\ \frac{dy}{dt} = Q(x, y, \lambda) \tag{1.1}$$

depending on the scalar parameter λ. Our goal is to derive conditions on P and Q such that system (1.1) has a unique limit cycle Γ_λ if λ belongs to some given interval $\Lambda = \{\lambda \in \mathbb{R} : \lambda_0 < \lambda < \lambda_1 \leq +\infty\}$. Additionally, we require that Γ_λ is ßtableïn the sense that Γ_λ is äsymptotically orbitally stable"with respect to the time t and also ßtructurally stable"with respect to small perturbations of P and Q. That means, we are looking for conditions guaranteeing the existence of a family of hyperbolic orbitally stable limit cycles Γ_λ of system (1.1) for $\lambda \in \Lambda$.
Our approach to treat this problem is based on the bifurcation theory of planar autonomous systems [1, 5, 6, 7, 10, 11] and on the existence of a Dulac-Cherkas function [4] which implies that any limit cycle of system (1.1) is hyperbolic, thus, there is no multiple limit cycle.
We recall that $\lambda = \lambda_b$ is said to be a bifurcation point of system (1.1) if there is some sufficiently small open interval $\tilde{\Lambda}$ having λ_b as boundary point such that the phase portraits of system (1.1) for $\lambda = \lambda_b$ and $\lambda \in \tilde{\Lambda}$ are topologically different.
The existence of a Dulac-Cherkas function $\Psi(x, y, \lambda)$ for system (1.1) implies that there is no bifurcation point connected with the bifurcation of limit cycles from a multiple limit cycle. If we additionally assume that the level set

$$\mathcal{W}_\lambda := \{(x, y) \in \mathbb{R}^2 : \Psi(x, y, \lambda) = 0\}$$

consists for $\lambda \in \Lambda$ of a unique simple closed curve (we call it oval), then there exists at most one limit cycle of system (1.1) in the phase plane, and if there is a limit cycle, it must surround the oval.
In order to exclude bifurcation points which are related to the bifurcation of a limit cycle from a homoclinic or heteroclinic orbit, we suppose that system (1.1) has only one equilibrium point in the finite part of the phase plane.
Under the assumption that system (1.1) has a unique equilibrium point and admits the existence of a Dulac-Cherkas function, it follows from the bifurcation theory of planar systems that there are exactly four types of bifurcation points connected with the bifurcation of a hyperbolic limit cycle:

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(i). bifurcation from an equilibrium point (Andronov-Hopf bifurcation),

(ii). bifurcation from a continuum of closed orbits,

(iii). bifurcation from infinity,

(iv). bifurcation from a discontinuous closed orbit (in case of a singularly perturbed system).

For our approach we additionally assume that the boundary points λ_0 and λ_1 of the interval Λ are bifurcation points and that there exists a planar system which is topologically equivalent to system (1.1) and represents a rotated vector field. The last assumption can be used to prove that there is no bifurcation point related with the bifurcation of a limit cycle from infinity.

Our paper is organized as follows. In Section 2 we introduce the notation of a Dulac-Cherkas function including its main properties and recall the definition of a rotated vector field and its importance for the bifurcation theory. A class of generalized van der Pol systems to which we apply our approach is described in Section 3. In Section 4 we derive important properties of the considered class of planar systems. The global bifurcation analysis of limit cycles together with the main result is presented in Section 5. Results on the bifurcation of a limit cycle of relaxation type in singularly perturbed systems are described in the Appendix.

2 Preliminaries

We denote by $X(\lambda)$ the vector field defined by system (1.1). For the following we assume $P, Q \in C^1(\mathbb{R}^2 \times \Lambda, \mathbb{R})$.

Definition 2.1. *A function $D \in C^1(\mathbb{R}^2 \times \Lambda, \mathbb{R})$ is called a Dulac function of system (1.1) in $\mathbb{R}^2$ for $\lambda \in \Lambda$ if $div(DX)$ does not change sign in $\mathbb{R}^2$ for $\lambda \in \Lambda$.*

Dulac functions can be used to establish the nonexistence of limit cycles or to estimate their maximum number in some connected regions [3]. The concept of Dulac functions has been extended by L. Cherkas [2]. The type of functions he has introduced is called Dulac-Cherkas function nowadays [4].

Definition 2.2. *A function $\Psi \in C^1(\mathbb{R}^2 \times \Lambda, \mathbb{R})$ is called a Dulac-Cherkas function of system (1.1) in $\mathbb{R}^2$ for $\lambda \in \Lambda$ if*

(i). The set

$$\mathcal{W}_\lambda := \{(x, y) \in \mathbb{R}^2 : \Psi(x, y, \lambda) = 0\} \tag{2.1}$$

does not contain a curve which is a trajectory of system (1.1).

(ii). There is a real number $k \neq 0$ such that

$$\Phi(x, y, \lambda, k) := (grad\Psi, X) + k\Psi divX \geq 0 \ (\leq 0) \quad \forall (x, y, \lambda) \in \mathbb{R}^2 \times \Lambda, \tag{2.2}$$

where the set

$$\mathcal{V}_\lambda := \{(x, y) \in \mathbb{R}^2 : \Phi(x, y, \lambda, k) = 0\}$$

has measure zero for $\lambda \in \Lambda$.

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The following two theorems describing the key properties of Dulac-Cherkas functions can be found in [4].

Theorem 2.3. *Let Ψ be a Dulac-Cherkas function of (1.1) in $\mathbb{R}^2$ for $\lambda \in \Lambda$. Then any limit cycle Γ_λ of (1.1) in $\mathbb{R}^2$ is hyperbolic and it is orbitally stable (unstable) if the condition $k\Phi\Psi < 0 (> 0)$ holds on Γ_λ.*

Theorem 2.4. *Let Ψ be a Dulac-Cherkas function of (1.1) in $\mathbb{R}^2$ for $\lambda \in \Lambda$ such that the set $\mathcal{W}_\lambda$ consists of s ovals in $\mathbb{R}^2$. Then in the case $k < 0$ system (1.1) has at most s limit cycles in $\mathbb{R}^2$ and any limit cycle surrounds an oval.*

In what follows we describe a class of parameter depending autonomous systems (1.1) exhibiting specific properties concerning the bifurcations and global behavior of limit cycles [9].

Definition 2.5. *System (1.1) is said to define a one-parameter family of negatively (positively) rotated vector fields for $\lambda \in \Lambda$ in $\mathbb{R}^2$ if the equilibria of system (1.1) are isolated and the inequality*

$$\Delta(x, y, \lambda) := P(x, y, \lambda)\frac{\partial Q(x, y, \lambda)}{\partial \lambda} - Q(x, y, \lambda)\frac{\partial P(x, y, \lambda)}{\partial \lambda} < 0 \quad (> 0) \tag{2.3}$$

holds at all ordinary points in $\mathbb{R}^2$ for $\lambda \in \Lambda$.

Remark 2.6. *This condition can be relaxed by assuming that Δ vanishes on a set of measure zero and that no closed curve of this set is a limit cycle of (1.1).*

Theorem 2.7. *Suppose that system (1.1) represents a one-parameter family of negatively rotated vector fields. Let $\{\Gamma_\lambda\}$ be a family of hyperbolic stable limit cycles of system (1.1) with positive (that means counterclockwise) orientation. Then Γ_λ contracts monotonically with decreasing λ, and the family terminates at $\lambda = \lambda_*$ where Γ_{λ_*} represents an equilibrium point.*

3 A class of generalized van der Pol systems

In what follows we describe the class of planar systems to which our approach for a global bifurcation analysis of limit cycles can be applied. We consider the Liénard system

$$\begin{aligned}\frac{dx}{dt} &= -y,\\ \frac{dy}{dt} &= x - \lambda(x^{2q} - 1)y\end{aligned} \tag{3.1}$$

in $\mathbb{R}^2$ for $\lambda \in \mathbb{R}$ and $q \in \mathcal{N}$. It is obvious that (3.1) has a unique equilibrium point in the finite part of the phase plane, namely the origin. The transformation $\lambda \to -\lambda, t \to -t, x \to -x$ leaves system (3.1) invariant. Hence, we can restrict ourselves to study system (3.1) for $\lambda \geq 0$. For $\lambda = 0$, system (3.1) represents a linear conservative system with the first integral $x^2 + y^2 = c^2$.
For $q = 1$ system (3.1) represents the famous van der Pol system

$$\begin{aligned}\frac{dx}{dt} &= -y,\\ \frac{dy}{dt} &= x - \lambda(x^2 - 1)y.\end{aligned} \tag{3.2}$$

It is well known that system (3.2) has the following properties [9]

(i). For $\lambda > 0$, system (3.2) has a unique limit cycle Γ_λ which is hyperbolic and stable.

(ii). The periodic solution $(x_\lambda(t), y_\lambda(t))$ of (3.2) corresponding to Γ_λ is for small λ sinusoidal like and for large λ of relaxation type.

Our goal is to to establish the same properties for the system (3.1) for $q \in \mathbb{N}$ by a global bifurcation analysis. For this purpose we first derive two systems which have the same phase portrait as system (3.1).
Using for $\lambda > 0$ the scaling

$$\bar{x} = \lambda^{\frac{1}{2q}} x, \quad \bar{y} = \lambda^{\frac{1}{2q}} y \tag{3.3}$$

then system (3.1) takes the form

$$\begin{aligned} \frac{d\bar{x}}{dt} &= -\,\bar{y}, \\ \frac{d\bar{y}}{dt} &= \bar{x} + \lambda \bar{y} - \bar{x}^{2q}\bar{y}. \end{aligned} \tag{3.4}$$

Since the transformation (3.3) is a diffeomorphism for $\lambda > 0$, system (3.1) and system (3.4) have the same topological structure of their trajectories for $\lambda > 0$. Especially, they have the same number of limit cycles with the same stability behavior. But we have to note that for $\lambda = 0$ the topological structure of the trajectories of system (3.1) and system (3.4) are different, in particular, system (3.1) is linear and conservative, whereas system (3.4) is nonlinear and has no nontrivial closed orbit.
Applying the transformation for $\lambda > 0$

$$t := \lambda\tau, \quad \eta := x, \quad \xi := -\frac{y}{\lambda} - x + \frac{x^{2q+1}}{2q+1} \tag{3.5}$$

we get from (3.1)

$$\begin{aligned} \frac{d\xi}{d\tau} &= -\eta, \\ \frac{1}{\lambda^2}\frac{d\eta}{d\tau} &= \xi + \eta - \frac{\eta^{2q+1}}{2q+1}. \end{aligned} \tag{3.6}$$

System (3.6) is of special interest in the case of large λ, since it represents a singularly perturbed system. The transformation (3.5) is a diffeomorphism, therefore, system (3.6) and system (3.1) are topologically equivalent for $\lambda > 0$.
Summarizing our investigations we have the following result

Lemma 3.1. *The systems (3.1), (3.4) and (3.6) are topologically equivalent for* $\lambda > 0$ *and* $q \in \mathbb{N}$.

4 Special properties of the class of generalized van der Pol systems

Lemma 4.1. *Systems (3.1), (3.4) and (3.6) have for* $\lambda \in \mathbb{R}$ *a unique equilibrium point in the finite part of the phase plane at the origin. It is an unstable focus for* $0 < \lambda < 2$ *and an unstable node for* $\lambda \geq 2$.

Theorem 4.2. *The function*

$$\Psi(x,y) \equiv x^2 + y^2 - 1 \tag{4.1}$$

is a Dulac-Cherkas function for system (3.1) in the phase plane for $\lambda > 0$.

Proof. From (3.1) we get

$$divX(\lambda) \equiv -\lambda(x^{2q} - 1), \quad (grad\Psi, X(\lambda)) \equiv 2\lambda(x^{2q} - 1)y^2.$$

By (2.2) we have for $k = -2$

$$\Phi(x, y, \lambda, -2) := (grad\Psi, X(\lambda)) - 2\Psi divX(\lambda)$$

$$= 2\lambda(x^2 - 1)^2(1 + x^2 + x^4 + ... + x^{2q-2}) \geq 0$$

for $\lambda > 0$, $(x, y) \in \mathbb{R}^2$, and $q \in \mathbb{N}$. According to Definition 2.2 , Theorem 4.2 is proved. □

Theorem 4.3. *System (3.1) has for $\lambda > 0$ at most one limit cycle Γ_λ. If Γ_λ exists, it is hyperbolic, stable and surrounds the unit circle.*

Proof. By (2.1) the zero-level set $\mathcal{W}_\lambda$ for the Dulac-Cherkas function Ψ defined in (4.1) consists of the unit circle. Thus, according to Theorem 2.4, system (3.1) has at most one limit cycle and, if it exists, it surrounds the unit circle. Since outside the unit cycle the relations $\Psi(x, y) > 0, \Phi(x, y, \lambda, -2) \geq 0$ hold, a possible limit cycle Γ_λ is stable according to Theorem 2.3. □

Corollary 4.4. *The systems (3.1), (3.4) and (3.6) have at most one limit cycle Γ_λ. If Γ_λ exist, it is hyperbolic, stable and positively oriented.*

Theorem 4.5. *If λ increases and crosses the value $\lambda = 0$, then Andronov-Hopf bifurcation takes place for system (3.4): a unique limit cycle $\hat{\Gamma}_\lambda$ bifurcates from the origin. $\hat{\Gamma}_\lambda$ is hyperbolic, stable and positively oriented.*

Proof. The linearization of system (3.4) at the origin takes the form

$$\begin{aligned} \frac{d\bar{x}}{dt} &= -\,\bar{y}, \\ \frac{d\bar{y}}{dt} &= \bar{x} + \lambda\bar{y}. \end{aligned} \tag{4.2}$$

The corresponding eigenvalues $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are conjugate complex for $\lambda > 0$ and can be written in the form

$$\mu_{1,2}(\lambda) = \frac{\lambda}{2} \pm i\frac{\sqrt{4-\lambda^2}}{2}. \tag{4.3}$$

Thus, the origin is an unstable focus for $0 < \lambda < 2$, it is an unstable node for $\lambda \geq 2$.

To study the stability and multiplicity of the origin for $\lambda = 0$ we represent the eigenvalues $\mu_{1,2}(\lambda)$ for $0 \leq \lambda < 2$ in the form

$$\mu_{1,2}(\lambda) = a(\lambda) \pm ib(\lambda), \quad a(\lambda) = \frac{\lambda}{2}, \quad b(\lambda) = \frac{\sqrt{4-\lambda^2}}{2}.$$

By means of the coordinate transformation

$$u = a(\lambda)\bar{x} + \bar{y}, \quad v = -b(\lambda)\bar{x}$$

system (3.4) takes the form

$$\begin{aligned} \frac{du}{dt} =& a(\lambda)u - b(\lambda)v - \frac{v^{2q}}{b(\lambda)^{2q}}\Big(u + \frac{a(\lambda)}{b(\lambda)}v\Big), \\ \frac{dv}{dt} =& b(\lambda)u + a(\lambda)v. \end{aligned} \tag{4.4}$$

Introducing polar coordinates

$$u = r\,\cos\varphi, \quad v = r\,\sin\varphi$$

system (4.4) is in a sufficiently small neighborhood of the origin and for $0 \leq \lambda < 2$ equivalent to the differential equation

$$\frac{dr}{d\varphi} = \sum_{i=1}^{\infty} k_i(\varphi,\lambda) r^i, \tag{4.5}$$

where

$$k_1(\varphi,\lambda) = \frac{a(\lambda)}{b(\lambda)} = \frac{\lambda}{\sqrt{4-\lambda^2}}. \tag{4.6}$$

For $\lambda = 0$ we have

$$\frac{dr}{d\varphi} = -r^{2q+1}(\sin\varphi)^{2q}(\cos\varphi)^2 + O(r^{4q+1}) \tag{4.7}$$

which implies

$$k_1(\varphi,0) \equiv 0, k_2(\varphi,0) \equiv 0, ..., k_{2q+1}(\varphi,0) \equiv -(\sin\varphi)^{2q}(\cos\varphi)^2. \tag{4.8}$$

We denote by $r(\varphi,\lambda,r_0)$ the solution of (4.5) satisfying $r(0,\lambda,r_0) = r_0 > 0$. For sufficiently small r_0, this solution can be represented as

$$r(\varphi,\lambda,r_0) = \sum_{i=1}^{\infty} h_i(\varphi,\lambda) r_0^i. \tag{4.9}$$

Substituting (4.9) into (4.5), we get a system of differential equations for determining the functions h_i. Using the initial conditions $h_1(0,\lambda) = 1, h_i(0,\lambda) = 0$ for $i = 2, ...$, the functions $h_i(\varphi,\lambda)$ can be determined uniquely. We obtain

$$\frac{dh_1}{d\varphi} = k_1(\varphi,\lambda)h_1,$$

which implies by (4.6)

$$h_1(\varphi,\lambda) = \exp\Big(\varphi\frac{\lambda}{\sqrt{4-\lambda^2}}\Big), \tag{4.10}$$

for $\lambda = 0$ we have by (4.8)

$$\frac{dh_1}{d\varphi} \equiv 0, \frac{dh_2}{d\varphi} \equiv 0, ..., \frac{dh_{2q+1}}{d\varphi} = k_{2q+1}(\varphi,0) = -(\sin\varphi)^{2q}(\cos\varphi)^2.$$

Thus, it holds

$$h_1(2\pi,0)=1, h_2(2\pi,0)=0,..., h_{2q}(2\pi,0)=0, h_{2q+1}(2\pi,0)=-\int_0^{2\pi}(\sin\varphi)^{2q}(\cos\varphi)^2 d\varphi<0. \tag{4.11}$$

The number of positive zeros of the function $d(r_0,\lambda):=r(2\pi,\lambda,r_0)-r_0$ determines the number of the limit cycles of system (3.4) near the origin. For sufficiently small r_0, $d(r_0,\lambda)$ can be represented in the form

$$d(r_0,\lambda)=\sum_{i=1}^{\infty}\alpha_i(\lambda)r_0^i. \tag{4.12}$$

For system (3.4) we get by (4.10)

$$\alpha_1(\lambda)=h_1(2\pi,\lambda)-1=\exp\left(2\pi\frac{\lambda}{\sqrt{4-\lambda^2}}\right)-1. \tag{4.13}$$

The coefficients $\alpha_i(0)$ are called Lyapunov numbers of the origin. They are determined by the relations

$$\alpha_1(0)=h_1(2\pi,0)-1,\quad \alpha_i(0)=h_i(2\pi,0),\quad i=2,.... \tag{4.14}$$

The first non-vanishing Lyapunov number determines the multiplicity of the origin. For system (3.4) we have

$$\alpha_1(0)=0,\quad \alpha_2(0)=0,...,\alpha_{2q}(0)=0,\quad \alpha_{2q+1}(0)=-\int_0^{2\pi}(\sin\varphi)^{2q}(\cos\varphi)^2 d\varphi<0, \tag{4.15}$$

that is, the origin is a focus of multiplicity q of system (3.4) for $\lambda=0$. Introducing the function

$$g(r_0,\lambda):=\sum_{i=1}^{\infty}\alpha_i(\lambda)r_0^{i-1}, \tag{4.16}$$

the function $d(r_0,\lambda)$ in (4.12) can be represented as

$$d(r_0,\lambda)=r_0 g(r_0,\lambda). \tag{4.17}$$

From (4.16) and (4.13) we get

$$g(0,0)=0,\quad g'_\lambda(0,0)=\alpha'_1(0)=\pi.$$

By the implicit function theorem there is a smooth function $h:\mathbb{R}_+\to\mathbb{R}$ such that $\lambda=h(r_0)$ solves the equation $g(r_0,\lambda)=0$ and satisfies

$$h(0)=0,\quad h'(0)=-\frac{g'_{r_0}(0,0)}{g'_\lambda(0,0)}=0.$$

From $g(h(r_0),r_0)\equiv 0$ and (4.15) we obtain

$$h''(0)=0,...,h^{(2q)}(0)=0,\quad h^{(2q+1)}(0)=-\frac{g_{r_0}^{(2q+1)}(0,0)}{g'_\lambda(0,0)}=-\frac{\alpha_{2q+1}(0)}{\pi}>0.$$

Thus, to each sufficiently small $r_0>0$ there exists a unique positive value $\lambda=h(r_0)$ such that $g(h(r_0),r_0)=0$. By (4.17) and Corollary 4.4 we can conclude that to each sufficiently small $\lambda>0$ there exists a unique small hyperbolic stable limit cycle Γ_λ of system (3.4) which is positively oriented. $\square$

Corollary 4.6. *According to Lemma, 3.1 system (3.1) has for sufficiently small λ a unique limit cycle which is hyperbolic, stable and positively oriented.*

Theorem 4.7. *System (3.4) represents a negatively rotated vector field.*

Proof. If we use the notation of system (1.1) for system (3.4) we obtain

$$P(\bar{x}, \bar{y}, \lambda) \equiv -\bar{y}, \quad Q(\bar{x}, \bar{y}, \lambda) \equiv \bar{x} + \lambda\bar{y} - \bar{x}^{2q}\bar{y}.$$

By (2.3) we have

$$\Delta(x, y, \lambda) \equiv -\bar{y}^2 \leq 0,$$

that is, by Definition 2.5 and Remark 2.6, system (3.4) represents a one-parameter family of negatively rotated vector fields. □

Remark 4.8. *We note that the systems (3.1) and (3.6) do not have the property of a rotated vector field. This property is not invariant under topological equivalence.*

As we already noted, system (3.6) represents for $\lambda \gg 1$ a singularly perturbed system. For this system we verify the validity of the assumptions $(C_1) - (C_3)$ in the Appendix for $\varepsilon = \frac{1}{\lambda^2}$. Identifying (3.6) and (6.1) we have

$$f(\xi, \eta) \equiv -\eta, \quad g(\xi, \eta) \equiv \xi + \eta - \frac{\eta^{2q+1}}{2q+1}. \tag{4.18}$$

Hence, assumption (C_1) is fulfilled. Also, the condition (C_2) is obviously valid. Setting $g(\xi, \eta) = 0$ in (4.18) we get

$$\xi = \varphi(\eta) := -\eta + \frac{\eta^{2q+1}}{2q+1}$$

satisfying $\varphi(0) = 0, \varphi'(0) = -1$. The equation $\varphi'(\eta) = 0$ has exactly two real roots $\eta = \pm 1$, where $\varphi''(-1) < 0, \varphi''(+1) > 0$. Thus, the conditions $(C_1) - (C_3)$ of Theorem 6.1 in the Appendix are satisfied. Applying this theorem to system (3.6) we obtain

Theorem 4.9. *System (3.6) has for sufficiently large $\lambda > 0$ a unique limit cycle $\bar{\Gamma}_\lambda$ which is hyperbolic, stable, positively oriented and located near the closed curve Z_0.*

5 Global bifurcation analysis

From Theorem 4.5 it follows that system (3.4) has for sufficiently small λ a hyperbolic stable limit cycle $\hat{\Gamma}_\lambda$ which shrinks to the origin as λ tends to zero. According to Theorem 4.7, system (3.4) represents a negatively rotated vector field. By Theorem 2.7 we can conclude that $\hat{\Gamma}_\lambda$ expands monotonically with increasing λ. Hence, there exists the possibility that $\hat{\Gamma}_\lambda$ tends to infinity if λ increases to some finite value $\bar{\lambda}$. In order to ensure that $\hat{\Gamma}_\lambda$ exists for any $\lambda > 0$, we consider the equivalent system (3.6). By Theorem 4.9, system (3.6) has for sufficiently large λ a unique stable hyperbolic limit cycle $\bar{\Gamma}_\lambda$. Since system (3.4) is topologically equivalent to system (3.6), we can conclude that system (3.4) has for sufficiently large λ a hyperbolic stable limit cycle $\tilde{\Gamma}_\lambda$ which is positively oriented. Taking into account that system (3.4) represents a rotated vector field, we obtain that $\tilde{\Gamma}_\lambda$ contracts monotonically as λ is decreasing. Since the origin is an unstable equilibrium point of system (3.4) for $\lambda > 0$, there is no

Andronov-Hopf bifurcation point for $\lambda > 0$, and we can conclude that $\tilde{\Gamma}_\lambda$ for decreasing λ terminates at an Andronov-Hopf bifurcation from the origin for $\lambda = 0$. From the property that system (3.4) has for all $\lambda > 0$ at most one limit cycle we get that the limit cycle $\hat{\Gamma}_\lambda$ of system (3.4) coincides with the limit cycle $\tilde{\Gamma}_\lambda$. Therefore, there exists a family Γ_λ of hyperbolic stable limit cycles of system (3.4) for $\lambda > 0$. Using the equivalence of the systems (3.1), (3.4) and (3.6) we have the result

Theorem 5.1. *System (3.1) has for all positive λ a unique limit cycle Γ_λ which is hyperbolic and stable.*

Summarizing our investigations we can state that the use of Dulac-Cherkas functions combined with transformations which preserve the topological structure of the phase portrait simplify the global bifurcation analysis of limit cycles.

6 Appendix. Bifurcation of a limit cycle in a singularly perturbed system

Consider the singularly perturbed system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \varepsilon\frac{dy}{dt} &= g(x,y) \end{aligned} \tag{6.1}$$

under the following assumptions
(C_1). *$f, g \in C^2(\mathbb{R}^2, \mathbb{R})$, ε is a small positive parameter.*
(C_2). *The origin is the unique equilibrium point of system* (6.1) *in the finite part of the phase plane. It is unstable for $\varepsilon > 0$. The trajectories surrounding the origin are positively oriented.*
(C_3). *$g(x,y) = 0$ has the unique simple solution $x = \varphi(y)$, where $\varphi \in C^2(\mathbb{R}^2, \mathbb{R})$ and satisfies*

$$\varphi(0) = 0, \varphi'(0) < 0.$$

$\varphi'(y) = 0$ has exactly two real roots y_- and y_+ satisfying

$$y_- < 0,\ \varphi''(y_-) < 0,\ y_+ > 0,\ \varphi''(y_+) > 0.$$

Using assumption (C_3) we can define a closed curve $\mathcal{Z}_0$ in the phase plane consisting of two finite segments of the curve $x = \varphi(y)$ bounded by the points $D = (y_{--}, \varphi(y_+)), A = (y_-, \varphi(y_-))$ and $C = (y_+, \varphi(y_+)), B = (y_{++}, \varphi(y_-))$ (see Fig.1) and of two finite segments of the straight lines $x = \varphi(y_-)$ and $x = \varphi(y_+)$ bounded by the points A, B and D, C respectively (see Fig.1).

The following theorem is a special case of a more general theorem by E.F. Mishchenko and N. Kh. Rozov in [8].

Theorem 6.1. *Under the assumptions $(C_1) - (C_3)$, system (6.1) has for sufficiently small ε a unique limit cycle Γ_ε in a small neighborhood of $\mathcal{Z}_0$ which is hyperbolic, stable and positively oriented.*

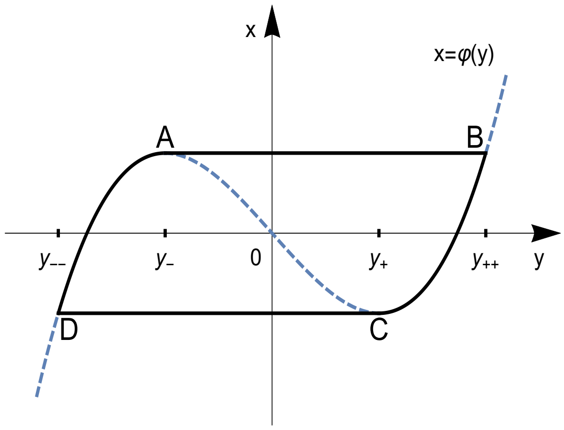

Fig.1. Closed curve $\mathbb{Z}_0$.

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